Blaschke's Rolling Theorem for Manifolds with Boundary

Ralph Howard
Department of Mathematics
University of South Carolina
Columbia, S.C. 29208
howard@math.scarolina.edu

Abstract

For a complete Riemannian manifold M with compact boundary ∂M denote by $\mathcal{C}_{\partial M}$ the cut locus of ∂M in M. The *rolling radius* of M is $\operatorname{Roll}(M) := \operatorname{dist}(\partial M, \mathcal{C}_{\partial M})$. (When M is a compact domain in Euclidean space this agrees with the definition given by Blaschke.) Let $\operatorname{Focal}(\partial M)$ be the focal distance of ∂M in M. When M is a strictly convex domain in Euclidean space Blaschke's rolling theorem is the equality $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$. In this note we give other conditions that imply $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$. In particular Blaschke's theorem holds if:

(1) The Ricci tensor Ric of M is non-negative and the mean curvature H of ∂M with respect to the inward normal is positive.

(2) The sectional curvature of M is non-negative and at every point of ∂M are least $(\dim M)/2$ of the principal curvatures of ∂M with respect to the inward normal are positive.

(3) M is the complement of a bounded star like domain D with Euclidean space.

Also in (1) if the condition on the mean curvature is weakened to just being non-negative there is a rigidity result: All counterexamples to Blaschke's theorem are either products $\partial M \times [0, b]$ or "generalized Möbius bands". These results extend to more general curvature conditions.