More problems on L^p spaces.

Problem 1. This problem is review, but we will be using it below. Let (X, μ) be a measure space and $f \colon X \to \mathbb{R}$ a measurable function. Show there is a sequence of non-negative simple functions $\langle \phi_n \rangle_{n=1}^{\infty}$ that increase pointwise to f. That is $0 \leq \phi_1 \leq \phi_2 \leq \phi_3 \leq \cdots$ and $\lim_{n\to\infty} \phi_n(x) = f(x$ for all x. *Hint:* For each pair of positive integers n, k let

$$E_{n,k} = \left\{ x \in X : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \right\}.$$

and set

$$\phi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{E_{n,k}}.$$

Verify this does the trick. $\square$

Recall the function $\operatorname{sgn} \colon \mathbb{R} \to \{-1, 0, 1\}$ is

$$\operatorname{sgn}(x) = \begin{cases} +1, & x > 0 \\ 0, & x = 0; \\ -1, & x < 0. \end{cases}$$

Problem 2. Let $X, \mu)$ be a measure space and $f \colon X \to \mathbb{R}$ a measurable function. Show there is a sequence of simple functions $\langle \psi_n \rangle_{n=1}^{\infty}$ such that

- $|\psi_n| \leq |\psi_{n+1}|$ for all n,
- $\psi_n(x) f(x) \geq 0$ for all $x \in X$,
- $|\psi_n| \leq |f|$, and
- $\lim_{n\to\infty} \psi_n(x) = f(x)$ for all x.

Hint: Using the function $|f|$ in place of f in the previous problem find a sequence of simple functions $0 \leq \phi_n \nearrow |f|$ and let $\psi_n = \operatorname{sgn}(f)\phi_n$. $\square$

Problem 3 (August 1984). Let (X, μ) be a measure space and let $1 < p < \infty$ and $1/p + 1/q = 1$. Let $g \in L^1(X)$ such that there is a constant M with

$$\left| \int_0^1 g s \, d\mu \right| \leq M \|s\|_{L^p}$$

for all simple functions s. Prove $g \in L^q(X)$ and $\|g\|_{L^q} \leq M$. *Hint:* Let $\psi_n \to g$ as in the last problem. Show

$$\begin{aligned} 0 \leq & |\psi_n|^q \leq |g|^q \\ |\psi_n|^q \leq & g \operatorname{sgn}(\psi_n) |\psi_n|^{q-1}. \end{aligned}$$

The function $\operatorname{sgn}(\psi_n)|\psi_n|^{q-1}$ is a simple function and thus

$$\begin{aligned}\|\psi_n\|_{L^q}^q &= \int_X |\psi_n|^q \, d\mu \\ &\le \int_X g \operatorname{sgn}(\psi_n)|\psi_n|^{q-1} \, d\mu \\ &\le M \big\| \operatorname{sgn}(\psi_n)|\psi_n|^{q-1} \big\|_{L^p}.\end{aligned}$$

Show this implies

$$\|\psi_n\|_{L^q} \le M$$

and then use the monotone convergence theorem to show

$$\|g\|_{L^q} = \lim_{n\to\infty} \|\psi_n\|_{L^q} \le M$$

to complete the proof. □

Problem 4 (August 2005). Let (X, μ) be a measure space and let $f \in L^1(X)$ with $f(x)$ for μ almost all x. Prove for all $\varepsilon > 0$ that $\inf\{\int_\Omega f \, d\mu : \mu(\Omega) \ge \varepsilon\} > 0$. *Hint:* Let S_n be the set

$$S_n := \{x \in X : f(x) < 1/n\}.$$

Then $S_{n+1} \subseteq S_n$ and $f > 0$ almost everywhere implies $\bigcap_{n=1}^\infty S_n$ has measure zero. This implies $\lim_{n\to\infty} \mu(S_n) = 0$. Therefore there exists an n_0 with

$$\mu(S_{n_0}) < \frac{\varepsilon}{2}.$$

Let $\mu(\Omega) \ge \varepsilon$. We wish to give a lower bound on $\int_\Omega f \, d\mu$. First show

$$\begin{aligned}\int_\Omega f \, d\mu \quad & \int_{S_{n_0}\cap\Omega} f \, d\mu + \int_{\Omega\setminus S_{n_0}} f \, d\mu \\ &\ge \int_{\Omega\setminus S_{n_0}} f \, d\mu \\ &\ge \frac{\mu(\Omega\setminus S_{n_0})}{n_0}.\end{aligned}$$

Then show

$$\mu(\Omega \setminus S_{n_0}) > \frac{\varepsilon}{2}.$$

Thus

$$\inf\left\{ \int_\Omega f \, d\mu : \mu(\Omega) \ge \varepsilon \right\} \ge \frac{\varepsilon}{2n_0} > 0.$$ □

Proposition 1. *Let (X, μ) be a measure space and let $f \in L^2(X)i$ with $f \ge 0$. Then*

$$\int_X f^2 \, d\mu = 2\int_0^\infty t\mu(\{x : f(x) > t\}) \, dt$$

Problem 5. Prove this by verifying the interchange of the order of integration in the following calculation:

$$\begin{aligned}\int_X f(x)^2\,\mu(x) &= 2\int_X \int_0^{f(x)} t\,dt\,d\mu(x)\\ &= 2\int_0^\infty \int_{\{x: f(x)>t\}} t\,d\mu(x)\,dt\\ &= 2\int_0^\infty t\int_{\{x: f(x)>t\}} d\mu(x)\,dt\\ &= 2\int_0^\infty \mu(\{x : f(x) > t\})\,dt.\end{aligned}$$

□

Problem 6. Let (X, μ) be a measure space and let $f \in L^p(X)$ with $1 \le p < \infty$ and $f \ge 0$. Prove

$$\int_X f^p\,d\mu = p\int_0^\infty t^{p-1}\mu(\{x : f(x) > t\})\,dt$$

□

Problem 7. Let (X, μ) be a measure space and f a measurable function $f\colon X \to \mathbb{R}$ such that e^f is integrable. Show

$$\int_X e^f\,d\mu = \int_{-\infty}^\infty e^t\mu(\{x : f(x) > t\})\,dt.$$

□