Some group theory problems.

Problem 1. This is just a warm up problem. Let a, b be elements of a group which commute with each other. Let $o(a) = m$ and $o(b) = n$ (that is the order of a is m and the orger of b is n.) Show that if $\gcd(m, n) = 1$ then $o(ab) = mn$. $\square$

Problem 2. Let $G = S_3$ and $a = (123)$ and $b = (12)$ show $bab^{-1} = a^2$, and therefore S_3 can be defined by generators and relations as

$$S_3 = \langle a, b : a^3 = b^2 = 1, bib^{-1} = a^2 \rangle.$$

$\square$

Problem 3. Show that up to isomorphism there are only two groups of order 6. *Hint:* Let $|G| = 6$. Then G has an element of order 3. Let $o(a) = 3$. Then the cyclic subgroup $\langle a \rangle = \{1, a, a^3\}$ has index two in G and thus it is normal. Let b be an element of G of order 2. Then as $\langle a \rangle$ is normal $bab^{-1} \in \langle a \rangle$ and therefore either $bab^{-1} = a$ or $bab^{-1} = a^2$. So G is one of the two groups

$$G = \langle a, b : a^3 = b^2 = 1, ab = ba \rangle$$
$$G = \langle a, b : a^3 = b^2 = 1, bab^{-1} = a^2 \rangle.$$

$\square$

Problem 4. Let G be a non-Abelian finite group of order pq with p and q distinct primes, $p < q$ and $p \mid (q - 1)$. Then G has generators a and b so that

$$G = \langle a, b : a^q = b^p = 1, bab^{-1} = a^r \rangle$$

where $r > 1$ and r is a generator of the subgroup of order p in the multiplicative group $(\mathbb{Z}_q)^\times$. *Hint:* Here is an outline of one way to do this:

(a) G has elements a and b with $o(a) = q$ and $o(b) = p$.
(b) The cyclic group $\langle a \rangle$ is normal in G.
(c) $bab^{-1} = a^r$ for some integer with $2 \le r \le q - 1$.
(d) For positive integers k

$$b^k ab^{-k} = a^{r^k}.$$

(e) Use that b has order p (so that $b^p = 1$) to show $a = a^{r^p}$ and thus $a^{r^p - 1} = 1$ and that this shows

$$r^p \equiv 1 \mod (q - 1).$$

(You may need to use the fact that $(\mathbb{Z}_q)^\times$ is cyclic of order $(q - 1)$.
(f) show that if $r^k \equiv 1 \mod (q - 1)$ then $p \mid k$. (If not there is a k with $1 \le k < p$ with $b^k ab^{-k} = a^{r^k} = a$ which implies $ab^k = b^k a$. Show this implies the group is Abelian.) $\square$

Problem 5. Show that a group of order 24 with no elements of order 6 is isomorphic to the symmetric group S_4. *Hint:* Here is an outline of one method to to this.

(a) Show that no element of order 2 commutes with an element of order 3.

(b) Show $|\operatorname{Syl}_3(G)| = 4$. (Here $\operatorname{Syl}_p(G)$ is the set of p=Sylow subgroups of the group G.)
(c) Let $P \in syl_3(G)$. Use that $|\operatorname{Syl}_3(G)| = 4$ to show $|N_G(P)| = 6$ for all $P \in \operatorname{Syl}_3(G)$. (Here $N_G(P) = \{g \in G : gPg^{-1} = P\}$ is the ***normalizer*** of P in G.)
(d) Show each $N_G(P)$ is isomorphic to S_3.
(e) Define a group homomorphism $\phi \colon G \operatorname{Perm}(\operatorname{Syl}_3(G))$ ($\operatorname{Perm}(X)$ is the group of permutations of the set X) by

$$\phi(g)(P) = gPg^{-1}.$$

Show the kernel of ϕ is

$$\ker(\phi) = \bigcap_{P \in \operatorname{Syl}_3(G)} N_G(P).$$

(f) Then $\ker(\phi)$ will be a normal subgroup of $N_G(P) \cong S_3$ and the only normal subgroups of S_3 are $\langle 1 \rangle$ and the subgroup of order 3. Show that the subgroup of order 3 is impossible.
(g) Show ϕ is an isomorphism. □

Problem 6. This is motivated by Problem 3 on McNulty's list. Let G be a group and H, K subgroups of G with

- $H, K \triangleleft G$
- $G = HK := \{hk : h \in H, k \in K\}$
- $H \cap K = \langle 1 \rangle$.

(a) Show G is isomorphic to the direct product: $G \cong H \times K$.
(b) Let N be a normal subgroup of G with $N \cap H = N \cap K = \langle 1 \rangle$. Show $N \leq Z(G)$. (That is N is a subset of the center of G.) *Hint:* Let $a \in H$ and $n \in N$ and view the commentator $ana^{-1}n^{-1}$ from a schizophrenic view point to see $ana^{-1}n^{-1} \in H \cap N = \langle 1 \rangle$ and therefore a and n commute. □