

## Some consequences of the Cauchy integral formula.

**Proposition 1.** *Let  $f(z)$  be analytic in the open set  $U$ . Then the derivative  $f'(z)$  is also analytic.*

**Problem 1.** Prove this. *Hint:* Here is an outline of my favorite proof. Like many of the proofs in complex analysis it is based on the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where  $D$  is a bounded open set with nice boundary,  $f$  is analytic in  $D$  and continuous on the closure  $\overline{D}$ .

(a) To start verify the identities algebraic identities

$$\begin{aligned} \frac{1}{\zeta - (z+h)} - \frac{1}{\zeta - z} &= \frac{h}{(\zeta - (z+h))(\zeta - z)} \\ \frac{1}{(\zeta - (z+h))^2} - \frac{1}{(\zeta - z)^2} &= \frac{h(2(\zeta - z) - h)}{(\zeta - (z+h))^2} \end{aligned}$$

(b) Use the first of these identities and the Cauchy integral formula to show that if  $z, z+h \in D$

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - (z+h))(\zeta - z)} d\zeta$$

(c) Take the limit as  $h \rightarrow 0$  in the previous equation to get the an integral formula for  $f'(z)$ :

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

(d) Use the second identity of Part (a) to and the formula for  $f'(z)$  to show

$$\frac{f'(z+h) - f'(z)}{h} = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)(2(\zeta - z) - h)}{(\zeta - (z+h))^2} d\zeta$$

(e) Use this formula so show that  $f''(z) = \lim_{h \rightarrow 0} \frac{f'(z+h) - f'(z)}{h}$  exists and has the integral formula

$$f''(z) = \frac{2}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^3} d\zeta.$$

This shows  $f'$  is analytic. □

We discussed the identity theorem:

**Theorem 2.** *Let  $U$  be a connected open set in  $\mathbb{C}$  and  $f, g$  analytic functions on  $U$  so that for some sequence  $\langle a_n \rangle_{n=1}^{\infty}$  distinct points in  $U$  with  $\lim_{n \rightarrow \infty} a_n = a$  for some  $a \in U$  we have*

$$f(a_n) = g(a_n)$$

*for all  $n$ . Then  $f \equiv g$  in  $U$ .* □

**Problem 2.** Let  $D = \{z : |z| < 1\}$  and let

$$f(z) = e^{\frac{i}{1-z}} - 1$$

and

$$a_n = 1 - \frac{1}{2\pi n}$$

for  $n = 1, 2, \dots$ . Show

$$f(a_n) = 0$$

and that

$$\lim_{n \rightarrow \infty} a_n = 1.$$

Why does this not contradict the identity theorem? □

**Problem 3.** Does there exist an analytic function  $f(z)$  on the unit disk  $D = \{z : |z| < 1\}$  such that for  $n = 2, 3, 4, \dots$

(a)

$$f(1/n) = \frac{3}{n^2}.$$

If it exists is it unique?

(b)

$$f(1/n) = \frac{n+1}{n+2}$$

If it exists is it unique?

(c)

$$f(1/n) = \frac{(-1)^n}{n^2}$$

If it exists is it unique? □

**Problem 4.** Let  $f$  be an analytic function on the unit disk  $D$  such that at each point  $z \in D$  at least one of the derivatives of  $f$  vanishes. (That is for each  $z \in D$  there is a  $n = n_z$  such that  $f^{(n)}(z) = 0$ .) Show  $f$  is a polynomial. □