ANALYSIS QUALIFYING EXAM JANUARY 1994.

Throughout this examination, unless otherwise specified, the terms measurable, a.e., refer to the Lebesgue measure m on the real line $\mathbb{R}$, and L^p of an interval to L^p of that interval with respect to Lebesgue measure on that interval. Integrals w.r.t. Lebesgue measure will be denoted by $\int f\,dx$. Problems one through eight are 10 points each. Problem 9 is 20 points.

1. Let f_n be measurable on $\mathbb{R}$. Let $E = \{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \text{ exists}\}$. Prove that E is measurable.

2. Let f be integrable over $\mathbb{R}$ and let $E_n = \{x : |f(x)| \geq n\}$. Prove that $n \cdot m(E_n) \to 0$ as $n \to \infty$.

3. Let f be a measurable function on $[0,1]$ such that $f(x) > 0$ a.e.. Let $E_n \subset [0,1]$ be measurable sets such that $\int_{E_n} f(x)\,dx \to 0$ as $n \to \infty$. Prove that $m(E_n) \to 0$ as $n \to \infty$.

4. Let f be an integrable function on $[a,b]$. Prove that for all $\epsilon > 0$ there exists a polynomial p such that $\int_a^b |f-p|\,dx < \epsilon$.

5. Let $1 < p < \infty$ and $\alpha = 1 - \frac{1}{p}$. Assume that f is absolutely continuous on $[a,b]$ and that $f' \in L^p([a,b])$. Prove that $f \in \mathrm{Lip}_\alpha$, i.e., that there exists a constant M such that $|f(x) - f(y)| \leq M|x-y|^\alpha$ for all $x, y \in [a,b]$.

6. Let $f(z)$ be analytic on a domain Ω, except for poles in Ω: Prove that the only singularities of

$$g(z) = \frac{f'(z)}{f(z) - A}$$

are simple poles at all the poles of f and all the points $z \in \Omega$ such that $f(z) = A$.

7. Compute

$$\oint_C \frac{z}{(z-1)(z-2)^2}\,dz,$$

where C is the circle $|z-2| = \frac{1}{2}$, traversed counterclockwise.

8. Let $\{f_n\}$ be a uniformly bounded sequence of analytic functions on Ω such that $f_n(z)$ converges pointwise for all $z \in \Omega$. Prove that $\{f_n\}$ converges uniformly on every compact subset of Ω. (Hint: Apply the Dominated Convergence Theorem to the Cauchy formula for $f_n - f_m$.)

9. <u>True or False. Prove, disprove or give a counterexample.</u>

T a. Let $f_n \in L_p([a,b])$ $(1 \le p < \infty)$ such that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. Then $f_n(x) \to 0$ a.e. on $[a,b]$.

T b. Let f be a continuous function on $[0,1]$ such that $f = 0$ a.e. Then $f(x) = 0$ for all x in $[0,1]$.

c. There exists a function $f(z)$ analytic in a neighborhood of 0 such that

$$f'\left(-\frac{1}{n}\right) = f'\left(\frac{1}{n}\right) = \frac{1}{n^3}.$$

True d. Let $f : [a,b] \to \mathbb{R}$ be a function such that for all $x \in [a,b]$ there exists a $\delta > 0$ such that f is bounded on $(x-\delta, x+\delta) \cap [a,b]$. Then f is bounded on $[a,b]$.