Mathematics 551 Final Homework.

This will be the last homework and will also be the last part of the final exam. It should be submitted as LaTeX output and is due via e-mail at 12:00pm (noon) on Wednesday, April 29 (which is when our final would have finished). Please send it using your school email address (as that is what I use to search for submissions) and use the subject line `Math 551 Final.`

The following problems are set up as much to be instructive as to do assessment. They concern some facts that are often covered in a complex variables class. The problems have enough hints that hope you can do the mathematics. What I will be most interested in is having well written arguments. In particular what I would consider an `A+` solution would be one that I could hand out to a student who has not seen the solution before and they could easily read your solution.

Let us recall some vector calculus. Let D be a bounded domain in $\mathbb{R}^2$ with nice boundary. Let $\mathbf{n}$ the outward pointing unit normal to the boundary, ∂D, of D. If

$$\mathbf{V} = (u, v)$$

then the ***divergence*** of V is

$$\operatorname{div}(V) = u_x + v_y.$$

is a vector field on defined on the closure, $\overline{D}$, of D, then the ***divergence theorem*** is that for any C^1 vector field on $\overline{D}$,

$$\iint_D \operatorname{div}(\mathbf{V})\, dA = \int_{\partial D} \mathbf{V} \cdot \mathbf{n}\, ds$$

where $dA = dx\, dy$ is the area measure and ds is arclength along the curve ∂D.

If $h \colon \overline{D} \to \mathbb{R}$ is a scalar valued function, then its ***gradient*** is the vector field

$$\nabla h = (h_x, h_y).$$

Problem 1. With this notation verify the product rule

$$\operatorname{div}(h\mathbf{V}) = \nabla h \cdot \mathbf{V} + h \operatorname{div}(\mathbf{V}).$$

Use this to prove the following two dimensional version of the integration by parts formula:

$$\iint_D \nabla h \cdot \mathbf{V}\, dA = \int_{\partial D} \mathbf{V}\, ds - \iint_D h \operatorname{div}(\mathbf{V})\, dA.$$

For smooth functions $f \colon \overline{D} \to \mathbb{R}$ define

$$\mathcal{A}[f] = \iint_D \sqrt{1 + \|\nabla f\|^2}\, dA.$$

As we have shown earlier in the term, this is the area of the graph

$$G_f = \{(x, y, f(x, y)) : (x, y) \in D\}.$$

Definition 1. The smooth function $f\colon \overline{D} \to \mathbb{R}$ is ***area minimizing*** if and only if for all other smooth functions $g\colon \overline{D} \to \mathbb{R}$ with

$$g\big|_{\partial D} = f\big|_{\partial D}$$

(that is $g(x, y) = f(x, y)$ for all $(x, y) \in \partial D$) we have

$$\mathcal{A}[f] \leq \mathcal{A}[g]. \qquad \square$$

We wish to see what being area minimizing implies about the differential geometry of the graph. We will use the same ideas what we used in finding the equation for geodesics.

To start let $h\colon \overline{D} \to \mathbb{R}$ be a smooth function such that

$$h(x, y) = 0 \qquad \text{for all } (x, y) \in \partial D.$$

Let f be area minimizing and let

$$f_\varepsilon = f + \varepsilon h.$$

Since $h(x, y) = 0$ on ∂D and f is area minimizing we have that

$$\mathcal{A}[f] \leq \mathcal{A}[f_\varepsilon]$$

for all real numbers ε. Therefore the function

$$\varphi(\varepsilon) = \mathcal{A}[f_\varepsilon]$$

has a minimum at $\varepsilon = 0$. Therefore, by the first derivative test form calculus,

$$\varphi'(0) = 0.$$

Problem 2. Compute $\varphi'(0)$ in the following steps:

(a) To start with the almost trival, note that $\nabla f_\varepsilon = \nabla f + \varepsilon \nabla h$ and thus

$$\frac{d}{d\varepsilon} \nabla f_\varepsilon = \nabla h.$$

(b) Next show

$$\frac{d}{d\varepsilon} \|f_\varepsilon\|^2 = \nabla f_\varepsilon \cdot \nabla h.$$

(c) Use this to show

$$\frac{d}{d\varepsilon} \sqrt{1 + \|\nabla f_\varepsilon\|^2} = \frac{\nabla f_\varepsilon \cdot \nabla h}{\sqrt{1 + \|\nabla f_\varepsilon\|^2}}.$$

(d) Now in the formula

$$\varphi(\varepsilon) = \iint_D \sqrt{1 + \|f_\varepsilon\|^2}\, dA$$

take $\frac{d}{d\varepsilon}$ and move the dirivative through the integral to get

$$\varphi'(\varepsilon) = \iint_D \frac{\nabla f_\varepsilon \cdot \nabla h}{\sqrt{1 + \|\nabla f_\varepsilon\|^2}}\, dA$$

(e) Set $\varepsilon = 0$ to get

$$\begin{aligned} 0 = \varphi'(0) &= \iint_D \frac{\nabla f \cdot \nabla h}{\sqrt{1 + \|\nabla f\|^2}}\, dA \\ &= \iint_D \nabla h \cdot \left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|^2}} \right) dA. \end{aligned}$$

(f) Finally use the integration by parts formula of Problem 1 to conclude that for all h with $h = 0$ on ∂D we have

$$\iint_D h \operatorname{div} \left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|^2}} \right) dA. \tag{1}$$

$\square$

It can be shown (cf. Math 555) that Equation (1) holding for all h with $h = 0$ on ∂D implies

Theorem 2. *If f is area minimizing then it satisfies the **minimal surface equation***

$$\operatorname{div} \left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|^2}} \right) = 0.$$

In a calculation that you did in homework (but with somewhat different notation) we found that that mean curvature of the graph G_f is

$$H = \frac{1}{2} \operatorname{div} \left(\frac{\nabla f}{\sqrt{1 + \|\nabla f\|^2}} \right)$$

and thus a surface that minimizes surface area for a given boundary we have that $H = 0$. This has lead to the terminology that a surface with $H \equiv 0$ is called a ***minimal surface***.

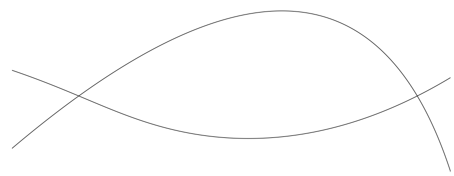
Recall that if D is a simply connected domain on a surface bounded by a curvilinear polygon, then the Gauss-Bonnet theorem tells us

$$\int_{\partial D} \kappa_g \, ds + \sum \text{Exterior angles} + \iint_D K \, dA = 2\pi,$$

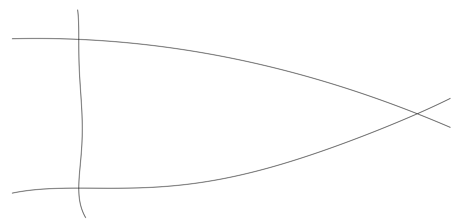
where κ_g is the geodesic curvature, and K is the Gaussian curvature.

Problem 3. This problem gives applications of the Gauss-Bonnet theorem to surfaces with non-positive curvature. Let $\mathcal{S}$ be the plane $\mathbb{R}^2$ with a Riemannian metric g such that the Gaussian curvature satisfies $K \le 0$ and also $\mathcal{S}$ is complete in the sense that all geodesics of $\mathcal{S}$ have infinite length. (That is if $\mathbf{u}$ is a unit vector tangent to $\mathcal{S}$ at the point p, then the geodesic $\boldsymbol{\gamma}(t)$ with $\boldsymbol{\gamma}(0) = p$ and $\dot{\boldsymbol{\gamma}}(0) = \mathbf{u}$ is defined for all $t \in \mathbb{R}$.)

(a) Show that two geodesics can intersect in at most one point. That is the following picture is impossible if both the curves are geodesics of $\mathcal{S}$.



(b) Show that if two geodesics, $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, are perpendicular to a third geodesic, $\boldsymbol{\gamma}_3$, at distinct points, then $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ do not intersect each other. That is the following figure is imposable when the curves are geodesics and two of the angles are right angles.



$\square$

Problem 4. Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{S}^2$ be a not necessarily unit speed curve where $\mathbb{S}^2$ is the unit sphere. Here we find a formula for the geodesic curvature of $\boldsymbol{\alpha}$ as curve in $\mathbb{S}^2$. The unit tangent to $\boldsymbol{\alpha}$ is

$$\mathbf{t} = \frac{1}{\|\dot{\boldsymbol{\alpha}}\|}\dot{\boldsymbol{\alpha}}$$

If we use the outward unit normal to $\mathbb{S}^2$, that is $\mathbf{n}(\mathbf{u}) = \mathbf{u}$, then the unit normal to $\boldsymbol{\alpha}$ in $\mathbb{S}^2$ is

$$\mathbf{N} = \boldsymbol{\alpha} \times \mathbf{t}.$$

By definition the geodesic curvature of $\boldsymbol{\alpha}$ is

$$\kappa_g = \frac{d\mathbf{t}}{ds} \cdot \mathbf{N}$$

where s is arclength along $\boldsymbol{\alpha}$. Use this to show

$$\kappa_g = \frac{(\boldsymbol{\alpha} \times \dot{\boldsymbol{\alpha}}) \cdot \ddot{\boldsymbol{\alpha}}}{\|\dot{\boldsymbol{\alpha}}\|^3}.$$

$\square$

Problem 5. Let $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ be a unit speed curve and assume that the curvature, κ, of $\boldsymbol{\gamma}$ never vanishes. We use t for the parameter along $\boldsymbol{\gamma}$, and thus $\left\|\frac{d\boldsymbol{\gamma}}{dt}\right\| = 1$ as $\boldsymbol{\alpha}$ is unit speed. The nonvanishing of κ implies the unit normal $\mathbf{n}$ of $\boldsymbol{\gamma}$ is defined along all of $\boldsymbol{\gamma}$. As $\mathbf{n}$ is a unit vector we can view $\mathbf{n}\colon (a,b) \to \mathbb{S}^2$ as a curve in $\mathbb{S}^2$.

(a) Let s be arclength along $\mathbf{n}$, so that $\left\|\frac{d\mathbf{n}}{ds}\right\| = 1$. Use the Frenet formulas for $\boldsymbol{\gamma}$ to show

$$\frac{ds}{dt} = \sqrt{\kappa^2 + \tau^2}.$$

where τ is the torsion of $\boldsymbol{\gamma}$.

(b) Use Problem 2 to show the geodesic curvature of $\mathbf{n}$ in $\mathbb{S}^2$ is

$$\kappa_g = \frac{\kappa\dot{\tau} - \tau\dot{\kappa}}{(\kappa^2+\tau^2)^{3/2}}.$$

(c) Show that the integral along $\mathbf{n}$ of the geodesic curvature, κ_g, of $\mathbf{n}$ with respect to arclength is

$$\int_{\mathbf{n}} \kappa_g\, ds = \int_a^b \frac{\kappa\dot{\tau} - \tau\dot{\kappa}}{(\kappa^2+\tau^2)^{3/2}} \sqrt{\kappa^2+\tau^2}\, dt = \arctan\left(\frac{\tau}{\kappa}\right)\Big|_{t=a}^{b}.$$ □

Problem 6*.* Use the last problem and the Gauss-Bonnet theorem to prove the following theorem of Jacobi: If $\boldsymbol{\gamma}\colon (a,b) \to \mathbb{R}^3$ is a closed unit speed curve with nonvanishing curvature such that the curve $\mathbf{n}\colon (a,b) \to \mathbb{S}^2$ is a simple closed curve in $\mathbb{S}^2$, then $\mathbf{n}$ divides $\mathbb{S}^2$ into two regions of equal area. □

Problem 7*.* Let D be a bounded domain in the plane, $\mathbb{R}^2$, which is nice enough that we can use the divergence theorem on D. Let the plane and $f(x,y)$ a function on $D \cup \partial D$ and let H be the mean curvature of the graph of $z = f(x,y)$. Assume

$$H \geq 1.$$

(a) Let A be the area of D and L the length of its boundary ∂D. Show

$$A \leq \frac{1}{2}L$$

Hint: The mean curvature of the graph is

$$H = \frac{1}{2}\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+\|\nabla f\|^2}}\right)$$

and

$$A = \iint_D dx\, dy \leq \iint_D H\, dx\, dy$$

(b) In the case that D is a disk of radius r show

$$r \leq 1$$

(c) If the Gaussian curvature, K, of the graph is positive, then by possibly replacing f by $-f$ (or just changing our choice of unit normal) we can assume the principle curvatures of the graph are positive. In this case show $\sqrt{K} \leq H$ and use this to show that the conclusions of (a) and (b) of this problem hold with the condition $H \geq 1$ replaced by $K \geq 1$. (The results of this problem are Theorems due to E. Heinz.) □