Qualifying Examination in Algebra

August 2012

Note! You must **prove** every assertion. Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**.

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample. If some problem is vague, then be sure to explain your interpretation of the problem.

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{Q}$ be the field of rational numbers, $\mathbb{Z}$ be the ring of integers, and if p is a prime integer, then let $\mathbb{Z}_p$ be the field of integers computed mod p. For each positive integer n and each commutative ring R, let $\mathrm{GL}_n(R)$ be the group of invertible matrices with entries from R.

There are 10 problems. The exam is worth 105 points.

1. (10 points) Let A and B be Abelian groups, $G = A \oplus B$, and H be a subgroup of G with H isomorphic to B. Does G/H have to be isomorphic to A? If yes, then prove the assertion. If no, then give a counterexample?

2. (10 points) Let G be a finite simple non-Abelian group, p be a prime integer which divides the order of G, and n be the number of Sylow p-subgroups of G. Prove that the order of G divides $n!$.

3. (10 points) Let n and p be positive integers with p prime. Find the number of Sylow p-subgroups of $\mathrm{GL}_n(\mathbb{Z}_p)$.

4. (10 points) Let R be the ring $R = \mathbb{Z}[x]/(2, x^4 + x^3 + x^2 + x + 1)$.
   (a) Is $R = \mathbb{Z}[x]/(2, x^4 + x^3 + x^2 + x + 1)$ a field?
   (b) Express x^{-1} as a polynomial of x in R, if this is possible.

5. (10 points) Let f and g be elements of the Unique Factorization Domain (UFD) D and let h be the greatest common divisor of f and g in D.
   (a) If D is a Principal Ideal Domain, then prove that there exist p and q in D with $pf + qg = h$.
   (b) Show that the conclusion of (a) is false if D is merely a UFD.

6. (10 points) Give an example of a commutative ring R and non-zero ideals $P \subsetneq M$ in R with P a prime ideal and M a maximal ideal.

7. (10 points) Let M be the ideal in $\mathbb{Z}[x]$ generated by 2 and x. Prove that M is not a direct sum of cyclic $\mathbb{Z}[x]$-modules.

8. (10 points) Is the regular 60-gon constructible with straight-edge and compass? Please justify your answer.

9. (15 points) For each field F listed below, let E be the splitting field of $f(x) = x^7 - 1$ over F. Find the Galois group $\mathrm{Gal}(E/F)$ in each case.
(a) $F = \mathbb{Q}$, (b) $F = \mathbb{Z}_7$, (c) $F = \mathbb{R}$ (d) $F = \mathbb{Z}_2$.

10. (10 points) Let $F \subseteq E$ be a finite Galois field extension. Suppose that the Galois group $\mathrm{Gal}(E/F)$ is the cyclic group of order 12. How many intermediate fields I are there with $F \subsetneq I \subsetneq E$?