Mathematics 552 Final Homework.

This will be the last homework and will also be the last part of the final exam. It should be submitted as LaTeX output and is due via e-mail at 2:00pm on Monday, May 1 (which is when our final would have finished). Please send it using your school email address (as that is what I use to search for submissions) and use the subject line `Math 552 Final`.

The following problems are set up as much to be instructive as to do assessment. They concern some facts that are often covered in a complex variables class. The problems have enough hints that hope you can do the mathematics. What I will be most interested in is having well written arguments. In particular what I would consider an `A+` solution would be one that I could hand out to a student who has not seen the solution before and they could easily read your solution.

We have seen earlier in the course that if $f = u + iv$ is analytic, then u satisfies $u_{xx} + u_{yy} = 0$. The following shows that on simply connected domains that the converse is true.

Theorem 1. *Let D be a simply connected domain in $\mathbb{C}$ and let $u\colon D \to \mathbb{R}$ be a function that satisfies*

$$u_{xx} + u_{yy} = 0$$

*(such a function is call **harmonic**). Then u is a real part of an analytic function. That is there is a real valued function v on D such that $f(z) = u(z) + iv(x)$ is analytic.*

Problem 1. Prove this along the following lines:

(a) Let

$$g = u_x - iu_y$$

and use the Cauchy-Riemann Equations to show $g(z)$ is analytic in D.

(b) Choose an arbitrary point $z_0 \in D$ and let $f(z)$ be an antiderivative of $g(z)$ (that is $f'(z) = g(z)$) with $f(z_0) = u(z_0)$. Explain how you know such an $f(z)$ exists.

(c) Show that $f(z)$ is the function we want. *Hint:* This is a little tricky. Write

$$f = U + iV$$

where U and V are the real and imagery parts of f and our goal is to show $U = u$. We know that the derivative of f is

$$f' = U_x + iV_x,$$

and (as f is an antiderivative of g)

$$f' = g = u_x - iu_y.$$

Comparing these gives

$$U_x = u_x$$
$$V_x = -u_y.$$

Since $f = U + iV$ is analytic we have, by the Cauchy-Riemann equations,

$$U_y = -V_x$$

and therefore

$$U_y = -V_x = u_y.$$

Also $f(z_0) = u(z_0)$ which implies

$$U(z_0) = u(z_0).$$

Put these facts together to show that if

$$h = U - u$$

then

$$h_x = 0, \quad h_y = 0, \quad h(z_0) = 0$$

The first two of these conditions implies h is constant and then $h(z_0) = 0$ implies this constant is zero. Explain why this finishes the proof.

(d) Where did we use that U is simply connected? □

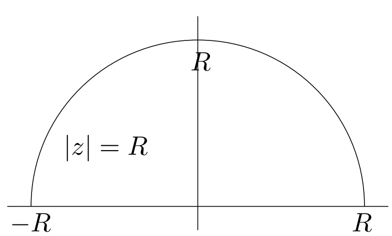
One of the most famous applications of complex analysis is to use the residue theorem to evaluate definite integrals that are hard or just about impossible to do by other means. Let us do an example (which is chosen because it illustrates several ideas): evaluate

$$\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3 + 9x}\, dx.$$

We will do this by evaluating the complex integral

(1) $$\int_{C_R} \frac{e^{2iz} - 1}{z^3 + 9z}.$$

over the curve:



which is the top half of the circle $|z| = R$ together with the part of the real axis between $x = -R$ and $x = R$.

Problem 2. Show that the function

$$f(z) = \frac{e^{2iz} - 1}{z^3 + 9z}$$

has a removable singularity at $z = 0$. *Hint:* There are several ways to do this, but what I find easiest is to first write the function as

$$f(z) = \left(\frac{e^{2iz} - 1}{z}\right)\left(\frac{1}{z^2 + 9}\right)$$

The fraction $\dfrac{1}{z^2+9}$ is clearly analytic at $z = 0$, so it is enough to show

$$g(z) = \frac{e^{2iz} - 1}{z}$$

has a removable singularity at $z = 0$. Now you should be able to show that

$$\lim_{z \to 0} g(z) = L$$

exists (and you should give the value of L). This shows that $g(z)$ is bounded near $z = 0$ and we have a theorem about functions bounded near an isolated singularity. □

Problem 3. Explain why the only singularities of

$$f(z) = \frac{e^{2iz} - 1}{z^3 + 9z}$$

are at $z = 3i$ and $z = -3i$ and that why in computing the integral in (1) only the singularity at $z = 3i$ matters. □

Problem 4. Compute the number

$$\rho_{3i} = \operatorname{Res}(f, 3i)$$

where $f(z)$ is as above. *Hint:* Your answer should be a positive real number. □

By the residue theorem we have

$$\int_{C_R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz = 2\pi i \operatorname{Res}(f, 3i) = 2\pi i \rho_{3i}. \tag{2}$$

Split the integral over C_R into two pieces, the the part on the real axis and the part on the circle, call the circular part S_R. Then

$$\int_{C_R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz = \int_{-R}^{R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz + \int_{S_R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz \tag{3}$$

Problem 5. Use the parameterization $z = x$ on the real axis to show

$$\begin{aligned}\int_{-R}^{R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz &= \int_{-R}^{R} \frac{\cos(2x) - 1 + i\sin(2x)}{x^3 + x}\, dx \\ &= \int_{-R}^{R} \frac{\cos(2x) - 1}{x^3 + x}\, dx + i\int_{-R}^{R} \frac{\sin(2x)}{x^3 + x}\, dx.\end{aligned}$$

□

Now we deal with the integral over S_R. Recall that we have proven

Proposition 2. *Let γ be a curve of length L and $f(z)$ a function defined on γ such that $|f(z)| \le M$ for some number M. Then*

$$\left| \int_\gamma f(z)\, dz \right| \le ML.$$

We will now apply this to the integral

$$\int_{S_R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz.$$

Since with have not done much with inequalities on class, I will do the estimate of $\frac{e^{2iz}-1}{z^3+9z}$ over the curve S_R. First observe that for $z = z + iy$ on S_R we have

$$\begin{aligned} |e^{2iz}| &= |e^{i(x+iy)}| \\ &= |e^{-y+ix}| \\ & |e^{-y}| && \text{(as } |e^{ix}| = 1\text{)} \\ & \le 1 && \text{(as on } S_R \text{ we have } y \ge 0 \text{ and so } e^{-y} \le 1\text{).} \end{aligned}$$

Therefore, when $R > 3$,

$$\begin{aligned} \left| \frac{e^{2iz} - 1}{z^3 + 9z} \right| &= \frac{|e^{2iz} - 1|}{|z^3 + 9z|} \\ &\le \frac{|e^{2iz}| + 1}{|z^3 + 9z|} && \text{(triangle inequality)} \\ &\le \frac{2}{|z^3 + 9z|} && \text{(using } |e^{iz}| \le 1\text{)} \\ &\le \frac{2}{|z^3| - 9|z|} && \text{(using the inequality } |a+b| \ge |a| - |b|\text{)} \\ &= \frac{2}{R^3 - 9R} && \text{(as } |z| = R \text{ on } S_R\text{).} \end{aligned}$$

Problem 6. Use Proposition 2 and the inequalities just given to show

$$\left| \int_{S_R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz \right| \le \frac{2\pi R}{R^3 - 9R} = \frac{2\pi}{R^2 - 9}$$

and then use this to show

$$\lim_{R \to \infty} \int_{S_R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz = 0.$$

Problem 7. Put together the last several problems to conclude

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\cos(2x)-1}{x^3+x}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3+x}\,dx \\
&\quad = \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{\cos(2x)-1}{x^3+x}\,dx + i\int_{-R}^{R} \frac{\sin(2x)}{x^3+x}\,dx\right) \\
&\quad = \lim_{R\to\infty}\int_{C_R} \frac{e^{2iz}-1}{z^3+9z}\,dz \\
&\quad = \lim_{R\to\infty} 2\pi i \rho_{3i} \\
&\quad = 2\pi i \rho_{3i}
\end{aligned}$$

where ρ_{3i} is the residue you computed in Problem 4. Since ρ_{3i} is a positive real number we can compare the real and imaginary parts of this to get our final result:

$$\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3+x}\,dx = 2\pi\rho_{3i}$$

$$\int_{-\infty}^{\infty} \frac{\cos(2x)-1}{x^3+x}\,dx = 0. \qquad \square$$

One of the final topics we covered was

Theorem 3 (Roché's Theorem)**.** *Let D be a bounded domain in $\mathbb{C}$ with nice boundary and let $f(z)$ and $g(z)$ be functions that are analytic on the closure, $\overline{D}$, of D. Assume*

$$|g(z)| < |f(z)|$$

on the boundary ∂D. Then $f(z)+g(z)=0$ and $f(z)=0$ have the same number of solutions in D (where the number of solutions is counted with multiplicity). $\square$

Note that since we are counting with multiplicity, the function $f(z)=z^n$ has n solutions inside the unit circle $|z|=1$ as the solution $z=0$ is counted n times.

Here is an example of using this. Let $a,b,c\in\mathbb{C}$ with

$$|a|+|b|+|c| < 1$$

then the polynomial

$$p(z) = z^3 + az^2 + bz + c$$

has three roots inside of the circle $|z|=1$. To see this let

$$f(z) = z^3$$
$$g(z) = az^2 + bz + c.$$

Let $D=\{z : |z|<1\}$. We wish to show that all three of the roots of $p(z)=f(z)+g(z)$ are in D and we will use Roché's Theorem to do this. Note that on ∂D we have $|z|=1$ and thus on ∂D

$$|f(z)| = |z|^3 = 1^3 = 1.$$

And for z on ∂D we can use the triangle inequality to conclude

$$|g(z)| = |az^2 + bz + c| \leq |a||z|^2 + |b||z| + |c| = |a| + |b| + |c| < 1 = |f(z)|.$$

Thus by Roché's $p(z) = f(z) + g(z)$ and $f(z) = z^3$ have the same number of roots in D. As $f(z) = z^3$ has three roots in D so does $p(z)$.

Here is a generalization of this example.

Proposition 4. *Let $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$ with*

$$|a_0| + |a_1| + \cdots + |a_{n-1}| < 1.$$

Then the polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

has all n of its roots inside the circle $|z| = 1$.

Problem 8*.* Prove this. $\square$

It is worth remarking that with only a little more work this argument can be used to give anther proof of the Fundamental Theorem of Algebra.