Projective modules and localization.

Definition 1. Let R be a ring and M an R module. Then a set $\mathcal{B} \subset M$ is a ***basis*** for M if and only if each element $x \in M$ can be uniquely expressed as a linear combination over R of a finite number of elements of $\mathcal{B}$. In the case where $\mathcal{B} = \{e_1, e_2, \dots, e_n\}$ is finite this just says that every element $x \in M$ can be written uniquely as

$$x = r_1 e + r_2 e_2 + \cdots + r_n e_n$$

with $r_1, r_2, \dots, r_n \in R$. $\square$

Problem 1. Show that $\mathcal{B}$ is a basis for M if and only if each element of M is a linear combination over R of a finite number of elements of $\mathcal{B}$ and $\mathcal{B}$ is ***linearly independent*** over R, that is if $e_1, e_2, \dots, e_n \in \mathcal{B}$ are distinct and

$$\sum_{j=1}^{n} r_j e_j = 0$$

with $r_1, r_2, \dots, r_n \in R$, then $r_1 = r_2 = \cdots = r_n = 0$. $\square$

Not every module has a basis:

Problem 2. Show that no finite group (considered as a $\mathbb{Z}$ module) has a basis. $\square$

Definition 2. An R module is a ***free*** module over R if and only if it has a basis. $\square$

Problem 3. Note that R is module over itself. Show that a subset M is a submodule of R if and only if it is an ideal of R. $\square$

Problem 4 (Off of some old qualifying exam). Let R be a commutative ring and I an ideal of R. Show that I is a free module if and only if $I = \langle a \rangle$ where $a \in R$ is not a zero divisor in R. $\square$

Proposition 3 (Universal mapping property of free modules.)**.** *Let M be a free module over R with basis $\mathcal{B}$. Let M' be a R module and $f \colon \mathcal{B} \to M'$. Then f has a unique extension $\hat{f} \colon M \to M'$ as a R module homomorphism. (Rephrased a bit there is a unique R module homomorphism $\hat{f} \colon M \to M'$ such that $\hat{f}(e) = f(e)$ for all $e \in \mathcal{B}$.*

Problem 5. Prove this. *Hint:* To make the basic idea a bit simpler let us assume $\mathcal{B} = \{e_1, e_1, \dots, e_n\}$ is finite. Then each $x \in M$ is uniquely expressible as $x = \sum_{j=1}^{n} r_j e_j$ with $r_j \in R$. Show that the map $\hat{f}(x) := \sum_{j=1}^{n} r_j f(e_j)$ is the required homomorphism. $\square$

Problem 6. Let P be a free R module and $f \colon M \to P$ a surjective R module homomorphism. Then there is homomorphism $g \colon P \to M$ such that $f \circ g = I_P$ (where I_P is the identity on P).

Problem 7. Prove this. *Hint:* Let $\mathcal{B}$ be a basis for P. As f is surjective for each $e \in \mathcal{B}$ we can choose $x_e \in M$ with $f(x_e) = e$. By the universal mapping property of free modules there is a module homomorphism $g \colon P \to M$ with $g(e) = x_e$. □

Let A, B, C be modules over a ring R. Then a sequence short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

splits if and only if there homomorphism $\rho \colon C \to B$ such that $\beta \circ \rho = I_C$ (where I_C is the identity on C).

Problem 8. If the short exact sequence above splits show that B is isomorphic to the direct sum $B \approx A \oplus C$. □

Proposition 4. *Let R be a commutative ring with $1 \in R$. Then the following conditions on a module, P, over R are equivalent.*

(a) Every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

splits.

(b) Given a module homomorphism $f \colon P \to M'$ and a surjective module homomorphism $g \colon M \to M'$, there exists a homomorphism $h \colon P \to M$ with $f = g \circ h$. Rephrased in the language of diagrams this says that any diagram of the following form can be completed to a commutative diagram as shown. (Solid arrows are the given maps and a dashed arrow means that such a map exists and makes the diagram commute.)

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "f"] & \\ M' \arrow[r, "g", "\text{surjective}"'] & M \arrow[r] & 0 \end{tikzcd}$$

(c) There is a module M such that $P \oplus M$ is a free module.

If these conditions hold P is a ***projective module****.*

Problem 9. Prove this. *Hint:* For (b) $\implies$ (a) given the set up of (a):

$$0 \longrightarrow A \longrightarrow B \xrightarrow{g} P \longrightarrow 0$$

use $M = P$, $f = I_P$ and $M' = B$ in (b).

For (a) $\implies$ (b) note that every module, P, is the homomorphic image of a some free module. (Use the universal mapping property to map the basis of a appropriate free module onto a set of generators of P.) In particular for a projective module P there is a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

where B is a free module. Use (a) to show that P isomorphic to a direct summand in B.

For (c) $\implies$ (b), starting with the set up of (b), use (c) to find a module N so that $P \oplus N$ is a free module. Then consider the diagram

$$\begin{tikzcd} & P \oplus N \arrow[dl, dashed, "h"'] \arrow[d, "f \oplus I_N"] & \\ M' \oplus N \arrow[r, "g \oplus I_N"] & M \oplus N \arrow[r] & 0 \end{tikzcd}$$

and use the universal mapping property of free modules to construct the map h. □

Problem 10*.* I do not know if you discussed functors, but there is characterization of projective modules in terms functorial terms. Show that M is projective if and only if the functor $M \mapsto \operatorname{Hom}_R(P, M)$ is exact. That is if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact, then so is

$$0 \longrightarrow \operatorname{Hom}_R(P, A) \longrightarrow \operatorname{Hom}_R(P, B) \longrightarrow \operatorname{Hom}_R(P, C) \longrightarrow 0.$$

Problem 11*.* Prove this. □

Recall that if R is a ring, then $S \subseteq R$ is a ***multiplicative set*** if and only if $s_1, s_2 \in S$ implies $s_1 s_2 \in S$. In what follows we will also assume that $1 \in S$ and that $0 \notin S$. Under these conditions on S and when R is an integral domain

$$S^{-1}R = \left\{ \frac{r}{s} : r \in R, s \in S \right\}$$

where

$$\frac{r_1}{s_1} = \frac{r_2}{s_2}$$

if and only if $r_1 s_2 = r_2 s_1$. (More precisely, $\frac{r}{s}$ is the equivalence class of the ordered pair (r, s) in $R \times S$ under the equivalence relation $(r_1, s_2) \sim (r_2, s_2)$ if and only if $r_1 s_2 = r_2 s_1$.) Then $S^{-1}R$ is a ring and the map $r \mapsto \frac{r}{1}$ given a natural embedding of R into $S^{-1}R$ as a subring.

Problem 12*.* Let R be an integral domain and $S \subseteq R$ a multiplicative set. Show the proper ideals of $S^{-1}R$ are all of the form $S^{-1}I$ where I is a proper ideal of R with $I \cap S = \varnothing$. □

Problem 13*.* Let R be an integral domain and M a R module. Let $S \subseteq R$ be a multiplicative set. Give a precise definition to

$$S^{-1}M = \left\{ \frac{x}{s} : x \in M, s \in S \right\}$$

and show that this is module over the ring $S^{-1}R$ which contains M in a natural way. □

Problem 14. Let R be an integral domain and P a prime ideal in R. Show that $S = R \setminus P$ is a multiplicative set in R. $\square$

Definition 5. If P is a prime ideal in the integral domain and $S = R \setminus P$, then

$$R_P = S^{-1}R = \left\{ \frac{r}{s} : r \in R, s \notin P \right\}$$

is the ***localization of*** R ***at*** P. Likewise if M is a R module, then

$$M_P = S^{-1}M = \left\{ \frac{x}{s} : x \in M, s \in R, s \notin P \right\}$$

is the ***localization of*** M ***at*** P. $\square$

Problem 15. Let R be an integral domain and $S \subseteq R$ a multiplicative set. Show that if Q is a projective R module, then $S^{-1}Q$ is a projective $S^{-1}R$ module. $\square$

Problem 16. Let R be an integral domain and P a prime ideal in R. Use Problem 12 so show that the ideals of the localization R_P are in a bijective correspondence with the ideals, I, of R with $I \subseteq P$. Use this to show that R_P has exactly one maximal ideal. $\square$

Problem 17. Let $S = \{1, 2, 2^2, 2^3, \ldots\}$. Given an explicit description of $S^{-1}\mathbb{Z}$ as a subset of the rational numbers $\mathbb{Q}$. $\square$

Problem 18. In the ring $\mathbb{Z}$ give an explicit description of the localization of $\mathbb{Z}_{\langle 2 \rangle}$, that is the localization of $\mathbb{Z}$ at the prime ideal $\langle 2 \rangle$ as a subset of the rational numbers. $\square$

Problem 19. In the ring $\mathbb{Q}[x]$ give a explicit description as a subring of the ring $\mathbb{Q}(x)$ of rational functions of the localization $\mathbb{Q}[x]_P$ where P is the prime ideal

(a) $P = \langle x - 1 \rangle$,
(b) $P = \langle x^2 + 1 \rangle$. $\square$