Admission to Candidacy Examination

in

Real Analysis

January 1988

Instructions: Answer all questions. Question # 1 is worth 20 points; the remaining problems are worth 10 points each.

Terminology: Unless otherwise specified, the terms measurable, a.e., refer to Lebesgue measure λ on the real line $\mathbb{R}$, and L^p of an interval with respect to Lebesgue measure on that interval.

1. Prove or provide a counterexample for each of the following.
   (a). If f is monotone on $[a,b]$ and f' exists a.e., then $f' \in L^1([a,b])$.
   (b). If f is monotone on $[a,b]$ and $f'(x) = 0$ a.e. on (a,b), then f is a constant on $[a,b]$.
   (c). Let μ and ν be measures on a measurable space $(X, \quad)$. If $\mu \ll \nu$, then there exists $f \in L^1(\nu)$ such that $\mu(E) = \int_E f\,d\nu$, for all $E \in \quad$.
   (d). If $\{f_n\}$ is a sequence in $L^1(X,\mu)$ and $\int |f_n|\,d\mu \to 0$ as $n \to \infty$, then $f_n \to 0$ μ-a.e. .

2. Suppose $g \in L^1([0,1])$, $g \geq 0$, and $f_n \to f$ in measure. If $|f_n| \leq g$, prove that $f \in L^1([0,1])$ and that

$$\lim_{n\to\infty} \int_0^1 f_n\,d\lambda = \int_0^1 f\,d\lambda .$$

3. Suppose $g(t)$ and $tg(t)$ are in $L^1((0,\infty))$. For x real, define $f(x)$ by

$$f(x) = \int_0^\infty g(t)\sin(xt)\,dt .$$

Prove that f is differentiable on $\mathbb{R}$, and that

$$f'(x) = \int_0^\infty tg(t)\cos(xt)\,dt .$$

4. Let $(X, \ , \mu)$ be a finite measure space and $\{f_n\}$ a sequence in $L^1(X, \ , \mu)$. If f is an -measurable function on X which is finite μ-a.e. with $f_n \to f$ μ-a.e., prove that

$$f \in L^1(X, \ , \mu) \text{ and } \lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$$

if and only if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\int_E |f_n| d\mu < \varepsilon$ for all $E \in$ with $\mu(E) < \delta$.

5. Suppose f is a measurable function on $[0,1]$. The distribution function for f is defined by $\mu_f(t) = \lambda(\{x: |f(x)| > t\})$. Suppose Φ is a nonnegative, absolutely continuous monotone increasing function on $[0,\infty)$ with $\Phi(0) = 0$. Prove that

$$\int_0^1 \Phi(|f(x)|)dx = \int_0^\infty \Phi'(t)\mu_f(t)dt \ .$$

6. Let $\mu \neq 0$ be a regular Borel measure on $[0,1]$ such that

$$\int_0^1 f\, g\, d\mu = \int_0^1 f\, d\mu \int_0^1 g\, d\mu$$

for all continuous functions f and g on $[0,1]$. Prove that there exist $a \in [0,1]$ such that $\int_0^1 f\, d\mu = f(a)$ for all continuous functions f on $[0,1]$.

7. Let μ and ν be measures on a measurable space $(X, \ )$. Suppose μ is σ-finite and $\nu \ll \mu$ with ν finite. If f_n, $n = 1,2,\ldots$ and f are measurable functions on X with $f_n \to f$ in measure $[\mu]$, prove that $f_n \to f$ in measure $[\nu]$.

8. Let μ be a positive measure on X. Let $K : X \times X \to [0,\infty)$ and $g : X \to (0,\infty)$ be measurable functions, $1 < p < \infty$ and $1/p + 1/q = 1$. Suppose there exist constants A and B so that

$$\int_X K(x,y)g(y)^q d\mu(y) \le [A\, g(x)]^q \quad \text{and} \quad \int_X K(x,y)g(x)^p d\mu(x) \le [B\, g(y)]^p.$$

Prove that T defined by

$$Tf(x) = \int_X K(x,y)f(y)d\mu(y)$$

is a bounded operator on $L^p(X,\mu)$ satisfying $||Tf||_p \le AB||f||_p$ for all $f \in L^p$.

9. Suppose $f \in L^p(\mathbb{R})$, $1 \le p < \infty$. Define the L^p modulus of continuity of f by

$$\omega_p(f,t) = \sup_{|h|\le t} \left[\int_R |f(x+h) - f(x)|^p dx \right]^{1/p}.$$

Prove that $\lim_{t\to 0^+} \omega_p(f,t) = 0$.

$$|Tf(x)|^p = \left| \int_X K(x,y)^{\frac{1}{p}} K(x,y)^{\frac{1}{q}} \frac{f(y)}{g(y)} g(y)\, d\mu(y) \right|^p$$

$$\le \left(\int_X K(x,y)\, g(y)^q d\mu\right)^{\frac{p}{q}} \cdot \left(\int_X K(x,y) \frac{f(y)^p}{g(y)^p} d\mu(y)\right)$$

$$\le (A\, g(x))^p \int_X K(x,y) \frac{f(y)^p}{g(y)^p} d\mu(y)$$

$$\therefore \int_X |Tf(x)|^p d\mu(x) \le \int_X \left(\int_X A^p K(x,y)(g(x))^p d\mu(x)\right) \frac{f(y)^p}{g(y)^p} d\mu(y)$$

$$\le \int_X A^p \cdot B^p g(y)^p \cdot \frac{f(y)^p}{g(y)^p} d\mu(y)$$

$$\le A^p B^p \|f\|_p^p$$