

Mathematics 555 Homework

1. THE DERIVATIVE.

1.1. The derivative at a point.

Definition 1. Let (α, β) be an open interval, $f: (\alpha, \beta) \rightarrow \mathbb{R}$ a function, and $a \in (\alpha, \beta)$. Then f is ***differentiable*** at a if and only if the limit

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists. When this limit exists it is the ***derivative*** of f at a and is denoted by $f'(a)$. \square

The limit defining $f'(a)$ can be rewritten in several ways. For example if we do the change of variable $x = a + h$ in the limit it becomes

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

which is the way it is often presented in calculus books. And sometimes, especially in older books, h is replaced by Δx and the limit is written as

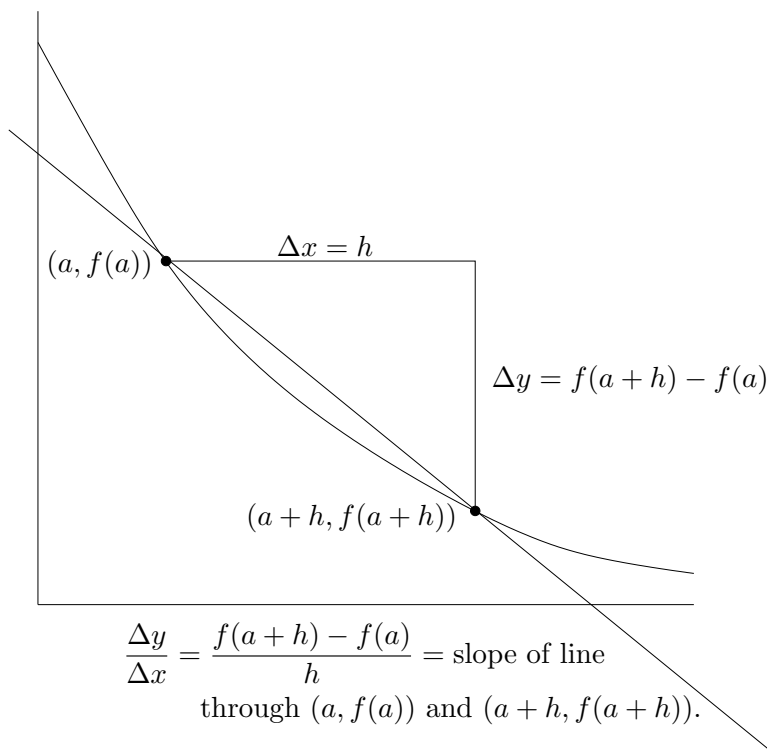
$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

And finally $f(a + \Delta x) - f(a)$ can be abbreviated as $\Delta y = f(a + \Delta x) - f(a)$ and then the limit becomes

$$f'(a) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

a notation meshes well with the Leibniz notation $\frac{dy}{dx}$ for the derivative.

I am now obligated to draw the standard picture that shows that the ***difference quotient*** $(f(a + h) - f(a))/h$ is the slope through the points $(a, f(a))$ and $(a + h, f(a + h))$ and therefore taking the limit as $h \rightarrow 0$ of this difference quotient is a reasonable definition of the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



We now do some examples of derivatives that you no doubt already know from calculus.

Let $f(x) = mx + b$ where m and b are constants. Then for any $a \in \mathbb{R}$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{m(x - a)}{x - a} = m.$$

For a slightly more complicated example consider $f(x) = x^2$ we have , using that $x^2 - a^2 = (x - a)(x + a)$:

$$\begin{aligned}
 f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\
 &= \lim_{x \rightarrow a} \frac{x^2 - a^2}{x - a} \\
 &= \lim_{x \rightarrow a} (x + a) \\
 &= 2a.
 \end{aligned}$$

Problem 1. Use the identities $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$ and $x^4 - a^4 = (x - a)(x^3 + ax^2 + a^2x + a^3)$ to prove that the functions $f(x) = x^3$ and $g(x) = x^4$ have the derivatives

$$\begin{aligned}
 f'(a) &= 3a^2 \\
 g'(a) &= 4a^3.
 \end{aligned}$$

□

The classic example of a function that does not have a derivative at a point is the absolute value function $f(x) = |x|$ which does not have a derivative at $x = 0$. Here are some other examples

Problem 2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function

$$f(x) = \max\{x, 2 - x\}.$$

Graph $y = f(x)$ and show that it is differentiable at every point other than $x = 1$. What is $f'(a)$ when $a \neq 1$? \square

Problem 3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function

$$f(x) = \min\{x^2, 1\}$$

Find the points where f is not differentiable and prove your result.

We now start proving the basic rules for derivatives you know from calculus.

Proposition 2 (Sum rule for derivatives). *Let f_1 and f_2 be defined on an interval containing the point a and assume that f_1 and f_2 are both differentiable at a . Let c_1 and c_2 be constants. Then the function $g = c_1f_1 + c_2f_2$ is differentiable at a and*

$$g'(a) = c_1f_1'(a) + c_2f_2'(a).$$

Problem 4. Prove this. \square

We extend this to sums with more terms:

Proposition 3. *Let f_1, f_2, \dots, f_n be functions defined on an interval containing a and with each f_j differentiable at a . Let c_1, c_2, \dots, c_n be constants. Then $g = c_1f_1 + c_2f_2 + \dots + c_nf_n$ is differentiable at a and*

$$g'(a) = c_1f_1'(a) + c_2f_2'(a) + \dots + c_nf_n'(a).$$

Proof. This is an easy induction proof. \square

Proposition 4. *Let f be defined on an interval containing a and assume that f is differentiable at a . Then f is continuous at a .*

Problem 5. Prove this. *Hint:* To show that f is continuous at a we need to show $\lim_{x \rightarrow a} f(x) = f(a)$. As f is differentiable we know that

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists. To use this write $f(x)$ as

$$f(x) = f(a) + \frac{f(x) - f(a)}{x - a}(x - a).$$

Now you can use standard results about limits (no ε , δ needed). \square

Proposition 5 (Product rule). *Let f and g be defined in an interval containing a and assume they are both differentiable at a . Then the product $p(x) = f(x)g(x)$ is differentiable at a and*

$$p'(a) = f'(a)g(a) + f(a)g'(a).$$

Problem 6. Prove this. *Hint:* One what is to do some adding and subtracting in the difference quotient for p :

$$\begin{aligned} \frac{p(x) - p(a)}{x - a} &= \frac{f(x)g(x) - f(a)g(a)}{x - a} \\ &= \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x - a} \\ &= f(x) \frac{g(x) - g(a)}{x - a} + \frac{f(x) - f(a)}{x - a} g(a) \end{aligned}$$

As f is continuous at a (why?) we have $\lim_{x \rightarrow a} f(x) = f(a)$. Finishing now should be easy. \square

Proposition 6. *Let g be defined in an interval containing a and assume g is differentiable at a and $g(a) \neq 0$. Then $h(x) = 1/g$ is differentiable at a and*

$$h'(a) = \frac{-g'(a)}{g(a)^2}.$$

Problem 7. Prove this. *Hint:* Write the difference quotient for h as

$$\begin{aligned} \frac{h(x) - h(a)}{x - a} &= \frac{1}{x - a} \left(\frac{1}{g(x)} - \frac{1}{g(a)} \right) \\ &= \frac{1}{x - a} \frac{g(a) - g(x)}{g(x)g(a)} \\ &= \frac{-1}{g(x)g(a)} \frac{g(x) - g(a)}{x - a} \end{aligned}$$

and now it should be easy to take the limit defining $h'(a)$. \square

Proposition 7 (Quotient rule). *Let f and g be defined on an interval containing a and with f and g differentiable at a . Also assume $g(a) \neq 0$. Then the quotient $q(x) = f(x)/g(x)$ is differentiable at a and*

$$q'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Problem 8. Prove this. *Hint:* We have already done most of the work for this. Write q as a product

$$q(x) = f(x) \left(\frac{1}{g(x)} \right)$$

and use Proposition 6 and the product rule. \square

Proposition 8 (The power rule for positive powers). *Let f be defined on an interval containing a and assume that f is differentiable at a . Then for any positive integer n the function $p(x) = f(x)^n$ is differentiable at a and*

$$p'(x) = nf(a)^{n-1}f'(a).$$

In particular letting $f(x) = x$ yields that when $p(x) = x^n$, then $p'(a) = na^{n-1}$.

Problem 9. Prove this. *Hint:* Induction. □

Problem 10.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial (thus a_0, a_1, \dots, a_n are constants) then f is differentiable at all points a and

$$f'(a) = na_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \cdots + 2a_2 x + a_1.$$

Proof. This follows by combining Propositions 2 and 8. □

The next result makes precise that the graph $y = f(x)$ of a function has a tangent line at the point $(a, f(a))$ if and only if the derivative $f'(a)$ exists.

Theorem 9. *Let f be a real valued function defined in a neighborhood of a . Then the following are equivalent.*

- (a) $f'(a)$ exists.
- (b) There is a constant m such that

$$f(x) = f(a) + m(x - a) + \rho(x; a)$$

where $\rho(x; a)$ satisfies

$$(1) \quad \lim_{x \rightarrow a} \frac{\rho(x; a)}{x - a} = 0.$$

Before going on with the proof, let us think a bit about what condition (b) of the theorem says. Rewrite (1) as

$$f(x) = f(a) + \left(m + \frac{\rho(x; a)}{x - a} \right) (x - a).$$

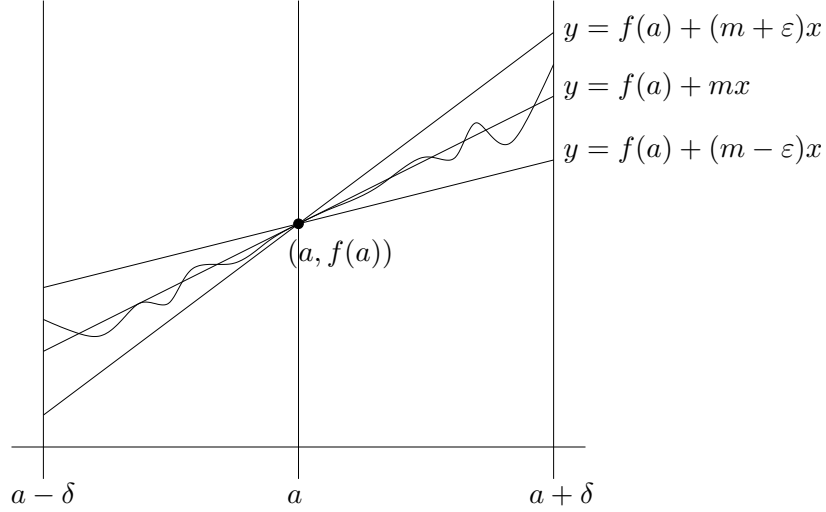
As $\lim_{x \rightarrow a} \rho(x; a)/(x - a) = 0$ for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \text{implies} \quad \frac{|\rho(x; a)|}{|x - a|} < \varepsilon.$$

Therefore

$$0 < |x - a| < \delta \quad \text{implies} \quad m - \varepsilon < m + \frac{\rho(x; a)}{x - a} < m + \varepsilon.$$

Problem 11. Show that these inequalities imply that on the interval $(a - \delta, a + \delta)$ that the graph of $y = f(x)$ stays between the graphs of the lines $y = f(a) + (m + \varepsilon)(x - a)$ and $y = f(a) + (m - \varepsilon)(x - a)$ as in the figure below. Therefore by making ε small we have that near $x = a$ the graph $y = f(x)$ is sandwiched between two line that have slope very close to m . □



Problem 12. Prove Theorem 9. □

Lemma 10. Let g be differentiable at a . Then there is a $\delta > 0$ such that

$$|x - a| < \delta \quad \text{implies} \quad |g(x) - g(a)| \leq (|g'(a)| + 1)|x - a|.$$

Theorem 11 (The chain rule). Let g be defined on an interval containing a and f defined and on an interval containing $g(a)$. Assume that g is differentiable at a and f is differentiable at $g(a)$. Then the composition $h = f \circ g$ is differentiable at a and

$$h'(a) = (f \circ g)'(a) = f'(g(a))g'(a).$$

Proof. As g is differentiable at a , by Theorem 9, we have

$$g(x) - g(a) = g'(a)(x - a) + \rho_1(x; a)$$

where

$$\lim_{x \rightarrow a} \frac{\rho_1(x; a)}{x - a} = 0.$$

Likewise as f is differentiable at $g(a)$ we have

$$f(y) - f(g(a)) = f'(g(a))(y - g(a)) + \rho_2(y; g(a))$$

where

$$\lim_{y \rightarrow g(a)} \frac{\rho_2(y; g(a))}{y - g(a)} = 0.$$

Therefore we have

$$\begin{aligned} f(g(x)) - f(g(a)) &= f'(g(a))(g(x) - g(a)) + \rho_2(g(x); g(a)) \\ &= f'(g(a)) (g'(a)(x - a) + \rho_1(x; a)) + \rho_2(g(x); g(a)) \\ &= f'(g(a))g'(a)(x - a) + \rho_2(g(x); g(a)) \end{aligned}$$

where

$$\rho(x; a) = f'(g(a))g'(a)\rho_1(x; a) + \rho_2(g(x); g(a)).$$

Therefore, by Theorem 9, to finish the proof it is enough to show

$$\lim_{x \rightarrow a} \frac{\rho(x; a)}{x - a} = 0.$$

The first term in the definition of is easy to deal with:

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f'(g(a))g'(a)\rho_1(x; a)}{x - a} &= f'(g(a))g'(a) \lim_{x \rightarrow a} \frac{\rho_1(x; a)}{x - a} \\ &= f'(g(a))g'(a)0 \\ &= 0 \end{aligned}$$

The second term takes a bit more work. Let $\varepsilon > 0$

$$\lim_{y \rightarrow g(a)} \frac{\rho_2(y; g(a))}{y - g(a)} = 0$$

there is a $\delta_1 > 0$ such that

$$0 < |y - g(a)| < \delta \quad \text{implies} \quad \left| \frac{\rho_2(y; g(a))}{y - g(a)} \right| < \frac{\varepsilon}{1 + |g'(a)|}$$

and therefore

$$|y - g(a)| < \delta \quad \text{implies} \quad |\rho_2(y; g(a))| \leq \frac{\varepsilon|y - g(a)|}{1 + |g'(a)|}.$$

Problem 13. If g is differentiable at x show that there is a $\delta_2 > 0$ such that

$$|x - a| < \delta_2 \quad \text{implies} \quad |g(x) - g(a)| \leq (|g'(a)| + 1)|x - a| \quad \square$$

Getting back to the proof of the chain rule, there is a if $|g(x) - g(a)| < \delta_1$ and $|x - a| < \delta_2$ then

$$|\rho_2(y; g(a))| \leq \frac{\varepsilon|y - g(a)|}{1 + |g'(a)|} \leq \frac{\varepsilon(1 + |g'(a)|)|x - a|}{1 + |g'(a)|} = \varepsilon|x - a|.$$

and therefore

$$\left| \frac{\rho_2(g(x); g(a))}{x - a} \right| \leq \varepsilon$$

Finally, as g is continuous at a , there is $\delta_3 > 0$ such that

$$|x - a| < \delta_1 \quad \text{implies} \quad |g(x) - g(a)| < \delta_1.$$

Whence if $\delta = \min\{\delta_2, \delta_2\}$,

$$0 < |x - a| < \delta \quad \text{implies} \quad \left| \frac{\rho_2(g(x); g(a))}{x - a} \right| \leq \varepsilon$$

and therefore

$$\lim_{x \rightarrow a} \frac{\rho_2(g(x); g(a))}{x - a} = 0$$

which completes the proof. \square

1.2. Functions differentiable on an interval, the first derivative test, and the mean value theorem.

Definition 12. Let f be defined on an open set U containing x_0 . Then f has a **local maximum** (respectively a **local minimum**) at x_0 if and only if there is a $\delta > 0$ such that

$$f(x) \leq f(x_0) \quad (\text{respectively } f(x) \geq f(x_0)) \quad \text{for } x \text{ with } |x - x_0| < \delta$$

In this case x_0 is a **local maximizer** (respectively a **local minimizer**) of f . The point x_0 is a **local extrema** if it is either a local maximizer or a local minimizer. \square

Theorem 13 (First Derivative Test). *If f is defined on an open U set containing the point x_0 and*

- f is differentiable at x_0
- f has a local extrema at x_0 .

then

$$f'(x_0) = 0.$$

Lemma 14. *Let f be differentiable at x_0 and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence with*

$$\lim_{n \rightarrow \infty} x_n = x_0 \quad \text{and for all } n \quad x_n \neq x_0.$$

Then

$$\lim_{n \rightarrow \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0)$$

Problem 14. Prove this. \square

Problem 15. Prove Theorem 13. *Hint:* You do not have to follow this hint, but here is one way to start. Without loss of generality we can assume f has a local maximum at x_0 . (If it has a local minimum, then replace f by $-f$.) Let

$$x_n = x_0 - \frac{1}{n} \quad \text{and} \quad y_n = x_0 + \frac{1}{n}.$$

Then show

$$\lim_{n \rightarrow \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \geq 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{f(y_n) - f(x_0)}{y_n - x_0} \leq 0$$

and use the lemma. \square

Theorem 15 (Rôle's Theorem). *Let f be a function that is continuous on $[a, b]$ and differentiable at all points of (a, b) . Assume*

$$f(a) = f(b).$$

Then there exists a point $\xi \in (a, b)$ such that

$$f'(\xi) = 0.$$

Problem 16. Prove this. *Hint:* Start by showing that either (or both) of the maximum or minimum of f occur in the open interval (a, b) . \square

Theorem 16 (Mean Value Theorem). *Let f be a function that is continuous on $[a, b]$ and differentiable at all points of (a, b) . Then there exists a point $\xi \in (a, b)$ such that*

$$f(b) - f(a) = f'(\xi)(b - a)$$

Problem 17. Prove this. *Hint:* One way to start is to show

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

satisfies the hypothesis of Rôlle's Theorem. \square

Definition 17. Let x_1, x_2 and ξ be three real numbers. Then ξ is **between** x_1 and x_2 if and only if one of the following three cases holds:

$$x_1 < \xi < x_2$$

$$x_2 < \xi < x_1$$

$$x_1 = \xi = x_2.$$

\square

Often we will use the Mean Value Theorem in the following slightly less general form:

Theorem 18 (Mean Value Theorem). *Let f be differentiable on the open interval (a, b) and let $x_1, x_2 \in (a, b)$. Then there is ξ between x_1 and x_2 such that*

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1).$$

Proof. If $x_1 = x_2$, then let $\xi = x_1$ and we have $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) = 0$. If $x_1 \neq x_2$, then by possibly changing the names of x_1 and x_2 we can assume that $x_1 < x_2$. Then f is continuous on $[x_1, x_2]$ and differentiable on $I(x_1, x_2)$. Therefore we can use our first form of the Mean Value Theorem to conclude there is a $\xi \in (x_1, x_2)$ with $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$. \square

Before using the Mean Value Theorem to prove theorems let us note that it can be used to prove interesting results about concrete functions. Here are a couple of examples.

Example 19. Assume that we know that the derivative of $\sin(x)$ is $\cos(x)$. Then for all $a, b \in \mathbb{R}$ we have

$$|\sin(b) - \sin(a)| \leq |b - a|.$$

To see this let $f(x) = \sin(x)$. Then the Mean Value Theorem tells us there is a ξ between b and a such that

$$|\sin(b) - \sin(a)| = |f(b) - f(a)| = |f'(\xi)(b - a)| = |\cos(\xi)(b - a)| \leq |b - a|$$

where at the last step we used that $|\cos(\xi)| \leq 1$. \square

Example 20. If $a, b \geq 2$, then

$$\left| \frac{a-1}{a+1} - \frac{b-1}{b+1} \right| \leq \frac{2}{9}|b-a|.$$

To see this let

$$f(x) = \frac{x-1}{x+1}.$$

Then if $\xi \geq 2$ we have

$$f'(\xi) = \frac{2}{(\xi+1)^2} \leq \frac{2}{(2+1)^2} = \frac{2}{9}.$$

Thus if $a, b \geq 2$ the Mean Value Theorem gives us a ξ between a and b (and therefore $\xi \geq 2$) such that

$$\left| \frac{a-1}{a+1} - \frac{b-1}{b+1} \right| = |f(b) - f(a)| = |f'(\xi)(b-a)| = \frac{2}{(\xi+1)^2} |b-a| \leq \frac{2}{9} |b-a|.$$

Problem 18. Use the ideas above to show the following

(a) For all $x, y \in \mathbb{R}$ the inequality

$$|\cos(4y) - \cos(4x)| \leq 4|y - x|.$$

(b) If $a, b > 1$ then

$$|\sqrt{b^2 - 1} - \sqrt{a^2 - 1}| \geq |b - a|.$$

(c) If $x > 0$ then

$$e^x - 1 > x.$$

Hint: $e^x - 1 = e^x - e^0$. □

Theorem 21. Let f be differentiable on the open interval (a, b) and assume

$$f'(x) = 0 \quad \text{for all } x \in (a, b).$$

Then f is constant.

Problem 19. Use the Mean Value Theorem to prove this. □

Definition 22. If f is a function defined on an interval I , then f is **increasing** if and only if for all $x_1, x_2 \in I$

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

Theorem 23. Let f be a function on the open interval and assume that f' exists on all of (a, b) and that $f'(x) > 0$ for all $x \in (a, b)$. Then f is increasing on (a, b) .

Problem 20. Use the Mean Value Theorem to prove this. □

Problem 21. Show that $f'(x_0)$ exists if and only if the limit

$$\lim_{h \rightarrow 0} \frac{f(x_0) - f(x_0 - h)}{h}$$

exists. When this limit exists what is its value? □

Problem 22. Show that if $f'(x_0)$ exists, then so does the limit

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

and its value is $f'(x_0)$. □

Problem 23. Let α be a positive real number and set

$$f(x) = \begin{cases} |x|^\alpha \cos(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

For what values of α does $f'(0)$ exist. When it does exist what is its value? \square

Proposition 24. The function $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$ and

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Problem 24. Prove this. *Hint:* The calculation

$$\begin{aligned} f(x+h) - f(x) &= \sqrt{x+h} - \sqrt{x} \\ &= \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{(x+h) - x}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{h}{\sqrt{x+h} + \sqrt{x}} \end{aligned}$$

might be useful. \square

Theorem 25 (Cauchy Mean Value Theorem). *Let f and g be functions that are differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$. Then there is a $\xi \in (a, b)$ such that*

$$g'(\xi)(f(b) - f(a)) = f'(\xi)(g(b) - g(a)).$$

(Note when g is the function $g(x) = x$ this reduces to the usual mean value theorem.)

Problem 25. Prove this. *Hint:* Let

$$h(x) = (g(b) - g(a))(f(x) - f(a)) - (f(b) - f(a))(g(x) - g(a))$$

and show $h(a) = h(b) = 0$. \square

We now wish to look at one of the other standard topics in differential calculus, l'hôpital's rule. Recall this involves evaluating limits of the type

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$$

where $f(x_0) = g(x_0) = 0$ which leads to

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$$

which, at least formally, does not make sense. Here is the basic result.

Theorem 26 (L'hôpital's rule). *Let f and g be differentiable in a neighborhood of x_0 with $g'(x) \neq 0$ for $x \neq x_0$. Assume that $f(x_0) = g(x_0) = 0$ and that*

$$\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = L$$

exists. Then $\lim_{x \rightarrow x_0} f(x)/g(x)$ exists and is given by

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = L$$

This is usually stated informally as that if $f(x_0) = g(x_0) = 0$ then

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}.$$

The important part is that the existence of the limit $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$ implies the existence of the limit $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$.

Problem 26. Prove Theorem 26 as follows. Let $\varepsilon > 0$ then as $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = L$ there is a $\delta > 0$ such that

$$0 < |x - x_0| < \delta \implies \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

Let x be so that $0 < |x - x_0| < \delta$. Then, by the Cauchy Mean Value Theorem, there is a ξ between x and x_0 such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - 0}{g(x) - 0} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)}.$$

Use this to show

$$0 < |x - x_0| < \delta \implies \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$$

and thus $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = L$. (A main point is that $0 < |\xi - x_0| < \delta$, so be sure to explain why this holds.) \square

Here is a standard application of l'hôpital's rule:

$$\lim_{x \rightarrow 0} \frac{\sin(2x)}{3x} = \lim_{x \rightarrow 0} \frac{\sin(2x)'}{(3x)'} = \lim_{x \rightarrow 0} \frac{2 \cos(2x)}{3} = \frac{2 \cos(0)}{3} = \frac{2}{3}.$$

It can also be applied several times in a row:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} &= \lim_{x \rightarrow 0} \frac{\sin(x)}{2x} && \text{(take } \frac{d}{dx} \text{ of top and bottom)} \\ &= \lim_{x \rightarrow 0} \frac{\cos(x)}{2} && \text{(take } \frac{d}{dx} \text{ of top and bottom)} \\ &= \frac{\cos(0)}{2} \\ &= \frac{1}{2}. \end{aligned}$$

So we have shown $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}$. Note that in terms of showing this limit exists, this should be read from the bottom up. That is l'hôpital's rule shows that $\lim_{x \rightarrow 0} \frac{\sin(x)}{2x}$ exists as $\lim_{x \rightarrow 0} = \frac{\sin(x)'}{(2x)'} = \lim_{x \rightarrow 0} \frac{\cos(0)}{2} = \frac{1}{2}$ exists.

Then another application of l'hôpital's rule shows that $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \rightarrow 0} \frac{(1 - \cos(x))'}{(x^2)'} = \lim_{x \rightarrow 0} \frac{\sin(x)}{2x}$ exists.

Problem 27. Here are some problems to practice the use of l'hôpital's rule. Compute the following

- (a) $\lim_{x \rightarrow 0} \frac{\sin(x) - x}{x^3}$
- (b) $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x}$
- (c) $\lim_{\theta \rightarrow \pi} \frac{\sin^3(x)}{x(\cos(x) + 1)}$

□

Now back to Rôlle's theorem. First a definition.

Definition 27. Let f be defined on an open interval I . Then f is **twice differentiable** on I if f' exists at all points of I and the function f' is differentiable on I . We denote the derivative of f' as f'' or $f^{(2)}$ and it is called the **second derivative** of f . If f'' exists at all points of I and f'' is differentiable on I its derivative is denoted by f''' or $f^{(3)}$ and is called the **third derivative** of f and f is said to be **three times differentiable**. Continuing recursively, if we have defined what it means for f to be n **times differentiable** on I and the n -th derivative, $f^{(n)}$, is differentiable on I then the derivative of $f^{(n)}$ is denoted by $f^{(n+1)}$ and f is $(n + 1)$ **times differentiable** on I . □

Remark 28. For consistency's sake we set $f^{(0)} = f$ and $f^{(1)} = f'$ □

Problem 28. Show that the function f on \mathbb{R} defined by

$$f(x) = \begin{cases} x^2, & x \geq 0; \\ -x^2, & x < 0. \end{cases}$$

is differentiable on \mathbb{R} but not twice differentiable. *Hint:* Show $f'(x) = 2|x|$. You may have to use the limit definition to compute $f'(0)$. □

Problem 29. Find a function that is twice differentiable on \mathbb{R} but not three times differentiable. More generally can you give an example of a function that is n times differentiable, but not $n + 1$ times differentiable. □

Proposition 29. Let I be an open interval and assume f is twice differentiable on I . Let $x_0, x_1 \in I$ with $x_0 \neq x_1$. Assume $f(x_0) = f'(x_0) = 0$ and $f(x_1) = 0$. Then there is a point ξ between x_0 and x_1 with $f''(\xi) = 0$.

Proof. As $f(x_0) = f(x_1) = 0$ by Rôlle's Theorem there is a ξ_1 between x_0 and x_1 with $f'(\xi_1) = 0$. But the function f' is differentiable on I and

$f'(x_0) = f'(\xi_1) = 0$ and thus another application of Rôlle's Theorem gives us a ξ between x_0 and ξ_1 with $f''(\xi) = (f')'(\xi) = 0$. As ξ_1 is between x_0 and x_1 and ξ is between x_0 and ξ_1 we have that ξ is between x_0 and x_1 . \square

This generalizes

Theorem 30. *Let f be $n + 1$ times differentiable on the open interval I . Let $x_0, x_1 \in I$ with $x_0 \neq x_1$. Assume that*

- $f(x_0) = f'(x_0) = \cdots = f^{(n)}(x_0) = 0$,
- $f(x_1) = 0$.

Then there is a point ξ between x_0 and x_1 with

$$f^{(n+1)}(\xi) = 0. \quad \square$$

Problem 30. Prove this. *Hint:* There are several ways to do this. One is to look at the proof of Proposition 29 and meditate upon induction. \square

Proposition 31. *Let f be twice differentiable on the open interval I and let $a, b \in I$ with $a \neq b$. Then there is a ξ between a and b with*

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(\xi)}{2}(b - a)^2.$$

Proof. Let h be defined on I by

$$h(x) = f(x) - f(a) - f'(a)(x - a) - c(x - a)^2$$

where c is a constant to be chosen shortly. Note

$$h(a) = 0$$

and

$$h'(x) = f'(x) - f'(a) - 2c(x - a),$$

and thus

$$h'(a) = 0.$$

With applying Proposition 29 in mind, we choose c so that $h(b) = 0$. That is

$$h(b) = f(b) - f(a) - f'(a)(b - a) - c(b - a)^2 = 0$$

which leads to

$$c = \frac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2}.$$

With this choice of c we have $h(a) = h'(a) = h(b) = 0$ and thus by Proposition 31 there is a ξ between a and b with

$$h''(\xi) = 0.$$

By direct calculation

$$h''(x) = f''(x) - 2c.$$

Then $h''(\xi) = 0$ yields

$$f''(\xi) - 2c = 0.$$

But using the formula for c above we find

$$f''(\xi) - 2 \left(\frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2} \right) = 0$$

which can be rearranged to give

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(\xi)}{2}(b-a)^2$$

as required. \square

As this was a more or less direct consequence of Proposition 29 it makes sense to look for a generalization that depends on Theorem 30. To make life a little easier on ourselves we first do the case of $n = 4$.

Lemma 32. *Let f be a function that is four times differentiable on an open interval I and let $a \in I$. Let $T(x)$ be the polynomial*

$$(2) \quad T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4,$$

and set

$$g(x) = f(x) - T(x).$$

Then

$$g(a) = g'(a) = g''(a) = g^{(3)}(a) = g^{(4)}(a) = 0.$$

Problem 31. Prove this. \square

Theorem 33. *Let f be five times differentiable on the open interval I and $a, b \in I$ with $a \neq b$. Then there is a ξ between a and b such that*

$$\begin{aligned} f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \frac{f^{(4)}(a)}{4!}(b-a)^4 \\ + \frac{f^{(5)}(\xi)}{5!}(b-a)^5. \end{aligned}$$

Or in different notation let $T(x)$ be the polynomial (2), then this is

$$f(b) = T(b) + \frac{f^{(5)}(\xi)}{5!}(b-a)^5.$$

Problem 32. Prove this. *Hint:* Let

$$h(x) = f(x) - T(x) - c(x-a)^5$$

where we choose c so that

$$h(b) = 0.$$

Show $h(a) = h'(a) = h''(a) = h^{(3)}(a) = h^{(4)}(a) = 0$. Now use Theorem 30 and now proceed as in the proof of Proposition 31. \square

Definition 34. Let f be n times differentiable on a neighborhood of a . Then the **degree n Taylor polynomial** of f at x is

$$T_n(x) := \sum_{k=0}^n f^{(k)}(a) \frac{(x-a)^k}{k!}. \quad \square$$

Problem 33. Show that if f is n times differentiable on an open interval I and T_n is its degree n Taylor polynomial at a , then for $0 \leq k \leq n$

$$T_n^{(k)}(a) = f^{(k)}(a).$$

That is the k -th derivatives of T_n and f agree at a for $0 \leq k \leq n$. \square

Theorem 35 (Taylor's Theorem with Lagrange's form of the remainder). *Let f be $(n+1)$ times differentiable on the open interval I and let $a, b \in I$ with $a \neq b$. Let T_n be the degree n Taylor polynomial of f at a . Then there is a ξ between a and b such that*

$$f(b) = T_n(b) + f^{(n+1)}(\xi) \frac{(b-a)^{n+1}}{(n+1)!}.$$

(The term $E_n(b) = f^{(n+1)}(\xi) \frac{(b-a)^{n+1}}{(n+1)!} = f(b) - T_n(b)$ is the **error term** or **remainder term** when approximating f by its Taylor polynomial T_n .)

Problem 34. Prove this. \square

We restate this with slightly different notation (just replacing a and b with x_0 and x .)

Theorem 36 (Taylor's Theorem with Lagrange's form of the remainder, form 2). *Let f be $(n+1)$ times differentiable on the open interval I and let $x, x_0 \in I$ with $x \neq x_0$. Then there is a ξ between x and x_0 such that*

$$f(x) = \sum_{k=0}^n f^{(k)}(x_0) \frac{(x-x_0)^k}{k!} + f^{(n+1)}(\xi) \frac{(x-x_0)^{n+1}}{(n+1)!}. \quad \square$$

Remark 37. In the case that $n = 0$ this becomes

$$f(x) = f(x_0) + f'(\xi)(x-x_0),$$

which can be rewritten as $f(x) - f(x_0) = f'(\xi)(x-x_0)$. That is for $n = 0$ we just get the mean value theorem. \square

One last restatement of Taylor's theorem. If we let $x = x_0 + h$ we get

$$f(x_0 + h) = \sum_{k=0}^n f^{(k)}(x_0) \frac{h^k}{k!} + f^{(n+1)}(\xi) \frac{h^{n+1}}{(n+1)!}$$

where ξ is between x_0 and $x_0 + h$.

As an examples of Taylor's theorem we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{e^\xi x^4}{4!} \quad (\text{Used } n = 3.)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{\cos(\xi)x^6}{6!} \quad (\text{Used } n = 5.)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{\cos(\xi)x^7}{7!} \quad (\text{Used } n = 6.)$$

where ξ is between x and 0 (and of course the value of ξ is different in each of the three equations).