Some complex analysis problems.

We have seen earlier in the course that if $f = u + iv$ is analytic, then u satisfies $u_{xx} + u_{yy} = 0$. The following shows that on simply connected domains that the converse is true.

Theorem 1. *Let D be a simply connected domain in $\mathbb{C}$ and let $u \colon D \to \mathbb{R}$ be a function that satisfies*

$$u_{xx} + u_{yy} = 0$$

(such a function is call ***harmonic****). Then u is a real part of an analytic function. That is there is a real valued function v on D such that $f(z) = u(z) + iv(x)$ is analytic.*

Problem 1. Prove this along the following lines:

(a) Let

$$g = u_x - iu_y$$

and use the Cauchy-Riemann Equations to show $g(z)$ is analytic in D.

(b) Choose an arbitrary point $z_0 \in D$ and let $f(z)$ be an antiderivative of $g(z)$ (that is $f'(z) = g(z)$) with $f(z_0) = u(z_0)$. Explain how you know such an $f(z)$ exists.

(c) Show that $f(z)$ is the function we want. *Hint:* This is a little tricky. Write

$$f = U + iV$$

where U and V are the real and imagery parts of f and our goal is to show $U = u$. We know that the derivative of f is

$$f' = U_x + iV_x,$$

and (as f is an antiderivative of g)

$$f' = g = u_x - iu_y.$$

Comparing these gives

$$\begin{aligned} U_x &= u_x \\ V_x &= -u_y. \end{aligned}$$

Since $f = U + iV$ is analytic we have, by the Cauchy-Riemann equations,

$$U_y = -V_x$$

and therefore

$$U_y = -V_x = u_y.$$

Also $f(z_0) = u(z_0)$ which implies

$$U(z_0) = u(z_0).$$

Put these facts together to show that if

$$h = U - u$$

then

$$h_x = 0, \quad h_y = 0, \quad h(z_0) = 0$$

The first two of these conditions implies h is constant and then $h(z_0) = 0$ implies this constant is zero. Explain why this finishes the proof.

(d) Where did we use that U is simply connected? $\square$