ANALYSIS QUALIFYING EXAMINATION

August, 2000

Questions 1 - 8 are worth ten points each and question nine is worth 20 points.

1) Suppose that $E, F \subset \mathbb{R}$ are compact. Prove that the set $E+F := \{x+y : x \in E,\ y \in F\}$ is compact.

2) Let $f_n \in L_2[a,b]$ be such that $\sum_{n=1}^{\infty}(\int_a^b |f_n|^2)^{1/2} < \infty$. Show that

(a) $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ for almost all $x \in [a,b]$.

(b) If $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then $f \in L_2[a,b]$.

(c) $(\int_a^b |f_n - f|^2)^{1/2} \to 0$.

3) Let f_n be measurable on $\mathbb{R}$, $f_n \geq 0$, and $f \in L_1(\mathbb{R})$. Prove that if $f_n \to f$ a.e. on $\mathbb{R}$, then

$$\int_{\mathbb{R}} (f - f_n)^+ \to 0, \quad \text{where} \quad x^+ := \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

4) State and prove Vitali's covering lemma.

5) Let $\langle g_n \rangle$ be a sequence of measurable functions on $\mathbb{R}$ such that there exists $M > 0$ with $|g_n| \leq M$ for $n = 1, 2, \ldots$. Suppose $\int_E g_n \to 0$ for every set $E \subset \mathbb{R}$ such that $m(E) < \infty$. Prove that for every $f \in L_1(\mathbb{R})$

$$\int_{\mathbb{R}} f g_n \to 0.$$

6) Suppose f is measurable on $[a,b] \times [c,d]$, $f(x,y) \geq 0$ on $[a,b] \times [c,d]$, and

$$\int\int_{[a,b]\times[c,d]} f(x,y)\, dx dy > b - a.$$

Show that there exists $x \in [a,b]$ such that

$$\int_c^d f(x,y)\, dy > 1.$$

7) Show that if $f \in L_p[0,1]$, $1 < p < \infty$, then $f \in L_1[0,1]$ and $\|f\|_{L_1} \leq \|f\|_{L_p}$.
Is the result true if $[0,1]$ is replaced by $\mathbb{R}$? Prove or give a counterexample.

8) (a) State and prove Liouville's theorem.
(b) Suppose f is an entire function and $|f(z)| \leq M e^{Re\, z}$ for every z from the complex plane, where M is a constant. Prove that there exists a constant C such that $f(z) = Ce^z$.

9) True or False? Prove or give a counterexample.

(a) Every uncountable set of real numbers has a non-measurable subset.

(b) For every $\varepsilon > 0$ there exists an open dense subset $\mathcal{O}$ of $[0,1]$ such that $m(\mathcal{O}) < \varepsilon$.

(c) If f_n are measurable on $[0,1]$, $f_n \geq 0$, $\int_0^1 f_n = 1$, and $f_n \to f$ a.e., then $\int_0^1 f = 1$.

(d) If f_n are measurable on $\mathbb{R}$, $0 \leq f_1 \leq f_2 \leq \ldots$, $f_n \to f$ a.e., and $\int_{\mathbb{R}} f_n \to 1$, then $f \in L_1(\mathbb{R})$.

(e) Suppose f and g are continuous on $[0,1]$, f' and g' exist a.e. on $[0,1]$, $f' = g'$ a.e., and $f(0) = g(0)$. Then $f = g$ on $[0,1]$.