CONSTRUCTING COMPLETE PROJECTIVELY FLAT CONNECTIONS

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The propose of this note is to tie up a couple of loose ends in the classical theory of linear connections. First, in [1, p. 395], Spivak rises the question of if, on a compact manifold with complete connection, any two points can be joined by a geodesic. The answer is "no" even when the connection is projectively flat and homogeneous:

Theorem 1. *Let T^2 be the two dimensional torus. Then for any positive integer m there is a complete torsion free projectively flat connection, ∇, on T^2 such that for any point $p \in T^2$ there is a point $q \in T^2$ with the property that any broken ∇-geodesic between p and q has at least m breaks. Moreover if T^2 is viewed as a Lie group in the usual manner, this connection is invariant under translations by elements of T^2.*

Another natural question is: For a connected open subset, U, of the Euclidean space, $\mathbf{R}^n$, is the usual flat connection restricted to U projectively equivalent to complete torsion free connection on U? This is true and is a special case of a more general result about connections on incomplete Riemannian manifolds.

Theorem 2. *Let (M, g) be a not necessarily complete Riemannian manifold. Then there is a complete torsion free connection on M that is projective with the metric connection on M. In particular any connected open subset M of the Euclidean space, $\mathbf{R}^n$, has a complete torsion free connection ∇ such that the geodesics of ∇ are reparameterizations of straight line segments of $M \subseteq \mathbf{R}^n$.*

The main tool is a roposition which gives an elementary method of constructing complete torsion free connections that are projective with a given torsion free connection.

References

1. M. Spivak, *A comprehensive introduction to differential geometry*, 2 ed., vol. 2, Publish or Perish Inc., Berkeley, 1979.

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