CIRCLES MINIMIZE MOST KNOT ENERGIES

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Abstract

For a curve $c\colon S^1 \to \mathbf{R}^n$ define

$$E_j^p[c] := \iint \left(\frac{1}{|c(t)-c(s)|^j} - \frac{1}{d(t,s)^j} \right)^p dt\, ds,$$

where $|c(t)-c(s)|$ is the distance between $c(t)$ and $c(s)$ in space, and $d(t,s)$ is the shortest distance between $c(t)$ and $c(s)$ along the curve. Up to some constants, these are basically the knot energies introduced by O'Hara in the early 1990's. These integrals converge if the curve c is smooth and embedded, $j < 2 + 1/p$.

Theorem 1. *Suppose $0 < j < 2 + 1/p$, while $p \geq 1$. Then for every closed unit-speed curve c in $\mathbf{R}^n$ with length 2π,*

$$E_j^p[c] \geq 2^{3-jp}\pi \int_0^{\frac{\pi}{2}} \left(\left(\frac{1}{\sin s} \right)^j - \left(\frac{1}{s} \right)^j \right)^p ds. \tag{0.1}$$

with equality if and only if c is the circle.

Our methods, based on earlier results of G. Lükő, also show that

$$\iint f(|c(s)-c(t)|^2)\, ds\, dt$$

is maximized uniquely by circles whenever f is increasing and concave. We also consider the case where f is convex, as in the case of the functional

$$\iint |c(s)-c(t)|^p\, ds\, dt$$

for $p > 2$. Numerical experiments suggest that the maximizing curve for this functional remains a circle for $p < \alpha$, with $3.3 < \alpha < 3.5721$, while for $p > 3.5721$, the maximizers form a family of stretched ovals converging to a doubly-covered line segment as $p \to \infty$.

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