Analysis Qualifying Exam
January 2010

Instructions: Write your name legibly on each sheet of paper. Write only on one side of of each sheet of paper. Try to answer all questions. Questions 1–8 are each worth 10 points and question 9 is worth 20 points.
Terminology: Measurability and integrability on $\mathbb{R}$ or a measurable subset of it will always refer to the Lebesgue measure, except if otherwise specified. Lebesgue measure will be denoted by m, dx or dy depending on the context. If A is a subset of $\mathbf{R}$ then $L^p(A)$ is considered with respect to the Lebesgue measure.

1. Let $E \subset \mathbb{R}^n$ and define $O_n = \{x : d(x, E) < \frac{1}{n}\}$, where d denotes the Euclidean metric.

(a) Prove O_n is open.

(b) If E is compact, show that $m(E) = \lim_{n\to\infty} m(O_n)$.

(c) Show that (b) can fail if either E is closed and unbounded; or E is open and bounded.

2. Let A, B be closed subsets of $\mathbb{R}^n$.

(a) Show $A + B = \{a + b : a \in A, b \in B\}$ is measurable.

(b) Give an example that $A + B$ does not need to be closed.

3. Let $f \in L^1(\mathbb{R})$ and define $f_n(x) = \frac{1}{n} f(nx)$

(a) Prove $f_n \in L^1(\mathbb{R})$ and $\|f_n\|_1 = \frac{1}{n^2}\|f\|$.

(b) Prove $f_n(x) \to 0$ a.e.

4. Let $F : \mathbb{R} \to \mathbb{R}$. Then there exists $f \in L^1(\mathbb{R})$ such that $F(x) = \int_{-\infty}^{x} f(t)\, dt$ if and only if

a. F is absolutely continuous on $[-M, M]$ for all $M > 0$ and

b. $\sup_M T_{-M}^{M}(F) < \infty$ (where $T_a^b(F)$ denotes the total variation of F over $[a, b]$) and

c. $\lim_{x\to-\infty} F(x) = 0$.

5. Let $E \subset \mathbb{R}$ be a measurable set and $0 \leq f : E \to \mathbb{R}$ a measurable function. Prove that for $p > 0$ we have

$$\int_E |f|^p \, dx = p \int_0^\infty t^{p-1} m(\{x \in E : f(x) > t\}) \, dt.$$

6. Let f and E be as in the previous problem and assume that in addition $0 \leq g : E \to \mathbb{R}$ is measurable such that

$$m(\{x \in E : f(x) > t\}) \leq \frac{1}{t} \int_{m(\{x \in E : f(x) > t\})} g(x) \, dx.$$

Prove that

$$\left(\int_E f^p \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\int_E g^p \right)^{\frac{1}{p}}$$

7. Evaluate $\int_0^\infty \frac{\cos x}{(1+x^2)^2} \, dx$.

8. Let G be bounded region and f a continuous function on $\overline{G}$ which is analytic on G.. Assume there exist a constant $c \geq 0$ such that $|f(z)| = c$ for all $z \in \partial G$, the boundary of G. Then either f is constant on G or f has a zero on G.

9. True or False. Prove, or give a counterexample.

- **a.** If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f_n(x) = f(x + \frac{1}{n})$, then f_n converges uniformly to f.
- **b.** There exists a holomorphic function f on $D(0;1)$ such that $f^{(n)}(0) = 2^n n!$ for all $n \geq 0$. (Here $D(0;1)$ denotes the open unit disk.)
- **c.** If $f : [0,1] \to \mathbb{R}$ is such that $|f|$ is measurable, then f is measurable.
- **d.** If $E \subset \mathbb{R}^d$ is measurable, then there exist compact $K_1 \subset K_2 \subset \cdots \subset E$ such that $K = \cup K_n$ satisfies $m(E \setminus K) = 0$.
- **e.** If F is increasing on $[0,1]$, $F(0) = 0$ and $F'(x) \geq 1$ a.e., then $F(x) \geq x$ on $[0,1]$.