Analysis Qualifying Exam
January 2006

Instructions: Write your name legibly on each sheet of paper. Write only on one side of each sheet of paper. Try to answer all questions. Prove all your claims. Questions 1-8 are worth 10 points each and question 9 is worth 20 points.

Terminology: Measurability and integrability on $\mathbb{R}$ or (Lebesgue) measurable subsets of $\mathbb{R}$ will always refer to the Lebesgue measure except if otherwise specified. Lebesgue measure will be denoted by m or dx depending on the context. If Ω is a measurable subset of $\mathbb{R}$ and $1 \leq p < \infty$ then recall that

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} | f \text{ is measurable and } \|f\|_p < \infty\}$$

where $\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{1/p}$. Also

$$\begin{aligned} L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} | f &\text{ is measurable and there exists } C < \infty \text{ such that } |f(x)| \leq C \\ &\text{for almost all } x \in \Omega\} \end{aligned}$$

and the infimum of such C's is denoted by $\|f\|_\infty$.

1. Let K be a compact subset of $\mathbb{R}$ and let (f_n) be a sequence of continuous real valued functions on K which converges pointwise to a continuous function f and which is pointwise monotone (i.e. either $f_{n+1}(x) \geq f_n(x)$ for all n and x, or $f_{n+1}(x) \leq f_n(x)$ for all n and x). Prove that (f_n) converges to f uniformly on K.
2. (a) Let $1 \leq p < q < r \leq \infty$. Show that for every $f \in L^q(\mathbb{R})$ there exist $g \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ and $h \in L^r(\mathbb{R}) \cap L^q(\mathbb{R})$ such that $f = g + h$ and $\|f\|_p = (\|g\|_p^p + \|h\|_p^p)^{1/p}$.
   (b) Compute $\sup\{x + 8y + 27z : x, y, z \in \mathbb{R} \text{ and } x^4 + y^4 + z^4 = 1\}$.
3. (a) Prove that if (A_n) is a sequence of measurable sets with $\sum_{n=1}^\infty m(A_n) < \infty$ then $m(\cap_{k=1}^\infty \cup_{n=k}^\infty A_n) = 0$.
   (b) Prove that if (A_n) is a sequence of measurable subsets of $[0, 1]$ such that for some $\delta > 0$, $m(A_n) \geq \delta$ for all n, then there is at least one point $x_0 \in [0, 1]$ which belongs to infinitely many A_n's.
4. (a) Prove that if $f_n : \mathbb{R} \to \mathbb{R}$ are measurable for all n, $f : \mathbb{R} \to \mathbb{R}$, $f_n \to f$ a.e., and $\sup_n |f_n| \in L^2(\mathbb{R})$, then $f \in L^2(\mathbb{R})$ and $f_n \to f$ in $L^2(\mathbb{R})$.
   (b) Give an example of measurable functions $f_n : [0, 1] \to [0, \infty)$ such that $f_n \to 0$ a.e., $\int f_n(x)dx \to 0$, yet $f_n \not\to 0$ in $L^2[0, 1]$.
5. For every $x \in [0, 1]$ let μ_x be a measure on the Lebesgue measurable sets of $[0, 1]$ such that for every measurable $A \subseteq [0, 1]$, the function $x \mapsto \mu_x(A)$ is measurable. For every Lebesgue measurable set $A \subseteq [0, 1]$ define $\mu(A) = \int_0^1 \mu_x(A)dx$. Prove that μ is a measure and that for every non-negative measurable function f,
$$\int_0^1 f(y)d\mu(y) = \int_0^1 \left[\int_0^1 f(y)d\mu_x(y)\right] dx.$$
6. (a) Prove that if f is absolutely continuous and g is Lipschitz then $g \circ f$ is absolutely continuous.
   (b) Prove that if f is absolutely continuous and strictly increasing and g is absolutely continuous, then $g \circ f$ is absolutely continuous.
   (c) Construct a Lipschitz function $f : [0, 1] \to [0, 1]$ such that $\sqrt{f(x)}$ is not absolutely continuous. Hint: $f(\frac{1}{n}) = \frac{1}{n^2}$ and $f(\frac{1}{2}(\frac{1}{n} + \frac{1}{n+1})) = 0$.

7. Suppose that f is analytic on $\{z\colon |z| < 2\}$, that $f(1/3) = 1 + i$, and that $|f(z)| > 2$ if $|z| = 1$. Prove carefully that f has a zero in $\{z\colon |z| < 1\}$. Standard results may be used without proof provided they are clearly stated.
8. Suppose that f is analytic on $\mathbb{C}$ except for a finite number of singularities and that $zf(z) \to 0$ as $z \to \infty$. Prove carefully that there exists $M > 0$ such that $|z^2 f(z)| \leq M$ for all z such that $|z| \geq M$. Standard results may be used without proof provided they are clearly stated. (Hint: consider $f(1/z)$.)
9. True or False. Prove, disprove, or give a counterexample, whichever is appropriate.
   a. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\phi(\int_0^1 f(x)dx) \leq \int_0^1 \phi(f)dx$ for all measurable bounded $f : [0,1] \to \mathbb{R}$. Then ϕ is convex.
   b. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions and $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f_n \to f$ a.e. Then there exists a partition of $\mathbb{R}$ into disjoint measurable sets $E_0,\ E_1,\ E_2, \ldots$ such that $m(E_0) = 0$ and $f_n \to f$ uniformly on E_i for all $i \geq 1$.
   c. If $f(x) = e^{-x} \sin x$ then $\sup\{T_a^b(f) : a, b \in \mathbb{R}, a < b\} = \infty$ (where $T_a^b(f)$ denotes the total variation of f from a to b).
   d. Suppose that $u + iv$ is analytic on a connected open set U, where $u(x, y)$ and $v(x, y)$ are real-valued functions. Then $e^u \sin v$ is harmonic on U.
   e. If f is analytic on a connected open set U then there exists F such that $F'(z) = f(z)$ for all $z \in U$.