Problems related to L^p spaces and/or Hölder's inequality.

Problem 1. Let (X, μ) be a measure space with $\mu(X) < \infty$. Prove

$$\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

for all measurable $f\colon X \to \mathbb{R}$. $\square$

Problem 2. Let $1 < p < \infty$. For $f \in L^p(\mathbb{R})$ and $h \in \mathbb{R}$ let

$$(\tau_h f)(x) = f(x_h).$$

Prove

$$\lim_{h\to 0} \|f - \tau_h f\|_{L^p} = 0$$

for all $f \in L^p(\mathbb{R})$. $\square$

Problem 3. Find the maximum of the function

$$f(x, y, z, w) = x - 2y + 3z - 4w$$

on the set defined by $x^4 + y^4 + z^4 + w^4 = 3$. *Hint:* This is a Hölder inequality problem in disguise. $\square$

Problem 4 (January 1984). Let $1 < p < \infty$ and $1/p + 1/q = 1$. Let $\langle g_n\rangle_{n=1}^{\infty}$ be a sequence in $L^q([0, 1])$ such that

(a) $M = \sup_n \|g_n\|_{L^q} < \infty$, and

(b) $\lim\limits_{n\to\infty} \displaystyle\int_E g_n\,dx = 0$ for all measurable subsets $E \subseteq [0, 1]$.

Prove for each $f \in L^p([0, 1])$ that $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f g_n\,dx = 0$. $\square$

Problem 5 (August 1984). Let $1 < p < \infty$ and $1/p + 1/q = 1$. Let $g \in L^1([0, 1])$ and that there is a constant M such that

$$\left| \int_0^1 g(x)s(x)\,dx \right| \le M\|s\|_{L^p}$$

for all simple functions s. Prove $g \in L^q([0, 1])$ and $\|g\|_{L^q} < \infty$. $\square$

Problem 6 (January 1985). Let (X, μ) be a measure space with $\mu(X) < \infty$. Show for $p_1 < p_2$ that $L^{p_2}(X) \subseteq L^{p_1}(X)$. Also show that if $p_1 < p_2$ there is a function $f \in L^{p_1}([0, 1])$ and $f \notin L^{p_2}([0, 1])$. $\square$

Problem 7 (January 1990). Let $1 < p < \infty$ and $f \in L^p(\mathbb{R})$. Prove

$$\lim_{h\to 0} h^{\frac{1}{p}-1} \int_x^{x+h} f(t)\,dt = 0 \quad \text{uniformly in } x. \qquad \square$$

Problem 8 (January 1991). Let $f \in L^p((0, \infty))$ where $1 < p < 2$ and set

$$\phi(y) = \int_0^\infty f(x)\frac{\sin(xy)}{\sqrt{x}}\,dx.$$

(a) Prove $\phi(y)$ is finite all y.

(b) Prove

$$\lim_{y\to\infty} y^{\frac{1}{2}-\frac{1}{p}}\phi(y) = 0.$$

Hint: Consider the integral over $[0, M]$ and $[M, \infty)$ separately, where M is appropriately large.

Hint: I did not see an easy way to use the hint to do this problem. Here is anther way to do it. We use the usual convention that $1/p + 1/1 = 1$. First use Hölder's inequality to get

$$\begin{aligned}
|\phi(y)| &\le \|f\|_{L^p}\left(\int_0^\infty \frac{|\sin(xy)|^q}{x^{\frac{q}{2}}}\,dx\right)^{\frac{1}{q}} \\
&= \|f\|_{L^p}\left(\int_0^\infty \frac{|\sin(t)|^q}{\left(\frac{t}{y}\right)^{\frac{q}{2}}}\,\frac{dt}{y}\right)^{\frac{1}{q}} && \text{Change of variable } x = t/y \\
&= \|f\|_{L^p} y^{\frac{1}{2}-\frac{1}{q}} C_q
\end{aligned}$$

where

$$C_q = \left(\int_0^\infty \frac{|\sin(t)|^q}{t^{\frac{q}{2}}}\,dt\right)^{\frac{1}{q}}.$$

Now show that C_q is finite (and I found this easiest to do by using $\int_0^\infty = \int_0^1 + \int_1^\infty$, using $|\sin(t)| \le |t|$ on $[0, 1]$, and on $[1, \infty)$ using that $q > 2$ so that $q/2 > 1$.)

So we now have

$$y^{\frac{1}{2}-\frac{1}{p}}|\phi(y)| \le y^{\frac{1}{2}-\frac{1}{p}}\|f\|_{L^p} y^{\frac{1}{2}-\frac{1}{q}} C_q = C_q y^{1-\frac{1}{p}-\frac{1}{q}}\|f\|_{L^p} = C_q\|f\|_{L^p}.$$

This implies that for any $f \in L^p((0, \infty))$ that

$$\limsup_{y\to\infty} y^{\frac{1}{2}-\frac{1}{p}}|\phi(y)| \le C_q\|f\|_{L^p}.$$

Now consider the special case of $f = \mathbb{1}_{[a,b]}$ with $0 < 1 < b$. In this case

$$\phi(y) = \int_a^b \frac{\sin(xy)}{\sqrt{x}}\,dx$$

and show for this function that $\lim_{y\to\infty} y^{\frac{1}{2}-\frac{1}{p}}\phi(y) = 0$. By linearity it follows that this holds for all step functions. Finally use that the step functions are dense to complete that proof. $\square$

Problem 9 (August 1991). Let $K(x, y)$ be a measurable function on $[0, 1] \times [0, 1]$ such that for some $M > 0$

$$\int_0^1 \int_0^1 K(x, y)^2\,dx\,dy \le M.$$

Let $f \in L^2([0,1])$ and set

$$F(x) = \int_0^1 K(x,y) f(y)\, dx.$$

Show

$$\|F\|_{L^2} \leq \sqrt{M}\, \|f\|_{L^2}.$$

Problem 10. Let (X,μ) and (Y,ν) be measure spaces and $K\colon X \times Y \to \mathbb{R}$ a measurable function such that there is a constant M such that

$$\int_X |K(x,y)|\, d\mu(x) \leq M$$

for almost all $y \in Y$ and

$$\int_y |K(x,y)|\, d\nu(y) \leq M$$

for almost all $x \in X$. Show that if $1 \leq p < \infty$ and $f \in L^p(X)$, then the function

$$(Tf)(y) = \int_X K(x,y)\, d\mu(x)$$

is in $L^p(Y)$ and

$$\|Tf\|_{L^p(Y)} \leq M \|f\|_{L^p(X)}. \qquad \square$$