Math 546 Test 2.

This is due on Monday, October 19 at midnight. You are to work alone in it. You can look up definitions and the statements of theorems we have covered in class. Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg. Please print your name on the first page of the test as this makes my book keeping easier.

We will use our usual notation for the dihedral D_n. That is it is the group generated by two elements a and b with

$$a^n = b^2 = 1, \quad ba = a^{-1}b.$$

Problem 1 (5 points). Let A, B, and C be subgroups of the group G. Give a detailed proof the intersection $A \cap B \cap C$ is a subgroup of G. $\square$

Problem 2 (5 points). Let A and B be subgroups of the group G. Show the union $A \cup B$ also a subgroup if and only if either $A \subseteq B$ or $B \subseteq A$. $\square$

Problem 3 (10 points). Let G be a group and H a subgroup of G. We have been working with the ***right cosets*** of H. That is the sets of the form

$$Hg = \{hg : h \in H\}$$

where g is an element of G. It is equally natural to look at ***left cosets*** of H, which are the sets of the form

$$gH = \{gh : h \in H\}.$$

In the group D_3 let $H = \langle b \rangle = \{1, b\}$ be the cynic subgroup generated by b. Show that the cosets aH and Ha are not equal to each other. What is $aH \cap Ha$?. $\square$

Problem 4 (10 points). Show that if $a, b \in G$ where G is a finite group, then $o(ab) = o(ba)$. *Hint:* Suppose $o(ab) = 3$. Then $(ab)^3 = e$, that is

$$(ab)(ab)(ab) = ababab = e.$$

Multiply this on the left by a^{-1} and on the right by a

$$a^{-1}(ababab)a = a^{-1}ea$$

which simplifies down to

$$bababa = e.$$

which can be rewritten as $(ba)^3 = e$. $\square$

Problem 5 (15 points). (a) Let a and b be elements of a group such that $a^m = b^m$ and $a^n = b^n$ where $\gcd(m, n) = 1$. Show $a = b$.

(b) Give an example of a group G and $a, b \in G$ with $a^2 = b^2$ and $a^4 = b^4$, but $a \neq b$. $\square$

Problem 6 (15 points). In the symmetric group S_n let H be the subset defined by

$$H = \{\sigma : \sigma(1) = 1\}.$$

(a) Show that H is a subgroup of S_n.
(b) If $\alpha, \beta \in S_n$ show that the cosets $H\alpha$ and $H\beta$ are equal if and only if $\alpha^{-1}(1) = \beta^{-1}(1)$. □

Problem 7 (10 points). Let G be a finite Abelian group and let A and B be subgroups of G with $A \cap B = \{1\}$. Prove directly that set

$$AB = \{ab : a \in A, b \in B\}$$

is a subgroup of G of order $|A||B|$. □

Definition. Let G be a group and $a, b \in G$. Say that a and b are ***friends*** if and only if there is $g \in G$ with $gag^{-1} = b$. We write this as $a \sim b$. □

Problem 8 (15 points). (a) Show that friendship is an equivalence relation. That is it is (i) *reflexive:* ($a \sim a$ for all $a \in G$), (ii) *symmetric:* ($a \sim b$ implies $b \sim a$), and (iii) *transitive:* ($a \sim b$ and $b \sim c$ implies $a \sim b$.)
(b) In D_4
  (i) The set of all elements x with $x \sim a$.
  (ii) The set of all elements x with $x \sim b$. □

Problem 9 (10 points). For $n \geq 3$ in the symmetric group S_n let σ be the three cycle $\sigma = (123)$ and τ an arbitrary element of S_n. Compute $\tau\sigma\tau^{-1}$ and give the answer in cycle notation. Show the details of your calculation. □

Problem 10 (10 points). These are harder and can be considered extra credit problems.
(a) Show that if G is a group such that $(ab)^k = a^k b^k$ for all a, b and for three consecutive values of k, then G is Abelian.
(b) If a, b are elements of a group such that

$$a^5 = 1, \quad aba^{-1} = b^2, \quad a, b \neq 1$$

then find the order of b. □