

Mathematics 554H/701I Homework

We now start come the the last big topic we will cover this term, which is continuous maps between metric spaces.

Definition 1. Let E and E' be metric spaces and $f: E \rightarrow E'$ a function from E to E' . Let $p_0 \in E$. Then f is **continuous** at p_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for $p \in E$

$$d(p, p_0) < \delta \quad \text{implies} \quad d(f(p), f(p_0)) < \varepsilon.$$

□

Example 2. Here is an example of showing something is continuous. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function

$$f(x) = 3x + 5$$

Then f is continuous at every point of \mathbb{R} . To see this let $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$. Let $\delta = \varepsilon/3$. Then if $|x - x_0| < \delta$ we have

$$\begin{aligned} |f(x) - f(x_0)| &= |3x + 5 - (3x_0 + 5)| \\ &= |3(x - x_0)| \\ &= 3|x - x_0| \\ &< 3\delta \\ &= \varepsilon. \end{aligned}$$

Proposition 3. Let E be a metric space and $f: E \rightarrow E$ the identity map, that is $f(p) = p$ for all $p \in E$. Then f is continuous.

Problem 1. Prove this. □

Problem 2. Let E be a metric space.

(a) Let $p, x_0, q \in E$ show that

$$|d(q, x_0) - d(p, x_0)| \leq d(p, q).$$

(b) Let $x_0 \in E$ and define $f(p)$ to be the distance of p from x_0 , that is $f(p) = d(p, x_0)$. Show that f is continuous at all points of E . *Hint:* Use part (a) to show $|f(p) - f(q)| \leq d(p, q)$. □

Recall that on \mathbb{R}^n we have defined the inner product

$$a \cdot b = \sum_{j=1}^n a_j b_j$$

where $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$. This was used to define the norm on \mathbb{R}^n as

$$\|a\| = \sqrt{a \cdot a}.$$

This in turn was used to define the distance function on \mathbb{R}^n by

$$d(p, q) = \|p - q\|.$$

Also recall that we have the Cauchy-Schwartz inequality:

$$|a \cdot b| \leq \|a\| \|b\|.$$

Problem 3. Let $a \in \mathbb{R}^n$ and define the function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$f(x) = a \cdot x.$$

Show that f is continuous at all points of \mathbb{R}^n . *Hint:* Let $x, x_0 \in \mathbb{R}^n$ then

$$f(x) - f(x_0) = a \cdot x - a \cdot x_0 = a(x - x_0).$$

Now use the Cauchy-Schwartz inequality to show $|f(x) - f(x_0)| \leq \|a\| \|x - x_0\|$. \square

Problem 4. Define the functions $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x, y) = x$ and $g(x, y) = y$. Show that f and g are continuous. *Hint:* As the two proofs are the same, it is enough to show that f is continuous. Let $a = (1, 0)$, the $f(x, y) = (x, y) \cdot a$ so one way to do this is to reduce it to the previous problem. \square

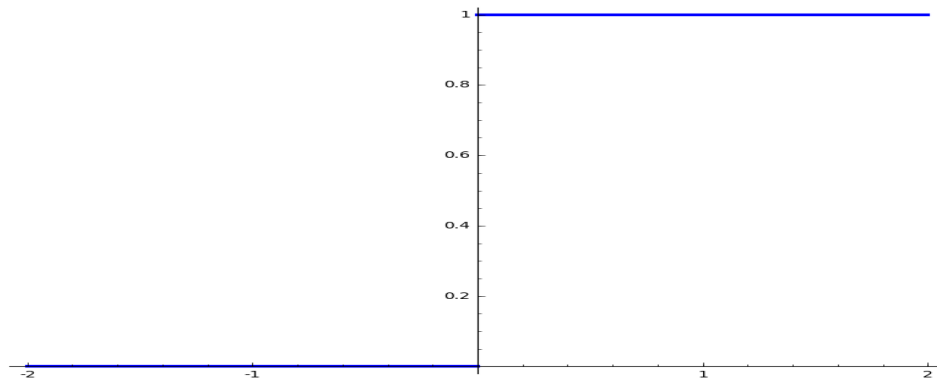
We now give examples of some functions that are not continuous. We first record what it means for a function to not be continuous at a point.

Negation of Definition of Continuity. 4. Let $f: E \rightarrow E'$ be a map between metric spaces. Let $p_0 \in E$. Then f is **discontinuous** at p_0 if and only if there is a $\varepsilon > 0$ such that for all $\delta > 0$ there is a $p \in E$ with $d(p, p_0) < \delta$ and $d'(f(p), f(p_0)) \geq \varepsilon$. \square

We now look at the function

$$f(x) = \begin{cases} 0, & x \leq 0; \\ 1, & 0 < x. \end{cases}$$

which has the graph:



We now show this is discontinuous at $x = 0$. Let $\varepsilon = 1/2$. Then for any $\delta > 0$ there is an $x > 0$ with $0 < x < \delta$. Then $x > 0$ and so $f(x) = 1$. As $f(0) = 0$ we have $|f(x) - f(0)| = |1 - 0| = 1 > \varepsilon$ as required.

Here is a more exotic example.

Problem 5. Define a function by

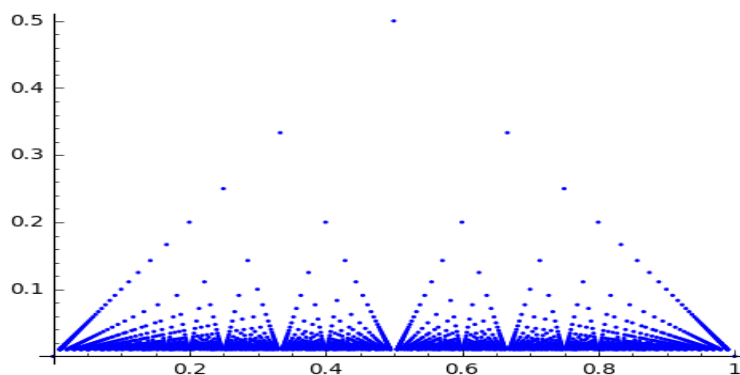
$$f(x) = \begin{cases} 0, & x \in \mathbb{Q}; \\ 1, & x \notin \mathbb{Q}. \end{cases}$$

That is $f(x)$ is one with x is a rational number, and $f(x)$ is zero when x is irrational. Show that f is discontinuous at all points of \mathbb{R} .

Problem 6 (Optional). Define a function $f: \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \text{ is rational in lowest terms;} \\ 0, & x \text{ is irrational.} \end{cases}$$

Here is the graph for rationals in $(0, 1)$ with denominators less than 100.



Show that f is continuous at all irrational points and discontinuous at all rational points. \square

Problem 7. Let $f: (0, \infty) \rightarrow \mathbb{R}$ be defined by

$$f(x) = \sqrt{x}$$

then show f is continuous at $x = 1$.

Solution: We first note that

$$f(x) - f(1) = \sqrt{x} - 1 = \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(\sqrt{x} + 1)} = \frac{x - 1}{\sqrt{x} + 1}.$$

Now let $\varepsilon > 0$ and let

$$\delta = \min \left\{ \frac{3}{4}, \frac{3\varepsilon}{2} \right\}.$$

Then $|x - 1| < \delta$ implies $|x - 1| < \frac{3}{4}$ and therefore

$$\frac{1}{4} = 1 - \frac{3}{4} < x$$

Therefore if $|x - 1| < \delta$ we have

$$\frac{1}{\sqrt{x} + 1} < \frac{1}{\sqrt{1/4} + 1} = \frac{1}{1/2 + 1} = \frac{2}{3}.$$

Thus, using the formula above for $f(x) - f(1)$ we have that if $|x - 1| < \delta$ then

$$\begin{aligned} |f(x) - f(1)| &= \left| \frac{x-1}{\sqrt{x}+1} \right| \\ &= \frac{1}{\sqrt{x}+1} |x-1| \\ &< \frac{2}{3} |x-1| \\ &< \frac{2}{3} \frac{3\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

□

Problem 8. Let $f: (0, \infty) \rightarrow \mathbb{R}$ be defined by

$$f(x) = \sqrt{x}.$$

Show f is continuous at $x = a$ for any $a > 0$.

□

Theorem 5. Let E be a metric space and $f, g: E \rightarrow \mathbb{R}$ be functions and $c_1, c_2 \in \mathbb{R}$ constants. Assume f and g are continuous at p_0 . Then the functions $c_1f + c_2g$ and fg are continuous at p_0 . If $g(p_0) \neq 0$ then the quotient $\frac{f}{g}$ is continuous at p_0 .

□

Proof. See the class notes or the text.

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□

Proposition 6. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Then f is continuous at all points of \mathbb{R} .

Problem 9. Prove this. *Hint:* Probably the easiest way is by induction on n . The base of the induction is $n = 0$ in which case $f(x) = a_0$ is a constant which is clearly continuous. Or we can use the base case of $n = 1$ in which case $f(x) = a_1 x + a_0$ is linear and we have done this in class.

Here is what the induction step from $n = 4$ to $n = 5$ looks like. Assume that we know that all polynomials of degree 4 are continuous and let

$$f(x) = a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

be a polynomial of degree 5. Write it as

$$\begin{aligned} f(x) &= x(a_5 x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_1) + a_0 \\ &= xg(x) + a_0 \end{aligned}$$

where $g(x) = a_5 x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_1$ is a polynomial of degree 4. By the induction hypothesis $g(x)$ is continuous and the function x is continuous. Therefore f is of the form

$$f = (\text{continuous function}) \times (\text{continuous function}) + (\text{constant})$$

and therefore f is continuous. Use this idea to do the general induction step. \square