Mathematics 546 Homework, September 30, 2020

Problem 1. For the two permutations σ and τ of problem 1 on page 88 of the text, write each of them as a product of transpositions in two different ways. Are they even or odd permutations? $\square$

Problem 2. (a) Show that any 3 cycle in S_n, and is an element of the form $\sigma = (abc)$, is even.
(b) Show that every 4 cycle is odd.
(c) Show that every 5 cycle is even.
(d) What can you say about the parity of a k cycle? $\square$

We have been looking at the dihedral group, D_n which is the group of symmetries of a regular n-gon. We have shown that D_n has elements a (a rotation of $360°/n$ about the center of the polygon) and b is a reflection in a line that goes through one of the vertices and the center of the polygon, then

$$a^n = 1, \qquad b^2 = 1, \qquad ba = a^{-1}b.$$

It is not hard to show that D_n has $2n$ elements and

$$D_n = \{1, a, a^2, \dots, a^{n-1}, b, ab, a^2b, , \dots, a^{n-1}b\}.$$

For example

$$D_5 = \{1, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}.$$

Problem 3. In D_n
(a) Show show for any positive integer k that $ba^k = a^{-k}b$.
(b) Use part (a) to show that $ba^k = a^{-k}$ for all integers k, positive or negative.
(c) Show that all the elements a^kb have order 2. (An element, x, has order 2 if and only if $x^2 = 1$.) $\square$

Anther group we looked at was the ***quaternion group*** which is the group with 8 elements:

$$Q = \{1, -1, i, -i, j, -j, k, -k\}$$

and the multiplication table

	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

This can be summarized by the rules

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

which should be familiar from vector calculus.

Definition 1. Let a is an element of a group G, then the ***order*** of a is the smallest positive integer n such that $a^n = 1$. We use the notation $o(a)$ for the order of a. If there is no positive integer n with $a^n = 1$, then we say the order of a is infinite and write $o(a) = \infty$. □

Problem 4. In the quaternion group Q find the order of the following elements. -1, i, and $-j$. □

Problem 5. In the symmetric group S_n find the order of the follwing elements

(a) (12),
(b) (123),
(c) (1234),
(d) (123)(45),
(e) (123)(456).

Problem 6. In D_4 find the order of the elements a^2 and a^3. □

Problem 7. Show that in a finite group that every element has finite order. *Hint:* Let G be finite and $a \in G$. As G is finite the element $a, a^2, a^3, \cdots$ can not all be distinct. So that are positive integers k and m with $k < m$ and $a^k = a^m$. Show $a^n = 1$ where $n = m - k$. □

Problem 8. Let a, b elements of the group G and let $c = bab^{-1}$.

(a) Show that for any positive integers k that $c^k = ba^k b^{-1}$.
(b) Show that a and c have the same order. □