Some consequences of the Cauchy integral formula.

Proposition 1. *Let $f(z)$ be analytic in the open set U. Then the derivative $f'(z)$ is also analytic.*

Problem 1. Prove this. *Hint:* Here is an outline of my favorite proof. Like may of the proves in complex analysis it is based on the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

where D is a bounded open set with nice boundary, f is analytic in D and continuous on the closure $\overline{D}$.

(a) To start verify the identities algebraic identities

$$\frac{1}{\zeta - (z+h)} - \frac{1}{\zeta - z} = \frac{h}{(\zeta - (z+h))(\zeta - z)}$$

$$\frac{1}{(\zeta - (z+h))^2} - \frac{1}{(\zeta - z)^2} = \frac{h(2(\zeta - z) - h)}{(\zeta - (z+h))^2}$$

(b) Use the first of these identities and the Cauchy integral formula to show that if $z, z+h \in D$

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - (z+h))(\zeta - z)}\, d\zeta$$

(c) Take the limit as $h \to 0$ in the previous equation to get the an integral formula for $f'(z)$:

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

(d) Use the second identity of Part (a) to and the formula for $f'(z)$ to show

$$\frac{f'(z+h) - f'(z_h)}{h} = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)(2(\zeta - z) - h)}{(\zeta - (z+h))^2}\, d\zeta$$

(e) Use this formula so show that $f''(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$ exists and has the integral formula

$$f''(z) = \frac{2}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta.$$

This shows f' is analytic. □

We discussed the identity theorem:

Theorem 2. *Let U be a connected open set in $\mathbb{C}$ and f, g analytic functions on U so that for some sequence $\langle a_n \rangle_{n=1}^{\infty}$ distinct points in U with $\lim_{n\to\infty} a_n = a$ for some $a \in U$ we have*

$$f(a_n) = g(a_n)$$

for all n. Then $f \equiv g$ in U. □

Problem 2. Let $D = \{z : |z| < 1\}$ and let

$$f(z) = e^{\frac{i}{1-z}} - 1$$

and

$$a_n = 1 - \frac{1}{2\pi n}$$

for $n = 1, 2, \ldots$. Show

$$f(a_n) = 0$$

and that

$$\lim_{n\to\infty} a_n = 1.$$

Why does this not contradict the identity theorem? □

Problem 3. Does there exist an analytic function $f(z)$ on the unit disk $D = \{z : |z| < 1\}$ such that for $n = 2, 3, 4, \ldots$

(a)

$$f(1/n) = \frac{3}{n^2}.$$

If it exists is it unique?

(b)

$$f(1/n) = \frac{1+n}{2+n}$$

If it exists is it unique?

(c)

$$f(1/n) = \frac{(-1)^n}{n^2}$$

If it exists is it unique? □

Problem 4. Let f be an analytic function on the unit disk D such that at each point $z \in D$ at least one of the derivatives of f vanishes. (That is for each $z \in D$ there is a $n = n_z$ such that $f^{(n)}(z) = 0$.) Show f is a polynomial. □