ANALYSIS QUALIFYING EXAMINATION JANUARY 1998.

Throughout this examination, unless otherwise specified, the terms measurable, a.e., refer to the Lebesgue measure m on the real line $\mathbb{R}$, and L^p of an interval to L^p of that interval with respect to Lebesgue measure on that interval. Integrals w.r.t. Lebesgue measure will be denoted by $\int f\,dx$. Problems one through eight are 10 points each. Problem 9 is 20 points.

1. Let $g \in L_1(\mathbb{R})$, f_n measurable functions such that $f_n \geq g$ and $f_n \uparrow f$ a.e. Prove that $\int f_n\,dx \uparrow \int f\,dx$. (Note that by definition $\int h\,dx = \int h^+\,dx - \int h^-\,dx$ as long as at least one of the two integrals on the right is finite.)

2. Let F, f and g be nondecreasing functions on $[a,b]$ such that $f + g = F$ and $F(a) = f(a) = g(a) = 0$. Prove that f is absolutely continuous whenever F is absolutely continuous.

3. Let $E \subset \mathbb{R}$ be such that there exist $a \in \mathbb{R}$ and $\delta > 0$ such that for all $|t| < \delta$ we have that $a - t \in E$ or $t - a \in E$. Prove that $m^*(E) \geq \delta$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists a $c > 0$ with $|f(x) - f(y)| \geq c|x - y|$ for all $x, y \in \mathbb{R}$.
   a. Show that $f(\mathbb{R})$ is closed in $\mathbb{R}$.
   b. Show that f is onto.

5. Let $f \in L_2([0,1])$. Let $g(x,y) = f(x)f(y)$.
   a. Prove that g is measurable with respect to the product Lebesgue measure.
   b. Prove that $g \in L_2([0,1] \times [0,1])$ and $\|g\|_2 = \int |f(x)|^2\,dx$.

6. Let f be a measurable function on $\mathbb{R}$ with $f \geq 0$. Prove that there exist measurable sets E_n and $\alpha_n \geq 0$ such that

$$f = \sum_1^\infty \alpha_n \chi_{E_n}.$$

7. Let $G \subset \mathbb{C}$ be an open set containing the closed disk $\overline{D_r(a)} = \{z : |z - a| \leq r\}$. Let $\langle f_n \rangle$ be a sequence of analytic functions on G such that $f_n(z) \to 0$ uniformly on $\{z : |z - a| = r\}$. Prove that $f_n(z) \to 0$ for all z in the open disk $D_r(a)$.

8. Let f be an analytic function on G, where G contains the closed unit disk $\{z : |z| \leq 1\}$ and assume that $|f(z)| > 2$ on $\{z : |z| = 1\}$ and $f(0) = 1$. Does f have to have a zero in the open unit disk?