Analysis Qualifying Exam
August 2005

Instructions: Write your name legibly on each sheet of paper. Write only on one side of each sheet of paper. Try to answer all questions. Prove all your claims. Questions 1-8 are worth 10 points each and question 9 is worth 20 points.

Terminology: Measurability and integrability on $\mathbb{R}$ or (Lebesgue) measurable subsets of $\mathbb{R}$ will always refer to the Lebesgue measure except if otherwise specified. Lebesgue measure will be denoted by m, dx or dt depending on the context. If Ω is a measurable subset of $\mathbb{R}$ and $1 \leq p < \infty$ then recall that

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} | f \text{ is measurable and } \|f\|_p < \infty\}$$

where $\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{1/p}$. Also

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} | f \text{ is measurable and there exists } C < \infty \text{ such that } |f(x)| \leq C \text{ for almost all } x \in \Omega\}$$

and the infimum of such C's is denoted by $\|f\|_\infty$.

1. Let A be a closed subset of $\mathbb{R}$ and B be a compact subset of $\mathbb{R}$ such that $A \cap B = \emptyset$. Prove that there exist $a \in A$ and $b \in B$ such that $|a - b| = \inf\{|a' - b'| : a' \in A \text{ and } b' \in B\}$. Give an example of closed disjoint sets A and B which do not satisfy the conclusion. Also examine whether the validity of the conclusion for two disjoint closed subsets of $\mathbb{R}$, implies that at least one of them must be compact.
2. Let Ω be a measurable subset of $\mathbb{R}$ (of finite or infinite measure) and $1 \leq p < q < r \leq \infty$. Prove that if $f \in L^p(\Omega) \cap L^r(\Omega)$ then $f \in L^q(\Omega)$.
3. Let Ω be a measurable subset of $\mathbb{R}$ and let $f_n, f \in L^2(\Omega)$ for all $n \in \mathbb{N}$, such that $f_n \to f$ pointwise a.e. in Ω as $n \to \infty$, and $\|f_n\|_2 \to \|f\|_2$ as $n \to \infty$. Show that $\|f_n - f\|_2 \to 0$ as $n \to \infty$.
4. Let $f \in L^1([0,1])$ such that $f(x) > 0$ for almost all $x \in [0,1]$. Prove that for every $\varepsilon > 0$, $\inf\{\int_\Omega f(x)dx : m(\Omega) \geq \varepsilon\} > 0$.
5. Prove that if $f : \mathbb{R} \to (0, \infty)$ is a measurable function then
$$\int_{\mathbb{R}} f(x)^2 dx = 2\int_0^\infty t m(\{x : f(x) > t\}) dt.$$
6. (a) Suppose that $f : [0,1] \to \mathbb{R}$ is continuous at 0 and that there exists $\varepsilon > 0$ such that $T_0^\delta(f) > \varepsilon$ for all $\delta > 0$ (recall that $T_a^b(f)$ denotes the total variation of f from a to b). Prove that f is not of bounded variation on $[0,1]$.
   (b) Let $f : [0,1] \to \mathbb{R}$ such that f is continuous at 0, f is of bounded variation on $[0,1]$, and f is absolutely continuous on $[\delta, 1]$ for any $0 < \delta < 1$. Prove that f is absolutely continuous on $[0,1]$.
7. Let C be the circle $|z| = 1$ traversed once counterclockwise. For $a > 1$, show that
$$\int_C \frac{dz}{z^2 + 2az + 1} = \frac{\pi i}{\sqrt{a^2 - 1}}.$$
8. Suppose that $f(z)$ is an entire function which satisfies $|f(z)| \leq C(1 + |z|^N)$, where $C > 0$ and $N \in \mathbb{N}$. Prove that $f(z)$ is a polynomial of degree at most N.
9. True or False. Prove or disprove, whichever is appropriate, in order to obtain credit.

a. If two continuous real-valued functions defined on $\mathbb{R}$ agree everywhere on the complement of a set of measure zero, then they agree everywhere.
b. There is no sequence $(a_{m,n})_{m,n\in\mathbb{N}}$ of real numbers such that $\sum_{n=1}^{\infty} a_{m,n} = 1$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} a_{m,n} = -1$ for all $n \in \mathbb{N}$.
c. Every absolutely continuous function $f : [0,1] \to \mathbb{R}$ is Lipschitz.
d. If $u + iv$ is analytic (where u and v are real-valued) then uv is harmonic.
e. Suppose that $f(z)$ is analytic inside and on a simple closed contour C (traversed once counterclockwise) which contains z_0 in its interior. Then

$$\int_C \frac{f''(z)}{(z-z_0)^2}\, dz = 6\int_C \frac{f(z)}{(z-z_0)^4}\, dz.$$