# ANALYSIS QUALIFYING EXAM
## AUGUST 2004

**DIRECTIONS**: Attempt all questions. Answer each question on a separate sheet. Questions 1-8 are worth 10 points each and question 9 is worth 20 points.

**NOTATION**: $m$ denotes Lebesgue measure on the real line $\mathbb{R}$.

1. (a) Suppose that $A \subset \mathbb{R}$ is compact and $B \subseteq \mathbb{R}$ is closed. Prove that $A + B = \{a + b \colon a \in A, b \in B\}$ is closed.
(b) Give an example of closed sets $A$ and $B$ such that $A + B$ is not closed.

2. (a) Define the *outer measure* $m^*(A)$ of $A \subseteq \mathbb{R}$.
(b) Suppose that $(A_n)_{n=1}^{\infty}$ is an increasing sequence of subsets of $\mathbb{R}$. Prove that there exists an increasing sequence $(G_n)_{n=1}^{\infty}$ of $G_\delta$-sets such that $A_n \subseteq G_n$ and $m^*(A_n) = m(G_n)$ for each $n \geq 1$. (Recall that a $G_\delta$ set is a countable intersection of open sets.)
(c) Deduce from (b) that $m^*(\cup_{n=1}^{\infty} A_n) = \lim m^*(A_n)$.

3. (a) State Fatou's Lemma for an arbitrary measure space $(X, \mathcal{M}, \mu)$.
(b) Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $(f_n)$ is a sequence of non-negative integrable functions which converges pointwise to an integrable function $f$. Suppose also that

$$\int_X f_n \, d\mu \to \int_X f \, d\mu < \infty.$$

Prove that

$$\int_E f_n \, d\mu \to \int_E f \, d\mu$$

for all $E \in \mathcal{M}$.

4. Supppose that $1 < p, q < \infty$, that $1/p + 1/q = 1$, and that $f \in L_p(\mathbb{R}) \cap L_q(\mathbb{R})$. Prove that

$$\int_{-\infty}^{\infty} f(x + y) f(x) \, dx \to 0$$

as $|y| \to \infty$.

5. (a) What is an *absolutely continuous* function from $[a, b]$ to $\mathbb{R}$.
(b) Suppose that $g : [a, b] \to [c, d]$ is *monotone increasing* and absolutely continuous and that $f : [c, d] \to \mathbb{R}$ is absolutely continuous. Prove that $f \circ g$ is absolutely continuous.

(c) Now suppose, in addition to the above, that $f$ is differentiable *everywhere*. Deduce that

$$f(g(x)) = f(g(a)) + \int_a^x f'(g(t))g'(t)\,dt \qquad (a \le x \le b).$$

6. Suppose that $E$ is an $m \otimes m$-measurable subset of $[0,1] \times [0,1]$ such that $m(\{x \in [0,1] \colon m(E_x) \ge 4/5\}) \ge 3/4$. Prove that $(m \otimes m)(E) \ge 3/5$ and deduce that $m(\{y \in [0,1] \colon m(E^y) \ge 1/2\}) \ge 1/5$. (Here $E_x = \{y \colon (x,y) \in E\}$ and $E^y = \{x \colon (x,y) \in E\}$.)

7. Suppose that $f(z)$ is analytic on a convex domain $U$ and that $\gamma(t)$ $(0 \le t \le 1)$ is a positively-oriented simple closed smooth parametrized curve contained in $U$.
(a) Write down an integral on [0,1] for the *length* $L(\gamma)$ of $\gamma$.
(b) Write down Cauchy's Integral Formula for $f(z)$ for a point $z$ inside $\gamma^*$.
(c) Deduce that

$$|f(z)| \le \frac{ML(\gamma)}{2\pi\,\mathrm{dist}(z,\gamma^*)},$$

where $M$ denotes the maximum value of $|f(z)|$ on $\gamma^*$ and $\mathrm{dist}(z,\gamma^*)$ denotes the distance from $z$ to $\gamma^*$.

8. (a) Suppose that $f : \Delta \to \Delta$ is an analytic mapping from the open unit disk $\Delta$ to itself such that $f(0) = 0$. By considering $f(z)/z$, show that $|f'(0)| \le 1$.
(b) Deduce that if $h : H \to \Delta$ is an analytic mapping from the upper half-plane $H = \{z = x + iy \colon y > 0\}$ into $\Delta$, with $h(i) = 0$, then $|h'(i)| \le 1/2$. (Hint: Consider $g(z) = (z - i)/(z + i)$.)

9. TRUE OR FALSE. Prove the result or find a counterexample.
(a) If $f \colon [a,b] \to \mathbb{R}$ is increasing and continuous then

$$f(b) - f(a) = \int_a^b f'(x)\,dx.$$

(b) If $f \colon \mathbb{R} \to \mathbb{R}$ is continuous and $B \subseteq \mathbb{R}$ is a Borel set, then $f^{-1}(B)$ is a Borel set.

(c) If $f_n$ $(n \ge 1)$ and $f$ are integrable functions on [0,1] and $f_n \to f$ pointwise then $\int_0^1 f_n\,dx \to \int_0^1 f\,dx$.

(d) Suppose that $U$ is a domain in the complex plane and that $f$ is analytic on $U$. Then $\int_\gamma f(z)\,dz = 0$ for every simple closed smooth curve $\gamma$ contained in $U$.

(e) If $f$ is an entire function and $|f(z)| \to \infty$ as $|z| \to \infty$ then $f$ is a polynomial.