ANALYSIS QUALIFYING EXAM AUGUST 1988.

Throughout this exam m and (in integrals) dx will denote Lebesgue measure on $\mathbb{R}$.

1) Let $f \in L_1([0,1],m)$. Prove that $\int_0^1 x^n f(x)dx \to 0$ as $n \to \infty$.

2) Let E be a measurable subset of [0,1] and assume there exists a constant $\alpha > 0$ such that $m(E \cap [a,b]) \geq \alpha(b-a)$ for all a and $b \in [0,1]$. Prove that $m(E) = 1$.

3) Let T be a measurable function from [0,1] into [0,1] such that there exists a constant $K > 0$ such that for all $x<y$ we have

$$m(T^{-1}((x,y)) \leq K\,(y-x).$$

Prove that there exists a bounded measurable function g such that

$$m(T^{-1}(E)) = \int_E g\,dx$$

for all Borel sets E in [0,1].

4) Let $I: L_\infty[0,1] \to \mathbb{R}$ be a linear functional such that $f \geq 0$ implies that $I(f) \geq 0$, and $f_n \downarrow 0$ a.e. implies $I(f_n) \to 0$.

a) Prove that there exists $g \in L_1[0,1]$ such that $I(f) = \int fg\,dx$.

b) If $I(x^n) = 1/(n+1)$ for $n = 0,1,2,\ldots$, then prove that $g = 1$ a.e. on [0,1], i.e. $I(f) = \int f\,dx$.

5) Let $1 < p < \infty$ and $g_n \in L_q([0,1],m)$ with $\|g_n\|_q \leq 1$, where $1/p + 1/q = 1$. Suppose $\int_E g_n\,dx \to 0$ as $n \to \infty$. Prove that $\int f g_n dx \to 0$ as $n \to \infty$ for all $f \in L_p([0,1],m)$.

6) a) Let F be absolutely continuous on $[\varepsilon,1]$ for all $\varepsilon > 0$, continuous at 0 and such that the total variation $T_0^1(F) < \infty$. Prove that F is absolutely continuous on [0,1].

b) If F is as in a) except that $T_0^1(F) = \infty$, is then the conclusion of a) still valid?

7) Let $0 \le f$ be Lebesgue measurable on $[0,1]$.

a) Prove that there exist simple functions $\varphi_n \ge 0$ such that $\varphi_n \uparrow f$.

b) Prove that there exist measurable sets E_n and $\alpha_n \ge 0$ such that $f(x) = \Sigma_1^\infty \alpha_n \chi_{E_n}(x)$ a.e. (Note the E_n's are not necessarily disjoint.)

c) Assume now that f is an integrable function on $[0,1]$. Prove that there exist measurable sets E_n and $\alpha_n \in \mathbb{R}$ such that $f(x) = \Sigma_1^\infty \alpha_n \chi_{E_n}(x)$ a.e. , $\int |f(x)|\, dx = \Sigma_1^\infty |\alpha_n|\, m(E_n)$ and $\int f(x)dx = \Sigma_1^\infty \alpha_n m(E_n)$.

8) Let $f \in L_p([0,1],m)$ with $1 < p < \infty$. Define f_h by

$$f_h(t) = \frac{1}{2h} \int_{t-h}^{t+h} f(x)\, dx$$

a) Prove that f_h is a continuous function of t.

b) Prove that $|f_h(t)| \le (2h)^{-1/p} \| f \|_p$.

c) Prove that $\| f_h \|_p \le \| f \|_p$.

9) <u>True or false. Prove or give a counterexample.</u>

a) Let E_n be measurable sets such that $E_n \downarrow E$. Then $m(E_n) \downarrow m(E)$.

b) Let E_n be measurable sets such that $E_n \uparrow E$. Then $m(E_n) \uparrow m(E)$.

c) If f is a nondecreasing function on $[0,1]$, then there exist $a < b$ in $[0,1]$ such that f is continuous on (a,b).

d) If $|f|$ is measurable, then f is measurable.

e) If $f_n \to f$ in $L_1[0,1]$, then $f_n \to f$ a.e.

f) If E is measurable with $m(E) < \infty$, then E is bounded.