NOTES ON ANALYSIS

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1. Metric Spaces.

Definition 1. A ***metric space*** is a nonempty set E with a function $d\colon E\times E\to[0,\infty)$ such that for all $p,q,r\in E$ the following hold

(a) $d(p,q)\geq 0$,
(b) $d(p,q)=0$ if and only if $p=q$,
(c) $d(p,q)=d(q,p)$, and
(d) $d(p,r)\leq d(p,q)+d(q,r)$. □

The function d is called the ***distance function*** on E. The condition $d(p,q)=d(q,p)$ is that the distance between points is ***symmetric***. The inequality $d(p,r)\leq d(p,q)+d(q,r)$ is the ***triangle inequality***.

The most basic example if a metric space is when $E\subseteq\mathbb{R}$ is a nonempty subset of the real numbers, $\mathbb{R}$, and the distance is defined by

$$d(p,q)=|p-q|.$$

Problem 1. Show that this makes E into a metric space. □

Solution. We need to show the four axioms for being a metric space hold. Let $p,q,r\in E$

(a) $d(p,q)=|p-q|\geq 0$ because $|x|\geq 0$ for all real numbers x.
(b) If $d(p,q)=|p-q|=0$, then $p=q$ because the only real number x with $|x|=0$ is $x=0$.
(c) $d(p,q)=|p-q|=|-(q-p)|=|q-p|=d(q,p)$ as $|-x|=|x|$ for all real numbers x.
(d) For the last axiom we use that for all real numbers, x, y, the inequality $|x+y|\leq|x|+|y|$ holds along with the basic adding and subtracting trick.

$$d(p,r)=|p-r|=|(p-q)-(q-r)|\leq|p-q|+|q-r|=d(p,q)+d(q,r).$$

Thus E with the distance function d is a metric space. □

We have seen that if $p=(p_1,\ldots,p_n)$ and $q=(q_1,\ldots,q_n)$ are points in $\mathbb{R}^n$ and we define the ***length*** or ***norm*** of p to be

$$\|p\|=\sqrt{p_1^2+\cdots+p_n^2}$$

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then the inequality

$$\|p+q\| \le \|p\| + \|q\|$$

holds.

Proposition 2. *Let E be a nonempty subset of $\mathbb{R}^n$ and for $p, q \in E$ let*

$$d(p,q) = \|p-q\|.$$

Then E with the distance function d is a metric space.

Problem 2. Prove this. □

Solution. This is almost exactly the same as the proof of the last problem. Let $p, q, r \in E \subseteq \mathbb{R}^n$

(a) $d(p,q) = \|p-q\| \ge 0$ because $\|x\| \ge 0$ for all vectors x.
(b) If $d(p,q) = \|p-q\| = 0$, then $p = q$ because the only vector x with $\|x\| = 0$ is $x = 0$.
(c) $d(p,q) = \|p-q\| = \|-(q-p)\| = \|q-p\| = d(q,p)$ as $\|-x\| = \|x\|$ for all vectors x.
(d) For the last axiom we use that for all vectors x, y, the inequality $\|x+y\| \le \|x\| + \|y\|$ holds along with the basic adding and subtracting trick.

$$d(p,r) = \|p-r\| = \|(p-q)-(q-r)\| \le \|p-q\| + \|q-r\| = d(p,q) + d(q,r).$$

Thus E with the distance function d is a metric space. □

Here are some inequalities that we will be using later.

Proposition 3 (Reverse triangle inequality)**.** *Let E be a metric space with distance function d and let $x, y, z \in E$. Then*

$$|d(x,y) - d(x,z)| \le d(y,z).$$

Problem 3. Prove this. Then draw a picture in the case of $\mathbb{R}^2$ with its standard distance function showing why this inequality is reasonable. □

Solution. From the triangle inequality

$$d(x,y) \le d(x,z) + d(z,y)$$

which can be rearranged as

$$d(x,y) - d(x,z) \le d(y,z)$$

Interchanging the roles of y and z gives $d(x,z) - d(x,y) \le d(y,z)$ which can be rewritten as

$$-d(x,y) \le d(x,y) - d(x,z).$$

Putting these inequalities together gives the required inequality: $|d(x,y) - d(x,z)| \le d(y,z)$. See Figure 1.

□

Proposition 4. *Let E be a metric space with distance function d and $x_1, \dots, x_n \in E$. Then*

$$d(x_1,x_n) \le d(x_1,x_2) + d(x_2,x_3) + \cdots + d(x_{n-1},x_n).$$

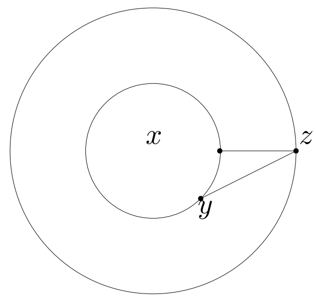


FIGURE 1. Figure illustrating Problem 3. The radii of the two circles are $d(x, y)$ and $d(x, z)$. The inequality tells us that the difference between the lengths of these radii at most the distance, $d(y, z)$, between y and z.

Problem 4. Prove this. Then draw a picture in the plane showing why this is reasonable. *Hint:* Induction. □

Solution. One way to do the proof is a straight forward induction. The base case is $n = 3$, $d(x_1, x_3) \le d(x_1, x_2) + d(x_2, x_3)$, which is just the triangle inequality. Assume that if is true for n, that is

$$d(x_1, x_n) \le d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

Then given $n + 1$ points $x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1}$ we apply the induction hypothesis to the n points and use that $d(x_{n-1}, x_{n+1}) \le d(x_{n-1}, x_n) + d(x_n, x_{n+1})$ $x_1, x_2, \ldots, x_{n-1}, x_{n+1}$ (that is we have just deleted x_n from our list of $n + 1$ points to get a list of n points). Thus

$$\begin{aligned} d(x_1, x_{n+1}) &\le d(x_1, x_2) + \cdots + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_{n+1}) \\ &\le d(x_1, x_2) + \cdots + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+1}). \end{aligned}$$

This closes the induction and completes the proof. See Figure 2 □

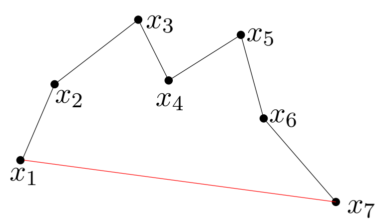


FIGURE 2. Figure illustrating Problem 4, which is the generalization of the triangle inequality to the n-gon inequality for $n \ge 3$. Here we have the 7-gon version, where the sum of the lengths of the six black segments is more than the length of the red segment.

Definition 5. Let E be a metric space with distance function d. Let $a \in E$, and $r > 0$.

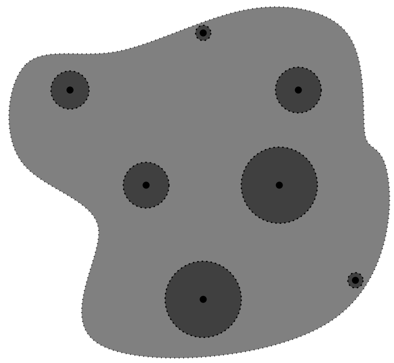


FIGURE 3. A set is open if and only if each of its points is the center of an open ball contained in the set.

(a) The ***open ball*** of radius r centered at x is

$$B(a, r) := \{x : d(a, x) < r\}.$$

(b) The ***closed ball*** or radius r centered at a is

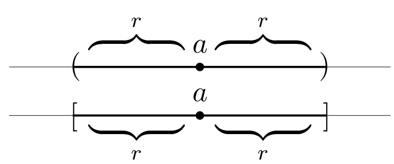
$$\overline{B}(a, r) := \{x : d(a, x) \leq r\}.$$

$\square$

In the real numbers with their usual metric $d(x, y) = |x - y|$ the open and closed balls about a are intervals with center a:

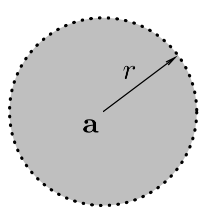
$B(a, r) = (a - r, a + r) =$

$\overline{B}(a, r) = [a - r, a + r] =$

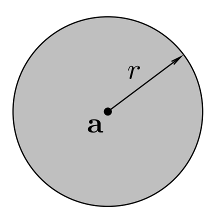


In the plane, $\mathbb{R}^2$, with its usual metric $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$, the open and closed balls about $\mathbf{a}$ are disks centered at $\mathbf{a}$.

$B(\mathbf{a}, r) =$



$\overline{B}(\mathbf{a}, r) =$



Definition 6. Let E be a metric space with distance function d. Then $S \subseteq E$ is an ***open set*** if and only if for all $x \in S$ there is an $r > 0$ such that $B(x, r) \subseteq S$. $\square$

This can be restated by saying that S is open if and only if each of its points is the center of an open ball contained in S. See Figure 1.

Proposition 7. *In any metric space E, the sets E and $\varnothing$ are open.* $\square$

Proof. Let $p \in E$, then for any $r > 0$ we have $B(p,r) = \{x \in E : d(x,p) < r\} \subseteq E$. Thus E contains not only some open ball about p, it contains every open ball about p. Therefore E is open.

That $\varnothing$ is open is a case of a vicious implication. To see this consider the statement

$$p \in \varnothing \text{ and } r > 0 \implies B(p,r) \subseteq \varnothing.$$

If this statement is true, then $\varnothing$ satisfies the definition of being open. But this is a true statement as an implication $P \implies Q$ is true whenever the hypotheses, P, is false. And the hypothesis "$p \in \varnothing$ and $r > 0$" is false as "$p \in \varnothing$" is false. □

Proposition 8. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the open ball $B(x,r)$ is an open set.*

Problem 5. Prove this. *Hint:* Let $x \in B(a,r)$. Then $d(a,x) < r$. Set $\rho := r - d(a,x) > 0$ and show $B(x,\rho) \subseteq B(a,r)$ □

Solution. Let $\rho := r - d(a,x)$, then $\rho > 0$ is as $x \in B(0,r)$ which implies $d(a,x) < r$. If $y \in B(x,\rho)$ then $d(x,y) < \rho$ and so

$$d(y,a) \leq d(a,x) + d(x,y) < d(a,x) + \rho = d(a,x) + r - d(a,x) = r.$$

This show $B(x,\rho) \subseteq B(a,r)$. Thus $B(a,r)$ contains a ball about x. As x was any point of $B(a,r)$ this shows $B(a,r)$ is open. See Figure 4 □

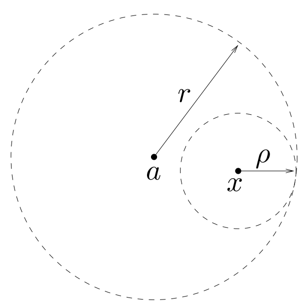


FIGURE 4. If $x \in B(x,r)$ then $B(a,r)$ contains the ball $B(x,\rho)$ where $\rho = r - d(a,x)$.

Proposition 9. *In the real numbers, $\mathbb{R}$, with their standard metric, the open intervals (a,b) are open.*

Problem 6. Prove this. □

Solution. Note for $x \in \mathbb{R}$ and $r > 0$ the ball $B(x,r)$ is just the interval $B(a,r) = (x-r, x+r)$. If $(a,b) = (-\infty,\infty)$, then then for any $x \in (a,b)$ we have $B(a,r) \subseteq (a,b)$. Now assume that at least one of a or b is not infinite. Let $x \in (a,b)$ and set

$$r := \min\{x-a, b-x\}.$$

Then if $y \in B(x,r)$ we have $x - r < y < x + r$. Thus

$$y < x + r \leq x + (b - x) = b$$

and

$$y > x - r \geq x - (x - a) = a$$

That is $y \in (a,b)$. This shows $B(x,r) \subseteq (a,b)$ and thus (a,b) contains a ball about any of its points, x. Thus (a,b) is open.

As anther solution in the case of a finite interval, note that the interval (a,b) is an open ball:

$$(a,b) = B((a+b)/2, r) \quad \text{where} \quad r = (b-a)/2.$$

But we have already seen that open balls are open sets. $\square$

Proposition 10. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the compliment, $\mathcal{C}(\overline{B}(a,r))$, of the closed ball $\overline{B}(a,r)$ is open.*

Problem 7. Prove this. *Hint:* If $x \in \mathcal{C}(B(a,r))$, then $d(x,a) > r$. Let $\rho := d(a,x) - r > 0$ and show $B(a,\rho) \subseteq \mathcal{C}(B(a,r))$. $\square$

Solution. Let $x \in \mathcal{C}(\overline{B}(a,r))$. We need to show that $\mathcal{C}(\overline{B}(a,r))$ contains a ball about x. That is we have to find $\rho > 0$ such that $B(a,r) \cap \overline{B}(x,\rho) = \varnothing$. Let

$$\rho := d(a,x) - r.$$

This is positive as $x \notin \overline{B}(a,r)$ and thus $d(a,x) > r$. Let $y \in B(x,\rho)$ then $d(x,y) < \rho$. By the triangle inequality

$$d(a,x) \leq d(a,y) + d(y,x).$$

This can be rearranged to give

$$d(a,y) \geq d(a,x) - d(x,y) > d(a,x) - \rho = d(a,x) - (r - d(a,x)) = r.$$

Therefore $y \in \overline{B}(a,r)$, that is $y \in \mathcal{C}\overline{B}(a,r)$. Thus $B(a,\rho) \subseteq \mathcal{C}\overline{B}(a,r)$ which shows that $\mathcal{C}\overline{B}(a,r)$ is open. See Figure 5. $\square$

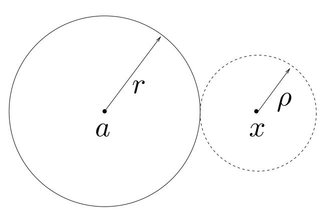


FIGURE 5. If $x \notin \overline{B}(a,r)$ and $\rho = d(a,x) - r$ then the ball $\overline{B}(a,r)$ and $B(x,\rho)$ are disjoint.

Proposition 11. *If U and V are open subsets of E, then so are $U \cup V$ and $U \cap V$.*

Proof. Let $x \in U \cup V$. Then $x \in U$ or $x \in V$. By symmetry we can assume that $x \in U$. Then, as U is open, there is an $r > 0$ such $B(x,r) \subseteq U$. But then $B(x,r) \subseteq U \subseteq U \cup V$. As x was any point of $U \cup V$ this shows that $U \cup V$ is open.

Let $x \in U \cap V$. Then $x \in U$ and $x \in V$. As $x \in U$ there is an $r_1 > 0$ such that $B(x,r_1) \subseteq U$. Likewise there is an $r_2 > 0$ such that $B(x,r_2) \subseteq V$. Let $r = \min\{r_1, r_2\}$. Then

$$B(x,r) \subseteq B(x,r_1) \subseteq U \quad \text{and} \quad B(x,r) \subseteq B(x,r_2) \subseteq V$$

and therefore $B(x,r) \subseteq U \cap V$. As x was any point of $U \cap V$ this shows that $U \cap V$ is open. □

Proposition 12. *Let E be a metric space.*

(a) Let $\{U_i : i \in I\}$ be a (possibly infinite) collection of open subsets of E. Then the union $\bigcup_{i \in I} U_i$ is open.
(b) Let $U_1, \ldots, U_n$ be a finite collection of open subsets of E. Then the intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is open.

Problem 8. Prove this. □

Solution. For (a) let $x \in \bigcup_{i \in I} U_i$. Then by the definition of the union that is at least one $i_0 \in I$ with $x \in U_{i_0}$. As U_{i_0} is open there is an $r > 0$ such that $B(x,r) \subseteq U_{i_0}$. But then

$$B(x,r) \subseteq U_{i_0} \subseteq \bigcup_{i \in I} U_i.$$

Thus $\bigcup_{i \in I} U_i$ contains a ball about any of its points and therefore is open.

For (b) let $x \in U_1 \cap U_2 \cap \cdots \cap U_n$ then by the definition of the intersection, $x \in U_i$ for each $i \in \{1, \ldots, n\}$. As U_i is open there is a $r_i > 0$ such that $B(x, r_i) \subseteq U_i$. Let

$$r = \min\{r_1, \ldots, r_n\}.$$

As these the minimum of a finite set of positive numbers it is is positive. For each i we gave $r \leq r_i$ and whence $B(x,r) \subset B(x,r_i)$. Thus holds for $i \in \{1, \ldots, \}$ and therefore

$$B(x,r) \subset U_1 \cap U_2 \cap \cdots \cap U_n.$$

Thus $U_1 \cap U_2 \cap \cdots \cap U_n$ contains a ball about any of its points and thus is open. □

Problem 9. In $\mathbb{R}$ let U_n be the open set $U_n := (-1/n, 1/n)$. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not open.

Solution. If $x \in \bigcap_{n=1}^{\infty} U_n$, then $|x| < 1/n$ for all positive integers n. By Archimedes' Axiom this implies $|x| = 0$. Therefore $\bigcap_{n=1}^{\infty} U_n = \{0\}$. But for any $r > 0$ the ball $B(0,r) = (-r,r)$ will contain nonzero points and thus is not contained in $\bigcap_{n=1}^{\infty} U_n = \{0\}$. So the point 0 is not in an open ball contained in $\bigcap_{n=1}^{\infty} U_n$. Therefore $\bigcap_{n=1}^{\infty} U_n$ is not open. □

Definition 13. Let E be a metric space. Then a subset S of E is ***closed*** if and only if its compliment, $\mathcal{C}(S)$ is open. $\square$

Because the compliment of the compliment is the original set this implies that a set, S, is open if and only if its compliment $\mathcal{C}(S)$ is closed. Likewise a set, S, is closed if and only if its compliment $\mathcal{C}(S)$ is open.

Proposition 14. *In any metric space E the sets $\varnothing$ and E are both closed.*

Proof. We have seen the sets E and $\varnothing$ are open, thus their compliments $\mathcal{C}(E) = \varnothing$ and $\mathcal{C}(\varnothing) = E$ are closed. $\square$

Proposition 15. *If E is a metric space, $a \in E$, and $r > 0$, then the closed ball $\overline{B}(a, r)$ is closed.* $\square$

Problem 10. Show that in $\mathbb{R}$ with its usual metric the closed intervals are closed. $\square$

Solution. The compliment of the closed interval $[a, b]$ is $(-\infty, b) \cup (a, \infty)$ which is the union of two open intervals and thus open. Therefore $[a, b]$ is the compliment of an open set and thus it is closed. $\square$

Proposition 16. *If E is a metric space, then every finite subset of E is closed.*

Problem 11. Prove this. $\square$

Solution. Let $F = \{x_1, \ldots, x_n\}$ be a finite set in the metric space E. Let U be the compliment of F. We wish to show that U is open. Let $x \in U$. Then $x \notin F = \{x_1, \ldots, x_n\}$ and therefore the number

$$r = \min\{d(x, x_1), d(x, x_2), \ldots, x_n\}$$

is positive. And if $x_i \in F$, then $d(x, x_i) \geq r$. Therefore $x \notin B(x, r)$. That is $B(x, r) \subseteq U$. Therefore U contains a ball about any of its points and thus is open, showing that F is closed. $\square$

Problem 12. In the real numbers show that the half open interval $[0, 1)$ is neither open or closed. $\square$

Solution. Let $r > 0$. Then ball of radius r about 0, that is $B(0, r) = (-r, r)$, contains negative numbers and thus contains points that are not in $[0, 1)$. Thus the point $0 \in [0, 1)$ is not contained in any open ball that is contained in $[0, 1)$. Therefore $[0, 1)$ is not open.

Let $r > 0$. The point 1 is in the compliment of $[0, 1)$. Therefore the The ball $B(1, r) = (1 - r, 1 + r)$ will contain points that are in $[0, 1)$ (that is points x with $1 - r < x < 1$). Therefore the compliment of $[0, 1)$ does not contain any open ball about 1. Therefore the compliment of $[0, 1)$ is not open and therefore $[0, 1)$ is not closed. $\square$

Problem 13. The integers, $\mathbb{Z}$, are a metric space with the metric $d(m, n) = |m - n|$. Note that for this metric space if $m \neq n$ that $d(m, n)$ is a nonzero positive integer and thus $d(m, n) \geq 1$. Assuming these facts prove the following

(a) Let $r = 1/2$, then for each $n \in \mathbb{Z}$ the open ball $B(n,r)$ is the one element set $B(n,r) = \{n\}$ and therefore $\{n\}$ is open.
(b) Every subset of $\mathbb{Z}$ is open. *Hint:* Let $S \subseteq \mathbb{Z}$, then $S = \bigcup_{n\in S}\{n\}$ and use Proposition 12 to conclude that S is open.
(c) Every subset of $\mathbb{Z}$ is closed. □

Solution. (a) If $x \in B(n,1/2)$ then $|x-n| < 1/2$ and both x and n are integers. Therefore $x = n$. Thus $B(x,r) = \{n\}$.
(b) Ignore the hint. Let S be a subset of $\mathbb{Z}$. Let $n \in S$. Then by Part (a) $B(n,1/2) = \{n\} \subseteq S$. Thus S contains a ball of radius $1/2$ about any of its point and therefore is open.
(c) Let S be any subset of $\mathbb{Z}$. Then by Part (b) its compliment is open. Therefore S is closed. □

Proposition 17. *Let E be a metric space.*

(a) Let $\{F_i : i \in I\}$ be a (possibly infinite) collection of closed subsets of E. Then the intersection $\bigcap_{i\in I} F_i$ is closed.
(b) Let $F_1, \dots, F_n$ be a finite collection of closed subsets of E, then the union $U_1 \cup \cdots \cup U_n$ is closed.

Problem 14. Prove this. *Hint:* The correct way to do this is to deduce it directly from Proposition 12. For example to show that the union of two closed sets is closed, let F_1 and F_2 be closed. Then the compliments $\mathcal{C}(F_1)$ and $\mathcal{C}(F_1)$ are open and the intersection of two open sets is open. Therefore $\mathcal{C}(F_1)\cap\mathcal{C}(F_2)$ is open and thus the compliment of this set is closed. That is

$$F_1 \cup F_2 = \mathcal{C}(\mathcal{C}(F_1)\cap\mathcal{C}(F_2))$$

is closed. □

Solution. (a) For each $i \in I$ set $U_i := \mathcal{F}_i$. That is U_i is the compliment of F_i. As F_i is close, each U_i is open. Therefore the union $\bigcup_{i\in I} U_i$ is open. Therefore the compliment of this set, is closed. That is

$$\mathcal{C}\left(\bigcup_{i\in I} U_i\right) = \bigcap_{i\in I}\mathcal{C}(U_i) = \bigcap_{i\in I} F_i$$

is closed, as required.
(b) Again let U_i be the compliment of F_i. Then each U_i is open and therefore the finite intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is open. Thus its compliment,

$$\mathcal{C}(U_1 \cap \cdots \cap U_n) = \mathcal{C}(U_1) \cup \cdots \cup \mathcal{C}(U_n) = F_1 \cup \cdots \cup F_n$$

is open. □

Let E be a metric space. Then a function $f\colon E \to \mathbb{R}$ is ***Lipschitz*** if and only if there is a constant $M \geq 0$ such that

$$|f(p) - f(q)| \leq M d(p,q) \quad \text{for all } p, q \in E.$$

Proposition 18. *Let E be a metric space and $f\colon E \to \mathbb{R}$ a Lipschitz function. Then for all $c \in \mathbb{R}$ the sets*

$$\begin{aligned} f^{-1}\big[(c,\infty)\big] &= \{p \in E : f(p) < c\} \\ f^{-1}\big[(-\infty,c)\big] &= \{p \in E : f(p) > c\} \end{aligned}$$

are open and the sets

$$\begin{aligned} f^{-1}\big[[c,\infty)\big] &= \{p \in E : f(p) \geq c\} \\ f^{-1}\big[(-\infty,c]\big] &= \{p \in E : f(p) \leq c\} \end{aligned}$$

are closed.

Half of the proof. Assume that f satisfies $|f(p) - f(q)| \leq M d(p,q)$ for $p, q \in E$. We will show that $f^{-1}\big[(-\infty,c)\big]$ is open. We need to show that for any $q \in f^{-1}\big[(-\infty,c)\big]$ the set $f^{-1}\big[(-\infty,c)\big]$ contains an open ball about q. As $q \in f^{-1}\big[(-\infty,c)\big]$ we have $f(q) < c$. Therefore

$$r = \frac{c - f(q)}{M}$$

is positive. Let $pe \in B(q,r)$. Then

$$\begin{aligned} f(p) &= f(q) + (f(p) - f(q)) \\ &\leq f(q) + |f(p) - f(q)| && (\text{as } (f(p) - f(q)) \leq |f(p) - f(q)|) \\ &\leq f(q) + M d(p,q) && (\text{as } f \text{ is Lipschitz}) \\ &< f(q) + Mr && (\text{as } p \in B(q,r), \text{ so } d(p,q) < r) \\ &= f(q) + M\left(\frac{c - f(q)}{M}\right) && (\text{from our definition of } r) \\ &= c. \end{aligned}$$

Therefore if $p \in B(q,r)$ we have $f(p) < c$ and thus $B(q,r) \subseteq f^{-1}\big[(-\infty,c)\big]$. Whence $f^{-1}\big[(-\infty,c)\big]$ contains an open ball about any of its points q and therefore $f^{-1}\big[(-\infty,c)\big]$ is open.

We now show $f^{-1}\big[[c,\infty\big] = \{p \in E : f(p) \geq c\}$ is closed. We know $f^{-1}\big[(-\infty,c)\big] = \{p \in E : f(p) < c\}$ is open. Its compliment is

$$\mathcal{C}\left(f^{-1}\big[(-\infty,c)\big]\right) = f^{-1}\big[[c,\infty)\big].$$

Therefore $f^{-1}\big[[c,\infty)\big]$ is the compliment of an open set, which means that $f^{-1}\big[[c,\infty)\big]$ is closed. $\square$

Problem 15. Prove the other half of Proposition 18, that is show $f^{-1}\big[(c,\infty)\big]$ is open and $f^{-1}\big[(-\infty,c]\big]$ is closed. $\square$

Proposition 19. *Let E be a metric space and $f\colon E \to \mathbb{R}$ a Lipschitz function. Then for any $c \in \mathbb{R}$ the set*

$$f^{-1}[c] = \{p \in E : f(p) = c\}$$

is a closed set.

Problem 16. Prove this. *Hint:* Write $f^{-1}[c]$ as the intersection of two closed sets. $\square$

We can now give some more examples of open and closed sets in the plane, $\mathbb{R}^2$ where we are using the distance function $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$. Let $\mathbf{a} = (a_1, a_2)$ be a vector in $\mathbb{R}^2$ define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2 + b$$

where b is a real number. If $\mathbf{x} = (x_1, x_2)$ this can be written in vector form as

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b.$$

If $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$ then

$$\begin{aligned} |f(\mathbf{p}) - f(\mathbf{q})| &= |\mathbf{a} \cdot \mathbf{p} + b - (\mathbf{a} \cdot \mathbf{q} + b)| \\ &= |\mathbf{a} \cdot (\mathbf{p} - \mathbf{q})| \\ &\leq \|\mathbf{a}\| \|\mathbf{p} - \mathbf{q}\| && \text{(Cauchy-Schwartz)} \\ &= M d(\mathbf{p}, \mathbf{q}) \end{aligned}$$

where $M = \|\mathbf{a}\|$. Thus f is a Lipschitz function.

Consider the case where $\mathbf{a} = (1, 0)$ and $b = 0$. Then for any $c \in \mathbb{R}$. In this case $f(x_1, x_2) = x_1$, or in slightly different notation $f(x, y) = x$. Therefore Proposition 18 implies the sets

$$\{(x, y) : x > c\}, \quad \{(x, y) : x < c\}$$

are open and that

$$\{(x, y) : x \geq c\}, \quad \{(x, y) : x \leq c\}$$

are closed.

Problem 17. Let $(a, b) \in \mathbb{R}^2$ be a nonzero vector and $c \in \mathbb{R}$.

(a) Show that the line

$$\{(x, y) \in \mathbb{R}^2 : ax + by = c\}$$

is closed.

(b) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by > c\}$$

is open (call such a half plane an ***open half plane***).

(c) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by \geq c\}$$

is closed (call such a half plane a ***closed half plane***).

(d) Show that the triangle

$$T = \{(x, y) \in \mathbb{R}^2 : x, y > 0, x + y < 0\}$$

is an open set. *Hint:* Write this as the intersection of three open half planes.

(e) Show that the triangle

$$S = \{(x, y) : x, y \geq 0, x + y \leq 0\}$$

is a closed subset of the plane. *Hint:* Write this as the interestion of three closed half planes. $\square$

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