ANALYSIS QUALIFYING EXAM AUGUST 1987.

Throughout this exam m and (in integrals) dx will denote Lebesgue measure on $\mathbb{R}$.

1) State and prove the Dominated Convergence Theorem. (You can assume Fatou's lemma.)

2) Let f be a nondecreasing function on [a,b]. Prove that there exist unique functions g and h on [a,b] such that
   a) $f=g+h$
   b) g and h are nondecreasing on [a,b]
   c) g is absolutely continuous on [a,b] and $g(a)=0$
   d) $h'(x)=0$ a.e.

3) Suppose g_n is a sequence of Lebesgue measurable functions on $\mathbb{R}$ such that $|g_n| \leq 1$ for $n= 1,2, \ldots$ and $\int_a^b g_n(x)\,dx \to 0$ as $n \to \infty$ for every interval [a,b]. Prove that $\int f g_n \to 0$ for all $f \in L^1(\mathbb{R},m)$.

4) Let μ be a measure on [a,b] defined on the Lebesgue measurable subsets of [a,b] such that there exists a constant K with $\mu(E) \leq K \cdot m(E)$ for all measurable subsets E of [a,b]. Prove that there exists $f \in L^\infty([a,b],m)$ such that $\| f \|_\infty \leq K$ and $\mu(E) = \int_E f(x)\,dx$.

5) Let $f \in L^1(\mathbb{R},m)$.
   a) Show that $\Sigma_{n=-\infty}^{+\infty} f(x+n)$ converges absolutely a.e. on [0,1].
   b) Let $F(x) = \Sigma_{n=-\infty}^{+\infty} f(x+n)$. Show that $\int_0^1 F(x)\,dx = \int_{\mathbb{R}} f(x)\,dx$.

6) Let $f \in L^1(\mathbb{R},m)$.
   a) Assume f is uniformly continuous on $\mathbb{R}$. Show $f(x) \to 0$ as $|x| \to \infty$.
   b) Give a counterexample to a), assuming that f is only continuous.

7) Let $f \in C([a,b])$ such that $\int_a^b x^n f(x)\,dx = 0$ for all n. Show that $f \equiv 0$.

8) Let f_n be a sequence of Lebesgue measurable functions on $[0,1]$ such that $f_n \to 0$ a.e. Show that there exist $t_n \geq 0$ with $\overline{\lim}\, t_n > 0$ such that $\sum_{n=1}^{\infty} t_n f_n(x)$ converges a.e. on $[0,1]$.

9) <u>True or false. Prove or give a counterexample:</u>

a) If $f \geq 0$ is Lebesgue integrable on $[0,1]$ and $\int_0^1 f(x)\,dx > 0$, then there exist $c > 0$ and a nonempty open interval $I \subset [0,1]$ such that $f(x) \geq c$ for all $x \in I$.

b) If $0 \leq f_1 \leq f_2 \leq \cdots$ and f_n converges to f in measure on $[0,1]$, then $f_n \to f$ a.e. on $[0,1]$.

c) If $f \in L^1(\mathbb{R},m) \cap L^\infty(\mathbb{R},m)$, then $f \in L^p(\mathbb{R},m)$ for all $1 < p < \infty$.

d) If E is a Lebesgue measurable set in $\mathbb{R}$ such that there exists a $c \in (0,1)$ with $m(E \cap I) \leq c\, m(I)$ for all intervals I, then $m(E) = 0$.

e) If f is a bounded Lebesgue integrable function on $[a,b]$, then f is Riemann integrable over $[a,b]$.