ADMISSION TO CANDIDACY EXAMINATION

IN REAL ANALYSIS

JANUARY 1985

NOTATION: R - real numbers; m - Lebesgue measure on R.

1. Let E be a measurable subset of R with $m(E) < \infty$. Prove that given $\varepsilon > 0$, there exists a compact set $F \subset E$ such that

$$m(E \sim F) < \varepsilon .$$

2. State and prove the Baire Category Theorem.

3. Let $0 < p < q \leq \infty$.

   a) Prove that $L^q([0,1],m) \subset L^p([0,1],m)$.

   b) Show that there exists $f \in L^p$ with $f \notin L^q$.

4. Let g be absolutely continuous and monotone on R. Prove that $g[E]$ has measure zero for every set E of measure zero.

5. Let $f(x,y)$ be continuous on $[0,1] \times [0,1]$. Prove that if

$$\int_0^1 \int_0^1 x^n y^k f(x,y)\,dx\,dy = 0$$

for all $n,k = 0,1,2,\ldots$, then $f(x,y) = 0$ for all $(x,y) \in [0,1] \times [0,1]$.

6. Let $\{ f_n \}$ be a sequence of Lebesgue integrable functions on R satisfying

$$\sum_{n=1}^{\infty} \int_R |f_n|\,dm < \infty .$$

   a) Show that $\int_R \sum_{n=1}^{\infty} |f_n|\,dm < \infty$.

   b) Show that $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ almost everywhere (a.e).

   c) Show that $\sum_{n=1}^{\infty} f_n \in L^1(R,m)$.

7. Suppose F is absolutely continuous on $[0,1]$, convex, and nonconstant with $F(0) = F(1) = 0$. Show that there exists an integrable function f on $[0,1]$ and $a \in (0,1)$ such that

a) $F(x) = \int_0^x f\, dm$,

b) $f(x) \leq 0$ a.e. on $[0,a]$ and $0 \leq f(x)$ a.e. on $[a,1]$,

c) $\int_0^1 |f|dm = -2F(a)$.

8. Let f be a nonnegative integrable function on R and let

$$\xi(t) = m(\{ x : f(x) > t \}).$$

Prove that ξ is decreasing and that

$$\int_R f(x)dx = \int_0^\infty \xi(t)dt.$$

9. Let (X,d) be a complete metric space and let $f: X \to X$ be a function satisfying

$$d(f(x),f(y)) \leq cd(x,y) \qquad (x,y \in X),$$

where c is a constant with $0 < c < 1$. Let $f^1 = f$ and for $n = 2,3,4,\ldots,$ let $f^n(x) = f(f^{n-1}(x))$.

a) For any $n = 1,2,\ldots,$ show that $d(f^n(x),f^n(y)) \leq c^n d(x,y)$, $(x,y \in X)$

b) For any $n,k = 1,2,\ldots,$ and $x \in X$ show that

$$d(f^n(x),f^{n+k}(x)) \leq (c^n + c^{n+1} + \ldots + c^{n+k-1})d(f(x),x).$$

c) For any $x \in X$, show that there exists a point $p \in X$ such that

$$\lim_{n\to\infty} f^n(x) = p \text{ and } f(p) = p.$$

d) Show that p is the unique fixed point of f, that is, if $f(q)=q$, then $q = p$.