Mathematics 574 Homework

Read sections 2.1 and 2.2 in the text.

Here is some review of Math 142 that we will be using. Let $f(x)$ be a from some open interval (a, b) containing 0. Then recall that $f(x)$ has a ***Taylor series***[1]

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

We now derive formulas for the coefficients x_n. To start with we will write $f(x)$ as

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

and first find a formula for a_0.

Problem 1. Let $x = 0$ if the formula for $f(x)$ to show that $a_0 = f(0)$. $\square$

Problem 2. We now take some derivatives of $f(x)$. Show the following

$$\begin{aligned}
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots \\
f''(x) &= 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + 6 \cdot 5a_6 x^5 + \cdots \\
f'''(x) &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + 5 \cdot 4 \cdot 3a_5 x^2 + 6 \cdot 5 \cdot 4a_6 x^3 + \cdots \\
f^{(4)}(x) &= 4 \cdot 3 \cdot 2a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5 x + 6 \cdot 5 \cdot 4 \cdot 3a_6 x^2 + \cdots \\
f^{(5)}(x) &= 5 \cdot 4 \cdot 3 \cdot 2a_5 + 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2a_6 x + \cdots
\end{aligned}$$

$\square$

Problem 3. If we let $x = 0$ in the formuala for $f'(x)$ we get $f'(x) = a_1 + 0 = a_1$. Thus $a_1 = f'(x)$. If we let $x = 0$ in the formula for $f''(x)$ we get

$$f''(x) = 2a_0 + 0 = 2a_2$$

and whence

$$a_2 = \frac{f''(0)}{2}$$

(a) Let $x = 0$ in the formula for $f'''(0)$ to get a formula for a_3.
(b) Let $x = 0$ in the formula for $f^{(4)}(x)$ to get a formula for a_4.
(c) Let $x = 0$ in the formula for $f^{(5)}(x)$ to get a formula for a_5. $\square$

Problem 4. After the last problem your can probably guess what this problem will be. Compute the n-th derivative $f^{(n)}(x)$ and let $x = 0$ in this formula to show that

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

$\square$

Putting these pieces together we have

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[1] Not every function has a Taylor series. We will only get working with ones that do, so we will just assume all functions that come up do have power series.

Theorem 1. *If the function $f(x)$ has a Taylor series around $x = 0$, then*

$$f(x) = \sum_{n=0}^{\infty} a_n x \qquad \textit{where} \qquad a_n = \frac{f^{(n)}(0)}{n!}.$$

□

Here is an example. Let

$$f(x) = \frac{1}{1+x} = (1+x)^{-1}.$$

Then we have

$$\begin{aligned}
f'(x) &= -(1+x)^{-2} \\
f''(x) &= (-1)(-2)(1+x)^{-3} \\
f'''(x) &= (-1)(-2)(-3)(1+x)^{-4} \\
f^{(4)}(x) &= (-1)(-2)(-3)(-4)(1+x)^{-5} \\
\vdots \quad &\quad \vdots \\
f^{(n)} &= (-1)(-2)(-3)\cdots(-n)(1+x)^{-n-1}.
\end{aligned}$$

Thus

$$f^{(n)}(0) = (-1)(-2)(-3)\cdots(-n)(1+0)^{-n-1} = (-1)^n n!$$

and therefore

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n n!}{n!} = (-1)^n.$$

This gives

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \cdots$$

Problem 5. Here is a generalization of this for you to do. Let α be any real number and set

$$f(x) = (1+x)^{\alpha}.$$

Then

$$\begin{aligned}
f'(x) &= \alpha(1+x)^{\alpha-1} \\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \\
f^{(4)}(x) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)(1+x)^{\alpha-4} \\
f^{(5)}(x) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(1+x)^{\alpha-5} \\
\vdots \quad &\quad \vdots
\end{aligned}$$

Use this start to get a formula for $f^{(n)}(x)$ and use it to find a formula for the coefficient a_n in the Taylor series of $f(x)$. □

As we saw in class the answer to the last question is

$$a_n = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} = \frac{\alpha^{\underline{n}}}{n!}.$$

This motivates the following:

Definition 2. Let α be any real number and k an integer. Then the ***binormal coefficient*** $\binom{\alpha}{k}$ is

$$\binom{\alpha}{k} = \frac{\alpha^{\underline{k}}}{k!}$$

for $k \geq 0$ and

$$\binom{\alpha}{k} = 0$$

when $k < 0$. □

We have at least formally proven the following

Theorem 3. *Let α be any real number and $|x| < 1$. Then*

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

For some values of α the binomial coefficient $\binom{\alpha}{k}$ can be simplified. This is particularly true if α is a negative integer. Here are some examples for the first few negative integers. First $\alpha = -1$.

$$\begin{aligned}\binom{-1}{k} &= \frac{(-1)(-2)\cdots(-k)}{k!} \\ &= (-1)^k \frac{k!}{k!} \\ &= (-1)^k.\end{aligned}$$

Next $\alpha = -2$.

$$\begin{aligned}\binom{-2}{k} &= \frac{(-2)(-3)\cdots(-k)(-k-1)}{k!} \\ &= (-1)^k \frac{(k+1)!}{k!} \\ &= (-1)^k (k+1)\end{aligned}$$

For $k = -3$

$$\begin{aligned}\binom{-3}{k} &= \frac{(-3)(-4)\cdots(-k-1)(-k-2)}{k!}\\ &= (-1)^k \frac{(k+2)!}{2!k!}\\ &= (-1)^k \frac{(k+1)(k+2)}{2!k!}\\ &= (-1)^k \binom{k+2}{2}\end{aligned}$$

For $k = -4$

$$\begin{aligned}\binom{-4}{k} &= \frac{(-4)(-5)\cdots(-k-2)(-k-3)}{k!}\\ &= (-1)^k \frac{(k+3)!}{3!k!}\\ &= (-1)^k \frac{(k+1)(k+2)}{3!k!}\\ &= (-1)^k \binom{k+3}{3}\end{aligned}$$

For $k = 5$

$$\begin{aligned}\binom{-5}{k} &= \frac{(-5)(-6)\cdots(-k-3)(-k-4)}{k!}\\ &= (-1)^k \frac{(k+4)!}{4!k!}\\ &= (-1)^k \frac{(k+1)(k+2)}{4!k!}\\ &= (-1)^k \binom{k+3}{4}\end{aligned}$$

At this point we see a pattern.

Proposition 4. *Let n be a positive integer. Then*

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

Problem 6. Prove this. □

We proved the following in class.

Proposition 5. *If*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \qquad \text{and} \qquad g(x) = \sum_{k=0}^{\infty} b_k x^k$$

then the product is

$$f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} a_i b_j \right) x^k.$$

Proposition 6. *The binomial coefficients satisfy the identity*

$$\binom{\alpha+\beta}{k} = \sum_{i+j=k} \binom{\alpha}{i}\binom{\beta}{j}.$$

Problem 7. Prove this. *Hint:* Use that $(1+x)^{\alpha}(1+x)^{\beta} = (1+x)^{\alpha+\beta}$ Now we know by Proposition 3 that.

$$(1+x)^{\alpha+\beta} = \sum_{k=0}^{\infty} \binom{\alpha+\beta}{k} x^k.$$

We can compute

$$(1+x)^{\alpha+\beta} = (1+x)^{\alpha}(1+x)^{\beta}$$

using Proposition 6. Then compare coefficients to get the result. □

We also have shown

Proposition 7. *If*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0} b_n x^n \qquad \text{and} \qquad h(x) = \sum_{n=0}^{\infty} c_n x^n$$

then the product of the three is

$$f(x)g(x)h(x) = \sum_{n=0}^{\infty} \left(\sum_{i+j+k=n} a_i b_j c_k \right) x^k.$$

□

Problem 8. Let α, β, and γ be three real numbers. Use

$$(1+x)^{\alpha}(1+x)^{\beta}(1+x)^{\gamma} = (1+x)^{\alpha+\beta+\gamma}$$

and the last proposition to to derive an identity involving $\binom{\alpha+\beta+\gamma}{n}$ similar to the identity to the identity for $\binom{\alpha+\beta}{k}$ given in Proposition 6. □

Problem 9. Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

and let

$$A_0 = a_0,\ A_1 = a_0 + a_1,\ A_2 = a_0 + a_1 + a_2,\ A_3 = a_0 + a_1 + a_2 + a_3, \ldots$$

and in general $A_n = a_0 + a_1 + \cdots + a_n$. Show that

$$(1+x+x^2+x^3+x^4+x^5+x^6+\cdots)f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 + \cdots$$

and that this can be rewritten as

$$(1-x)^{-1}f(x) = \sum_{n=0}^{\infty} A_n x^n.$$

$\square$