Analysis Qualifying Exam
August 2002

Instructions: Write your name legibly on each sheet of paper. Write only on one side of of each sheet of paper. Try to answer all questions. Questions 1–8 are each worth 10 points and question 9 is worth 20 points.
Terminology: f increasing means $x \le y$ implies $f(x) \le f(y)$. Measurability and inegrability on $\mathbb{R}$ or an interval will always refer to the Lebesgue measure. except if otherwise specified. Lebesgue measure will be denoted by m, dx or dy depending on the context.

1. Let $f : [a, b] \to [c, d]$ be an increasing absolutely continuous function and let $g : [c, d] \to \mathbb{R}$ be an absolutely continuous function. Prove that the composition $g \circ f : [a, b] \to \mathbb{R}$ is absolutely continuous.

2. let f be an integrable function $\mathbb{R}$. Prove that
$$\int_{\mathbb{R}} \cos(xt) f(x)\, dx \to 0$$
as $t \to \infty$.

3. Let $0 \le f : [0, \infty) \to \mathbb{R}$ be Lebesgue integrable. Prove that
$$\lim_{n\to\infty} \frac{1}{n} \int_0^n x f(x)\, dx = 0.$$

4. Let $f \in L_2([0, \infty))$. Prove that
$$\frac{1}{\sqrt{x}} \int_0^x f(t)\, dt \to 0$$
as $x \downarrow 0$.

5. Let $\mu(X) < \infty$ and $1 < p \le \infty$. Let $f_n, f \in L_p(X, \mu)$ with $\|f_n\|_p \le 1$ for all n such that $f_n \to f$ a.e.. Let $g_n \in L_q$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that $g_n \to g$ in norm in L_q. Prove that $f_n g_n \to fg$ in L_1.

6. Let $f : [0, 1] \to \mathbb{R}$ be a Lebesgue measurable function such that the function $F(x, y) = f(x) - f(y)$ is integrable over $[0, 1]^2$. Prove that f is integrable over $[0, 1]$ and also compute $\int\int F(x, y)\, dxdy$.

7. Let $\Omega \subset \mathbb{C}$ be an open set containing the unit disk $\{z : |z| \le 1\}$ and let $f : \Omega \to \mathbb{C}$ be a holomorphic function such that $|f(z)| > |f(0)|$ for all $|z| = 1$. Prove that f has a zero in $|z| < 1$.

8. Let $f, g : \{z : |z| < 1\} \to \mathbb{C}$ be holomorphic functions such that $|f(z)| = |g(z)|$ for all $|z| < 1$. Prove that every zero of g is also a zero of f of the same multiplicity and that thus $f = \lambda g$ for some λ with modulus one.

9. True or False. Prove, or give a counterexample.

   a. There exists a compact set $K \subset [0,1] \setminus \mathbb{Q}$ with $m(K) > \frac{1}{2}$.

   b. If $f(x) = \sqrt{x}$, then f is uniformly continuous on $[0, \infty)$.

   c. If $m(E) > 0$, then E has a non-empty interior.

   d. There exists $M > 0$ such that $|\sin z| \le M$ for all $z \in \mathbb{C}$.

   e. Let f_n be integrable functions on $\mathbb{R}$ such that $\lim_{n\to\infty} \int f_n \, dx = 0$. then there exists an integrable function g such that $|f_n| \le g$ a.e. for all n.

$\cancel{\delta = \varepsilon^2 \quad |x-y| < \delta \Rightarrow |\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{|\sqrt{x}+\sqrt{y}|} < \varepsilon}$

$[1, \infty) \quad \delta = \varepsilon$

$[0, 1]$ compact set

$|\sin z| = \frac{|e^{iz} - e^{-iz}|}{2}$ let $z = -iM$, $|\sin z| = \frac{e^{M} - e^{-M}}{2}$

M

$\ge \frac{e^{M} - e}{2} > \frac{e^{M}}{2} - 1$

But $M > \frac{e^{M}}{2} - 1$

$\Rightarrow 2M + 2 > e^{M}$

$\rightarrow\leftarrow$ for $M \gg 1$