Some Galois theory examples.

In class the question came up as to if $\mathbb{Q}(\sqrt[4]{2}+i) = \mathbb{Q}(\sqrt[4]{2}, i)$. Here are a couple of arguments showing this is the case.

Problem 1. Let $t \in \mathbb{Q}$, $t \neq 0$. Show $\mathbb{Q}(\sqrt[4]{2}+ti) = \mathbb{Q}(\sqrt[4]{2}, i)$. *Hint:* Let $b = \sqrt[4]{2}+ti$. Then take the fourth power of the equation $b - ti = \sqrt[4]{2}$ to get

$$b^4 - 4tib^3 - 6t^2b^2 + 4t^3ib + t^4 = 2.$$

Solve this for i

$$i = \frac{2 - b^4 + 6t^2b^2 - t^4}{4tb(t^2 - b^2)} \in \mathbb{Q}(\sqrt[4]{2}+ti).$$

From this it is easy to see $\mathbb{Q}(\sqrt[4]{2}+ti) = \mathbb{Q}(\sqrt[4]{2}, i)$. □

Problem 2. Let n be an integer, $n \geq 0$ and let a be an integer such that $x^4 - a$ is irreducible. Let $t \in \mathbb{Q}$ with $t \neq 0$. Use the idea of the previous question to show $\mathbb{Q}(\sqrt[4]{a}+ti) = \mathbb{Q}(\sqrt[4]{a}, i)$. □

But it would be nice to use Galois theory to do these problems. We do this in the special case of the last problem in the case $a > 0$, so that $\sqrt[4]{a}$ is a positive real number. Let $L = \mathbb{Q}(\sqrt[4]{a}, i)$. This is the splitting field of $x^4 - a$ and therefore is a Galois extension of $\mathbb{Q}$. Let G be the Galois group. Then the order of G is

$$|G| = [L : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{a}, i) : \mathbb{Q}(\sqrt[4]{a})]\,[\mathbb{Q}(\sqrt[4]{a}) : \mathbb{Q}] = 2 \cdot 4 = 8.$$

We wish to find generators for G. Note that $\sqrt[4]{a}$ and i generate L and therefore an element of G is determined by what it does to $\sqrt[4]{a}$ and i. One element of L is easy to come by: complex conjugation. Since $\sqrt[4]{a}$ is real this is given by

$$\sigma_1(\sqrt[4]{a}) = \sqrt[4]{a}, \qquad \sigma_1(i) = -i.$$

Because the Galois group is transitive on the roots of $x^4 - a$, there is a $\sigma_2 \in G$ with $\sigma_2(\sqrt[4]{a}) = i\sqrt[4]{a}$. Then

$$\sigma_2^2(\sqrt[4]{a}) = \sigma_2(\sigma_2(\sqrt[4]{a})) = \sigma_2(i\sqrt[4]{a}) = \sigma_2(i)\sigma_2(\sqrt[4]{a}) = \sigma_2(i)i\sqrt[4]{a}.$$

But $(\sigma_2(i))^2 = \sigma_2(i^2) = \sigma_2(-1) = -1$. Therefore $\sigma_2(i) = \pm i$. If $\sigma_2 = -i$, then

$$(\sigma_2\sigma_1)(i) = \sigma_2(-i) = -\sigma_2(i) = -(-i) = i.$$

And $(\sigma_2\sigma_1)(\sqrt[4]{a}) = \sigma_2(\sqrt[4]{a}) = i\sqrt[4]{a}$. So if $\sigma_2(i) = -i$, we replace it by $\sigma_2\sigma_1$ so that we have

$$\sigma_2(\sqrt[4]{a}) = i\sqrt[4]{a}, \qquad \sigma_2(i) = i.$$

Then it is not hard to check that

$$\sigma_2^2(\sqrt[4]{a}) = -\sqrt[4]{a}, \quad \sigma_2(\sqrt[4]{a}) = -i\sqrt[4]{a}, \quad \sigma_2^4(\sqrt[4]{a}) = \sqrt[4]{a}.$$

Therefore σ_2 has order 4. More calculations along these lines show

$$\begin{aligned} \sigma_1^2 &= 1 \\ \sigma_2^4 &= 1 \\ \sigma_1\sigma_2\sigma_1 &= \sigma_2^{-1}. \end{aligned}$$

Therefore G is the dihedral group D_4.

Problem 3. Use what we have just shown about the Galois group G to give anther proof that $\mathbb{Q}(\sqrt[4]{a}+ti) = \mathbb{Q}(\sqrt[4]{a}, i)$ for $t \in Q$ and $t \neq 0$. *Hint:* (This is following an idea e-mailed to me by Brandon). If $\mathbb{Q}(\sqrt[4]{a}+ti) \neq \mathbb{Q}(\sqrt[4]{a}, i)$ then by the fundamental theorem of Galois theory there is a non-trivial subgroup of G that fixes $\mathbb{Q}(\sqrt[4]{a}+ti)$. The elements of G are all of the form

$$\sigma_2^k, \qquad \sigma_2^k\sigma_1$$

for $0 \leq k \leq 4$. We now check that none these other than $1 = \sigma_2^0$ fixes $\sqrt[4]{a}+ti$.

$$\begin{aligned} \sigma_2^k(\sqrt[4]{a}+ti) &= i^k\sqrt[4]{a}+ti \neq \sqrt[4]{a}+ti && (1 \leq k \leq 3), \\ \sigma_2^k\sigma_1(\sqrt[4]{a}+ti) &= i^k\sqrt[4]{a}-ti \neq \sqrt[4]{a}+ti && (0 \leq k \leq 3). \end{aligned}$$

Thus $\sqrt[4]{a}+ta$ is not fixed by any non-trivial element of G and so $\mathbb{Q}(\sqrt[4]{a}+ta) = L = \mathbb{Q}(\sqrt[4]{a}, i)$. $\square$