

Absolutely continuity and bounded variation.

We have talked a bit about *absolutely continuous* functions and functions of *bounded variation*. I will not write out the definitions here.

We start with some easy observations about functions of bounded variation. Let $V_a^b(f)$ be the *total variation* of f on $[a, b]$. That is

$$V_a^b(f) = \sup_{a=t_0 \leq t_1 \leq \dots \leq t_{n-1} \leq t_n=b} \sum_{j=1}^n |f(t_{j-1}) - f(t_j)|$$

where the supremum is over all partitions of $[a, b]$. Using this definition it is not hard to show that if $a < b < c$ and f is continuous at b that

$$V_a^b(f) + V_b^c(f) = V_a^c(f).$$

That is V is additive over intervals. Also if f is monotone increasing on $[a, b]$ then then for any partition $|f(t_j) - f(t_{j-1})| = f(t_j) - f(t_{j-1})$ and therefore

$$\sum_{j=1}^n |f(t_{j-1}) - f(t_j)| = \sum_{j=1}^n (f(t_{j-1}) - f(t_j)) = f(b) - f(a)$$

as the series is telescoping. Thus in this case

$$V_a^b(f) = f(b) - f(a).$$

Note that f does not have to be continuous for this to hold. Likewise if f is monotone decreasing

$$V_a^b(f) = f(a) - f(b) = |f(b) - f(a)|.$$

Now assume that f is piecewise monotone on $[a, b]$ with the changes of monotonicity at $a < t_1 < t_2 < \dots < t_{n-1} < b$ and set $t_0 = a$, $t_n = b$, see Figure 1.

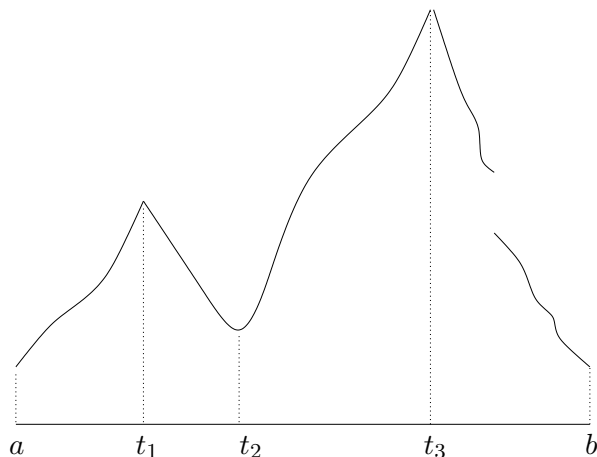


FIGURE 1.

Problem 1. For a function as in Figure 1 assume that f is continuous at t_1, \dots, t_{n-1} . Show

$$V_a^b(f) = \sum_{j=1}^n |f(t_j) - f(t_{j-1})|.$$

What, if anything, goes wrong if f is not continuous at one of the t_j 's? \square

The big theorem for absolutely continuous functions is that $f \in \text{AC}([a, b])$ if and only if f' exists almost everywhere on $[a, b]$ with $f' \in L^1[a, b]$ and for all $x \in [a, b]$

$$f(x) - f(a) = \int_a^x f'(t) dt.$$

Problem 2. Assume that $f \in \text{AC}([a, b])$ and that f is monotone on $[a, b]$. Show

$$\int_a^b |f'(t)| dt = |f(b) - f(a)| = V_a^b(f).$$

Now use that $V_a^b(f)$ is additive on intervals to show that if $f \in \text{AC}([a, b])$ and is piecewise monotone in the sense of Figure 1 that

$$(1) \quad \int_a^b |f'(t)| dt = V_a^b(f). \quad \square$$

We would like Equation 1 to hold for all $f \in \text{AC}([a, b])$. Here is a start on this.

Problem 3. Let $f \in \text{AC}([a, b])$ and $\varepsilon > 0$. Then $f' \in L^1([a, b])$ and therefore there is a step function ϕ on $[a, b]$ with

$$\int_a^b |f'(t) - \phi(t)| dt < \varepsilon.$$

Define g on $[a, b]$ by

$$g(x) = f(a) + \int_a^x \phi(t) dt.$$

(a) Show that g is piecewise linear and continuous. Thus g is piecewise monotone and continuous and thus

$$V_a^b(g) = \int_a^b |g'(t)| dt = \int_a^b |\phi(t)| dt.$$

(b) Let $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ be a partition of $[a, b]$. Then show

$$\begin{aligned} \left| |f(t_j) - f(t_{j-1})| - |g(t_j) - g(t_{j-1})| \right| &\leq \left| (f(t_j) - f(t_{j-1})) - (g(t_j) - g(t_{j-1})) \right| \\ &= \left| \int_{t_{j-1}}^{t_j} (f'(t) - \phi(t)) dt \right| \\ &\leq \int_{t_{j-1}}^{t_j} |f'(t) - \phi(t)| dt \end{aligned}$$

and use this to show

$$|V_a^b(f) - V_a^b(g)| \leq \int_a^b |f'(t) - \phi(t)| dt < \varepsilon. \quad \square$$

Problem 4. Let $f \in \text{AC}([a, b])$ and let ϕ_n be a step function on $[a, b]$ with

$$\int_a^b |f'(t) - \phi_n(t)| dt \leq \frac{1}{n}.$$

Let

$$g_n(x) = f(a) + \int_a^x \phi_n(t) dt.$$

(a) Show for $x \in [a, b]$

$$|f(x) - g_n(x)| \leq \frac{(b-a)}{n}$$

and therefore $g_n \rightarrow f$ uniformly on $[a, b]$.

(b) Show

$$\lim_{n \rightarrow \infty} V_a^b(g_n) = V_a^b(f).$$

(c) Show

$$\lim_{n \rightarrow \infty} \int_a^b |g'_n(t)| dt = \int_a^b |\phi_n(t)| dt = \int_a^b |f'(t)| dt. \quad \square.$$

Theorem 1. Let $f \in \text{AC}([a, b])$ then the total variation of f is

$$V_a^b(f) = \int_a^b |f'(t)| dt.$$

Problem 5. Prove the last theorem. \square

The next problem is has appeared in some form or another on at least one of the qualifying exams.

Problem 6. Let $f \in \text{BV}([a, b])$ and assume that

- (a) The function $x \mapsto V_x^b(f)$ is continuous at $x = 0$,
- (b) for each $x \in (a, b]$ the restriction $f|_{[x, b]}$ is in $\text{AC}([x, b])$.

Then $f \in \text{AC}([a, b])$. \square

Now standard example. For $\alpha, \beta > 0$ let $f_{\alpha, \beta}$ be defined on $[0, 1]$ by

$$f_{\alpha, \beta}(x) = \begin{cases} x^\alpha \sin(1/x^\beta), & x > 0; \\ 0, & x = 0. \end{cases}$$

Problem 7. Use the Theorem above to show that $f_{\alpha, \beta} \in \text{AC}([0, 1])$ if and only if $f_{\alpha, \beta} \in \text{BV}([0, 1])$. \square

Problem 8. For what α, β are the functions $f_{\alpha, \beta}$ of bounded variation? \square

Problem 9. Show that $f_{\alpha, \beta}$ is differentiable at $x = 0$ if and only if $\alpha > 1$. \square