

NOTES ON ANALYSIS

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1. THE REAL NUMBERS.

1.1. **Fields.** A field is an algebraic object where we can do the usual operations of high school algebra. That is addition, subtraction, multiplication, and division.

Definition 1. A *field* is a set F with operations¹ $+$, called *addition*, and \cdot , called *multiplication*, such that

- (a) both operations are *associative*:

$$(x + y) + z = x + (y + z) \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

for all $x, y, z \in F$.

- (b) both operations are *commutative*:

$$x + y = y + x \quad x \cdot y = y \cdot x$$

for all $x, y \in F$.

- (c) Multiplication *distributes over addition*:

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

for all $x, y, z \in F$.

- (d) There are additive and multiplicative *identities*. That is there are $0 \in F$ and $1 \in F$ such that

$$x + 0 = x \quad x \cdot 1 = x$$

for all $x \in F$.

- (e) Every element has an *additive inverse*. That is for every $x \in F$ there is an element $y \in F$ such that

$$x + y = 0.$$

- (f) Every nonzero element has a *multiplicative inverse*. There is for $x \in F$, with $x \neq 0$, there is an element $z \in F$ such that

$$xz = 1.$$

- (g) F has at least two elements.

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¹To be a bit more precise we should call these *binary operations* in that they take an ordered pair of elements of F , say (x, y) , and each gives a unique output $x + y$ or $x \cdot y$.

This definition requires a bit of comment. First as to the additive identity the definition as it stands does not rule out the possibility that there are two additive identities. that is there are $0, 0' \in F$ with

$$x + 0 = x \quad \text{and} \quad x + 0' = x$$

for all $x \in F$. In this case

$$\begin{aligned} 0' &= 0' + 0 && (x + 0 = x \text{ with } x = 0') \\ &= 0 + 0' && (\text{addition is commutative}) \\ &= 0 && (x + 0' = x \text{ with } x = 0). \end{aligned}$$

So 0 and $0'$ are the same element.

Problem 1. Use a variant on this argument to show that if $1, 1' \in F$ satisfy

$$x \cdot 1 = x \quad \text{and} \quad x \cdot 1' = x$$

for all $x \in F$ that $1 = 1'$. Thus the multiplicative identity is unique. \square

We also have that additive inverses are unique. Let $x \in F$ and assume that $y, y' \in F$ such that

$$x + y = 0 \quad \text{and} \quad x + y' = 0.$$

Then

$$\begin{aligned} y' &= y' + 0 && (x + 0 = x \text{ with } x = y') \\ &= y' + (x + y) && (x + y = 0) \\ &= (y' + x) + y && (\text{associativity of addition}) \\ &= (x + y') + y && (\text{commutative of addition}) \\ &= 0 + y && (x + y' = 0) \\ &= y + 0 && (\text{commutative of addition}) \\ &= y && (x + 0 = x \text{ with } x = y) \end{aligned}$$

Thus the additive inverse of any element any element is unique. From now on we denote the additive inverse of $x \in F$ as $-x$ and use the abbreviation

$$x - y := x + (-y).$$

Proposition 2. For any $x \in F$ the equality

$$-(-x) = x$$

holds.

Proof. By definition $-(-x)$ is the additive inverse of $-x$. But we also have

$$\begin{aligned} -x + x &= x + (-x) && (\text{commutative of additive}) \\ &= 0 && (-x \text{ is additive inverse of } x) \end{aligned}$$

This shows that x is also an additive inverse of $-x$. As additive inverses are unique we have $-(-x) = x$. \square

Problem 2. Modify the argument above to show that the multiplicative inverse of $x \in F$ with $x \neq 0$ is unique. \square

If $x \in F$ and $x \neq 0$ we now denote the unique multiplicative inverse of x by either of the two notations.

$$\text{multiplicative inverse of } x = \frac{1}{x} = x^{-1}.$$

and write

$$yx^{-1} := \frac{y}{x}.$$

Problem 3. Modify one of the arguments above to show if $x \in F$ with $x \neq 0$ then

$$(x^{-1})^{-1} = x.$$

That is

$$\frac{1}{\left(\frac{1}{x}\right)} = x. \quad \square$$

Here are several results that we are so use to seeing that it seems irritating to have to prove them.

Proposition 3. In a field $-0 = 0$.

Problem 4. Prove this. *Hint:* $0 + 0 = 0$ so 0 is the additive inverse of 0 . \square

Problem 5. In a field, F ,

$$x \cdot 0 = 0$$

for all $x \in F$.

Problem 6. Prove this. *Hint:* First show $x \cdot 0 = x \cdot 0 + x \cdot 0$ by justifying the steps in the following.

$$\begin{aligned} x \cdot 0 &= x \cdot (0 + 0) \\ &= x \cdot 0 + x \cdot 0. \end{aligned}$$

Now add the additive inverse of $x \cdot 0$ to both sides of $x \cdot 0 = x \cdot 0 + x \cdot 0$. \square

The associativity law implies that for any three elements $x_1, x_2, x_3 \in F$ that

$$(x_1 x_2) x_3 = x_1 (x_2 x_3).$$

As this is the only two ways to group the product of three elements we can write the product of three elements as

$$x_1 x_2 x_3$$

without ambiguity. There are five ways to group four elements in a product

$$x_1(x_2(x_3 x_4)), x_1((x_2 x_3) x_4), (x_1 x_2)(x_3 x_4), (x_1(x_2 x_3)) x_4, ((x_1 x_2) x_3) x_4$$

These are all equivalent. We see this by showing they are all the same as $x_1(x_2(x_3x_4))$.

$$\begin{aligned}
x_1((x_2x_3)x_4) &= x_1(x_2(x_3x_4)) && \text{as } (x_2x_3)x_4 = x_2(x_3x_4) \\
(x_1x_2)(x_3x_4) &= x_1(x_2(x_3x_4)) && \text{as } (x_1x_2)y = x_1(x_2y) \text{ with } y = x_3x_4 \\
(x_1(x_2x_3))x_4 &= x_1((x_2x_3)x_4) && \text{as } (x_1y)x_4 = x_1(yx_4) \text{ with } y = x_2x_3 \\
&= x_1(x_2(x_3x_4)) && \text{as } (x_2x_3)x_4 = x_2(x_3x_4) \\
((x_1x_2)x_3)x_4 &= (x_1x_2)(x_3x_4) && \text{as } (yx_3)x_4 = y(x_3x_4) \text{ with } y = x_1x_2 \\
&= x_1(x_2(x_3x_4)) && \text{as } (x_1x_2)y = x_1(x_2y) \text{ with } y = x_3x_4
\end{aligned}$$

So again we can write the product

$$x_1x_2x_3x_4$$

without ambiguity as all the groupings are equal. In light of this the following will most likely not surprise you.

Proposition 4. *Let x_1x_2, \dots, x_n be elements of the field. Then the associativity law implies that any two groupings of the product $x_1x_2 \cdots x_n$ are equal.*

Problem 7. Prove this. *Hint:* Use induction to show that any grouping is equal to the grouping

$$x_1(x_2(x_3(x_4 \cdots x_n) \cdots)).$$

This is the grouping where the parenthesis are moved as far to the right as possible. For the rest of this problem call this the **standard form** of the product.

Here is the induction step in going from $n = 5$ to $n = 6$. For $n = 5$ the standard form is

$$x_1(x_2(x_3(x_4x_5))).$$

Let p be some grouping of x_1, x_2, \dots, x_6 . We first consider the case that p is of the form

$$p = x_1(p_2)$$

where p_2 is a product of x_2, \dots, x_6 . Then p_2 is a product of $n = 5$ elements and thus by the induction hypothesis $p_2 = x_2(x_3(x_4(x_5x_6)))$. But then $p = x_1(p_2) = x_1(x_2(x_3(x_4(x_5x_6))))$ can be put in standard form.

This leaves the case where $p = (p_1)(p_2)$ where for some k with $2 \leq k \leq 5$ we have

$$p = (p_1)p_2$$

where p_1 is a product of x_1, \dots, x_k and p_2 is a product of x_{k+1}, \dots, x_6 . Then, as p_1 has less than $n = 6$ factors it can be put in standard. This implies that $p_1 = x_1(q)$ where q is a product of x_2, \dots, x_k . Therefore

$$p = (p_1)p_2 = (x_1q)p_2 = x_1(qp_2).$$

But qp_2 only involves the variables x_2, \dots, x_6 so another application of the induction hypothesis implies that qp_2 can be put in standard form. But then $p = x_1(qp_2)$ is in standard form.

To complete the proof you show that this argument can be used to show that if it is true for n variables, then it is true of $n + 1$ variables. \square

In light of this from now on we write products $x_1x_2 \cdots x_n$ without putting in the parenthesis. There is a similar proposition about parenthesis and sums, and we will also write sums as $x_1 + x_2 + \cdots + x_n$ without parenthesis.

The following could be summarized by saying that much of the basic results you know from basic algebra still holds in fields.

Proposition 5. *Let F be a field. Then*

(a) *If $a, b, c, d \in F$ and $b, c \neq 0$ then*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

(b) *If $a, b \in F$ and $ab = 0$, then $a = 0$ or $b = 0$.*

(c) *(This is just a useful restatement of part (b).) If $a, b \in F$ and $a, b \neq 0$ then $ab \neq 0$.*

(d) *If the elements $a_1, a_2, \dots, a_n \in F$ are all nonzero, then so is the product and*

$$(a_1a_2 \cdots a_{n-1}a_n)^{-1} = a_n^{-1}a_{n-1}^{-1} \cdots a_2^{-1}a_1^{-1}.$$

(e) *If $a, b \in F$ and $a^2 = b^2$, then $a = \pm b$.*

Problem 8. Prove this. \square

Problem 9. Here is another fact that will likely come up at least once during the term. Let a, b, c, d, e, f be elements of the field F with

$$ad - bc \neq 0.$$

Then the equations

$$ax + by = e$$

$$cx + dy = f$$

have a unique solution. This solution is

$$x = \frac{ed - bf}{ad - bc}, \quad y = \frac{af - ec}{ad - bd}.$$

Hint: To find x multiply the first equation by d and the second by b and then subtract the two. A similar trick works to find y . \square

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