CONVEX BODIES OF CONSTANT WIDTH AND CONSTANT BRIGHTNESS

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A *convex body* in the n-dimensional Euclidean space $\mathbf{R}^n$ is a compact convex set with non-empty interior. A convex body K in three dimensional Euclidean space has *constant width* w iff the orthogonal projection of K onto every line is an interval of length w. It has *constant brightness* b iff the orthogonal projection of K onto every plane is a region of area b.

Theorem 1. *Any convex body in $\mathbf{R}^3$ of constant width and constant brightness is a Euclidean ball.*

Under the extra assumption that the boundary is of class C^2 this was proven by S. Nakajima (= A. Matsumura) in 1926 Theorem 1 solves this problem.

For convex bodies with C^2 boundaries and positive curvature Nakajima's result was generalized by Chakerian [**?**] in 1967 to "relative geometry" where the width and brightness are measured with with respect to some convex body K_0 symmetric about the origin called the *gauge body*. While the main result of this paper is Theorem 1, Chakerian's methods generalize and simplify parts of our original proof. The following isolates the properties required of the gauge body. Recall the *Minkowski sum* of two subsets A and B of $\mathbf{R}^n$ is $A + B = \{a + b : a \in A, b \in B\}$.

Definition. A convex body K_0 is a *regular gauge* iff it is centrally symmetric about the origin and there are convex sets K_1, K_2 and Euclidean balls B_r and B_R such that $K_0 = K_1 + B_r$ and $B_R = K_0 + K_2$.

Any convex body symmetric about the origin with C^2 boundary and positive Gaussian curvature is a regular gauge (Corollary **??** below). For any linear subspace P of $\mathbf{R}^n$ let $K|P$ be the projection of K onto P (all projections in this paper are orthogonal). For a unit vector u let $w_K(u)$ be the width in the direction of u. For each positive integer k and any Borel subset A of $\mathbf{R}^n$ let be $V_k(A)$ be the k-dimensional volume of A (which in this paper is the k-dimensional Hausdorff measure of A). Two subsets A and B of $\mathbf{R}^n$ are *homothetic* iff there is a positive scalar λ and a vector v_0 such that $B = v_0 + \lambda A$.

Theorem 2. *Let K_0 be a regular gauge in $\mathbf{R}^3$ and let K be any convex body in $\mathbf{R}^3$ such that for some constants α, β the equalities $w_K(u) = \alpha w_{K_0}(u)$ and $V_2(K|u^\perp) = \beta V_2(K_0|u^\perp)$ hold for all $u \in \mathbb{S}^2$. Then K is homothetic to K_0.*

Letting K_0 be a Euclidean ball recovers Theorem 1. While we are assuming some regularity on the gauge body K_0, the main point is that no assumptions, other than convexity, are being put on K.

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