Some group theory.

We have talked some about group actions. Here are some applications. Many of these use the following set up: let H be a subgroup of the finite group G and let $G/H = \{\xi H : \xi \in G\}$ be set of left cosets of H in G and let $\mathrm{Perm}(G/H)$ be the group of permutations of G/H. Then there is homomorphism $\phi\colon G \to \mathrm{Perm}(G/H)$ given by

$$\phi(g)(\xi H) = g\xi H.$$

Proposition 1. *The kernel of this homomorphism is*

$$\ker(\phi) = \bigcap_{g \in G} gHg^{-1}.$$

Problem 1. Prove this. □

An easy application of this is

Proposition 2. *Let G be a finite simple group. Then for any subgroup H of G*

$$|G| \,\Big|\, [G : H]!$$

Problem 2. Prove this. □

As an example not that if $|G| = 16 \cdot 5$, then either a 2-Sylow is normal of it has 5 conjugates. As the number of conjugates is the index of the normalizer of the subgroup we let H be the normalizer of one of the 2-Sylow subgroups. By Proposition 2, if G is simple we would have that $|G| = 16 \cdot 5$ divides evenly into $[G : H]! = 5! = 120$. As this this is not the case we see that there are no simple group of order $16 \cdot 5$.

Problem 3. (These are all old qualifying exams.) Show that there are no simple groups G with

(a) $|G| = 200$.
(b) $|G| = 56$.
(c) $|G| = 30$. □

Problem 4. Show that every group of order $3^4 7^3$ is solvable. □

Problem 5. Let p be the smallest prime dividing the order of the finite group G. Show every subgroup of G of index p is normal. □

Problem 6. Let $|G| = 2m$ where m is an odd number. Show G has a subgroup of order m. □

Problem 7. How many elements of order 7 does a simple group of order 168 have? (An example of a simple group of this order is the group of non-singular 3×3 matrices over the two element field.) □

Problem 8. Prove that every group of over 24 with no elements of order 6 is isomorphic to the symmetric group S_4. □