ANALYSIS QUALIFYING EXAMINATION
January, 1997

DIRECTIONS: 1. Questions 1-8 are worth ten points each and question 9 is worth 20 points.
2. Write your solution to each problem on a separate sheet.

1. (a) Suppose that $A \subset \mathbb{R}$ is compact and that $B \subset \mathbb{R}$ is closed. Prove that the set $A + B = \{a + b : a \in A, b \in B\}$ is closed.
(b) Give an example of two closed sets A and B such that $A + B$ is not closed.

2. In this question m denotes Lebesgue measure and m^* denotes Lebesgue outer measure on the line.
(i) Let $A \subset \mathbb{R}$. Prove that there exists a Borel set G such that $A \subset G$ and $m(G) = m^*(A)$.
(ii) Suppose that $0 < m^*(A) < \infty$ and that A is non-measurable. Prove that $m(F) < m^*(A)$ for every measurable set $F \subset A$.
(iii) Now suppose that $A \subset [a, b]$. Prove that A is measurable if and only if $m^*(A) + m^*([a, b] \setminus A) = b - a$.

3. (i) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere. Prove that f' is measurable.
(ii) Is f' necessarily integrable on $[0, 1]$? Prove or give a counterexample.

4. Suppose that f, f_n $(n \geq 1)$ are integrable functions defined on the measure space (X, Σ, μ) such that $\langle f_n(x) \rangle$ **decreases** to $f(x)$ a.e. Prove carefully that

$$\int_X f \, d\mu = \lim \int_X f_n \, d\mu.$$

5. Suppose that $F : [a, b] \to \mathbb{R}$ is increasing.
(i) Prove that there exist unique real-valued functions G and H on $[a, b]$ satisfying the following:

(a) $F(x) = G(x) + H(x)$ for all $x \in [a, b]$;
(b) G is absolutely continuous and $G(a) = F(a)$;
(c) $H'(x) = 0$ a.e.

(ii) Prove that G and H are increasing.

6. Let (X, Σ, μ) be a **finite** measure space, let $1 < p < \infty$, and let f_n $(n \geq 1)$ belong to $L^p(\mu)$. Suppose that $\|f_n\|_p \leq 1$ and that $f_n \to 0$ a.e. Prove that $\|f_n\|_1 \to 0$.

7. Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^2} \, dx.$$

8. In this question Δ denotes the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Suppose that $f : \Delta \to \Delta$ is holomorphic and that $f(0) = 0$.
(i) Prove that $g(z) = f(z)/z$ has a removable singularity at $z = 0$ and that $|g(z)| \leq 1$ $(z \in \Delta \setminus \{0\})$.
(ii) Deduce that either $|f'(0)| < 1$ or $f(z) = cz$ $(z \in \Delta)$ for some complex number c with $|c| = 1$.

9 True or False? Prove or construct a counterexample in each case.

(i) Suppose that f is continuous on $[a, b]$, that g is bounded on $[a, b]$, and that $g(x) = f(x)$ a.e. Then g is Riemann-integrable on $[a, b]$?

$\Rightarrow$ every pt in U is a limit pt

(ii) Suppose that U is a dense open subset of $\mathbb{R}$. Then U has infinite Lebesgue measure?

(iii) Suppose that $f(z)$ and $g(z)$ are holomorphic on $\mathbb{C} \setminus \{0\}$, that both have poles at $z = 0$, and that $f(1/n) = g(1/n)$ for $n \geq 1$. Then $f(z) = g(z)$ for all z?

(iv) Suppose that $f \in L^2(\mathbb{R})$ and that $\|f\|_2 \leq 1$. Then

$$\int_{-\infty}^{\infty} \frac{|f(x)|^{1/2}}{(1 + x^2)^{3/4}}\, dx \leq \pi^{3/4}?$$

(ii) False take $U = (r_n - \frac{1}{n^2}, r_n + 1/n^2)$