Analysis Qualifying Examination **February 2009**

INSTRUCTIONS:

(1) Write your solutions on only one side of your paper.
(2) Start each new problem on a separate page.
(3) Write your name (or just your initials) on the top of each page.
(4) Before handing in the exam, put the problems in order and then consecutively number your pages.
(5) Each of the 8 problems is worth 12 points. Following the instructions is worth 4 points.

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

Signature / Date : ______________________________

Name (printed) : ______________________________

Problem 1. Let (X, ρ) be a metric space. Throughout this problem, A and B are *nonempty*, *closed*, *disjoint* subsets of X. Define the distance $d(A, B)$ between A and B by

$$d(A,B) \;=\; \inf\left\{\rho(x,y)\colon x \in A \;\text{ and }\; y \in B\right\} \;. \tag{1}$$

(a) Given an example of two such subsets A and B of some metric space X such that $d(A, B) = 0$.

(b) Now assume, furthermore, that B is compact. Show that $d(A, B) > 0$.

Problem 2. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show *Young's inequality*, i.e. show that if $x, y \geq 0$ then

$$xy \quad \leq \quad \frac{x^p}{p} + \frac{y^q}{q} \;. \tag{2}$$

You may use, without proving, the fact that $\varphi(x) = -\ln x$ is a convex function on $(0, \infty)$.

(b) Show *Hölder's inequality* for sequence spaces, i.e.
show that if $x = \{x_i\}_{i=1}^{\infty} \in \ell_p$ and $y = \{y_i\}_{i=1}^{\infty} \in \ell_q$ then $\{x_i \; y_i\}_{i=1}^{\infty} \in \ell_1$ and

$$\|\{x_i \; y_i\}_{i=1}^{\infty}\|_{\ell_1} \;\leq\; \|\{x_i\}_{i=1}^{\infty}\|_{\ell_p} \;\cdot\; \|\{y_i\}_{i=1}^{\infty}\|_{\ell_q} \;. \tag{3}$$

(c) Show *Hölder's inequality* for function spaces, i.e.
show that if $f \in L_p$ and $g \in L_q$ then $fg \in L_1$ and

$$\|fg\|_{L_1} \;\leq\; \|f\|_{L_p} \;\cdot\; \|g\|_{L_q} \;. \tag{4}$$

Problem 3. Let C denote the Cantor "middle-thirds" set.

(a) Show directly that C has outer measure equal to zero.

(b) Show that C contains a point which is not an endpoint of any of the closed intervals that remain at any stage of the construction.

(c) Show that C is uncountable.

Problem 4. Let $g\colon [a,b] \to [c,d]$ and $f\colon [c,d] \to \mathbb{R}$ be *absolutely continuous* functions.

(a) Define what it means for a function $h\colon [a,b] \to \mathbb{R}$ to be *absolutely continuous*.

(b) Assume, furthermore, that g is *monotone increasing*. Show that $f \circ g$ is absolutely continuous.

Problem 5. Let $L_1 = \{f\colon \mathbb{R} \to \mathbb{R} \mid \ f \text{ is Lebesgue integrable}\}$.
Establish the Riemann-Lebesgue Theorem: if $f \in L_1$ then $\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\ \cos(nx)\, dx = 0$.
You may use, without proving, that step functions (i.e. functions that are finite linear combinations of characteristic functions of intervals of finite length) are dense in L_1.

Problem 6. State Rouché's Theorem. Use it to show that all five zeros of the polynomial

$$p(z) = z^5 + 3z + 1$$

lie in the disk of radius 2 centered at the origin.

Problem 7. (a) State and prove Liouville's Theorem.

(b) State and prove the Fundamental Theorem of Algebra, using Liouville's Theorem.

Problem 8. Use contour integration to evaluate the integral

$$\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\, d\theta.$$