Some problems on metric spaces.

I am assuming you know the definition of a ***metric space***, ***open and closed sets***, the ***closure*** and ***interior*** of a set, what it means for a function to be ***continuous*** or ***uniformly continuous***, and the definitions of ***compactness***, ***connectedness***, and ***completeness***.

Problem 1. This is a good exercise in working with some of the definitions. Let (X, d) and (Y, ρ) be metric spaces and $A \subseteq X$. Let $f \colon A \to Y$ be a continuous function.

(a) Show that if f is uniformly continuous on A and Y is complete, then it has a continuous extension $\widehat{f} \colon \overline{A} \to Y$.

(b) Show that this is false if either the uniform continuity condition on f or the completeness condition on Y is dropped. $\square$

Problem 2. Let $f \colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Show there exists constants A and B such that $|f(x)| \leq A + B|X|$. Use this to explain why no polynomial of degree ≥ 2 is uniformly continuous on $\mathbb{R}$. $\square$

Problem 3. Let $A \subseteq \mathbb{R}$ be a subgroup of the additive group of $\mathbb{R}$ such that A is a closed subset of $\mathbb{R}$. Show that either $A = \mathbb{R}$ or A is cyclic (that is for some $a \in \mathbb{R}$ we have $A = \{na : n \in \mathbb{Z}\}$. $\square$

And here are some problems from the old analysis exams that are worth looking at:

- January 2019, Problem 1.
- August 2018, Problem 1.
- August 2017, Problem 4.
- January 2017, Problem 1.
- August 2016, Problem 1.
- August 2012, Problem 1. (In the problems notation K_ε^c is the compliment of K_ε in X.)