Problems related to L^p spaces and/or Hölder's inequality.

Problem 1. Let (X, μ) be a measure space with $\mu(X) < \infty$. Prove

$$\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$$

for all measurable $f\colon X \to \mathbb{R}$. □

Problem 2. Let $1 < p < \infty$. For $f \in L^p(\mathbb{R})$ and $h \in \mathbb{R}$ let

$$(\tau_h f)(x) = f(x_h).$$

Prove

$$\lim_{h\to 0} \|f - \tau_h f\|_{L^p} = 0$$

for all $f \in L^p(\mathbb{R})$. □

Problem 3. Find the maximum of the function

$$f(x, y, z, w) = x - 2y + 3z - 4w$$

on the set defined by $x^4 + y^4 + z^4 + w^4 = 3$. *Hint:* This is a Hölder inequality problem in disguise. □

Problem 4 (January 1984). Let $1 < p < \infty$ and $1/p + 1/q = 1$. Let $\langle g_n \rangle_{n=1}^{\infty}$ be a sequence in $L^q([0, 1])$ such that

(a) $M = \sup_n \|g_n\|_{L^q} < \infty$, and

(b) $\displaystyle\lim_{n\to\infty} \int_E g_n\, dx = 0$ for all measurable subsets $E \subseteq [0, 1]$.

Prove for each $f \in L^p([0, 1])$ that $\displaystyle\lim_{n\to\infty} \int_0^1 f g_n\, dx = 0$. □

Problem 5 (August 1984). Let $1 < p < \infty$ and $1/p + 1/q = 1$. Let $g \in L^1([0, 1])$ and that there is a constant M such that

$$\left| \int_0^1 g(x)s(x)\, dx \right| \le M\|s\|_{L^p}$$

for all simple functions s. Prove $g \in L^q([0, 1])$ and $\|g\|_{L^q} < \infty$. □

Problem 6 (January 1985). Let (X, μ) be a measure space with $\mu(X) < \infty$. Show for $p_1 < p_2$ that $L^{p_2}(X) \subseteq L^{p_1}(X)$. Also show that if $p_1 < p_2$ there is a function $f \in L^{p_1}([0, 1])$ and $f \notin L^{p_2}([0, 1])$. □

Problem 7 (January 1990). Let $1 < p < \infty$ and $f \in L^p(\mathbb{R})$. Prove

$$\lim_{h\to 0} h^{\frac{1}{p}-1} \int_x^{x+h} f(t)\, dt = 0 \quad \text{uniformly in } x.$$ □

Problem 8 (January 1991). Let $f \in L^p((0, \infty))$ where $1 < p < 2$ and set

$$\phi(y) = \int_0^\infty f(x) \frac{\sin(xy)}{\sqrt{x}}\, dx.$$

(a) Prove $\phi(y)$ is finite all y.

(b) Prove

$$\lim_{y\to\infty} y^{\frac{1}{2}-\frac{1}{p}}\phi(y) = 0.$$

Hint: Consider the integral over $[0, M]$ and $[M, \infty)$ separately, where M is appropriately large.

Better Hint: I did not see an easy way to use the given hint to do this problem. Here is anther way to do it, which is as about as natural. We use the usual convention that $1/p + 1/q = 1$. Since $f \in L^p((0,\infty))$ it is natural to try Hölder's inequality and simplify a bit:

$$\begin{aligned}
|\phi(y)| &\le \|f\|_{L^p} \left(\int_0^\infty \frac{|\sin(xy)|^q}{x^{\frac{q}{2}}}\, dx \right)^{\frac{1}{q}} \\
&= \|f\|_{L^p} \left(\int_0^\infty \frac{|\sin(t)|^q}{\left(\frac{t}{y}\right)^{\frac{q}{2}}} \frac{dt}{y} \right)^{\frac{1}{q}} && \text{Change of variable } x = t/y \\
&= \|f\|_{L^p} y^{\frac{1}{2}-\frac{1}{q}} C_q
\end{aligned}$$

where

$$C_q = \left(\int_0^\infty \frac{|\sin(t)|^q}{t^{\frac{q}{2}}}\, dt \right)^{\frac{1}{q}}.$$

Now show that C_q is finite (and I found this easiest to do by splitting the integral as $\int_0^\infty = \int_0^1 + \int_1^\infty$, using $|\sin(t)| \le |t|$ on $[0, 1]$, and on $[1, \infty)$ using that $q > 2$ so that $q/2 > 1$.)

So we now have

$$y^{\frac{1}{2}-\frac{1}{p}}|\phi(y)| \le y^{\frac{1}{2}-\frac{1}{p}}\|f\|_{L^p} y^{\frac{1}{2}-\frac{1}{q}} C_q = C_q y^{1-\frac{1}{p}-\frac{1}{q}}\|f\|_{L^p} = C_q\|f\|_{L^p}.$$

This implies that for any $f \in L^p((0,\infty))$ that

$$\limsup_{y\to\infty} y^{\frac{1}{2}-\frac{1}{p}}|\phi(y)| \le C_q\|f\|_{L^p}.$$

Now consider the special case of $f = \mathbb{1}_{[a,b]}$ with $0 < 1 < b$. In this case

$$\phi(y) = \int_a^b \frac{\sin(xy)}{\sqrt{x}}\, dx$$

and show for this function that $\lim_{y\to\infty} y^{\frac{1}{2}-\frac{1}{p}}\phi(y) = 0$. By linearity it follows that this holds for all step functions. Finally use that the step functions are dense to complete that proof. $\square$

Problem 9 (August 1991). Let $K(x, y)$ be a measurable function on $[0, 1] \times [0, 1]$ such that for some $M > 0$

$$\int_0^1 \int_0^1 K(x, y)^2\, dx\, dy \le M.$$

Let $f \in L^2([0,1])$ and set

$$F(x) = \int_0^1 K(x,y) f(y)\, dx.$$

Show

$$\|F\|_{L^2} \leq \sqrt{M}\, \|f\|_{L^2}.$$

Problem 10. Let (X,μ) and (Y,ν) be measure spaces and $K\colon X \times Y \to \mathbb{R}$ a measurable function such that there is a constant M such that

$$\int_X |K(x,y)|\, d\mu(x) \leq M$$

for almost all $y \in Y$ and

$$\int_y |K(x,y)|\, d\nu(y) \leq M$$

for almost all $x \in X$. Show that if $1 \leq p < \infty$ and $f \in L^p(X)$, then the function

$$(Tf)(y) = \int_X K(x,y)\, d\mu(x)$$

is in $L^p(Y)$ and

$$\|Tf\|_{L^p(Y)} \leq M \|f\|_{L^p(X)}. \qquad \square$$