ANALYSIS QUALIFYING EXAMINATION

August 1998

Instructions:

(1) Write your solutions on only one side of your paper.

(2) Start each problem on a separate page.

(3) There are 8 problems on this exam; each problem is worth 10 points.

Notation/Conventions:

(1) $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.

(2) $(X, \mathcal{F}, \mu)$ is an arbitrary (complete) measure space and:

$$L_0(X, \mathcal{F}, \mu) = \{f \colon X \to \mathbb{R} \mid f \text{ is } \mathcal{F}\text{-measurable}\}$$
$$L_p(X, \mathcal{F}, \mu) = \{f \in L_0(X, \mathcal{F}, \mu) \mid \|f\|_p < \infty\} .$$

So we are only considering finite–valued functions.

(3) $\mathcal{B}$ is the collection of Borel subsets of $\mathbb{R}$.

---

1) Let (X_i, ρ_i) be metric spaces and $f \colon X_1 \to X_2$ be a continuous map. Let $Y_1 \subset X_1$ and $Y_2 = f(Y_1)$.

1a) Show that a continuous image of a compact set is compact. Namely, show that if Y_1 is compact, then Y_2 is compact.

1b) Prove or give a counterexample to the following converse: *If Y_2 is compact, then Y_1 is compact.*

2) Let $f \in L_\infty(X, \mathcal{F}, \mu)$ with $\|f\|_\infty = M$. Consider the property $(*)$:

$$\lim_{p \to \infty} \|f\|_p \overset{(*)}{=} \|f\|_\infty .$$

2a) If $\mu(X) = 1$, does $(*)$ necesessarily hold? Prove or give a counterexample.

2b) If $0 < \mu(X) < \infty$, does $(*)$ necesessarily hold? Prove or give a counterexample.

2c) If $\mu(X) = \infty$, does $(*)$ necesessarily hold? Prove or give a counterexample.

3) Let $(\mathbb{R}, \mathcal{M}, m)$ be the Lebesgue measure space on $\mathbb{R}$. Establish the Riemann–Lebesgue Theorem: If $f \in L_1(\mathbb{R}, \mathcal{M}, m)$, then $\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\ \cos(nx)\, dm = 0$.

4) Let $([0,1], \mathcal{M}, m)$ be the Lebesgue measure space on $[0,1]$ and $\varepsilon > 0$.
Consider a Lebesgue measurable subset E of $[0,1]$ with the property:

$$m\,(E \cap [a,b]) \overset{(\star)}{\geqslant} \varepsilon\, m\,([a,b])$$

for each $0 \leqslant a \leqslant b \leqslant 1$. Show that $m(E) = 1$.
You may use, without proving, Lebesgue's Differentiation Theorem.

5) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a continuous function.
Show that f^{-1} of a Borel set is again a Borel set.

6) Consider a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $L_1(X, \mathcal{F}, \mu)$ along with some function $f \colon X \to \mathbb{R}$. Assume that:

(1) $f_n \to f$ almost everywhere,

(2) for each $\varepsilon > 0$ there is a set $A_\varepsilon \in \mathcal{F}$ with $m(A_\varepsilon) < \infty$ such that

$$\int_{X \setminus A_\varepsilon} |\, f_n \,|\; d\mu < \varepsilon$$

for each $n \in \mathbb{N}$,

(3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \in \mathcal{F}$ and $m(E) < \delta$ then

$$\int_E |\, f_n \,|\; d\mu < \varepsilon$$

for each $n \in \mathbb{N}$.

Show that $f_n \to f$ in $L_1(X, \mathcal{F}, \mu)$, ie. show that $\lim_n \int_X |\, f - f_n \,|\; d\mu = 0$.

7) Let G be an open connected subset of $\mathbb{C}$. Let f be a continuous function on G such that e^f is constant on G, say $e^{f(z)} = c$ for each $z \in G$ for some $c \in \mathbb{C}$. Show that f is constant on G.

8) Show Liouville's Theorem: If f is an holomorphic bounded function on $\mathbb{C}$, then f is constant on $\mathbb{C}$. HINT: it suffices to show that f' is zero on $\mathbb{C}$.