Qualifying Exam in Analysis
August 1999

F | E ⊆ A which fails the infim prop.

Notation: $\mathbb{N}$ denotes the natural numbers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ denotes the real numbers, and λ Lebesgue measure on $\mathbb{R}$.

Note: All problems except number 9 are worth 10 points each

1. a. If A is a subset of $[a,b]$, then there exists a measurable set $E \subset A$ such that if F is measurable and $F \subset A$, then $\lambda(F \setminus E) = 0$.
   b. Is the result still true if we only assume that $A \subset \mathbb{R}$?

NO $m(\mathbb{R}) = \infty$

2. Suppose that for each $n \in \mathbb{N}$, F_n is a nondecreasing absolutely continuous functions on $[a,b]$ such that
   (1) $F_n(a) = 0$ for each n, and
   (2) the sequence $\{F_n'(x)\}$ is decreasing for a.e. x.
   Prove that
   (a) $\{F_n(x)\}$ is decreasing for each x, and
   (b) If $F(x) = \lim_{n\to\infty} F_n(x)$, then $F(x)$ is absolutely continuous.

3. Suppose $\{E_n\}$ is a sequence of measurable subset of $\mathbb{R}$ such that for every interval I, $\lim_{n\to\infty} \lambda(E_n \cap I) = \alpha\lambda(I)$, where α is a constant with $\alpha \in [0,1]$. If f is Lebesgue integrable, prove that
$$\lim_{n\to\infty} \int_{E_n} f\,d\lambda = \alpha \int_{\mathbb{R}} f\,d\lambda.$$

4. Let $\{p_n\}$ be a sequence of $2\pi-$ periodic measurable functions on $\mathbb{R}$ satisfying
   (a) $p_n(t) \geq 0$ for all n and t,
   (b) $\int_{-\pi}^{\pi} p_n(t)dt = 1$,
   (c) For each $\delta > 0$, $\lim_{n\to\infty} \int_{\delta\leq|t|\leq\pi} p_n(t)dt = 0.$
For f continuous and $2\pi-$ periodic on $\mathbb{R}$, set
$$f_n(x) = \int_{-\pi}^{\pi} p_n(x-t)f(t)dt.$$
Prove that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly on $\mathbb{R}$.

5. Let $\{g_n\}$ be a sequence in $L^q([0,1],\lambda)$, $1 < q < \infty$, such that $\|g_n\|_q \leq M$ for some $M > 0$ and all n. Suppose also that
$$\lim_{n\to\infty} \int_0^1 f g_n d\lambda$$
exists for every $f \in L^\infty([0,1],\lambda)$. Prove that
   (a) if $f \in L^p([0,1],\lambda)$, where $\frac{1}{p} + \frac{1}{q} = 1$, then $\lim_{n\to\infty} \int_0^1 f g_n\,d\lambda$ exists, and
   (b) there exists $g \in L^q([0,1],\lambda)$ with $\|g\|_q \leq M$ and
$$\lim_{n\to\infty} \int_0^1 f g_n d\lambda = \int_0^1 f g\,d\lambda.$$

6. Let f be analytic in $B(0,1)$ and let γ be a closed path in $B(0,1)$. For any $z_o \in B(0,1)$, $z_o \notin \gamma$, prove that

$$\int_\gamma \frac{f'(\zeta)}{\zeta - z_o}\, d\zeta = \int_\gamma \frac{f(\zeta)}{(\zeta - z_o)^2}\, d\zeta$$

7. Prove that $\int_0^\pi e^{\cos\theta}\cos(\sin\theta)d\theta = \pi$. (Hint: Consider $\int_\gamma (e^z/z)dz$ where γ is the unit circle)

8. a. For each positive integer n, show that $F_n(z) = \int_{1/n}^{1} e^{-t} t^{(z-1)} dt$ is analytic in $\operatorname{Re} z > 0$, where $t^z = e^{z \ln t}$. (Hint: Consider Morera's theorem.)
   b. Show that for every $\delta > 0$, $F_n(z)$ converges uniformly to an analytic function $F(z)$ on $\operatorname{Re} z \geq \delta$.

9. (20) True or False! If the result is true, prove it; if the result is false, provide a counterexample.
   a. If f is monotone increasing on $[a,b]$ and continuous with $f'(x) = 0$ a.e. on $[a,b]$, then f is constant on $[a,b]$.
   b. If f is continuous on $[0,1]$ and absolutely continuous on $[c,1]$ for every $c > 0$, then f is absolutely continuous on $[0,1]$.
   c. If f is differentiable on (a,b), then f' is continuous on (a,b).
   d. The set of functions $f \in L^1([0,1],\lambda)$ with $\|f\|_1 = 1$ is sequentially compact in the norm topology of L^1.
   e. If f is analytic in $\mathbb{C}$ satisfying $|f(z)| \leq M|z|^n$ for some constant M and all z sufficiently large, then f is a polynomial of degree less than or equal to N.