Mathematics 554H/701I Homework

The topic we have started since the last test is the convergence of sequences.

Definition 1. Let E be a metric space and $\langle p_n\rangle_{n=1}^{\infty} = \langle p_1, p_2, p_3, \ldots\rangle$ a sequence in E. Then

$$\lim_{n\to\infty} p_n = p$$

if and only if for all $\varepsilon > 0$ there is a $N > 0$ such that

$$n > N \qquad \Longrightarrow \qquad d(p_n, p) < \varepsilon.$$

In the case we say that the sequence $\langle p_n\rangle_{n=1}^{\infty}$ ***converges*** to p. $\square$

1. Let $\lim_{n\to\infty} p_n = p$ in the metric space E. Let $a_n = p_{2n}$. Show that $\lim_{n\to\infty} a_n = p$ also holds. $\square$

2. Write out the proof from the definition that if $\lim_{n\to\infty} x_n = x$ in $\mathbb{R}$, that $\lim_{n\to\infty} -5x_n = -5x$. $\square$

3. Write out the proof that if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in $\mathbb{R}$ that

$$\lim_{n\to\infty} (10x_n - 12y_n) = 10x - 12y.$$

$\square$

We did a proof of the following in class.

Proposition 2. *If $\langle x_n\rangle$ be a convergent sequence in $\mathbb{R}$. Then there is a constant M such that $|x_n| < M$ for all M.* $\square$

Theorem 3. *Let*

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

in $\mathbb{R}$. Then

$$\lim_{n\to\infty} x_n y_n = xy.$$

4. Prove this. *Hint:* Start with

> **Scratch work that the no one else needs to see:** Our goal is to make $|x_n y_n - xy|$ small. We compute

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \qquad \text{(Adding and subtracting trick.)} \\ &\le |x_n y_n - x y_n| + |x y_n - xy| \\ &= |x_n - x||y_n| + |x||y_n - y| \end{aligned}$$

> The factors $|x_n - x|$ and $|y_n - y|$ are both good in that we can make them small. The factor $|x|$ is independent of n and thus is not a problem. The sequence $\langle y_n\rangle_{n=1}^{\infty}$ is convergent and thus bounded, so we bound the factor $|y_n|$. We now return to our regularly scheduled proof.

Let $\varepsilon > 0$. The sequence $\langle y_n \rangle_{n=1}^{\infty}$ is convergent thus it is bounded. Therefore there is an M so that

$$|y_n| \leq M \quad forall \quad n.$$

As $\lim_{n\to\infty} x_n = x$ there There is a $N_n > 0$ such that

$$n > N_2 \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{2(M+1)}$$

and as $\lim_{n\to\infty} y_n = y$ there is a $N_2 > 0$ such that

$$n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{2(|x|+1)}.$$

Now let $N = \max\{N_1, N_2\}$ and use the calculation from our scratch work to show

$$n > N \quad \text{implies} \quad |x_n y_n - xy| < \varepsilon$$

which completes the proof. $\square$