Mathematics 739 Homework 1: Basics about fiber bundles.

As a review of vector bundles we look at a generalization, fiber bundles. These are spaces that are locally products in a nice way.

1. Definitions, examples, and basic properties.

Definition 1. A map $p\colon E \to B$ is a ***fiber bundle*** with ***fiber*** F, if the following hold:

(a) E and B are topological spaces and p is a continuous map.
(b) For each $x \in B$ the preimage $p^{-1}[x]$ is homeomorphic to F.
(c) p is surjective.
(d) $p\colon E \to B$ is locally a product in the following sense. For each $x \in B$ there is an open neighborhood, U_x, of x in B and a homeomorphism $\Psi_U : p^{-1}[U] \to U \times F$ such that the following diagram commutes.

$$\begin{tikzcd} p^{-1}[U] \arrow[r, "\Psi_U"] \arrow[d, "p"] & U \times F \arrow[d, "\text{projection}"] \\ U \arrow[r, "\text{Identity}"] & U \end{tikzcd}$$

Such a map Ψ_U is a ***local trivialization*** of over U. $\square$

In this set up the B is called the ***base space*** and E the ***total space*** of the bundle.

The most obvious example of a fiber bundle is the product bundle $E = B \times F$.

The following can be used to give examples that are not products.

Proposition 2. *Let $p\colon E \to B$ be a covering space with B connected. Then $p\colon E \to B$ is a fiber bundle. Conversely any fiber bundle where the fiber has the discrete topology and the base is locally connected is a covering space.*

Problem 1. If you know the definition of a covering space, prove this. $\square$

Recall that $\mathbb{CP}^n$ is the space of one dimensional linear subspaces of $\mathbb{C}^{n+1}$. Let S^{2n+1} be the set of unit vectors in $\mathbb{C}^{n+1}$. Define a $p\colon S^{2n+1} \to \mathbb{CP}^n$ by

$$p(u) = \{zu : z \in \mathbb{C}\} = \text{Linear subspace spanned by } u.$$

Problem 2. Show that $p\colon S^{2n+1} \to \mathbb{CP}^n$ is a circle bundle. (A ***circle bundle*** is a fiber bundle where the fiber is the circle S^1.)

Problem 3. Let $\psi\colon F \to F$ be a homeomorphism. Let E be the quotient space $[0,1] \times F/\sim$ where $\sim$ is the equivalence relation such that

$$(0, v) \sim (1, \psi(v)).$$

Let $S^1 = [0,1]/0,1$ (that is $[0,1]$ with the end points identified). Let $p\colon E \to S^1$ be the map

$$p([t, v]) = t/\{0,1\}$$

where $[t, v]$ is the equivalence class of (t, v). Show that this is a fiber bundle over S^1 with fiber F. $\square$

Problem 4. Here is a problem for those that know some differential topology. Let $f\colon M \to N$ be a smooth map between connected compact smooth manifolds. Assume that f is a submersion. (That is for all $x \in M$ the derivative $f'(x)\colon T_xM \to T_{f(x)}N$ is surjective.) Then $f\colon M \to N$ is a fiber bundle. $\square$

Problem 5. Let $B = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half plane. For each $z \in B$ let E_z be the torus

$$E_z = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\, z).$$

Set

$$E = \bigcup_{z \in B} E_z.$$

Show that E is a fiber bundle over B. $\square$

Let $p\colon E \to B$ be a fiber bundle. Choose an open cover $\{U_\alpha\}_{\alpha \in A}$ of B such that for each U_α we have a trivialization

$$\Psi_\alpha\colon p^{-1}[U_\alpha] \to U_\alpha \times F.$$

For each ordered pair $(\alpha, \beta) \in A \times A$ let $U_{\alpha\beta} = U_\alpha \cap U_\beta$ (this may be empty) define a map

$$g_{\alpha\beta}\colon U_{\alpha\beta} \to \mathcal{G}(F)$$

where $\mathcal{G}(F)$ is the group of homeomorphisms of F by

$$g_{\alpha\beta}(x) = \left(\Psi_\alpha \Big|_{p^{-1}(x)} \right) \circ \left(\Psi_\beta \Big|_{p^{-1}(x)} \right)^{-1}.$$

Proposition 3. *The functions $g_{\alpha\beta}$ satisfy the following*

(a) $g_{\alpha\alpha}(x) = \mathrm{Id}_F$ for all $x \in U_\alpha$.
(b) One the intersection $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ we have

$$g_{\alpha\beta}(x) \circ g_{\beta\gamma}(x) = g_{\alpha\gamma}(x).$$

This is the ***cocycle condition****.*
(c) The maps $g_{\alpha\beta}$ are continuous with respect to the natural topology on $\mathcal{G}(F)$.

Proposition 4. *Prove this.* Hint: *Since I have not told you what the topology on $\mathcal{G}(F)$ is you can ignore part (c). (One natural topology is the* ***compact open topology*** *and if you know what this is, then you can do the problem.)*

Theorem 5. *Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of B by open sets and let $g_{\alpha\beta}$ be functions that satisfy the conditions of Proposition 3. Then there is fiber bundle $p\colon E \to B$ and local trivializations $\Psi_\alpha\colon p^{-1}[U_\alpha] \to U_\alpha \times F$ such that the $g_{\alpha\beta}$'s come from the bundle as above.*

Problem 6. Prove this. *Hint:* Let $\mathcal{E}$ be the disjoint union

$$\mathcal{E} := \coprod_{\alpha\in A} U_\alpha \times F.$$

Define an equivalence relation on $\mathcal{E}$ by

$$(x_\alpha, v_\alpha) \sim (x_\beta, v_\beta) \qquad \Longleftrightarrow \qquad x_\alpha = x_\beta \text{ and } g_{\alpha\beta}(x_\alpha)(v_\alpha) = v_\beta$$

where $(x_\alpha, v_\alpha) \in U_\alpha \times F$. Let E to be the quotient space $E := \mathcal{E}/\sim$. Let $[x, v]$ be the equivalence class of $(x, v) \in \mathcal{E}$ and define $p([x, v]) = x$. Now show that $p\colon E \to B$ is the bundle we want. $\square$

We now relate this back to vector bundles. Let $GL(\mathbb{C}^n)$ be the general linear group. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of the space B. Assume that for each pair $(\alpha, \beta) \in A^2$ that there is a continuous map

$$g_{\alpha\beta} U_{\alpha\beta} \to GL(\mathbb{C}^n)$$

and that the functions $g_{\alpha\beta}$ satisfy the cocycle condition. Then we can use the construction of Theorem 5 to construct a fiber bundle $p\colon E \to B$ with fiber $\mathbb{C}^n$.

Problem 7. Show that the construction just outlined gives a vector bundle in the sense that you know and love. $\square$

Problem 8. In the construction of the last problem assume that B is a complex analytic manifold and that the function $g_{\alpha\beta}$ are holomorphic. Then show that $p\colon E \to B$ is a holomorphic vector bundle. That is E is a complex analytic manifold, and p is a holomorphic map. $\square$

Problem 9. Let M be a n-dimensional complex analytic manifold and let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in A}$ be a covering of M by holomorphic coordinate charts. That is each U_α is an open subset of M and each $\phi_\alpha\colon U_\alpha \to C^n$ is a map such that $\phi_\alpha[U_\alpha]$ is an open subset of $\mathbb{C}^n$ and on any overlap $U_{\alpha\beta}$ the map $\psi_{\alpha\beta}\colon \phi_\beta[U_{\alpha\beta}] \to \phi_\alpha[U_{\alpha\beta}]$ by

$$\psi_{\alpha\beta} := \left(\phi_\alpha\big|_{U\alpha\beta}\right) \circ \left(\phi_\beta\big|_{U\alpha\beta}\right)^{-1}.$$

Then on each $U_{\alpha\beta}$ define functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to GL(\mathbb{C}^n)$ by

$$g_{\alpha\beta}(x) = \psi'_{\alpha\beta}(x).$$

(a) Show that the functions $g_{\alpha\beta}$ satisfy the cocycle condition and therefore define a vector bundle.
(b) Show this vector bundle is the holomorphic tangent bundle of M. $\square$

Problem 10. Using the notation of the last problem, let $1 \le k \le n$. Then on each $U_{\alpha\beta}$ define $f_{\alpha\beta}\colon U_{\alpha\beta} \to GL(\bigwedge^k(\mathbb{C}^n))$ by

$$f_{\alpha\beta}(x) = \wedge^k(g_{\alpha\beta}(x)).$$

Show this is a holomorphic vector bundle and the fiber at $x \in B$ is the k-th exterior power of tangent space $T_x(M)$. $\square$

Problem 11. A variant on the last problem and still using the notation of Problem 9 let $f_{\alpha\beta}\colon U_{\alpha\beta} \to GL((\mathbb{C}^n)^*)$ (where $(\mathbb{C}^n)^*$ is the dual space to $\mathbb{C}^n$) by

$$f_{\alpha\beta}(x) = (g_{\alpha\beta}(x)^{-1})^t$$

where A^t is the transpose of the linear map A. Show set of transition functions defines the holomorphic cotangent bundle. $\square$

Problem 12. Let $\mathbb{CP}^n$ be the space of all one dimensional subspaces of $\mathbb{C}^{n+1}$. Then the group $GL(\mathbb{C}^n)$ acts on $\mathbb{CP}^n$ by

$$A\langle v\rangle = \langle Av\rangle.$$

This action has a kernel. It is not hard to see that $A\rangle v\langle = \rangle v\rangle$ for all $v \in \mathbb{C}^{n+1}$ if and only if $A = \lambda I$ for some $\lambda \in \mathbb{C}^*$. Let $G = GL(\mathbb{C}^{n+1})/\{\lambda I\}$. This is the automorphism group of $\mathbb{CP}^n$.

(a) Show that G is a complex analytic manifold.
(b) Let $\{U_\alpha\}_{a\in A}$ be an open cover of the complex manifold M and $g_{\alpha\beta}\colon U_{\alpha\beta} \to G$ be holomorphic maps that satisfy the cocycle condition. Show that the result fiber bundle is a complex analytic manifold and that the fibers are all isomorphic to $\mathbb{CP}^n$. $\square$

Theorem 6. *Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of the space B. Let F be anther space. Assume that for each $\alpha \in A$ there is a continuous map $h_\alpha\colon U_\alpha\mathcal{G}(F)$. On $U_{\alpha\beta}$ define*

$$g_{\alpha\beta} := h_\alpha\big|_{U_{\alpha\beta}} \circ h_\beta\big|_{U_{\alpha\beta}}^{-1}.$$

Show that these function satisfy the cocycle condition and that the resulting fiber bundle is isomorphic to the product bundle $B \times F$. $\square$

A ***section*** of the bundle $p\colon E \to B$ is a continuous map $s\colon B \to E$ such that $p \circ s = \mathrm{Id}_B$. Note every bundle will have any sections. For example let TS^2 be the tangent bundle of the sphere S^2. Then a section of $p\colon TS^2 \to S^2$ is a vector field on S^2. But we know that any vector vanishes for at least one point. Let E be the bundle of unit vectors in TS^2. Then E is a circle bundle over S^2. A section of E would be a vector field that does not vanish at any point. No such vector field exists and therefore E does not have any sections.

Problem 13. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of B and $\{g_{\alpha\beta}\}$ transition functions that define a fiber bundle $p\colon E \to B$ with fiber F. Assume that there are functions $s_\alpha\colon U_\alpha \to F$ such that on each $U_{\alpha\beta}$ they are related by

$$s_\alpha(x) = g_{\alpha\beta}(x)s_\beta(x).$$

Show that this data defines a section of the bundle. $\square$

Problem 14. Let $p\colon E \to B$ be a fiber bundle with fiber F defined by transition functions $\{g_{\alpha\beta}\}$ for the open cover $\{U_\alpha\}_{\alpha\in A}$. Let $h_\alpha\colon \mathcal{G}(F)$ be continuous

2. A classification result.

We fix a base space B and a fiber. Let G be a subgroup of $\mathcal{G}(F)$. That is G is some group of homeomorphisms of F. In most of the examples we will be looking at the G will be a group of matrices, or a bit more generally a Lie group.

For the rest of this section we fix an open cover, $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of the space B. We now do a construction that looks a lot like Čech cohomology. Let $C^0_{\mathcal{U}}(B)$ be collection of all sets $\{c_\alpha\}_{\alpha\in A}$ where $c_\alpha\colon U_\alpha \to G$ is a continuous function.