1. Compute the following integral using residues

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx$$

2. Let f and g be be analytic and nonzero-valued on the open disk $B(0,1) = \{z \in C : |z| < 1\}$ and

$$\frac{f'(\frac{1}{n})}{f(\frac{1}{n})} = \frac{g'(\frac{1}{n})}{g(\frac{1}{n})} \text{ for each } n \in \mathbb{N} \setminus \{1\}$$

Show that f is a constant multiple of g on $B(0,1)$, that is, show there exists a $k \in \mathbb{C} \setminus \{0\}$ such that, for each $z \in B(0,1)$, $f(z) = kg(z)$

3. Let A and B be nonempty subsets of a metric space (X, ρ).

Define the distance $d(A, B)$ between A and B by

$$d(A, B) = \inf\{\rho(a, b) : a \in A, b \in B\}$$

Show that if A is compact and B is closed, then $d(A, B) = 0$ if and only if $A \cap B = \emptyset$

4. Let $f : X \to Y$ where (X, d_X) and (Y, d_Y) are nonempty metric spaces.

Show that the following are equivalent.

(1) For each open subset V in Y, one has $f^{-1}(V)$ is open in X

(2) For each subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$

If you use any other characterization on continuity you must prove equivalence between that characterization and (1)

5. Let A and B be subsets of a separable metric space (D, d)

(1) Define what it means for B to be separable.

(2) Show that A is separable.

6. Let (Ω, Σ, μ) be a nonnegative finite measure space.

Let $f : \Omega \to \mathbb{R}$ be a μ-essentially bounded Σ-measurable function. Show that

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty$$

7. Let a Lebesgue measurable function $f : [0, \infty) \to \mathbb{R}$ and $c \in \mathbb{R}$ satisfy

(1) f is Lebesgue integrable over each subinterval I of $[0, \infty)$ with $\mu(I) < \infty$

(2) $\lim_{t\to\infty} f(t) = c$

Show that

$$\lim_{a\to\infty} \frac{1}{a} \int_{[0,1]} f d\mu = c$$

8. Let (Ω, Σ, μ) be a nonnegative finite measure space.

Let $f \in L_1((\Omega, \Sigma, \mu); \mathbb{R})$ and the sequence $\{f_n\}$ from L_1 satisfy

(a) $\lim_{n\to\infty} f_n = f$ almost everywhere

(b) $\lim_{n\to\infty} ||f_n||_1 = ||f||_1$

Show that

(1) $\lim_{n\to\infty} \int_E |f_n| d\mu = \int_E |f| d\mu$ for each $E \in \Sigma$

(2) $\lim_{n\to\infty} ||f - f_n||_1 = 0$

Remark: You may use Egoroff's without proof provided you state it, and define each involved mode of convergence.

9. TRUE or FALSE. Then either prove or give a counterexample.

(a) For the f from problem 8, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $E \in \Sigma$ and $\mu(E) < \delta$ then $\int_E |f| d\mu < \varepsilon$

(b) The statement from problem 6, if you omit the word finite.

(c) The statement from problem 3, replacing A is compact with A is closed

(d) Let (Ω, Σ, μ) be a nonnegative measure space and $f, f_n : \Omega \to \mathbb{R}$ for each $n \in \mathbb{N}$.

If $\{f_n\}$ is a sequence of Σ-measurable functions converging almost everywhere to f, then f is Σ-measurable.

(e) Let G be an open and connected subset of $\mathbb{C}$

If $f, g : G \to \mathbb{C}$ are analytic on G and $f(z)g(z) = 0$ for each $z \in G$, then $f \equiv 0$ or $g \equiv 0$ on G.