# Mathematics 554 Homework.

This homework is about some practice with inequalities. To start let use list a few inequalities we will be using repeatedly. The first is that squares are non-negative: for any real number $x$

$$x^2 \geq 0 \qquad \text{with equality if and only if } x = 0.$$

*Example* 1. Show for all $x$ that

$$x^2 + 4x + 6 \geq 2$$

with equality if and only if $x = -2$.

*Solution:* This is just completing the square:

$$x^2 + 4x + 6 = x^2 + 4x + 4 + 2 = (x+2)^2 + 2 \geq 0 + 2 = 2$$

and equality holds if and only if $(x + 2) = 0$, that is if $x = -2$. $\square$

**Problem** 1. Show that $3x^2 - 6x - 7 \geq -10$ with equality if and only if $x = 1$. $\square$

**Proposition 2** (Sum of squares is non-negative)**.** *If $x_1, x_2 cd x_n$ are real numbers, than*

$$x_1^2 + x^2 + \cdots + x_n^2 \geq 0$$

*with equality if and only if $x_1 = x_2 = \cdots = x_n = 0$. (In English: A sum of squares of real numbers is non-negative and is zero if and only if all of the numbers are zero.)*

*Proof.* If we are going to be really precise we can prove this by induction on $n$. But I am assuming you have all done enough induction proofs, so we will skip this one. $\square$

We will often use a slight generalization of this proposition. To state the case when $n = 2$, let $p_1, p_2$ be positive numbers, and $x, y$ any real numbers. Then

$$p_1 x^2 + p_2 y^2 \geq 0$$

with equality if and only if $x = y = 0$. For example $3x^2 + 4y^2 \geq 0$ with equality if and only if $x = y = 0$. The general case is

**Proposition 3** (Positive combination of squares is non-negative)**.** *If $x_1, x_2 cd x_n$ are real numbers and $p_1, p_2, \ldots, p_n$ are positive than*

$$p_1 x_1^2 + p_2 x^2 + \cdots + p_n x_n^2 \geq 0$$

*with equality if and only if $x_1 = x_2 = \cdots = x_n = 0$. (In English: A linear combination of squares real numbers with positive coefficients is non-negative and is zero if and only if all the numbers are zero.)*

*Proof.* This is anther induction proof we are going to skip. $\square$

*Example* 4. Show that $4x^2 + 4xy + 8y^2 \geq 0$ with equality if and only if $x = y = 0$.

*Solution:* This is a also a completing the square problem:

$$\begin{aligned} 4x^2 + 4xy + 7y^2 &= 4x^2 + 4xy + y^2 + 6y^2 \\ &= (2x + y)^2 + 3y^3 \\ &\geq 0 \qquad \text{(positive combination of squares.)} \end{aligned}$$

If equality holds than

$$\begin{aligned} 2x + y &= 0 \\ y &= 0 \end{aligned}$$

Using $y = 0$ in $2x + y = 0$ gives $2x = 0$ and so if equality holds, then $x = y = 0$.

Note in $4x^2 + 4xy + 7y^2$ we can complete the squares in several ways

$$4x^2 + 4xy + 7y^2 = 4x^2 + 2(x + y)^2 + 5y^2$$

so there are many ways to do this problem. $\square$

**Problem** 2. Show that for all real numbers $x, y$ that $x^2 + xy + y^2 \geq 0$ and equality holds if and only if $x = y = 0$. $\square$

**Problem** 3. Use the previous problem and that $y^3 - x^3$ can be factored as

$$y^3 - x^3 = (y - x)(x^2 + xy + y^2)$$

to show that $x < y$ implies $x^3 < y^3$. $\square$

One of the most famous consequences of the fact that squares are non-negative is

**Theorem 5** (The Arithmetic-Geometric Mean Inequality)**.** *For any positive real numbers*

$$\sqrt{ab} \leq \frac{a + b}{2}$$

*with equality if and only if* $a = b$.

**Problem** 4. Prove this. *Hint:* We are assuming in this that all positive numbers have square roots, something will will prove shorty. Probably the easiest way to start the proof is by showing

$$\left(\frac{a + b}{2}\right) - \sqrt{ab} = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2$$

and taking it from there. $\square$

The next basic inequality we discuss is the triangle: for any real numbers

$$|a + b| \leq |a| + |b|.$$

And this holds for sums of more than just two numbers. That is let $a_1, a_1, \ldots, a_n$ be real numbers than

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

In summation notation this is

$$\left| \sum_{k=1}^{n} a_k \right| \leq \sum_{k=1}^{n} |a_k|.$$

*Example* 6. Show that if $|a| \leq 4$ and $|b| \leq 5$, then

$$|a^4 - b^4| \leq 369|a - b|$$

*Solution:* This combines factoring with the triangle inequality.

$$\begin{aligned}
|a^4 - b^4| &= \big|(a-b)(a^3 + a^2 b + ab^2 + b^3)\big| \\
&= |a-b||a^3 + a^2 b + ab^2 + b^3| \\
&\leq |a-b| \left(|a|^3 + |a|^2|b| + |a||b|^2 + |b|^3\right) \quad \text{(by triangle inequality)} \\
&= |a-b| \left(4^3 + (4)^2 5 + 4(5)^2 + 5^3\right) \\
&= 369|a-b|.
\end{aligned}$$

□

**Problem** 5. Let $f(x) = 3x^3 - 2x + 4$ and let $|a| \leq 10$, $|b| \leq 11$. Show

$$|f(b) - f(a)| \leq 333\,|b - a|.$$

Anther basic fact is that in a fraction

$$f = \frac{y}{x}$$

with $x$ and $y$ positive if we increase $x$, then $f$ decreases and if $x$ is decreased, then $f$ in increased.

*Example* 7. If $a > 10$ and $b > 20$, show

$$\left| \frac{1}{a} - \frac{1}{b} \right| \leq \frac{|b-a|}{200}.$$

*Solution:*

$$\begin{aligned}
\left| \frac{1}{a} - \frac{1}{b} \right| &= \left| \frac{b-a}{ab} \right| \\
&= \frac{|b-a|}{ab} \\
&\leq \frac{|b-a|}{(10)(20)} \qquad\qquad \text{(as } a \geq 10 \text{ and } b \geq 20.) \\
&= \frac{|b-a|}{200}.
\end{aligned}$$

□

*Example* 8. Here is a related, but slightly tricker problem. If $|x| \geq 5$ and $|y| \geq 6$ show

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \leq \frac{11}{900}|y-x|.$$

*Solution:* Start as in the last problem:

$$\begin{aligned}
\left|\frac{1}{x^2} - \frac{1}{y^2}\right| &= \left|\frac{y^2 - x^2}{x^2y^2}\right| \\
&= \frac{|y-x||y+x|}{|x|^2|y|^2} \\
&\leq \frac{|y-x|\left(|x|+|y|\right)}{x^2y^2} && \text{(by triangle inequality)} \\
&= \frac{|y-x||x|}{|x|^2|y|^2} + \frac{|y-x||y|}{|x|^2|y|^2} \\
&= \frac{|y-x|}{|x||y|^2} + \frac{|y-x|}{|x|^2|y|} \\
&\leq \frac{|y-x|}{5(6)^2} + \frac{|y-x|}{(5)^2 6} && \text{(as } |x| \geq 5 \text{ and } |y| \geq 6.\text{)} \\
&= \frac{11}{900}\,|y-x|
\end{aligned}$$

**Problem** 6. Let $|a| \geq 1$ and $|b| \geq 2$, show

$$\left|\frac{1}{a^3} - \frac{1}{b^3}\right| \leq \frac{7}{8}\,|b-a|.$$

We now come to the adding and subtracting trick and related tricks involving absolute values.

*Example* 9. Assume that $|a-5| < 2$. Show

$$|a| < 7.$$

*Solution:* One way to do this is adding and subtracting along with the triangle inequality

$$|a| = |5 + (a-5)| \leq |5| + |a-5| = 5 + |a-5| < 5 + 2 = 7.$$

Here is anther, maybe more natural method. The inequality $|a-5| < 2$ is equivalent to

$$-2 < a - 5 < 2.$$

Add 5 to these inequalities to get

$$-2 + 5 < a - 5 + 5 < 2 + 5$$

which gives

$$3 < a < 7$$

which implies $|a| < 7$. $\square$

*Example* 10. Let $a > 0$. Show that

$$|x - a| < \frac{a}{3}$$

then

$$\frac{2a}{3} < x < \frac{4a}{3}$$

and

$$\frac{3}{4a} < \frac{1}{x} < \frac{3}{2a}. \tag{1}$$

*Solution:* The given inequality is equivalent to

$$-\frac{a}{3} < x - a < \frac{a}{3}.$$

Add $a$ to these inequalities to get

$$-\frac{a}{3} + a < x - a + a < \frac{a}{3} + a.$$

This reduces to

$$\frac{2a}{3} < x < \frac{4a}{3}.$$

Recall that for positive numbers taking reciprocals reverses inequalities we get the inequalities (1). □

**Problem** 7. Let $c > 0$. Show that

$$|x - c| < \frac{c}{5}$$

implies

$$\frac{4c}{5} < x < \frac{6c}{5}$$

and

$$\frac{5}{6c} < \frac{1}{x} < \frac{5}{4c}.$$ □

*Example* 11. Here is an example where we really do need the adding and substracting trick. Assume that $|a - b| < 1$ and $|b - c| < 1$. Show

$$|a - c| < 2.$$

*Solution:* Add and subtract $b$ and use the triangle inequality

$$|a - c| = |a - b + b - c| \leq |a - b| + |b - c| < 1 + 1 = 2.$$ □

**Problem** 8. Assume $|x_1 - x_0| < 1$, $|x_2 - x_1| < 1/2$, and $|x_3 - x_2| < 1/4$. Show

$$|x_3 - x_0| < \frac{7}{4}.$$ □

*Example* 12. Assume $|a|, |b|, |x|, |y| < 10$. Show

$$|xy - ab| < 10|x - a| + 10|y - b|.$$

*Solution:* Add and subtract $ay$

$$\begin{aligned}
|xy - ab| &= |xy - ay + ay - ab| \\
&\leq |xy - ay| + |ay - ab| && \text{(triangle inequality)} \\
&= |x - a||y| + |a||y - b| \\
&\leq |x - a|(10) + 10|y - b| \\
&= 10|x - a| + 10|y - b|.
\end{aligned}$$

Or add and subtract $bx$.

$$\begin{aligned}
|xy - ab| &= |xy - bx + bx - ab| \\
&\leq |xy - bx| + |bx - ab| && \text{(triangle inequality)} \\
&= |x|(y - b| + |b||x - a| \\
&\leq 10|y - b| + 10|x - a| \\
&= 10|x - a| + 10|y - b|.
\end{aligned}$$

**Problem** 9. If $|a|, |b|, |c|, |x|, |y|, |z| \leq 5$ show

$$|xyz - abc| \leq 25|x - a| + 25|y - b| + 25|z - c|. \qquad \square$$

**Problem** 10. Let $\delta > 0$ and assume $|x - a| < \delta$.
(a) Show $|x| < |a| + \delta$.
(b) Use this to show

$$|x^2 - a^2| \leq \big(2|a| + \delta\big)|x - a|. \qquad \square$$

**Problem** 11. Let $\delta > 0$ and let

$$f(x) = ax^2 + bx + c$$

where $a, b$, and $c$ are constants. Assume $|x - x_0| < \delta$. Show

$$|f(x) - f(x_0)| \leq \Big(|a|\big(2|x_0| + \delta\big) + |b|\Big)|x - x_0|.$$