QUALIFYING EXAM IN ANALYSIS

(AUGUST 2001)

Name :
S.S. # :

Throughout this examination the term measurable refers to the Lebesgue measure m on the real line. Integrals with respect to Lebesgue measure will be denoted by $\int f$. Problems are 10 points each.

1. Let $A \subset (0,1)$ be a Lebesgue measurable set such that for some $0 \leq q < 1$ and any interval I we have $m(A \cap I) \leq qm(I)$. Prove that then $m(A) = 0$.

3. Assume that for a sequence of measurable functions $\{f_n\}$ we have

$$\lim_{n\to\infty} \int_0^1 \frac{|f_n|}{1+|f_n|} = 0.$$

Prove that $\{f_n\}$ converges to 0 in measure.

4. Let A be a measurable subset of $[0,1]$ and $mA = a > 0$. Prove that for any $0 \leq b < a$ there exists a closed set $B \subset A$ such that $mB = b$.

5. Let f be increasing on $[0,1]$ and

$$\int_0^1 f' = f(1) - f(0).$$

Prove that f is absolutely continuous on $[0,1]$.

6. Compare the following four types of convergence of measurable functions on $[0,1]$:

a). $\{f_n\}$ converges to f almost everywhere;
b). $\{f_n\}$ converges to f in measure;
c). $\{f_n\}$ converges to f in the L_1-norm;
d). $\{f_n\}$ converges to f in the L_2-norm.

Give an answer in the form: i). $\Rightarrow$ j). (explain) or i). $\not\Rightarrow$ j). (provide a counterexample).

7. Let $p \geq 3$. Prove that if f_n converges to f in L_p then f_n^3 converges to f^3 in $L_{p/3}$.

8. Let F be a bounded linear functional on $L_p(0,1)$, $1 \leq p < \infty$. Prove that a function $\Phi(s) := F(\chi_{[0,s]})$, $\chi_{[0,s]}$ is a characteristic function of the interval $[0,s]$, $0 \leq s \leq 1$, is absolutely continuous.

9. Prove Liouville's Theorem: A bounded entire function on $\mathbb{C}$ is a constant.

10. Evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^2}$$

by contour integration.