Mathematics 554H/703I Test 1 Name: Answer Key

1. (a) Define the binomial coefficient $\binom{n}{k}$

Solution. The binomial coefficient is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$\square$

(b) State the ***binomial theorem***.

Solution. For any positive integer n and real numbers x and y

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-x}.$$

$\square$

(c) Simplify $\dfrac{(x+h)^4 - x^4 - 4x^3h}{h^2}$ (the answer should have no h in the denominator).

Solution. We use the binomial theorem to expand $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$:

$$\begin{aligned}
\frac{(x+h)^4 - x^4 - 4x^3h}{h^2} &= \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4 - 4x^3h}{h^2} \\
&= \frac{6x^2h^2 + 4xh^3 + h^4}{h^2} \\
&= \frac{h^2(6x^2 + 4xh + h^2)}{h^2} \\
&= 6x^2 + 4xh + h^2.
\end{aligned}$$

$\square$

2. What is the sum of $2 + 2x^3 + 2x^6 + \cdots + 2x^{30}$?

Solution. This is a geometric series so we have

$$\begin{aligned}
2 + 2x^3 + 2x^6 + \cdots + 2x^{30} 2 + 2x^3 + 2x^6 + \cdots + 2x^{30} &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\
&= \frac{2 - 2x^{32}}{1 - x^2}.
\end{aligned}$$

This holds for $x \neq 1$. When $x = 1$ each term is 2 and there are 16 of them, so in this case the sume is $2 \cdot 16 = 32$. $\square$

3. Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ and subsets $A, B \subseteq \mathbb{R}$ such that $f(A \cap B) \neq f(A) \cap f(B)$.

Solution. Let $f(x) = x^2$ and for the sets let $A = (-1, 0)$ and $B = (0, 1)$. Then $f[A] = f[B] = (0, 1)$ and so $f[A] \cap f[B] = (0, 1)$. But $A \cap B = \varnothing$ and this $f[A \cap B] = f[\varnothing] = \varnothing$. Thus $f(A \cap B) \neq f(A) \cap f(B)$. $\square$

4. Let $f\colon X \to Y$ be a function between the two sets X and Y.

(a) If $U \subseteq Y$ define $f^{-1}(U)$.

Solution. This is given by

$$f^{-1}(U) = \{x \in X : f(x) \in U\}.$$

$\square$

(b) If $\{U_\alpha : \alpha \in I\}$ is a collection of subsets of Y define $\displaystyle\bigcup_{\alpha\in I} U_\alpha$.

Solution. The set $\displaystyle\bigcup_{\alpha\in I} U_\alpha$ is the set of elements that are in at least one of the sets U_α. Explicitly:

$$\bigcup_{\alpha\in I} U_\alpha = \{x : x \in U_\alpha \text{ for at least one } \alpha \in I\}.$$

$\square$

(c) Prove $\displaystyle f^{-1}\Big(\bigcup_{\alpha\in I} U_\alpha\Big) = \bigcup_{\alpha\in I} f^{-1}(U_\alpha)$.

Solution.

$$\begin{aligned} x \in f^{-1}\Big(\bigcup_{\alpha\in I} U_\alpha\Big) &\iff f(x) \in \bigcup_{\alpha\in I} U_\alpha \\ &\iff f(x) \in U_\alpha && \text{for at least one } \alpha \in I \\ &\iff x \in f^{-1}[U_\alpha] && \text{for at least one } \alpha \in I \\ &\iff x \in \bigcup_{\alpha\in I} f^{-1}(U_\alpha). \end{aligned}$$

Thus $f^{-1}\Big(\bigcup_{\alpha\in I} U_\alpha\Big)$ and $\bigcup_{\alpha\in I} f^{-1}(U_\alpha)$ have the same elements and therefore they are equal. $\square$

5. Let E be a metric space, $\langle p_n\rangle_{n=1}^{\infty}$ a sequence of points in E and $p \in E$. Define what

$$\lim_{n\to\infty} p_n = p$$

means.

Solution. This means that for any $\varepsilon > 0$ there is a $N > 0$ such that if $n > N$, then $d(p_n, p) < \varepsilon$. $\square$

6. (a) State the ***least upper bound axiom***.

Solution. Let $S \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$ that is bounded from above. Then S has a least upper bound.

More explicitly this means that if there is a number b such that $x \leq b$ for all $x \in S$ (i.e. b is an upper bound for S), then there is a number $\beta = \sup(S)$ such that β is an upper bound for S and if b is any upper bound for S, then $\beta \leq b$. $\square$

(b) State ***Archimedes' axiom***.

Solution. For any positive real number b there is an natural number n with $n > b$. $\square$

(c) Use the least upper bound axiom to prove Archimedes's axiom.

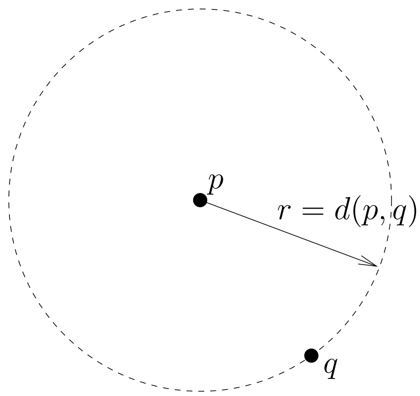
Solution. Towards a contradiction assume that there is a positive real number b such that $n \leq b$ for all natural numbers. Then $\mathbb{N} = \{1, 2, 3, \ldots\}$ (the set of natural numbers) is bounded above. Therefore $\mathbb{N}$ has a least upper bound $\beta = \sup(\mathbb{N})$. For any natural number n the number $n+1$ is also a natural number and therefore $n + 1 \leq \beta$, which implies that $n \leq \beta - 1$. This shows that $\beta - 1 < \beta$ is also an upper bound for $\mathbb{N}$ contradicting that β was the least upper bound. $\square$

7. (a) Define what it means for the set U to be ***open*** in the metric space E.

Solution. The set U is open in E if and only if for all $p \in U$ there is an $r > 0$ such that $B(p, r) \subseteq U$. (Here $B(p, r) = \{x \in E : d(x, p) < r\}$ is the open ball of radius r about p.) $\square$

(b) Let E be a metric space and $p \in E$. Let $U = \{x \in E : x \neq p\}$ be the compliment of p in E. Show U is an open set in E.

Solution. Let $q \in U$. Then $q \neq p$ and thererfore $r = d(p, q) > 0$. If $x \in B(q, r)$, then $d(x, q) < r = d(p, q)$ and thus $x \neq p$. This shows that $B(q, r) \subseteq U$. Therefore U contains an open ball about any of its points and thus it is open.



The set U is the complement of the point p. If $q \in U$ and $r = d(p,q)$ then the open ball $B(q,r)$ is entirely contained in U showing that U is open. $\square$

(c) Let U and V be open sets. Prove that $U \cap V$ also open.

Solution. Let $p \in U \cap V$. Then $p \in U$ and $p \in V$. As U is open there is a $r_1 > 0$ such that $B(p, r_1) \subseteq U$. As V is open there is a $r_2 > 0$ such that $B(p, r_2) \subseteq V$. Let $r = \min\{r_1, r_2\}$. Then $B(x,r) \subseteq B(p, r_1) \subseteq U$ and $B(x,r) \subseteq B(p,r_2) \subseteq V$. Therefore $B(x,r) \subseteq U \cap V$. This shows that $\cap V$ contains an open ball about any of its points and thus is open. $\square$

8. Let $f\colon [0,1] \to \mathbb{R}$ be a function such that for all $x_1, x_2 \in [0,1]$

$x_1 < x_2$ implies $f(x_1) < f(x_2)$ and $|f(x_2) - f(x_1)| \leq 3|x_2 - x_1|$

Assume that $f(0) = -1$ and $f(1) = 1$. Set

$$S = \{x \in [0,1] : f(x) \leq 0\}.$$

This set is bounded above by 1 and thus by the least upper bound axiom

$$r = \sup(S)$$

exists. Show that $f(r) = 0$. *Hint:* Let $\varepsilon > 0$. Then

$$r - \varepsilon < r < r + \varepsilon.$$

(a) Show $r - \varepsilon \in S$ and $r + \varepsilon \notin S$ and that this implies

$$f(r - \varepsilon) \leq 0 \leq f(r + \varepsilon).$$

Solution. As $r - \sup(S)$ there is an $x \in S$ with $r - \varepsilon < x < r$. By the definition of S we have $f(x) \leq 0$. But $r - \varepsilon < x$ and f is an increasing function. Therefore

$$f(r - \varepsilon) < f(x) \leq 0.$$

Note $r+\varepsilon > 0$ and $r = \sup(S)$, which implies that $r+\varepsilon \notin S$. From the definition of S this implies that $f(r+\varepsilon) > 0$. Putting these inequalities together gives

$$f(r-\varepsilon) < 0 < f(r+\varepsilon)$$

which implies the required inequality $f(r-\varepsilon) \le 0 \le f(r+\varepsilon)$. $\square$

(b) Show

$$|f(r+\varepsilon) - f(r-\varepsilon)| \le 6\varepsilon.$$

and use this to show

$$-6 \le f(r-\varepsilon) \qquad \text{and} \qquad f(r+\varepsilon) \le 6\varepsilon.$$

Solution. From $|f(x_2)-f(x_1)| \le 3|x_2-x_1|$ with $x_1 = r-\varepsilon$ and $x_2 = r+\varepsilon$ we have

$$|f(r+\varepsilon) - f(r-\varepsilon)| \le 3|(r+\varepsilon) - (r-\varepsilon)| = 6\varepsilon.$$

Then

$$f(r+\varepsilon) = f(r-\varepsilon) + (f(r+\varepsilon) - f(r-\varepsilon)) \le 0 + 6\varepsilon = 6\varepsilon$$

and

$$f(r-\varepsilon) = f(r+\varepsilon) + (f(r-\varepsilon) - f(r+\varepsilon)) \ge 0 - 6\varepsilon = -6\varepsilon$$

which can be rearranged to give $-6\varepsilon \le f(r-\varepsilon)$. $\square$

(c) Finally show that

$$-6\varepsilon \le f(r) \le 6\varepsilon$$

and use this to show that $f(r) = 0$.

Solution. Using that f is increasing and using Part (b) of this problem.

$$-6\varepsilon \le f(r-\varepsilon) \le f(r) \le f(r+\varepsilon) \le 6\varepsilon.$$

This shows that for all $\varepsilon > 0$ that

$$|f(r)| \le 6\varepsilon.$$

But as this holds for all $\varepsilon > 0$ this implies $|f(r)| = 0$. $\square$

Remark. Now that we know the Intermediate Value Theorem the conclusion of the last problem becomes clear. For $|f(x_2) - f(x_1)| \le 3|x_2 - x_1|$ implies that f is continuous. Also $f(0) = -1 < 0$ and $f(1) = 0$ and thus 0 is between $f(0)$ and $f(1)$. Thus by the Intermediate Value Theorem there is a $r \in (0,1)$ with $f(r) = 0$. $\square$