Modes of Convergence.

Let X be a set with a measure μ. Let $f_n, f\colon X \to \mathbb{R}$ measurable functions. Then there are some notions of convergence that do not involve the measure. The main ones are $\lim_{n\to\infty} f_n = f$ pointwise and $\lim_{n\to\infty} f_n = f$ uniformly.

Of more interest to us are the notions that involve the measure. The main ones are

AE ***Almost everywhere:*** There is a set $B \subseteq X$ with $\mu(B) = 0$ and for all $x \in X \setminus B$ we have $\lim_{n\to\infty} f_n(x) = f(x)$.

AU ***Almost uniform:*** For all $\varepsilon > 0$ there is a set $B \subseteq X$ with $\mu(B) < \varepsilon$ and such that $f_n \to f$ uniformly on $X \setminus B$.

AE ***Almost everywhere:*** There is a set $B \subseteq X$ with $\mu(B) = 0$ such that $f_n \to f$ pointwise on $X \setminus B$.

M ***In measure:*** For every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mu\left(\{x : |f_n(x) - f(x)| > \varepsilon\}\right) = 0.$$

(It is a good exercise to rewrite this as ε-N statement.)

$\boldsymbol{L^p}$ ***In*** $\boldsymbol{L^p}$***:*** This is that

$$\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0.$$

Here we assume that $1 \le p < \infty$.

Some of these imply others. For general measure spaces the following diagram summarizes the implosions. A solid arrow means is a general implication. A dotted form condition **A** to condition **B** means that if $f_n \to f$ in the sense of **A** then there is a subsequence of $\langle f_{n_k}\rangle_{k=1}^{\infty}$ of $\langle f_n\rangle_{n=1}^{\infty}$ that converges in the sense of **B**. The absence of an arrow means there is a counterexample.

$$\begin{tikzcd}
\mathbf{AE} & \mathbf{AU} \arrow[l] \arrow[d, shift right] \\
\boldsymbol{L^p} \arrow[u, dashed] \arrow[r] \arrow[ur, dashed] & \mathbf{M} \arrow[u, dashed, shift right] \arrow[ul, dashed]
\end{tikzcd}$$

In the case $\mu(X) < \infty$ more can be said the corresponding diagrams is

$$\begin{tikzcd}
\mathbf{AE} \arrow[r, shift left] \arrow[dr] & \mathbf{AU} \arrow[l, shift left] \arrow[d, shift right] \\
\boldsymbol{L^p} \arrow[u, dashed] \arrow[ur, dashed] \arrow[r] & \mathbf{M} \arrow[u, dashed, shift right] \arrow[ul, dashed, shift right]
\end{tikzcd}$$

In class today I made a bit of a hash of proving the following:

Proposition 1. *On the measure space (X, μ) if $f_n \to f$ almost uniformly, then $f_n \to f$ almost everywhere.*

Here is shorter argument.

Proof. Assume that $f_n \to f$ almost uniformly and we wish to show that $f_n \to f$ almost everywhere. Note that for each $x \in X$

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{ if and only if } \limsup_{n\to\infty} |f_n(x) - f(x)| = 0.$$

Therefore the set of points, x, where $\lim_{n\to\infty} f_n(x) \neq f(x)$ is

$$B = \{x \in X : \limsup_{n\to\infty} |f_n(x) - f(x)| > 0.\}$$

So our goal is to show that $\mu(B) = 0$. For each positive integer let

$$B_k = \{x \in X : \limsup_{n\to\infty} |f_n(x) - f(x)| > 1/k.\}$$

Then

$$B = \bigcup_{k=1}^{\infty} B_k$$

and thus if we can show

$$\mu(B_k) = 0$$

for all k we will have shown that B is a countable union of sets of measure zero and therefore $\mu(B) = 0$.

We now use $f_n \to f$ almost uniformly. Let $\varepsilon > 0$. There there is a set $A \subseteq X$ such that $f_n \to f$ uniformly on $X \smallsetminus A$ and with $\mu(A) < \varepsilon$. Using the uniform convergence we see there is a N such that

$$n \geq N \quad \text{implies} \quad |f_n(x) - f(x)| < 1/k \text{ for all } x \in X \smallsetminus A.$$

This implies that if $x \in X \smallsetminus A$, then $\limsup_{n\to\infty} |f_n(x) - f(x)| \leq 1/k$ and therefore $x \in X \smallsetminus B_k$. That is $(X \smallsetminus A) \subseteq (X \smallsetminus B_k)$. This in turn implies $B_k \subseteq A$. Thus

$$\mu(B_k) \leq \mu(A) < \varepsilon.$$

This holds for all $\varepsilon > 0$ and thus $\mu(B_k) = 0$ which completes the proof. $\square$

Here are some proofs related to this that I wrote up for some of the review sessions in summer past.

Theorem 2 (Egoroff's Theorem)**.** *Let (X, μ) be a measure space with $\mu(X) < \infty$. Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of measurable functions with $\lim_{n\to\infty} f_n = f$ almost everywhere. Then $f_n \to f$* ***almost uniformly****. That is for all $\varepsilon > 0$ there is a measurable set $E \subseteq X$ with $\mu(E) < 0$ and $f_n \to f$ uniformly on $X \setminus E$.*

Proof. Let Z be the set of measure zero of points, x, where $f_n(x)$ does not converge to $f(x)$. For each pair of positive integers n, k let $E_{n,k}$ be the measurable set

$$\begin{aligned} E_{n,k} &= \bigcup_{j=k}^{\infty} \{x : |f_j(x) - f(x)| \geq 1/n\} \\ &= \{x : \text{for some } j \geq k \text{ the inequality } |f_j(x) - f(x)| \geq 1/n \text{ holds}\} \end{aligned}$$

and set

$$Z_n = \bigcap_{k=1}^{\infty} E_{n,k}.$$

For a fixed n, if $x \in Z_n$, then $x \in E_{n,k}$ for all k and so for any k there is a $j \geq k$ with $|f_j(x) - f(x)| \geq 1/n$ and thus $\langle f_j \rangle_{j=1}^{\infty}$ does not converge to $f(x)$. This shows that $Z_n \subseteq Z$ which implies that $\mu(Z_n) = 0$. As $E_{n,k+1} \subseteq E_{n,k}$ we have for each n that

$$\lim_{k\to\infty} \mu(E_{n,k}) = 0.$$

(This is the step where we use $\mu(X) < \infty$.)

Now let $\varepsilon > 0$. Then for each n there is a k_n such that

$$\mu(E_{n,k_n}) < \frac{\varepsilon}{2^n}.$$

Let

$$E = \bigcup_{n=1}^{\infty} E_{n,k_n}.$$

Then

$$\mu(E) \leq \sum_{n=1}^{\infty} \mu(E_{n,k_n}) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Finally if $x \notin E$, then $x \notin E_{n,k_n}$ for all n. But this implies

$$|f_j(x) - f(x)| < \frac{1}{n} \qquad \text{for all} \qquad j \geq k_n.$$

Thus $f_j \to f$ uniformly on $X \setminus E$. $\square$

We do have that if f_n converges to f in measure on a finite measure space, then a subsequence of the sequence converges to f almost uniformly.

Theorem 3. *Let (X, μ) be a measure space with $\mu(X) < \infty$. Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of measurable functions such that $f_n \to f$ in measure. Then there is a subsequence $\langle f_{n_k} \rangle_{k=1}^{\infty}$ with $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ almost uniformly (and thus almost everywhere).*

Proof. By definition $f_n \to f$ in measure means that for all $\varepsilon > 0$ that

$$\lim_{k\to\infty} \mu(\{x : |f_k(x) - f(x)| \geq \varepsilon\}) = 0.$$

Fix $\varepsilon > 0$. Then for each $n > 0$ there is a k_n such that

$$\mu(\{x : |f_{k_n}(x) - f(x)| \geq 1/n\}) < \frac{\varepsilon}{2^n}.$$

Let

$$E = \bigcup_{n=1}^{\infty} \{x : |f_{k_n}(x) - f(x)| \geq 1/n\}.$$

Without loss of generality we can assume that $\langle k_n \rangle_{n=1}^{\infty}$ is an increasing sequence. Using subadditivity we see

$$\mu(E) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

If $x \notin E$, then for each n we have $|f_{k_n}(x) - f(x)| < 1/n$ and thus $f_{n_k} \to f$ uniformly on $X \setminus E$.

We are not quite done, as this subsequence we have found works for ε, but if we choose a smaller value $\varepsilon' < \varepsilon$ this subsequence may not converge uniformly on the compliment of a set of measure less that ε'. One way around this is to use the Cantor diagonalization method. First find a subsequence that converges uniformly on the compliment of a set, E_1, with $\mu(E_1) < 1/2$. Then choose a subsequence of the first subsequence that converges uniformly on the compliment of a set $E_2 \subset E_1$ with $\mu(E_2) < 1/2^2$. Choose a subsequence this second subsequence that converges uniformly on $X \setminus E_3$ and with $E_3 \subseteq E_2$ and $\mu(E_3) < 1/2^3$. Continue in this manner. Then the diagonal sequence will converge uniform on the compliment of each E_n and $\mu(E_n) < 1/2^n$. Thus the diagonal converges almost uniformly to f. $\square$

Problem 1. These results cover most of the hard implications in the diagrams. Several of the counterexamples are based on the following functions. Let $\alpha > 0$ and set

$$f_{n,\alpha} = n^{\alpha} \mathbb{1}_{(0,1/n)}$$

or a bit more explicitly

$$f_{n,\alpha}(x) = \begin{cases} n^{\alpha}, & 0 < x < 1/n; \\ 0, & \text{otherwise.} \end{cases}$$

Whow $f_{n,\alpha} \to 0$ pointwise (that thus almost everywhere) and this convergence is also almost uniform and in measure and that the L^p norms are

$$\|f_{n,\alpha}\|_{L^p} = n^{\alpha - \frac{1}{p}}.$$