Analysis Qualifying Examination **August 2008**

INSTRUCTIONS:

(1) Write your solutions on only one side of your paper.
(2) Start each new problem on a separate page.
(3) Write your name (or just your initials) on the top of each page.
(4) Before handing in the exam, put the problems in order and then consecutively number your pages.
(5) Each of the 8 problems is worth 12 points. Following the instructions is worth 4 points.

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

Signature / Date : ______________________________

Name (printed) : ______________________________

Notation:
$(X, \mathcal{M}, \mu)$ is an arbitrary (complete) measure space.
$\mathbb{K}$ is the field of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers.
$L_0(X, \mathcal{M}, \mu)$ is the space of μ-measurable functions from X to $\mathbb{R}$.
$L_p(X, \mathcal{M}, \mu) = \{f \in L_0(X, \mathcal{M}, \mu) \colon \|f\|_{L_p} < \infty\}$ for $1 \le p \le \infty$.
If confusion is unlikely, $L_p(X, \mathcal{M}, \mu)$ is denoted by just L_p.

1. Let A and B be subsets of $\mathbb{R}$. Define the subset $A + B$ of $\mathbb{R}$ by

$$A + B = \{a + b \in \mathbb{R} \colon a \in A \text{ and } b \in B\} .$$

1a. Prove that if A is compact and B is closed, then $A + B$ is closed.

1b. Give an example of closed sets A and B such that $A + B$ is not closed.

2. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces and let $T \colon X \to Y$ be a linear mapping. Prove that the following 3 conditions are equivalent.

(1) There exists a point $a \in X$ such that T is continuous at a.
(2) T is continuous on X.
(3) There exists a constant $M \in \mathbb{R}$ such that for each $x \in X$

$$\|Tx\|_Y \le M \, \|x\|_X .$$

3. In this problem, $(E, \mathcal{M}, m)$ is the Lebesgue measure space on a measurable subset E of $\mathbb{R}$.
Let $1 \le p, q, r \le \infty$. Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, with the usual convension that $\frac{1}{\infty} = 0$.
The Generalized Hölder's Inequality says that if $f \in L_p(E, \mathcal{M}, m)$ and $g \in L_q(E, \mathcal{M}, m)$, then $fg \in L_r(E, \mathcal{M}, m)$ and

$$\|fg\|_{L_r} \le \|f\|_{L_p} \, \|g\|_{L_q} \ . \tag{H}$$

3a. Prove Hölder's Inequality, i.e., prove the Generalized Hölder's Inequality for the special case $r = 1$.

3b. Prove the Generalized Hölder's Inequality. (You may use part 3a here.)

4. In this problem, we are working on the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$.

4a. Let $f \in L_1(\mathbb{R}, \mathcal{M}, m)$. Show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{M}$ and $m(A) < \delta$ then

$$\int_A |f| \, dm < \varepsilon \ .$$

4b. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence from $L_1(\mathbb{R}, \mathcal{M}, m)$ that converges in the L_1-norm. Show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{M}$ and $m(A) < \delta$ then

$$\sup_{n \in \mathbb{N}} \quad \int_A |f_n| \, dm \ < \ \varepsilon \ .$$

5. In this problem, $(X, \mathcal{F}, \mu)$ is an arbitrary (complete) measure space and $1 \le p < \infty$. This problem is a Lebesgue Dominated Convergence Theorem for L_p. You may use, without proving, Lebesgue's Dominated Convergence Theorem for L_1.

5a. Let $\{f_n\}$ be a sequence from $L_p(X, \mathcal{F}, \mu)$ which converges almost everywhere to a function $f \in L_0(X, \mathcal{F}, \mu)$. Show that if there exists a function $g \in L_p(X, \mathcal{F}, \mu)$ such that

$$|f_n(x)| \le g(x) \qquad \text{for all} \qquad x \in X, \ n \in \mathbb{N}$$

then $f \in L_p(X, \mathcal{F}, \mu)$ and $\{f_n\}$ converges to f in L_p-norm.

5b. For each $p \in [1, \infty)$, give an example (on a finite measure space of your choice) of a sequence $\{f_n\}$ of L_p functions that converge pointwise to a function $f \in L_p$ but the sequence $\{f_n\}$ does not converge in the L_p-norm.

6. Here we are working on the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$ over the field $\mathbb{K} = \mathbb{C}$ of scalars.
The Fourier transform $\hat{f} \colon \mathbb{R} \to \mathbb{C}$ of a function $f \in L_1(\mathbb{R}, \mathcal{M}, m)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx \ . \tag{FT}$$

6a. Let $f \in L_1(\mathbb{R}, \mathcal{M}, m)$. Show that, for each $\xi \in \mathbb{R}$, the integral in (FT) exists (and thus the function $\hat{f}$ is indeed defined). Then show that $\hat{f}$ is continuous and bounded on $\mathbb{R}$.

6b. Let f and g belong to $L_1(\mathbb{R}, \mathcal{M}, m)$. Carefully show that

$$\int_{\mathbb{R}} \hat{f}(\xi) g(\xi) \, d\xi \ = \ \int_{\mathbb{R}} f(x) \hat{g}(x) \, dx \ .$$

Be sure to justify that the 2 above integrals exists.

7. Prove Liouville's Theorem: A bounded entire function on the complex plane $\mathbb{C}$ must be constant.

8. The family $\{P_r\}_{0\le r<1}$ of Poisson kernels, of functions $P_r\colon \mathbb{R}\to\mathbb{R}$, is given by

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

and they satisfy

$$\text{if } 0 \le r < 1, \text{ then} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_r(\theta)|\,d\theta \;=\; 1 \tag{8.1}$$

$$\text{if } 0 < \delta < \pi, \text{ then} \qquad \lim_{r\to 1-}\int_{\delta<|\theta|<\pi} |P_r(\theta)|\,d\theta = 0\,. \tag{8.2}$$

8a. Let $f = f(\theta)$ be a continuous 2π-periodic function on the real line. Show that if $0 \le \theta \le 2\pi$ then

$$\lim_{r\to 1-} (f \star P_r)(\theta) \;=\; f(\theta)\,.$$

Here, by definition,

$$(f \star P_r)(\theta) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)P_r(t)\,dt\,.$$

8b. Is the convergence in part 6a uniform in θ? That is, does

$$\lim_{r\to 1-}\ \sup_{0\le\theta\le 2\pi} |(f\star P_r)(\theta) - f(\theta)| \;=\; 0\,?$$

Explain your answer.

8c. Given a real-valued continuous function f on the unit circle, *Dirichlet's problem* is to find a continuous function on the closed unit disk, harmonic in the interior, that coincides with f on the boundary. Use Féjer's theorem (i.e. 6a) to provide a solution to the Dirichlet problem for the unit disk.