Qualifying Examination in Analysis
August 2013

Please use only one side of the paper and start each problem on a new page.

1. Find all continuous functions $f\colon [0,1] \to \mathbb{R}$ such that

$$\int_0^1 f(x)x^n\,dx = \frac{1}{n+4} \quad \text{for} \quad n = 0,1,2,3,\ldots$$

Hint: One such function is $f(x) = x^3$.

2. Let $A \subseteq \mathbb{R}$ be a (not necessarily measurable) set with $0 < \mu^*(A) < \infty$. Show there is an interval I such that $\mu^*(I \cap A) \geq (1/2)\mu^*(I)$. (Here μ^* is Lebesgue outer measure on the subsets of $\mathbb{R}$.)

3. Let $\langle r_k \rangle_{k=1}^{\infty}$ be an enumeration of the rational numbers in $[0,1]$. Show the series

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^n\sqrt{|x - r_k|}}$$

converges for almost all $x \in [0,1]$.

4. Let $f\colon [0,1] \to \mathbb{R}$.

(a) If f is absolutely continuous show

$$|f(1) - f(0)| \leq \left(\int_0^1 f'(t)^2\,dt \right)^{1/2}.$$

(b) Give an example of a continuous function $f\colon [0,1] \to \mathbb{R}$ where $f'(t) = 0$ for almost all $t \in [0,1]$ but the inequality of (a) is false.

5. Let $f\colon \mathbb{R} \to \mathbb{R}$ be measurable. Show

$$\int_{\mathbb{R}} |f|^2\,d\mu = 2\int_0^{\infty} t\,\mu\{x : |f(x)| > t\}\,dt$$

where μ is Lebesgue on $\mathbb{R}$.

6. Let $f(z)$ be an entire function such that $|f(z)| \neq 1$ for all $z \in \mathbb{C}$. Show that f is constant.

7. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$.

8. Let f_n be a sequence of analytic functions on an open set U such that there is function $f(z)$ on U with $f_n(z) \to f(z)$ uniformly on compact subsets of U. Then show $f(z)$ is analytic in U.

9. True or False. Either give a proof or a counter example.

(a) If U is an open subset of $(0,1)$, that contains all the rational numbers in (0,1), then $\mu(U) = 1$ (where μ is Lebesgue measure).

(b) If f is a function on $[0,1]$ such that f^2 is measurable, then f is also measurable.

(c) There is an entire function $f(z)$ such that $f(1/n) = \dfrac{n^2+1}{n^2-1}$ for $n = 1,2,3,\ldots$.

(d) If $f \in L^1([0,\infty))$, then $\lim_{n\to\infty} f(x+n) = 0$ for almost all $x \in [0,1]$.

(e) If (X,d) is a complete metric space and $f\colon X \to X$ is a function such that $d(f(x),f(y)) < d(x,y)$ for all $x,y \in X$ with $x \neq y$, then f has a fixed point on X. (Recall that x is a *fixed point* of f iff $f(x) = x$.)