DETERMINING CYLINDERS BY THE PERIMETERS OF SECTIONS WITH PLANES.

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Abstract. Let C_1 and C_2 be cylinders in $\mathbf{R}^3$ which are parallel to the z-axis and centrally symmetric about the origin. Let ℓ_1 and ℓ_2 be distinct lines in $\mathbf{R}^2$ that pass through the origin. If $\operatorname{Length}(P \cap \partial C_1) = \operatorname{Length}(P \cap \partial C_2)$ for all planes P in $\mathbf{R}^3$ that contain either ℓ_1 or ℓ_2, then $C_1 = C_2$.

1. Introduction.

This note is motivated by a question [1, Prob. 7.6 p. 289] from Richard Gardner's book, *Geometric Tomography*, which asks if a centrally symmetric convex body in $\mathbf{R}^3$ is determined by the perimeters of its central sections. While we can not answer that question, we are able to show that centrally symmetric cylinders in $\mathbf{R}^3$ are determined by the perimeters of their sections with planes constrained to contain one of two non-parallel lines in the plane. To be precise if K is a convex body in $\mathbf{R}^2$ let

$$\operatorname{Cyl} K := \{(x, y, z) : (x, y) \in K, z \in \mathbf{R}\}$$

be the cylinder over K with generators parallel to the z axis. If P is a plane in $\mathbf{R}^3$ that does not contain a line parallel to the z-axis, let $\mathbf{n}(P)$ be the upward pointing unit normal to P and e_3 the unit vector pointing in the direction of the positive z-axis. For nonzero vectors $u, v \in \mathbf{R}^3$ let $\angle(u, v)$ be the angle between u and v. For a line, ℓ, through the origin of $\mathbf{R}^2$ and $\varepsilon > 0$ let

$$\mathcal{P}(\ell, \varepsilon) := \text{ Set of planes, } P \subset \mathbf{R}^3, \text{ with } \ell \subset P \text{ and } \angle(e_3, \mathbf{n}(P)) < \varepsilon.$$

Theorem. *Let K_1 and K_2 be convex bodies in $\mathbf{R}^2$ centrally symmetric about the origin, ℓ_1 and ℓ_2 distinct lines of $\mathbf{R}^2$ through the origin, and $\varepsilon > 0$. If*

$$\operatorname{Length}(P \cap \partial(\operatorname{Cyl} K_1)) = \operatorname{Length}(P \cap \partial(\operatorname{Cyl} K_2))$$

for all $P \in \mathcal{P}(\ell_1, \varepsilon) \cup \mathcal{P}(\ell_2, \varepsilon)$, then $K_1 = K_2$.

Without central symmetry there is no uniqueness (cf. Section 4).

The proof is based on a formula for $\operatorname{Length}(P \cap \partial(\operatorname{Cyl} K))$ that has the surprising, at least to us, property that it is linear in the support function

Date: December 17, 2008.
2000 *Mathematics Subject Classification.* Primary: 52A38. Secondary: 52A10, 52A15.
Key words and phrases. Convex cylinders, perimeters of sections.

of K and in fact is just the convolution of the support function with a continuous function. This allows the proof to be reduced to more or less standard calculations with Fourier series.

2. A formula for $\operatorname{Length}(P \cap (\partial \operatorname{Cyl} K))$.

To parametrize the planes in $\mathbf{R}^3$, or at least the planes that do not contain a line parallel to the z-axis, let $a \in \mathbf{R}^2$ and set

$$P_a := \{(v, \langle v, a\rangle) : v \in \mathbf{R}^2\}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbf{R}^2$. (This is the graph of $z = \langle v, a\rangle$ where $v = (x, y) \in \mathbf{R}^2$).) Set

$$e(\theta) := (\cos\theta, \sin\theta).$$

For each fixed θ the vectors $e(\theta)$ and its derivative, $e'(\theta)$, form an orthonormal basis of $\mathbf{R}^2$.

Proposition 1. *Let $a \in \mathbf{R}^2$ and write it in 'polar form' as $a = re'(\alpha)$ with $r \geq 0$ and $0 \leq \alpha < 2\pi$. If K is a convex body in $\mathbf{R}^2$ with support function h, then*

$$\operatorname{Length}(P_a \cap \partial(\operatorname{Cyl} K)) = (1 + r^2) \int_0^{2\pi} \frac{h(\theta)\, d\theta}{(1 + r^2 \cos^2(\theta - \alpha))^{3/2}} \tag{1}$$

Proof. We first assume that K has smooth boundary and strictly positive boundary curvature. Let h be the support function of K. As ∂K has positive curvature $h'' + h > 0$ (cf. [2, p. 3]) and ∂K is parametrized by

$$c(\theta) = h(\theta)e(\theta) + h'(\theta)e'(\theta).$$

An elementary calculation using $e''(\theta) = -e(\theta)$ (cf. [2, pp. 2–3]) yields

$$c'(\theta) = (h''(\theta) + h(\theta))e'(\theta). \tag{2}$$

The curve $P_a \cap \operatorname{Cyl}(K)$ is parametrized by

$$\gamma_a(\theta) = (c(\theta), \langle a, c(\theta)\rangle)$$

so, using (2),

$$\gamma_a'(\theta) = (h''(\theta) + h(\theta))(e'(\theta), \langle a, e'(\theta)\rangle).$$

As $h'' + h > 0$ and $e'(\theta)$ is a unit vector this and integration by parts gives

$$\begin{aligned}
\operatorname{Length}(P_a \cap \partial(\operatorname{Cyl} K)) &= \int_0^{2\pi} |\gamma_a'(\theta)|\, d\theta \\
&= \int_0^{2\pi} (h''(\theta) + h(\theta))\sqrt{1 + \langle a, e'(\theta)\rangle^2}\, d\theta \\
&= \int_0^{2\pi} (h''(\theta) + h(\theta))\sqrt{1 + r^2 \cos^2(\theta - \alpha)}\, d\theta \\
&= \int_0^{2\pi} h(\theta) \left(\frac{d^2}{d\theta^2} + 1\right) \sqrt{1 + r^2 \cos^2(\theta - \alpha)}\, d\theta \\
&= \int_0^{2\pi} h(\theta) \frac{1 + r^2}{(1 + r^2 \cos^2(\theta - \alpha))^{3/2}}\, d\theta
\end{aligned}$$

which shows that (1) holds when K has a smooth positively curved boundary.

For an arbitrary convex body in $\mathbf{R}^2$ with support function h choose a sequence of convex bodies with smooth positively curved boundaries $\{K_n\}_{n=1}^{\infty}$ with $K_n \to K$ in the Hausdorff metric (cf. [3, pp. 160–161]). Then the support functions, $\{h_n\}_{n=1}^{\infty}$, of the sequence converge uniformly to h (cf. [3, Thm 1.8.11, p. 53]). Therefore replacing h by h_n in (1) and taking a limit gives the general result. □

Corollary 1. *With notation as in the last proposition, if $|a| = r < 1$, then*

$$\frac{\operatorname{Length}(P_a \cap (\operatorname{Cyl} K))}{1 + |a|^2} = \sum_{k=0}^{\infty} \binom{-3/2}{k} |a|^{2k} \int_0^{2\pi} h(\theta) \cos^{2k}(\theta - \alpha)\, d\theta$$

Proof. This follows from (1) by using the binomial expansion of $(1+x)^{-3/2}$ with $x = r^2 \cos^2(\theta - \alpha)$ integrating the resulting series termwise. □

3. Proof of the Theorem

For $j = 1, 2$ let h_j be the support function of K_j. The condition that K_1 and K_2 are symmetric about the origin is equivalent to $h_j(\theta + \pi) = h_j(\theta)$ for $j = 1, 2$. Let $\alpha_j \in [0, \pi)$ be so that $e'(\alpha_j)$ is a unit normal to ℓ_j in $\mathbf{R}^2$. . As ℓ_1 and ℓ_2 are distinct $0 < |\alpha_2 - \alpha_1| < \pi$. Let $a(t, \alpha) = te'(\alpha)$. Then

$$\mathcal{P}(\ell_j, \varepsilon) = \{P_{a(t,\alpha_j)} : |t| < \arctan \varepsilon\}.$$

For $|t| < 1$ we use $|a(t, \alpha)| = |t|$ and Corollary 1 to get

$$\begin{aligned}
0 &= \frac{\operatorname{Length}(P_{a(t,\alpha_j)} \cap \partial(\operatorname{Cyl} K_2)) - \operatorname{Length}(P_{a(t,\alpha_j)} \cap \partial(\operatorname{Cyl} K_1))}{1 + t^2} \\
&= \sum_{k=0}^{\infty} \binom{-3/2}{k} t^{2k} \int_0^{2\pi} (h_2(\theta) - h_1(\theta)) \cos^{2k}(\theta - \alpha_j)\, d\theta
\end{aligned}$$

which implies

$$\int_0^{2\pi} (h_2(\theta) - h_1(\theta)) \cos^{2k}(\theta - \alpha_j)\, d\theta = 0 \tag{3}$$

for $k = 0, 1, 2, \ldots$ and $j = 1, 2$. The following is elementary and the proof is left to the reader.

Lemma 1. *For k a non-negative integer and $0 < |\alpha_2 - \alpha_1| < \pi$*

$$\begin{aligned}\operatorname{Span}&\left(\{1\} \cup \bigcup_{j=1}^{k} \Big\{ \cos^{2j}(\theta - \alpha_1), \cos^{2j}(\theta - \alpha_2) \Big\} \right) \\ &= \operatorname{Span}\left(\{1\} \cup \bigcup_{j=1}^{k} \Big\{ \cos(2j\theta), \sin(2j\theta) \Big\} \right)\end{aligned}$$

□

By this and Equation (3) it follows that for $k = 0, 1, 2, \ldots$

$$\int_0^{2\pi} (h_2(\theta) - h_1(\theta)) \cos(2k\theta)\, d\theta = \int_0^{2\pi} (h_2(\theta) - h_1(\theta)) \sin(2k\theta)\, d\theta = 0$$

Therefore all the even Fourier coefficients of $h_2 - h_1$ varnish. As $h := h_2 - h_1$ satisfies $h(\theta + \pi) = h(\theta)$, all its odd Fourier coefficients vanish. But a continuous function with all its Fourier coefficients vanishing is the zero function. Whence $h_2 - h_1 = 0$, which in turn implies $K_1 = K_2$. □

4. Non-uniqueness without central symmetry

For a convex body K, let $-K := \{-a : a \in K\}$ be the reflection of K in the origin and $\Delta K = \frac{1}{2}(K + (-K)) := \{\frac{1}{2}(a + b) : a \in K, b \in (-K)\}$ the ***central symmetral*** of K (cf. [1, p. 106]).

Proposition 2. *Let K_1, K_2 be convex bodies in $\mathbf{R}^2$. Then for all $a \in \mathbf{R}^2$*

$$\operatorname{Length}(P_a \cap \partial(\operatorname{Cyl} K_1)) = \operatorname{Length}(P_a \cap \partial(\operatorname{Cyl} K_2)). \tag{4}$$

for all $a \in \mathbf{R}^2$ if and only if $\Delta K_1 = \Delta K_2$. Therefore, if K is not centrally symmetric about some point, then K is not determined (even up to translation) by the function $a \mapsto \operatorname{Length}(P_a \cap \partial(\operatorname{Cyl} K))$.

Proof. If h_j is the support function of K_j write $h_j = p_j + q_j$ where

$$p_j(\theta) = \frac{1}{2}\big(h_j(\theta) + h_j(\theta + \pi)\big), \qquad q_j(\theta) = \frac{1}{2}\big(h_j(\theta) - h_j(\theta + \pi)\big).$$

Then $p_j(\theta + \pi) = p_j(\theta)$ and $q_j(\theta + \pi) = -q_j(\theta)$. Also (cf. [1, p. 106]) p_j is the support function of the central symmetral ΔK_j. Using $\cos^2(\theta + \pi - \alpha) = \cos^2(\theta - \alpha)$ and letting $a = re'(\alpha)$ as in Proposition 1, the change of variable $\theta \mapsto \theta + \pi$ gives

$$\int_0^{2\pi} \frac{(1 + r^2) q_j(\theta)\, d\theta}{(1 + r^2 \cos^2(\theta - \alpha))^{3/2}} = -\int_0^{2\pi} \frac{(1 + r^2) q_j(\theta)\, d\theta}{(1 + r^2 \cos^2(\theta - \alpha))^{3/2}}$$

and thus this integral vanishes. It follows form this and Proposition 1 that (4) holds for all $a \in \mathbf{R}^2$ if and only if

$$\int_0^{2\pi} \frac{(1+r^2)p_1(\theta)\,d\theta}{(1+r^2\cos^2(\theta-\alpha))^{3/2}} = \int_0^{2\pi} \frac{(1+r^2)p_2(\theta)\,d\theta}{(1+r^2\cos^2(\theta-\alpha))^{3/2}}$$

for all α and r. As p_j is the support function of ΔK_j another application of Proposition 1 yields that (4) equivalent to

$$\operatorname{Length}(P_a \cap \partial(\operatorname{Cyl}\Delta K_1)) = \operatorname{Length}(P_a \cap \partial(\operatorname{Cyl}\Delta K_2))$$

for all $a \in \mathbf{R}^2$. As ΔK_1 and ΔK_2 are symmetric about the origin Theorem 1 implies that (4) holds for all a if and only if $\Delta K_1 = \Delta K_2$. $\square$

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