The Vitali Covering Theorem for Doubling Measures

Let $(\mathbf{X}, d)$ be a metric space and for $x \in \mathbf{X}$ and $r > 0$ let $B(x,r) = \{y \in \mathbf{X} : d(x,y) \leq r\}$ be the closed ball of radius r about x. Let μ^* be an outer measure on $\mathbf{X}$ such that all the Borel subsets of $\mathbf{X}$ are measurable. The restriction of μ^* to the measurable sets will be denote by μ. The measure is a ***doubling measure*** with ***doubling constant*** C iff

$$\mu B(x, 2r) \leq C \mu B(x, r)$$

holds for all $x \in \mathbf{X}$ and $r > 0$. If m is a positive integer, then

$$\mu B(x, 2^m r) \leq C^m \mu B(x, r).$$

Therefore if for some ball $B(x,r)$ we have $\mu B(x,r) = 0$, then for all $m \geq 1$ there holds $\mu B(x, 2^m) = 0$ and thus

$$\mu \mathbf{X} = \mu \left(\bigcup_{m=1}^{\infty} B(x, 2^m r) \right) \leq \sum_{m=1}^{\infty} \mu B(x, 2^m r) = 0.$$

Thus if the measure of any ball is zero, then μ is the zero measure. As any open set contains a ball we have

Proposition 1. *If μ is a non-zero doubling measure on $\mathbf{X}$, then $\mu U > 0$ for every open subset of $\mathbf{X}$.* $\square$

The basic example of a doubling measure is the Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$. For if $B(x,r)$ is ball in $\mathbb{R}^n$ the equality $\mathcal{L}^n B(x, 2r) = 2^n \mathcal{L}^n B(x, r)$ and thus $\mathcal{L}^n$ is a doubling measure on $\mathbb{R}^n$ with doubling constant $C = 2^n$.

Let $(\mathbf{X}, d)$ be a metric space such that closed bounded sets are compact and let μ^* a nonzero outer measure on $\mathbf{X}$ such that all Borel subsets of $\mathbf{X}$ are measurable with respect to μ^*. The restriction of μ^* to the measurable subsets of $\mathbf{X}$ will be denoted by μ. Let $B(x,r) := \{y \in \mathbf{X} : d(x,y) \leq r\}$ be the closed ball of radius r about x. We assume that μ satisfies a ***doubling condition***:

$$\mu B(x, 2r) \leq C \mu B(x, r)$$

where the constant C is independent of r and x. This impies that if m is a positive integer that

$$\mu B(x, 2^m r) \leq C^m \mu B(x, r)$$

If I is a bounded subset of $\mathbf{X}$, then set

$$r(I) = \inf\{\rho : \text{there exists } x \in \mathbf{X} \text{ with } I \subseteq B(x, \rho)\}.$$

That is $r(I)$ is the smallest radius of a ball containing I.

Let E be a subset of $\mathbf{X}$ then a collection $\mathcal{V}$ of subset I of $\mathbf{X}$ is a ***Vitali cover*** of E iff

(1) Each $I \in \mathcal{V}$ is a closed set,
(2) For all $x \in E$ and $\varepsilon > 0$ there is an $I \in \mathcal{V}$ such with $x \in I$ and $r(I) < \varepsilon$, and

(3) There is an $\alpha > 0$ such for each $I \in \mathcal{V}$ there is a ball $B(x_I, r_I)$ with $I \subset B(x_I, R_I)$ and
$$\mu I \geq \alpha \mu B(x_I, r_I).$$

If $\mathcal{V}$ is a Vitali cover of some set let
$$r(\mathcal{V}) = \sup_{I \in \mathcal{V}} r_I.$$

Theorem 2 (Vitali Covering Theorem)**.** *Let $U \subset \mathbf{X}$ be a bounded open set with $\mu U < \infty$ and let $E \subseteq U$ be any set (not necessarily measurable). Let $\mathcal{V}$ be Vitali cover of E such that $I \subset U$ for all $I \in \mathcal{V}$. Let $\rho \in (0,1)$ and let a sequence $\{I_k\}_{k=1}^{\infty}$ of elements of $\mathcal{V}$ be chosen so that I_1 is any element of $\mathcal{V}$ such that*
$$r_{I_1} \geq \rho r(\mathcal{V}).$$
If $I_1, \ldots, I_k$ have been chosen let
$$\mathcal{V}_k := \{I \in \mathcal{V} : I \cap (I_1 \cup \cdots \cup I_k) = \varnothing, \text{ and } E \cap I \neq \varnothing\}$$
and let I_{k+1} be any element of $\mathcal{V}_k$ such that
$$r_{I_{k+1}} \geq \rho r(\mathcal{V}_k).$$
Then
$$\mu^* \left(E \setminus \bigcup_{k=1}^{\infty} I_k \right) = 0$$
and
$$E \subseteq \bigcup_{k=1}^{\infty} B(x_{I_k}, (1 + 2\rho^{-1}) r_{I_k}).$$

Remark. Here is an informal description of this. We wish to cover E, or as much of E as possible, with a pairwise disjoint sequence of elements from $\mathcal{V}$. A natural idea would be to use a *greedy algorithm*, that is Choose I_1 to be the element of $\mathcal{V}$ that is largest in the sense it maximizes $r(I)$ over all elements $I \in \mathcal{V}$, then choose I_2 to maximize $r(I)$ over all elements of $\mathcal{V} \setminus \{I_1\}$ and continuing in this manner. But this does not work as $\mathcal{V}$ need not have a element that maximizes $r(I)$. This is where ρ comes into play. Since we can not be absolutely greedy, we use ρ as a measure of how greedy we will be. So let us be 90% greedy. That is $\rho = .9$. We may not be able to find an I with $r(I) = r(\mathcal{V}) = \sup\{r(I) : I \in \mathcal{V}\}$, but we can get 90% of the way to this supremum. That is we choose I_1 with $r(I_1) > .9r(\mathcal{V})$. Then choose I_2 with $r(I_2) > .9 \sup\{r(I) : I \in \mathcal{V}, I \cap I_1 = \varnothing\}$, then choose I_3 with $r(I_2) > .9 \sum\{r(I) : I \in \mathcal{V}, I \cap (I_1 \cup I_2) = \varnothing\}$ etc. The content of the Vitali Covering Theorem is that this almost greedy algorithm covers almost all of E in the sense that $\mu^*(E \setminus \bigcup_{k=1}^{\infty} I_k) = 0$. Even more remarkable we do not even have to be all that greedy, we still cover almost all of E even if we only are 1% greedy, that is $\rho = .01$. $\square$

Here are some problems that outline a proof. To simplify notation let $r_k = r_{I_k}$ and $x_k = x_{I_k}$.

Problem 1. $\lim_{k\to\infty} r_k = 0$. *Hint:* As the sets $\{I_k\}_{k=1}^\infty$ are pairwise disjoint we have

$$\mu U \ge \mu\left(\bigcup_{k=1}^\infty I_k\right) = \sum_{k=1}^\infty \mu I_k.$$

Therefore $\sum_{k=1}^\infty \mu I_k < \infty$ which implies that

$$0 = \lim_{k\to\infty} \mu I_k \ge \lim_{k\to\infty} \alpha\mu B(x_k, r_k) \ge 0.$$

Thus $\lim_{k\to\infty} \mu B(x_k, r_k) = 0$. Now assume, toward a contradiction, that $\lim_{k\to\infty} r_k \ne 0$. Then there is a $\delta > 0$ and a subsequence r_{k_ℓ} such that $r_{k_\ell} \ge 2\delta$ for all ℓ and, by local complactness, by going to a subsequence we can assume that $\lim_{\ell\to\infty} x_{k_\ell} = x_*$ exists. Then for all sufficiently large ℓ we have that $B(x_*, \delta) \subset B(x_{k_\ell}, r_\ell)$. Use this to show that $\mu B(x_*, \delta) = 0$ and then use the doubling condition to show that this implies $\mu\mathbf{X} = 0$, which contradicts that μ is not the zero measure. $\square$

Problem 2. Let $x \in E\smallsetminus\bigcup_{j=1}^k I_j$. Show that $x \in \bigcup_{j=k+1}^\infty B(x_j, (2\rho^{-1}+1)r_j)$. *Hint:* The set $\bigcup_{j=1}^k I_j$ is closed and $\mathcal{V}$ is a Vitali cover thus there is an $I \in \mathcal{V}$ with $x \in I$ and $B(x_I, r_I) \cap \left(\bigcup_{j=1}^k I_j\right) = \varnothing$. Let $n \ge k$ and assume that $B(x_I, r_I) \cap \left(\bigcup_{j=1}^n I_j\right) = \varnothing$. Show

$$r_{n+1} \ge \rho r(\mathcal{V}_n) \ge \rho r_I$$

and this implies that $B(x_I, r_I) \cap \left(\bigcup_{j=1}^{n+1} I_j\right) \ne \varnothing$ for some $n \ge k$ (as $r_{n+1} \to 0$). Let n be the smallest integer where $B(x_I, r_I) \cap \left(\bigcup_{j=1}^{n+1} I_j\right) \ne \varnothing$. Then explain why $r_{n+1} \ge \rho r(\mathcal{V}_n) \ge \rho r_I$ and

$$B(x_I, r_I) \cap B(x_{n+1}, r_{n+1}) \ne \varnothing.$$

Show these facts and that $x \in B(x_I, r_I)$ implies that $x \in B(x_{n+1}, (2\rho^{-1} + 1)r_{n+1})$ which completes the proof. $\square$

Problem 3. Complete the proof of Theorem 2. *Hint:* Let $F := E\smallsetminus\bigcup_{k=1}^\infty I_k$. We wish to show that $\mu^* F = 0$. By the last problem we have for each postive integer n that $F \subset \bigcup_{k=n+1}^\infty B(x_k, (2\rho^{-1}+1)r_k)$. Therefore by sub-additivity

$$\mu^* F \subset \sum_{k=n+1}^\infty \mu B(x_k, (2\rho^{-1} + 1)r_k).$$

Whence if the series

$$\sum_{k=1}^\infty \mu B(x_k, (2\rho^{-1} + 1)r_k)$$

converges we are done. Explain why it does converge.

Problem 4. Let U be a bounded open set in the plane. For any such set let

$$r(U) = \sup\{r : \ U \text{ contians a disk of radius } r.\}.$$

Define a sequence of closed disks as follows. D_1 is any closed disk in U with radius at least $\frac{1}{2}r(U)$. If $D_1, \ldots, D_n$ have been defined let $U_n := U \setminus (D_1 \cup \cdots \cup D_n)$ and D_{n+1} be any disk contained in U_n with radius at least $\frac{1}{2}r(U_n)$. Show that

$$\sum_{k=1}^{\infty} \text{Area}(D_k) = \text{Area}(U).$$

Hint: Let $E = U$ in the Vitali Covering Theorem. $\square$