PH.D. QUALIFYING EXAMINATION
IN ALGEBRA
JANUARY 2010

PROBLEM 1.
Prove that no finite field is algebraically closed.

PROBLEM 2.
Prove that $\ln u$ and $\sin u$ are transcendental over the field of rational numbers, whenever u is a positive algebraic real number.

PROBLEM 3.
Let $\mathbf{R}$ be a commutative ring and let I be a nontrivial ideal of $\mathbf{R}$. Prove that $I = \langle a \rangle$ where a is not a zero divisor if and only if I, considered as a unitary left $\mathbf{R}$-module, is free.

PROBLEM 4.
Let $\mathbf{G}$ and $\mathbf{H}$ be finite Abelian groups such that $\mathbf{G} \times \mathbf{G} \times \mathbf{H} \cong \mathbf{G} \times \mathbf{H} \times \mathbf{H}$. Prove that $\mathbf{G} \cong \mathbf{H}$.

PROBLEM 5.
Let $\mathbf{G}$ be a finite group such that any two elements that have prime power orders commute. Prove that $\mathbf{G}$ is Abelian.

PROBLEM 6.
Prove that every group of order $3^4 \cdot 7^3$ is solvable.

PROBLEM 7.
Let $\mathbf{F}$ be a field and let $\mathbf{R}$ be the subring of $\mathbf{F}[t]$ consisting of those polynomials with t-coefficient 0. Prove that $\mathbf{R}$ has elements which are irreducible but not prime and that the ideal of $\mathbf{R}$ consisting of those members of $\mathbf{R}$ with constant term 0 is not a principal ideal. Prove that $\mathbf{R}$ is a Noetherian ring. [Hint: Consider a connection between $\mathbf{F}[x, y]$ and $\mathbf{R}$.]

PROBLEM 8.
Is the polynomial $x^5 - 80x + 2$ solvable by radicals over $\mathbb{Q}$? Be sure to justify your answer with a proof.

PROBLEM 9.
Find all the prime ideals of $\mathbb{Z}[x]$ that are minimal with respect to containing the polynomial

$$f(x) = 6(x-1)(x^3 + 3x - 2).$$

Also describe explicitly three distinct maximal ideals of $\mathbb{Z}[x]$, each containing at least two of the minimal prime ideals containing $f(x)$.

PROBLEM 10.
Consider $\mathbb{Q}$ and $\mathbb{Z}$, under addition, as $\mathbb{Z}$-modules. Then $\mathbb{Q}/\mathbb{Z}$ is also a $\mathbb{Z}$-module. Suppose that $\mathbf{N} \leq \mathbf{M}$ are $\mathbb{Z}$-modules and that $f : \mathbf{N} \to \mathbb{Q}/\mathbb{Z}$ is a homomorphism. Prove that f can be extended to a homomorphism $g : \mathbf{M} \to \mathbb{Q}/\mathbb{Z}$.

PROBLEM 11.
Let $\mathbf{F}, \mathbf{K}$, and $\mathbf{L}$ be fields so that $\mathbf{K}$ is a finite separable extension of $\mathbf{F}$ and $\mathbf{L}$ is a finite separable extension $\mathbf{K}$. Prove that $\mathbf{L}$ is a finite separable extension of $\mathbf{F}$.