Homework assigned Wednesday, February 29.

The following problems are to prepare for results that we are about to prove.

Problem 1. Let $f(z)$ be analytic in a domain D and let $a \in K$. Let $r > 0$ be so small that the disk $|z-a| \le r$ is contained in D.

(a) Parametrize $|z-a| = r$ by $z = a + re^{it}$ with $0 \le t \le 2\pi$. Use this reparameterizing to show that
$$\int_{|z-a|=r} \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a+re^{it})\, dt.$$

(b) Use part (a) to show
$$\lim_{r\to 0^+} \int_{|z-a|=r} \frac{f(z)}{z-a}\, dz = 2\pi i f(a).$$

FIGURE 1

In the next problem we will be using Cauchy's theorem, which we now recall.

Theorem 1 (Cauchy's Theorem). *Let D be a bounded domain with nice boundary and $f(z)$ a function that is analytic on the closure of D. Then*
$$\int_{\partial D} f(z)\, dz = 0$$
where, as usual, we orient ∂D so as we move with the inside on our left. $\square$

Problem 2. In Figure 1 we have a bounded domain with nice boundary and a point a inside. Let $f(z)$ be a function that is analytic on the closure of D. Let r be a small positive number and D_r the domain D with the inside of the circle $|z-a| = r$ removed. That is D_r is the region inside of D and outside of $|z-a| = r$.

(a) Explain why
$$\int_{\partial D_r} \frac{f(z)}{z-a}\, dz = 0.$$
Hint: The function $g(z) = \frac{f(z)}{z-a}$ is analytic in D_r.

(b) The boundary of ∂D_r has two pieces. First there is the boundary, ∂D, of the original domain and second there is the circle $|z-a| = r$. Thus
$$\int_{\partial D_r} \frac{f(z)}{z-a}\, dz = \int_{\partial D} \frac{f(z)}{z-a}\, dz - \int_{|z-a|=r} \frac{f(z)}{z-a}\, dz.$$
Explain why the sign on the second integral is negative. *Hint:* We always move along the boundary with the inside on our left.

(c) Combine parts (a) and (b) to conclude
$$\int_{\partial D} \frac{f(z)}{z-a}\, dz = \int_{|z-a|=r} \frac{f(z)}{z-a}\, dz.$$

(d) In the last equation take the limit at r goes to 0 and part (b) of Problem 1 to conclude
$$\int_{\partial D} \frac{f(z)}{z-a}\, dz = 2\pi i f(a).$$

We have thus proven the following, which is maybe the most important result in complex analysis.

Theorem 2 (Cauchy Integral Formula). *Let D be a bounded domain with nice boundary and $f(z)$ be analytic on the closure of D. Then for any point $a \in D$*

$$f(a) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)\,dz}{z-a}.$$

□

Example. Let γ be the path in Figure 2.

$10i$

πi

-10 10

$-\pi i$

FIGURE 2

We now use the Cauchy Integral formula to evaluate

$$\int_\gamma \frac{e^z}{z^2+\pi^2}\,dz.$$

This function is analytic except where the denominator becomes zero. That is where $z^2 + \pi^2 = 0$. Note that $z^2 + \pi^2 = (z - \pi i)(z + \pi i)$. So that the bad points are $z = \pi i$ and $z = -\pi i$. Thus our integral becomes

$$\int_\gamma \frac{e^z}{(z-\pi i)(z+\pi i)}\,dz.$$

We only need to work about the point πi as it is the only non-analytic point inside of γ. Rewrite the integral as

$$\int_\gamma \frac{e^z/(z+\pi i)}{(z-\pi i)}\,dz = \int_\gamma \frac{f(z)}{(z-\pi i)}\,dz$$

where

$$f(z) = \frac{e^z}{z+\pi i}.$$

The function $f(z)$ is analytic inside of γ. So by the Cauchy integral formula

$$\int_\gamma \frac{e^z}{z^2+\pi^2}\,dz = \int_\gamma \frac{f(z)}{(z-\pi i)}\,dz = 2\pi i f(\pi i) = 2\pi i \frac{e^{\pi i}}{\pi i + \pi i} = e^{\pi i} = -1.$$

□

Problem 3. Let z_1 be a complex number and γ a simple closed curve that does not pass through z_1. Show

$$\int_\gamma \frac{dz}{z-z_1} = \begin{cases} 2\pi i, & \text{if } z_1 \text{ is inside of } \gamma, \\ 0, & \text{if } z_1 \text{ is outside of } \gamma. \end{cases}$$

Hint: Use part (d) of Problem 2, or the Cauchy Integral Formula, with $f(z) = 1$, D the region inside of γ, and $z = z_1$.

Problem 4. Figure 3 shows the points i, $-i$, 0, and 4 along with three paths α, β, and γ. Use either part (d) or Problem 2 or the Cauchy integral formula to

(a) Evaluate $\displaystyle\int_\alpha \frac{2z+1}{z(z-4)(z^2+1)}\,dz$,

(b) Evaluate $\displaystyle\int_\beta \frac{2z+1}{z(z-4)(z^2+1)}\,dz$, and

(c) Evaluate $\displaystyle\int_\gamma \frac{2z+1}{z(z-4)(z^2+1)}\,dz$.

i

α

0 4

β

$-i$ γ

FIGURE 3