ANALYSIS QUALIFYING EXAM JANUARY 9, 1991.

Throughout this exam, unless otherwise specified, the terms measurable, a.e.,refer to the Lebesgue measure λ (or dx) on the real line $\mathbb{R}$, and L^p of an interval to L^p of that interval with respect to Lebesgue measure on that interval.

1. a. Let μ be a finite Borel measure on $\mathbb{R}$. Prove that there exists a smallest closed set F, denoted by $\operatorname{supp}(\mu)$, such that $\mu(F) = \mu(\mathbb{R})$.

   b. Let F be a closed subset of $\mathbb{R}$. Prove that there exists a finite Borel measure μ such that $F = \operatorname{supp}(\mu)$. (Hint: Consider a countable dense subset of F.)

2. Let f be a bounded and uniformly continuous function on $\mathbb{R}$ and G a continuous function on $\mathbb{R}$ such that $G(0) = 0$ and $G(x) > 0$ if $x \neq 0$. Prove that if

$$\int_{-\infty}^{\infty} G(f(x))dx < \infty,$$

then $\lim_{x\to\infty} f(x) = 0$.

3. Let $f \in L^1(\mathbb{R})$.

   a. Prove that

$$\sum_{n=-\infty}^{\infty} f(\frac{x}{2} + n)$$

   converges absolutely a.e.

   b. Let $F(x)$ denote the sum of this series (where we put $F(x) = 0$ whenever the series diverges). Prove that $F \in L^1([0,2])$ and that

$$\frac{1}{2}\int_0^2 F(x)dx = \int_{-\infty}^{\infty} f(x)dx.$$

4. Let $E \subset [0,1]$ be a measurable set. Define

$$f_n(x) = n\int_0^{\frac{1}{n}} \chi_E(x+t)dt.$$

   a. Prove that each f_n is absolutely continuous on $\mathbb{R}$.

   b. Prove that $f_n(x) \to \chi_E(x)$ a.e. as $n \to \infty$.

   c. Prove that

$$\int_{\mathbb{R}} |f_n - \chi_E|dx \to 0$$

   as $n \to \infty$.

5. Suppose that $f \in L^p((0,\infty))$, where $1 < p < 2$. Let

$$\phi(y) = \int_0^{\infty} f(x)\frac{\sin xy}{\sqrt{x}}dx.$$

   a. Prove that $\phi(y)$ is finite everywhere.

   b. Prove that $y^{\frac{1}{2}-\frac{1}{p}}\phi(y) \to 0$ as $y \to 0$. (Hint: Consider the integrals over $[0,M]$ and $[M,\infty]$ separately, where M is appropriately large.)

6. Let $f_n \in L^p[0,1]$, where $1 < p < \infty$, such that $\|f_n\|_p \leq 1$ and $f_n(x) \to 0$ a.e. Prove that $\int f_n(t)g(t)dt \to 0$ for all $g \in L^{p'}[0,1]$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

7. Let μ and ν be two finite measures on a measurable space $(X, \mathcal{B})$. Prove that there exist a partition (A_1, A_2, A_3) of X and a measurable function f defined on A_3 such that
(i) $\mu(A_1) = 0$,
(ii) $\nu(A_2) = 0$,
(iii) $\mu(E) = \int_E f d\nu$ and $\nu(E) = \int_E \frac{1}{f} d\mu$ for every measurable $E \subset A_3$.

8. Let $f \in L^1(\mathbb{R}, \lambda)$ and $g \in L^1(\mathbb{R}, \lambda) \cap L^\infty(\mathbb{R}, \lambda)$.
   a. Prove that
   $$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$
   exists for almost all x and that $f * g \in L^1(\mathbb{R})$. (You can assume the measurablity of the integrand as an function of (x,t))
   b. Show that there exists a uniformly continuous function $h(x)$ such that $h(x) = f * g(x)$ a.e.
   c. Deduce that $h(x) \to 0$ as $|x| \to \infty$

9. <u>True or False. Prove or give a counterexample</u>
   a. If f is continuous a.e. on $[a,b]$, then there exists a continuous function g on $[a,b]$ such that $f = g$ a.e.
   b. If f is absolutely continuous on $[a,b]$ and $f' = g$ a.e, where g is a continuous function on $[a,b]$, then $f' = g$ everywhere on $[a,b]$.
   c. If $f' = 0$ a.e., then f is of bounded variation.
   d. If f is continuous on $[0,1]$ and $\int_0^1 t^n f(t)dt = \frac{1}{n+2}$ for all $n \geq 0$, then $f(t) = t$ for all $t \in [0,1]$.
   e. If $f_n \in L^1[0,1]$ such that $\|f_n\|_1 \leq 1$ and $f_n(x) \to 0$ a.e., then $\int f_n(t)g(t)dt \to 0$ for all $g \in L^\infty[0,1]$.