The convergence theorems.

Theorem 1 (Bounded Convergence Theorem)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space and assume $f_1, f_2, f_3, \ldots$ are measurable functions on X and*

$$\lim_{n\to\infty} f_n = f$$

almost everywhere. Also assume $\mu(X) < \infty$ and that there is a constant B with $|f_n| \le B$ almost everywhere for all n. Then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu \qquad \square$$

This is special case of

Theorem 2 (Dominated Convergence Theorem)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space and assume $f_1, f_2, f_3, \ldots$ are measurable functions on X and*

$$\lim_{n\to\infty} f_n = f$$

almost everywhere. Also assume there is a function $g \in L^1(X, \mu)$ with

$$|f_n| \le g$$

for all n. Then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu \qquad \square$$

Problem 1. Show the existence of the dominating function g is required by giving an example of functions $f_n \in L^1([0,1])$ with $\lim_{n\to\infty} f_n = 0$ almost everywhere, but $\lim_{n\to\infty} \int_0^1 f_n \, d\mathfrak{m} = 1$. $\square$

Problem 2. Show that there is no sequence of real numbers λ_n such that $\lim_{n\to\infty} \lambda_n = \infty$ and $\lim_{n\to\infty} \sin(\lambda_n x) = 0$ almost everywhere on $[0,1]$. $\square$

Theorem 3 (Monotone Convergence Theorem)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space that assume $f_1, f_2, f_3, \ldots$ are measurable functions on X with*

$$f_n \ge 0$$

almost everywhere for all n and that for almost all x

$$f_1(x) \le f_2(x) \le f_3(x) \le f_4(x) \le \cdots$$

then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu \qquad \square$$

One advantage of the Monotone Convergence Theorem over the Dominated Convergence Theorem is in cases where there is no dominating function, g. Also it allows on to conclude the limit function $f = \lim_{n\to\infty}$ is integrable by just showing that the sequence of numbers $\left\langle \int_X f_n \, d\mu \right\rangle_{n=1}^{\infty}$ is bounded. Finally Monotone Convergence Theorem is also called Beppo Levi's Theorem.

The following is useful corollary to the Monotone Convergence Theorem.

Theorem 4 (Convergence of L^1 sums)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space that assume $f_1, f_2, f_3, \ldots$ are measurable functions on X such that*

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty.$$

Then series

$$f(x) := \sum_{n=1}^{\infty} f_n(x)$$

converges absolutely for almost all $x \in X$, $f \in L^1(X, \mu)$, and

$$\sum_{n=1}^{\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Problem 3*.* Prove this by applying the Monotone Convergence Theorem to the sequence $F_1, F_2, F_3, \ldots$ where $F_N(x) = \sum_{n=1}^{N} |f_n(x)|$. □

The final one of the main convergence theorems is

Theorem 5 (Fatou's Lemma)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space that assume $f_1, f_2, f_3, \ldots$ are measurable functions on X with $f_n \geq 0$ for all n. Then*

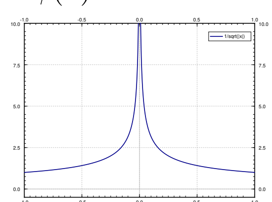
$$\int_X \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$ □

The following problems can be solved by using one or more of these theorems.

Problem 4*.* Let f be the function defined on $\mathbb{R}$ by

$$\phi(x) = \begin{cases} 1/\sqrt{|x|}, & 0 < |x| < 1; \\ 0, & \text{otherwise.} \end{cases}$$

For $-1 < x < 1$ the graph of $\phi(x)$ looks like



Let $\langle r_n \rangle_{n=1}^{\infty}$ be an enumeration of the rational numbers $\mathbb{Q}$ and define

$$f(x) = \sum_{n=1}^{\infty} \frac{\phi(x - r_n)}{2^n}.$$

(a) Show this series converges for almost all $x \in \mathbb{R}$.
(b) Show that $f(x)$ be becomes unbounded on every interval (a, b).
(c) Compute $\int_{-\infty}^{\infty} f(x) \, dx$. □

Problem 5. Let $f\colon [0,\infty) \to \mathbb{R}$ be a measurable function with

$$\int_0^\infty |f(x)|\,dx < \infty.$$

Show

$$\lim_{n\to\infty} |f(n^2x)| = 0$$

for almost all $x \in [0,\infty)$. □

Problem 6. Let $f, f_1, f_2, f_3, \ldots \in L^2(\mathbb{R})$ such that

$$\lim f_n(x) = f(x)$$

almost everywhere and

$$\lim_{n\to\infty} \|f_n\|_{L^2} = \|f\|_{L^2}.$$

Show

$$\lim_{n\to\infty} \|f_n - f\|_{L^2} = 0.$$ □