Mathematics 555 Homework

1. Continuous functions.

1.1. Uniformly continuous functions.

Definition 1. Let $f\colon E \to E'$ be a function between metric spaces. Then f is ***uniformly*** continuous if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $p, q \in E$,

$$d(p,q) < \delta \quad \text{implies} \quad d(f(p), f(q)) < \varepsilon. \qquad \square$$

Recall that a function $f\colon E \to E'$ between is Lipschitz if and only if there is a constant $C \geq 0$ such that $d(f(p), f(q)) \leq C d(p,q)$ for all $p, q \in E$. Last term we saw several examples of Lipschitz functions.

Problem 1. Show that every Lipschitz function is uniformly continuous. $\square$

Proposition 2. *Every uniformly continuous function is continuous.*

Problem 2. Prove this. $\square$

Problem 3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2$. Show that f is not uniformly continuous. *Hint:* Towards a contradiction assume that f is uniformly continuous. Let $\varepsilon = 1$, then there is a $\delta > 0$ such that for all $x, y \in \mathbb{R}$

$$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| < 1.$$

Show this leads to a contradiction. $\square$

Problem 4. Let $f\colon (0,1) \to \mathbb{R}$ be the continuous function

$$f(x) = \frac{1}{x}.$$

Show that f is not uniformly continuous. $\square$

Problem 5. On let $f\colon [0,1] \to \mathbb{R}$ be the functions

$$f(x) = \sqrt{x}.$$

Prove directly from the definition that f is uniformly continuous. $\square$

Here is a bit of review in using the triangle inequality in metric spaces. If E is a metric space and $y_0, y_1, y_2 \in E$, then

$$d(y_0, y_2) \leq d(y_0, y_1) + d(y_1, y_2).$$

If $y_0, y_1, y_2, y_3 \in E$, then

$$d(y_0, y_3) \leq d(y_0, y_2) + d(y_2, y_3) \leq d(y_0, y_1) + d(y_1, y_2) + d(y_2, y_3).$$

And by now you may have guessed the pattern which is given by the following:

Proposition 3. *Let E be a metric space and $y_0, y_1, \ldots, y_n \in E$. Then*

$$\begin{aligned} d(y_0, y_1) &\leq d(y_0, y_1) + d(y_1, y_2) + \cdots + d(y_{n-1}, y_n) \\ &= \sum_{j=1}^{n} d(y_j, y_{j-1}). \end{aligned}$$

Problem 6. Prove this. *Hint:* Induction. □

The following will let us use Proposition 3 to derive some properties of uniformly continuous function.

Definition 4. Let E be a metric space and $\delta > 0$ a positive real number. Then a finite sequence $x_0, x_1, \ldots, x_n \in E$ is a ***δ-sequence*** if and only if for each $j \in \{1, 2, \ldots, n\}$ the inequality $d(x_{j-1}, x_j) < \delta$. □

Lemma 5. *Let $f \colon E \to E'$ be a map between metric spaces and $\delta, \varepsilon > 0$. Assume that for all $p, q \in E$ that*

$$d(p, q) < \delta \quad \textit{implies} \quad d(f(p), f(q)).$$

Then for any δ-sequence $x_0, x_1, \ldots, x_n \in E$ in E we have

$$d(f(x_0), f(x_n)) \leq n\varepsilon.$$

Problem 7. Prove this. *Hint:* Letting $y_j = f(x_j)$ in Lemma 3 we have

$$d(f(x_0), f(x_n)) \leq \sum_{j=1}^{n} d(f(x_{j-1}), f(x_j)).$$

□

For the last lemma to be useful we need to be able to find some δ-sequences. In $\mathbb{R}$, or more generally in $\mathbb{R}^n$ this is easy.

Lemma 6. *Let $p, q \in \mathbb{R}^n$ and $\delta > 0$. Let n be the unique positive integer with*

$$n - 1 \leq \frac{\|p - q\|}{\delta} < n$$

(this is the same as choosing n to be the smallest positive integer with $\|p - q\|/n < \delta$). For $0 \leq j \neq n$ let

$$x_j = p + \frac{j}{n}(q - p).$$

Then $x_0, x_1, \ldots, x_n$ is a δ-sequence with $x_0 = p$ and $x_n = q$.

Problem 8. In the case of $n = 5$ and $p, q \in \mathbb{R}^2$ draw the picture of what these points look like. Then prove the result in the general case. □

Problem 9. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be uniformly continuous. Show there are constants $A, B > 0$ such that

$$|F(x)| \leq A + B\|x\|$$

for all $x \in \mathbb{R}^n$. □

Problem 10. Prove this. *Hint:* Start by letting $\varepsilon = 1$ in the definition of uniform continuity. Then there is a δ such that

$$\|p - q\| < \delta \quad \text{imlies} \quad |f(p) - f(q)| < 1.$$

Let $x \in \mathbb{R}^n$. By Lemma 6 there is a δ-sequence $x_0, x_1, \ldots, x_n \in \mathbb{R}^n$ with $x_0 = 0$ and $x_n = x$ and

$$n - 1 \leq \frac{\|x - 0\|}{\delta} < n.$$

Now use Lemma 5 to show

$$\|f(x) - f(0)\| \leq n$$

and then use this to show

$$|f(x)| \leq |f(0)| + 1 + \frac{\|x\|}{\delta}$$

and explain why this completes the proof. □

Problem 11. Use the previous problem to show that no polynomial of degree greater than 1 is uniformly continuous on $\mathbb{R}$. □

1.2. **Continuous functions on compact sets.**

Theorem 7. *Let $f\colon E \to E'$ be a map between metric spaces with E compact. Let $f\colon E \to E'$ be a continuous function. Then f is uniformly continuous.*

Problem 12. Prove this. *Hint:* We use the open cover definition of compactness. Want to find a open open cover reduce to a finite subcover. Let $\varepsilon > 0$. As E is compact, for each $x \in E$ there a $\delta_x > 0$ such that for all $y \in E$

$$d(x, y) < \delta_x \quad \text{implies} \quad d(f(x), f(y)) < \frac{\varepsilon}{2}.$$

Then

$$\mathcal{U} = \{B(x, \delta_x/2) : x \in E\}$$

is an open cover of E. By compactness it has a finite subcover

$$\mathcal{U}_0 = \{B(x_1, \delta_{x_1}/2), B(x_2, \delta_{x_2}/2, \ldots, B(x_n, \delta_{x_n}/2)\}.$$

Let

$$\delta = \min_{1 \leq j \leq n} \frac{\delta_{x_j}}{2}.$$

The goal now is to show that for all $x, y \in E$

$$d(x, y) < \delta \quad \text{implies} \quad d(f(x), f(y)) < \varepsilon.$$

So let $x, y \in E$ with

$$d(x, y) < \delta.$$

Then as $\mathcal{U}_0$ is a cover of E, there there is a $B(x_j, \delta_{x_j}/2)$ with $x \in B(x_j, \delta_{x_j}/2)$.

(a) Show $y \in B(x_j, \delta_{x_j})$.

(b) Use that $x, y \in B(x_j, \delta_j)$ to show

$$d(x_j, x) < \frac{\varepsilon}{2} \quad \text{and} \quad d(x_j, y) < \frac{\varepsilon}{2}.$$

(c) Now show

$$d(x, y) < \varepsilon$$

which completes the proof. $\square$

Note this implies that for any positive integer n that the function

$$f(x) = x^{1/n}$$

is uniformly continuous on $[0, 1]$. Compare this with Problem 5 where I had to work as hard to do the special case as doing this general result.

1.3. Uniform convergence.

Definition 8. Let E and E' be metric spaces and $f, f_1, f_2, \cdots : E \to E'$ functions. Then

(a) $\lim_{n\to\infty} f_n = f$ ***pointwise*** if and only if for all $x \in E$ we have

$$\lim_{n\to\infty} f_n(x) = f(x).$$

(b) $\lim_{n\to\infty} f_n = f$ ***uniformly*** if and only if for all $\varepsilon > 0$ there is a N such that

$$n \geq N \quad \text{implies} \quad d(f_n(x), f(x)) < \varepsilon.$$

$\square$

To be explicit about the difference between the two modes of convergence, here is a restatement of each with all the dependencies made explicit.

Pointwise convergence: We have $\lim_{n\to\infty} f_n = f$ pointwise if and only if

$$\forall x \in E \; \forall \varepsilon > 0 \, \exists N_{x,\varepsilon} \big[n \geq N_{x,\varepsilon} \text{ implies } d(f_n(x), f(x)) < \varepsilon \big].$$

Uniform convergence: We have $\lim_{n\to\infty} f_n = f$ uniformly if and only if

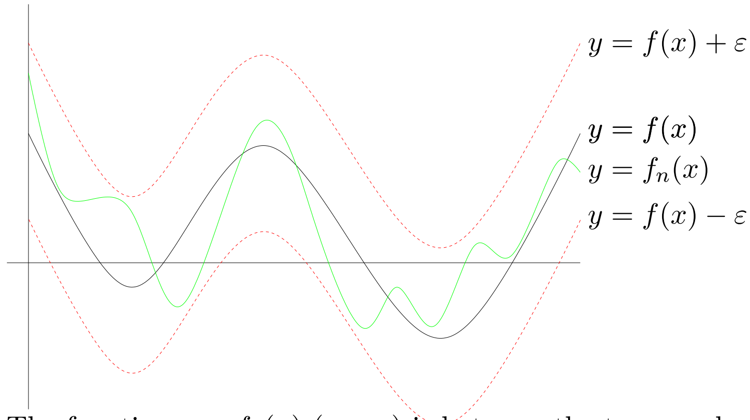
$$\forall \varepsilon > 0 \, \exists N_\varepsilon \, \forall x \in E \big[n \geq N_\varepsilon \text{ implies } d(f_n(x), f(x)) \big].$$

The following picture shows what this looks like in the case E is an interval.

In the case the functions $f, f_1, f_2, \cdots : E \to \mathbb{R}$ (this the functions are real valued) then the condition that $|f_n(x) - f(x)| < \varepsilon$ is equivalent to

$$f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon \quad \text{for all } x \in E.$$

Here is the picture in the case where E is an interval.



The function $y = f_n(x)$ (green) is between the two graphs $y = f(x) - \varepsilon$ and $y = f(x) + \varepsilon$ (both in red).

Let us now look at an example where there is pointwise convergence, but no uniform convergence. This is a variant on the teepee example we did in class. Let $g\colon [0,\infty) \to \mathbb{R}$ be the function

$$g(x) = \frac{2x}{1+x^2}.$$

This function has a maximum at $x = 1$. One way to see this is to use the adding and subtracting trick to get

$$g(x) = 1 + g(x) - 1 = 1 + \frac{2x - 1 - x^2}{1+x^2} = 1 - \frac{(x-1)^2}{1+x^2} \le 1.$$

And equality will only hold when $x = 1$. Let

$$f_n(x) = g(nx) = \frac{2nx}{1+n^2x^2}.$$

Then

$$f_n(1/n) = g(n(1/n)) = g(1) = 1.$$

If $x > 0$ we have

$$0 \le f_n(x) = \frac{xn}{1+(nx)^2} \le \frac{nx}{n^2x^2} = \frac{1}{nx}$$

and so

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{for} \quad x > 0$$

But since $f_n(0) = 0$ for all x more generally have

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{for} \quad x \ge 0.$$

Therefore $\lim_{n\to\infty} f_n = 0$ pointwise on $[0,\infty)$. But the convergence is not uniform:

Problem 13. For this example show that the convergence is not uniform. *Hint:* Let $\varepsilon \le 1$. Then show that for all N there is a $n \ge N$ and a x_n such that $f_n(x_n) = 1$ and thus $|f_n(x_n) - 0| \not< \varepsilon$. $\square$

Problem 14. This problem is important as it gives an example of what can go wrong when the convergence of a sequence of functions is not uniform. Let

$$g(x) = \frac{x^2}{1+x^2}$$

and set

$$f_n(x) = g(nx)$$

and let

$$f(x) = \begin{cases} 0, & x = 0; \\ 1, & x \neq 0. \end{cases}$$

(a) Show that for all $x \in \mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = f(x).$$

(b) Let $x_m = 1/m$. Show that for each n that

$$\lim_{m\to\infty} f_n(x_m) = 0.$$

(c) Show

$$\lim_{n\to\infty} \lim_{m\to\infty} f_n(x_m) = 0.$$

(d) Show

$$\lim_{m\to\infty} \lim_{n\to\infty} f_n(x_m) = 1. \qquad \square$$

1.4. **Uniform limits of continuous functions.**

Theorem 9. *Let $f, f_1, f_2, \cdots : E \to E'$ be maps between metric spaces. Assume that each of the f_n's is continuous and that*

$$\lim_{n\to\infty} f_n = f \quad \textit{uniformly.}$$

Then f is also continuous. (This is open restated by saying "The uniform limit of continuous functions is continuous".)

Problem 15. Prove this. *Hint:* Let $\varepsilon > 0$ and let $p \in E$. You need to show that there is a $\delta > 0$ such that $d(p, q) < \delta$ implies $d(f(p), f(q)) < \varepsilon$.

So let $\varepsilon > 0$. Then, using that $f_n \to f$ uniformly, there is a $N > 0$ such that for all $x \in E$

$$n \geq N \quad \text{implies} \quad d(f_n(x), f(x)) < \frac{\varepsilon}{3}.$$

Choose any $n_0 \geq N$. Then f_{n_0} is continuous at p and therefore there is a $\delta > 0$ such that

$$d(p, q) < \delta \quad \text{implies} \quad d(f_{n_0}(p), f_{n_0}(q)) < \frac{\varepsilon}{3}.$$

Now use the triangle inequality to put these pieces together and complete the proof. $\square$

We now show that the pathology of Problem 14 does not occur when the limit function is the uniform limit of continuous functions.

Theorem 10. *Let $f, f_1, f_2, \cdots : E \to E'$ be maps between metric spaces. Assume that each of the f_k's is continuous and that*

$$\lim_{k\to\infty} f_k = f \quad \textit{uniformly.}$$

Let $\langle p_m \rangle_{m=1}^{\infty}$ be a sequence in E with

$$\lim_{m\to\infty} p_m = p.$$

Then

$$\lim_{m\to\infty} \lim_{n\to\infty} f_n(p_m) = \lim_{n\to\infty} \lim_{n\to\infty} f_n(p_m) = f(p).$$

Problem 16. Prove this. *Hint:* As $f_n \to f$ uniformly there is a N such that

$$n \geq N \quad \text{implies} \quad d(f_n(x), f(x)) < \frac{\varepsilon}{2}$$

for all $x \in E$. By Theorem 9 the function f is continuous and therefore there is a $\delta > 0$ such that for $q \in E$

$$d(p, q) < \delta \quad \text{implies} \quad d(f(p), f(q)) < \frac{\varepsilon}{2}.$$

Becasue $\lim_{m\to\infty} p_m = p$ there is a M so that

$$m \geq M \quad \text{implies} \quad d(p_m, p) < \frac{\varepsilon}{2}.$$

Now use the triangle inequality to show

$$n \geq N,\ m \geq M \quad \text{implies} \quad d(f_n(p_m), f(p)) < \varepsilon$$

and use this to complete the proof. $\square$