ANALYSIS QUALIFYING EXAMINATION

January 1999

Instructions:

(1) Please write your solutions on only one side of your paper.

(2) Start each problem on a separate page.

(3) There are 8 problems on this exam; each problem is worth 10 points.

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[1] Let (X,ρ) and (Y,σ) be metric spaces and $f\colon X \to Y$ be a uniformly continuous function. Show that if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X,ρ), then $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in (Y,σ).

[2] Let (X,Σ,μ) be a finite complete measure space and L_0 be the collection of all μ-measurable functions from X into $\mathbb{R}$.

[2a] Prove Egoroff's Theorem, namely: Let $\{f_n\}$ be a sequence of functions from L_0 that converge almost everywhere to $f_0 \in L_0$. Show that $f_n \to f_0$ almost uniformly.

[2b] Does the statement of [2a] remain true if (X,Σ,μ) is an arbitrary (i.e., not necessarily finite) complete measure space? Prove or give a counterexample.

[3] Let (X,Σ,μ) be a complete measure space and L_1 be the collection of all μ-integrable functions from X into $\mathbb{R}$.

[3a] Let $f \in L_1$. Show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $E \in \Sigma$ and $\mu(E) < \delta$, then

$$\int_E |f|\, d\mu < \varepsilon .$$

[3b] Recall that a subset K of L_1 is *uniformly integrable* if :
for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \in \Sigma$ and $\mu(E) < \delta$ then for each $f \in K$

$$\int_E |\, f\,|\, d\mu < \varepsilon .$$

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in L_1 that convergences in the L_1-norm. Show that the set $\{f_n\colon n \in \mathbb{N}\}$ is uniformly integrable.

[4] Let $(\mathbb{R}, \Sigma, \mu)$ be the Lebesgue measure space on $\mathbb{R}$.

[4a] Let $A \in \Sigma$ satisfy $0 < \delta < \mu(A)$. Show that if there exists $K > 0$ such that

$$A \subset [-K, K] \qquad (*)$$

then there exists $B \in \Sigma$ that satisfies $B \subset A$ and $\mu(B) = \delta$.

HINT: Consider $f : [-K, K] \to \mathbb{R}$ defined by $f(t) = \mu(A \cap [-K, t])$.

[4b] Does the statement of [4a] remain true if the condition $(*)$ is removed? Prove or give a counterexample.

[5] Let $([0,1], \Sigma, \mu)$ be the Lebesgue measure space on $[0,1]$. Let $f, g \in L_1$ be two positive functions satisfying $f(x)\, g(x) \geqslant 1$ for almost all $x \in [0,1]$. Show that

$$1 \leqslant \left(\int f \, d\mu\right) \cdot \left(\int g \, d\mu\right) .$$

[6] Let $([0,1], \Sigma, \mu)$ be the Lebesgue measure space on $[0,1]$. Let $f\colon [0,1] \to \mathbb{R}$ be a nondecreasing function. Recall the definition of the following Dini Derivate D^+ of f:

$$D^+ f(x) = \overline{\lim_{h \to 0^+}} \frac{f(x+h) - f(x)}{h} .$$

Let $\delta > 0$ and $A \subset [0,1]$ be such that $D^+ f(x) \geqslant \delta$ for each $x \in A$. Show that $\mu^*(f(A)) \geqslant \delta \mu^*(A)$.

HINT: Consider an appropriate Vitali cover of the set $\{x \in A\colon f \text{ is continuous at } x\}$.

[7] Onto Complex!

[7a] Finish this statement of Cauchy's Integral Formula:

If $f(z)$ is analytic inside and on a simple closed curve C and a is any point inside C, then

$$f(a) = \boxed{}$$

and for $n = 1, 2, 3, \ldots$

$$f^{(n)}(a) = \boxed{}$$

where C is traversed in the positive (counterclockwise) sense.

[7b] Prove Liouville's Theorem:

A bounded entire function (on the complex plane) must be constant.

[8] The Fundamental Theorem of Algebra says that if $p\colon \mathbb{C} \to \mathbb{C}$ is a polynominal of degree $n \in \mathbb{N}$, say $p(z) = \sum_{k=0}^{n} a_k \, z^k$ where $a_k \in \mathbb{C}$ and $a_n \neq 0$, then $w = p(z)$ has exactly n complex roots, counting multiplicity. Prove the Fundamental Theorem of Algebra using Liouville's Theorem.