Analysis Qualifying Exam
August 2006

Instructions: Write your name legibly on each sheet of paper. Write only on one side of of each sheet of paper. Try to answer all questions. Questions 1–8 are each worth 10 points and question 9 is worth 20 points.
Terminology: Measurability and integrability on $\mathbb{R}$ or a measurable subset of it will always refer to the Lebesgue measure, except if otherwise specified. Lebesgue measure will be denoted by m, dx or dy depending on the context.

1. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of functions such that for all $x \in [0,1]$ there exists an open interval I_x such that $x \in I_x$ and $\{f_n\}$ converges uniformly on $I_x \cap [0,1]$. Prove that $\{f_n\}$ converges uniformly on $[0,1]$.

2. Let f_n and f be non-negative measurable functions on $\mathbb{R}$ such that $0 \le f_n \le f$ and $f_n(x) \to f(x)$ a.e.

   a. Prove that
   $$\lim_{n\to\infty} \int f_n \, dx = \int f \, dx.$$
   (Note: the sequence $\{f_n(x)\}$ does not have to converge monotonically to $f(x)$.)

   b. Does it follow that $\lim_{n\to\infty} \int |f - f_n| \, dx = 0$?

3. Let $A \subset \mathbb{R}$ be a measurable set with $m(A) = 0$. Prove that $m(A^2) = 0$, where $A^2 = \{x : x = y^2 \text{ for some } y \in A\}$.

4. Let f and g be absolutely continuous functions on $[a,b]$

   a. Prove that the product fg is absolutely continuous on $[a,b]$.

   b. Prove that
   $$\int_a^b f' g \, dx = f(b)g(b) - f(a)g(a) - \int_a^b f g' \, dx.$$

5. Let $1 \le p < \infty$ and $f_n \in L^p[1,\infty)$ such that f_n converges in norm to f in $L^p[1,\infty)$. Prove that
$$\lim_{n\to\infty} \int_1^\infty \frac{f_n(x)}{x} \, dx = \int_1^\infty \frac{f(x)}{x} \, dx.$$

6. Let f be a holomorphic function on $|z| < 1$.

a. Let $g(z) = \overline{f(\overline{z})}$ on $|z| < 1$. Prove g is holomorphic on $|z| < 1$.

b. Assume $f(\frac{1}{n}) \in \mathbb{R}$ for $n \geq 2$. Prove $f(x) \in \mathbb{R}$ for all $-1 < x < 1$.

7. Compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}}\, dx$$

by using a contour integral over the rectangle bounded by $x = R$, $x = -R$, $y = 0$, and $y = \pi$.

8. Let f be a holomorphic function on $|z| < R$ and assume that for all $|a| < R$ the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ of f around $z = a$ has at least one coefficient $c_n = 0$ (where n depends on a). Prove that f is a polynomial.

9. True or False. Prove, or give a counterexample.

a. If f is of bounded variation on $[0, 1]$, then $\lim_{x \to a+} f(x)$ exists for all $0 \leq a < 1$.

b. If f_n and f are integrable over $[0, 1]$ and $\int_0^1 |f - f_n|\, dx \to 0$ as $n \to \infty$, then $f_n(x) \to f(x)$ a.e.

c. If f is uniformly continuous and of bounded variation on $[a, b]$, then f is absolutely continuous.

d. There exists a holomorphic function f on $|z| < r$ for some $r > 0$ such that $f^{(n)}(0) = 2^n n!$ for all $n \geq 0$.

e. If $z = 0$ is an essential singularity for g, where $g \in H(D(0;1)\backslash\{0\})$, then $z = 0$ is an essential singularity for e^g.