Some analysis problems.

The following is not on the qualifying exam syllabus, but it came up in class today and is a standard analysis technique and so it worth looking at. It will also be good practice in working with uniform convergence. The question that was ask was if we have a series

$$\sum_{n=1}^{\infty} f_n$$

of functions that we wish to show is uniformly convergent and the Weierstrass M test does not work, what other method can we use. One method is ***summation by parts*** also call ***Abelian summation***. It is based on:

Proposition 1. *Let $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ be two sequences of complex numbers and let*

$$A_n = \sum_{k=1}^{n} a_k$$

and for $m < n$

$$S_{m,n} = \sum_{k=m+1}^{n} a_k b_k$$

then

$$S_{m,n} = A_n b_n - A_m b_{m+1} + \sum_{k=m+1}^{n-1} A_k (b_k - b_{k+1}).$$

Problem 1. Prove this. *Hint:* Use that $a_k = A_k - A_{k-1}$ so that

$$S_{m,n} = \sum_{k=m+1}^{n} (A_k - A_{k-1}) b_k$$

split this into two sums and do a change of index one of the sums and them simplifying. $\square$

Theorem 2. *Let X be a metric space and $f_n \colon X \to \mathbb{C}$ for $n = 1, 2, \ldots$ be functions such that the partial sums*

$$F_n(x) = \sum_{k=1}^{n} f_n(x)$$

are uniformly bounded (that is there is a constant M such that $|F_n(x)| \leq M$ for all n and x). Let $b_1, b_2, b_3, \ldots$ be a sequence of real constants that decrease monotonically to 0 (i.e. $\lim_{n\to\infty} b_n = 0$ and $b_1 \geq b_2 \leq b_2 \geq \cdots$.) Then the series

$$f(x) = \sum_{n=1}^{\infty} b_n f_n(x)$$

converges uniformly.

Problem 2. Prove this. *Hint:* Let

$$S_n(x) = \sum_{k=1}^{n} b_n f_n(x)$$

be the n-th partial sum. Use Proposition 1 to show that for $m < n$ that

$$S_n(x) - S_m(x) = b_n F_n(x) - b_{m+1} F_m(x) + \sum_{k=m+1}^{n-1} F_k(x)(b_k - b_{k+1}).$$

Now use that $|F_k x)| \leq M$ and $(b_k - b_{k+1}) \geq 0$ to show

$$\begin{aligned} |S_m(x) - S_n(x)| &\leq M b_n + M b_{m+1} + \sum_{k=m+1}^{n-1} M(b_k - b_{k+1}) \\ &= M(b_n + b_{m+1} + b_{m+1} - b_n) \\ &= 2M b_{m+1}. \end{aligned}$$

This can be used to show that for any $\varepsilon > 0$ there is a N such that if $n, m \geq N$, then

$$|S_n(x) - S_m(x)| < \varepsilon$$

and thus the sequence $\langle S_n(x) \rangle_{n=1}^{\infty}$ is uniformly continuous and therefore converges uniformly. □

Proposition 3. *If*

$$f_n(x) = e^{inx}$$

then

$$\sum_{n=0}^{n} f_k(x) = \sum_{k=0}^{n} e^{ikx} = \frac{1 - e^{i(n+1)x} - e^{-ix} + e^{inx}}{2(1 - \cos(x))}$$

and therefore if $\cos(x) \neq 1$ *the inequality*

$$\left| \sum_{k=1}^{n} e^{ikx} \right| \leq \frac{2}{1 - \cos(x)}$$

holds.

Proof. This is an exercise in summing a finite geometric series. □ □

Theorem 4. *Let* $\langle b_k \rangle_{k=1}^{\infty}$ *be a sequence of real numbers that decrease to* 0. *Then for any* $\delta > 0$ *the series*

$$f(x) = \sum_{n=1}^{\infty} b_n e^{inx}$$

converges uniformly on the interval $[\delta, 2\pi - \delta]$.

Problem 3. Prove this. □

Therefore the series

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{inx}}{n}$$

converges uniformly on each interval $[\delta, 2\pi - \delta]$, and this is a case where the Weierstrass M test does not apply (at least in any direct way).

Theorem 5 (Banach Fixed Point Theorem)**.** *Let X be a complete metric space and let $f\colon X \to X$ be a map such that for some ρ with $0 \le \rho < 1$ the inequality*

$$d(f(x,y) \le \rho d(x,y)$$

(such a map is called a ***contraction****). Then f has a unique fixed point in X. That is there is a unique point $x_* \in X$ with $f(x_*) = x_*$.*

Problem 4. Prove this. *Hint:* Start with any point $x_0 \in X$ and recursively define a sequence by

$$\begin{aligned}
x_1 &= f(x_0) \\
x_2 &= f(x_1) \\
x_3 &= f(x_2) \\
&\vdots \quad \vdots \\
x_n &= f(x_{n-1})
\end{aligned}$$

Use induction to show for $k \ge 1$ that

$$d(x_k, x_{k-1}) \le \rho^{k-1} d(x_1, x_0).$$

Then for $1 < m < n$

$$d(x_n, x_m) \le \sum_{k=m+1}^{n} d(x_k, x_{k-1}) \le \sum_{k=m+1}^{n} \rho^{k-1} d(x_1, x_0) \le d(x_1, x_0) \sum_{k=m+1}^{\infty} \rho^{k-1}$$

and this geometric series can be sum to give

$$d(x_n, x_m) \le \frac{\rho^k d(x_1, x_0)}{1-\rho}.$$

Use this to show the sequence is Cauchy and thus converges. Let $x_* = \lim_{n\to\infty} x_n$ then take the limit as $n \to \infty$ in

$$x_n = f(x_{n-1})$$

to see that x_* is fixed point.

To show uniqueness let x_* and x_{**} be fixed points. Then

$$0 \le d(x_*, x_{**}) = d(f(x_*), f(x_{**}) \le \rho d(x_*, x_{**}),$$

which implies $d(x_*, x_{**}) = 0$. $\square$

In the case of compact spaces the hypothesis of the Banach Fixed Point Theorem can be weakened a bit.

Problem 5. Let X be a compact metric space and $f\colon X \to X$ a map with

$$d(f(x), f(y)) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$. Then f has a unique fixed point in X. □

Problem 6. In the previous problem compactness is required. Show there is a map $f\colon \mathbb{R} \to \mathbb{R}$

$$|f(x) - f(y)| < |x - y|$$

for all $x \neq y$, but with no fixed point. □

Here is a non-trivial application of the Banach Fixed Point Theorem. We would like to show that the initial value problem

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0$$

has a solution. This can be integrated to get

$$y(x) - y_0) = y(x) - y(x_0) = \int_{x_0}^{x} y(t)\, dt = \int_{x_0}^{x} f(t, y(t))\, dt$$

so that

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

Note that if $y(x)$ satisfies this integral equation, then $y(x_0) = y_0$, and the fundamental theorem of calculus shows that

$$y'(x) = \frac{d}{dx}\left(y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \right) = f(x, y(x)).$$

Therefore solving this integral equation is equivalent to solving the differential equation. One advantage of working with the integral equation is that it is of the form of a fixed point. We just have to find the right metric space. Let $\delta > 0$ and let

$$C([x_0 - \delta, x_0 + \delta]) = \{f\colon [x_0 - \delta, x_0 + \delta] \to \mathbb{R} :\ f \text{ is continuous.}\}$$

Define a norm on this space by

$$\|f\| = \max_{x \in [x_0 - \delta, x_0 + \delta]} |f(x)|$$

and make $C[x_0 - \delta, x_0 + \delta])$ into a metric space using the distance function

$$d(f, g) = \|f - g\|.$$

Problem 7. If you have not already done so, show that $C([x_0 - \delta, x_0 + \delta])$ is a complete metric space. □

We define a map $T\colon C([x_0 - \delta, x_0 + \delta]) \to C([x_0 - \delta, x_0 + \delta])$ by

$$T[u](x) = y_0 + \int_{x_0}^{x} f(t, u(t))\, dt.$$

Problem 8. Let $f\colon [x_0-\delta, x_0+\delta]\times\mathbb{R}\to\mathbb{R}$ be a continuous function that satisfies the Lipschitz condition

$$|f(x,y_2)-f(x,y_1)|\le M|y_2-y_1|$$

for some positive constant M. Let $\delta>0$ be so that

$$\rho := \delta M < 1.$$

Show that the integral operator T defined above is is a contraction on $C([x_0-\delta, x_0+\delta])$ that there for T has a unique fixed point in $C([x_0-\delta, x_0+\delta])$. $\square$