NOTES ON ANALYSIS

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1. Metric Spaces.

Definition 1. A ***metric space*** is a nonempty set E with a function $d\colon E \times E \to [0,\infty)$ such that for all $p,q,r \in E$ the following hold

(a) $d(p,q) \geq 0$,
(b) $d(p,q) = 0$ if and only if $p = q$,
(c) $d(p,q) = d(q,p)$, and
(d) $d(p,r) \leq d(p,q) + d(q,r)$. $\square$

The function d is called the ***distance function*** on E. The condition $d(p,q) = d(q,p)$ is that the distance between points is ***symmetric***. The inequality $d(p,r) \leq d(p,q) + d(q,r)$ is the ***triangle inequality***.

The most basic example if a metric space is when $E \subseteq \mathbb{R}$ is a nonempty subset of the real numbers, $\mathbb{R}$, and the distance is defined by

$$d(p,q) = |p - q|.$$

Problem 1. Show that this makes E into a metric space. $\square$

We have seen that if $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ are points in $\mathbb{R}^n$ and we define the ***length*** or ***norm*** of p to be

$$\|p\| = \sqrt{p_1^2 + \cdots + p_n^2}$$

then the inequality

$$\|p + q\| \leq \|p\| + \|q\|$$

holds.

Proposition 2. *Let E be a nonempty subset of $\mathbb{R}^n$ and for $p,q \in E$ let*

$$d(p,q) = \|p - q\|.$$

Then E with the distance function d is a metric space.

Problem 2. Prove this. $\square$

Here are some inequalities that we will be using later.

Proposition 3 (Reverse triangle inequality)**.** *Let E be a metric space with distance function d and let $x,y,z \in E$. Then*

$$|d(x,y) - d(x,z)| \leq d(y,z).$$

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Problem 3. Prove this. Then draw a picture in the case of $\mathbb{R}^2$ with its standard distance function showing why this inequality is reasonable. $\square$

Proposition 4. *Let E be a metric space with distance function d and $x_1, \ldots, x_n \in E$. Then*

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

Problem 4. Prove this. Then draw a picture in the plane showing why this is reasonable. *Hint:* Induction. $\square$

Definition 5. Let E be a metric space with distance function d. Let $a \in E$, and $r > 0$.

(a) The ***open ball*** of radius r centered at x is

$$B(a, r) := \{x : d(a, x) < r\}.$$

(b) The ***closed ball*** or radius r centered at a is

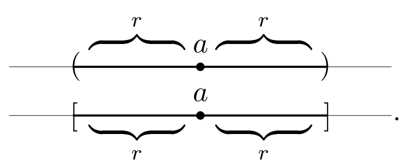
$$\overline{B}(a, r) := \{x : d(a, x) \leq r\}.$$

$\square$

In the real numbers with their usual metric $d(x, y) = |x - y|$ the open and closed balls about a are intervals with center a:

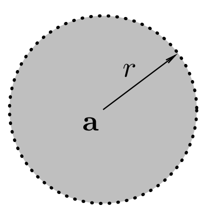
$B(a, r) = (a - r, a + r) =$

$\overline{B}(a, r) = [a - r, a + r] =$

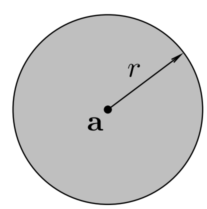


In the plane, $\mathbb{R}^2$, with its usual metric $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$, the open and closed balls about $\mathbf{a}$ are disks centered at $\mathbf{a}$.

$B(\mathbf{a}, r) =$



$\overline{B}(\mathbf{a}, r) =$



Definition 6. Let E be a metric space with distance function d. Then $S \subseteq E$ is an ***open set*** if and only if for all $x \in S$ there is an $r > 0$ such that $B(x, r) \subseteq S$. $\square$

This can be restated by saying that S is open if and only if each of its points is the center of an open ball contained in S. See Figure 1.

Proposition 7. *In any metric space E, the sets E and $\varnothing$ are open.* $\square$

Proof. Let $p \in E$, then for any $r > 0$ we have $B(p, r) = \{x \in E : d(x, p) < r\} \subseteq E$. Thus E contains not only some open ball about p, it contains every open ball about p. Therefore E is open.

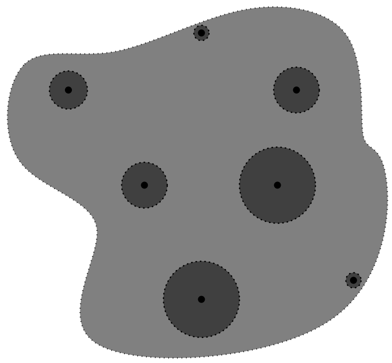


FIGURE 1. A set is open if and only if each of its points is the center of an open ball contained in the set.

That $\varnothing$ is open is a case of a vicious implication. To see this consider the statement

$$p \in \varnothing \text{ and } r > 0 \implies B(p,r) \subseteq \varnothing.$$

If this statement is true, then $\varnothing$ satisfies the definition of beiing open. But this is a true statement as an implication $P \implies Q$ is true whenever the hypotheses, P, is false. And the hypothesis "$p \in \varnothing$ and $r > 0$" is false as "$p \in \varnothing$" is false. $\square$

Proposition 8. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the open ball $B(x,r)$ is an open set.*

Problem 5. Prove this. *Hint:* Let $x \in B(a,r)$. Then $d(a,x) < r$. Set $\rho := r - d(a,x) > 0$ and show $B(x,\rho) \subseteq B(a,r)$ $\square$

Proposition 9. *In the real numbers, $\mathbb{R}$, with their standard metric, the open intervals (a,b) are open.*

Problem 6. Prove this. $\square$

Proposition 10. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the compliment, $\mathcal{C}(\overline{B}(a,r))$, of the closed ball $\overline{B}(a,r)$ is open.*

Proposition 11. *Prove this.* Hint: *If $x \in \mathcal{C}(B(a,r))$, then $d(x,a) > r$. Let $\rho := d(a,x) - r > 0$ and show $B(a,\rho) \subseteq \mathcal{C}(B(a,r))$.* $\square$

Proposition 12. *If U and V are open subsets of E, then so are $U \cup V$ and $U \cap V$.*

Proof. Let $x \in U \cup V$. Then $x \in U$ or $x \in V$. By symmetry we can assume that $x \in U$. Then, as U is open, there is an $r > 0$ such $B(x,r) \subseteq U$. But then $B(x,r) \subseteq U \subseteq U \cup V$. As x was any point of $U \cup V$ this shows that $U \cup V$ is open.

Let $x \in U \cap V$. Then $x \in U$ and $x \in V$. As $x \in U$ there is an $r_1 > 0$ such that $B(x,r_1) \subseteq U$. Likewise there is an $r_2 > 0$ such that $B(x,r_2) \subseteq V$. Let

$r = \min\{r_1, r_2\}$. Then

$$B(x,r) \subseteq B(x,r_1) \subseteq U \quad \text{and} \quad B(x,r) \subseteq B(x,r_2) \subseteq V$$

and therefore $B(x,r) \subseteq U \cap V$. As x was any point of $U \cap V$ this shows that $U \cap V$ is open. □

Proposition 13. *Let E be a metric space.*

(a) Let $\{U_i : i \in I\}$ be a (possibly infinite) collection of open subsets of E. Then the union $\bigcup_{i \in I} U_i$ is open.
(b) Let $U_1, \dots, U_n$ be a finite collection of open subsets of E. Then the intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is open.

Problem 7*.* Prove this. □

Problem 8*.* In $\mathbb{R}$ let U_n be the open set $U_n := (-1/n, 1/n)$. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not open.

Definition 14. Let E be a metric space. Then a subset S of E is ***closed*** if and only if its compliment, $\mathcal{C}(S)$ is open. □

Because the compliment of the compliment is the original set this implies that a set, S, is open if and only if its compliment $\mathcal{C}(S)$ is closed. Likewise a set, S, is closed if and only if its compliment $\mathcal{C}(S)$ is open.

Proposition 15. *In any metric space E the sets $\varnothing$ and E are both closed.*

Proof. We have seen the sets E and $\varnothing$ are open, thus their compliments $\mathcal{C}(E) = \varnothing$ and $\mathcal{C}(\varnothing) = E$ are closed. □

Proposition 16. *If E is a metric space, $a \in E$, and $r > 0$, then the closed ball $\overline{B}(a,r)$ is closed.* □

Problem 9*.* Show that in $\mathbb{R}$ with its usual metric the closed intervals are closed. □

Proposition 17. *If E is a metric space, then every finite subset of E is closed.*

Problem 10*.* Prove this. □

Problem 11*.* In the real numbers show that the half open interval $[0,1)$ is neither open or closed. □

Problem 12*.* The integers, $\mathbb{Z}$, are a metric space with the metric $d(m,n) = |m-n|$. Note that for this metric space if $m \neq n$ that $d(m,n)$ is a nonzero positive integer and thus $d(m,n) \geq 1$. Assuming these facts prove the following

(a) Let $r = 1/2$, then for each $n \in \mathbb{Z}$ the open ball $B(n,r)$ is the one element set $B(n,r) = \{n\}$ and therefore $\{n\}$ is open.
(b) Every subset of $\mathbb{Z}$ is open. *Hint:* Let $S \subseteq \mathbb{Z}$, then $S = \bigcup_{n \in S}\{n\}$ and use Proposition 13 to conclude that S is open.
(c) Every subset of $\mathbb{Z}$ is closed. □

Proposition 18. *Let E be a metric space.*

(a) Let $\{F_i : i \in I\}$ be a (possibly infinite) collection of closed subsets of E. Then the intersection $\bigcap_{i\in I} F_i$ is closed.
(b) Let $F_1, \ldots, F_n$ be a finite collection of closed subsets of E, then the union $U_1 \cup \cdots \cup U_n$ is closed.

Problem 13. Prove this. *Hint:* The correct way to do this is to deduce it directly from Proposition 13. For example to show that the union of two closed sets is closed, let F_1 and F_2 be closed. Then the compliments $\mathcal{C}(F_1)$ and $\mathcal{C}(F_1)$ are open and the intersection of two open sets is open. Therefore $\mathcal{C}(F_1) \cap \mathcal{C}(F_2)$ is open and thus the compliment of this set is closed. That is

$$F_1 \cup F_2 = \mathcal{C}(\mathcal{C}(F_1) \cap \mathcal{C}(F_2))$$

is closed. $\square$

Let E be a metric space. Then a function $f\colon E \to \mathbb{R}$ is ***Lipschitz*** if and only if there is a constant $M \geq 0$ such that

$$|f(p) - f(q)| \leq M d(p, q) \quad \text{for all } p, q \in E.$$

Proposition 19. *Let E be a metric space and $f\colon E \to \mathbb{R}$ a Lipschitz function. Then for all $c \in \mathbb{R}$ the sets*

$$\begin{aligned} f^{-1}\big[(c,\infty)\big] &= \{p \in E : f(p) < c\} \\ f^{-1}\big[(-\infty,c)\big] &= \{p \in E : f(p) > c\} \end{aligned}$$

are open and the sets

$$\begin{aligned} f^{-1}\big[[c,\infty)\big] &= \{p \in E : f(p) \geq c\} \\ f^{-1}\big[(-\infty,c]\big] &= \{p \in E : f(p) \leq c\} \end{aligned}$$

are closed.

Half of the proof. Assume that f satisfies $|f(p) - f(q)| \leq M d(p,q)$ for $p, q \in E$. We will show that $f^{-1}\big[(-\infty,c)\big]$ is open. We need to show that for any $q \in f^{-1}\big[(-\infty,c)\big]$ the set $f^{-1}\big[(-\infty,c)\big]$ contains an open ball about q. As $q \in f^{-1}\big[(-\infty,c)\big]$ we have $f(q) < c$. Therefore

$$r = \frac{c - f(q)}{M}$$

is positive. Let $pe \in B(q, r)$. Then

$$\begin{aligned}
f(p) &= f(q) + (f(p) - f(q)) \\
&\leq f(q) + |f(p) - f(q)| && (\text{as } (f(p) - f(q)) \leq |f(p) - f(q)|) \\
&\leq f(q) + Md(p, q) && (\text{as } f \text{ is Lipschitz}) \\
&< f(q) + Mr && (\text{as } p \in B(q, r), \text{ so } d(p, q) < r) \\
&= f(q) + M\left(\frac{c - f(q)}{M}\right) && (\text{from our definition of } r) \\
&= c.
\end{aligned}$$

Therefore if $p \in B(q, r)$ we have $f(p) < c$ and thus $B(q, r) \subseteq f^{-1}\big[(-\infty, c)\big]$. Whence $f^{-1}\big[(-\infty, c)\big]$ contains an open ball about any of its points q and therefore $f^{-1}\big[(-\infty, c)\big]$ is open.

We now show $f^{-1}\big[[c, \infty]\big] = \{p \in E : f(p) \geq c\}$ is closed. We know $f^{-1}\big[(-\infty, c)\big] = \{p \in E : f(p) < c\}$ is open. Its compliment is

$$\mathcal{C}\left(f^{-1}\big[(-\infty, c)\big]\right) = f^{-1}\big[[c, \infty)\big].$$

Therefore $f^{-1}\big[[c, \infty)\big]$ is the compliment of an open set, which means that $f^{-1}\big[[c, \infty)\big]$ is closed. □

Problem 14. Prove the other half of Proposition 19, that is show $f^{-1}\big[(c, \infty)\big]$ is open and $f^{-1}\big[(-\infty, c]\big]$ is closed. □

Proposition 20. *Let E be a metric space and $f\colon E \to \mathbb{R}$ a Lipschitz function. Then for any $c \in \mathbb{R}$ the set*

$$f^{-1}[c] = \{p \in E : f(p) = c\}$$

is a closed set.

Problem 15. Prove this. *Hint:* Write $f^{-1}[c]$ as the intersection of two closed sets. □

We can now give some more examples of open and closed sets in the plane, $\mathbb{R}^2$ where we are using the distance function $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$. Let $\mathbf{a} = (a_1, a_2)$ be a vector in $\mathbb{R}^2$ define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x_1, x_2) = a_1x_1 + a_2x_2 + b$$

where b is a real number. If $\mathbf{x} = (x_1, x_2)$ this can be written in vector form as

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b.$$

If $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$ then

$$\begin{aligned}
|f(\mathbf{p}) - f(\mathbf{q})| &= |\mathbf{a} \cdot \mathbf{p} + b - (\mathbf{a} \cdot \mathbf{q} + b)| \\
&= |\mathbf{a} \cdot (\mathbf{p} - \mathbf{q})| \\
&\leq \|\mathbf{a}\| \|\mathbf{p} - \mathbf{q}\| && (\text{Cauchy-Schwartz}) \\
&= Md(\mathbf{p}, \mathbf{q})
\end{aligned}$$

where $M = \|\mathbf{a}\|$. Thus f is a Lipschitz function.

Consider the case where $\mathbf{a} = (1, 0)$ and $b = 0$. Then for any $c \in \mathbb{R}$. In this case $f(x_1, x_2) = x_1$, or in slightly different notation $f(x, y) = x$. Therefore Proposition 19 implies the sets

$$\{(x, y) : x > c\}, \quad \{(x, y) : x < c\}$$

are open and that

$$\{(x, y) : x \geq c\}, \quad \{(x, y) : x \leq c\}$$

are closed.

Problem 16. Let $(a, b) \in \mathbb{R}^2$ be a nonzero vector and $c \in \mathbb{R}$.

(a) Show that the line

$$\{(x, y) \in \mathbb{R}^2 : ax + by = c\}$$

is closed.

(b) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by > c\}$$

is open (call such a half plane an ***open half plane***).

(c) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by \geq c\}$$

is closed (call such a half plane a ***closed half plane***).

(d) Show that the triangle

$$T = \{(x, y) \in \mathbb{R}^2 : x, y > 0, x + y < 0\}$$

is an open set. *Hint:* Write this as the intersection of three open half planes.

(e) Show that the triangle

$$S = \{(x, y) : x, y \geq 0, x + y \leq 0\}$$

is a closed subset of the plane. *Hint:* Write this as the interestion of three closed half planes. □

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