Take home portion of final

We start by extending our definition of $\sup(S)$ and $\inf(S)$ for nonempty subsets of $\mathbf{R}$. If S is bounded above, the $\sup(S)$ is the usual ***least upper bound*** or ***supremum***. If S is not bounded above set $\sup(S) = +\infty$. Likewise if S is bounded below, then $\inf(S)$ is the usual ***greatest lower bound*** or ***infinmum***, if S is not bounded below, then $\inf(S) = -\infty$.

We also want to extend the definition of the limit of a sequence.

Definition 1. Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of real numbers. Then

$$\lim_{k\to\infty} a_k = +\infty$$

if and only if for all real numbers B there is a N such that

$$k \geq N \quad \text{implies} \quad a_k > B.$$

□

Definition 2. Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of real numbers. Then

$$\lim_{k\to\infty} a_k = -\infty$$

if and only if for all real numbers A there is a N such that

$$k \geq N \quad \text{implies} \quad a_k < A.$$

□

Definition 3. The ***extended real numbers*** is the set $\mathbf{R} \cup \{-\infty, +\infty\}$. That is the extended real numbers is just the usual real numbers with the two values $-\infty$ and $+\infty$ thrown in. □

Note that now every nonempty subset of $\mathbf{R}$ has an sup and inf in the extended real numbers.

One of the more useful results from last term was that any bounded monotone sequence is convergent. With the above definition we can drop the requirement that the sequence be bounded.

Proposition 4. *Let $\langle a_k \rangle_{k=1}^{\infty}$ be a monotone sequence. Then it has a limit in the extended real numbers.*

Proof. Assume that $\langle a_k \rangle_{k=1}^{\infty}$ is monotone increasing. If the sequence is bounded above, then we saw last term that it it converged to $\sup\{a_1, a_2, a_3, \ldots\}$. So assume that it is not bounded above. Then for any real number B, there is some N with $a_N > B$. But $\langle a_k \rangle_{k=1}^{\infty}$ is monotone increasing and therefore if $k \geq N$ we have $a_k \geq a_N > B$, and therefore $\lim_{k\to\infty} a_k = +\infty$. □

Lemma 5. *Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of read numbers and set*

$$S_n = \sup\left(\{a_n, a_{n+1}, a_{n+2}, \ldots\}\right).$$

Then $\langle S_n \rangle_{n=1}^{\infty}$ is a monotone decreasing sequence.

Proof. The number S_n is upper bound for the set

$$\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

and

$$\{a_{n+1}, a_{n+1}, a_{n+3}, \dots\} \subseteq \{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

Therefore S_n is also an upper bound for

$$\{a_{n+1}, a_{n+2}, a_{n+3}, \dots\}$$

But $S_{n+1} = \sup(\{a_{n+1}, a_{n+2}, a_{n+3}, \dots\})$ is the least upper bound for this set and therefore $S_{n+1} \le S_n$. □

Likewise we have

Lemma 6. *Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of read numbers and set*

$$I_n = \sup\left(\{a_n, a_{n+1}, a_{n+2}, \dots\}\right).$$

Then $\langle I_n \rangle_{n=1}^{\infty}$ is a monotone increasing sequence.

Problem 1. Prove this. □

The last two lemmas and Proposition 4 imply that the following two limits exist in the extended real numbers.

$$\lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}$$
$$\lim_{n\to\infty} \inf\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

Therefore the following definition makes sense.

Definition 7. Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of real numbers.

$$\begin{aligned} \limsup_{k\to\infty} a_k &= \lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+1}, \dots\} \\ \liminf_{k\to\infty} a_k &= \lim_{n\to\infty} \inf\{a_n, a_{n+1}, a_{n+2}, \dots\}. \end{aligned}$$

□

Proposition 8. *For any sequence $\langle a_k \rangle_{k=1}^{\infty}$ show that*

$$\liminf_{k\to\infty} a_k \le \limsup_{k\to\infty} a_k.$$

Problem 2. Prove this. □

Problem 3. Find the following:

(a) $\liminf_{n\to\infty}(3 + (-1)^n)$.
(b) $\limsup_{n\to\infty}(3 + (-1)^n)$.
(c) $\liminf_{n\to\infty}(-2)^n$.

Problem 4. Give examples of sequences $\langle a_k \rangle_{k=1}^{\infty}$ such that

(a) $\liminf_{k\to\infty} a_k = -1$ and $\limsup_{k\to\infty} a_k = +1$.
(b) $\liminf_{k\to\infty} a_k = 0$ and $\limsup_{k\to\infty} a_k = +\infty$.

Proposition 9. *If* $\langle a_k \rangle_{k=1}^{\infty}$ *is a sequence with such that*

$$\lim_{k\to\infty} a_k = L$$

exists, then

$$\limsup_{n\to\infty} a_k = \liminf_{k\to\infty} a_k = L.$$

Problem 5. Prove this. □

Proposition 10. *If*

$$\liminf_{k\to\infty} a_k = \limsup_{k\to\infty} a_k$$

and this number is finite, then

$$\lim_{k\to\infty} a_k$$

exists.

Problem 6. Prove this. □

Proposition 11. *If* $\langle a_k \rangle_{k=1}^{\infty}$ *and* $\langle b_k \rangle_{k=1}^{\infty}$ *are two sequences of real numbers, then*

$$\limsup_{k\to\infty}(a_k + b_k) \leq \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k.$$

Problem 7. Prove this and give an example where

$$\limsup_{k\to\infty}(a_k + b_k) < \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k.$$ □