Some group and ring theory problems.

Problem 1. This is Problem 11 off of McNulty's list of group theory problems: Prove that a group G cannot have four distinct proper subgroups H_0, H_1, H_2, and H_3 so that $H_0 \le H_1 \le H_2 \le G$, $H_0 = H_2 \cap H_3$, and $H_1H_3 = G$. *Hint:* Under these assumptions show that $H_1 = H_2$, which will contradict that the subgroups are distinct. As $H_1 \le H_2$ it is enough to show $H_2 \le H_1$. So let $h_2 \in H_2$ with the goal of showing $h_2 \in H_1$. As $G = H_1H_3$ we have

$$h_2 = h_1h_3 \quad \text{for some } h_1 \in H_1 \text{ and } h_3 \in H_3.$$

Then show

$$h_3 = h_1^{-1}h_2 \in H_2 \cap H_3 = H_0 \subseteq H_1$$

and conclude $h_2 = h_1h_3 \in H_1$. □

Problem 2. Let G be a finite group and $A, B \le G$ subgroups and define

$$AB = \{ab : a \in A, b \in B\}.$$

(a) Give an example of G, A, and B such that the set AB is not a subgroup of G.
(b) Show the size of AB is

$$|AB| = \frac{|A||B|}{|A \cap B|}$$

(c) Show that AB is a subgroup of G if and only if $AB = BA$. □

Problem 3. Let the finite group G act on the finite set X. For each $g \in G$ let

$$X^g = \{x \in X : gx = x\}$$

be the set of points fixed by g. Show

$$\text{Number of orbits of } G \text{ on } X = \frac{1}{|G|}\sum_{g\in G} |X^g|.$$

That is the number of orbits is the average number of fixed points of the action. □

Problem 4. This problem is just to review some definitions. Let R be a ring and I and J ideals of R. Then let

$$I + J = \{i + j : i \in I, j \in J\}$$
$$IJ = \left\{ \sum_{\ell} i_\ell j_\ell : i_\ell \in I, j_\ell \in J, \text{the sum has only finitely terms.} \right\}$$

A more verbal description of IJ is that it is the set of all finite sums of products ij with $i \in I$ and $j \in J$.
(a) Show that $I + J$ and IJ are ideals of R.
(b) Show

$$IJ \subseteq I \cap J.$$

(c) Give an example where $IJ \neq I \cap J$. $\square$

Problem 5. Let $R = \mathbb{Z}$ (or more generally an principal ideal domain) and let $I = \langle a \rangle$ and $J = \langle b \rangle$. Prove the following

(a) $I + J = \langle \gcd(a, b) \rangle$. In particular a and b are relatively prime if and only if $I + J = 1$.
(b) $IJ = \langle ab \rangle$.
(c) $I \cap J = \langle \operatorname{lcm}(a, b) \rangle$, where $\operatorname{lcm}(a, b)$ is least common multiple of a and b. $\square$

Problem 6. Let I, J be ideals in the ring R with $I + J = R$ (in which case, motivated by the previous problem, we say I and J are relatively prime.) Show $IJI \cap J$. $\square$

Problem 7. Let I and J be ideals in the ring R with $I + J = R$. Show that for any positive integers m and n that $I^m + J^n = R$. $\square$

Problem 8. Let R be a ring, I and J ideals in R with $I + J = 1$. Define $\phi \colon R/(I \cap J) \to R/I \times R/J$ by

$$\phi(r + I \cap J) = (r + I, r + J).$$

Show ϕ is an isomorphism. $\square$