Mathematics 242 Homework.

Here are some practice problems. I took some of these problems from the online text *Elementary Differential Equations* by William F. Trench which can be found at

`http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_FREE_DIFFEQ_I.PDF`

It is a good source for worked examples.

Problem 1. Is $y = e^x$ a solution to $y' = \frac{1}{y^2}$? Explain how you got your answer. $\square$

Problem 2. Assume

$$\frac{dy}{dx} = \frac{1 + x - y}{1 + x^2 + y^2}$$

(a) If $y(0) = 2$ what is $y'(0)$?
(b) If $y(2) = 4$ compute $y'(2)$ and use it to estimate $y(2.05)$. $\square$

Problem 3. Find the critical points of the equation

$$y' = .1y(y - 5)(15 - y).$$

and sketch graphs of the critical solutions along with the solutions with $y(0) = 20$, $y(0) = 10$, $y(0) = 3$, and $y(0) = -2$ all on the same axis. If $y(0) = 3$ estimate $y(100)$. $\square$

Problem 4. Find the general solution to the following:

(a) $\dfrac{du}{dt} - ku = 0$ where k is constant.
(b) $y' + 3y = 1$.
(c) $y' + (\tan x)y = \cos x$.
(d) $y' = \dfrac{3x^2 + 2x + 1}{2y - 3}$.
(e) $x^2 y y' = (y^2 - 1)^{3/2}$.
(f) $7xy' - 2y = -\frac{x^2}{y^6}$.
(g) $y' - xy = x^3 y^3$.
(h) $y' = \dfrac{y + x}{x}$.
(i) $y' = \dfrac{y^2 + 2xy}{x^2}$.

Problem 5. Find the solutions to the following initial value problems.

(a) $y' = \dfrac{xy + y^2}{x^3}, \quad y(-1) = 2$.
(b) $\dfrac{du}{dt} - tu = tu^{3/2}, \quad u(1) = 4$.
(c) $y' + ky = 1, \quad y(0) = 1$ where $k > 0$ is a constant.
(d) $y' + x(y^2 + y) = 0, \quad y(2) = 1$.

Problem 6. A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C. After 4 minutes its temperature is 200°C. What is its temperature after 8 minutes? □

Problem 7. A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting $t = 0$, water that contains 1/2 pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate.

(a) Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$ and solve the equation to determine $Q(t)$.

(b) Find $\lim_{t\to\infty} Q(t)$. □

Problem 8. A large cask of wine will hold 1,000 gallons of wine. It originally contains 100 gallons of wine of which 10 gallons are alcohol (that it is 10% alcohol by volume). At some point new wine is pumped into the tank at a rate of 2 gallons/hour and this wine is 15% alcohol by volume. At the same time a mistreated and under payed employee of the winery starts siphoning wine out of the cask at a rate of .1 gallon/hour to save and serve at the next meeting of her book club.

Let $w(t)$ be the total volume of alcohol in the cask t hours after the start of filling and siphoning. We wish to compute $w(t)$ and also how much alcohol is in the cask when filled.

(a) What is the volume of the tank after t hours?

(b) How long until the tank is filled.

(c) What is the rate of alcohol coming into the tank in gallons/hour?

(d) What is the concentration of alcohol in the tank in (gallons of alcohol)/(gallon of wine).

(e) At what rate is alcohol leaving the tank? *Hint:* This is (flow rate)× (concentration).

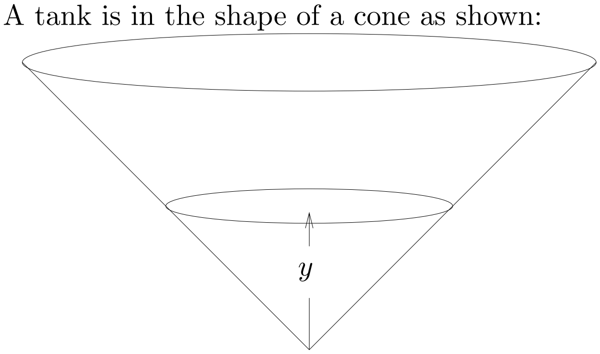
(f) Use that

$$\frac{dw}{dt} = (\text{rate in}) - (\text{rate out})$$

to derive a differential equation for $w(t)$. Note we know $w(0) = 10$.

(g) Solve for $w(t)$ and compute the amount of alcohol in the cask when it is full. □

Problem 9. A tank is in the shape of a cone as shown:



The total height of the tank is 10 feet and the radius of the top is also 10 feet. Water is draining out of the bottom of the tank. We assume that Torricelli's law holds, that is that the flow rate is proportional to the depth of the liquid. Assume the tank is full at time $t = 0$ and that after 5 minutes the depth is 9 feet.

(a) What is the volume of liquid in the tank when the depth is y feet? *Hint:* A good first step would be to find the radius of the surface of the liquid when the depth is y feet.

(b) Torricelli tells us that for some constant of proportionality k that

$$\frac{d}{dt}(\text{Volume}) = -k\sqrt{y}.$$

Use this and part (a) to derive a differential equation for y.

(c) Use this differential equation and that $y(0) = 10$ and $y(5) = 9$ to give an explicit formula for $y(t)$.

(d) How long does it take the tank to completely drain?