# Mathematics 551 Homework, March 4, 2020

**Problem** 1. In this problem you will derive some standard formulas for the first and second fundamental forms of graphs. Let $U \subseteq \mathbb{R}^2$ be an open set and let $f\colon U \to \mathbb{R}$ be a smooth function. Define a $\mathbf{x}\colon U \to \mathbb{R}^3$ by

$$\mathbf{x}(u,v) = (u, v, f(u,v)).$$

Let $M$ be the surface parameterized by this function $\mathbf{x}$. That is $M$ is the graph of the function $z = f(x,y)$.

(a) Show

$$\begin{aligned} \mathbf{x}_u &= (1, 0, f_u) \\ \mathbf{x}_v &= (0, 1, f_v) \end{aligned}$$

(b) Show that the first fundamental form is

$$I = (1 + f_u^2)\, du^2 + 2 f_u f_v \, du dv + (1 + f_v^2)\, dv^2.$$

(c) Show the unit normal is

$$\mathbf{n}(u,v) = (1 + f_u^2 + f_v^2)^{-1/2}(-f_u, -f_v, 1).$$

(d) Find the second fundamental form of $\mathbf{x}$. $\square$

**Problem** 2. In the last problem let us consider the special case were

$$f(0,0) = f_u(0,0) = f_v(0,0) = 0$$

and let $M$ be the surface which is the graph of $z = f(x,y)$. Then the graph will be tangent to the $x$-$y$ plane at the origin. Assume

$$\begin{aligned} f_{uu}(0,0) &= k_1 \\ f_{uv}(0,0) &= 0 \\ f_{vv}(0,0) &= k_2 \end{aligned}$$

where $k_1$ and $k_2$ are constants. As in the previous problem let

$$\mathbf{x}(u,v) = (u, v, f(u,v)).$$

(a) Show that the first and second fundamental forms of $\mathbf{x}$ at the origin are

$$\begin{aligned} I_{(0,0,0)} &= du^2 + dv^2 \\ I\!I_{(0,0,0)} &= k_1\, du^2 + k_2\, dv^2 \end{aligned}$$

and that the normal at the origin is

$$\mathbf{n}(0,0) = (0,0,1).$$

(This should follow at once from Problem 1.)

(b) Show that at the origin the shape operator $S = S_{(0,0,0)}$ satisfies satisfies

$$S\mathbf{x}_u(0,0) = k_1 \mathbf{x}(u,v), \qquad S\mathbf{x}_v(0,0) = k_2 \mathbf{x}_v(0,0).$$

Thus $k_1$ and $k_2$ are the eigenvalues of $S$.

(c) One way to understand how a surface is curved is to intersect it with planes and look at the curvature of the resulting curve. Let us look at an example of this. Let $\mathcal{P}_\theta$ be the plane spanned by

$$E_1(\theta) = (\cos(\theta), \sin(\theta), 0), \qquad E_3 = (0, 0, 1) = \mathbf{n}(0, 0).$$

Show that the curve of intersection $\mathcal{P}_\theta \cap M$ is parameterized by

$$\begin{aligned} \boldsymbol{\gamma}(t) &= (t\cos(\theta), t\sin(\theta), f(t\cos(\theta), t\sin(\theta))) \\ &= tE_1(\theta) + f(t\cos(\theta), t\sin(\theta))E_3. \end{aligned}$$

Show that the curvature of this curve (viewed as a curve in $\mathcal{P}_\theta$) at the origin is

$$\kappa(0) = k_1\cos^2(\theta) + k_2\sin^2(\theta).$$

□