NOTES ON ANALYSIS

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Contents

1. The real numbers.

Our short term goal is to give a precise description of the real numbers, $\mathbb{R}$. This will involve three aspects of them. The first is the usual algebraic properties (addition, multiplication, etc.), order properties (the basic properties of inequalities) and finally a completeness property (the least upper bound axiom) which in one form says that there are no "holes" in the real numbers.

1.1. Fields.

Here we deal with the algebraic properties of the real numbers. A ***field*** is an algebraic object where we can do the usual operations of high school algebra. That is addition, subtraction, multiplication, and division.

Definition 1. A ***field*** is a set F with operations[1] $+$, called ***addition***, and $\cdot$, called ***multiplication***, such that

(a) both operations are ***associative***:

$$(x+y)+z = x+(y+z) \qquad x\cdot(y\cdot z) = (x\cdot y)\cdot z$$

for all $x, y, z \in F$.

(b) both operations are ***commutative***:

$$x+y = y+x \qquad x\cdot y = y\cdot x$$

for all $x, y \in F$.

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[1]To be a bit more precise we should call these ***binary operations*** in that that they take an ordered pair of elements of F, say (x, y), and each gives a unique output $x+y$ or $x\cdot y$.

(c) Multiplication ***distributes over addition***:

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

for all $x, y, z \in F$.

(d) There are additive and multiplicative ***identities***. That is there are $0 \in F$ and $1 \in F$ such that

$$x + 0 = x \qquad x \cdot 1 = x$$

for all $x \in F$.

(e) Every element has an ***additive inverse***. That is for every $x \in F$ there is an element $y \in F$ such that

$$x + y = 0.$$

(f) Every nonzero element has a ***multiplicative inverse***. There is for $x \in F$, with $x \neq 0$, there is an element $z \in F$ such that

$$xz = 1.$$

(g) F has at least two elements.

This definition requires a bit of comment. First as to the additive identity the definition as it stands does not rule out the possibility that there are two additive identities. that is there are $0, 0' \in F$ with

$$x + 0 = x \quad \text{and} \quad x + 0' = x$$

for all $x \in F$. In this case

$$\begin{aligned} 0' &= 0' + 0 && (x + 0 = x \text{ with } x = 0') \\ &= 0 + 0' && (\text{addition is commutative}) \\ &= 0 && (x + 0' = x \text{ with } x = 0). \end{aligned}$$

So 0 and $0'$ are the same element.

Problem 1. Use a variant on this argument to show that if $1, 1' \in F$ satisfy

$$x \cdot 1 = x \quad \text{and} \quad x \cdot 1' = x$$

for all $x \in F$ that $1 = 1'$. Thus the multiplicative identity is unique. □

We also have that additive inverses are unique. Let $x \in F$ and assume that $y, y' \in F$ such that

$$x + y = 0 \quad \text{and} \quad x + y' = 0.$$

Then

$$\begin{aligned}
y' &= y' + 0 && (x + 0 = x \text{ with } x = y') \\
&= y' + (x + y) && (x + y = 0) \\
&= (y' + x) + y && (\text{associativity of addition}) \\
&= (x + y') + y && (\text{commutative of addition}) \\
&= 0 + y && (x + y' = 0) \\
&= y + 0 && (\text{commutative of addition}) \\
&= y && (x + 0 = x \text{ with } x = y)
\end{aligned}$$

Thus the additive inverse of any element any element is unique. From now on we denote the additive inverse of $x \in F$ as $-x$ and use the abbreviation

$$x - y := x + (-y).$$

Proposition 2. *For any $x \in F$ the equality*

$$-(-x) = x$$

holds.

Proof. By definition $-(-x)$ is the additive inverse of $-x$. But we also have

$$\begin{aligned}
-x + x &= x + (-x) && (\text{commutative of additive}) \\
&= 0 && (-x \text{ is additive inverse of } x)
\end{aligned}$$

This shows that x is also an additive inverse of $-x$. As additive inverses are unique we have $-(-x) = x$. □

Problem 2. Modify the argument above to show that the multiplicative inverse of $x \in F$ with $x \neq 0$ is unique. □

If $x \in F$ and $x \neq 0$ we now denote the unique multiplicative inverse of x by either of the two notations.

$$\text{multiplicative inverse of } x = \frac{1}{x} = x^{-1}.$$

and write

$$yx^{-1} := \frac{y}{x}.$$

Problem 3. Modify one of the arguments above to show if $x \in F$ with $x \neq 0$ then

$$(x^{-1})^{-1} = x.$$

That is

$$\frac{1}{\left(\dfrac{1}{x}\right)} = x.$$

□

Here are several results that we are so use to seeing that it seems irritating to have to prove them.

Proposition 3. *In a field* $-0 = 0$*.*

Problem 4. Prove this. *Hint:* $0+0 = 0$ so 0 is the additive inverse of 0. $\square$

Problem 5. In a field, F,

$$x \cdot 0 = 0$$

for all $x \in F$.

Problem 6. Prove this. *Hint:* First show $x \cdot 0 = x \cdot 0 + x \cdot 0$ by justifying the steps in the following.

$$\begin{aligned} x \cdot 0 &= x \cdot (0+0) \\ &= x \cdot 0 + x \cdot 0. \end{aligned}$$

Now add the additive inverse of $x \cdot 0$ to both sides of $x \cdot 0 = x \cdot 0 + x \cdot 0$. $\square$

The associativity law implies that for any three elements $x_1, x_2, x_3 \in F$ that

$$(x_1x_2)x_3 = x_1(x_2x_3).$$

As this is the only two ways to group the product of three elements we can write the product of three elements as

$$x_1x_2x_3$$

without ambiguity. There are five ways to group four elements in a product

$$x_1(x_2(x_3x_4)),\ x_1((x_2x_3)x_4),\ (x_1x_2)(x_3x_4),\ (x_1(x_2x_2))x_4,\ ((x_1x_2)x_3)x_4$$

These are all equivalent. We see this by showing they are all the same as $x_1(x_2(x_3x_4))$.

$$\begin{aligned} x_1((x_2x_3)x_4) &= x_1(x_2(x_3x_4)) && \text{as} \quad (x_2x_3)x_4 = x_2(x_3x_4) \\ (x_1x_2)(x_3x_4) &= x_1(x_2(x_3x_4)) && \text{as} \quad (x_1x_2)y = x_1(x_2y) \text{ with } y = x_3x_4 \\ (x_1(x_2x_3))x_4 &= x_1((x_2x_3)x_4) && \text{as} \quad (x_1y)x_4 = x_1(yx_4) \text{ with } y = x_2x_3 \\ &= x_1(x_2(x_3x_4)) && \text{as} \quad (x_2x_3)x_4 = x_2(x_3x_4) \\ ((x_1x_2)x_3)x_4 &= (x_1x_2)(x_3x_4) && \text{as} \quad (yx_3)x_4 = y(x_3x_4) \text{ with } y = x_1x_2 \\ &= x_1(x_2(x_3x_4)) && \text{as} \quad (x_1x_2)y = x_1(x_2y) \text{ with } y = x_3x_4 \end{aligned}$$

So again we can write the product

$$x_1x_2x_3x_4$$

without ambiguity as all the groupings are equal. In light of this the following will most likely not surprise you.

Proposition 4. *Let* $x_1x_2, \ldots, x_n$ *be elements of the field. Then the associativity law implies that any two groupings of the product* $x_1x_2 \cdots x_n$ *are equal.*

Problem 7. Prove this. *Hint:* Use induction to show that any grouping is equal to the grouping

$$x_1(x_2(x_3(x_4\cdots x_n)\cdots)).$$

This is the grouping where the parenthesis are moved as far to the right as possible. For the rest of this problem call this the ***standard form*** of the product.

Here is the induction step in going from $n=5$ to $n=6$. For $n=5$ the standard form is

$$x_1(x_2(x_3(x_4x_5))).$$

Let p be some grouping of $x_1, x_2, \ldots, x_6$. We first consider the case that p is of the form

$$p = x_1(p_2)$$

where p_2 is a product of $x_2, \ldots, x_6$. Then p_2 is a product of $n=5$ elements and thus by the induction hypothesis $p_2 = x_2(x_3(x_4(x_5x_6)))$. But then $p = x_1(p_2) = x_1(x_2(x_3(x_4(x_5x_6))))$ can be put in standard form.

This leaves the case where $p = (p_1)(p_2)$ where for some k with $2 \le k \le 5$ we have

$$p = (p_1)p_2$$

where p_1 is a product of $x_1, \ldots, x_k$ and p_2 is a product of $x_{k+1}, \ldots, x_6$. Then, as p_1 has less than $n=6$ factors it can be put in standard form. This implies that $p_1 = x_1(q)$ where q is a product of $x_2, \ldots, x_k$. Therefore

$$p = (p_1)p_2 = (x_1q)p_2 = x_1(qp_2).$$

But qp_2 only involves the variables $x_2, \ldots, x_6$ so anther application of the induction hypothesis implies that qp_2 can be put standard form. But then $p = x_1(qp_2)$ is in standard form.

To complete the proof you show show that this argument can be used to show that if it is true for n variables, then it is true of $n+1$ variables. $\square$

From now on we write products $x_1x_2\cdots x_n$ without putting the the parenthesis. There is a similar proposition about parenthesis and and sums, and we will write also write sums as $x_1 + x_2 + \cdots + x_n$ without parenthesis.

The following could be summarized by saying that much of the basic results you know from basic algebra still holds in fields.

Proposition 5. *Let F be a field. Then*

(a) If $a, b, c, d \in F$ and $b, c \neq 0$ then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

(b) If $a, b \in F$ and $a, b = 0$, then $a = 0$ or $b = 0$.

(c) (This is just a useful restatement of part (b).) If $a, b \in F$ and $a, b \neq 0$ then $ab \neq 0$.

(d) If the elements $a_1, a_2, \ldots, a_n$ F are all nonzero, then so is the product and
$$(a_1 a_2 \cdots a_{n-1} a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_2^{-1} a_1^{-1}.$$
(e) If $a, b \in F$ and $a^2 = b^2$, then $a = \pm b$.

Problem 8. Prove this. □

Problem 9 (Cramer's rule for solving linear systems.). Here is anther fact that will likely come up at least one during the term. Let a, b, c, d, e, f be elements of the field F with
$$ad - bc \neq 0.$$
Then the equations
$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$
have a unique solution. This solution is
$$x = \frac{ed - bf}{ad - bc}, \qquad y = \frac{af - ec}{ad - bd}.$$
Hint: To find x multiply the first equation by d and the second by b and then subtract the two. A similar trick works to find y. □

1.1.1. *Some examples of fields. The rational numbers.*

We first recall some sets of numbers that occur often enough that they have earned names. First there is the set of ***natural numbers***,
$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$
and the set of ***integers***
$$\mathbb{Z} = \{\ldots - 4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$
Thus the natural numbers are just the positive integers.[2] Then the ***rational numbers*** are
$$\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$
That is rational numbers are the quotients of integers where the denominators are not zero. It is not hard to check that $\mathbb{Q}$ is a field.

The rational numbers are fine for doing some parts of algebra, for example if $a, b \in \mathbb{Q}$ with $a \neq 0$ we can always solve the equation
$$ax + b = 0$$
for x. More generally if $a, b, c, d, e, f \in \mathbb{Q}$ and $ad - bc \neq 0$ then we can solve
$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

[2]Some books and some mathematicians include 0 in the set of natural numbers, so that for them $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$. I am going to use the same convention our text uses.

for x and y and get rational numbers as solutions. In your linear algebra class you saw that in any field that you can solve consistent systems of linear equations of any size and get solution that are in the same field as the coefficients.

However there are natural equations that do not have solutions in the rational number. For example $x^2 - 2 = 0$ has no rational solution as $\sqrt{2}$ is irrational. More generally we have:

Theorem 6. *If m is a positive integer that is not a perfect square (that is $m \neq k^2$ for any integer k) then the equation*

$$x^2 = m$$

has no solution in the rational numbers. (That is $\sqrt{m}$ is irrational.)

Proof. Towards a contradiction assume $x = \sqrt{m}$ is a rational number that is not an integer. Let n be the smallest positive integer such that the product nx is an integer.

Let $\lfloor x \rfloor$ be the greatest integer in x. Then $0 \leq x - \lfloor x \rfloor < 1$. And as x is not an integer $x \neq \lfloor x \rfloor$ and so $0 < x - \lfloor x \rfloor < 1$. Let $p = n(x - \lfloor x \rfloor)$. Then $0 < p < n$ and $p = nx - n\lfloor x \rfloor$ is an integer. But, using that $x^2 = m$,

$$px = n(x - \lfloor x \rfloor)x = nx^2 - (nx)\lfloor x \rfloor) = nm - (nx)\lfloor x \rfloor$$

which is an integer. As $p < n$ this contradicts that n was the smallest positive integer such that the product nx is an integer. $\square$

The last theorem shows that all the square roots that you expect to be irrational ($\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, etc.) are irrational. In some vague sense this tells use that the rational numbers are not "complete" in the sense that there is a missing number or a "hole" where $\sqrt{2}$ should be.

1.1.2. *An aside: Some other examples of fields.* While these will not come up again in this course, it is interesting to see a couple of other examples of fields.

The first example is the integers modulo a prime, p. One description of this is the numbers $0, 1, \ldots, (p-1)$ and then when adding or multiplying them reduce them modulo p. (If n is an integer, then n reduced mod p is the remainder when p is divided into n.) For $p = 5$ here are what the addition and multiplication tables.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

+	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Thus there are fields that only have a finite number of elements.

The second example is the rational numbers extended by and irrational square root of an integer. Let m be a positive integer that is not a perfect square. Then by Theorem 6 the number $\sqrt{m}$ is irrational. Let

$$\mathbb{Q}(\sqrt{m}) := \{a + b\sqrt{m} : a, b \in \mathbb{Q}\}.$$

This set is closed under addition and subtraction. Also as

$$(a + b\sqrt{m})(c + d\sqrt{m}) = (ac + bdm) + (ad + bc)\sqrt{m}$$

we see that $\mathbb{Q}(\sqrt{m})$ is closed under multiplication. For it to be a field still need that multiplicative inverses exist.

$$\begin{aligned} \frac{1}{a + b\sqrt{m}} &= \frac{a - b\sqrt{m}}{(a + b\sqrt{m})(a - b\sqrt{m})} \\ &= \frac{a - b\sqrt{m}}{a^2 - b^2 m} \\ &= \frac{a}{a^2 - b^2 m} + \frac{-b}{a^2 - b^2 m}\sqrt{m} \\ &= \alpha + \beta\sqrt{m} \end{aligned}$$

where

$$\alpha = \frac{a}{a^2 - b^2 m}, \qquad \beta = \frac{-b}{a^2 - b^2 m}$$

are rational numbers, provided the denominator $a^2 - b^2 m \neq 0$. But if $a^2 - b^2 m = 0$, then $m = (a/b)^2$, contradicting that $\sqrt{m}$ is irrational. Thus we can add, subtract, multiply, and divide in $\mathbb{Q}(\sqrt{m})$ and so $\mathbb{Q}(\sqrt{m})$ is a field. These fields are useful in number theory.

1.2. **The order axioms.** Let $\mathbb{F}$ be a field. Then an ***ordering*** of $\mathbb{F}$ is a subset $\mathbb{F}_+$, called the set of ***positive elements***, such that the following hold.

Pos 1: The set of positive elements is closed under addition and multiplication. That is if $a, b \in \mathbb{F}_+$ then $a + b \in \mathbb{F}_+$ and $ab \in \mathbb{R}_+$.

Pos 2: (The trichotomy principle.) For any $a \in \mathbb{F}$ exactly one of the following holds:

$$\begin{aligned} a &\in \mathbb{F}_+ \\ a &= 0 \\ -a &\in \mathbb{F}_+. \end{aligned}$$

We define the set ***negative elements*** of $\mathbb{F}$ as

$$\mathbb{F}_- = \{a : -a \in \mathbb{F}_+\}.$$

Now define the relation $<$ on $\mathbb{F}$ by

$$a < b \quad \text{if and only if} \quad b - a \in \mathbb{F}_+.$$

(Likewise $b < a$ is defined by $b - a \in \mathbb{F}_+$, which is the same as $a > b$.) We define $a \leq b$ by

$$a \leq b \quad \text{if and only if} \quad a = b \text{ or } a < b.$$

with a similar definition for $a \geq b$. As expected $a <$ is read as "a is less than b" and $a \leq b$ is real as "a is less than or equal to b".

With this definition and terminology we have that the set of positive elements is the same as the set of elements that are greater than 0.

Proposition 7 (Trichotomy for inequalities.)**.** *If $a, b \in \mathbb{F}$, then exactly one of the following holds:*

$$a < b, \quad a = b, \quad a > b.$$

Problem 10. Prove this. *Hint:* Apply **Pos: 1** to $b - a$.

Proposition 8 (Transitivity)**.** *If $a < b$ and $b < c$ then $a < c$.*

Problem 11. Prove this. □

Proposition 9. *If $a < b$ and $c < d$ then $a + c < b + d$.*

Problem 12. Prove this. □

Proposition 10. *If $a < b$ and $c > 0$, then $ac < bc$.*

Problem 13. Prove this. □

Proposition 11. *If $a < b$ and $c < 0$, then $ac > bc$.*

Problem 14. Prove this. □

Proposition 12. *If $a < b$ and $c < d$, then $a + c < b + d$.*

Problem 15. Prove this. □

Proposition 13. *If $0 < a < b$ and $0 < c \leq d$, then $ac < bd$.*

Problem 16. Prove this. □

Proposition 14. *If $a_1, a_2, \ldots, a_n > 0$ then $a_1 a_2 \cdots a_n > 0$ and and $a_1 + a_2 + \cdots + a_n > 0$. (That is the sum and product of positive numbers is positive.) Thus if $a > 0$ then a^n and $na > 0$.*

Problem 17. Prove this. *Hint:* Use this problem to practice using induction. □

Proposition 15. *If $a \neq 0$, then $a^2 > 0$. That is the square of any nonzero element is positive. In particular $1 = 1^2$ is positive.*

Problem 18. Prove this. *Hint:* By trichotomy we have $a > 0$, $a = 0$, or $a < 0$. We are assuming that $a \neq 0$. So this leaves two cases: If $a > 0$, then as the positive numbers are closed under multiplication $a^2 = aa > 0$. If $-a > 0$ then $a^2 = (-a)^2 = (-a)(-a) > 0$. □

Proposition 16. *Let* $a_1, a_2, \ldots, a_n \in \mathbb{F}$. *Then*
$$a_1^2 + a^2 + \cdots + a_n \geq 0$$
with equality if and only if $a_1 = a_2 = \cdots = a_n = 0$. *That is the sum of squares of elements from* $\mathbb{F}$ *is positive unless all the elements are zero.*

Problem 19. Prove this. *Hint:* Use induction. □

Proposition 17. *If* $a > 0$*, then* $1/a > 0$*. If* $a < 0$ *then* $1/a < 0$*.*

Problem 20. Prove this. *Hint:* Towards a contradiction, assume that $a > 0$ and $1/a < 0$. Then use $1 = a(1/a)$ to get the contradiction. □

Proposition 18. *If* $0 < a < b$*, then* $1/b < 1/a$*.*

Problem 21. Prove this. *Hint:* Multiply the inequality $a < b$ by the positive number $1/ab$. □

If $a \in \mathbb{F}$ we define the ***absolute value*** of a by
$$|a| = \begin{cases} a, & a > 0; \\ 0, & a = 0; \\ -a, & a < 0. \end{cases}$$
One way to think of $|a|$ is that it is the distance of a from the origin. Inequalities involving absolute values will come up repeatedly in what follows. Here are some of the basic ones. Note that a direct consequence of the defintion is that
$$|-a| = |a|.$$

Proposition 19. *For* $a \in \mathbb{F}$,
$$|a| \geq 0$$
with equality if and only if $a = 0$*.*

Problem 22. Prove this. *Hint:* Consider the three case $a > 0$, $a = 0$, and $a > 0$. □

Proposition 20. *For* $a \in \mathbb{F}$ *we have* $a \leq |a|$*.*

Problem 23. Prove this. □

Proposition 21. *For* $a \in \mathbb{F}$ *we have* $a^2 = |a|^2$*.*

Problem 24. Prove this. □

Proposition 22. *If* $a, b \in \mathbb{F}$*, then the following are equivalent:*
(a) $|a| = |b|$,
(b) $a = \pm b$,
(c) $a^2 = b^2$.

Problem 25. Prove this. □

The following can be useful in proving inequalities about absolute values:

Proposition 23. *If* $a, b \in \mathbb{F}$*,*

$$|a| < |b| \quad \textit{if and only if} \quad a^2 < b^2$$

Proof. First assume that $|a| < |b|$. Then $|b| - |a| > 0$. Multiply this inequality by the positive element $|b| + |a|$ to get

$$(|b| + |a|)(|b| - |a|) > 0.$$

This simplifies to

$$|b|^2 - |a|^2 > 0.$$

But $|a|^2 = a^2$ and $|b|^2 = b^2$ and thus

$$b^2 - a^2 > 0$$

which implies $a^2 < b^2$. This shows $|a| < |b|$ implies $a^2 < b^2$.

Conversely assume $a^2 < b^2$. Then $0 < b^2 - a^2$. Again using that $|a|^2 = a^2$ and $|b|^2 = b^2$ we have

$$0 < b^2 - a^2 = |b|^2 - |a|^2 = \big(|b| + |a|\big)\big(|b| - |a|\big).$$

Multiply this inequality by the positive element $1/(|b| + |a|)$ to get

$$|b| - |a| > 0$$

which implies $|a| < |b|$. Thus we have shown $a^2 < b^2$ implies $|a| < |b|$, which completes the proof. $\square$

Here is a slight variant on the last proposition.

Proposition 24. *If* $a, b \in \mathbb{F}$*, then*

$$|a| \leq |b| \quad \textit{if and only if} \quad a^2 \leq b^2.$$

Proof. This follows from the last proposition by splitting into the two cases $|a| = |b|$ and $|a| < |b|$. $\square$

Proposition 25. *Let* $a > 0$*. Then for* $x \in \mathbb{F}$

$$|x| < a \quad \textit{if and only if} \quad -a < x < a.$$

Problem 26*.* Prove this. $\square$

Proposition 26 (The triangle inequality)**.** *If* $a, b \in \mathbb{F}$*, then*

$$|a + b| \leq |a| + |b|.$$

Proof. The most straightforward proof of this is by considering cases. But the number of cases is large enough that this is no fun. Here is a somewhat more insightful method idea.

$$\begin{aligned}
(a + b)^2 &= a^2 + 2ab + b^2 \\
&= |a|^2 + 2ab + |b|^2 && (\text{as } a^2 = |a|^2 \text{ and } b^2 = |b|^2) \\
&\leq |a|^2 + 2|a||b| + |b|^2 && (\text{as } ab \leq |ab| = |a||b|) \\
&\leq (|a| + |b|)^2
\end{aligned}$$

That is $(a+b)^2 \le (|a|+|b|)^2$. Therefore by Proposition 24

$$|a+b| \le \big||a|+|b|\big| = |a|+|b|$$

where $\big||a|+|b|\big| = |a|+|b|$ because $|a|+|b|$ is positive. □

Proposition 27 (The reverse triangle inequality.)**.** *If $a, b \in \mathbb{F}$, then*

$$\big||a|-|b|\big| \le |a-b|.$$

Understanding the absolute value of products and quotients is easier:

Proposition 28. *If $a, b \in \mathbb{F}$, then*

$$|ab| = |a||b|$$

and if $b \neq 0$,

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

Problem 27*.* Prove this. □

We will use the usual notation to define intervals in $\mathbb{F}$. That is

$$\begin{aligned}
(a,b) &= \{x \in \mathbb{F} : a < x < b\} \\
[a,b) &= \{x \in \mathbb{F} : a \le x < b\} \\
(a,b] &= \{x \in \mathbb{F} : a < x \le b\} \\
[a,b] &= \{x \in \mathbb{F} : a \le x \le b\}
\end{aligned}$$

where $a, b \in \mathbb{F}$ and $a < b$. The length of any of these intervals is $b - a$.

For any $x_1, x_2 \in \mathbb{F}$, we can think of $|x_2 - x_1|$ as the distance between x_1 and x_2. If $x_1, x_2 \in [a,b]$ then the distance between x_1 and x_2 should be at most the length of the interval. The following makes this precise.

Proposition 29. *If $a < b$ and $x_1, x_2 \in [a,b]$, then*

$$|x_2 - x_1| \le (b-a).$$

Problem 28*.* Prove this. □

1.2.1. *Some practice with inequalities.* Being able to work with inequalities is a one of the main tools in analysis. In this subsection we give some examples and problems using inequalities. We will be using the following generalization of the triangle inequality:

Proposition 30. *If $a_1, a_2, \ldots, a_n \in \mathbb{F}$, then*

$$|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots |a_n|.$$

Proof. This is an easy induction proof and is left to the reader. □

Example 31. Let $\delta > 0$ and let $|x - a| < \delta$. Then

$$|(3x + 2) - (3a + 2)| < 3\delta.$$

Solution: This one is just a matter of regrouping:

$$\begin{aligned} |(3x + 2) - (3a + 2)| &= |3(x - a)| \\ &= 3|x - a| \\ &< 3\delta. \end{aligned}$$

□

Problem 29. Let $\delta > 0$. Assume that $|y - c| < \delta$. Show that

$$|(-5y + 42) - (-5c + 42)| < 5\delta.$$

Example 32. If $|x - a| < \delta \leq 1$, then

$$|x^3 - a^3| < 3(1 + |a|)^2\delta.$$

Solution: We start with a bit of algebra:

$$\begin{aligned} |x^3 - a^3| &= |(x - a)(x^2 + ax + a^2)| \\ &= |x - a||x^2 + ax + a^2| \\ &\leq |x^2 + ax + a^2|\delta && \text{(Using that } |x - a| < \delta\text{)} \\ &\leq \left(|x^2| + |ax| + |a^2|\right)\delta && \text{(Triangle inequality)} \\ &= \left(|x|^2 + |a||x| + |a|^2\right)\delta && \text{(Using } |AB| = |A||B|\text{)} \end{aligned} \tag{1}$$

We now use use that "add and subtract" trick, which will come up at least 173 times during the term.

$$\begin{aligned} |x| &= |x - a + a| \\ &\leq |x - a| + |a| && \text{(Triangle inequality)} \\ &< \delta + |a| && \text{(As } |x - a| < \delta\text{)} \\ &\leq 1 + |a| && \text{(As } \delta \leq 1\text{)} \end{aligned}$$

We also have the trivial inequality

$$|a| < |a| + 1.$$

Using these inequalities we have

$$\begin{aligned} |x|^2 + |a||x| + |a|^2 &\leq (1 + |a|)^2 + (1 + |a|)(1 + |a|) + (1 + |a|)^2 \\ &= 3(1 + |a|)^2. \end{aligned}$$

Finally using this in (1) we get

$$|x^3 - a^3| < 3(1 + |a|)^2\delta$$

as required. □

Problem 30. Show that if $|y-c|<\delta\le 2$, then

$$|y^4-c^4|\le 4(2+|c|)^3\delta$$

□

Problem 31. Assume $|a-x|<\delta$, $|y-b|<\delta$, and $\delta\le 1$. Show that

$$|xy-ab|<(1+|a|+|b|)\delta$$

Hint: Here is a start

$$\begin{aligned}
|xy-ab| &= |xy-ay+ay-ab| && \text{(Adding and subtracting trick)}\\
&=|(x-a)y+a(y-b)| &&\\
&\le|x-a||y|+|a||y-b| && \text{(Triangle inequality)}\\
&\le\delta|y|+|a|\delta && (|x-a|<\delta \text{ and } |y-b|<\delta)\\
&=(|y|+|a|)\delta &&
\end{aligned}$$

and the rest is left to you. □

Example 33. If $c\ne 0$ and $|z-c|<\delta<|c|/2$, show $z\ne 0$ and that

$$\left|\frac{1}{z}-\frac{1}{c}\right|<\frac{2\delta}{|c|^2}$$

Solution: First

$$\begin{aligned}
|z| &= |c+(z-c)| && \text{(Add and subtract trick)}\\
&\ge |c|-|z-c| && \text{(reverse triangle inequality)}
\end{aligned}$$

We are given that

$$|z-c|<\frac{|c|}{2}.$$

Thus

$$-|z-c|\ge-\frac{|c|}{2}.$$

Using this with what we have already done gives

$$|z|\ge|c|-|z-c|\ge|c|-\frac{|c|}{2}=\frac{|c|}{2}.$$

This implies $z\ne 0$. For future use note this also implies

$$\frac{1}{|z|}\le\frac{2}{|c|}.$$

Now

$$\begin{aligned}
\left|\frac{1}{z}-\frac{1}{c}\right| &= \left|\frac{c-z}{zc}\right| \\
&= \frac{1}{|c|}\frac{1}{|z|}|c-z| \\
&< \frac{1}{|c|}\frac{1}{|z|}\delta && \text{(As } |z-c|<\delta\text{)} \\
&\le \frac{1}{|c|}\frac{2}{|c|}\delta && \text{(As } 1/|z|<2/|c|\text{)} \\
&= \frac{2\delta}{|c|^2}
\end{aligned}$$

as required. □

Problem 32. Assume $|x-a|<\delta\le|a|/2$. Show

$$\frac{|a|}{2}<|x|<\frac{3|a|}{2}$$

and

$$\left|\frac{1}{x^2}-\frac{1}{a^2}\right|<\frac{6\delta}{|a|^3}$$

□

Let $a,b\in\mathbb{F}$. Then define

$$\max(a,b)=\begin{cases} a, & \text{if } a\ge b;\\ b, & \text{if } b>a.\end{cases}$$

$$\min(a,b)=\begin{cases} a, & \text{if } a\le b;\\ b, & \text{if } b<a.\end{cases}$$

Proposition 34. *For* $a,b\in\mathbb{F}$

$$\max(a,b)=\frac{a+b+|a-b|}{2}$$

$$\min(a,b)=\frac{a+b-|a-b|}{2}$$

Problem 33. Prove this. □

Example 35. Solve the inequality $x^2+6x+5\le 0$.

Solution: Like many things involving quadratic polynomials completing the square is a smart thing to do. Adding 4 to both sides of the inequality gives

$$x^2+3x+9\le 4.$$

This can be rewritten as

$$(x+3)^3\le 2^2$$

which is equivalent to

$$|x+3|\le 2.$$

This in turn is equivalent to

$$-3 - 2 \leq x \leq -3 + 2$$

and therefore the solution set is the interval

$$(-5, -1) = \{x : -5 \leq x \leq -1\}.$$

□

Problem 34. Solve the following inequalities

(a) $5x - 9 < 7x + 21$.
(b) $x^2 - 10x + 9 < 16$.
(c) $\dfrac{x+2}{x-2} \leq 5$. □

1.3. **The least upper bound axiom.** Let $\mathbb{F}$ be an ordered field and let $S \subseteq \mathbb{F}$ and assume $S \neq \varnothing$. Then S is ***bounded above*** if and only if there is a $b \in S$ such that

$$s \leq b \quad \text{for all} \quad s \in S.$$

Any such b is an ***upper bound*** for S. Likewise S is ***bounded below*** if there is an $a \in \mathbb{F}$ with

$$s \geq a \quad \text{for all} \quad s \in S,$$

and a is a ***lower bound*** for S.

Note that not every subset of $\mathbb{F}$ will have an upper bound or a lower bound. For example if $S = \mathbb{F}$, then S has no upper or lower bound. And in the rational numbers the integers, $\mathbb{Z}$, has no upper or lower bounds.

Definition 36. If $\varnothing \neq S \subseteq \mathbb{F}$, then c is a ***least upper bound*** (or ***supremum***) of S if c is an upper bound for S and $c \leq b$ for all upper bounds, b, of S. □

Proposition 37. *If S has a supremum, then it is unique.*

Proof. Let c and c' be supremums (i.e. least upper bounds) for S. Then by definition both the inequalities $c \leq c'$ and $c' \leq c$ hold. This implies $c = c'$. □

If S has a supremum, then we denote it by $\sup(S)$. In older books and articles, such as our text, this is often written as $\operatorname{lub}(S)$ or $\text{l.u.b.}(S)$.

Problem 35. Let $S \subseteq \mathbb{F}$, with $S \neq \varnothing$.

(a) Define what it means for a to be a ***lower bound*** for S.
(b) Define what it means for c to be a ***greatest lower bound*** (or ***infinmum***) of S.
(c) Prove that if S has an infinmum, that it is unique. It will be denoted by $\inf(S)$ (or $\operatorname{glb}(S)$, or $\text{g.l.b.}(S)$). □

If $\varnothing \neq S \subseteq \mathbb{F}$, then b is a ***largest***, or ***maximum***, of S if and only if $b \in S$ and $s \leq b$ for all $s \in S$. A ***smallest*** or ***minimum*** of S is defined similarly. If S has a maximum, then it is denoted by $\max(S)$. Likewise if it has a minimum, it is denoted by $\min(S)$.

Problem 36. Show that if S has a maximum, then it it has a supremum and

$$\sup(S) = \max(S).$$

(And there is a similar result for sets with a minimum.)

Proposition 38. *If $\varnothing \neq S \subseteq \mathbb{F}$ is finite, then S has both a maximum and a minimum.*

Proof. This is an easy induction on the cardinality of S. $\square$

Theorem 39 (Least upper bound property)**.** *In the real numbers, $\mathbb{R}$, every nonempty set that is bounded above has a least upper bound.* $\square$

Unfortunately we will not prove this result in this class. If is not much harder than results that we will prove, but is long and drawn out. The longest part of this is giving a precise construction of the real numbers. This can be done in several ways. The Wikipedia article

`https://en.wikipedia.org/wiki/Construction_of_the_real_numbers`

has good discussion of several of these constructions. This article closes with the quote, which applies to most of the presentations of these constructions, "The details are all included, but as usual they are tedious and not too instructive".

One ideation that the least upper bound property is central is the following theorem, which we will also not prove.

Theorem 40 (Uniquness of the real numbers)**.** *Let $\mathbb{F}$ be an ordered field where every set that is bounded above has a least upper bound. Then $\mathbb{F}$ is isomorphic to the real numbers.* $\square$

Thus the least upper bound property is in some sense the defining property of the real numbers.

There is nothing special about upper bounds:

Theorem 41 (Greatest lower bound property)**.** *In the real numbers every nonempty set that is bounded below has a greatest lower bound.*

Problem 37. Prove this. *Hint:* Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Let b be a lower bound. Let $-S$ be

$$-S := \{-s : s \in S\}.$$

Show $-b$ is an upper bound for $-S$ and therefore S is bounded above. Therefore by the least upper bound property of the real numbers $-S$ has a least upper bound $c = \sup(-S)$. Show that $-c$ is a greatest lower bound for S. $\square$

Proposition 42 (Archimedes' axiom (big version))**.** *For any real number x there is a natural n with $x < n$.*

Problem 38. Prove this. *Hint:* Toward a contradiction assume that there is a real number x such that for all $n \in \mathbb{N}$ we have $n \leq x$. This means that the set $\mathbb{N}$ is bounded above and therefore by the least upper bound property that $\mathbb{N}$ has a least upper bound $b = \sup(\mathbb{N})$. Thus for all natural numbers n

$$n \leq b.$$

But for $n \in \mathbb{N}$ the number $n+1$ is also a natural number and thus for all $n \in \mathbb{N}$

$$n + 1 \leq b,$$

and therefore for all $n \in \mathbb{N}$ we have

$$n \leq b - 1.$$

Use this to derive a contradiction. □

Proposition 43. *Let $a > 1$ be a real number. Show that for any real number x, there is a natural number n such that $a^n > x$.*

Problem 39. Prove this. *Hint:* If this is false, then the set $S = \{a^n : n \in \mathbb{N}\}$ has an upper bound. Derive a contradiction along the lines of the proof of Proposition 42. □

Proposition 44 (Archimedes' axiom (small version))**.** *Let $a > 0$ be a positive real number. There there is a natural number, n, such that $1/n < a$.*

Proof. By the first version of Archimedes's axiom there is a natural number n with $n > 1/a$. But then $1/n < a$. □

Proposition 45. *Let a be a real number with $0 < a < 1$. Then for any real number x, there is a natural number n such that $a^n < x$.*

Problem 40. Prove this as a corollary to Proposition 43. □

Proposition 46 (Existance of greatest integer)**.** *For any real number x there is a unique integer n such that*

$$n \leq x < n + 1.$$

Problem 41. Prove this along the following lines:

(a) The obvious way to choose n is to choose the largest integer n with $n \leq x$. That is choose n to be the maximum of the set $S = \{k \in \mathbb{Z} : k \leq x\}$. This set is bounded above (by x) but it is infinite so there is no guarantee that it has a largest element. To get around this problem use instead the set

$$S_1 = \{k \in \mathbb{Z} : |k| \leq |x| + 1 \text{ and } k \leq x\}.$$

This is a finite set and so by Proposition 38, S_1 has a maximum. Let n be this maximum. Show that $n \leq x < n+1$, so that n is as required.

(b) Prove uniqueness. That is if m is an integer with $m \leq x < m+1$ show $m = n$. *Hint:* One way to do this is to show that m and n satisfy $x - 1 < n \leq x$ and $x - 1 < m \leq x$. That is $m, n \in (x-1, x]$. Therefore $|m - n| < 1$ (cf. the proof of Proposition 29). As $|m - n|$ is an integer this implies $|m - n| = 0$. □

From now one we use the notation

$$\lfloor x \rfloor = \text{The unique integer } n \text{ with } n \leq x < n+1.$$

The integer $\lfloor x \rfloor$ is called the ***greatest integer*** in x or the ***floor*** of x. Let

Proposition 47. *between any two real numbers there is a rational number. (That is if $a < b$ are real numbers, there is a rational number, r, with $a < r < b$.)*

Problem 42. Prove this. *Hint:* By one form of Archimedes axiom there is a natural number N with

$$\frac{1}{N} < (b - a).$$

Let $n = \lfloor Na \rfloor$. Then

$$n \leq Na < n + 1.$$

Show

$$a < \frac{n+1}{N} < b$$

and therefore the rational number

$$r = \frac{n+1}{N}$$

does the trick. □

Proposition 48. *Between any two rational numbers there is an irrational number.*

Problem 43. Prove this. □

The following may look a little silly at first, but variants on it will be used repeatedly during the term to show that two numbers are equal. The first example of this is the proof of Theorem 50 below.

Proposition 49. *Let $y_0, y_1 \in \mathbb{R}$ and assume that there is a number $M > 0$ such that for all $\varepsilon > 0$*

$$|y_1 - y_0| \leq M\varepsilon. \tag{2}$$

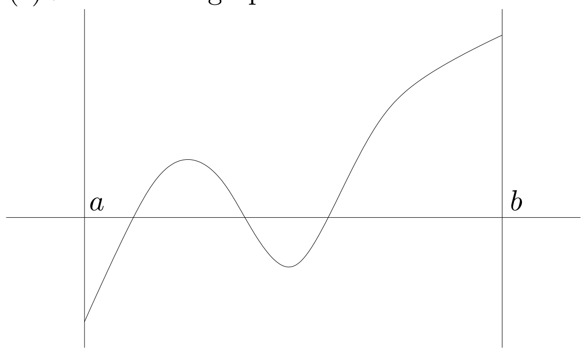
Then $y_0 = y_1$.

Problem 44. Prove this. *Hint:* Towards a contradiction assume that $y_0 \neq y_1$. Then let

$$\varepsilon = \frac{|y_1 - y_0|}{2M}$$

in the inequality (2) and show this leads to a contradiction. □

We now show that the least upper bound principle lets us show that reasonable equations have solution. Consider a function $f\colon [a,b] \to \mathbb{R}$ with $f(a) < 0$ and $f(b) > 0$. So the graph looks like

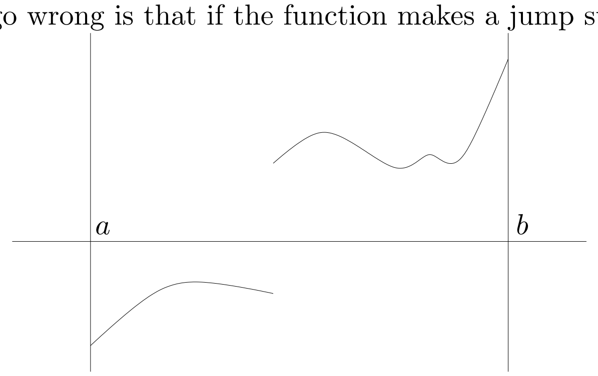


where the graph starts below the x-axis (at $x = a$) and ends up above the axis (at $x = b$). Then if there is any justice the graph will cross the axis somewhere between a and b and thus there is at least one point $\xi \in (a,b)$ with $f(\xi) = 0$. That is the equation

$$f(x) = 0$$

has a solution in the interval (a,b).

What can go wrong is that if the function makes a jump such as



So we need a condition that rules out jumps. Later in the term we will define what it means for a function to be continuous and show that for continuous functions that there are no jumps. Here we will use a less general condition.

Theorem 50 (Baby intermediate value theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be a function such that*

$$f(a) < 0 \quad \textit{and} \quad f(b) > 0$$

and such that there is a $M > 0$ such that for all $x_1, x_2 \in [a,b]$ the inequality

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$$

holds. Then there is a number $\xi \in (a,b)$ with

$$f(\xi) = 0.$$

Problem 45. Prove this. *Hint:* First we use the least upper bound principle to get a candidate for ξ. Let

$$S = \{x \in [a,b] : f(x) \le 0\}.$$

Then $S \ne \varnothing$ (as $a \in S$) and S is bounded above (by b) and therefore

$$\xi = \sup(S)$$

exists. Let $\varepsilon > 0$. (Very many of our proofs during the term will start with the phase "Let $\varepsilon > 0$".)

(a) Start by showing $\xi \ne a$. Let $x \in [a, a + |f(a)|/2]$. Then $|x - a| \le |f(a)|/(2M)$. Thus

$$\begin{aligned} f(x) &= f(a) + (f(x) - f(a)) \\ &= -|f(a)| + (f(x) - f(a)) \qquad \text{(as } f(a) < 0\text{)} \\ &\le -|f(a)| + |f(x) - f(a)| \end{aligned}$$

Use this and

$$|f(x) - f(a)| \le M|x - a| \le M\frac{|f(a)|}{2M}$$

to show

$$f(x) \le \frac{-|f(a)|}{2} < 0 \quad \text{for} \quad x \in [a, a + |f(a)|/(2M)]$$

and use this to show

$$a < a + \frac{|f(a)|}{2M} \le \xi.$$

(b) Do a similar argument to show that for $x \in [b - f(b)/(2M), b]$

$$f(x) \ge \frac{f(b)}{2} > 0 \quad \text{for} \quad x \in [b - f(b)/(2M), b]$$

and thus

$$\xi \le b - \frac{f(b)}{2M} < b.$$

(c) Explain why there is a x_1 with

$$\xi - \varepsilon < x_1 \le \xi \quad \text{and} \quad f(x_1) \le 0.$$

(*Hint:* $[\xi - \varepsilon] \cap S \ne \varnothing$, for if not $\xi - \varepsilon$ would be an upper bound for S.)

(d) Note

$$f(\xi) = f(x_1) + (f(\xi) - f(x_1)) \le 0 + |f(\xi) - f(x_1)|$$

and use this to show

$$f(\xi) \le M\varepsilon.$$

(e) Explain why there is a $x_2 \in [\xi, \xi + \varepsilon]$ with

$$f(x_2) > 0.$$

(*Hint:* If $x_2 > \xi$, then $\xi \notin S$ and thus $f(x_2) > 0$.)

(f) Use that $f(x_2) > 0$ and $|\xi - x_2| < \varepsilon$ to show
$$f(\xi) \geq -M\varepsilon.$$
(g) Combine steps (d) and (f) to get that
$$|f(\xi)| = |f(\xi) - 0| \leq M\varepsilon.$$
(h) This works for all $\varepsilon > 0$. Now use Proposition 49 to finish the proof. □

We now use this to show that positive real numbers have square roots.

Proposition 51. *Let $c > 0$ be a positive real number. Then c has a positive square root.*

Proof. Let f be the function
$$f(x) = x^2 - c$$
and let
$$a = 0 \quad \text{and} \quad b = c + 1.$$
Then
$$f(a) = f(0) = -c < 0$$
and
$$f(b) = (c+1)^2 - c = c^2 + c + 1 > 0.$$
Also if $x_1, x_2 \in [a, b]$ we have
$$\begin{aligned} |f(x_2) - f(x_2)| &= |x_2^2 - x_1^2| \\ &= |x_2 + x_1||x_2 - x_1| \\ &\leq (|x_2| + |x_1|)|x_2 - x_1| && \text{(triangle inequality)} \\ &\leq 2b|x_2 - x_1| && \text{(as } |x_1|, |x_2| \leq b\text{)} \\ &= M|x_2 - x_1| \end{aligned}$$
where $M = 2b$. Therefore $f(x)$ satisfies the hypothesis of our version of the intermediate value theorem and so there is a $\xi \in (a, b)$ with $f(\xi) = 0$. But then $\xi^2 = c$ and therefore c has a square root. □

Proposition 52. *Let $c > 0$. Then c has a cube root.*

Problem 46. Prove this along the lines of the proof of Proposition 51. □

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