Mathematics 554H/701I Homework

The goal of this homework set is to review some algebra that will be useful during the course of this term.

Contents

1. Summation notation.

Summation notation will be used a great deal in this class. We recall the basics about it. The notation is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

Thus

$$\sum_{k=0}^{5} ar^k = a + ar + ar^2 + ar^3 + ar^4 + ar^5.$$

There is nothing special about using k for the index:

$$\sum_{k=1}^{100} a_k = \sum_{j=1}^{100} a_j = \sum_{\alpha=1}^{100} a_\alpha = \sum_{\smiley=1}^{100} a_{\smiley} = \sum_{\frownie=1}^{100} a_{\frownie}.$$

A basic property of sums is

$$c_1 \sum_{k=m}^{n} a_k + c_2 \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (c_1 a_k + c_2 b_k).$$

We will also want to do changes of index in sum. For example

$$\begin{aligned}\sum_{k=m}^{n} a_k x^{k+3} &= a_m x^{m+3} + a_{m+1} xm + 4 + \cdots + a_{n-1} x^{n+2} + a_n x^{n+3} \\ &= \sum_{k=m+3}^{n+3} a_{k-3} x^k.\end{aligned}$$

2. Geometric series.

A (finite) ***geometric series*** is a finite sum of the form

$$S = a + ar + ar^2 + \cdots + ar^n.$$

In summation notation this is

$$S = \sum_{k=0}^{n} ar^k.$$

Such sums occur naturally in many contexts and fortunately it is easy give a formula for their sum. We first look at the case of $n = 2$. Then

$$S = a + ar + ar^2.$$

Multiply this by r to get

$$rS = ar + ar^2 + ar^3.$$

Note that the sums for S and rS have the terms ar and ar^2 in common, which suggests subtracting to cancel these terms out:

$$\begin{aligned} S &= a + ar + ar^2 \\ -rS &= \quad - ar - ar^2 - ar^3 \\ S - rS &= a - ar^3. \end{aligned}$$

Therefore

$$(1-r)S = a - ar^3$$

which, when $r \neq 1$, we can solve for S to get

$$S = \frac{a - ar^3}{1 - r}$$

For $n = 5$ the calculation looks like

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + ar^4 + ar^5 \\ -rS &= \quad - ar - ar^2 - ar^3 - ar^4 - ar^5 - ar^6 \\ S - rS &= a - ar^6 \end{aligned}$$

and therefore

$$(1-r)S = a - ar^6.$$

So when $r \neq 1$ we have

$$S = \frac{a - ar^6}{1 - r}.$$

At this point you have likely guessed the general pattern:

Theorem 1. *Let a and r be real numbers with $r \neq 1$ and $n \geq 2$ and integer. Then the sum of the geometric series*

$$S = a + ar + ar^2 + \cdots + ar^n$$

is

$$S = \frac{a - ar^{n+1}}{1 - r}.$$

Problem 1. Prove this. □

Problem 2. What happens in the theorem when $r = 1$? □

The way I find easiest to remember and apply this is to note that if the series $a + ar + ar^2 + \cdots + ar^n$ is continued that the next term would be ar^{n+1}. Therefore if we call the number r the ***ratio*** then

$$a + ar + ar^2 + \cdots + ar^n = \frac{1 - \text{next term}}{1 - \text{ratio}}.$$

Here are some examples

$$x^2 + x^4 + x^6 + \cdots + x^{20} = \frac{\text{first} - \text{next term}}{1 - \text{ratio}} = \frac{x^2 - x^{22}}{1 - x^2}$$

holds when $x \neq \pm 1$.

Let

$$S = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}.$$

Then

$$\begin{aligned} S &= \frac{1 - \text{next term}}{1 - \text{ratio}} \\ &= \frac{1 - (-1/128)}{1 - (-1/2)} = \frac{128 + 1}{128 + 64} = \frac{129}{192}. \end{aligned}$$

Let

$$\alpha = \overbrace{.333\cdots 3}^{n \text{ digits}}.$$

Then

$$\begin{aligned} \alpha &= 3(.1) + 3(.1)^2 + 3(.1)^3 + \cdots + 3(.1)^n \\ &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{3(.1) - 3(.1)^{n+1}}{1 - .1} \\ &= \frac{.3 - .3(.1)^n}{.3(3)} \\ &= \frac{1}{3} - \frac{1}{3(10)^n} \end{aligned}$$

There is anther natural way to find α:

$$9\alpha = 10\alpha - \alpha = (3.33\cdots 3) - (.333\cdots 3)$$
$$= 3 - \underbrace{.000\cdots 3}_{\text{10 decmal places}}$$

Therefore

$$\alpha = \frac{3 - .000\cdots 3}{9} = \frac{1}{3} - \frac{.000\cdots 1}{3} = \frac{1}{3} - \frac{1}{3(10)^n}$$

For the classical problem[1] of putting one grain rice on the first square of a chess broad, two on the second square, four on the third square, eight on the fourth square: that is doubling the number on each square up until the 64th square, then the total number of grains is

$$1 + 2 + 4 + \cdots + 2^{63} = \frac{1 - 2^{64}}{1 - 2} = 2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615.$$

Remark 2. The internet tells me that "A single long grain of rice weighs an average of 0.001 ounces (29 mg)." Thus the total weight of the rice on the chess board is $(2^{64}-1)/(1{,}000)$ onces. The number of onces $(2^{64}-1)/(1{,}000)$ in a ton is $2{,}000 \times 16 = 32{,}000$. Therefore the weight in tons of the rice

$$W = (2^{64} - 1)/(1{,}000 \times 32{,}000) = 5.76460752303423 \times 10^{11}.$$

The internet also says that the current rate of world rice production is about $P = 7.385477 \times 10^8$ tones/year. At this rate is would take about

$$\frac{W}{P} \approx 780.533$$

years to cover the chess board. □

Problem 3. (a) Find the sum of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$

(b) Find the sum of $P_0(1+r) + P_0(1+r)^2 + \cdots + P_0(1+r)^n$. (If at the beginning of each year you put P_0 in a bank account that pays interest at a rate of $100r\%$ per year, then this sum is the total after n years. As a check on your answer when $P_0 = 1{,}000$ and $r = .05$, (that is a 5%

[1] From the Wikipedia article on putting on grains of rice (or wheat) *Wheat and chessboard problem* `https://en.wikipedia.org/wiki/Wheat_and_chessboard_problem` The problem appears in different stories about the invention of chess. One of them includes the geometric progression problem. The story is first known to have been recorded in 1256 by Ibn Khallikan.[1] Another version has the inventor of chess (in some tellings Sessa, an ancient Indian Minister) request his ruler give him wheat according to the wheat and chessboard problem. The ruler laughs it off as a meager prize for a brilliant invention, only to have court treasurers report the unexpectedly huge number of wheat grains would outstrip the ruler's resources. Versions differ as to whether the inventor becomes a high-ranking advisor or is executed.

simple interest) then after 20 years the total is, to the nearest penny, 35,719.25.) $\square$

3. Some useful factoring formulas.

You recall that

$$x^2 - y^2 = (x-y)(x+y)$$

and may recall that

$$x^3 - y^3 = (x-y)(x^2+xy+y^2).$$

These generalize. To see how let us look at the right hand side of the last equation. If we multiple this out we get

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x(x^2+xy+y^2) - y(x^2+xy+y^2) \\ &= x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3 \qquad \text{(most terms cancel)} \\ &= x^3 - y^3.\end{aligned}$$

Let us look at a similar product:

$$\begin{aligned}(x-y)(x^3+x^2y+xy^2+y^3) &= x(x^3+x^2y+xy^2+y^3) - y(x^3+x^2y+xy^2+y^3) \\ &= x^4 + x^3y + x^2y^2 + xy^3 \\ &\qquad - x^3y - x^2y^2 - xy^3 - y^4 \\ &= x^4 - y^4.\end{aligned}$$

And just to be sure we see the pattern let us look at the next case

$$\begin{aligned}(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4) &= x(x^4+x^3y+x^2y^2+xy^3+y^4) \\ &\qquad - y(x^4+x^3y+x^2y^2+xy^3+y^4) \\ &= x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 \\ &\qquad - x^4y - x^3y^2 - x^2y^3 - xy^4 - y^5 \\ &= x^5 - y^5.\end{aligned}$$

The pattern is now clear:

Theorem 3. *Let n be any positive integer and let x and y be any two real numbers. Then*

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}).$$

In summation notation this is

$$x^n - y^n = (x-y)\left(\sum_{k=0}^{n-1} x^{n-1-k}y^k\right) = (x-y)\left(\sum_{\substack{j+k=n-1 \\ 0\le j,k\le n-1}} x^jy^k\right)$$

Problem 4. Prove this by multiplying out $(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1})$ and seeing that all but two terms cancel. $\square$

Problem 5. The proof of Theorem 3 may remind you of the proof of Theorem 1 because both rely on a lot of cancellation. This is because there is a geometric series hidden in the proof of Theorem 3. Let use consider the case of $n = 5$ and set

$$S = x^4 + x^3y + x^2y^2 + xy^3 + y^4.$$

This can be written as

$$S = x^4 + x^4\left(\frac{y}{x}\right) + x^4\left(\frac{y}{x}\right)^2 + x^4\left(\frac{y}{x}\right)^3 + x^4\left(\frac{y}{x}\right)^4$$

which is a geometric series. Thus

$$\begin{aligned} S &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{x^4 - x^4\left(\frac{y}{x}\right)^5}{1 - \frac{y}{x}} \\ &= \frac{x^5 - y^5}{x - y}. \end{aligned}$$

Recalling the definition of S this is

$$x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \frac{x^5 - y^5}{x - y}$$

which is equivalent to the $n = 5$ version of Theorem 3. Give a proof of general case of Theorem 3 using the method just given. $\square$

Theorem 3 will be useful when we talk about difference quotients of polynomials (and more generally rational functions). We will be interested in simplifying expressions of the form

$$\frac{f(x) - f(a)}{x - a}$$

by trying to cancel the $(x - a)$ out of the denominator. This is becasue we will be wanting to compute limits

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

As an example let $f(x) = x^3 - 2x^2$. Then

$$\begin{aligned}
\frac{f(x)-f(a)}{x-a} &= \frac{x^3 - 2x^2 - (a^3 - 2a^2)}{x-a} \\
&= \frac{(x^3-a^3) - 2(x^2-a^2)}{(x-a)} \\
&= \frac{(x^3-a^3) - 2(x^2-a^2)}{x-a} \\
&= \frac{(x-a)(x^2+xa+a^2) - 2(x-a)(x+a)}{x-a} \\
&= (x^2+xa+a^2) - 2(x+a).
\end{aligned}$$

and therefore

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a}((x^2+xa+a^2) - 2(x+a)) = 3a^2 - 4a.$$

Problem 6. Let

$$f(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

where c_0, c_1, c_2, and c_3 are constants. Simplify

$$\frac{f(x)-f(a)}{x-a}$$

by showing that $(x-a)$ can be canceled out of the denominator and use this to compute $\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$. □

4. Sums of arthritic series.

Anther sum that occurs naturally is

$$S = 1 + 2 + 3 + \cdots + n.$$

Let us compute this in some special cases. If

$$S = 1 + 2 + 3 + 4 + 5 + 6,$$

then also

$$S = 6 + 5 + 4 + 3 + 2 + 1.$$

We will add these together

$$\begin{aligned}
S &= 1 + 2 + 3 + 4 + 5 + 6 \\
S &= 6 + 5 + 4 + 3 + 2 + 1 \\
2S &= (1+6) + (2+5) + (3+4) + (4+3) + (5+2) + (6+1) \\
&= 7 + 7 + 7 + 7 + 7 + 7 \qquad\qquad \text{(6 terms in the sum)} \\
&= 6 \cdot 7
\end{aligned}$$

Therefore

$$S = \frac{6 \cdot 7}{2} = 21.$$

This method works in general. If

$$S = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

then we can reverse the sum

$$S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.$$

Adding these together gives

$$\begin{aligned} 2S &= (1+n) + (2+(n-1)) + (3+(n-2)) + \cdots + ((n-2)+3) + ((n-1)+2) + (n+1) \\ &= \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)}_{n \text{ terms}} \\ &= n(n+1). \end{aligned}$$

Dividing by 2 gives

$$S = \frac{n(n+1)}{2}.$$

This gives a proof of

Theorem 4. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

□

This can be generalized a bit. In general a finite ***arithmetic series*** is a sum of the form

$$\begin{aligned} S &= a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-1)d) \\ &= \sum_{k=0}^{n-1} (a+kd). \end{aligned} \tag{1}$$

This sum has n terms. The number d is the ***common difference*** (or just the ***difference***) of the series.

Problem 7. (This problem as much about leaning to use summation notation as it is about the result.) Use summation notation and the generalization of the argument given here for $n = 5$ to derive a formula for the sum of the series (1). When $n = 5$ we have

$$S = a + (a+d) + (a+2d) + (a+3d) + (a+4d) = \sum_{k=0}^{4} (a+kd)$$

Writing this sum in the reverse order

$$S = (a+4d) + (a+3d) + (a+2d) + (a+d) + a = \sum_{k=0}^{4} \big(a + (4-k)d\big)$$

Therefore

$$\begin{aligned}
2S &= S + S \\
&= \sum_{k=0}^{4}(a+kd) + \sum_{k=0}^{4}\big(a+(4-k)d\big) \\
&= \sum_{k=0}^{4}\big((a+kd+a+(4-k)d\big) \\
&= \sum_{k=0}^{4}(2a+4d) \\
&= 4(2a+4d).
\end{aligned}$$

Dividing by 2 then gives

$$S = 2(2a+4d) = 4a+8d.$$

Here is a way to rewrite this to make is seem more intuitive.

$$S = 4(a+2d) = 4\left(\frac{a+(a+4d)}{2}\right) = (\text{number of terms})\left(\frac{\text{first}+\text{last}}{2}\right)$$

Thus the sum is the number of terms times the average of the first and last terms. Now you should do this argument in the case of general n. $\square$

5. The binomial theorem

5.1. **Factorials and binomial coefficients.** We recall the definition of the ***factorials***. If n is a non-negative integer $n!$ is defined by

$$0! = 1 \quad \text{and for } n \geq 1 \quad n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

For small values of n we have

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5,040
8	40,320
9	362,880

n	$n!$
10	3,628,800
11	39,916,800
12	47,9001,600
13	622,7020,800
14	87,178,291,200
15	1,307,674,368,000
16	20,922,789,888,000
17	3556,87,428,096,000
18	6,402,373,705,728,000
19	121,645,100,408,832,000

n	$n!$
20	2,432,902,008,176,640,000
21	51,090,942,171,709,440,000
22	1,124,000,727,777,607,680,000
23	25,852,016,738,884,976,640,000
24	620,448,401,733,239,439,360,000
25	15,511,210,043,330,985,984,000,000
26	403,291,461,126,605,635,584,000,000
27	10,888,869,450,418,352,160,768,000,000
28	304,888,344,611,713,860,501,504,000,000
29	8,841,761,993,739,701,954,543,616,000,000
30	265,252,859,812,191,058,636,308,480,000,000

Problem 8. Show that for $n \geq 10$ that $n! \geq 3.6288(10)^{n-4}$. *Hint:* Use that $10! = 3{,}628{,}800 = 3.6288(10)^6$. For example if $n = 15$

$$\begin{aligned}
15! &= 10!(11)(12)(13)(14)(15) \\
&\geq 10!(10)(10)(10)(10)(10) \\
&= 10!(10)^5 \\
&= 3.6288(10)^6(10)^5 \\
&= 3.6288(10)^{11}.
\end{aligned}$$

This idea works in general. □

Remark 5. These tables and the last problem make it clear that $n!$ grows very fast. There is a well known approximation, ***Stirling's formula***,

$$n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

which shows that $n!$ grows faster than any exponential function. A more precise form of this was given by Herbert Robbins in 1955:

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} e^{\frac{1}{12n}}.$$

for all positive integers n. Time permitting we will prove some form of Stirling's formula either this term or next term. □

An elementary property of factorials we will use many times is that we get $n!$ by multiplying $(n-1)!$ by n. Thus

$$\begin{aligned}
n! &= n((n-1)!) \\
&= n(n-1)((n-2)!) \\
&= n(n-1)(n-2)((n-3)!)
\end{aligned}$$

and so on. This especially useful when dealing with fractions involving factorials. For example:

$$\frac{(n-1)!}{(n+2)!} = \frac{(n-1)!}{(n+2)(n+1)n((n-1)!)} = \frac{1}{(n+2)(n+1)n}.$$

Let $n, k \geq 0$ be integers with $0 \leq k \leq n$. Then the ***binomial coefficient*** $\binom{n}{k}$ is defined by

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}.$$

This is read as "n choose k".

Problem 9. Show this this definition implies

$$\binom{n}{k} = \binom{n}{n-k}.$$

□

Also we generally do not have to compute $n!$ to find $\binom{n}{k}$ as lots of terms cancel. For example

$$\binom{100}{3} = \frac{100!}{3! \cdot 97!} = \frac{100 \cdot 99 \cdot 98 \cdot 97!}{3! \cdot 97!} = \frac{100 \cdot 99 \cdot 98}{3!} = 161{,}700.$$

Proposition 6. *The following hold*

$$\begin{aligned}
\binom{n}{0} &= \binom{n}{n} = 1, \\
\binom{n}{1} &= \binom{n}{n-1} = n, \\
\binom{n}{2} &= \binom{n}{n-2} = \frac{n(n-1)}{2}, \\
\binom{n}{3} &= \binom{n}{n-3} = \frac{n(n-1)(n-2)}{6}.
\end{aligned}$$

Problem 10. Prove this. □

The expression $n(n-1)\cdots(n-k+1)$ comes up often enough that it is worth giving a name. Let $x^{\underline{k}}$ be the ***k-th falling power*** of x. That is

$$x^{\underline{k}} := \begin{cases} 1, & k = 0 \\ x(x-1)\cdots(x-k+1), & k \geq 1. \end{cases}$$

Thus

$$\begin{aligned}
x^{\underline{0}} &= 1 \\
x^{\underline{1}} &= x \\
x^{\underline{2}} &= x(x-1) \\
x^{\underline{3}} &= x(x-1)(x-2) \\
&\vdots \\
x^{\underline{k}} &= \underbrace{x(x-1)(x-2)\cdots(x-k+1)}_{k \text{ factors}}
\end{aligned}$$

Proposition 7. *The equality*

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n^{\underline{k}}}{k!}$$

holds.

Problem 11. Prove this. $\square$

Here is anther basic property of the binomial coefficients.

Proposition 8 (Pascal Identity)**.** *For* $1 \leq k \leq n$ *with* k, n *integers the equality*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

holds.

Problem 12. Prove this. *Hint:* Here is a special case

$$\begin{aligned}
\binom{12}{7} + \binom{12}{8} &= \frac{12!}{7!\,5!} + \frac{12!}{8!\,4!} \\
&= \frac{12!}{7!\,4!}\left(\frac{1}{5} + \frac{1}{8}\right) \\
&= \frac{12!}{7!\,4!}\left(\frac{8+5}{5\cdot 8}\right) \\
&= \frac{12!}{7!\,4!}\left(\frac{13}{5\cdot 8}\right) \\
&= \frac{13!}{8!\,5!} \\
&= \binom{13}{8}
\end{aligned}$$

where we have used $13! = 12! \cdot 13$, $8! = 7! \cdot 8$, and $5! = 4! \cdot 5$. $\square$

If we put the binomial coefficients in a triangular table (***Pascal's triangle***):

$$\begin{array}{ccccccccccc}
 & & & & & \binom{1}{1} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

the relation $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ tells us that any entry is the sum of the two entries directly above. This can be used to compute $\binom{n}{k}$ for small values of n. For example up to $n = 5$ the binomial coefficients are given by:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The following problem is both interesting in its own right and is a chance to review induction.

Problem 13. Let k, n be nonnegative integers with $0 \le k \le n$. Prove the binomial coefficient $\binom{n}{k}$ is an integer. *Hint:* We use induction on n. We can either use $n = 0$ or $n = 1$ as the base case as

$$\binom{0}{0} = 1, \text{ and } \binom{1}{0} = \binom{1}{1} = 1$$

and 1 is an integer. We now use induction on n. Assume that

$$\binom{n}{k} \text{is an integer for } 0 \le k \le n. \tag{2}$$

To complete the induction step we need to show

$$\binom{n+1}{k} \text{is an integer for } 0 \le k \le n+1. \tag{3}$$

This is true for the values $k = 0$ and $k = n + 1$ as

$$\binom{n+1}{0} = \binom{n+1}{n+1} = 1.$$

Thus we can assume $1 \le k \le n$. By the Pascal Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

and now use (2) to show that (3) holds to complete the induction. $\square$

5.2. **The binomial theorem.** One reason the binomial coefficients are important is

Theorem 9 (Binormal Theorem)**.** *For any positive integer n and $x, y \in \mathbb{R}$*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

In summation notation this is

$$(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

We will prove this shortly. For $n = 5$ we have

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Let $x = y = 1$ in this to get

$$\begin{aligned} 2^5 &= (1+1)^5 \\ &= (1)^5 + 5(1)^4(1) + 10(1)^3(1)^2 + 10(1)^2(1)^3 + 5(1)(1)^4 + (1)^5 \\ &= 1 + 5 + 10 + 10 + 5 + 1, \end{aligned}$$

which may not be that interesting of a fact, but the argument lets us see a pattern for something that is interesting.

Problem 14. Use this idea to show the sum of the numbers $\binom{n}{k}$ for $k = 0, 1, \ldots, n$ is 2^n. That is for all positive integers n

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

□

Problem 15. Prove for any positive integer n that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Hint: $(1-1) = 0$. □

Here is a bit of practice in using the binomial theorem.

Problem 16. Expand the following:

(a) $(1+2x^3)^4$,
(b) $(x^2 - y^5)^3$.

Problem 17. Use induction and the Pascal Identity to prove the Binomial Theorem. *Hint:* Use for the base case that $(x+y)^1 = x + y$. Here is what the induction step from $n = 4$ to $n = 5$ looks like. Assume that we know that

$$\begin{aligned} (x+y)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}y^4 \\ &= x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4 \end{aligned}$$

where we have used that $\binom{4}{0} = \binom{4}{4} = 1$. We now want to show the theorem holds for $n = 5$.

$$\begin{aligned}
(x+y^5) &= (x+y)(x+y)^4 \\
&= (x+y)\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&= x\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&\quad + y\left(x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4\right) \\
&= x^5 + \binom{4}{1}x^4y + \binom{4}{2}x^3y^2 + \binom{4}{3}x^2y^3 + xy^4 \\
&\quad + x^4y + \binom{4}{1}x^3y^2 + \binom{4}{2}x^2y^3 + \binom{4}{3}x^1y^4 + y^5 \\
&= x^5 + \left(\binom{4}{0} + \binom{4}{1}\right)x^4y + \left(\binom{4}{1} + \binom{4}{2}\right)x^3y^2 \\
&\quad + \left(\binom{4}{2} + \binom{4}{3}\right)x^2y^3 + \left(\binom{4}{3} + \binom{4}{4}\right)xy^4 + y^5 \\
&= x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^3 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + y^5 \\
&= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^3 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5
\end{aligned}$$

If you don't like this long hand what of doing it, here is what the same calculation looks like using summation notation. Assume that

$$(x+y)^4 = \sum_{k=0}^{4} \binom{4}{k} x^k y^{4-k}.$$

Then

$$
\begin{aligned}
(x+y)^5 &= (x+y)(x+y)^4 \\
&= x(x+y)^4 + y(x+y)^4 \\
&= x\sum_{k=0}^{4}\binom{4}{k}x^k y^{4-k} + y\sum_{k=0}^{4}\binom{4}{k}x^k y^{4-k} \\
&= \sum_{k=0}^{4}\binom{4}{k}x^{k+1} y^{4-k} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k} \\
&= \sum_{k=1}^{5}\binom{4}{k-1}x^{k} y^{4-(k-1)} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k} \\
&= \sum_{k=1}^{5}\binom{4}{k-1}x^{k} y^{5-k} + \sum_{k=0}^{4}\binom{4}{k}x^k y^{5-k} \\
&= \binom{4}{4}x^5 + \binom{4}{0}y^5 + \sum_{k=1}^{4}\left(\binom{4}{k-1}+\binom{4}{k}\right)x^k y^{5-k} \\
&= \binom{5}{5}x^5 + \binom{5}{0}y^5 + \sum_{k=1}^{4}\left(\binom{4}{k-1}+\binom{4}{k}\right)x^k y^{5-k} \\
&= \binom{5}{5}x^5 + \binom{5}{0}y^5 + \sum_{k=1}^{4}\binom{5}{k}x^k y^{5-k} \\
&= \sum_{k=0}^{5}\binom{5}{k}x^k y^{5-k}.
\end{aligned}
$$

where we have done the change of variable $k \mapsto k-1$ in the first sum on line 5, used the Pascal Identity to get to the second to the last line, and used that $\binom{4}{0} = \binom{5}{0} = 1$ and $\binom{4}{4} = \binom{5}{5} = 1$.

Either of these two calculations shows that if the Binomial Theorem holds for $n = 4$ then it holds for $n = 5$. Use a similar calculation to show that if the theorem holds for n, then in holds for $n + 1$. $\square$

Here is an example of one (of the many) ways we will be using the binomial theorem. Similar to some examples given above we will want to simplify expressions of the form

$$\frac{f(x+h)-f(x)}{h}$$

by cancelling the h out of the denominator so that we can compute the limit

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Here is an example. Let $f(x) = x^4$. Then

$$\begin{aligned}\frac{f(x+a)-f(x)}{h} &= \frac{(x+h)^4 - y^4}{h} \\ &= \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h} \\ &= 4x^3 + 6x^2h + 4xh^2 + h^3.\end{aligned}$$

Whence

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3.$$

6. Aside: Elements of the Theory of Finite Differences.

Let $f\colon \mathbb{Z} \to \mathbb{R}$ be a function from the integers, $\mathbb{Z}$, to the real numbers, $\mathbb{R}$. We wish to find methods to evaluate sums of the form

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

and in particular the special case

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n).$$

For example we will be able to show

$$\begin{aligned} 1 + 2 + \cdots + n &= \frac{n(n+1)}{2} \\ 1^2 + 2^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^3 + 2^3 + \cdots + n^3 &= \frac{n^2(n+1)^2}{4}. \end{aligned}$$

6.1. The difference operator and the Fundamental Theorem of Summation Theory.

Definition 10. Let $f\colon \mathbb{Z} \to \mathbb{R}$. Then the ***diffenence***, Δf, of f is the function

$$\Delta f(x) = f(x+1) - f(x).$$

The operator Δ is called the ***difference operator***. □

For example if $f(x) = 3x + 2$, then

$$\Delta f(x) = f(x+1) - f(x) = (3(x+1)+2) - (3x+2) = 3.$$

If $f(x) = x^2$, then

$$\Delta f(x) = f(x+1) - f(x) = (x+1)^2 - x^2 = 2x + 1$$

In the following table a, b, c, r are constants.

$f(x)$	$\Delta f(x)$
c	0
$ax+b$	a
ar^x	$a(r-1)r^x$

Problem 18. Verify these. $\square$

Theorem 11 (**Fundamental Theorem of Summation Theory**)**.** *Let $f\colon \mathbb{Z} \to \mathbb{R}$ and let F be an **anti-difference** of f. That is $\Delta F = f$. Then for $a, b \in \mathbb{Z}$ with $a < b$*

$$\sum_{k=a}^{b} f(k) = F(b+1) - F(a).$$

In particular

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1).$$

Proof. This uses the basic trick about telescoping sums:

$$\begin{aligned}
\sum_{k=a}^{b} f(k) &= \sum_{k=a}^{b} (F(k+1) - F(k)) \\
&= \sum_{k=a}^{b} F(k+1) - \sum_{k=a}^{b} F(k) \\
&= \big(F(a+1) + F(a+2) + \cdots + F(b) + F(b+1)\big) \\
&\qquad - \big(F(a) + F(a+1) + \cdots + F(b-1) + F(b)\big) \\
&= F(b+1) - F(a)
\end{aligned}$$

as required. $\square$

Theorem 11 makes it interesting to find anti-differences of functions. Here are some basic examples of functions $f(x)$ defined on the integers and their anti-differences (a, r and b are constants).

$f(x)$	$F(x)$
$ax+b$	$a\dfrac{x(x-1)}{2} + bx$
ar^x	$\dfrac{ar^x}{r-1}$

Problem 19. Verify these. (You just need to check $F(x+1) - F(x) = f(x)$). $\square$

Problem 20. Use that $\dfrac{ar^x}{r-1}$ is the anti-difference of ar^x and Theorem 11 give anther proof of

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r} = \frac{\text{first} - \text{next}}{1 - \text{ratio}}.$$

□

6.2. **Falling factorial powers and sums of powers.** For Theorem 11 to be useful we need more functions $f(x)$ where we know the anti-difference $F(x)$. As a start we give

Definition 12. For natural number p define the ***falling factorial power*** of $x \in \mathbb{R}$ as $x^{\underline{0}} = 1$ and for $p \geq 1$

$$x^{\underline{p}} = x(x-1)(x-2)\cdots(x-(p-1)).$$

(This product has p terms.) □

For small values of p this becomes

$$\begin{aligned}
x^{\underline{0}} &= 1 \\
x^{\underline{1}} &= x \\
x^{\underline{2}} &= x(x-1) \\
x^{\underline{3}} &= x(x-1)(x-2) \\
x^{\underline{4}} &= x(x-1)(x-2)(x-3) \\
x^{\underline{5}} &= x(x-1)(x-2)(x-3)(x-4).
\end{aligned}$$

Proposition 13. *If $f(x) = x^{\underline{p}}$ where p is a natural number, then $\Delta f(x) = px^{\underline{p-1}}$. That is*

$$\Delta x^{\underline{p}} = px^{\underline{p-1}}.$$

Problem 21. Prove this. *Hint:* Here is what the calculation looks like when $p = 5$.

$$\begin{aligned}
\Delta x^{\underline{5}} &= (x+1)^{\underline{5}} - x^{\underline{5}} \\
&= (x+1)x(x-1)(x-2)(x-3) - x(x-1)(x-2)(x-3)(x-4) \\
&= \big((x+1)-(x-4)\big)x(x-1)(x-2)(x-3) \\
&= 5x^{\underline{4}}.
\end{aligned}$$

□

Remark 14. The formula should remind you of the formula $\frac{d}{dx}x^p = px^{p-1}$ for derivatives. □

Proposition 15. *If $f(x) = x^{\underline{p}}$ where p is a non-negative integer, then $F(x) = \dfrac{1}{p+1}x^{\underline{p+1}}$ is an anti-difference of f.*

Problem 22. Prove this as a corollary of Proposition 13 by noting (by replacing p by $p+1$), that $\Delta x^{\underline{p+1}} = (p+1)x^{\underline{p}}$ and dividing by $(p+1)$. □

Problem 23. Show that if $p \geq 2$ that $1^{\underline{p}} = 0$. (For example $1^{\underline{3}} = 1(1-1)(1-2) = 0$.) □

Proposition 16. *If p is a positive integer, then*

$$\sum_{k=1}^{n} k^{\underline{p}} = \frac{(n+1)^{\underline{p+1}}}{p+1}.$$

Remark 17. This should remind you of the formula $\displaystyle\int_0^x t^p\,dt = \frac{x^{p+1}}{p+1}$.

Problem 24*.* Prove this. HINT: Let $f(x) = x^{\underline{p}}$. Then $F(x) = \dfrac{x^{\underline{p+1}}}{p+1}$ is an anti-difference of $f(x)$ and thus by Theorem 11

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1)$$

and use Problem 23 to see that $F(1) = 0$. □

Proposition 18. *The equalities*

$$\begin{aligned}
x &= x^{\underline{1}}\\
x^2 &= x^{\underline{2}} + x^{\underline{1}}\\
x^3 &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}}\\
x^4 &= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}}\\
x^5 &= x^{\underline{5}} + 10x^{\underline{4}} + 25x^{\underline{3}} + 15x^{\underline{2}} + x^{\underline{1}}
\end{aligned}$$

hold.

Problem 25*.* Verify the first three of these. □

Problem 26*.* Find formulas for

$$\sum_{k=1}^{n} k^2, \qquad \sum_{k=1}^{n} k^3.$$

HINT: Here is the idea for $\sum_{k=1}^{n} k^2$. Using the last problem and Proposition 16

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} (k^{\underline{2}} + k^{\underline{1}})\\
&= \sum_{k=1}^{n} k^{\underline{2}} + \sum_{k=1}^{n} k^{\underline{1}}\\
&= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}\\
&= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}.
\end{aligned}$$

We can leave the answer like this, or expand and factor to get

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Do similar calculations for $\sum_{k=1}^{n} k^3$. $\square$

6.3. **A couple of trigonometric sums.** For your convenience we recall some trig identities:

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\\
\cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\\
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\\
\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)
\end{aligned}$$

Problem 27. Let θ be a constant with $\sin(\frac{\theta}{2}) \neq 0$. Use the identities above to show

$$\sin\left(\theta\left(x+\frac{1}{2}\right)\right) - \sin\left(\theta\left(x-\frac{1}{2}\right)\right) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\theta x\right)$$

and therefore

$$F(x) = \frac{\sin\left(\theta\left(x-\frac{1}{2}\right)\right)}{2\sin(\frac{\theta}{2})}.$$

is an anti-difference of

$$f(x) = \cos(\theta x).$$

$\square$

Proposition 19. *If* $\sin\left(\frac{\theta}{2}\right) \neq 0$, *then*

$$\sum_{k=1}^{n} \cos(k\theta) = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} - \frac{1}{2}.$$

Problem 28. Use Problem 27 and Theorem 11 to prove this. $\square$

There is a similar formula for sums for the sine function.

Proposition 20. *If* $\sin(\frac{\theta}{2}) \neq 0$, *then*

$$\sum_{k=1}^{n} \sin(k\theta) = \frac{\cos(\frac{\theta}{2}) - \cos((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}.$$

Problem 29. Prove this. $\square$