Analysis Qualifying Exam
January 2007

Instructions: Write your name legibly on each sheet of paper. Write only on one side of of each sheet of paper. Try to answer all questions. Questions 1–8 are each worth 10 points and question 9 is worth 20 points.
Terminology: Measurability and integrability on $\mathbb{R}$ or a measurable subset of it will always refer to the Lebesgue measure, except if otherwise specified. Lebesgue measure will be denoted by m, dx or dy depending on the context.

1. Let $E, F \subset \mathbb{R}$ with E closed and bounded and F closed. Define $E \cdot F = \{xy : x \in E, y \in F\}$.

a. Assume $0 \notin E$. Prove that $E \cdot F$ is closed.

b. Show that in general $E \cdot F$ does not need to be closed.

2. Let $f \geq 0$ be a measurable function on $\mathbb{R}$ and assume that $F(x) = \sum_{n=1}^{\infty} \frac{f(x+n)}{n}$ is integrable. Prove that $f(x) = 0$ a.e.

3. Let $f, g \geq 0$ be measurable functions on $\mathbb{R}$. Let $h = f * g$ be the measurable function defined by $h(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$. Show, justifying each step, that

$$\int_{-\infty}^{\infty} e^x h(x)\,dx = \left(\int_{-\infty}^{\infty} e^x f(x)\,dx\right)\left(\int_{-\infty}^{\infty} e^x g(x)\,dx\right).$$

4. Let $f : [0,1] \to \mathbb{R}$. Prove that the following are equivalent.

a. f is absolutely continuous, $f'(x) \in \{0,1\}$ a.e., and $f(0) = 0$.

b. There exists a measurable set $A \subset [0,1]$ such that $f(x) = m\,(A \cap (0,x))$.

5. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $f'(0)$ exists. Define $g(x) = x^{-\frac{3}{2}} f(x)$.

a. Prove that $g \in L^p([0,1], m)$ for all $1 \leq p < 2$.

b. Give an example, which shows that the conclusion can fail, if we drop the condition that $f'(0)$ exists.

6. Let $A \subset \mathbb{R}$. Prove that there exists a measurable set $E \supset A$ such that for all measurable $F \supset A$ we have $m(E \setminus F) = 0$.

7. Let $G \subset \mathbb{C}$ be a region and assume D is an open disc such that $\overline{D} \subset G$. Assume f is a non-constant holomorphic function on G and $|f|$ is constant on the boundary of D. Prove that f has at least one zero on D.

8. Let $G \subset \mathbb{C}$ be a region. Assume f, g are holomorphic on G and that also $\overline{f}g$ is holomorphic on G. Prove that either f is constant on G or g is identical zero on G.

9. True or False. Prove, or give a counterexample.

   a. If $f : (0,1) \to \mathbb{R}$ is continuous and bounded, then f is uniformly continuous.

   b. If $f, g \in L^2([0,1])$ with $\int_0^1 f\,dx = 0$, then

   $$\left(\int_0^1 fg\,dx\right)^2 \leq \left(\int_0^1 g^2\,dx - \left(\int_0^1 g\,dx\right)^2\right)\left(\int_0^1 f^2\,dx\right).$$

   c. If $\|f_n\|_1 \leq \frac{1}{n^2}$, then $f_n(x) \to 0$ a.e.

   d. If f' has a pole at $z = a$, then f has a pole at $z = a$.

   e. If $f, f_n \in L^1([0,1])$ such that $\int_A f_n\,dx \to \int_A f\,dx$ for all measurable sets $A \subset [0,1]$, then $\int_0^1 |f_n - f|\,dx \to 0$