REAL ANALYSIS QUALIFYING EXAM
AUGUST 1990

<u>Directions</u>: Answer all nine questions. Questions one to eight are each worth <u>ten</u> points, while question nine is worth <u>twenty</u> points. Throughout, λ denotes Lebesgue measure on $\mathbb{R}$.

1. (a) Let $f: E \to \mathbb{R}$ be a real-valued uniformly continuous function defined on a subset E of $\mathbb{R}$. Prove that f has a unique extension to a uniformly continuous function $g: \bar{E} \to \mathbb{R}$, where $\bar{E}$ denotes the closure of E.

   (b) Give an example of a bounded continuous real-valued function on $(0,1]$ which has no continuous extension to $[0,1]$.

2. Recall that a subset of $\mathbb{R}$ is said to be an F_σ-set if it is a countable union of closed sets.

   (a) Let E be a Lebesgue measurable set. Prove that there exists an F_σ-set A contained in E such that $\lambda(E\setminus A) = 0$.

   (b) Let $f: \mathbb{R} \to \mathbb{R}$ be continuous. Prove that if A is an F_σ-set, then $f(A)$ is also an F_σ-set.

3. (a) Give an example of a finitely additive set function ν defined on a measurable space (X,B) and taking values in $[0,\infty]$ (possibly taking the value ∞) which is not σ-additive.

   (b) Let (X,B,μ) be a finite measure space. Suppose that ν is a finitely additive set function on (X,B) taking values in $[0,\infty]$ and satisfying the following condition: given $\varepsilon > 0$ there exists $\delta > 0$ such that, for every E belonging to B, if $\mu(E) < \delta$ then $\nu(E) < \varepsilon$. Prove that ν is σ-additive.

   (c) Is the result of (b) still true if (X,B,μ) is merely σ-finite? Explain your answer.

4. Let (X,B,μ) be a finite measure space. Suppose that $\langle E_n \rangle$ is a sequence of measurable sets such that $\{n \in \mathbb{N}: x \in E_n\}$ is a finite set for every $x \in X$. Prove that $\mu(E_n) \to 0$.

   <u>Hint</u>: Prove that for every $\varepsilon > 0$ there exists a measurable set A such that $\mu(A) < \frac{\varepsilon}{2}$ and $\int_{X\setminus A} (\sum_{n=1}^{\infty} \chi_{E_n}) \, d\mu < \infty$.

5. (a) Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable everywhere and satisfy (for some $M > 0$) $|f'(x)| \le M$ for all x. Prove that

$(U_\alpha)_{\alpha\in A}$ be a (possibly uncountable) collection of open sets such that $\mu(U_\alpha) = 0$ for each α. Then $\mu(\bigcup_{\alpha\in A} U_\alpha) = 0$? Lindelöf Property of Reals.

(c) $\lim\limits_{n\to\infty} \int_1^\infty \frac{e^{nx}}{e^{nx}x^2 + 1}\, dx = 1$?

(d) Let $\langle E_n\rangle$ be a decreasing sequence of Lebesgue measurable subsets of R with empty intersection. Then $\lambda(E_n) \to 0$?

(e) Let E be a $(\lambda\times\lambda)$-measurable subset of $[0,1]\times[0,1]$ such that $\lambda\{x \in [0,1]: \lambda(E_x) \geq \frac{1}{2}\} \geq \frac{7}{8}$. Then $\lambda\{y \in [0,1]: \lambda(E^y) \geq \frac{1}{4}\} \geq \frac{1}{4}$?

($E_x = \{y \in [0,1]: (x,y) \in E\}$ and $E^y = \{x \in [0,1]: (x,y) \in E\}$.)