Admission to Candidacy Examination

in Real Analysis

August 20, 1984

Notation and Terminology: m = Lebesgue measure on $\mathbb{R}$. The word "measurable" applied to a set or a function means "Lebesgue measurable". All integrals are to be interpreted as Lebesgue integrals. Finally, L^p denotes the usual space of functions defined on $[0,1]$.

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1. Let X be a compact metric space, Y be a metric space, and $f: X \to Y$ be a continuous function.

   (a) Prove that if C is an open cover of X, then there exists a $\delta > 0$ such that for each $x \in X$ there is a $U \in C$ with $B(x,\delta) \subseteq U$.

   (b) Prove that f is uniformly continuous.

2. Prove that the metric space L^1 is separable.

3. Let f be a bounded measurable function defined on a measurable set E. Prove that for each $\varepsilon > 0$ there exists a simple function s on E such that

$$|f(x) - s(x)| < \varepsilon \quad \text{for each } x \in E.$$

4. Let E be a measurable set of finite measure, and let (f_n) be a sequence of real-valued measurable functions on E. Suppose $f(x) = \lim f_n(x)$ exists and is finite a.e. on E.

   (a) Prove that for given $\varepsilon > 0$ and $\delta > 0$, there exists a measurable set $A \subseteq E$ with $m(A) < \delta$ and an integer N such that

$$|f_n(x) - f(x)| < \varepsilon \text{ when } x \in E - A \text{ and } n \geq N.$$

   (b) Prove Egoroff's Theorem: Given $\delta > 0$ there exists a measurable set $A \subseteq E$ with $m(A) < \delta$ such that (f_n) converges to f uniformly on $E - A$.

5. Let (f_n) be a sequence of measurable functions on $\mathbb{R}$, and let f be an integrable function on $\mathbb{R}$. Suppose that for each n, $|f_n| \leq f$ a.e. and that for each finite interval I, $\int_I f_n \to 0$. Prove that

$\int_{\mathbb{R}} f_n \to 0$.

6. Let E be a measurable subset of $[0,1]$, and define $F(x) = m(E \cap [0,x])$ for each $x \in [0,1]$. Prove that F is differentiable a.e. and that for a.e. $x \in [0,1]$

$$F'(x) = \begin{cases} 1 \text{ if } & x \in E \\ 0 \text{ if } & x \notin E \end{cases} .$$

7. Let (f_n) be a sequence of measurable functions on a measurable set E, and let g be an integrable function on E. Suppose that for each n, $f_n \le g$ a.e. and that $f_n \to f$ a.e. Prove that

$$\overline{\lim} \int_E f_n \le \int_E f.$$

8. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $g \in L^1$ and there is a constant M such that for each simple function s

$$\left|\int_0^1 sg\right| \le M||s||_p.$$

Prove that $g \in L^q$ and $||g||_q \le M$.

9. True or False. Either prove the statement or give a counterexample.

(a) If $f: [a,b] \to \mathbb{R}$ is increasing and continuous, $E \subseteq [a,b]$ and $m(E) = 0$, then $m(f(E)) = 0$.

(b) If (f_n) is a sequence of measurable functions on $\mathbb{R}$ and $f_n \to f$ a.e. then $f_n \to f$ in measure.

(c) Given $\varepsilon > 0$ there exists a closed nowhere dense set E with $m(\mathbb{R} - E) \le \varepsilon$.