Qualifying Exam in Analysis

August 1991

Lebesgue measure, defined on the set of measurable subsets of the real numbers, will be denoted by m. Lebesgue outer measure, defined on the set of all subsets of the real numbers, will be denoted by m^*. The integral $\int_{[a,b]} f\,dm$ will also be written as $\int_a^b f(x)\,dx$.

1 (10 points) Let $E \subset [0,1]$ with $1 \in E$ and so that $E \cap [0,x]$ is compact for all $x < 1$. Show E is compact.

2 (10 points) Let $f_n, f, g \in L^1[0,1]$ with $|f|, |f_n| \leq g$. Assume that for all intervals $I \subset [0,1]$ that

$$\lim_{n\to\infty} \int_I f_n\,dm = \int_I f\,dm.$$

Show that if h is a measurable function with $h, gh \in L^1[0,1]$ then

$$\lim_{n\to\infty} \int_0^1 f_n(x)h(x)\,dx = \int_0^1 f(x)h(x)\,dx.$$

3 (10 points) Let $\langle r_i\rangle_{i=1}^\infty$ be an enumeration of the rational numbers in $[0,1]$. Define a function f on $[0,1]$ by

$$f(x) = \sum_{i=1}^\infty \frac{1}{2^i\sqrt{|x-r_i|}}$$

for irrational values of x, and $f(x) = 0$ when x is rational.

a) Show that f becomes unbounded in every interval $(a,b) \subset [0,1]$.

b) Show the series defining f converges almost everywhere.

4 (20 points) For each of the following give a proof or a counterexample.

a) If $\langle f_n\rangle_{n=1}^\infty$ is a sequence of measurable functions on $[0,1]$ with $0 \leq f_n \leq 1$ for all n and $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$, then $\langle f_n\rangle_{n=1}^\infty$ converges to 0 in measure.

b) If $\langle f_n\rangle_{n=1}^\infty$ is a sequence of measurable functions on $[0,1]$ with $0 \leq f_n \leq 1$ for all n and $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 0$, then $\langle f_n\rangle_{n=1}^\infty$ converges to 0 almost everywhere.

c) If U is an open dense subset of $(0,1)$ then $m(U) = 1$.

d) There is a sequence of measurable functions $\langle f_n\rangle_{n=1}^\infty$ with $\int_0^1 f_n(x)\,dx = 1$ for all n but $f_n \to 0$ almost everywhere.

5 (10 points) Let f be a nonnegative nondecreasing singular function on $[0,1]$ with $f(0) = 0$ and let μ be the measure with $\mu([0,x]) = f(x)$. Show μ is singular with respect to m. (HINT: Apply the Lebesgue decomposition lemma to μ.)

6 (10 points) Let $K(x,y)$ be a measurable function on $[0,1]\times[0,1]$ so that for some $M>0$,

$$\int_0^1\int_0^1 K(x,y)^2\,dx\,dy \le M.$$

Let $f\in L^2[0,1]$ and set

$$F(x)=\int_0^1 K(x,y)f(y)\,dy.$$

Show $\|F\|_2 \le \sqrt{M}\|f\|_2$. (Here $\|h\|_2=(\int_0^1 h(x)^2\,dx))^{\frac{1}{2}}$.)

7 (10 points) Define a measure on $[0,1]$ by $\mu(A)=m\{x : x^3\in A\}$.

a) Show $\mu \ll m$.

b) Compute the Radon-Nikodym derivative $\frac{d\mu}{dm}$

8 (10 points) Let $f(x,y)$ be a measurable function defined on $[0,1]\times[0,1]$ so that

$$|f(x_2,y)-f(x_1,y)|\le 5|x_2-x_1| \qquad \text{and} \qquad |f(x,y)|\le 13.$$

Show the function

$$F(x)=\int_0^1 f(x,y)\,dy$$

is differentiable almost everywhere.

9 (10 points) Let f be a bounded measurable function defined on $[0,1]$ so that

$$\int_0^1 f(x)x^k\,dx=\frac{1}{(k+1)(k+2)}=\frac{1}{k+1}-\frac{1}{k+2}$$

for $k=0,1,2,\ldots$ Show $f(x)=x(1-x)=x-x^2$ almost everywhere.

10 (10 points) Give examples of:

a) Subsets E_i $(i=1,2,\ldots)$ of the real numbers with $E_i\cap E_j=\emptyset$ when $i\neq j$, but

$$m^*(\cup_{i=1}^{\infty}E_i)<\sum_{i=1}^{\infty}m^*(E_i).$$

b) Subsets $F_1\supset F_2\supset F_3\supset\cdots$ of the real numbers with $m^*(F_1)<\infty$ and $\cap_{i=1}^{\infty}F_i=\emptyset$, but $\lim_{i\to\infty}m^*(F_i)\neq 0$.