Analysis Qualifying Exam
August 2012

Instructions: Write your name legibly on each sheet of paper. Write only on one side of of each sheet of paper. Try to answer all questions. Questions 1–8 are each worth 10 points and question 9 is worth 20 points.
Terminology: Measurability and integrability on $\mathbb{R}$ or a measurable subset of it will always refer to the Lebesgue measure, except if otherwise specified. Lebesgue measure will be denoted by m, dx or dy depending on the context. If A is a subset of $\mathbf{R}$ then $L^p(A)$ is considered with respect to the Lebesgue measure.

1. Let (X, ρ) and (Y, σ) be metric spaces and let $f : X \to Y$ be a continuous mapping. Assume that for all $\epsilon > 0$ there exists a compact subset K_ϵ of X such that $\sigma(f(x), f(y)) < \epsilon$ for all $x, y \in K_\epsilon^c$. Prove that f is uniformly continuous.

2. Let $E_n \subset \mathbb{R}$ be measurable sets and $f : \mathbb{R} \to \mathbb{R}$ be integrable on $\mathbb{R}$ such that $\|\chi_{E_n} - f\|_1 \to 0$. Prove that there exists a measurable set E such that $f = \chi_E$ a.e.

3. Let f be integrable on $\mathbb{R}$ and $\alpha > 0$. Let $f_n(x) = \frac{f(nx)}{n^\alpha}$. Prove that the series $\sum_{n=1}^\infty f_n(x)$ converges a.e.

4. Let $f_n : [0, 1] \to \mathbb{R}$ be absolutely continuous functions with $f_n(0) = 0$. Assume
$$\sum_{n=1}^\infty \int_0^1 |f_n'(x)|\, dx < \infty.$$
   (a) Prove that $\sum_{n=1}^\infty f_n(x)$ converges for all $x \in [0, 1]$.
   (b) Let $f(x) = \sum_{n=1}^\infty f_n(x)$. Prove that f is absolutely continuous on $[0, 1]$.
   (c) Prove
$$f'(x) = \sum_{n=1}^\infty f_n'(x)$$
   a.e.

5. Let $A \subset E \subset B \subset \mathbb{R}$. Assume A, B are measurable and that $m(A) = m(B) < \infty$. Prove that E is measurable. Does the conclusion still hold, if we drop the condition that $m(A) < \infty$?

6. Let $f \in L^2(\mathbb{R})$.

   a. Prove $\|f\chi_{[-1,1]}\|_1 \leq \sqrt{2}\|f\chi_{[-1,1]}\|_2$.

   b. Assume in addition that g defined by $g(x) = xf(x)$ is in $L^2(\mathbb{R})$. Prove that $f \in L^1(\mathbb{R})$ and

$$\|f\|_1 \leq \sqrt{2}(\|f\|_2 + \|g\|_2).$$

7. Let f be holomorphic on $\mathbb{C}\setminus\{p_n\}$ where $\lim |p_n| = \infty$. Assume $|f(z)| \geq C > 0$ on $\mathbb{C} \setminus \{p_n\}$. Prove that f is constant.

8. The function $f(z) = \frac{1}{\cos(\pi z)}$ has a power series expansion

$$\sum_{n=0}^{\infty} a_n(z-i)^n.$$

   Find $\limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

9. True or False. Prove, or give a counterexample.

   a. Let f be 2π-periodic function which is integrable over $[-\pi, \pi]$. Then $\int_I f(x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx$ for any interval I of length 2π.

   b. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + 1$ be a polynomial. Then $\max_{|z|=1} |P(z)| \geq 1$.

   c. There exist a function f, holomorphic on the unit disk, such that $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3}$.

   d. If $f_n : \mathbb{R} \to \mathbb{R}$ are measurable functions such that $\int_{\mathbb{R}} f_n^2\,dx \to 0$, then $\int_{\mathbb{R}} |f_n|\,dx \to 0$.

   e. Let $f_n \geq 0$ and f be measurable functions on $\mathbb{R}$ such that $|f_n(x) - f(x)| < \frac{1}{n}$ for all $x \in \mathbb{R}$ and all $n \geq 1$. If all f_n are integrable, then $\int_{\mathbb{R}} f_n(x)\,dx \to \int_{\mathbb{R}} f(x)\,dx$.