Analysis Admission to Candidacy Exam

January 11, 1984

The term "measurable" applied to sets or functions means "Lebesgue measurable". If E is a measurable set, then $m(E)$ denotes the Lebesgue measure of E. The term "increasing" applied to functions means that $x \leq y$ implies $f(x) \leq f(y)$.

1. Let X be a compact metric space and let $T: X \to X$ be a continuous mapping which satisfies $d(Tx,Ty) < d(x,y)$ when $x \neq y$. Prove that there exists $x_0 \in X$ with $Tx_0 = x_0$.

2. Suppose $f: \mathbb{R} \to \mathbb{R}$ is uniformly continuous and Lebesgue integrable over $\mathbb{R}$. Prove that $f(x) \to 0$ as $x \to \infty$.

3. Prove that Q, the set of all rational numbers, is not a G_δ-set.

4. Suppose that f is an increasing function on $[a,b]$. Prove that there exist unique functions g and h on $[a,b]$ such that

   (a) g is absolutely continuous and $g(a) = f(a)$,

   (b) h is singular, and

   (c) $f = g + h$ on $[a,b]$.

5. Suppose E is a measurable set, $m(E) < \infty$, $\langle f_n \rangle$ is a sequence of measurable functions on E and $f_n \to f$ a.e. on E. Prove that for each $\varepsilon > 0$

$$m(\{x \in E: |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \quad \text{as} \quad n \to \infty.$$

   What happens if $m(E) = \infty$?

6. Suppose g is a measurable function on $(0,\infty)$, $\int_0^\infty |g(t)|dt < \infty$ and $\int_0^\infty t|g(t)|dt < \infty$

   (a) Show that $f(x) = \int_0^\infty g(t)\sin(xt)dt$ defines a bounded function f on $(0,\infty)$.

   (b) Prove that f is differentiable and $f'(x) = \int_0^\infty tg(t)\cos(xt)dt$.

   (Hint: $|\sin b - \sin a| \leq |b - a|$.)

7. Suppose $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. let $\langle g_n \rangle$ be a sequence in $L^q([0,1])$ such that

(a) $M = \sup_n ||g_n||_q < \infty$ and

(b) $\int_E g_n \to 0$ as $n \to \infty$ for each measurable set $E \subseteq [0,1]$.

Prove that for each $f \in L^p([0,1])$ $\int_0^1 fg_n \to 0$.

8. Let E be a measurable set of finite measure.

(a) Prove that for each $\varepsilon > 0$ there exists a finite disjoint union of open intervals U such that $m(E \Delta U) < \varepsilon$. (Recall that $E \Delta U = (E-U) \cup (U-E)$.)

(b) Let ϕ be a measurable simple function on $[a,b]$ and $\varepsilon > 0$. Prove there exists a step function f on $[a,b]$ with $m(\{x \in E: f(x) \neq \phi(x)\}) < \varepsilon$.

9. True or False. Either prove the statement or give a counter example.

(a) Every bounded measurable function on $[a,b]$ is Riemann integrable.

(b) If f is continuous and increasing on $[0,1]$, then f is absolutely continuous.

(c) If $f \in L^\infty([0,1])$, then $||f||_\infty = \lim_{p\to\infty} ||f||_p$.

(d) If f is a measurable function on $\mathbb{R}$ and $f > 0$ a.e. then there exist $\delta > 0$ and a measurable set E with $m(E) > 0$ and $f(x) \geq \delta$ for $x \in E$.