# Mathematics 172 Homework, February 4, 2019.

Read over the last homework and make sure you understand about equilibrium points. One of the things we mentioned in class is that if $N_*$ is an equilibrium point of

$$N_{t+1} = f(N_t)$$

then it is stable if $|f'(N_*)| < 1$ and unstable if $|f'(N_*)| > 1$. We will explain why this holds in class.

**Problem 1.** Let $r, K > 0$. Then the discrete logistic with pre capita growth rate of $r$ and carrying capacity $K$ is

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right) = f(N)$$

where

$$f(N) = N + rN\left(1 - \frac{N}{K}\right)$$

We wish to find the equilibrium points and see if they are stable.

(a) We first look at the special case where $r = .2$ and $K = 100$. Find the equilibrium points and determine if they are stable. *Solution:* In this case we wish to solve

$$f(N) = N + .2N\left(1 - \frac{N}{100}\right) = N.$$

This reduces to

$$.2N\left(1 - \frac{N}{100}\right) = 0$$

and we see the equilibrium are

$$N_* = 0, 100.$$

Now compute the derivative of $f$. To start it is a bit easier if we first rewrite $f$ a bit.

$$f(N) = N + .2N - \frac{.2N^2}{100}.$$

This

$$f'(N) = 1 + .2 - \frac{.4N}{100}.$$

At $N_* = 0$ we have

$$f'(0) = 1 + .2 - \frac{.4(0)}{100} = 1.2 > 1$$

and therefore $N_* = 0$ is unstable. At $N_* = 100$ we have

$$f'(100) = 100 + .2 - \frac{.4(100)}{100} = .8$$

which shows that this point is also stable.

(b) Now do the general case where

$$f(N) = N + rN\left(1 - \frac{N}{K}\right) f(N)$$

*Solution:* The equilibrium points are $N_* = 0$ and $N_* = K$ A calculation like the ones done above yield that

$$f'(0) = 1 + r > 1$$

and so for the logistic equation $N_* = 0$ is always unstable. We also have

$$f'(K) = 1 - 2r$$

This in this case $N_* = K$ is stable when $0 < r < 2$ (which implies $|1-2r| < 1$ and it is unstable when $2 < r$ (which implies $|1 - 2r| > 1$). □

Putting this all together gives

**Theorem 1.** *For the discrete logistic equation with per capita growth rate $r$ and carrying capacity $K$,*

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right)$$

*there are two equilibrium points $N_* = 0$ and $N_* = K$.*

- *$N_* = 0$ is always unstable for the discrete logistic.*
- *$N_* = K$ is stable for $0 < r < 2$ and unstable for $r > 2$.*

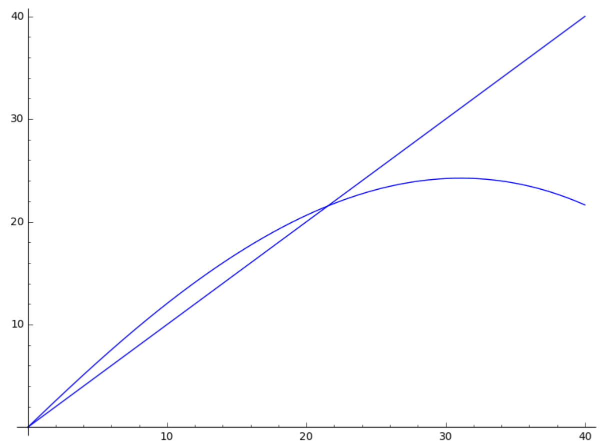
Here are some problem for finding equilibrium points in discrete dynamical and determining if they are stable using your calculator.

**Problem 2.** For the dynamical system

$$P_{t+1} = P_t + .3P_t\left(1 - \frac{P_t^{1.5}}{100}\right)$$

Find the equilibrium points and determine if they are stable.

*Solution:* First enter
`\Y1=X+.3X(1-X^(1.5)/100)`
`\Y2=X`
And by some trial and error I found that good size for the window is
`Xmin=0`
`Xmax=40`
Now plot by using `ZOOM` and then `0:ZoomFit` The result should look something like:




From the picture we see that there are two equilibrium points.

To find them use `2nd CALC 5:intesect` It will now ask your `First curve`. Just hit `ENTER`. It will now ask `Second curve`. Again just hit `ENTER`. The next (and last) question is `Guess?`. This time move the cursor to be as close as possible to the intersection point you want to find and hit `ENTER`. It you moved the cursor to 0 (which is clearly and equilibrium point from the picture) we get that `x=0` and `Y=0` is an intersection point. To find if this is stable we hit `2nd CALC 6:dy/dx` which will give us the value of the derivative of the `\Y1` curve at the current $x$ value, which is $x = 0$. This will tell use that $f'(0) = 1.3$. As $|f'(0)| = 1.3 > 1$ the point $P_* = 0$ is unstable.

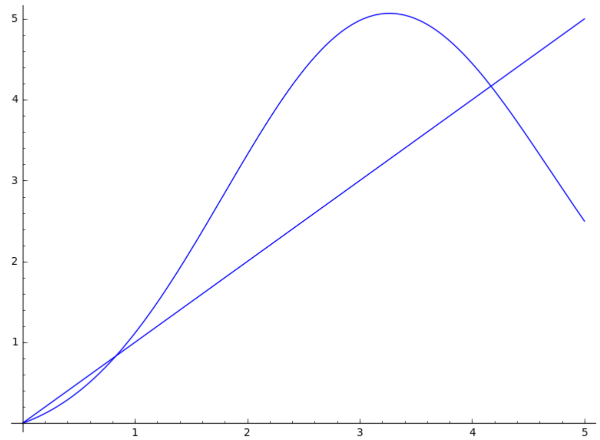
To find the second equilibrium point go through the steps above with the difference that when you are ask `Guess?`, you move the cursor to get as close as you can to the other point of intersection. This time you will get that `X=21.544347` and that `Y` has the same value. Therefore $P_* = 21.544347$ is the second equilibrium point. Now use the calculator to find the value of $f'(21.544347) = .55$ so this point is stable.

**Problem 3.** For the dynamical system

$$N_{t+1} = .5N_t e^{N_t - .2N_t^2}$$

(a) Plot $y = f(x) = .5xe^{x-.2x^2}$ and $y = x$ on your calculator for $0 \le x \le 5$.

*Solution:* Your picture should look like:




(b) So we see there are three equilibrium points. Now find them and determine if they are stable or unstable.

*Solution:* The first is $P_* = 0$. As $f'(0) = .500$ this one is stable.

The second is $P_* = .83138857$ and at this point $f'(.83138857) = 1.5549057$ so this one is unstable.

The third is $P_* = 4.1686114$ and at this point $f'(4.1686114) = -1.759351$ so this one is unstable.

**Problem 4.** Let

$$f(P) = \frac{5 + 20P}{1 + P^2}$$

and consider the discrete dynamical

$$P_{t+1} = f(P_t).$$

(a) Graph $y = f(x)$ and $y = x$ for with $0 \leq x \leq 10$ and use the calculator to find the where these graphs intersect. *Solution:* There is only one point of intersection and it is $P_* = 4.48495684796404$.

(b) Use the calculator to find $f'(P_*)$. *Solution:* $f'(P_*) = -0.958078670794696$.

(c) Is $P_*$ stable or unstable? *Solution:* since $|f'(P_*)| = 0.958078670794696 < 1$ the point is stable.