ANALYSIS QUALIFYING EXAMINATION
August 21, 1996

DIRECTIONS: 1. Questions 1-8 are worth ten points each and question 9 is worth 20 points.
2. Write your solution to each problem on a separate sheet.

1. Let $\langle f_n \rangle$ be a sequence of continuous functions on $[0,1]$ such that $\langle f_n(x) \rangle$ **decreases** to zero for each $x \in [0,1]$. Prove that $\langle f_n \rangle$ converges uniformly to zero.

2. In this question m denotes Lebesgue measure and m^* denotes Lebesgue outer measure on the line.
(i) Let $A \subset \mathbb{R}$. Prove that there exists a G_δ set G such that $A \subset G$ and $m(G) = m^*(A)$.
(ii) Suppose that $m^*(A) < \infty$. Prove that A is measurable if and only if $m^*(G \setminus A) = 0$.

3. Let (X, Σ, μ) be a measure space and let $\langle E_n \rangle_{n \geq 1}$ be a sequence of distinct measurable sets such that $\Sigma\mu(E_n) < \infty$. An extended real-valued function f is defined thus:

$$f(x) = \begin{cases} k, & \text{if } x \text{ belongs to exactly } k \text{ of the } E_n\text{'s} \\ \infty, & \text{if } x \text{ belongs to infinitely many } E_n\text{'s.} \end{cases}$$

(i) Prove that f is integrable and evaluate $\int f\,d\mu$.
(ii) Now suppose, in addition, that $\mu(E_m \cap E_n) = \mu(E_m)\mu(E_n)$ for all $m \neq n$. Prove that $f \in L^2(\mu)$ and evaluate $\|f\|_2$.

4. Let $f : [0,1] \to \mathbb{R}$ be absolutely continuous and let $g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable (i.e. g is differentiable everywhere and its derivative is continuous). Prove that $g \circ f$ is absolutely continuous. Deduce that

$$g(f(x)) = g(f(0)) + \int_0^x g'(f(t))f'(t)\,dt \qquad (x \in [0,1]).$$

5. Let f, f_n $(n \geq 1)$ be integrable functions that are defined on a σ-finite measure space (X, Σ, μ). Suppose that $f_n(x) \to f(x)$ a.e. Prove that, for each $0 \leq \alpha < \mu(X)$, there exists a measurable set E_α such that $m(E_\alpha) \gtrsim \alpha$ and

$$\int_{E_\alpha} f\,d\mu = \lim \int_{E_\alpha} f_n\,d\mu.$$

Does this result hold also for $\alpha = \mu(X)$? Prove or construct a counterexample.

6. Let (X, Σ, μ) be a measure space, let $1 \leq p < \infty$, and let f, f_n $(n \geq 1)$ belong to $L^p(\mu)$. Suppose that

$$\int_X fg\,d\mu = \lim \int_X f_n g\,d\mu$$

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for every $g \in L^q(\mu)$, where $1/p + 1/q = 1$. Prove that $\|f\|_p \leq \liminf \|f_n\|_p$. (HINT: Consider $g = |f|^{p-1} \operatorname{sgn} f$.)

7. Evaluate

$$\int_0^\infty \frac{x^2}{1+x^4}\, dx.$$

8. **EITHER:** Let f be an entire function. Prove carefully that f is a polynomial of degree n if and only if there exists a non-zero $\alpha \in \mathbb{C}$ such that $\lim_{z\to 0} z^n f(1/z) = \alpha$.

OR: (i) What is meant by the "principle of isolated zeros"?
(ii) Let f be holomorphic and not identically zero on the region Ω. Prove that $Z(f) = \{z \in \Omega : f(z) = 0\}$ is countable and that $Z(f) \cap K$ is finite for every compact set K wholly contained in Ω.

9 True or False? Prove or construct a counterexample in each case.

(i) If f is holomorphic in the region $\mathbb{C} \setminus \{0\}$ and γ is a simple closed path in $\mathbb{C} \setminus \{0\}$, then $\int_\gamma f(z)\, dz = 0$.

(ii) Every uncountable set of real numbers has a non-measurable subset.

(iii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ has the property that $f^{-1}(U)$ is a Borel set whenever U is an open set. Then $f^{-1}(B)$ is a Borel set whenever B is a Borel set.

(iv) Let (X, Σ, μ) be a finite measure space and let f be a non-negative measurable function such that $f(x) < \infty$ a.e. Then the measure ν defined by $\nu(E) = \int_E f\, d\mu$ is σ-finite.