

Math 552, February 26, 2020.

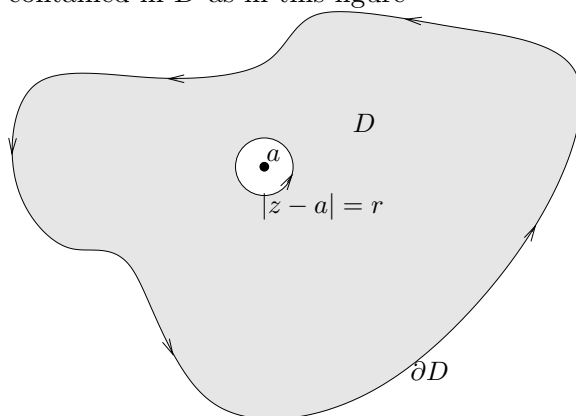
The following problems are to prepare for results that we are about to prove. We have just proven

Theorem 1 (Cauchy's Theorem). *Let D be a bounded domain with nice boundary and $f(z)$ a function that is analytic on the closure of D . Then*

$$\int_{\partial D} f(z) dz = 0$$

where, as usual, we orient ∂D so that we move with the inside on our left. \square

Problem 1. Let $f(z)$ be analytic in a domain D and let $a \in K$. Let $r > 0$ be so small that the disk $|z - a| \leq r$ is contained in D as in this figure



(a) Use the Cauchy Integral Theorem to show

$$\int_{\partial D} \frac{f(z)}{z - a} dz = \int_{|z-a|=r} \frac{f(z)}{z - a} dz.$$

Be sure to say why Cauchy Integral Formula applies.

(b) Use Part (a) and the parameterization of $|z - a| = r$ given by $z = a + re^{it}$ with $0 \leq t \leq 2\pi$ to show

$$\int_{\partial D} \frac{f(z)}{z - a} dz = \int_{|z-a|=r} \frac{f(z)}{z - a} dz = i \int_0^{2\pi} f(a + re^{it}) dt.$$

Problem 2. With the same set up as in Problem 1 explain why

$$\lim_{r \rightarrow 0^+} \int_0^{2\pi} f(a + re^{it}) dt = 2\pi f(a)$$

and use this to show

$$\int_{\partial D} \frac{f(z)}{z - a} dz = 2\pi f(a).$$

\square

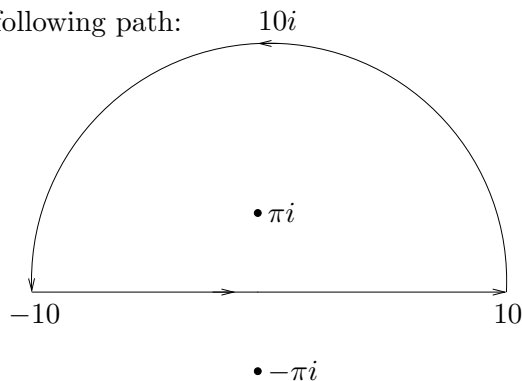
You have just proven what may be the most important result in Complex Analysis:

Theorem 2 (Cauchy Integral Formula). *Let D be a bounded domain with nice boundary and $f(z)$ be analytic on the closure of D . Then for any point $a \in D$*

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z) dz}{z - a}.$$

\square

Example. Consider the following path:



We now use the Cauchy Integral formula to evaluate

$$\int_{\gamma} \frac{e^z}{z^2 + \pi^2} dz.$$

This function is analytic except where the denominator becomes zero. That is where $z^2 + \pi^2 = 0$. Note that $z^2 + \pi^2 = (z - \pi i)(z + \pi i)$. So that the bad points are $z = \pi i$ and $z = -\pi i$. Thus our integral becomes

$$\int_{\gamma} \frac{e^z}{(z - \pi i)(z + \pi i)} dz.$$

We only need to work about the point πi as it is the only non-analytic point inside of γ . Rewrite the integral as

$$\int_{\gamma} \frac{e^z/(z + \pi i)}{(z - \pi i)} dz = \int_{\gamma} \frac{f(z)}{(z - \pi i)} dz$$

where

$$f(z) = \frac{e^z}{z + \pi i}.$$

The function $f(z)$ is analytic inside of γ . So by the Cauchy integral formula

$$\int_{\gamma} \frac{e^z}{z^2 + \pi^2} dz = \int_{\gamma} \frac{f(z)}{(z - \pi i)} dz = 2\pi i f(\pi i) = 2\pi i \frac{e^{\pi i}}{\pi i + \pi i} = e^{\pi i} = -1.$$

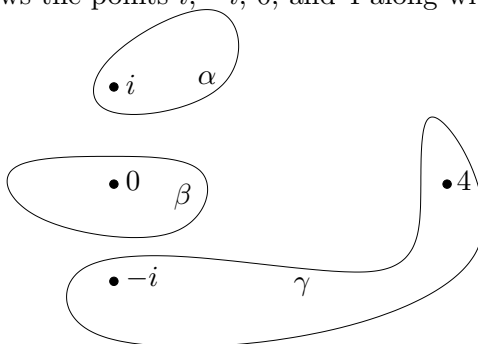
□

Problem 3. Let z_1 be a complex number and γ a simple closed curve that does not pass through z_1 . Show

$$\int_{\gamma} \frac{dz}{z - z_1} = \begin{cases} 2\pi i, & \text{if } z_1 \text{ is inside of } \gamma, \\ 0, & \text{if } z_1 \text{ is outside of } \gamma. \end{cases}$$

Hint: Use part (d) of Problem 2, or the Cauchy Integral Formula, with $f(z) = 1$, D the region inside of γ , and $z = z_1$.

Problem 4. Figure 3 shows the points i , $-i$, 0 , and 4 along with three paths α , β , and γ .



Use the Cauchy integral formula to

(a) Evaluate $\int_{\alpha} \frac{2z+1}{z(z-4)(z^2+1)} dz$,

(b) Evaluate $\int_{\beta} \frac{2z+1}{z(z-4)(z^2+1)} dz$,

(c) Evaluate $\int_{\gamma} \frac{2z+1}{z(z-4)(z^2+1)} dz$.