ANALYSIS QUALIFYING EXAMINATION AUGUST 1994.

Throughout this examination, unless otherwise specified, the terms measurable, a.e., refer to the Lebesgue measure m on the real line $\mathbb{R}$, and L^p of an interval to L^p of that interval with respect to Lebesgue measure on that interval. Integrals w.r.t. Lebesgue measure will be denoted by $\int f\,dx$. Problems one through eight are 10 points each. Problem 9 is 20 points.

closed compact

1. Let $E, K \subset \mathbb{R}^2$ with E closed and K compact with respect to the Euclidean metric d and assume that $E \cap K = \emptyset$.
   a. Let $f(x) = d(x, E) = \inf\{d(x,y) : y \in E\}$. Prove that $|f(x) - f(y)| \leq d(x,y)$.
   b. Show that there exists $\lambda > 0$ such that $d(x,y) \geq \lambda$ for all $x \in K$ and all $y \in E$.

2. Let $E \subset \mathbb{R}$ with $m^*(E) > 0$ and let $0 < \lambda < 1$. Prove that there exists $a < b$ such that $m^*(E \cap (a,b)) > \lambda(b-a)$.

3. Let f be an increasing, continuous function on $[a,b]$.
   a. Show that $f(x) = F(x) + g(x)$, where F and g are increasing, F is absolutely continuous, g is singular and both F and g are continuous.
   b. Show that $\int_a^b f'(x)\,dx = \int_a^b F'(x)\,dx$.

4. Suppose $\langle g_n \rangle$ is a sequence of measurable functions on $[0,1]$ such that $0 \leq g_n \leq$ and for $k = 0, 1, \ldots$ we have

$$\lim_{n\to\infty} \int_0^1 x^k g_n(x)\,dx = \frac{1}{2}\int_0^1 x^k\,dx = \frac{1}{2(k+1)}.$$

   Prove that

$$\lim_{n\to\infty} \int_0^1 f(x) g_n(x)\,dx = \frac{1}{2}\int_0^1 f(x)dx$$

   for all $f \in L_1([0,1])$.

5. Let $f, g \in L_1([0,1])$. Prove that $h(x,y) = f(x)g(y)$ is measurable with respect to tl product measure, $h \in L_1([0,1]\times[0,1])$ and $\iint h(x,y)\,dxdy = \left[\int f(x)\,dx\right]\left[\int g(y)\,dy\right.$

6. Let $\langle f_n \rangle$ be a sequence in $L_1(\mathbb{R})$ such that $\sum_{n=1}^\infty \|f_n\|_1 < \infty$. Prove th $\sum_{n=1}^\infty |f_n(x)| < \infty$ a.e. and if $f = \sum_{n=1}^\infty f_n$, then $f \in L_1(\mathbb{R})$.

7. Let $G \subset \mathbb{C}$ be a region and let $\langle f_n \rangle$ be a sequence of holomorphic functions on (which converges uniformly on every compact subset of G to a function f. Prove th f is holomorphic on G.

8. Let f be an entire function on $\mathbb{C}$ and assume that $|f(z)| \leq A|z|^k + B$ for so constants A, B, integer k and all $z \in \mathbb{C}$. Prove that f is a polynomial.

9. True or False. Prove, disprove or give a counterexample.

a. Let f be an entire function such that $\operatorname{Re} f(z) \geq 0$ for all z. Then f is constant.

b. $L_2(\mathbb{R}) \subset \{g + h : g \in L_1(\mathbb{R}), h \in L_\infty(\mathbb{R})\}$.

c. If $f_n \in L_1[0,1]$ and $\int_0^1 |f_n| dx \to 0$, then there exists $g \in L_1[0,1]$ such that $|f_n| \leq g$.

d. If γ is the curve in the picture below, $G = \mathbb{C} \setminus \{0,1\}$, and f is holomorphic on G, then $\int_\gamma f(z)\, dz = 0$.