

## Mathematics 551 Homework, February 5, 2020

Let  $c: [a, b] \rightarrow \mathbb{R}^2$  be a  $C^3$  curve with curvature  $\kappa$  everywhere positive. Then the radius of curvature for the curve at  $\mathbf{c}(s)$  is

$$\rho(s) = \frac{1}{\kappa(s)}.$$

The point

$$\mathbf{E}(s) = \mathbf{c}(s) + \rho(s)\mathbf{n}(s)$$

is the *center of curvature* of  $\mathbf{c}$  at  $\mathbf{c}(s)$  and the *osculating circle* at  $\mathbf{c}(s)$  is the circle with center  $\mathbf{E}(s)$  and radius  $\rho(s)$ . This is the circle that is tangent to  $\mathbf{c}$  at  $\mathbf{c}(s)$  and has the same curvature as the curve and therefore is the circle that “best fits”  $\mathbf{c}$  at  $\mathbf{c}(s)$ . The curve  $\mathbf{E}: [a, b] \rightarrow \mathbb{R}^2$  is the *evolute* of  $\mathbf{c}$ .

**Problem 1.** Use the Frenet formulas to show that

$$\mathbf{E}'(s) = \rho'(s)\mathbf{n}(s).$$

Then use this to show that the unit normal  $\mathbf{n}_E(s)$  to  $\mathbf{E}$  is

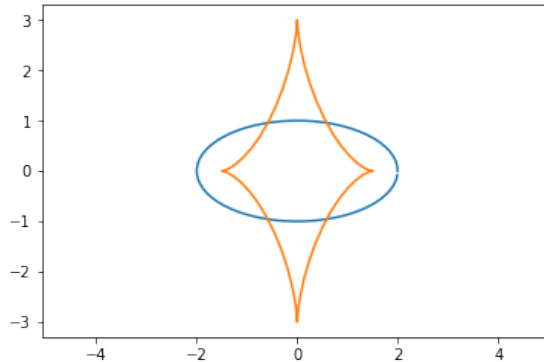
$$\mathbf{n}_E(s) = \begin{cases} \mathbf{n}(s), & \rho'(s) > 0; \\ -\mathbf{n}(s), & \rho'(s) < 0. \end{cases}$$

Use this to show that  $\mathbf{n}_E$  flips direction by  $\pi$  radians at any point where  $\rho$  has a local maximum or minimum. That is at the vertices of  $\mathbf{c}$ .  $\square$

Here is a picture of the ellipse

$$\frac{x^2}{2^2} + y^2 = 1$$

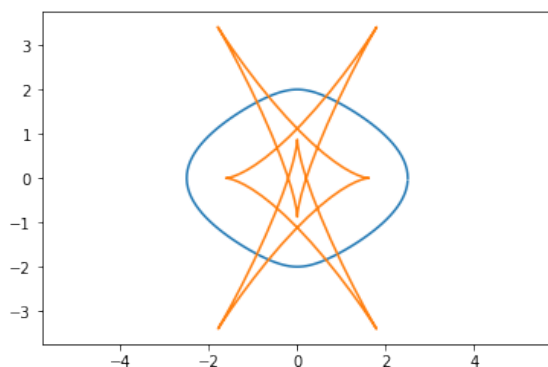
together with its evolute.



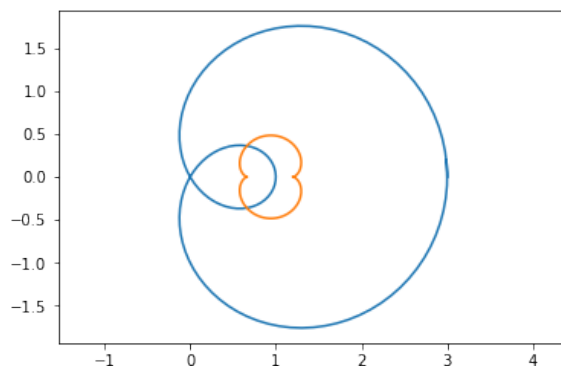
For a more exotic example, or at least one with more vertices, here is the graph of the curve

$$\mathbf{c}(t) = ((2 + .5 \cos(2 * t)) \cos(t), 2 \sin(t)) \quad 0 \leq t \leq 2\pi.$$

together with its evolute.



As one last example recall that curve with polar equation  $r = 1 + 2 \cos(\theta)$  was our example of a closed, but not simple, curve that only has two vertices. Thus its evolute should only have two cusps. Here is the picture showing this is the case:



**Problem 2.** Let  $\mathbf{c}: [a, b] \rightarrow \mathbb{R}^2$  be a curve where  $\kappa > 0$  and is monotone (that is either increasing or decreasing) on the interval. Show that the length of the evolute  $\mathbf{E}$  is

$$\text{Length}(\mathbf{E}) = |\rho(b) - \rho(a)|.$$

*Hint:* The arclength formula is

$$\text{Length}(\mathbf{E}) = \int_a^b \|\mathbf{E}'(s)\| ds.$$

Now use that  $\mathbf{E}'(s) = \rho'(s)$  and that since  $\kappa$ , and therefore also  $\rho$ , is monotone that  $\rho'$  is either always positive or always negative.  $\square$

**Problem 3.** With the same hypothesis as the previous problem, show

$$\|\mathbf{E}(b) - \mathbf{E}(a)\| < |\rho(b) - \rho(a)|.$$

*Hint:* For a curve that is not a line segment, its length is greater than the distance between its endpoints. Or put more succinctly, the shortest path between two points is a straight line.  $\square$

**Problem 4.** Let  $\mathbf{P}_1$  and  $\mathbf{P}_2$  be points in the plane,  $R_1, R_2$  positive real numbers, and let  $\mathcal{C}_j$  be the circle with center  $\mathbf{P}_j$  and radius  $R_j$ . Show that if  $\|\mathbf{P}_1 - \mathbf{P}_2\| < |R_1 - R_2|$  then one of the circles  $\mathcal{C}_1$  or  $\mathcal{C}_2$  is contained in the other one. *Hint:* Proof by picture is fine, and even preferred.  $\square$

**Theorem 1** (Tait-Kneser Theorem<sup>1</sup>). *Let  $\mathbf{c}: [a, b] \rightarrow \mathbb{R}^3$  be a  $C^3$  unit speed curve that has positive curvature. Also assume that  $\kappa$  is monotone. Then the osculating circles of the curve are nested. That is for any pair of them, one is contained inside the other.*

**Problem 5.** Prove this. *Hint:* Follow the outline of what we did in class.  $\square$

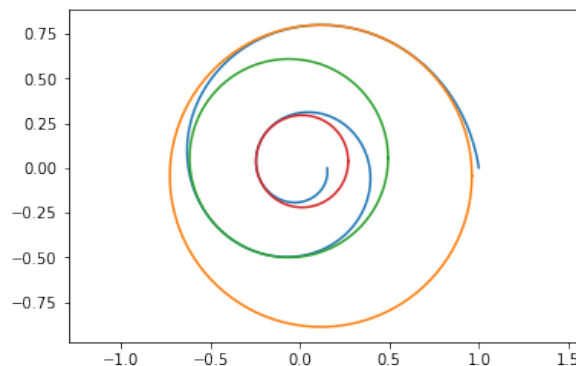
The following figure shows the curve  $\mathbf{c}: [0, 4\pi] \rightarrow \mathbb{R}^2$  given by

$$\mathbf{c}(t) = (e^{-.15t} \cos(t), e^{-.15t} \sin(t))$$

(which has polar equation  $r = e^{-.15\theta}$ ) which has curvature

$$\kappa(t) = \frac{20e^{.15t}}{\sqrt{409}}$$

which is increasing. Three of the osculating circles of the curve are shown.



The Tait-Kneser Theorem has some nice consequences. Note that a curve with positive curvature can cross itself many times.

**Problem 6.** Draw a curve with positive curvature that crosses itself four times.  $\square$

**Problem 7.** Show that a curve  $\mathbf{c}: [a, b] \rightarrow \mathbb{R}^2$  that has positive increasing curvature is embedded (the term **embedded** in this context just means that the curve does not cross itself).  $\square$

**Problem 8.** We have shown that for a simple closed curve of length  $L$  that encloses an area  $A$  and with inradius  $R_{\text{in}}$  and circumradius  $R_{\text{out}}$  that the

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<sup>1</sup> This result was originally proven by Peter Tait in a paper published in 1896. Adolf Kneser rediscovered it and published a proof in 1912.

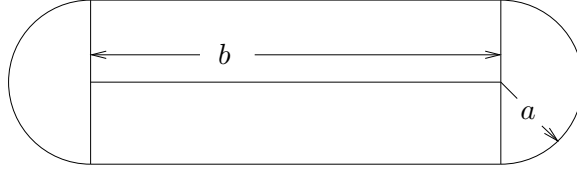
***Bonnesen<sup>2</sup> inequalities***

$$\frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \leq R_{\text{in}} \leq R_{\text{out}} \leq \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi}$$

and

$$\pi^2(R_{\text{out}} - R_{\text{in}})^2 \leq L^2 - 4\pi A$$

hold.



In the figure above we have  $R_{\text{in}} = a$ . Compute  $A$  and  $L$  for this shape and show

$$R_{\text{in}} = a = \frac{L - \sqrt{L^2 - 4\pi A}}{2\pi}$$

and therefore the lower bound for  $R_{\text{in}}$  is best possible. □

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<sup>2</sup>These were first appeared in a paper by the Danish mathematician Tommy Bonnesen in 1921