Admission to Candidacy Examination

in

Real Analysis

January 1990

Terminology: Unless otherwise specified, the terms measurable, a.e., refer to Lebesgue measure λ on the real line $\mathbb{R}$, and L^p of an interval with respect to Lebesgue measure on that interval.

1. (a) Define Riemann integrability of a bounded function f on $[a,b]$.

   (b) Prove that if f is Riemann integrable on $[a,b]$, then f is Lebesgue integrable on $[a,b]$ and

$$(R)\int_a^b f(x)dx = \int_a^b f\, d\lambda .$$

2. (a) Let $\langle f_n \rangle$ be a sequence of continuous functions on a compact subset K of $\mathbb{R}$ such that $f_n(x) \geq f_{n+1}(x)$ for all $n = 1,2,3,\ldots$ and all $x \in K$, and $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in K$. Prove that $f_n \to 0$ uniformly on K.

   (b) Show by example that the result is false if K is not compact.

3. (a) Prove that $\left(1 - \frac{x}{n}\right)^n < e^{-x}$ for $0 < x < n$, and that

$$\lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^n = e^{-x} \quad \text{for } 0 < x < \infty .$$

   (b) Prove that $\displaystyle\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{\alpha-1}dx = \int_0^\infty e^{-x}\, x^{\alpha-1}\, dx$ for $\alpha > 0$.

4. (a) Define absolute continuity of a function f on $[0,1]$.

   (b) Let f be absolutely continuous on $[0,1]$. Prove that for $1 \leq p < \infty$, $|f|^p$ is absolutely continuous on $[0,1]$.

   (c) Prove that the product of two absolutely continuous functions is absolutely continuous.

non-negative integer n. Prove that $f(t) = 0$ a.e..

6. Let $1 < p < \infty$ and $f \in L^p(\mathbb{R})$. Prove that

$$\lim_{h \to 0} h^{\frac{1}{p} - 1} \int_x^{x+h} f(t)\,dt = 0 \quad \underline{\text{uniformly}} \text{ in } x .$$

7. Let Σ be a σ-field of subsets of the set X and let $\langle \mu_n \rangle$ be a sequence of positive measures defined on Σ such that $\mu_n(X) = 1$ for all n. Define μ on Σ by

$$\mu(E) = \sum_{n=1}^{\infty} 2^{-n} \mu_n(E) \qquad (E \in \Sigma).$$

(a) Prove that μ is a measure on Σ.

(b) Prove that for each n there exists $f_n \in L^1(\mu)$ with

$$\mu_n(E) = \int_E f_n \, d\mu \qquad (E \in \Sigma).$$

8. Let $f \in L^p((0,\infty))$, $1 < p < \infty$, and let

$$F(x) = \frac{1}{x} \int_0^x f(t)\,dt .$$

Prove that $||F||_p \le \frac{p}{p-1} ||f||_p$.

Hint: First assume f is continuous with compact support and apply integration by parts.

$\{K_n\}$ be a sequence of non-negative continuous functions on $[-1,1]$

satisfying (a) $\displaystyle\int_{-1}^{1} K_n(x)dx = 1$ for all n, and

(b) $\displaystyle\lim_{n\to\infty} \int_{-\delta}^{\delta} K_n(x)dx = 1$ for every $\delta > 0$.

Let f be a continuous periodic function on $\mathbb{R}$ with period 2.

Define $P_n(x)$ on $\mathbb{R}$ by $P_n(x) = \displaystyle\int_{-1}^{1} f(x+t)K_n(t)dt$.

Prove that $\displaystyle\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $\mathbb{R}$.