A GENERALIZED AFFINE ISOPERIMETRIC INEQUALITY

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ABSTRACT. A purely analytic proof is given for an inequality that has as a direct consequence the two most important affine isoperimetric inequalities of plane convex geometry: The Blaschke-Santalo inequality and the affine isoperimetric inequality of affine differential geometry.

1. INTRODUCTION.

In [3], Harrell showed how an analytic approach could be used to obtain a well-known *Euclidean* inequality of plane convex geometry – the Blaschke-Lebesgue inequality. In this article we show how a purely analytic approach can be used to establish the best known *affine* inequalities of plane convex geometry. To be precise, we will use a purely analytic approach to establish an analytic inequality that has as an immediately consequence both the affine isoperimetric inequality of affine differential geometry and the Blaschke-Santaló inequality.

Let $C \subset \mathbf{R}^2$ be a compact convex set. Let $\mathbf{S}$ be the unit circle parameterized by the angular coordinate θ (corresponding to $(\cos\theta, \sin\theta) \in \mathbf{S}$). We will use the notation

$$e(\theta) := (\cos\theta, \sin\theta).$$

Then $h = h_C : \mathbf{S} \to \mathbf{R}$ defined by

$$h(\theta) := \max_{x \in C} e(\theta) \cdot x$$

is the *support function* of C.

The affine isoperimetric inequality of affine differential geometry states that if a plane convex figure has support function $h \in C^2(\mathbf{S})$, the twice continuously differentiable functions on $\mathbf{S}$, then

$$4\pi^2 \int_{\mathbf{S}} h(h + h'')\, d\theta \geq \left(\int_{\mathbf{S}} (h + h'')^{2/3}\, d\theta \right)^3 \tag{1.1}$$

with equality if and only if the figure is an ellipse.

The integral on the left is twice the area of the figure, while the integral on the right is the so called *affine perimeter* of the figure.

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The Blaschke-Santaló inequality states that if a convex figure is *positioned* so that its support function h is positive and

$$\int_{\mathbf{S}} \frac{\sin\theta}{h(\theta)^3}\, d\theta = 0 = \int_{\mathbf{S}} \frac{\cos\theta}{h(\theta)^3}\, d\theta, \tag{1.2}$$

then

$$4\pi^2\left(\int_{\mathbf{S}} h^{-2} d\theta\right)^{-1} \geq \int_{\mathbf{S}} h(h+h'')\, d\theta, \tag{1.3}$$

with equality if and only if the figure is an ellipse.

The integral on the left is twice the area of the *polar reciprocal* of the figure. The Hölder inequality tells us that the Blaschke-Santaló inequality is stronger (i.e., directly implies) the classical isoperimetric inequality between perimeter and area. To see this note that the Hölder inequality shows that

$$\left(\int_{\mathbf{S}} h\, d\theta\right)^2 \geq 8\pi^3\left(\int_{\mathbf{S}} h^{-2} d\theta\right)^{-1}$$

with equality if and only if the figure is a circle.

The integral on the left is the perimeter of the figure. When this inequality is combined with (1.3) the result is the classical isoperimetric inequality in the plane.

In [7], it was shown that both inequalities (1.1) and (1.3) are encoded in the following inequality: If K and L are convex figures whose support functions are such that $h_L \in C^2(\mathbf{S})$ and h_K arbitrary then

$$4\pi^2\left(\int_{\mathbf{S}} (h_L + h_L'')h_K\, d\theta\right)^2 \geq \left(\int_{\mathbf{S}} (h_L + h_L'')^{2/3} d\theta\right)^3 \int_{\mathbf{S}} [h_K^2 - (h_K')^2]\, d\theta \tag{1.4}$$

with equality if and only if K and L are homothetic ellipsoids.

When h_K is also C^2 then we can use integration by parts and replace the integral $\int_{\mathbf{S}} [h_K^2 - (h_K')^2]\, d\theta$ by $\int_{\mathbf{S}} h_K(h_K + h_K'')\, d\theta$ in this inequality. More generally if h_K is the support of an arbitrary planar convex body, then h_K is Lipschitz and so the the first derivative h_K' exists almost everywhere in the classical sense. The second derivative h_K'' exists as a distribution and $h_K'' + h_K$ is a non-negative measure (cf. Proposition 3.1). Thus the formula for integration by parts

$$\int_{\mathbf{S}} [h_K^2 - (h_K')^2]\, d\theta = \int_{\mathbf{S}} (h_K + h_K'')h_K\, d\theta$$

still holds, where the integral on the right is interpreted as the integral of the function h_K with respect to the measure $(h_K + h_K'')\, d\theta$.

Using this version of the inequality, we see that choosing $L = K$ in (1.4) immediately gives (1.1). To see how (1.4) gives (1.3) choose the figure L so that h_L satisfies the equation $y'' + y = h_K^{-3}$. (This is always possible as h_K satisfies (1.2) and therefore the equation $y'' + y = h_K^{-3}$ can be solved for a 2π periodic function, for example by the method of Fourier series. Then

$h_L := y$ will be the support function of a convex set by Proposition 3.1 below.)

In [7], it was shown that (1.4) is a consequence of (1.1) and the *mixed area* inequality. The aim of this paper is to establish an analytic inequality that extends inequality (1.4). Our proof of this new analytic inequality uses none of the tools of convex geometry.

2. The main inequality.

Let $H^1(\mathbf{S})$ be the space of functions $u\colon \mathbf{S} \to \mathbf{R}$ such that u is absolutely continuous and $u' \in L^2(\mathbf{S})$. We use the norm

$$\|u\|_{H^1} = \left(\int [u^2 + (u')^2]\, d\theta\right)^{\frac{1}{2}}.$$

The space $H^1(\mathbf{S})$ can also be described as the space of functions whose first distributional derivative is in L^2. The norm is a Hilbert space norm with corresponding inner product $\langle u, v\rangle_{H^1} = \int_{\mathbf{S}} (uv + u'v')\, d\theta$.

Theorem 1 (Two Dimensional Analytic Affine Isoperimetric Inequality)**.** *Assume*

i) F and h are non-negative 2π periodic functions that do not vanish identically.

ii) F is measurable and satisfies the integrability condition

$$\int_{\mathbf{S}} F^{1/3}(\theta)\, d\theta < \infty$$

and the orthogonality conditions

$$\int_{\mathbf{S}} F^{1/2}(\theta) \cos\theta\, d\theta = 0 = \int_{\mathbf{S}} F^{1/2}(\theta) \sin\theta\, d\theta. \tag{2.1}$$

iii) $h \in H^1(\mathbf{S})$.

Then

$$\left(\int_{\mathbf{S}} F^{1/2}(\theta) h(\theta)\, d\theta\right)^2 \geq \frac{1}{4\pi^2} \left(\int_{\mathbf{S}} F^{1/3}\, d\theta\right)^3 \left(\int_{\mathbf{S}} [h^2 - (h')^2]\, d\theta\right). \tag{2.2}$$

Equality holds if and only if there exist $k_1, k_2, a > 0$, and $\alpha \in \mathbf{R}$ such that

$$h(\theta) = k_1 \sqrt{a^2 \cos^2(\theta - \alpha) + a^{-2} \sin^2(\theta - \alpha)} \tag{2.3}$$

and F is given almost everywhere by

$$F(\theta) = k_2 (a^2 \cos^2(\theta - \alpha) + a^{-2} \sin^2(\theta - \alpha))^{-3}. \tag{2.4}$$

Remark 2.1. The functions $h(\theta)$ of the form (2.3) are exactly support functions of the ellipses centered at the origin.

The main ingredient in the proof of Theorem 1 is a family of transforms, that leave a few key integrals invariant and which let us construct maximizing sequences. We will introduce the transforms and study their properties in Section 5. In Section 6, we prove the inequality. In Sections 3 we study some regularity results for support functions of planar convex sets. These regularity results are used in Section 4 to derive the affine isoperimetric inequality for general planar sets.

3. Function spaces associated with the inequality.

Because of the integral $\int_{\mathbf{S}}[h^2 - (h')^2]\,d\theta$ that appears in Theorem 1, the natural function space for the functions h in the theorem is $H^1(\mathbf{S})$. Moreover in the geometric applications Theorem 1 a natural choice for the function h is to be a support function of a bounded convex set and $H^1(\mathbf{S})$ contains all the support functions. However, for the geometric applications mentioned in the introduction, the function F is taken to be $F = (h + h'')^2$ where h is a support function. For a general support function the second derivative only exists in a generalized sense, say as a distribution, and therefore the expression $(h + h'')^2$ is not necessarily defined. In fact for a figure as simple as a polygon the $h+h''$ is a sum of point masses (i.e. delta functions) and so $(h + h'')^2$ is undefined. The following characterizes the support functions of bounded convex sets. It seems to be a folk theorem, but as we can not find an explicit reference we include a short proof. Recall that a distribution u on $\mathbf{S}$ is positive iff, when viewed as a linear functional on $C^\infty(\mathbf{S})$, we have $u(\phi) \geq 0$ for all $\phi \in C^\infty(\mathbf{S})$ with $\phi \geq 0$.

Proposition 3.1. *A continuous function $h\colon \mathbf{S} \to \mathbf{R}$ is the support function of a bounded convex set if and only if the second distributional derivative h'' satisfies $(h'' + h) \geq 0$ as a distribution.*

Proof. It will be convenient to view functions on $\mathbf{S}$ as 2π periodic functions on $\mathbf{R}$. Let $\phi \geq 0$ be C^∞ on $\mathbf{R}$ with support in $[-1, 1]$ and $\int_{-1}^{1} \phi(\theta)\,d\theta = 1$. For $\epsilon > 0$ let $\phi_\epsilon := \epsilon^{-1}\phi(\theta/\epsilon)$. Then ϕ_ϵ is supported in $[-\epsilon, \epsilon]$ and $\int_{-\epsilon}^{\epsilon} \phi_\epsilon(\theta)\,d\theta = 1$. If $(h''+h) \geq 0$ as a distribution, then the same is true of the convolution $\phi_\epsilon * h(\theta) := \int_{-\infty}^{\infty} \phi_\epsilon(t)h(\theta - t)\,dt$ (as $(h_\epsilon'' + h_\epsilon) = \phi_\epsilon * (h'' + h)$ and the convolution of a non-negative function and a non-negative distribution is non-negative) and $\phi_\epsilon * h$ is 2π periodic. Thus the 2π periodic function $h_\epsilon := \epsilon + \phi_\epsilon * h$ is C^∞ and satisfies $(h_\epsilon'' + h_\epsilon) \geq \epsilon$. Therefore the curve $\gamma_\epsilon(\theta) := h_\epsilon(\theta)e(\theta) + h_\epsilon'(\theta)e'(\theta)$ has curvature $1/(h_\epsilon'' + h_\epsilon) > 0$ (cf. [10, p. 3].) Thus γ_ϵ is a convex curve and therefore h_ϵ is the support function of a bounded convex set. The set of support functions is closed with respect to uniform convergence and $h_\epsilon \to h$ uniformly, so h is a support function.

Conversely if h is a support function, then $\phi_\epsilon * h$ is C^∞ and is also a support function (as the set of support functions is closed under convex combinations and, by considering Riemannian sums, we see that $\phi_\epsilon * h$ is a limit of convex combinations of translates of h). Therefore $(\phi_\epsilon * h)'' + \phi_\epsilon * h$

is a non-negative function and, by standard properties of convolutions and distributions (cf. [5, Thm 4.1.4 p. 89]), $(\phi_\epsilon * h)'' + \phi_\epsilon * h \to h'' + h$ in the sense of distributions as $\epsilon \downarrow 0$. Therefore $(h''+h)$ is the limit of non-negative distributions and thus is also a non-negative distribution. $\square$

As a distribution is positive if and only if it is represented by a non-negative measure we see that the distribution $h'' + h$ is represented by a non-negative measure if and only if h is a support function. We now describe the smallest function space that contains the support functions of convex sets. Let $\mathcal{D}$ be the set of 2π periodic functions u such that the distributional derivative u'' is a signed measure (by Riesz's characterization of the dual of $C(\mathbf{S})$ as space of signed measures this is the same as u'' being a continuous linear functional on $C(\mathbf{S})$). The total variation $\|\mu\|_{TV}$ of a signed measure μ on $\mathbf{S}$ is its norm as a linear functional on $C(\mathbf{S})$. That is

$$\|\mu\|_{TV} := \sup\left\{ \int_{\mathbf{S}} \phi(\theta)\, d\mu(\theta) : \phi \in C(\mathbf{S}), |\phi(\theta)| \le 1 \right\}.$$

The standard norm on $\mathcal{D}$ is $\|h\|_{L^\infty} + \|h''\|_{TV}$, but, for geometric reasons, we use the equivalent norm

$$\|h\|_{\mathcal{D}} := \|h\|_{L^\infty} + \|h'' + h\|_{TV}.$$

The space $\mathcal{D}$ can also be defined as the functions h on $\mathbf{S}$ that are absolutely continuous and such that the first derivative h' is of bounded variation. As functions of bounded variation are bounded this implies all elements of $\mathcal{D}$ are Lipschitz. Therefore the imbedding $\mathcal{D} \subset C^\alpha(\mathbf{S})$ is compact for $\alpha \in [0,1)$, where $C^\alpha(\mathbf{S})$ is the space of Hölder continuous functions u such that the norm $\|u\|_{C^\alpha} := \sup_{\theta\in\mathbf{S}} |u(\theta)| + \sup_{\theta_1\neq\theta_2} |\theta_2 - \theta_1|^{-\alpha} |u(\theta_2) - u(\theta_1)|$ is finite.

Theorem 3.2. *The space $\mathcal{D}(\mathbf{S})$ contains all the support functions of bounded convex sets. Moreover every element of $\mathcal{D}(\mathbf{S})$ is a difference of two support functions. Thus $\mathcal{D}(\mathbf{S})$ is the smallest function space containing all the support functions. More precisely if $f \in \mathcal{D}$ there are support functions $h_1, h_2 \in \mathcal{D}$ with $f = h_1 - h_2$ and*

$$\|h_1'' + h_1\|_{TV},\ \|h_2'' + h_2\|_{TV} \le 3\|f'' + f\|_{TV}. \tag{3.1}$$

Proof. We have already seen that $\mathcal{D}$ contains all the support functions of bounded convex sets. Let $f \in \mathcal{D}(\mathbf{S})$. Then $f'' + f$ is a signed measure. We now claim that we can write $f'' + f = \mu_+ - \mu_-$ where μ_+ and μ_- are non-negative measures with the extra conditions that

$$\int_{\mathbf{S}} \cos\theta\, d\mu_+ = \int_{\mathbf{S}} \cos\theta\, d\mu_- = \int_{\mathbf{S}} \sin\theta\, d\mu_+ = \int_{\mathbf{S}} \sin\theta\, d\mu_- = 0 \tag{3.2}$$

and

$$\|\mu_+\|_{TV},\ \|\mu_-\|_{TV} \le 3\|f'' + f\|_{TV}. \tag{3.3}$$

To start let $f'' + f = \nu_+ - \nu_-$ be the Jordan decomposition (cf. [9, p. 274]) of $f'' + f$. Then ν_+ and ν_- are non-negative measures and $\|f'' + f\|_{TV} =$

$\|\nu_+\|_{TV}+\|\nu_-\|_{TV}$. From the definition of the second distributional derivative (which is formally just integration by parts)

$$\int_{\mathbf{S}}(f''+f)\cos\theta\,d\theta = -\int_{\mathbf{S}} f\cos\theta\,d\theta + \int_{\mathbf{S}} f\cos\theta\,d\theta = 0$$

and likewise $\int_{\mathbf{S}}(f''+f)\sin\theta\,d\theta = 0$. Using this in $f''+f=\nu_+-\nu_-$ gives

$$\int_{\mathbf{S}}\cos\theta\,d\nu_+ = \int_{\mathbf{S}}\cos\theta\,d\nu_-, \qquad \int_{\mathbf{S}}\sin\theta\,d\nu_+ = \int_{\mathbf{S}}\sin\theta\,d\nu_-.$$

Set

$$\begin{aligned} a &:= \frac{1}{\pi}\int_{\mathbf{S}}\cos\theta\,d\nu_+ = \frac{1}{\pi}\int_{\mathbf{S}}\cos\theta\,d\nu_-, \\ b &:= \frac{1}{\pi}\int_{\mathbf{S}}\sin\theta\,d\nu_+ = \frac{1}{\pi}\int_{\mathbf{S}}\sin\theta\,d\nu_-. \end{aligned}$$

Let $C>0$, to be chosen shortly, and set

$$\begin{aligned} \mu_+ &= \nu_+ + (C - a\cos\theta - b\sin\theta)\,d\theta \\ \mu_- &= \nu_- + (C - a\cos\theta - b\sin\theta)\,d\theta. \end{aligned}$$

There is an α so that $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta+\alpha)$. Thus if $C := \sqrt{a^2+b^2}$ the measures μ_+ and μ_- are non-negative. Using that in L^2 the function $\cos\theta$ is orthogonal to $\sin\theta$ and to the constants and that $\int_{\mathbf{S}}\cos^2\theta\,d\theta = \pi$

$$\begin{aligned} \int_{\mathbf{S}}\cos\theta\,d\mu_+ &= \int_{\mathbf{S}}\cos\theta\,d\nu_+ + \int_{\mathbf{S}}\cos\theta(C - a\cos\theta - b\sin\theta)\,d\theta \\ &= \int_{\mathbf{S}}\cos\theta\,d\nu_+ - a\int_{\mathbf{S}}\cos^2\theta\,d\theta = 0 \end{aligned}$$

and likewise all the other conditions of (3.2) hold. As $\|f''+f\|_{TV} = \|\nu_+\|_{TV}+\|\nu_-\|_{TV}$ we have $\min\{\|\nu_+\|_{TV},\|\nu_-\|_{TV}\} \le \frac{1}{2}\|f''+f\|_{TV}$. The formulas for a imply

$$\begin{aligned} |a| &\le \min\left\{\frac{1}{\pi}\int_{\mathbf{S}}|\cos\theta|\,d\nu_+(\theta),\ \frac{1}{\pi}\int_{\mathbf{S}}|\cos\theta|\,d\nu_-(\theta)\right\} \\ &\le \frac{1}{\pi}\min\left\{\|\nu_+\|_{TV},\ \|\nu_-\|_{TV}\right\} \le \frac{1}{2\pi}\|f''+f\|_{TV}. \end{aligned}$$

Likewise $|b| \le (2\pi)^{-1}\|f''+f\|_{TV}$. Using this in the definition of C gives $C \le \sqrt{2}(2\pi)^{-1}\|f''+f\|_{TV}$. As μ_+ and μ_- are non-negative measures their total variation is just their total mass. Thus

$$\begin{aligned} \|\mu_\pm\|_{TV} &= \int_{\mathbf{S}} 1\,d\mu_\pm = \int_{\mathbf{S}} 1\,\nu_\pm + \int_{\mathbf{S}}(C - a\cos\theta - b\sin\theta)\,d\theta \\ &= \|\nu_\pm\| + 2\pi C \le (1+\sqrt{2})\|f''+f\|_{TV} \\ &\le 3\|f''+f\|_{TV} \end{aligned}$$

This shows that (3.3) holds.

We claim that there is a function h_+ so that $h_+'' + h_+ = \mu_+$. To see this expand μ_+ in a Fourier series and use the equations (3.2) to see that the coefficients of sin and cos vanish.

$$\mu_+ = \frac{a_0}{2} + \sum_{k=2}^{\infty}(a_k \cos(k\theta) + b_k \sin(k\theta)).$$

Then h_+ is given explicitly by

$$h_+(\theta) = \frac{a_0}{2} + \sum_{k=2}^{\infty} \frac{a_k \cos(k\theta) + b_k \sin(k\theta)}{1 - k^2}.$$

The formulas $a_k = \pi^{-1} \int_{\mathbf{S}} \cos(k\theta)\, d\mu_+(\theta)$, $b_k = \pi^{-1} \int_{\mathbf{S}} \sin(k\theta)\, d\mu_+(\theta)$ imply that $|a_k|, |b_k| \leq 2\|\mu_+\|_{TV}$. Therefore the series defining h_+ converges uniformly and thus h_+ is continuous. Likewise there is a continuous function h_- with $h_-'' + h_- = \mu_-$. As μ_+ and μ_- are non-negative measures and formal differentiation of Fourier series corresponds to taking distributional derivatives, both h_+ and h_- are support functions.

Let $y = f - (h_+ - h_-)$. Then $y'' + y = 0$. This implies $y = \alpha \cos\theta + \beta \sin\theta$ for some constants α and β. Thus $f = (h_+ + \alpha\cos\theta + \beta\sin\theta) - h_-$. But $\alpha\cos\theta + \beta\sin\theta$ is the support function of the point (α, β). So $(h_+ + \alpha\cos\theta + \beta\sin\theta)$ is a support function, and f is a difference of support functions as required. Letting $h_1 = h_+ + \alpha\cos\theta + \beta\sin\theta$ and $h_2 = h_-$ then $f = h_1 - h_2$ and for $i = 1, 2$ and by (3.3) $\|h_i'' + h_i\|_{TV} = \|\mu_\pm\| \leq 3\|f'' + f\|_{TV}$. □

4. The affine isoperimetric inequality for arbitrary planar convex sets.

If h is a support function, then h is Lipschitz and therefore absolutely continuous. Therefore the distributional derivative h' of h is just the classical derivative which exists almost everywhere. As h is a support function then by Proposition 3.1 the second distributional derivative h'' is a measure and therefore h' is of bounded variation. By a theorem of Lebesgue, the function h' will be differentiable (in the classical sense) almost everywhere. Denote this derivative of h' by Dh' to distinguish it from the distributional derivative. In what follows we will denote classical derivatives of a function f by Df. As the first distributional derivative of h agrees with the classical derivative we have $Dh' = D^2h$ so that Dh' is the second classical derivative.

Recall, by a theorem of Alexandrov, a convex function on an n dimensional space, and thus a support function, has a generalized second derivative, called the Alexandrov second derivative, almost everywhere and in the one dimensional case the Alexandrov second derivative is just D^2h.

Various authors [6, 11, 8, 12] have extended the definition of affine arclength (and more generally higher dimensional affine surface area) from convex sets with C^2 boundary to general convex sets. It was eventually shown all these definitions are equivalent see i.e. [1] and, for two dimensional convex sets, are given in terms of the support function are given by

$$\int_{\mathbf{S}} (D^2 h + h)^{2/3}\, d\theta.$$

The following is the general form of the affine isoperimetric inequality in the plane.

Theorem 4.1. *Let K be an compact convex body in the plane with area A and affine perimeter Ω. Then*

$$\Omega \le 8\pi^2 A \tag{4.1}$$

with equality if and only if K is an ellipse.

Our proof is based on Theorem 1 and the following result which compares the distributional and classical derivatives of a support function.

Proposition 4.2. *Let $h\colon \mathbf{S} \to \mathbf{R}$ be the support function of a bounded convex set. Then distribution $h'' + h$ is of the form*

$$h'' + h = (D^2 h + h)\, d\theta + d\mu \tag{4.2}$$

where $d\theta$ is Lebesgue measure, the function $D^2h + h$ is in $L^1(\mathbf{S})$ and μ is a non-negative measure that is singular with respect to Lebesgue measure (i.e. there is a set N of Lebesgue measure zero with $\mu(S \setminus N) = 0$).

Proof. As $Dh = h'\colon [0, 2\pi] \to \mathbf{R}$ is of bounded variation it is a difference $h' = f_1 - f_2$ of monotone increasing functions f_1 and f_2. Then, [9, Thm. 3, p. 100], Df_i exists almost everywhere on $[0, 2\pi]$ and $Df_i \in L^1([0, 2\pi])$. For $i = 1, 2$ define $F_i\colon [0, 2\pi] \to \mathbf{R}$ by

$$F_i(\theta) = f_i(\theta) - \int_0^\theta Df_i(t)\, dt.$$

The function F_i is monotone increasing (this follows from another application of [9, Thm. 3 p. 100]) and by Lebesgue's theorem on the differentiability of indefinite integrals $DF_i = 0$ almost everywhere. Let μ_i be the Stieltjes measure defined by F_i. That is for $\phi \in C[0, 2\pi]$ we have $\int_{[0,2\pi]} \phi(\theta)\, d\mu_i(\theta) = \int_{[0,2\pi]} \phi(\theta)\, dF_i(\theta)$ where $\int_{[0,2\pi]} \phi(\theta)\, dF_i(\theta)$ is the Riemann-Stieltjes integral. The measure μ_i is also determined by $\mu_i([0, a]) = F_i(a) - F_i(0)$ at all continuity points of F_i. (As F_i is increasing it has at most countably may discontinuity points.) But then, [4, Thm. 19.60 p. 336], $DF_i = 0$ almost everywhere implies that μ_i is singular with respect to the Lebesgue measure. That is there is a Borel set N_i of Lebesgue measure zero such that $\mu_i([0, 2\pi] \setminus N_i) = 0$. Let $N = N_1 \cup N_2$ and $\mu = \mu_1 - \mu_2$. Note that for any subset A of $[0, 2\pi] \setminus N$ we have $\mu(A) = 0$.

Let ϕ be a smooth 2π periodic function. We look at how the measure μ_i acts as a distribution on ϕ by use of the integration by parts formula for

Stieltjes integrals,

$$\begin{aligned}
\int_{\mathbf{S}} \phi(\theta)\, d\mu_i(\theta) &= \int_0^{2\pi} \phi(\theta)\, dF_i(\theta) \\
&= F_i(\theta)\phi(\theta)\Big|_0^{2\pi} - \int_0^{2\pi} F_i(\theta)\phi'(\theta)\, d\theta \\
&= F_i(\theta)\phi(\theta)\Big|_0^{2\pi} - \int_0^{2\pi} \left(f_i(\theta) - \int_0^{\theta} Df_i(t)\, dt \right) \phi'(\theta)\, d\theta \\
&= F_i(\theta)\phi(\theta)\Big|_0^{2\pi} - \left[\left(f_i(\theta) - \int_0^{\theta} Df_i(t)\, dt \right) \phi(\theta) \right]_0^{2\pi} \\
&\qquad + \int_0^{2\pi} [f_i'(\theta) - Df_i(\theta)]\phi(\theta)\, d\theta \\
&= F_i(\theta)\phi(\theta)\Big|_0^{2\pi} - F_i(\theta)\phi(\theta)\Big|_0^{2\pi} + \int_0^{2\pi} [f_i'(\theta) - Df_i(\theta)]\phi(\theta)\, d\theta \\
&= \int_0^{2\pi} [f_i'(\theta) - Df_i(\theta)]\phi(\theta)\, d\theta.
\end{aligned}$$

This shows that as distributions

$$f_i' = Df_i\, d\theta + d\mu_i.$$

As $h' = f_1 - f_2$ and the distributional and classical derivatives are linear this implies

$$h'' = (f_1 - f_2)' = Dh'\, d\theta + d\mu = D^2 h\, d\theta + d\mu$$

Adding $h\, d\theta$ to both sides of this gives (4.2).

All that remains is to show that μ is a non-negative measure. If μ is not non-negative, then, by the inner regularity of Borel measures, there is a compact set $M \subset N$ with $\mu(M) < 0$. As N has measure zero, the set M also has Lebesgue measure zero. Let $g_k \colon \mathbf{S} \to \mathbf{R}$ be the continuous function defined by

$$g_k(\theta) = \max\{0, 1 - k \operatorname{distance}(\theta, M)\}.$$

Then $g_k(\theta) = 1$ for $\theta \in M$ and $g_k(\theta) \to 0$ when $\theta \notin M$. So if χ_M is the characteristic function of M we have $g_k \to \chi_M$ pointwise. Thus by Lebesgue's dominated convergence theorem (using that $D^2 h + h \geq 0$ is integrable)

$$\lim_{k\to\infty} \int_{\mathbf{S}} (D^2h+h) g_k\, d\theta = \int_M (D^2h+h)\, d\theta = 0, \quad \lim_{k\to\infty} \int_{\mathbf{S}} g_k\, d\mu(\theta) = \mu(M) < 0.$$

As h is a support function, $h'' + h$ is a non-negative measure. Therefore

$$\begin{aligned}
0 &\leq \lim_{k\to\infty} \int_{\mathbf{S}} (h'' + h) g_k\, d\theta \\
&= \lim_{k\to\infty} \int_{\mathbf{S}} (D^2 h + h) g_k\, d\theta + \lim_{k\to\infty} \int_{\mathbf{S}} g_k\, d\mu(\theta) = \mu(M) < 0.
\end{aligned}$$

This contradiction completes the proof. $\square$

Proof of Theorem 4.1. Let h_o be the support function of a planar convex body K. In the proof of Lemma 6.3 we will see that it is possible to choose a_o and b_o so that $h(\theta) := h_o(\theta) + a_o \cos\theta + b_o \sin\theta$ is positive on $\mathbf{S}$ and

$$\int_{\mathbf{S}} \frac{\cos\theta}{h^3(\theta)}\, d\theta = \int_{\mathbf{S}} \frac{\sin\theta}{h^3(\theta)}\, d\theta = 0. \tag{4.3}$$

(The idea is to choose (a_o, b_o) to minimize $f(a,b) = \int_{\mathbf{S}} (h_o(\theta) + a\cos\theta + b\sin\theta)^{-2}\, d\theta$ over the set of $(a,b) \in \mathbf{R}^2$ such that the origin is in the interior of $K + (a,b)$.) Using the relation between $D^2h + h$ and $h'' + h$ given by Proposition 4.2 we have

$$\int_{\mathbf{S}} h(D^2h + h)\, d\theta \le \int_{\mathbf{S}} h(h'' + h)\, d\theta.$$

This observation, preceded by Hölder's inequality, gives,

$$\begin{aligned}
\int_{\mathbf{S}} (D^2h + h)^{2/3}\, d\theta &= \int_{\mathbf{S}} h^{-2/3} h^{2/3} (D^2h + h)^{2/3}\, d\theta \\
&\le \left(\int_{\mathbf{S}} \frac{d\theta}{h^2}\right)^{1/3} \left(\int_{\mathbf{S}} h(D^2h + h)\, d\theta\right)^{2/3} \\
&\le \left(\int_{\mathbf{S}} \frac{d\theta}{h^2}\right)^{1/3} \left(\int_{\mathbf{S}} h(h'' + h)\, d\theta\right)^{2/3}.
\end{aligned}$$

Thus

$$\left(\int_{\mathbf{S}} (D^2h + h)^{2/3}\, d\theta\right)^3 \le \left(\int_{\mathbf{S}} \frac{d\theta}{h^2}\right) \left(\int_{\mathbf{S}} h(h'' + h)\, d\theta\right)^2. \tag{4.4}$$

In Theorem 1 take $F = h^{-6}$. Then (4.3) shows that conditions (2.1) are satisfied. Therefore

$$\begin{aligned}
\left(\int_{\mathbf{S}} \frac{d\theta}{h^2}\right)^3 \left(\int_{\mathbf{S}} h(h'' + h)\, d\theta\right) &= \left(\int_{\mathbf{S}} \frac{d\theta}{h^2}\right)^3 \left(\int_{\mathbf{S}} [h^2 - (h')^2]\, d\theta\right) \\
&\le 4\pi^2 \left(\int_{\mathbf{S}} \frac{d\theta}{h^2}\right)^2.
\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) and using the fact that $D^2h + h = D^2h_o + h_o$ and $\int_{\mathbf{S}} h(h'' + h)\, d\theta = \int_{\mathbf{S}} h_o(h_o'' + h_o)\, d\theta$ gives

$$\left(\int_{\mathbf{S}} (D^2h_o + h_o)^{2/3}\, d\theta\right)^3 \le 4\pi^2 \int_{\mathbf{S}} h_o(h_o'' + h_o)\, d\theta.$$

This is the affine isoperimetric inequality for K.

If equality holds, then the equality conditions of Theorem 1 imply h is the support function of an ellipse centered at the origin. Thus $h_o = h - a\cos\theta - b\sin\theta$ is the support function of an ellipse centered at $(-a, -b)$. $\square$

5. A family of transforms.

Let $\mathbf{S}$ be the unit circle in $\mathbf{R}^2$ with coordinate θ as above. For each $\lambda \in (0, \infty)$, let

$$\psi_\lambda(\theta) = \sqrt{\lambda^2 \cos^2\theta + \frac{1}{\lambda^2}\sin^2\theta}.$$

Define on $\mathbf{S}$ a family of mappings

$$m_\lambda(\theta) = \int_0^\theta \frac{dt}{\psi_\lambda^2(t)}.$$

When $\lambda = 1$, this is the identity map. For $0 \le \theta < \frac{\pi}{2}$ it is easy to verify that

$$m_\lambda(\theta) = \arctan\left(\frac{1}{\lambda^2}\tan\theta\right). \tag{5.1}$$

For any measurable function u on $\mathbf{S}$, define the transform

$$(T_\lambda u)(\theta) = u(m_\lambda(\theta))\psi_\lambda(\theta).$$

Lemma 5.1. *Let u and v be measurable functions on $\mathbf{S}$ for which the integrals below exist. Then*

(i) The mappings $m_\lambda(\cdot)$ each leave four points fixed:

$$m_\lambda(0) = 0, \quad m_\lambda\left(\frac{\pi}{2}\right) = \frac{\pi}{2}, \quad m_\lambda(\pi) = \pi, \quad m_\lambda\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2}.$$

(ii) The transforms leave the following integrals invariant:

$$\int \frac{d\theta}{(T_\lambda u)^2} = \int \frac{d\theta}{u^2}\ , \tag{5.2}$$

$$\int \frac{T_\lambda u}{(T_\lambda v)^3}\, d\theta = \int \frac{u}{v^3}\, d\theta, \tag{5.3}$$

$$\int_{\mathbf{S}} \{(T_\lambda u)^2 - [(T_\lambda u)']^2\}\, d\theta = \int_{\mathbf{S}} [u^2 - (u')^2]\, d\theta, \tag{5.4}$$

$$\int \frac{\cos\theta}{(T_\lambda u)^3(\theta)}\, d\theta = \frac{1}{\lambda}\int \frac{\cos\theta}{u^3(\theta)}\, d\theta, \tag{5.5}$$

$$\int \frac{\sin\theta}{(T_\lambda u)^3(\theta)}\, d\theta = \lambda\int \frac{\sin\theta}{u^3(\theta)}\, d\theta. \tag{5.6}$$

Here "$\int$" represents the integral with respect to $d\theta$ on any of the intervals $[0, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \pi]$, $[\pi, \frac{3\pi}{2}]$, $[\frac{3\pi}{2}, 2\pi]$, or $[0, 2\pi]$. For (5.2), (5.3), (5.5), *and* (5.6) *u and v can be any measurable functions for which the integrals converge. In* (5.4) *$u \in H^1(\mathbf{S})$.*

Proof. (i) From (5.1), one can see that $m_\lambda(0) = 0$, $m_\lambda(\frac{\pi}{2}) = \frac{\pi}{2}$. Since the integrand is symmetric about $\theta = \frac{\pi}{2}$ on $[0, \pi]$, and is π-periodic, in follows that π and $\frac{3\pi}{2}$ are also fixed points of m_λ.

(ii) We verify invariance for the integrals on the interval $[0, \frac{\pi}{2}]$. Then by the results in (i) and the symmetry of $\psi_\lambda(\theta), \cos\theta$, and $\sin\theta$, the invariance of the integrals on the other intervals follows.

Equations (5.2) and (5.3) are direct consequences from the substitution $\tilde{\theta} = m_\lambda(\theta)$.

As $C^2(\mathbf{S})$ is dense in $H^1(\mathbf{S})$ it is enough to verify (5.4) in the case $u \in C^2(\mathbf{S})$. After integrating by parts, we only need to show

$$\int_{\mathbf{S}} T_\lambda u[T_\lambda u + (T_\lambda u)''] \, d\theta = \int_{\mathbf{S}} u(u + u'') \, d\theta.$$

We employ the fact that ψ_λ is a solution of the equation

$$\psi_\lambda'' + \psi_\lambda = \frac{1}{\psi_\lambda^3}. \tag{5.7}$$

It follows from (5.7) and a straightforward calculation that

$$T_\lambda u[T_\lambda u + (T_\lambda u)''] = u(m_\lambda(\theta))[u(m_\lambda(\theta)) + u''(m_\lambda(\theta))] \frac{1}{\psi_\lambda^2(\theta)}.$$

Again using the change of variable $\tilde{\theta} = m_\lambda(\theta)$, we see (5.4) holds.

To obtain (5.5) and (5.6), we write $\tan\theta = \lambda^2 \tan\tilde{\theta}$. It follows that

$$\frac{\cos\theta}{\psi_\lambda(\theta)} = \frac{1}{\sqrt{\lambda^2 + \frac{1}{\lambda^2}\tan^2\theta}} = \frac{1}{\lambda}\cos\tilde{\theta}, \quad \frac{\sin\theta}{\psi_\lambda(\theta)} = \frac{1}{\sqrt{\frac{\lambda^2}{\tan^2\theta} + \frac{1}{\lambda}}} = \lambda \sin\tilde{\theta}$$

and another application of the substitution $\tilde{\theta} = m_\lambda(\theta)$ completes the proofs of (5.5) and (5.6). □

Remark 5.2. Let u be a positive continuous function on $\mathbf{S}$, then the integral of $\frac{d\theta}{(T_\lambda u)^2}$ is independent of λ. It is not hard to check that as $\lambda \to \infty$ the mass of $\frac{d\theta}{(T_\lambda u)^2}$ concentrates about the points $\pi/2$ and $3\pi/2$ and when $\lambda \to 0$ the mass concentrates about 0 and π.

6. Proof of the main inequality.

6.1. Some lemmas.

Lemma 6.1. *If $\{u_k\}$ is a bounded sequence in $H^1(\mathbf{S})$, then there exists a subsequence (still denoted by $\{u_k\}$) and $u_o \in H^1(\mathbf{S})$, such that $u_k \to u_o$ in the weak topology of $H^1(\mathbf{S})$,*

$$u_k \to u_o \quad \text{in } C^\beta(\mathbf{S}) \quad \text{for all} \quad \beta < \frac{1}{2}. \tag{6.1}$$

This implies

$$\limsup_{k\to\infty} \int_{\mathbf{S}} [u_k^2 - (u_k')^2] \, d\theta \le \int_{\mathbf{S}} [u_o^2 - (u_o')^2] \, d\theta. \tag{6.2}$$

Moreover, if $u_o(\theta_0) = 0$ *at some point* θ_0*, then for* $\delta > 0$

$$\int_{\theta_0}^{\theta_0+\delta} \frac{d\theta}{u_o^2} = \infty \quad \textit{and} \quad \int_{\mathbf{S}} \frac{d\theta}{u_k^2} \to \infty \quad \textit{as} \quad k \to \infty. \tag{6.3}$$

Lemma 6.2. *Assume* $\{u_k\}$ *is a bounded sequence in* $H^1(\mathbf{S})$ *with* $u_k > 0$*,* $u_k \to u_o \in H^1(\mathbf{S})$ *in the weak topology,*

$$\int_{\mathbf{S}} \frac{\cos\theta}{u_k^3}\, d\theta = 0 = \int_{\mathbf{S}} \frac{\sin\theta}{u_k^3}\, d\theta, \tag{6.4}$$

and that u_o *has at least one zero. Then, viewing the zeros of* u_o *as a subset of* $\mathbf{S} \subset \mathbf{R}^2$*,*

$$(0,0) \in \textit{convex hull of the zeros of } u_o. \tag{6.5}$$

If u_o *has three or more zeros then*

$$\int_{\mathbf{S}} [u_k^2 - (u_k')^2]\, d\theta < 0 \quad \textit{for sufficiently large } k. \tag{6.6}$$

Lemma 6.3. *Suppose the inequality* (2.2) *holds under the stronger conditions:*

i) F *is measurable and positive on* $\mathbf{S}$ *and* $h \in H^1(\mathbf{S})$ *is positive;*

ii) F *satisfies the orthogonality conditions* (2.1) *and* h *satisfies orthogonality conditions*

$$\int_{\mathbf{S}} \frac{\cos\theta}{h^3}\, d\theta = 0 = \int_{\mathbf{S}} \frac{\sin\theta}{h^3}\, d\theta. \tag{6.7}$$

Then the same inequality (2.2) *holds without the orthogonality conditions* (6.7) *on* h *and the strict positivity of* F*.*

Proof of Lemma 6.1. That there is a $u_o \in H^1(\mathbf{S})$ and a subsequence with $u_k \to u_o$ in the weak topology follows from the weak compactness of the closed balls in a Hilbert space. Then (6.1) is a direct consequence of the compact Sobolev imbedding of $H^1(\mathbf{S})$ into $C^\beta(\mathbf{S})$ for any $\beta < \frac{1}{2}$. To prove (6.2) use the fact that the norm of a Hilbert space is lower semi-continuous with respect to weak convergence and thus $\liminf_{k\to\infty} \int_{\mathbf{S}} [u_k']^2\, d\theta \geq \int_{\mathbf{S}} [u_o']^2\, d\theta$. From (6.1) $\lim_{k\to\infty} \int_{\mathbf{S}} u_k^2\, d\theta = \int_{\mathbf{S}} u_o^2\, d\theta$. Together these imply (6.2).

Assume that u_o vanishes at θ_0. Then by the Sobolev imbedding $H^1(\mathbf{S}) \subset C^{\frac{1}{2}}(\mathbf{S})$, or an elementary Hölder inequality argument, $u_o \in C^{\frac{1}{2}}(\mathbf{S})$ and therefore $|u_o(\theta)| = |u_o(\theta) - u_o(\theta_0)| \leq C_1\sqrt{|\theta - \theta_0|}$ which implies the divergence of the integral $\int_{\theta_0}^{\theta_0+\delta} u_o^{-2} d\theta$. As $u_k \to u_o$ uniformly this implies the second part of (6.3) and completes the proof of the Lemma. □

Proof of Lemma 6.2. Because the imbedding of $H^1(\mathbf{S})$ into $C^\beta(\mathbf{S})$ is compact for $\beta \in [0, 1/2)$ the weak convergence $u_k \to u_o$ implies $\{u_k\}$ converges to u_o uniformly. By Lemma 6.1 the integral $\int_{\mathbf{S}} u_o^{-2}\, d\theta$ diverges and therefore $\int_{\mathbf{S}} u_o^{-3}\, d\theta$ also diverges. Thus $\int_{\mathbf{S}} u_k^{-3}\, d\theta \to \infty$ as $k \to \infty$. Let $c_k := \left(\int_{\mathbf{S}} u_k^{-3}\, d\theta\right)^{-1}$. Then $c_k u_k^{-3}(\theta) d\theta$ is a probability measure on $\mathbf{S}$ and the conditions (6.4) imply the center of mass of this measure is $(0,0)$. But as $k \to \infty$ the masses of the measures $c_k u_k^{-3}(\theta)d\theta$ concentrate at the zeros of u_0. This implies (6.5).

If u_o has three or more zeros, then the convex hull property (6.5) implies there are three zeros θ_1, θ_2, θ_3 of u_o such that

(6.8) The zeros $\theta_1, \theta_2, \theta_3$ of u_o are not on an arc of length less than π.

We will show this implies

$$\int_{\mathbf{S}} [u_o^2 - (u_o')^2]\, d\theta < 0. \tag{6.9}$$

Which, by (6.2) of Lemma 6.1, implies (6.6).

To see (6.9), we write the integral in three parts:

$$\int_{\mathbf{S}} [u_o^2 - (u_o')^2]\, d\theta = \left\{ \int_{\theta_1}^{\theta_2} + \int_{\theta_2}^{\theta_3} + \int_{\theta_3}^{\theta_1} \right\} [u_o^2 - (u_o')^2]\, d\theta = I_1 + I_2 + I_3. \tag{6.10}$$

From (6.8), we see that lengths of intervals of integration in (6.10) are all less than or equal to π, and at least two of them are strictly less than π. But, [2, p. 185], if $\theta_{i+1} - \theta_i \le \pi$, then $u_o(\theta_i) = u_o(\theta_{i+1}) = 0$ implies $\int_{\theta_i}^{\theta_{i+1}} [u_o^2 - (u_o')^2]\, d\theta < 0$ unless $\theta_{i+1} - \theta_i = \pi$ and $u_o = C \sin\theta$ on $[\theta_i, \theta_{i+1}]$. This proves (6.9) and completes the proof of the lemma. □

Proof of Lemma 6.3. We assume that the inequality (2.2) holds under the assumptions i) and ii) of Lemma 6.3. We first claim that for each positive function $h \in C^2(\mathbf{S})$, there exists an $h_o(\theta) = a_o \cos\theta + b_o \sin\theta + h(\theta)$ that satisfies the orthogonality conditions (6.7). To see this minimize the function

$$f(a,b) = \int_{\mathbf{S}} \frac{1}{(a\cos\theta + b\sin\theta + h(\theta))^2} d\theta$$

for any real numbers a and b, such that $a\cos\theta + b\sin\theta + h(\theta) > 0$ for all θ. It is obvious that $f(a,b)$ is bounded from below by zero. Let $\{h_k(\theta) = a_k \cos\theta + b_k \sin\theta + h(\theta)\}$ be a minimizing sequence. From $h_k(t) > 0$, one can easily see that $\{a_k\}$ and $\{b_k\}$ are bounded, and hence there exist subsequences converging to some $a_o, b_o \in \mathbf{R}$. Then (a_o, b_o) is a minimizer of f.

Moreover, from Lemma 6.1, we can see that $h_o(\theta) = a_o \cos\theta + b_o \sin\theta + h(\theta) > 0$. (Otherwise h_o has a zero and (by Lemma 6.1) $\int_{\mathbf{S}} h_o^{-2} d\theta = \infty$, contradicting that h_o is a minimizer.) Consequently, at (a_o, b_o), we have $\partial f/\partial a = 0 = \partial f/\partial b$. This implies the orthogonality conditions (6.7) on h_o.

We now show that if inequality (2.2) holds for $h_o = a_o \cos\theta + b_o \sin\theta + h(\theta)$, then it is also holds for h. By the orthogonality conditions (2.1) on F.

$$\int_{\mathbf{S}} F^{\frac{1}{2}}(\theta) h_o(\theta)\, d\theta = \int_{\mathbf{S}} F^{\frac{1}{2}}(\theta) h(\theta)\, d\theta, \tag{6.11}$$

and if h is of class C^2 we can use use integration by parts and the fact that both $\sin\theta$ and $\cos\theta$ are in the kernel of the differential operator $d^2/d\theta^2 + 1$

to get

$$\begin{aligned}\int_{\mathbf{S}}[h_o^2-(h_o')^2]\,d\theta &= \int_{\mathbf{S}} h_o(h_o''+h_o)\,d\theta \\ &= \int_{\mathbf{S}} h(h''+h)\,d\theta = \int_{\mathbf{S}}[h^2-(h')^2]\,d\theta. \qquad (6.12)\end{aligned}$$

This will also hold for $h \in H^1(\mathbf{S})$ by approximating by C^2 functions. So if (2.2) holds for h_o and F, then (6.11) and (6.12) show it holds for h and F.

To see that inequality (2.2) holds also for non-negative continuous functions F and non-negative $h \in H^1(\mathbf{S})$, we let

$$F_\epsilon = (F^{\frac{1}{2}}+\epsilon)^2 \quad \text{and} \quad h_\epsilon = h+\epsilon.$$

Then obviously, for each $\epsilon > 0$, both F_ϵ and h_ϵ are positive, and F_ϵ satisfies the orthogonality conditions (2.1). Therefore inequality (2.2) holds for F_ϵ and h_ϵ. Take the limit as $\epsilon \to 0$ to see that see that (2.2) is also holds for F and h.

Finally the extensions to F non-negative and measurable follows by approximating F by positive functions satisfying the orthogonality conditions (2.1) and taking limits. □

6.2. **Outline of the Proof.** Let

$$G := \left\{ u \in H^1(\mathbf{S}) : u > 0, \int_{\mathbf{S}} \frac{d\theta}{u^3} < \infty, \int_{\mathbf{S}} \frac{\cos\theta}{u^3}\,d\theta = \int_{\mathbf{S}} \frac{\sin\theta}{u^3}\,d\theta = 0 \right\}$$

Then by substituting $F = v^{-6}$ and $h = u$ and using Lemma 6.3 we see that Theorem 1 is equivalent to showing that for all $u \in G$ and measurable $v > 0$ with

$$\int_{\mathbf{S}} \frac{d\theta}{v^3} < \infty \qquad (6.13)$$

the inequality

$$\left(\int_{\mathbf{S}} \frac{d\theta}{v^2}\right)^3 \left(\int_{\mathbf{S}} [u^2-(u')^2]d\theta\right) \le 4\pi^2 \left(\int_{\mathbf{S}} \frac{u}{v^3} d\theta\right)^2 \qquad (6.14)$$

holds with equality if and only if

$$u = k_1\sqrt{\lambda^2\cos^2(\theta-\alpha)+\frac{1}{\lambda^2}\sin^2(\theta-\alpha)} \qquad (6.15)$$

and

$$v = k_2\sqrt{\lambda^2\cos^2(\theta-\alpha)+\frac{1}{\lambda^2}\sin^2(\theta-\alpha)} \qquad (6.16)$$

for any positive constants constants k_1, k_2, and λ and any $\alpha \in \mathbf{R}$. This follows from:

Proposition 6.4. *For all $u \in G$*

$$\left(\int_{\mathbf{S}} \frac{d\theta}{u^2}\right)\left(\int_{\mathbf{S}} [u^2 - (u')^2]\, d\theta\right) \le 4\pi^2 \tag{6.17}$$

with equality if and only if u is of the form (6.15).

To see this implies inequality (6.14), and thus Theorem 1, assume the proposition holds. Then for any $u \in G$ and positive measurable v such that (6.13) holds the integal $\int_{\mathbf{S}} uv^{-3}\, d\theta$ is finite because of (6.13) and that u is bounded (as it is in $H^1(\mathbf{S})$ and therefore continuous). So by Hölder's inequality

$$\int_{\mathbf{S}} \frac{d\theta}{v^2} = \int_{\mathbf{S}} \frac{1}{u^{\frac{2}{3}}} \frac{u^{\frac{2}{3}}}{v^2}\, d\theta \le \left(\int_{\mathbf{S}} \frac{d\theta}{u^2}\right)^{\frac{1}{3}} \left(\int_{\mathbf{S}} \frac{u}{v^3}\, d\theta\right)^{\frac{2}{3}}$$

with equality if and only if $v = ku$ for some $k > 0$. This implies

$$\left(\int_{\mathbf{S}} \frac{d\theta}{v^2}\right)^3 \le \left(\int_{\mathbf{S}} \frac{d\theta}{u^2}\right)\left(\int_{\mathbf{S}} \frac{u}{v^3}\, d\theta\right)^2 \tag{6.18}$$

with equality if and only if $v = ku$. Multiply both sides of (6.17) by $\left(\int_{\mathbf{S}} uv^{-3}\, d\theta\right)^3$ and use (6.18) to arrive at (6.14) with equality if and only if u and v are of the form (6.15) and (6.16).

Define a functional J on G by

$$J[u] := \left(\int_{\mathbf{S}} \frac{d\theta}{u^2}\right)\left(\int_{\mathbf{S}} [u^2 - (u')^2]\, d\theta\right).$$

Then we wish to show that for $u \in G$, that $J[u] \le 4\pi^2$ with equality if and only if u is of the form (6.15). Note that for any positive constant s that

$$J[su] = J[u]. \tag{6.19}$$

The proof proceeds in two steps. First we argue that there is a finite constant $C > 0$ so that $J[u] \le C$ for all $u \in G$. This is done by assuming there is a sequence $\{u_k\} \subset G$ with $J[u_k] \to \infty$, and using the transforms T_λ to replace $\{u_k\}$ with a sequence $\{w_k\} \subset G$ with properties that lead to a contradiction. Second we study a maximizing sequence $\{u_k\}$ for the functional $J[u]$. Such a sequence may be unbounded, but the same argument used to construct the sequence $\{w_k\}$ allows us to replace $\{u_k\}$ by a convergent sequence $\{w_k\}$. This shows that maximizers exist. We then use the Euler-Lagrange equations for the maximizer to show the maximizers are of the form (6.15) to complete the proof.

6.3. **Part I: Existence of a finite upper bound for $J[u]$.** In this part, we show that there exists a constant $C < \infty$, such that

$$J[u] \le C, \quad \text{for all} \quad u \in G. \tag{6.20}$$

Assume, toward a contradiction, that there is a sequence $\{\tilde{u}_k\} \subset G$ with $J[\tilde{u}_k] \to \infty$ as $k \to \infty$. Let $u_k = \|\tilde{u}_k\|_{H^1}^{-1}\, \tilde{u}_k$. By (6.19) $J[u_k] = J[\tilde{u}_k]$, so

$$\|u_k\|_{H^1} = 1, \quad \text{and} \quad J[u_k] \to \infty. \tag{6.21}$$

By Lemma 6.1 we can replace $\{u_k\}$ by a subsequence and assume there is $u_o \in H^1(\mathbf{S})$ with $u_k \to u_o$ in the weak topology of $H^1(\mathbf{S})$ and

$$u_k \to u_o \quad \text{in } C^\beta(\mathbf{S}) \text{ for all } \beta < \frac{1}{2}. \tag{6.22}$$

The first part of (6.21) implies

$$\int_{\mathbf{S}} [u_k^2 - (u_k')^2]\, d\theta \le \int_{\mathbf{S}} [u_k^2 + (u_k')^2]\, d\theta = \|u_k\|_{H^1}^2 = 1. \tag{6.23}$$

Using this in the definition of J gives

$$J[u_k] \le \int_{\mathbf{S}} \frac{d\theta}{u_k^2}. \tag{6.24}$$

As $u_k \to u_o$ uniformly, if u_o has no zeros, then $\limsup_{k\to\infty} J[u_k] \le J[u_o]$ contradicting that $J[u_k] \to \infty$.

Without loss of generality we assume that $J[u_k] > 0$ for all k. We have shown that u_o has at least one zero and therefore by the convex hull property of Lemma 6.2 the point $(0, 0)$ is in the convex hull of the zeros of u_o. This implies that u_0 has at least two zeros. If u_o has three or more zeros then (6.6) of Lemma 6.2 implies that $J[u_k] < 0$ which is not the case. Thus u_o has exactly two zeros.

As u_o has exactly two zeros, the convex hull property (6.5) implies the two zeros must be antipodal, say they are at $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. Obviously, at these two points, $u_k^{-2} \to \infty$. For each u_k, pick a point p_k near $\pi/2$, such that

$$\int_{p_k-\frac{\pi}{2}}^{p_k} \frac{1}{u_k^2}\, d\theta = \int_{p_k}^{p_k+\frac{\pi}{2}} \frac{1}{u_k^2}\, d\theta. \tag{6.25}$$

Then, $p_k \to \pi/2$. For a number $\delta > 0$ (to be concrete $\delta = \pi/4$ will work), let

$$D_k^1 = \left\{\theta \mid p_k - \frac{\pi}{2} \le \theta \le p_k - \delta\right\}, \quad D_k^2 = \left\{\theta \mid p_k + \delta \le \theta \le p_k + \frac{\pi}{2}\right\}$$

$$D_k = D_k^1 \cup D_k^2.$$

We will apply the family of transforms T_λ introduced in Section 5. We say that the family of transforms T_λ in Lemma 6.1 are centered at $\frac{\pi}{2}$, and write $T_\lambda = T_{\lambda,\frac{\pi}{2}}$. Similarly, one can define transforms centered at any point q, and denote them by $T_{\lambda,q}$.

By Lemma 5.1 and Remark 5.2 for each u_k, one can choose a transform $T_k = T_{\lambda_k,p_k}$, such that

$$\int_{B_\delta(p_k)} \frac{d\theta}{(T_k u_k)^2} = \int_{D_k} \frac{d\theta}{(T_k u_k)^2}, \tag{6.26}$$

where $B_\delta(p_k) = (p_k - \delta, p_k + \delta)$. Let $w_k = \|T_k u_k\|_{H^1}^{-1} T_k u_k$. Then $\|w_k\|_{H^1} = 1$ and we can apply Lemma 6.1 to the sequence $\{w_k\}$ and find a subsequence, still denoted by $\{w_k\}$, and a $w_o \in H^1(\mathbf{S})$ such that $w_k \to w_o$ in the weak topology of H^1 and $w_k(\theta) \to w_o(\theta)$ in C^β for all $\beta < \frac{1}{2}$.

The lemma is used both in the proof that J is bounded above on G and later in the proof that J has maximizers in G.

Lemma 6.5. *If $\{u_k\} \subset G$ is a sequence with $J[u_k] > 0$ for all k and $w_k := \|T_k u_k\|_{H^1}^{-1} T_k u_k$ (so that $\|w_k\|_{H^1} = 1$ and $J[w_k] = J[u_k]$) where the transforms T_k are chosen so that* (6.26) *holds, then any weak limit of a subsequence of $\{w_k\}$ is non-vanishing.*

Assuming this lemma we show that J is bounded above on G. For if not, there is a sequence $\{u_k\} \subset G$ with $J[u_k] \to \infty$, and we can assume $J[u_k] > 0$ for all k. Let $w_k = \|T_k u_k\|_{H^1}^{-1} T_k u_k$ be as in Lemma 6.5. By Lemma 6.1, by going to a subsequence, we can assume there is a $w_o \in H^1(\mathbf{S})$ such that $w_k \to w_o$ in the weak topology of $H^1(\mathbf{S})$ and $w_k \to w_o$ in $C^\beta(\mathbf{S})$ for all $\beta < 1/2$. Because $\|w_k\|_{H^1} = 1$, we have, as in (6.24), that $J[w_k] \leq \int_{\mathbf{S}} w_k^{-2}\, d\theta \to \int_{\mathbf{S}} w_o^{-2}\, d\theta$. But w_o non-vanishing implies $\int_{\mathbf{S}} w_o^{-2}\, d\theta < \infty$ which would contradict that $J[w_k] = J[u_k] \to \infty$ and completes the proof that J is bounded above on G.

Proof of Lemma 6.5. Assume, toward a contradiction, that w_o has at least one zero. By the convex hull property of Lemma 6.2

$$(0,0) \in \text{convex hull of the zeros of } w_o. \tag{6.27}$$

This implies that w_o has at least two zeros and if w_o has three or more zeros then Lemma 6.2 implies $J[w_k] < 0$ for large k, which again would contradict that the assumption that $J[u_k] = J[w_k] > 0$. Therefore w_o has exactly two zeros and by the convex hull property (6.27) these zeros are antipodal. Let the zeros be θ_0 and θ_1 and we can assume that $\theta_0 \in [0, \pi]$. Then $\int_0^\pi w_o^{-2}\, d\theta = \infty$ (by (6.3)), $w_k \to w_o$ uniformly, and $p_k \to \pi/2$ imply

$$\int_{p_k - \frac{\pi}{2}}^{p_k + \frac{\pi}{2}} \frac{d\theta}{w_k^2} \to \infty. \tag{6.28}$$

From (6.25), (6.26) and the properties of the transforms, we also have

$$\int_{p_k - \frac{\pi}{2}}^{p_k} \frac{d\theta}{w_k^2} = \int_{p_k}^{p_k + \frac{\pi}{2}} \frac{d\theta}{w_k^2}, \tag{6.29}$$

and

$$\int_{B_\delta(p_k)} \frac{d\theta}{w_k^2} = \int_{D_k} \frac{d\theta}{w_k^2}. \tag{6.30}$$

Now (6.28), (6.29), and (6.30) imply that the integrals of w_k^{-2} on all the four sets

$$[p_k - \frac{\pi}{2}, p_k],\ [p_k, p_k + \frac{\pi}{2}],\ B_\delta(p_k),\ \text{and } D_k$$

approach infinity. Therefore w_o has at least one zero on each of the following sets

$$[0, \frac{\pi}{2}],\ [\frac{\pi}{2}, \pi],\ B_\delta(\frac{\pi}{2}),\ \text{and } [0,\pi] \setminus B_\delta(\frac{\pi}{2}). \tag{6.31}$$

From this we see that w_o has at least two zeros on the closed upper half circle. As the two zeros of w_o are antipodal they must be 0 and π. But as $B_\delta(\pi/2)$ also contains a zero this implies that w_o has three zeros, a contradiction. This finishes the proof of Lemma 6.5 and therefore also the proof that $J[u]$ is bounded above on G. $\square$

6.4. **Part II: Existence and characterization of maximizers.** In Part I, we have shown that there is a $C > 0$ with $J[u] \leq C$ for all $\in G$. We now prove the existence of maximizers for $J[u]$. Let $\{\tilde{u}_k\} \subset H^1(\mathbf{S})$ be a maximizing sequence for $J[u]$. We can assume that $J[\tilde{u}_k] > 0$ for all k. By going to a subsequence we can assume that $\tilde{u}_k \to u_o$ in the weak topology of $H^1(\mathbf{S})$ and that $\tilde{u}_k \to u_o$ uniformly. If u_o is non-vanishing, then let $u_k = \|\tilde{u}_k\|_{H^1}^{-1} \tilde{u}_k$.

If u_o has a zero, then the convex hull property (6.5) implies that u_o has at least two zeros, and Lemma 6.2 implies that if u_o has three or more zeros that $J[\tilde{u}_k] < 0$ for large k which is not the case. Therefore u_o has exactly two zeros which are antipodal. Without lose of generality we can assume they are $\pi/2$ and $3\pi/2$. Then we can construct a sequence $\left\{w_k = \|T_k\tilde{u}_k\|_{H^1}^{-1} T_k\tilde{u}_k\right\}$ as in Lemma 6.5. Then using Lemma 5.2 and Lemma 6.5 we can replace $\{w_k\}$ by one of its subsequences and assume $w_k \to w_o$ and that w_o is non-vanishing.

Putting these two cases together we can either let $u_k := \|\tilde{u}_k\|_{H^1}^{-1} \tilde{u}_k$, or $u_k := \|T_k\tilde{u}_k\|_{H^1}^{-1} T_k\tilde{u}_k$ and pass to a subsequence and assume there is a non-vanishing $u_o \in H^1(\mathbf{S})$ such that $u_k \to u_o$ in the weak topology of $H^1(\mathbf{S})$ and $u_k \to u_o$ in $C^\beta(\mathbf{S})$ for all $\beta \in [0, 1/2)$. Also $J[u_k] = J[\tilde{u}_k]$ and therefore $\{u_k\}$ is also a maximizing sequence. Because u_o is non-vanishing and $u_k \to u_o$ uniformly we have that (6.2) of Lemma 6.1 implies

$$\lim_{k\to\infty} J[u_k] \leq J[u_o].$$

Therefore u_o is a maximizer of J on G. This proves the existence of maximizers for J on G.

We need to show that any maximizer of $J[u_o]$ on G is of the from (6.15). Let u_o be a maximizer of $J[u]$ on G. The Lagrange multiplier equation for the variational problem of maximizing $J[u]$ subject to $\int_{\mathbf{S}} \frac{1}{u^3} \cos\theta \, d\theta = 0 = \int_{\mathbf{S}} \frac{1}{u^3} \sin\theta \, d\theta$ is

$$J_1[u_o](u_o'' + u_o) - J_2[u_o]\frac{1}{u_o^3} + a\frac{\cos\theta}{u_o^4} + b\frac{\sin\theta}{u_o^4} = 0 \tag{6.32}$$

where

$$J_1[u_o] = \int_{\mathbf{S}} \frac{dt}{u_0^3}, \quad J_2[u_0] = \int_{\mathbf{S}} [u_o^2 - (u_o')^2] \, dt,$$

and a and b are constants. The coefficient $J_1[u_o]$ is clearly positive. The coefficient $J_2[u_o]$ is a factor in the definition of $J[u_o]$ and $J[u_o] > 0$ as u_o is a maximizer. Thus $J_2[u_o]$ is also positive. As J_1 and J_2 are positive and homogeneous of different degrees there is a positive number s so that $J_1[su_o] = J_2[su_o]$. Because $J[su_o] = J[u_o]$ we can replace u_o by su_o and assume that $J_1[u_o] = J_2[u_o]$. Using this in (6.32) gives (for possibly different constants a and b)

$$u_o'' + u_o - \frac{1}{u_o^3} + a\frac{\cos\theta}{u_o^4} + b\frac{\sin\theta}{u_o^4} = 0 \tag{6.33}$$

To determine the constants a and b, we multiply both sides of (6.33) by $\cos\theta$ and $\sin\theta$ respectively, then integrate over $\mathbf{S}$ and use the fact that $u_o'' + u_o$ and $1/u_o^3$ are orthogonal to $\cos\theta$ and $\sin\theta$ to obtain

$$a\int_{\mathbf{S}} \frac{\cos^2\theta}{u_o^4}\,d\theta + b\int_{\mathbf{S}} \frac{\sin\theta\cos\theta}{u_o^4}\,d\theta = 0, \tag{6.34}$$

$$a\int_{\mathbf{S}} \frac{\cos\theta\sin\theta}{u_o^4}\,d\theta + b\int_{\mathbf{S}} \frac{\sin^2\theta}{u_o^4}\,d\theta = 0. \tag{6.35}$$

By Hölder's inequality

$$\begin{vmatrix} \displaystyle\int_{\mathbf{S}} \frac{\cos^2\theta}{u_o^4}\,d\theta & \displaystyle\int_{\mathbf{S}} \frac{\sin\theta\cos\theta}{u_o^4}\,d\theta \\ \displaystyle\int_{\mathbf{S}} \frac{\sin\theta\cos\theta}{u_o^4}\,d\theta & \displaystyle\int_{\mathbf{S}} \frac{\sin^2\theta}{u_o^4}\,d\theta \end{vmatrix} > 0. \tag{6.36}$$

Therefore the algebraic system (6.34) and (6.35) has only the trivial solution $a = b = 0$. Consequently, u_o satisfies

$$u_o'' + u_o = \frac{1}{u_o^3}. \tag{6.37}$$

To solve this equation let α be where the maximum of u_0 occurs and let $y(\theta) = u_o(\theta + \alpha)$. Then y has a maximum at $\theta = 0$ and therefore

$$y'' + y = \frac{1}{y^3}, \quad y > 0, \quad y'(0) = 0. \tag{6.38}$$

Multiple this equation by y' and integrate to get

$$(y')^2 + y^2 + \frac{1}{y^2} = 2a$$

for some constant $a > 0$. These two equations can be used to show that

$$(y^2)'' + 4y^2 = 4a.$$

This, along with $y'(0) = 0$, yields

$$y^2 = a + b\cos(2t)$$

for some $b \in \mathbf{R}$. Thus $y(0) = a + b$. Using this and $y'(0) = 0$ in (6.38) gives $\frac{1}{a+b} + (a+b) = 2a$ which can be rearranged to give $a^2 - b^2 = 1$. This implies there is a $\lambda > 0$ so that $a = \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda}^2\right)$ and $b = \frac{1}{2}\left(\lambda^2 - \frac{1}{\lambda^2}\right)$. Therefore

$$y(\theta)^2 = \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda}^2\right) + \frac{1}{2}\left(\lambda^2 - \frac{1}{\lambda}^2\right)\cos(2\theta)$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)\left(\cos^2\theta + \sin^2\theta\right) + \frac{1}{2}\left(\lambda^2 - \frac{1}{\lambda^2}\right)\left(\cos^2\theta - \sin^2\theta\right) \\
&= \lambda^2\cos^2\theta + \frac{1}{\lambda^2}\sin^2\theta
\end{aligned}
$$

But $u_o(\theta) = y(\theta - \alpha)$ and so

$$u_o(\theta) = \sqrt{\lambda^2\cos^2(\theta-\alpha) + \frac{1}{\lambda^2}\sin^2(\theta-\alpha)}.$$

The maximum of $I(u, v)$ is $I(u_o, u_o)$.

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