Mathematics 551 Homework, February 24, 2020

Problem 1. This problem is about rotating a quadratic to eliminate the x-y term. Let

$$q(x,y) = Ax^2 + 2Bxy + Cy^2.$$

The substation

$$\begin{aligned} x &= \overline{x}\sin\theta + \overline{y}\cos\theta \\ y &= -\overline{x}\cos\theta + \overline{y}\sin\theta \end{aligned}$$

corresponds to rotating the axis by an angle of θ and where $\overline{x}$ and $\overline{y}$ are new variables.

(a) Using this substation show

$$q(x,y) = q\big(\overline{x}\sin\theta + \overline{y}\cos\theta, -\overline{x}\cos\theta + \overline{y}\sin\theta\big) = \overline{A}\overline{x}^2 + 2\overline{B}\,\overline{x}\,\overline{y} + \overline{C}\overline{y}^2$$

where

$$\begin{aligned} \overline{A} &= A\cos^2\theta - 2B\cos\theta\sin\theta + C\sin^2\theta \\ \overline{B} &= A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) - C\cos\theta\sin\theta \\ \overline{C} &= A\sin^2\theta s + 2B\cos\theta\sin\theta + C\cos^2\theta \end{aligned}$$

(b) Use the trigonometric identities $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ and $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$ to rewrite the equation for $\overline{B}$ as

$$\overline{B} = \frac{1}{2}(A - C)\sin(2\theta) + B\cos(2\theta).$$

(c) Show that if $B \neq 0$ and θ is chosen so that

$$\cot(2\theta) = \frac{C - A}{2B}$$

then $\overline{B} = 0$. Thus if we set $\overline{q}(\overline{x},\overline{y}) = q\big(\overline{x}\sin\theta + \overline{y}\cos\theta, -\overline{x}\cos\theta + \overline{y}\sin\theta\big)$ we have

$$\overline{q}(\overline{x},\overline{y}) = \overline{A}\overline{x}^2 + \overline{C}\overline{y}^2$$

That is this rotation eliminates the cross term $\overline{x}\,\overline{y}$. $\square$

In Shifrin's book read the first part of Section 2.1, page 35 to the top of page 39. We will use the notation

$$\begin{aligned} E_1(\theta) &= (\cos\theta, \sin\theta, 0) \\ E_2(\theta) &= (-\sin\theta, \cos\theta, 0) \\ E_3(\theta) &= (0, 0, 1). \end{aligned}$$

Then for all θ these are an orthonormal basis of $\mathbb{R}^3$, that is

$$\begin{aligned}
E_i \cdot E_i &= 1 && \text{for } 1 \le i \le 3 \\
E_i \cdot E_j &= 0 && \text{for } 1 \le i < j \le 3 \\
E_1 \times E_2 &= E_3 \\
E_2 \times E_3 &= E_1 \\
E_3 \times E_1 &= E_2.
\end{aligned}$$

Problem 2. In this notation Shifrin's Example 1 (c) on page 36 (the torus) has the parameterization

$$\mathbf{x}(u,v) = (a + b\cos u)E_1(v) + b\sin u E_3 \qquad 0 \le u \le 2\pi, \quad 0 \le v \le 2\pi.$$

(a) Draw a picture of the image of $\mathbf{x}$ (which is just Figure 1.3 in Shifrin) and draw the curve

$$\boldsymbol{\gamma}(t) = \mathbf{x}(t,t) \qquad 0 \le t \le 2\pi.$$

(b) Find the first fundamental form of $\mathbf{x}$.

(c) Set up the integral for the length of $\boldsymbol{\gamma}$. (Don't try to evaluate this integral.) □

Problem 3. Let $\mathbf{c}(s) = (x(s), y(s))$ with $0 \le s \le L$ be a unit speed curve in the plane. Define $\mathbf{x}\colon [0,L] \times (-\infty,\infty) \to \mathbb{R}^3$ by

$$\mathbf{x}(u,v) = (x(u), y(u), v).$$

Find the first fundamental form of $\mathbf{x}$. □

Problem 4. For $u > 0$ and $0 \le v \le 2\pi$ and m a positive constant define

$$\mathbf{f}(u,v) = uE_1(v) + muE_3.$$

Draw a picture of the image of $\mathbf{f}$ and find its first fundamental form. □

Problem 5 (Optional). With the notation of Problem 1 show for any θ that

$$\overline{A}\overline{C} - \overline{B}^2 = AC - B^2 \qquad \text{and} \qquad \overline{A} + \overline{C} = A + C$$ □