Qualifying exam in Analysis

August 2014

You are allowed the statement of any problem even if you have not solved it, if you need it to solve another problem.

1)a) (1 point) Define what it means for a subset A of a topological space X to be compact. Hint: your answer should contain and define the term "open covering".

b) (4 points) Let X_1, X_2 be topological spaces and $f : X_1 \to X_2$ be a continuous function. Prove that if A is a compact subset of X_1 then $f(A)$ is a compact subset of X_2.

2)a) (2 points) Let $[0, 1]$ equipped with its usual metric obtained from the absolute value, and let A be a non-empty set. Let a sequence of functions $(g_n)_{n\in\mathbb{N}}$ with $g_n \in [0, 1]^A$. What does it mean that $(g_n)_n$ converges in the product topology of $[0, 1]^A$? (does there exist another name for this convergence?)

$[0,1]^{\mathbb{Q}}$

b) (4 points) Let $[0, 1]$ be equipped with its usual metric and a sequence of functions $(f_n)_{n\in\mathbb{N}}$ with $f_n \in [0, 1]^{\mathbb{Q}}$. Prove that (f_n) has a converging subsequence in the product topology of $[0, 1]^{\mathbb{Q}}$. compact? $f_n : \mathbb{Q} \to [0,1]$

c) (4 points) Recall that if (X_1, d_1), (X_2, d_2) are metric spaces and $(f_i)_{i\in I}$ is a family of functions with $f_i : X_1 \to X_2$ for all $i \in I$, then the family $(f_i)_{i\in I}$ is called equicontinuous if for all $x \in X_1$ and all $\epsilon > 0$ there exists $\delta > 0$ such that for all $y \in X_1$ with $d_1(x, y) < \delta$ we have that $d_2(f_i(x), f_i(y)) < \epsilon$ for all $i \in I$. Now assume that $[0, 1]$ and $\mathbb{R}$ are equipped with their usual metrics obtained from the absolute value, and prove that if $(f_n)_{n\in\mathbb{N}}$ is an

f x V_c

f is continuous

equicontinuity $\forall x, y$ $|f_n - f| < \frac{\varepsilon}{2}$ $\forall n$ x δ

$f_n : \mathbb{R} \to [0,1]$ $|f_n(x) - f_n(y)| < \frac{1}{n}$

Construct $\mathbb{Q} \to \mathbb{R}$ Compact $|f_n(x) - f(x)| \leq |f_n(x)$

equicontinuous family of functions with $f_n \in [0,1]^{\mathbb{R}}$ for $n \in \mathbb{N}$, then $(f_n)_n$ has a converging subsequence in the product topology of $[0,1]^{\mathbb{R}}$. Be very careful with the order that you write the quantifiers! $[0,1]^{[0,1]}$ Compact

3) a) (5 points) Let X be a topological space and for $x \in X$ let N_x denote the family of open neighborhoods of x. If $f : X \to \mathbb{R}$ is a function then define the oscillation of f at the point $x \in X$ to be

$$\omega_f(x) = \inf_{V \in N_x} \left(\sup_{z,y \in V} |f(z) - f(y)| \right).$$

a note card

Prove that f is continuous at x if and only if $\omega_f(x) = 0$.

b) (5 points) Let X be a topological space and $f : X \to \mathbb{R}$ be a function. Prove that the set of discontinuities of f is an F_σ set. 闭集的并 不是 Countable 并

4) Recall that a fractional linear transformation (f.l.t.) on the extended complex plane $\mathbb{C} \cup \{\infty\}$ is a map f defined by $f(z) = \frac{az+b}{cz+d}$ where a, b, c, d are fixed complex numbers such that $ad - bc \neq 0$, (with the usual conventions about infinity).

a) (4 points) Prove that every f.l.t. has an inverse function which is also a f.l.t.

说明 1-1 对应

b) (1 point) Which subsets of the extended complex plane are the possible ranges of a f.l.t.? 不确定 the exteded complex plane 1-1

c) (5 points) Find all f.l.t. f such that $f \circ f = f$. check ?

5) (10 points) Compute $\int_C \frac{dz}{z^2+4}$ for all simple closed rectifiable positively oriented curves C which do not pass through the points $z = \pm 2i$. 既然是all 各种情况都要讨论

z 如果 $\int \frac{dz}{cz^2+4}$ 就 $= 0$ $f(z_1)$ on

$A_n \subset A$ $\mu(A_n) =$

6) (5 points) Recall that if $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of a set X, then $\liminf_n A_n = \bigcup_n \left(\bigcap_{m\geq n} A_m\right)$. Prove that if (X, Σ, μ) is a measure space and $A_n \in \Sigma$ for all $n \in \mathbb{N}$ then $\mu(\liminf_n A_n) \leq \liminf_n \mu(A_n)$.

7) Let (X, μ) be a measure space and for $p \in (0, \infty)$ let $\|\cdot\|_p$ denote the $L_p(X, \mu)$ norm.

a) (2 points) State Hölder's inequality for functions defined on the measure space (X, μ). $\|fg\|_1 \leq \|f\|_p \|g\|_q$ $1 < p, q < +\infty$ $\frac{1}{p} + \frac{1}{q}$

b) (8 points) Now let $p, q, r \in (1, \infty)$ satisfying $1/p + 1/q = 1/r$. Suppose that $f \in L^p(X)$ and $g \in L^q(X)$. Show that $fg \in L^r(X)$ and $\|fg\|_r \leq \|f\|_p \|g\|_q$. $p=1$

8) (10 points) Let $f : [a, b] \to \mathbb{R}$ be a differentiable function at every point of $[a, b]$, such that f' is uniformly bounded on $[a, b]$. Prove that f' is Lebesgue integrable and $\int_{[a,b]} f'(x)d\lambda(x) = f(b) - f(a)$, (where $d\lambda$ denotes the Lebesgue measure). $|f'| \leq M$

9) a) (5 points). Let (X, Σ, μ) be a measure space and $f \in L_1(\mu)$. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $A \in \Sigma$ with $\mu(A) < \delta$ we have that $\int_A |f| d\mu < \epsilon$. simple functions

b) (5 points) Let (X, Σ, μ) be a measure space, $f \in L_1(\mu)$, $(f_n)_n \subset L_1(\mu)$ and $f_n \to f$ in $L_1(\mu)$. Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $A \in \Sigma$ with $\mu(A) < \delta$ we have that $\int_A |f_n| d\mu < \epsilon$ for all $n \in \mathbb{N}$.

10) True or False? (Prove or give counterexample as needed). Each is worth 5 points.

True **a)** If A, B are subsets of a topological space then $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

b) Assume that f is holomorphic on the unit disc and for some $z_0 \in \mathbb{C}$ with $|z_0| < 1$ we have that $|f(z_0)| = \max_{|z|\leq 1} |f(z)|$. Then $f'(z_0) = 0$.

True

f Maximum Modulus $f(z) =$

$A_n = [n, n+1]$

c) If (X, Σ, μ) is a measure space and $\mu(\liminf_n A_n) = 0$ then $\liminf_n \mu(A_n) = 0$. False

$\bigcup_{n=1}^{\infty} \bigcap_{m \geq n} A_m$ $\mu(A_n) = 1$

d) For $\boxed{\text{every}}$ closed subset F of $[0, 1]$ the characteristic function of F, χ_F, is Riemann integrable.

Borel measurable set
Lebesgue
$\Rightarrow$ measurable

Cantor set

closed set

χ_F is me

( )

$L \int \chi_F \, dm(x) = m(F) < +\infty$