Mathematics 739 Homework 2: Some examples of vector bundles.

1. Divisors and line bundles.

In this section M will be a compact complex manifold. By a ***hypersurface*** in M we mean a closed subset of M such that for each $p \in V$ there is $f \in \mathcal{O}_p$ such that f does not vanish identically and near p the set V is given by $\{f = 0\}$. To be more precise, since f is the germ of a function, this can be restated by saying that for all $p \in V$ there is a connected open set U with $p \in U$ and a holomorphic function, not identically zero, $f \colon U \to \mathbb{C}$ such that $V \cap U = \{x \in U : f(x) = 0\}$. The hypersurface is ***irreducible*** if and only if it is not the union of two distinct hypersurfaces. The ***divisor group*** of M is the set of formal sums

$$\sum_V a_V V$$

where the sum is over all irreducible hypersurfaces of M, the a_V'x are integers and all but finitely many of the a_V'x are zero. If V is an irreducible and $f \colon M \to \mathbb{C}$ is a meromorphic function then the ***order*** of f along V is well defined.

If V is an irreducible hypersurface and $f \colon M \to \mathbb{C}$ is meromorphic, then we can define the ***order***, $\operatorname{ord}_V(f)$ along V. To be just a little bit more explicit, if $p \in V$, then near p we can write

$$f = \frac{g}{h}$$

where $g, h \in \mathcal{O}_p$ and g and h are relatively prime in $\mathcal{O}_p$. (Recall that $\mathcal{O}_p$ is a UFD.) Then let $\operatorname{ord}_V(g)$ is the order that g vanishes along V and $\operatorname{ord}_V(h)$ the order that h vanishes along V. Set

$$\operatorname{ord}_V(f) = \operatorname{ord}_V(g) - \operatorname{ord}_V(h).$$

Problem 1. Review the definitions involved here and convince yourself this is all well defined. $\square$

Now given a meromorphic function f on M we can define a divisor

$$(f) = \sum_V \operatorname{ord}_V(f) V.$$

Proposition 1. *If f, g are meromorphic on M show that*

$$(fg) = (f) + (g)$$

and therefore $\{(f) : f$ is meromorphic on M is a subgroup of $\operatorname{Div}(M)$. $\square$

Problem 2. Prove this. $\square$

Given a divisor D on M near each point $p \in M$ there is an open neighborhood, U, of p and a meromorphic function f on U such that

$$D \cap U = (f) \cap U.$$

(If $p \notin D$, then choose U with $D \cap U = \varnothing$ and f to be nonvanishing on U.) If f_1 and f_2 are both meromorphic on U and define $D \cap U$, then f_1/f_2 is holomorphic and nonvanishing on U. Therefore given a divisor D we can cover M with open set $\{U_\alpha\}_{\alpha \in A}$ such that on each U_α there is a meromorphic function f_α that defines $D \cap U_\alpha$. On each overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$ define nonvanishing holomorphic functions $g_{\alpha\beta}$ by

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}.$$

Proposition 2. *This data, that is the cover $\{U_\alpha\}$, the meromorphic functions $\{f_\alpha\}$, and the functions $\{g_{\alpha\beta}\}$ define a holomorphic line bundle, L_D, over M and that this bundle has meromorphic section which has D as its divisor.*

Problem 3. Give a precise statement of the last proposition (this includes defining what it means for a section of a holomorphic vector bundle to be meromorphic) and prove it. □

Proposition 3. *If $D_1, D_2 \in \operatorname{Div}(M)$, then $L_{D_1+D_2} = L_{D_1} \otimes L_{D_2}$.*

Problem 4. Prove this. □

Proposition 4. *Let $D \in \operatorname{Div}(M)$. Then L_D is the trivial bundle (that is the product bundle $M \times \mathbb{C}$) if and only if $D = (f)$ for some meromorphic function f on M.*

Problem 5. Prove this. □

Let $\operatorname{Pic}(M)$ be the set of isomorphism classes of holomorphic line bundles over M. Make this into a group using tensor product for the group operation. This is the ***Picard group*** of M.

Proposition 5. *There is a group isomorphism*

$$\operatorname{Pic}(M) \approx \operatorname{Div}(M)/\{(f) : f \textit{ is holomorphic on } M.\}$$

Problem 6. Prove this. □

2. Using line bundles to embed complex manifolds into projective spaces.

Let M be a compact complex manifold. Let $pE \to M$ be a holomorphic vector bundle over M. Let $\Gamma(M, E)$ be the vector space of all holomorphic sections of M. For many vector bundles this will just be the trivial vector space $\{o\}$. The following is known and follows from some facts about partial differential equations.

Proposition 6. *If $p\colon E \to M$ is a holomorphic vector bundle over a compact manifold, then the space of holomorphic sections $\Gamma(M, E)$ is finite dimensional.* □

Proposition 7. *Let $p\colon L \to M$ be a holomorphic line bundle over a compact complex manifold M. Assume that $\dim_{\mathbb{C}}(\Gamma(M, L)) > 1$ and that for each $p \in M$ there is a $s \in \Gamma(M, L)$ with $s(p) \neq 0$. Let $\mathbb{P}^*(\Gamma(M, L))$ be the projective space of all codimension one linear subspaces of $\Gamma(M, L)$. Define a map $\phi\colon M \to \mathbb{P}^*(\Gamma(M, L))$ by*

$$\phi(p) = \{s \in \mathbb{P}^*(\Gamma(M, L)) : s(p) = 0\}.$$

Then ϕ is holomorphic.

Problem 7. Prove this. □