Jan '87

Analysis Qualifying Exam.

Throughout the exam $\lambda$ will denote the Lebesgue measure on $\mathbb{R}$ (the real numbers). Integrals with respect to $\lambda$ will however be denoted by $\int f(x)dx$ or $\int f(t)dt$. Measurable will always mean Lebesgue measurable.

1) Give examples of :

a) A sequence $(E_n)$ of measurable subsets of $\mathbb{R}$ so that $E_1 \supset E_2 \supset \cdots$ and

$$\lim_{n\to\infty} \lambda(E_n) \neq \lambda\left(\bigcap_{n=1}^{\infty} E_n\right)$$

b) A continuous function on $\mathbb{R}$, which is not uniformly continuous.

c) A measurable function on $[0,1]$, which is in $L^1[0,1]$, but not in $L^2[0,1]$.

d) A sequence $f_n$ of measurable functions on $[0,1]$ with $\int |f_n(x)|\,dx \to 0$, but with $f_n$ not converging to zero a.e..

e) A sequence $f_n$ of measurable functions on $[0,1]$ with $f_n \to 0$ a.e., but with $\int |f_n(x)|\,dx$ not converging to zero.

f) An open dense subset $E$ of $\mathbb{R}$ with $\lambda(E) \le 1$.

2) a) State the Dominated Convergence theorem.

b) Compute

$$\lim_{n\to\infty} \int_0^{\infty} \frac{x^2 - n^2}{x^2 + n^2}\, e^{-x}\, dx$$

Justify the steps in your calculations!

3) a) Define convergence in measure.

b) Show that $\int |f_n| \to 0$ implies $f_n \to 0$ in measure.

4) Let $E$ be a measurable subset of $\mathbb{R}$ and assume that $m(E) < \infty$. Let $f_n$ and $f$ be measurable functions on $E$ and assume that

i) $f_n(x) \to f(x)$ a.e. on $E$

ii) for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\lambda(F) < \delta$ implies that $\int_F |f_n(t)|dt < \varepsilon$ for all $n$.

Prove that $\int_E |f_n - f|\, dt \to 0$ as $n \to \infty$.

5) Define

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin \dfrac{1}{x^{\alpha}} & \text{if } x \in (0,1] \end{cases}$$

a) Define absolute continuity of a function.

b) Show $f$ is absolutely continuous on $[0,1]$ if $\alpha < 2$.

c) Define when a function is Lipschitz.

d) Show $f$ is Lipschitz on $[0,1]$ if $\alpha \leq 1$.

6) For any measurable subset $B$ of $[0,1]$ define

$\mu(B) = \lambda(\{t^3 : t \in B\})$.

You may assume that $\mu$ is defined for all measurable sets $B$ and that $\mu$ is a measure on the $\sigma$-algebra of measurable sets.

a) Show that $\mu$ is absolutely continuous with respect to $\lambda$.

b) Show that for any open interval $(a,b) \subset [0,1]$ we have

$$\mu((a,b)) = \int_a^b 3x^2 \, dx$$

c) Find the the Radon-Nikodym derivative of $\mu$ with respect to $\lambda$.

7) Let $E$ be a Lebesgue measurable subset of $[0,1]\times[0,1]$. Denote by $E_x$ the set $\{y: (x,y) \in E\}$ and by $E^y$ the set $\{x: (x,y) \in E\}$. Prove or disprove: If $\lambda(E_x) \leq 1/2$ a.e., then $\lambda(\{y: \lambda(E^y) = 1\}) \leq 1/2$.

8) a) Let $f \in L^1([a,b])$ such that $\int_a^x f(t)\,dt = 0$ for all $x \in [a,b]$. Show that $f = 0$ a.e.

b) Let $f$ be a bounded measurable function on $\mathbb{R}$ and assume that

$$\int_x^{x+1} f(t)\,dt = 0$$

for all $x$. Prove that $f(x+1) = f(x)$ a.e.