# QUALIFYING EXAM IN ANALYSIS

(January 11, 2008)

**Name :**
**S.S. # :**

Throughout this examination the term measurable refers to the Lebesgue measure $m$ on the real line. Integrals with respect to Lebesgue measure will be denoted by $\int f$. Problems are 10 points each.

**1.** For $n = 1, 2, \ldots$, let

$$s_n := \sum_{k=1}^{n} k^{-k}.$$

Prove that the sequence $\{s_n\}$ is a Cauchy sequence.

**2.** Prove the Banach's fixed point principle.

**Theorem.** *If $\Omega$ is a contraction mapping on a complete metric space $(X, \rho)$, then the equation $\Omega(x) = x$ has one and only one solution (i.e. the mapping $\Omega : X \to X$ leaves one and only one point unchanged).*

**3.** For each positive integer $n$ define

$$f_n(x) = \frac{x^n e^{-x}}{(2n)!}.$$

Determine whether or not the sequence $\{f_n\}$ converges uniformly on $[0, \infty)$.

**4.** Evaluate the integral

$$\int_\gamma \frac{dz}{1+z^3}$$

where $\gamma$ is the circuit : $z = -1 + e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

**5.** Use the Hölder inequality to prove that for a continuous on $[a, b]$ function $f$

$$\|f\|_2 \leq \|f\|_1^{1/3} \|f\|_4^{2/3},$$

where

$$\|f\|_p := (\int_a^b |f(x)|^p dx)^{1/p}.$$

**6.** Prove the Lebesgue Convergence Theorem: Let $g$ be integrable over $E$ and let $\{f_n\}$ be a sequence of measurable functions such that $f_n(x) \to f(x)$ almost everywhere on a set $E$ and $|f_n| \leq g$, then

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

**7.** Let $A$ be a measurable subset of $[0, 1]$ and $mA = a > 0$. Prove that for any $0 \leq b < a$ there exists a closed set $B \subset A$ such that $mB = b$.

**8.** Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $(-\infty, \infty)$ such that $f_n \to f$ a.e., and suppose that $\int f_n \to \int f < \infty$. Prove that for each measurable set $E$ we have $\int_E f_n \to \int_E f$.

**9.** Let $f > 0$ be continuous and of bounded variation on a finite interval $[a, b]$. Prove that $1/f$ is of bounded variation on $[a, b]$.

**10.** Let $f$ be absolutely continuous on $[0,1]$ with $f' \in L_p([0,1])$, $1 < p < \infty$. Show that there is a constant $C$ such that

$$|f(b) - f(a)| \leq C|b-a|^{1-1/p}$$

for all $a, b \in [0,1]$.