Mathematics 555 Test 1, Take Home Portion.

The problems are 10 points each. You can choose any three to use as the take home part of the test. If you turn in more than three, I will use the three best for the grade.

1. Let $f\colon (a,b) \to \mathbf{R}$ be twice differentiable with f' and f'' both continuous. Show for $x \in (a,b)$ that

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

$\square$

2. Let E be a compact metric space and $f, f_1, f_2, f_3, \ldots : E \to \mathbf{R}$ continuous functions. Assume for all $x \in E$ that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

and that

$$f_1(x) \geq f_2(x) \geq f_3(x) \geq f_4(x) \geq \cdots$$

(that is the sequence $\langle f_n(x)\rangle_{n=1}^{\infty}$ is monotone decreasing). Show $\lim_{n\to\infty} f_n = f$ uniformly. *Hint:* Let $\varepsilon > 0$ and let

$$U_n = \{x \in E : f_n(x) - f(x) < \varepsilon\}.$$

Quote a theorem from last semester that tells us that U_n is open. Then show

$$U_n \subseteq U_{n+1}$$

and that $\mathcal{U} = \{U_1, U_2, U_3, \ldots\}$ is an open cover of E. $\square$

3. Let $f\colon \mathbf{R} \to \mathbf{R}$ be differentiable at all points and let $a < b$. Assume $f'(a) > 0$ and $f'(b) < 0$. Prove there is c between a and b with $f'(c) = 0$. *Remark:* The derivative f' need not be continuous and therefore this does not follow from the intermediate value theorem. $\square$

4. Let E and E' be a metric space and $f\colon E \to E'$ a function. Let $\alpha > 0$. Then f satisfies a ***Hölder condition*** of order α if and only if there is a constant $C \geq 0$ such that

$$d(f(p), f(q)) \leq C d(p,q)^{\alpha}$$

for all $p, q \in E$.

(a) Show that if f satisfies a Hölder condition, then f is uniformly continuous.

(b) Let $f\colon \mathbf{R} \to \mathbf{R}$ satisfy a Hölder condition of order $\alpha > 1$. Show f is constant. (If you want to be a bit more definite it is ok to assume that $\alpha = 2$ in this problem.) $\square$

5. Let $f, f_1, f_2, f_3, \ldots E \to E'$ be maps between metric the metric spaces E and E'. Assume that $\lim_{n\to\infty} f_n = f$ uniformly and that each f_n is uniformly continuous. Show that f is also uniformly continuous. $\square$