Mathematics 555 Homework

1. The derivative.

1.1. The derivative at a point.

Definition 1. Let (α, β) be an open interval, $f\colon (\alpha,\beta) \to \mathbb{R}$ a function, and $a \in (\alpha, \beta)$. Then f is ***differentiable*** at if and only if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. When this limit exists it is the ***derivative*** of f at a and is denoted by $f'(a)$. $\square$

The limit defining $f'(a)$ can be rewritten in several ways. For example if we do the change of variable $x = a + h$ in the limit it becomes

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

which is the way it is often presented in calculus books. And sometimes, especially in older books, h is replaced by Δx and the limit is written as

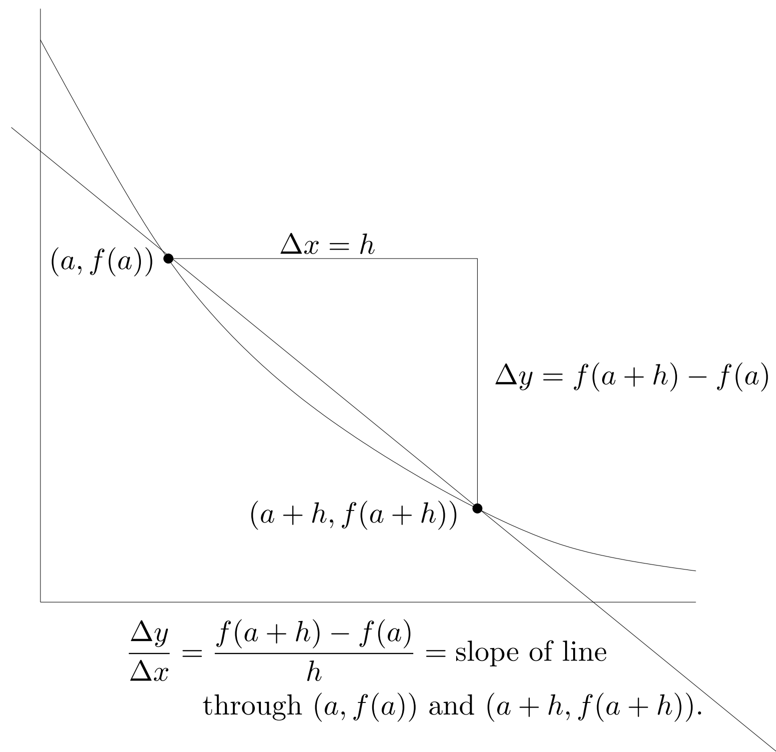
$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

And finally $f(a+\Delta x) - f(a)$ can be abbreviated as $\Delta y = f(a+\Delta x) - f(a)$ and then the limit becoves

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

a notation meshes well with the Leibniz notation $\dfrac{dy}{dx}$ for the derivative.

I am now obligated to draw the standard picture that shows that the ***difference quotient*** $(f(a+h) - f(a))/h$ is the slope through the points $(a, f(a))$ and $(a+h, f(a+h))$ and therefore taking the limit as $h \to 0$ of this difference quotient is a reasonable definition of the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



$$\frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{h} = \text{slope of line through } (a, f(a)) \text{ and } (a+h, f(a+h)).$$

We now do some examples of derivatives that you no doubt already know from calculus.

Let $f(x) = mx + b$ where m and b are constants. Then for any $a \in \mathbb{R}$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{m(x-a)}{x-a} = m.$$

For a slightly more complicated example consider $f(x) = x^2$ we have , using that $x^2 - a^2 = (x-a)(x+a)$:

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{x^2 - a^2}{x - a} \\ &= \lim_{x \to a} (x + a) \\ &= 2a. \end{aligned}$$

Problem 1. Use the identities $x^3 - a^3 = (x-a)(x^2 + ax + a^2$ and $x^4 - a^4 = (x-a)(x^3 + ax^2 + a^2x + a^3$ to prove that the functions $f(x) = x^3$ and $g(x) = x^4$ have the derivatives

$$\begin{aligned} f'(a) &= 3a^2 \\ g'(a) &= 4a^3. \end{aligned}$$

□

The classic example of a function that does not have a derivative at a point is the absolute value function $f(x) = |x|$ which does not have a derivative at $x = 0$. Here are some other examples

Problem 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \max\{x, 2 - x\}.$$

Graph $y = f(x)$ and show that it is differentiable at every point other than $x = 1$. What is $f'(a)$ when $a \neq 1$? □

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \min\{x^2, 1\}$$

Find the points where f is not differentiable and prove your result.

We now start proving the basic rules for derivatives you know from calculus.

Proposition 2 (Sum rule for derivatives)**.** *Let f_1 and f_2 be defined on an interval containing the point a and assume that f_1 and f_2 are both differentiable at a. Let c_1 and c_2 be constants. Then the function $g = c_1 f_1 + c_2 f_2$ is differentiable at a and*

$$g'(a) = c_1 f_1'(a) + c_2 f_2'(a).$$

Problem 4. Prove this. □

We extend this to sums with more terms:

Proposition 3. *Let $f_1, f_2, \ldots, f_n$ be functions defined on an interval containing a and with each f_j differentiable at a. Let $c_1, c_2, \ldots, c_n$ be constants. Then $g = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$ is differentiable at a and*

$$g'(a) = c_1 f_1'(a) + c_2 f_2'(a) + \cdots + c_n f_n'(a).$$

Proof. This is an easy induction proof. □

Proposition 4. *Let f be defined on an interval containing a and assume that f is differentiable at a. Then f is continuous at a.*

Problem 5. Prove this. *Hint:* To show that f is continuous at a we need to show $\lim_{x\to a} f(x) = f(a)$. As f is differentiable we know that

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists. To use this write $f(x)$ as

$$f(x) = f(a) + \frac{f(x) - f(a)}{x - a}(x - a).$$

Now you can use standard results about limits (no ε, δ needed). □

Proposition 5 (Product rule)**.** *Let f and g be defined in an interval containing a and assume they are both differentiable at a. Then the product $p(x) = f(x)g(x)$ is differentiable at a and*

$$p'(a) = f'(a)g(a) + f(a)g'(a).$$

Problem 6*.* Prove this. *Hint:* One what is to do some adding and subtracting in the difference quotient for p:

$$\begin{aligned}
\frac{p(x) - p(a)}{x - a} &= \frac{f(x)g(x) - f(a)g(a)}{x - a} \\
&= \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x - a} \\
&= f(x)\frac{g(x) - g(a)}{x - a} + \frac{f(x) - f(a)}{x - a}g(a)
\end{aligned}$$

As f is continuous at a (why?) we have $\lim_{x\to a} f(x) = f(a)$. Finishing now should be easy. $\square$

Proposition 6. *Let g be defined in an interval containing a and assume g is differentiable at a and $g(a) \neq 0$. Then $h(x) = 1/g$ is differentiable at a and*

$$h'(a) = \frac{-g'(a)}{g(a)^2}.$$

Problem 7*.* Prove this. *Hint:* Write the difference quotient for h as

$$\begin{aligned}
\frac{h(x) - h(a)}{x - a} &= \frac{1}{x - a}\left(\frac{1}{g(x)} - \frac{1}{g(a)}\right) \\
&= \frac{1}{x - a}\frac{g(a) - g(x)}{g(x)g(a)} \\
&= \frac{-1}{g(x)g(a)}\frac{g(x) - g(a)}{x - a}
\end{aligned}$$

and now it should be easy to take the limit defining $h'(a)$. $\square$

Proposition 7 (Quotient rule)**.** *Let f and g be defined on an interval containing a and with f and g differentiable at a. Also assume $g(a) \neq 0$. Then the quotient $q(x) = f(x)/g(x)$ is differentiable at a and*

$$q'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Problem 8*.* Prove this. *Hint:* We have already done most of the work for this. Write q as a product

$$q(x) = f(x)\left(\frac{1}{g(x)}\right)$$

and use Proposition 6 and the product rule. $\square$

Proposition 8 (The power rule for positive powers)**.** *Let f be defined on an interval containing a and assume that f is differentiable at a. Then for any positive integer n the function $p(x) = f(x)^n$ is differentiable at a and*

$$p'(x) = nf(a)^{n-1}f'(a).$$

In particular letting $f(x) = x$ yields that the when $p(x) = x^n$, then $p'(a) = na^{n-1}$.

Problem 9. Prove this. *Hint:* Induction. □

Proposition 9.

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial (thus $a_0, a_1, \ldots, a_n$ are constants) then f is differentiable at all points a and

$$f'(a) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1.$$

Proof. This follows by combining Propositions 2 and 8. □

The next result makes precise that the graph $y = f(x)$ of a function has a tangent line at the point $(a, f(a))$ if and only if the derivative $f'(a)$ exists.

Theorem 10. *Let f be a real valued function defined in a neighborhood of a. Then the following are equivalent.*

(a) $f'(a)$ exists.

(b) There is a constant m such that

$$f(x) = f(a) + m(x-a) + \rho(x;a)$$

where $\rho(x;a)$ satisfies

$$\lim_{x\to a} \frac{\rho(x;a)}{x-a} = 0. \tag{1}$$

Before going on with the proof, let us think a bit about what condition (b) of the theorem says. Rewrite (1) as

$$f(x) = f(a) + \left(m + \frac{\rho(x;a)}{x-a}\right)(x-a).$$

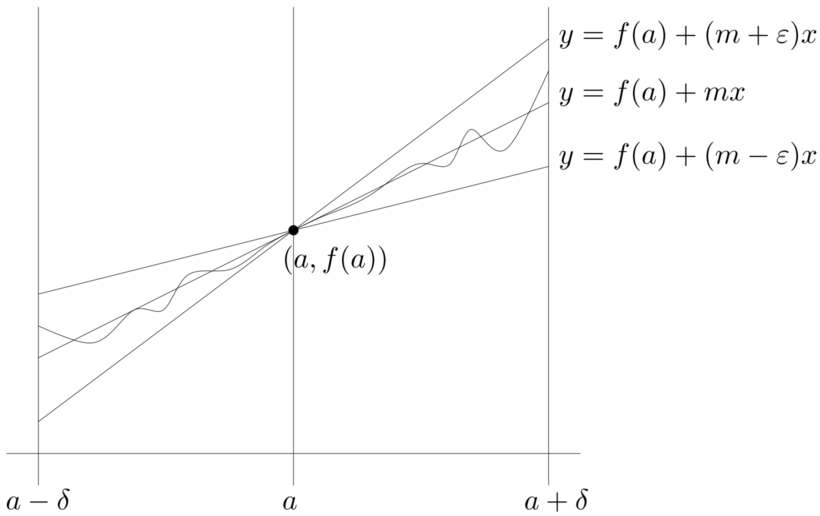
As $\lim_{x\to a} \rho(x;a)/(x-a) = 0$ for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \text{implies} \quad \frac{|\rho(x;a)|}{|x-a|} < \varepsilon.$$

Therefore

$$0 < |x-a| < \delta \quad \text{implies} \quad m - \varepsilon < m + \frac{\rho(x:a)}{x-a} < m + \varepsilon.$$

Problem 10. Show that these inequalities imply that on the interval $(a-\delta, a+\delta)$ that the graph of $y = f(x)$ stays between the graphs of the lines $y = f(a)+(m+\varepsilon)(x-a)$ and $y = f(a)+(m-\varepsilon)(x-a)$ as in the figure below. Therefore by making ε small we have that near $x = a$ the graph $y = f(x)$ is sandwiched between two line that have slope very close to m. □



Problem 11. Prove Theorem 10. $\square$

Lemma 11. *Let g be differentiable at a. Then there is a $\delta > 0$ such that*

$$|x-a| < \delta \quad \textit{implies} \quad |g(x) - g(a)| \le (|g'(a)|+1)|x-a|.$$

Theorem 12 (The chain rule)**.** *Let g be defined on an interval containing a and f defined and on an interval containing $g(a)$. Assume that g is differentiable at a and f is differentiable at $g(a)$. Then the composition $h = f \circ g$ is differentiable at a and*

$$h'(a) = (f \circ g)'(a) = f'(g(a))g'(a).$$

Proof. As g is differentiable at a, by Theorem 10, we have

$$g(x) - g(a) = g'(a)(x-a) + \rho_1(x;a)$$

where

$$\lim_{x \to a} \frac{\rho_1(x;a)}{x-a} = 0.$$

Likewise as f is differentiable at $g(a)$ we have

$$f(y) - f(g(a)) = f'(g(a))(y - g(a))(x-a) + \rho_2(y; g(a))$$

where

$$\lim_{y \to g(a)} \frac{\rho_2(y;g(a))}{y - g(a)} = 0.$$

Therefore we have

$$\begin{aligned} f(g(x)) - f(g(a)) &= f'(g(a))(g(x) - g(a)) + \rho_2(g(x); g(a)) \\ &= f'(g(a))\big(g'(a)(x-a) + \rho_1(x;a)\big) + \rho_2(g(x); g(a)) \\ &= f'(g(a))g'(a)(x-a) + \rho_2(g(x); g(a)) \end{aligned}$$

where

$$\rho(x;a) = f'(g(a))g'(a)\rho_1(x;a) + \rho_2(g(x); g(a)).$$

Therefore, by Theorem 10, to finish the proof it is enough to show

$$\lim_{x\to a}\frac{\rho(x:a)}{x-a}=0.$$

The first term in the definition of is easy to deal with:

$$\begin{aligned}\lim_{x\to a}\frac{f'(g(a))g'(a)\rho_1(x;a)}{x-a}&=f'(g(a))g'(a)\lim_{x\to a}\frac{\rho_1(x;a)}{x-a}\\&=f'(g(a))g'(a)0\\&=0\end{aligned}$$

The second term takes a bit more work. Let $\varepsilon>0$

$$\lim_{y\to g(a)}\frac{\rho_2(y;g(a))}{y-g(a)}=0$$

there is a $\delta_1>0$ such that

$$0<|y-g(a)|<\delta \quad \text{implies} \quad \left|\frac{\rho_2(y;g(a))}{y-g(a)}\right|<\frac{\varepsilon}{1+|g'(a)|}$$

and therefore

$$|y-g(a)|<\delta \quad \text{implies} \quad |\rho_2(y;g(a))|\le\frac{\varepsilon|y-g(a)|}{1+|g'(a)|}.$$

Problem 12. If g is differentiable at x show that there is a $\delta_2>0$ such that

$$|x-a|<\delta_2 \quad \text{implies} \quad |g(x)-g(a)|\le(|g'(a)|+1)|x-a| \qquad \square$$

Getting back to the proof of the chain rule, there is a if $|g(x)-g(a)|<\delta_1$ and $|x-a|<\delta_2$ then

$$|\rho_2(y;g(a))|\le\frac{\varepsilon|y-g(a)|}{1+|g'(a)|}\le\frac{\varepsilon(1+|g'(a)|)|x-a|}{1+|g'(a)|}=\varepsilon|x-a|.$$

and therefore

$$\left|\frac{\rho_2(g(x);g(a))}{x-a}\right|\le\varepsilon$$

Finally, as g is continuous at a, there is $\delta_3>0$ such that

$$|x-a|<\delta_1 \quad \text{implies} \quad |g(x)-g(a)|<\delta_1.$$

Whence if $\delta=\min\{\delta_2,\delta_2\}$,

$$0<|x-a|<\delta \quad \text{implies} \quad \left|\frac{\rho_2(g(x);g(a))}{x-a}\right|\le\varepsilon$$

and therefore

$$\lim_{x\to a}\frac{\rho_2(g(x);g(a))}{x-a}=0$$

which completes the proof. $\square$

1.2. **Functions differentiable on an interval, the first derivative test, and the mean value theorem.**

Definition 13. Let f be defined on an open set U containing x_0. Then f has a ***local maximum*** (respectively a ***local minimum***) at x_0 if and only if there is a $\delta > 0$ such that

$$f(x) \leq f(x_0) \quad \text{(respectively } f(x) \geq f(x_0)) \quad \text{for } x \text{ with } |x - x_0| < \delta$$

In this case x_0 is a ***local maximizer*** (respectively a ***local minimizer***) of f. The point x_0 is a ***local extrema*** if it is either a local maximizer or a local minimizer. $\square$

Theorem 14 (First Derivative Test)**.** *If f is defined on an open U set containing the point x_0 and*

- *f is differentiable at x_0*
- *f has a local extrema at x_0.*

then

$$f'(x_0) = 0.$$

Lemma 15. *Let f be differentiable at x_0 and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence with*

$$\lim_{n\to\infty} x_n = x_0 \quad \textit{and for all } n \quad x_n \neq x_0.$$

Then

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0)$$

Problem 13. Prove this. $\square$

Problem 14. Prove Theorem 14. *Hint:* You do not have to follow this hint, but here is one way to start. Without loss of generality we can assume f has a local maximum at x_0. (If it has a local minimum, then replace f by $-f$.) Let

$$x_n = x_0 - \frac{1}{n} \quad \text{and} \quad y_n = x_0 + \frac{1}{n}.$$

Then show

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \geq 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \leq 0$$

and use the lemma. $\square$

Theorem 16 (Rôlle's Theorem)**.** *Let f be a function that is continuous on $[a, b]$ and differentable at all points of (a, b). Assume*

$$f(a) = f(b).$$

Then there exists a point $\xi \in (a, b)$ such that

$$f'(\xi) = 0.$$

Problem 15. Prove this. *Hint:* Start by showing that either (or both) of the maximum or minimum of f occur in the open interval (a, b). $\square$

Theorem 17 (Mean Value Theorem)**.** *Let f be a function that is continuous on $[a, b]$ and differentiable at all points of (a, b). There there exists a point $\xi \in (a, b)$ such that*

$$f(b) - f(a) = f'(\xi)(b - a)$$

Problem 16*.* Prove this. *Hint:* One way to start is to show

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

satisfies the hypothesis of Rôlle's Theorem. □

Definition 18. Let x_1, x_2 and ξ be three real numbers. Then ξ is ***between*** x_1 and x_2 if and only if one of the following three cases holds:

$$\begin{aligned} x_1 <& \xi < x_2 \\ x_2 <& \xi < x_1 \\ x_1 =& \xi = x_2. \end{aligned}$$

□

Often we will use the Mean Value Theorem in the following slightly less general form:

Theorem 19 (Mean Value Theorem)**.** *Let f be differentiable on the open interval (a, b) and let $x_1, x_2 \in (a, b)$. There there is ξ between x_1 and x_2 such that*

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1).$$

Proof. If $x_1 = x_2$, then let $\xi = x_1$ and we have $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) = 0$. If $x_1 \neq x_2$, then by possibly changing the names of x_1 and x_2 we can assume that $x_1 < x_2$. Then f is continuous on $[x_1, x_2]$ and differentiable on $I(x_1, x_2)$. Therefore we can use our first form of the Mean Value Theorem to conclude there is a $\xi \in (x_1, x_2)$ with $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$. □

Before using the Mean Value Theorem to prove theorems let us note that it can be use to prove interesting results about concrete functions. Here are a couple of examples.

Example 20*.* Assume that we know that the derivative of $\sin(x)$ is $\cos(x)$. Then for all $a, b \in \mathbb{R}$ we have

$$|\sin(b) - \sin(a)| \leq |b - a|.$$

To see this let $f(x) = \sin(x)$. Then the Mean Value Theorem tells us there is a ξ between b and a such that

$$|\sin(b) - \sin(a)| = |f(b) - f(a)| = |f'(\xi)(b - a)| = |\cos(\xi)(b - a)| \leq |b - a|$$

where at the last step we used that $|\cos(\xi)| \leq 1$. □

Example 21*.* If $a, b \geq 2$, then

$$\left|\frac{a - 1}{a + 1} - \frac{b - 1}{b + 1}\right| \leq \frac{2}{9}|b - a|.$$

To see this let
$$f(x) = \frac{x-1}{x+1}.$$
Then if $\xi \geq 2$ we have
$$f'(\xi) = \frac{2}{(\xi+1)^2} \leq \frac{2}{(2+1)^2} = \frac{2}{9}.$$
Thus if $a, b \geq 2$ the Mean Value Theorem gives us a ξ between a and b (and therefore $\xi \geq 2$ such that
$$\left|\frac{a-1}{a+1} - \frac{b-1}{b+1}\right| = |f(b) - f(a)| = |f'(\xi)(b-a)| = \frac{2}{(\xi+1)^2}|b-a| \leq \frac{2}{9}|b-a|. \square$$

Problem 17. Use the ideas above to show the following

(a) For all $x, y \in \mathbb{R}$ the inequality
$$|\cos(4y) - \cos(4x)| \leq 4|y-x|.$$

(b) If $a, b > 1$ then
$$|\sqrt{b^2-1} - \sqrt{a^2-1}| \geq |b-a|.$$

(c) If $x > 0$ then
$$e^x - 1 > x.$$

Hint: $e^x - 1 = e^x - e^0$. □

Theorem 22. *Let f be differentiable on the open interval (a, b) and assume*
$$f'(x) = 0 \quad \textit{for all} \quad x \in (a, b).$$
Then f is constant.

Problem 18. Use the Mean Value Theorem to prove this. □

Definition 23. If f is a function defined on an interval I, then f is ***increasing*** if and only if for all $x_1, x_2 \in I$
$$x_1 < x_2 \quad \Longrightarrow \quad f(x_1) < f(x_2).$$

Theorem 24. *Let f be a function on the open interval and assume that f' exists on all of (a, b) and that $f'(x) > 0$ for all $x \in (a, b)$. Then f is increasing on (a, b).*

Problem 19. Use the Mean Value Theorem to prove this. □

Problem 20. Show that $f'(x_0)$ exists if and only if the limit
$$\lim_{h \to 0} \frac{f(x_0) - f(x_0 - h)}{h}$$
exists. When this limit exists what is its value? □

Problem 21. Show that if $f'(x_0)$ exists, then so does the limit
$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$
and its value is $f'(x_0)$. □

Problem 22*.* Let α be a positive real number and set

$$f(x) = \begin{cases} |x|^\alpha \cos(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

For what values of α does $f'(0)$ exist. When it doses exist what is its value? $\square$

Proposition 25. *The function $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$ and*

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Problem 23*.* Prove this. *Hint:* The calculation

$$\begin{aligned} f(x+h) - f(x) &= \sqrt{x+h} - \sqrt{x} \\ &= \frac{(\sqrt{x+h} - \sqrt{x}\,)(\sqrt{x+h} + \sqrt{x}\,)}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{(x+h) - x}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{h}{\sqrt{x+h} + \sqrt{x}} \end{aligned}$$

might be useful. $\square$

Theorem 26 (Cauchy Mean Value Theorem)**.** *Let f and g be functions that are differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$. Then there is a $\xi \in (a, b)$ such that*

$$g'(\xi)\big(f(b) - f(a)\big) = f'(\xi)\big(g(b) - g(a)\big).$$

(Note when g is the function $g(x) = x$ this reduces to the usual mean value theorem.

Problem 24*.* Prove this. *Hint:* Let

$$h(x) = (g(b) - g(a))(f(x) - f(a)) - (f(b) - f(a))(g(x) - g(a))$$

and show $h(a) = h(b) = 0$. $\square$

We now wish to look at one of the other standard topics in differential calculus, l'hôpital's rule. Recall this involves evaluating limits of the type

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

where $f(x_0) = g(x_0) = 0$ which leads to

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$$

which, at least formally, does not make sense. Here is the basic result.

Theorem 27 (L'hôpital's rule)**.** *Let f and g be differentiable in a neighborhood of x_0 with $g'(x)] \neq 0$ for $x \neq x_0$. Assume that $f(x_0) = g(x_0) = 0$ and that*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

exists. Then $\lim_{x \to x_0} f(x)/g(x)$ exists and is given by

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = L$$

This is usually stated informally as that if $f(x_0) = g(x_0) = 0$ then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

The important part is that the existence of the limit $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ implies the existence of the limit $\lim_{x \to x_0} \frac{f(x)}{g(x)}$.

Problem 25. Prove Theorem 27 as follows. Let $\varepsilon > 0$ then as $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$ there is a $\delta > 0$ such that

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

Let x be so that $0 < |x - x_0| < \delta$. Then, by the Cauchy Mean Value Theorem, there is a ξ between x and x_0 such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - 0}{g(x) - 0} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)}.$$

Use this to show

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$$

and thus $\lim_{x \to x_0} \frac{f(x)}{g(x)} = L$. (A main point is that $0 < |\xi - x_0| < \delta$, so be sure to explain why this holds.) $\square$

Here is a standard application of l'hôpital's rule:

$$\lim_{x \to 0} \frac{\sin(2x)}{3x} = \lim_{x \to 0} \frac{\sin(2x)'}{(3x)'} = \lim_{x \to 0} \frac{2\cos(2x)}{3} = \frac{2\cos(0)}{3} = \frac{2}{3}.$$

It can also be applied several times in a row:

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} &= \lim_{x \to 0} \frac{\sin(x)}{2x} && \left(\text{take } \frac{d}{dx} \text{ of top and bottom}\right) \\
&= \lim_{x \to 0} \frac{\cos(x)}{2} && \left(\text{take } \frac{d}{dx} \text{ of top and bottom}\right) \\
&= \frac{\cos(0)}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

So we have shown $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$. Note that in terms of showing this limit exists, this should be read from the bottom up. That is l'hôpital's rule shows that $\lim_{x\to 0} \frac{\sin(x)}{2x}$ exists as $\lim_{x\to 0} = \frac{\sin(x)'}{(2x)'} = \lim_{x\to 0} \frac{\cos(0)}{2} = \frac{1}{2}$ exists. Then anther application of l'hôpital's rule shows that $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{(1-\cos(x))'}{(x^2)'} = \lim_{x\to 0} \frac{\sin(x)}{2x}$ exists.

Problem 26. Here are some problems to practice the use of l'hôpital's rule. Compute the following

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$

(b) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x}}{x}$

(c) $\displaystyle\lim_{\theta\to \pi} \frac{\sin^3(x)}{x(\cos(x)+1)}$

$\square$

Now back to Rôlle's theorem. First a definition.

Definition 28. Let f be defined on an open integral I. Then f is ***twice differentiable*** on I if f' exsits at all points of I and the function f' is differentiable on I. We denote the derivative of f' as f'' or $f^{(2)}$ and it is called the ***second derivative*** of f. If f'' exists at all points of I and f'' is differentiable on I its derivative is denoted by f''' or $f^{(3)}$ and is called the ***third derivative*** of f and f is said to be ***three times differentiable***. Continuing recursively, if we have defined what it means for f to be n ***times differentiable*** on I and the n***-th derivative***, $f^{(n)}$, is differentiable on I then the derivative of $f^{(n)}$ is denoted by $f^{(n+1)}$ and f is $(n+1)$ ***times differentiable*** on I. $\square$

Remark 29. For consistency's sake we set $f^{(0)} = f$ and $f^{(1)} = f'$ $\square$

Problem 27. Show that the function f on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2, & x \geq 0; \\ -x^2, & x < 0. \end{cases}$$

is differentiable on $\mathbb{R}$ but not twice differentiable. *Hint:* Show $f'(x) = 2|x|$. You may have to use the limit definition to compute $f'(0)$. $\square$

Problem 28. Find a function that is twice differentiable on $\mathbb{R}$ but not three times differentiable. More generally can you give an example of a function that is n times differentiable, but not $n+1$ times differentiable. $\square$

Proposition 30. *Let I be an open interval and assume f is twice differentiable on I. Let $x_0, x_1 \in I$ with $x_0 \neq x_1$. Assume $f(x_0) = f'(x_0) = 0$ and $f(x_1) = 0$. Then there is a point ξ between x_0 and x_1 with $f''(\xi) = 0$.*

Proof. As $f(x_0) = f(x_1) = 0$ by Rôlle's Theorem there is a ξ_1 between x_0 and x_1 with $f'(\xi_1) = 0$. But the function f' is differentiable on I and

$f'(x_0) = f'(\xi_1) = 0$ and thus anther application of Rôlle's Theorem gives us a ξ between x_0 and ξ_1 with $f''(\xi) = (f')'(\xi) = 0$. As ξ_1 is between x_0 and x_1 and ξ is between x_0 and ξ_1 we have that ξ is between x_0 and x_1. □

This generalizes

Theorem 31. *Let f be $n+1$ times differentiable on the open interval I. Let $x_0, x_1 \in I$ with $x_0 \neq x_1$. Assume that*

- $f(x_0) = f'(x_0) = \cdots = f^{(n)}(x_0) = 0$,
- $f(x_1) = 0$.

Then there is a point ξ between x_0 and x_1 with

$$f^{(n+1)}(\xi) = 0.$$

□

Problem 29. Prove this. *Hint:* There are several ways to do this. One is to look at the proof of Proposition 30 and meditate upon induction. □

Proposition 32. *Let f be twice differentiable on the open interval I and let $a, b \in I$ with $a \neq b$. Then there is a ξ between a and b with*

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(\xi)}{2}(b-a)^2.$$

Proof. Let h be defined on I by

$$h(x) = f(x) - f(a) - f'(a)(x-a) - c(x-a)^2$$

where c is a constant to be chosen shortly. Note

$$h(a) = 0$$

and

$$h'(x) = f'(x) - f'(a) - 2c(x-a),$$

and thus

$$h'(a) = 0.$$

With applying Proposition 30 in mind, we choose c so that $h(b) = 0$. That is

$$h(b) = f(b) - f(a) - f'(a)(b-a) - c(b-a)^2 = 0$$

which leads to

$$c = \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}.$$

With this choice of c we have $h(a) = h'(a) = h(b) = 0$ and thus by Proposition 32 there is a ξ between a and b with

$$h''(\xi) = 0.$$

By direct calculation

$$h''(x) = f''(x) - 2c.$$

Then $h''(\xi) = 0$ yields

$$f''(\xi) - 2c = 0.$$

But using the formula for c above we find

$$f''(\xi) - 2\left(\frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}\right) = 0$$

which can be rearranged to give

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(\xi)}{2}(b-a)^2$$

as required. □

As this was a more or less direct consequence of Proposition 30 it makes sense to look for a generalization that depends on Theorem 31. To make life a little easier on ourselves we first do the case of $n = 4$.

Lemma 33. *Let f be a function that is four times differentiable on an open interval I and let $a \in I$. Let $T(x)$ be the polynomial*
(2)

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4,$$

and set

$$g(x) = f(x) - T(x).$$

Then

$$g(a) = g'(a) = g''(a) = g^{(3)}(a) = g^{(4)}(a) = 0.$$

Problem 30. Prove this. □

Theorem 34. *Let f be five times differentiable on the open interval I and $a, b \in I$ with $a \neq b$. Then there is a ξ between a and b such that*

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \frac{f^{(4)}(a)}{4!}(b-a)^4 + \frac{f^{(5)}(\xi)}{5!}(b-a)^5.$$

Or in different notation let $T(x)$ be the polynomial (2)*, then this is*

$$f(b) = T(b) + \frac{f^{(5)}(\xi)}{5!}(b-a)^5.$$

Problem 31. Prove this. *Hint:* Let

$$h(x) = f(x) - T(x) - c(x-a)^5$$

where we choose c so that

$$h(b) = 0.$$

Show $h(a) = h'(a) = h''(a) = h^{(3)}(a) = h^{(4)}(a) = 0$. Now use Theorem 31 and now proceed as in the proof of Proposition 32. □

Definition 35. Let f be n times differentiable on a neighborhood of a. Then the ***degree*** n ***Taylor polynomial*** of f at x is

$$T_n(x) := \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}. \qquad \square$$

Problem 32. Show that if f is n times differentiable on an open interval I and T_n is its degree n Taylor polynomial at a, then for $0 \le k \le n$

$$T_n^{(k)}(a) = f^{(k)}(a).$$

That is the k-th derivatives of T_n and f agree at a for $0 \le k \le n$. $\square$

Theorem 36 (Taylor's Theorem with Lagrange's form of the remainder)**.** *Let f be $(n+1)$ times differentiable on the open interval I and let $a, b \in I$ with $a \neq b$. Let T_n be the degree n Taylor polynomial of f at a. Then there is a ξ between a and b such that*

$$f(b) = T_n(b) + f^{(n+1)}(\xi)\frac{(b-a)^{n+1}}{(n+1)!}.$$

*(The term $E_n(b) = f^{(n+1)}(\xi)\frac{(b-a)^{n+1}}{(n+1)!} = f(b) - T_n(b)$ is the **error term** or **remainder term** when approximating f by its Taylor polynomial T_n.)*

Problem 33. Prove this. $\square$

We restate this with slightly different notation (just replacing a and b with x_0 and x.)

Theorem 37 (Taylor's Theorem with Lagrange's form of the remainder, form 2)**.** *Let f be $(n+1)$ times differentiable on the open interval I and let $x, x_0 \in I$ with $x \neq x_0$. There there is a ξ betweem x and x_0 such that*

$$f(x) = \sum_{k=0}^{n} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!} + f^{(n+1)}(\xi)\frac{(x-x_0)^{n+1}}{(n+1)!}. \qquad \square$$

Remark 38. In the case that $n = 0$ this becomes

$$f(x) = f(x_0) + f'(\xi)(x - x_0),$$

which can be rewritten as $f(x) - f(x_0) = f'(\xi)(x - x_0)$. That is for $n = 0$ we just get the mean value theorem. $\square$

One last restatement of Taylor's theorem. If we let $x = x_0 + h$ we get

$$f(x_0 + h) = \sum_{k=0}^{n} f^{(k)}(x_0)\frac{h^k}{k!} + f^{(n+1)}(\xi)\frac{h^{n+1}}{(n+1)!}$$

where ξ is between x_0 and $x_0 + h$.

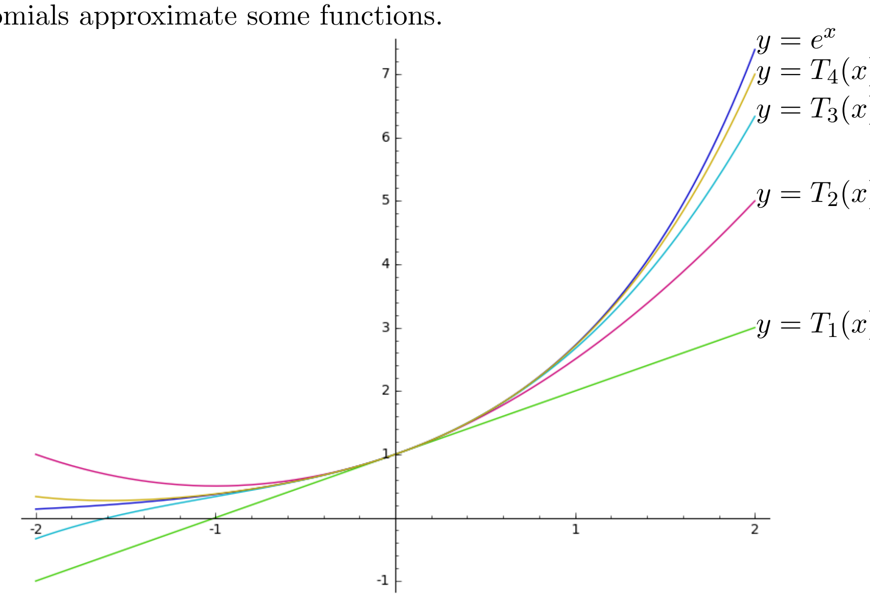
As an examples of Taylor's theorem we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{e^\xi x^4}{4!} \qquad \text{(Used } n = 3.)$$

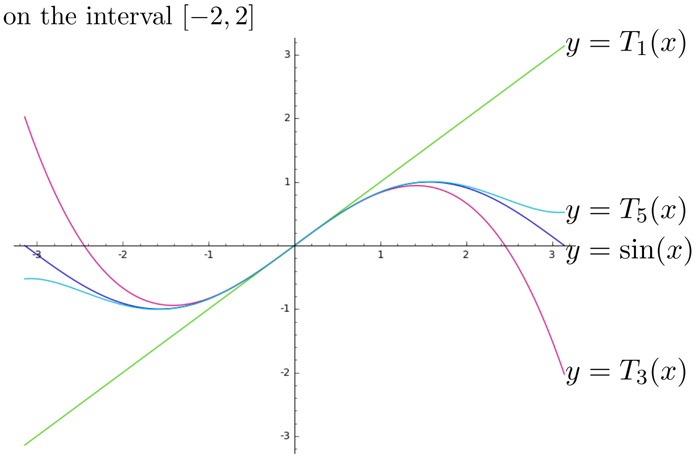
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{\cos(\xi)x^6}{6!} \qquad \text{(Used } n = 5.)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{\cos(\xi)x^7}{7!} \qquad \text{(Used } n = 6.)$$

where ξ is between x and 0 (and of course the value of ξ is different in each of the three equations). Here are some graphs that show how closely Taylor polynomials approximate some functions.



Graphs of $y = e^x$ and the Taylor polynomials $y = T_n(x)$ for $k = 1, 2, 3, 4$ on the interval $[-2, 2]$



Graphs of $y = \sin(x)$ and the Taylor polynomials $y = T_1(x)$, $y = T_2(x) = T_3(x)$, and $y = T_4(x) = T_5(x)$. on the interval $[-\pi, \pi]$. (If $f(x) = \sin(x)$, then for odd positive integers n we have $f^n(0) = 0$. Therefore $T_{2k}(x) = T_{2k+1}(x)$.)