

# Constructing Approximations to Functions

Given a function,  $f$ , it is often useful to approximate it by “nicer” functions. For example, given a continuous function,  $f$ , it can be useful to find a sequence of differentiable functions  $f_1, f_2, f_3, \dots$  that converge to  $f$  uniformly. Here we give one of the basic methods for doing this.

**Definition 1.** A sequence of functions  $K_1, K_2, K_3, \dots$  defined on  $\mathbf{R}$  is a **Dirac sequence**, or an **approximation to the identity** iff it satisfies the following conditions.

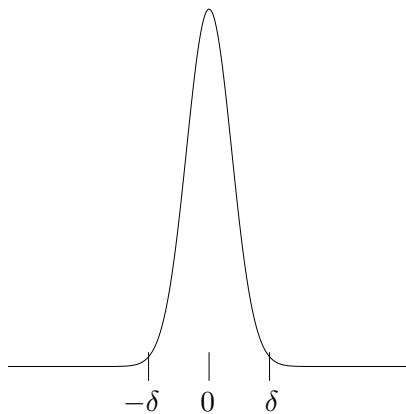
- (a)  $K_n \geq 0$  for all  $n$ ,
- (b) For all  $n$

$$\int_{-\infty}^{\infty} K_n(x) dx = 1.$$

- (c) For all  $\delta > 0$

$$\lim_{n \rightarrow \infty} \int_{|x| \geq \delta} K_n(x) dx = 0. \quad \square$$

The condition (c) says that almost all of the mass of  $K_n$  is in  $(-\delta, \delta)$ .



For large  $n$  almost all of the area under the graph of  $y = K_n(x)$  is between  $-\delta$  and  $\delta$ .

Here is a standard method of constructing Dirac sequences.

**Proposition 2.** Let  $\varphi: \mathbf{R} \rightarrow \mathbf{R}$  be a Riemann integrable function with

$$\varphi \geq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \varphi(x) dx = 1.$$

Then

$$K_n(x) = n\varphi(nx)$$

is a Dirac sequence.

**Problem 1.** Prove this.  $\square$

**Theorem 3.** Let  $f$  be a bounded continuous function on  $\mathbf{R}$  and  $\langle K_n \rangle_{n=1}^\infty$  be a Dirac sequence. Let

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y)K_n(y) dy$$

then

$$\lim_{n \rightarrow \infty} f_n(x) = f(x)$$

pointwise.

**Problem 2.** Prove this. *Hint:* The basic trick is to note that as  $\int_{-\infty}^{\infty} K_n(y) dy = 1$  we have

$$f(x) = f(x) \cdot 1 = f(x) \int_{-\infty}^{\infty} K_n(y) dy = \int_{-\infty}^{\infty} f(x)K_n(y) dy.$$

Therefore for any  $\delta > 0$  we have

$$\begin{aligned} f(x) - f_n(x) &= \int_{-\infty}^{\infty} f(x)K_n(y) dy - \int_{-\infty}^{\infty} f(x-y)K_n(y) dy \\ &= \int_{-\infty}^{\infty} (f(x) - f(x-y))K_n(y) dy \\ &= \int_{|y| < \delta} (f(x) - f(x-y))K_n(y) dy + \int_{|y| \geq \delta} (f(x) - f(x-y))K_n(y) dy \\ (1) \quad &= I_{\delta,n}(x) + J_{\delta,n}(x). \end{aligned}$$

Now let  $\varepsilon > 0$ . Then as  $f$  is continuous at  $x$  there is a  $\delta > 0$  such that

$$|y - x| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Explain why the following holds

$$|I_{\delta,n}(x)| \leq \int_{|y| < \delta} |f(x) - f(x-y)|K_n(y) dy < \int_{|y| < \delta} \left(\frac{\varepsilon}{2}\right) K_n(y) dy \leq \frac{\varepsilon}{2}.$$

Using this in the displayed sequence of equalities (1) gives

$$|f(x) - f_n(x)| \leq |I_{\delta,n}(x)| + |J_{\delta,n}(x)| < \frac{\varepsilon}{2} + |J_{\delta,n}(x)|.$$

This holds for all  $n$ . The function  $f$  is bounded thus there is a constant  $B$  such that  $|f(x)| \leq B$  for all  $x$ . It follows that for all  $x, y \in \mathbf{R}$  that

$$|f(x) - f(x-y)| \leq |f(x)| + |f(x-y)| \leq 2B.$$

Therefore

$$|J_{\delta,n}| \leq \int_{|y| \geq \delta} |f(x) - f(x-y)|K_n(y) dy \leq 2B \int_{|y| \geq \delta} K_n(y) dy.$$

If you now look back at the definition of a Dirac sequence you should be able to use the last inequality to show

$$\lim_{n \rightarrow \infty} |J_{\delta,n}(x)| = 0$$

and thus there is a  $N$  such that  $n > N$  implies  $|J_{\delta,n}(x)| < \varepsilon/2$ .  $\square$

We can do a bit better.

**Theorem 4.** Let  $f$  function on  $\mathbf{R}$  that is both bounded and uniformly continuous and let  $\langle K_n \rangle_{n=1}^\infty$  be a Dirac sequence. Define

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y)K_n(y) dy.$$

Then

$$\lim_{n \rightarrow \infty} f_n(x) = f(x)$$

uniformly on  $\mathbf{R}$ .

**Problem 3.** Prove this. *Hint:* This is just a matter of rewriting the proof of Theorem 3 and making sure that you can make the choices of quantities such as  $\delta$  and  $N$  in a way that is independent of  $x$ .  $\square$

The following gives a large number of examples of functions where Theorem 4 applies.

**Proposition 5.** Let  $f$  be a continuous function such that for some interval  $[\alpha, \beta]$  we have  $f(x) = 0$  for all  $x \notin [\alpha, \beta]$ . Then  $f$  is bounded and uniformly continuous.

**Problem 4.** Prove this. *Hint:* This is a good problem to review several of the results we have been working with. (Continuous on closed bounded intervals are bounded and uniformly continuous).  $\square$

**Proposition 6.** Let  $f$  be bounded and continuous on  $\mathbf{R}$  and let  $\langle K_n \rangle_{n=1}^\infty$  be a Dirac sequence and

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y)K_n(y) dy.$$

Then  $f_n$  can be rewritten as

$$f_n(x) = \int_{-\infty}^{\infty} f(y)K_n(x-y) dy$$

**Problem 5.** Prove this. *Hint:* As far as  $y$  is concerned,  $x$  is a constant. So if we do the change of variable  $z = x - y$  we have  $dz = -dy$ .  $\square$

**Remark 7.** In what follows we will use whichever formula for  $f_n$  given by Proposition 6 that is convenient without referring Proposition 6.

We are now in a position to prove one of the most famous theorems in analysis, the *Weierstrass Approximation Theorem*, which says that a continuous function on a closed bounded interval can be uniformly approximated by a polynomial. To start we need a Dirac sequence that is constructed from polynomials.

**Lemma 8.** Let

$$K_n(x) := \begin{cases} c_n(1-x^2)^n, & |x| \leq 1; \\ 0, & |x| > 1. \end{cases}$$

where

$$c_n := \frac{1}{\int_{-1}^1 (1-x^2)^n dx}.$$

Then  $\langle K_n \rangle_{n=1}^\infty$  is a Dirac sequence.

*Proof.* That  $K_n \geq 0$  and  $\int_{-\infty}^\infty K_n(x) dx = 1$  are easy, so it remains to show that for  $\delta > 0$  the limit  $\lim_{n \rightarrow \infty} \int_{|x| \geq \delta} K_n(x) dx = 0$ . We first give a bound on  $c_n$ .

$$\int_{-1}^1 (1-x^2)^n dx = 2 \int_0^1 (1+x)^n (1-x)^n dx \geq 2 \int_0^1 (1-x)^n dx = \frac{2}{n+1}.$$

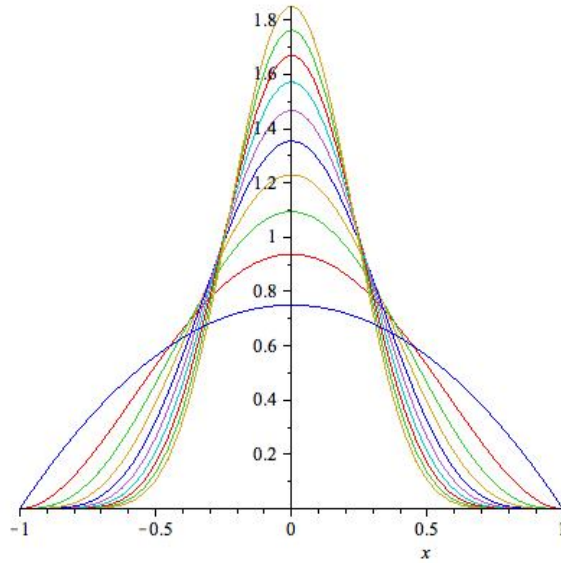
Thus

$$c_n \leq \frac{n+1}{2}.$$

Let  $0 < \delta < 1$ . Then

$$\int_{|x| \geq \delta} K_n(x) dx = 2c_n \int_\delta^1 (1-x^2)^n dx \leq 2c_n \int_\delta^1 (1-\delta)^n dx \leq (n+1)(1-\delta^2)^n.$$

But  $(1-\delta^2) < 1$  so  $\lim_{n \rightarrow \infty} (n+1)(1-\delta^2)^n = 0$  which completes the proof.  $\square$



The graphs of  $y = K_n(x)$  for  $n = 1, 2, \dots, 10$ .

**Problem 6.** While the exact value of  $\int_{-1}^1 (1-x^2)^n dx$  is not needed in the last proof, it is fun to compute it. So find  $\int_{-1}^1 (1-x^2)^n dx$ . *Hint:* This is a case where it pays to generalize. Let

$$I(m, n) := \int_{-1}^1 (1-x)^m (1+x)^n dx.$$

Then we are trying to compute  $I(n, n)$ . Use integration by parts to show

$$I(m, n) = \frac{m}{n+1} I(m-1, n+1)$$

when  $m \geq 1$  and  $n \geq 0$  and note that  $I(0, k) = \int_{-1}^1 (1+x)^k dx$  is easy to compute.  $\square$

**Proposition 9.** *Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a continuous function so such that  $f(x) = 0$  for all  $x \notin [0, 1]$  and let  $K_n$  be as in Lemma 8. Set*

$$p_n(x) = \int_{-1}^1 K_n(x-y)f(y) dy$$

*then  $p_n \rightarrow f$  uniformly and the restriction of  $p_n$  to  $[0, 1]$  is a polynomial.*

*Proof.* By Proposition 5  $f$  is bounded and uniformly continuous. Let  $B$  be a bound for  $f$ , that is  $|f(x)| \leq B$  for all  $x \in \mathbf{R}$ . As  $f$  is uniformly continuous for  $\varepsilon > 0$  here is a  $\delta > 0$ , such that

$$|x-y| \leq \delta \text{ and } x, y \in [0, 1] \implies |f(x) - f(y)| \leq \varepsilon.$$

As  $f$  is bounded and uniformly continuous, Theorem 4 implies  $p_n \rightarrow f$  uniformly. All that remains is to show that  $p_n$  restricted to  $[0, 1]$  is a polynomial. If  $x, y \in [0, 1]$ , then  $x-y \in [-1, 1]$  and therefore

$$\begin{aligned} K_n(x-y) &= c_n(1-(x-y)^2)^n \\ &= g_0(y) + g_1(y)x + g_2(y)x^2 + \cdots + g_{2n}(y)x^{2n} \\ &= \sum_{k=0}^{2n} g_k(y)x^k \end{aligned}$$

where we have just expanded  $c_n(1-(x-y)^2)^n$  and grouped by powers of  $x$ . (Each  $g_k(y)$  is a polynomial in  $y$ , but this does not really matter for us.) As  $f(y) = 0$  for  $y \notin [0, 1]$  if  $x \in [0, 1]$  we have

$$\begin{aligned} f_n(x) &:= \int_0^1 K_n(x-y)f(y) dy \\ &= \int_0^1 \sum_{k=0}^{2n} g_k(y)x^k f(y) dy \\ &= \sum_{k=0}^{2n} \left( \int_0^1 g_k(y) dy \right) x^k \end{aligned}$$

which is clearly a polynomial.  $\square$

**Lemma 10.** *Let  $f: [\alpha, \beta] \rightarrow \mathbf{R}$  be a continuous function with  $f(x) = 0$  for  $x \notin [\alpha, \beta]$ . Define  $F: [0, 1] \rightarrow \mathbf{R}$  to be the function*

$$F(x) := f(\alpha + (\beta - \alpha)x)$$

and let  $P_n: [0, 1] \rightarrow \mathbf{R}$  be polynomials such that  $P_n \rightarrow f$  uniformly and set

$$p_n(x) = P_n\left(\frac{x - \alpha}{\beta - \alpha}\right).$$

Then each  $p_n$  is a polynomial and  $p_n \rightarrow f$  uniformly.

**Problem 7.** Prove this. *Hint:* This is not hard, so don't be long winded.

**Theorem 11 (Weierstrass Approximation Theorem).** Let  $f: [a, b] \rightarrow \mathbf{R}$  be continuous. Then there is a sequence of polynomial  $p_n: [a, b] \rightarrow \mathbf{R}$  with  $p_n \rightarrow f$  uniformly.

**Problem 8.** Prove this. *Hint:* Extend  $f$  to  $\mathbf{R}$  (we still denote the extended function by  $f$ ) by

$$f(x) := \begin{cases} 0, & x < a - 1; \\ (x - (a - 1))f(a), & a - 1 \leq x < a; \\ f(x), & a \leq x \leq b; \\ ((b + 1) - x)f(b), & b < x \leq b + 1; \\ 0, & b + 1 < x. \end{cases}$$

This is continuous (don't prove this, just draw the picture and say it is clear). Let  $\alpha := a - 1$  and  $\beta = b + 1$ . Then use Proposition 9 and Proposition 5 to complete the proof.

We now give some applications of these results.

**Problem 9.** Let  $f: [a, b] \rightarrow \mathbf{R}$  be continuous and assume that

$$\int_a^b f(x)x^n dx = 0$$

for all  $n = 0, 1, 2, 3, \dots$ . Then show  $f(x) = 0$  for all  $x \in [a, b]$ . *Hint:* Show that  $\int_a^b f(x)p(x) dx = 0$  all polynomials. Then choose a sequence of polynomials  $p_n \rightarrow f$  uniformly. Use this sequence to conclude  $\int_a^b f(x)^2 dx = 0$ .  $\square$

**Problem 10.** Let  $f, g: [a, b] \rightarrow \mathbf{R}$  be continuous functions such that

$$\int_a^b f(x)x^n dx = \int_a^b g(x)x^n dx$$

for all  $n = 0, 1, 2, 3, \dots$ . Show that  $f(x) = g(x)$  for  $x \in [a, b]$ . *Hint:* Reduce this to the last problem.  $\square$

**Convention.** For the rest of this homework  $f: \mathbf{R} \rightarrow \mathbf{R}$  is a function such that for some  $b > 0$  we have  $f(x) = 0$  for all  $x$  with  $|x| \geq b$  and  $f$  is Riemann integrable on  $[-b, b]$  and that there is a constant  $B$  such that  $|f(x)| \leq B$  for all  $x$ .  $\square$

**Theorem 12.** If  $\langle K_k \rangle_{k=1}^\infty$  is a Dirac sequence and

$$f_n(x) = \int_{-\infty}^{\infty} K_n(y) f(x-y) dy = \int_{-\infty}^{\infty} K_n(x-y) f(y) dy$$

then at any point  $x$  where  $f$  is continuous

$$\lim_{n \rightarrow \infty} f_n(x) = f(x).$$

**Problem 11.** Prove this. *Hint:* This is an easier version of an earlier theorem.

**Definition 13.** A Dirac sequence  $\langle K_n \rangle_{n=1}^\infty$  is **differentiable** iff for each  $n$   $K_n$  is differentiable and

$$\lim_{h \rightarrow 0} \frac{K_n(x+h) - K_n(x)}{h} = K'_n(x)$$

uniformly. Explicitly this means that for each  $n$  and  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$(2) \quad |h| \leq \delta \quad \implies \quad \left| \frac{K_n(x+h) - K_n(x)}{h} - K'_n(x) \right| \leq \varepsilon$$

for all  $x \in \mathbf{R}$  □

**Proposition 14.** Let  $f$  be as in the convention and  $\langle K_k \rangle_{k=1}^\infty$  a differentiable Dirac sequence. Then for each  $n$

$$f_n(x) = \int_{-\infty}^{\infty} K_n(x-y) f(y) dy$$

is differentiable and

$$f'_n(x) = \int_{-\infty}^{\infty} K'_n(x-y) f(y) dy.$$

(It is not being assumed that  $f$  is differentiable.)

**Problem 12.** Prove this. *Hint:* First show

$$\begin{aligned} & \left( \frac{f_n(x+h) - f_n(x)}{h} \right) - \int_{-\infty}^{\infty} K'_n(x-y) f(y) dy \\ &= \int_{-\infty}^{\infty} \left( \frac{K_n(x-y+h) - K_n(x-y)}{h} - K'_n(x-y) \right) f(y) dy \\ &= \int_{-b}^b \left( \frac{K_n(x-y+h) - K_n(x-y)}{h} - K'_n(x-y) \right) f(y) dy \end{aligned}$$

take absolute values and then use (2).

**Lemma 15.** Let  $f$  be as in the convention and also assume that  $f$  is differentiable with  $f'$  uniformly continuous and let  $\langle K_k \rangle_{k=1}^\infty$  be a differentiable Dirac sequence. Then the derivative of

$$f_n(x) = \int_{-\infty}^{\infty} K_n(x-y) f(y) dy$$

can be written as

$$f'_n(x) = \int_{-\infty}^{\infty} K_n(x-y)f'(y) dy$$

**Problem 13.** Prove this. *Hint:* Starting with Proposition 7 show

$$\begin{aligned} f'_n(x) &= \int_{-\infty}^{\infty} K'_n(x-y)f(y) dy \\ &= - \int_{-\infty}^{\infty} \left( \frac{d}{dy} K_n(x-y) \right) f(y) dy \\ &= - \int_{-b}^b \left( \frac{d}{dy} K_n(x-y) \right) f(y) dy \end{aligned}$$

and use integration by parts along with  $f(-b) = f(b) = 0$ . □

**Theorem 16.** Let  $f$  be as in the convention and also assume that  $f$  is differentiable with  $f'$  uniformly continuous and let  $\langle K_k \rangle_{k=1}^{\infty}$  be a differentiable Dirac sequence. Then if

$$f_n(x) = \int_{-\infty}^{\infty} K_n(x-y)f(y) dy$$

the limit

$$\lim_{n \rightarrow \infty} f'_n = f'$$

holds uniformly.

**Problem 14.** Prove this. □