Some problems on Fubini's and Tonelli's Theorems, convolutions, and Fourier Transforms.

For the exact statements of the Fubini's and Tonelli's Theorems see the Wikipedia page

https://en.wikipedia.org/wiki/Fubini%27s_theorem.

The statements I gave in class were correct, but a bit informal. One of the applications of Fubini's theorem is to proving the basic properties of convolutions. If $f, g \in L^1(\mathbb{R})$, the ***convolution*** of f and g is

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, dy.$$

The change of variable $y \mapsto x - y$ shows this is also given by

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\, dy.$$

Note these definitions make sense if $\mathbb{R}$ is replaced by $\mathbb{R}^n$ and dy is n-dimensional measure on $\mathbb{R}^n$. If you are planning to do applied mathematics, or some sort of partial differential equations, you should probably do this problems working with $\mathbb{R}^n$, rather than $\mathbb{R}$.

The basic properties of the convolution are

Proposition 1. *If $f, g \in L^1(\mathbb{R})$ then $f * g \in L^1(\mathbb{R})$ and*

$$\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}.$$

Also

$$f * g = g * f$$
$$f * (g * h) = (f * g * h$$
$$\int_{-\infty}^{\infty} f * g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx \int_{-\infty}^{\infty} g(x)\, dx$$

Problem 1. Prove this. □

Note these properties show that $L^1(\mathbb{R})$ with the operations of addition (usual sum of functions) and using convolution as a product is a ring in the sense of abstract algebra. This is a natural example (at least for an analyst) of a ring that does not contain a multiplicative identity.

Proposition 2. *Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ where $1 \le p \le \infty$. Then $f * g \in L^p(\mathbb{R})$ and*

$$\|f * g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}.$$

Problem 2. Prove this. □

The convolution is closely related to the ***Fourier transform***. If $f \in L^1(\mathbb{R})$ then its transform is defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{ix\xi}\, dx.$$

In the case we are working over $\mathbb{R}^n$ this definition is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix\cdot\xi}\, dx$$

where $x \cdot \xi$ is the usual dot product of the vectors x and ξ.

Proposition 3. *For $f \in L^1(\mathbb{R})$ the Fourier transform $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ is a bounded (in fact $|\hat{f}(\xi)| \le \|f\|_{L^1}$) continuous function and $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.*

Problem 3. Prove this. □

Proposition 4. *If $f, g \in L^1(\mathbb{R})$, then*

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

Problem 4. Prove this. □

Remark 5. Let $C(\mathbb{R})$ be the continuous functions $f\colon \mathbb{R} \to \mathbb{C}$ which is a ring using pointwise operations. The previous proposition shows that the Fourier transform is a ring homomorphism from $L^1(\mathbb{R})$ to $C(\mathbb{R})$. □

Proposition 6 (August 2000, Problem 6)**.** *Suppose f is measurable on $[a, b] \times [c, d]$ such that*

$$\int_c^d f(x, y)\, dx\, dy > b - a.$$

Show there exists $x \in [a, b]$ such that

$$\int_a^b f(x, y)\, dy > 1.$$

□

Problem 5 (August 2002, Problem 6). Let $f\colon [0, 1] \to \mathbb{R}$ be Lebesgue measurable such that $F(x, y) := f(x) - f(x)$ is integrable over $[0, 1]^2$. Prove f is integrable over $[0, 1]$ and compute $\iint F(x, y)\, dx\, dy$.

Problem 6 (August 2003, Problem 4). Let f be a non-negative measurable function on a σ-finite measure space (X, Σ, μ). Prove $\{(x, y) \in X \times \mathbb{R} : 0 \le y \le f(x)\}$ is $\mu \times \lambda$ measurable (where λ is Lebesgue measure) and that

$$\int_X f d\mu = (\mu \times \lambda)\left(\{(x, y) \in X \times \mathbb{R} : 0 \le y \le f(x)\}\right)$$

□

Problem 7 (January 2014, Problem 5). Let E be a measurable subset of $[0, 1] \times [0, 1]$ and let λ be Lebesgue measure on $[0, 1]$. For each $x \in [0, 1]$ let

$$E_x = \{y : (x, y) \in E\}.$$

If

$$\lambda\left(\left\{x : \lambda(E_x) \ge \frac{1}{2}\right\}\right) \ge \frac{3}{4}$$

show

$$\lambda \times \lambda(E) \ge \frac{3}{8}.$$