Mathematics 546 Homework.

We start with some problems related to subgroups.

Definition 1. Let G be a group. Then the ***center*** of G, denoted by $Z(G)$, is the set of elements of G that commute with all the elements of G. That is

$$Z(G) = \{a \in G : ax = xa \text{ for all } x \in G\}.$$

Problem 1. Show that $Z(G)$ is a subgroup of G. $\square$

Recall that the dihedral group D_n is the group generated by two elements a and b with

$$a^n = b^2 = 1, \quad ba = a^{-1}b.$$

Here (again) is the multiplication table for the quaternion group Q:

	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

Problem 2. (a) Show that the center of the dihedral D_3 is trivial, that is $Z(D_3)$ is just the one element subgroup $\{1\}$.
(b) Show center of the dihedral group D_4 is $Z(D_4) = \{1, a^2\}$.
(c) Find the center of Q. $\square$

Definition 2. Let G be a group and $a \in G$. Then the ***centralizer*** of a, denoted $C(a)$, is the set of all element of G that commute with a. That is

$$C(a) = \{x \in G : ax = za\}.$$

$\square$

Problem 3. (a) In D_4 find $C(a)$ and $C(b)$.
(b) In Q find $C(i)$.
(c) In $\mathrm{GL}(2, \mathbb{R})$ (the group of invertible 2×2 matrices) find $C(A)$ where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\square$

If G is a group and $a \in G$ then a has ***finite order*** if and only if there is a positive integers k with $a^k = e$ (where e is the identity of G). The ***order***, denoted $o(a)$ of a is then the smallest positive integer n with $a^n = 1$.

Proposition 3. *If a is a group element with finite order and $a^k = e$, then $o(a) \mid k$.*

Proof. We proved this in class: here is a recap of the argument. Let $n = o(a)$. Then n is the smallest positive integer with $a^n = e$. Use the division algorithm to divide n into k:

$$k = qn + r \qquad \text{with} \qquad 0 \leq r < n.$$

Then

$$e = a^k = a^{qn+r} = (a^n)^q a^r = e^q a^r = a^r.$$

Since $0 \leq r < n$ and n is the smallest positive integer with $a^n = e$ this implies $r = 0$. But then $k = qn + r = qn$ which implies $k \mid n$. $\square$

We have also defined the ***cyclic subgroup*** generated a as

$$\langle a \rangle = \{a^k : k \in \mathbb{Z}\}.$$

In class we proved

Proposition 4. *If a has finite order then $|\langle a \rangle| = o(a)$. (Here $|S|$ is the number of elements in the set S.)* $\square$

Problem 4. Let $a \in G$ have $o(a) = n$ and assume that $\gcd(k, n) = 1$. Show that $o(a^k) = o(a) = n$. *Hint:* First note

$$(a^k)^n = (a^n)^k = e^k = e$$

and $o(a^k)$ is the smallest positive integers m with $(a^k)^m = e$ thus $o(a^k) \leq n$. Let $m = o(a^k)$. Then $(a^k)^m = a^{km} = e$ and by Proposition 3 this implies $n \mid km$. Now use that $\gcd(n, k) = 1$ to explain why $n \mid m$ and use this to finish the proof. $\square$

Let G be a group a H a subgroup of G. Then the ***right cosets*** of H are the sets

$$Hg = \{hg : h \in H\}$$

where $g \in G$.

Problem 5. (a) In D_3 list all the cosets of $H = \langle a \rangle = \{1, a, a^2\}$ (there are two of them).

(b) In D_4 list all the cosets of $H = \langle a^2 \rangle = \{1, a^2\}$ (there are four of them).

(c) In D_4 list all the cosets of $H = \{1, ab\} = \langle ab \rangle$ (there are four of them).

(d) In Q list all the cosets of $\langle k \rangle = \{1, k, -1, -k\}$ (there are two of them).