Mathematics 552 Homework, January 15, 2020

Problem 1. First some practice with doing arithmetic with complex numbers. Compute the following:

(a) $(3-4i)(2+5i)$
(b) $\dfrac{2+5i}{4-3i}$
(c) $z^2 - 2z + 2$ where $z = 1+i$
(d) $(1+i)^2$
(e) $(1+i)^3$ $\square$

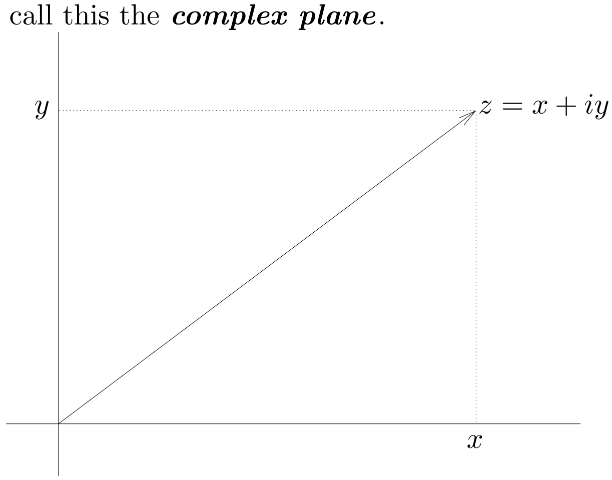
Problem 2. If $z = 4 - 3i$ compute the following:

(a) $\overline{z}$
(b) $|z|$

Powers of i will come up repeatedly. The next problem shows us the pattern.

Problem 3. Compute i^2, i^3, i^4, i^5, i^6, i^7, and i^8. Then give a formula for i^n when n is a positive integer. $\square$

We can view a complex number $z = x + iy$ as a two dimensional vector in a obvious way: z corresponds to the point (x, y) in the plane. Then the addition of complex numbers corresponds to the vector of the vectors in the usual way. We call this the ***complex plane***.



Problem 4. Let $a = 1 + 2i$ and $b = 3 - i$. Draw a picture showing, and labeling, a, b, $a + b$, $a - b$ and $2a$. $\square$

Problem 5. Let a be a complex number and r a positive real number. Explain why the set of points z such that

$$|z = a| = r$$

is a circle with center a and radius r. $\square$

For any real number θ define a complex number $e^{i\theta}$ by

$$\operatorname{cis}(\theta) = \cos(\theta) + i\sin(\theta).$$

Problem 6. Show for all real numbers α and β that

$$\operatorname{cis}(\alpha)\operatorname{cis}(\beta) = \operatorname{cis}(\alpha+\beta)$$

Hint: Recall the addition formulas for sin and cos:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\\ \sin(\alpha+\beta) &= \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta).\end{aligned}$$ □

Problem 7. Find the following:

(a) All the cube roots of i. Draw a picture of them.
(b) All the fourth roots of -16. Draw a picture of them.

Problem 8. Draw a picture showing all the fifth roots of $-4+4i$.

Problem 9. Let $p(z) = a_3z^3 + a_1z^2 + a_1z + a_0$ where a_0, a_1, a_2, a_3 are real numbers. Show that if z_0 is a complex number with $p(z_0) = 0$, then also $p(\overline{z}_0) = 0$. That is if a complex number is a root of $p(z)$, then so is its complex conjugate. *Hint:* We know that $p(z_0) = a_3z_0^3 + a_1z_0^2 + a_1z_0 + a_0 = 0$. Take the complete conjugate of this equation and use that $\overline{a}_j = a_j$ as the a_j's are real.

Problem 10. Generalize the last problem to polynomials or arbitrary degree.

Problem 11. Use the De Moivre's formula

$$\operatorname{cis}(\theta)^n = \operatorname{cis}(n\theta)$$

to find formulas for $\cos(2\theta)$, $\sin(2\theta)$, $\cos(3\theta)$, and $\sin(3\theta)$.