Mathematics 555 Homework

1. Continuous functions.

Definition 1. Let $f\colon E \to E'$ be a function between metric spaces. Then f is ***uniformly*** continuous if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $p, q \in E$,

$$d(p,q) < \delta \quad \text{implies} \quad d(f(p), f(q)) < \varepsilon. \qquad \square$$

Recall that a function $f\colon E \to E'$ between is Lipschitz if and only if there is a constant $C \geq 0$ such that $d(f(p), f(q)) \leq C d(p,q)$ for all $p, q \in E$. Last term we saw several examples of Lipschitz functions.

Problem 1. Show that every Lipschitz function is uniformly continuous. $\square$

Proposition 2. *Every uniformly continuous function is continuous.*

Problem 2. Prove this. $\square$

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2$. Show that f is not uniformly continuous. *Hint:* Towards a contradiction assume that f is uniformly continuous. Let $\varepsilon = 1$, then there is a $\delta > 0$ such that for all $x, y \in \mathbb{R}$

$$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| < 1.$$

Show this leads to a contradiction. $\square$

Problem 4. Let $f\colon (0,1) \to \mathbb{R}$ be the continuous function

$$f(x) = \frac{1}{x}.$$

Show that f is not uniformly continuous. $\square$

Problem 5. On let $f\colon [0,1] \to \mathbb{R}$ be the functions

$$f(x) = \sqrt{x}.$$

Prove directly from the definition that f is uniformly continuous. $\square$

Here is a bit of review in using the triangle inequality in metric spaces. If E is a metric space and $y_0, y_1, y_2 \in E$, then

$$d(y_0, y_2) \leq d(y_0, y_1) + d(y_1, y_2).$$

If $y_0, y_1, y_2, y_3 \in E$, then

$$d(y_0, y_3) \leq d(y_0, y_2) + d(y_2, y_3) \leq d(y_0, y_1) + d(y_1, y_2) + d(y_2, y_3).$$

And by now you may have guessed the pattern which is given by the following:

Proposition 3. *Let E be a metric space and $y_0, y_1, \ldots, y_n \in E$. Then*

$$\begin{aligned} d(y_0, y_1) &\le d(y_0, y_1) + d(y_1, y_2) + \cdots + d(y_{n-1}, y_n) \\ &= \sum_{j=1}^{n} d(y_j, y_{j-1}). \end{aligned}$$

Problem 6. Prove this. *Hint:* Induction. □

The following will let us use Proposition 3 to derive some properties of uniformly continuous function.

Definition 4. Let E be a metric space and $\delta > 0$ a positive real number. Then a finite sequence $x_0, x_1, \ldots, x_n \in E$ is a ***δ-sequence*** if and only if for each $j \in \{1, 2, \ldots, n\}$ the inequality $d(x_{j-1}, x_j) < \delta$. □

Lemma 5. *Let $f \colon E \to E'$ be a map between metric spaces and $\delta, \varepsilon > 0$. Assume that for all $p, q \in E$ that*

$$d(p, q) < \delta \quad \textit{implies} \quad d(f(p), f(q)).$$

Then for any δ-sequence $x_0, x_1, \ldots, x_n \in E$ in E we have

$$d(f(x_0), f(x_n)) \le n\varepsilon.$$

Problem 7. Prove this. *Hint:* Letting $y_j = f(x_j)$ in Lemma 3 we have

$$d(f(x_0), f(x_n)) \le \sum_{j=1}^{n} d(f(x_{j-1}), f(x_j)).$$

□

For the last lemma to be useful we need to be able to find some δ-sequences. In $\mathbb{R}$, or more generally in $\mathbb{R}^n$ this is easy.

Lemma 6. *Let $p, q \in \mathbb{R}^n$ and $\delta > 0$. Let n be the unique positive integer with*

$$n - 1 \le \frac{\|p - q\|}{\delta} < n$$

(this is the same as choosing n to be the smallest positive integer with $\|p - q\|/n < \delta$). For $0 \le j \ne n$ let

$$x_j = p + \frac{j}{n}(q - p).$$

Then $x_0, x_1, \ldots, x_n$ is a δ-sequence with $x_0 = p$ and $x_n = q$.

Problem 8. In the case of $n = 5$ and $p, q \in \mathbb{R}^2$ draw the picture of what these points look like. Then prove the result in the general case. □

Problem 9. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be uniformly continuous. Show there are constants $A, B > 0$ such that

$$|F(x)| \le A + B\|x\|$$

for all $x \in \mathbb{R}^n$. □

Problem 10. Prove this. *Hint:* Start by letting $\varepsilon = 1$ in the definition of uniform continuity. Then there is a δ such that

$$\|p - q\| < \delta \quad \text{imlies} \quad |f(p) - f(q)| < 1.$$

Let $x \in \mathbb{R}^n$. By Lemma 6 there is a δ-sequence $x_0, x_1, \ldots, x_n \in \mathbb{R}^n$ with $x_0 = 0$ and $x_n = x$ and

$$n - 1 \leq \frac{\|x - 0\|}{\delta} < n.$$

Now use Lemma 5 to show

$$\|f(x) - f(0)\| \leq n$$

and then use this to show

$$|f(x)| \leq |f(0)| + 1 + \frac{\|x\|}{\delta}$$

and explain why this completes the proof. $\square$

Problem 11. Use the previous problem to show that no polynomial of degree greater than 1 is uniformly continuous on $\mathbb{R}$. $\square$