ANALYSIS QUALIFYING EXAM
JANUARY 2000

DIRECTIONS: Answer all questions. Questions 1-8 are worth 10 points each and question 9 is worth 20 points.
NOTATION: λ denotes Lebesgue measure on the line.

1. Let $S \subseteq \mathbb{C}$. A mapping $f : S \to S$ is *contractive* if $|f(x) - f(y)| < |x - y|$ for all $x, y \in S$.
(a) Prove that if f is contractive and S is compact then there exists $x \in S$ with $f(x) = x$.
(b) Does the same result hold for $S = [0, \infty)$? Prove or give a counterexample.

2. Let μ be a finite measure defined on the Borel σ-field of $\mathbb{R}$. Prove that there exists a unique closed set F such that $\mu(F) = \mu(\mathbb{R})$ and such that if F_1 is any closed set satisfying $\mu(F_1) = \mu(\mathbb{R})$, then $F \subseteq F_1$.

3. Let $(A_n)_{n \geq 1}$ be a *decreasing* sequence of measurable subsets of $\mathbb{R}$. Construct a nonnegative extended real-valued measurable function f such that $A_n = \{f \geq n\}$ $(n \geq 1)$ and such that

$$\int f \, d\lambda = \sum_{n \geq 1} \lambda(A_n).$$

Show also that f is unique (up to a.e. equivalence) when $\sum_{n \geq 1} \lambda(A_n) < \infty$.

4. The function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and integrable. Prove that $f(x) \to 0$ as $|x| \to \infty$.

5. Suppose that $f \in L^p(\mathbb{R})$, where $1 < p < \infty$. Prove that

$$\lim_{h \to 0} h^{(1/p)-1} \int_x^{x+h} f(t) \, dt = 0$$

uniformly in x.

6. (a) State without proof Lebesgue's Differentiation Theorem.
(b) Prove or disprove the existence of a measurable set $A \subset [0, 1]$ such that $\lambda(A \cap I) = (1/2)\lambda(I)$ for every *interval* I contained in $[0, 1]$.

7. Suppose that g is an *entire* function satisfying $|g(z)| \geq M|z|^N$ for all sufficiently large z, where N is a positive integer and $M > 0$. Prove that g is a polynomial of degree at least N.

8. (a) Suppose that f is analytic on the disk $D = \{|z| < 2\}$ and has no zeroes on the unit circle $C = \{|z| = 1\}$. Using the Residue Theorem, show that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N,$$

where C is positively oriented and N is the number of zeroes of f inside C counted according to multiplicity.

(b) Now suppose that g is also analytic on D and that $(1-t)f(z)+tg(z) \neq 0$ for all $t \in [0,1]$ and $z \in C$. Deduce that

$$\frac{1}{2\pi i}\int_C \frac{g'(z)}{g(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz.$$

9. TRUE OR FALSE. Prove or disprove the result or find a counterexample.

(a) Suppose that $(f_n)_{n\geq 1}$ is a sequence of pointwise decreasing (i.e. $f_n(x) \geq f_{n+1}(x)$ for all x and for all n) nonnegative integrable functions. Then

$$\lim \int f_n\,d\lambda = \int (\lim f_n)\,d\lambda.$$

(b) If $f : [a,b] \to \mathbb{R}$ is increasing and continuous then

$$f(b) - f(a) = \int_a^b f'\,d\lambda.$$

(c) If A is closed and $\lambda(A) > 0$ then A contains an interval I of positive length.

(d) There exists an entire function $f(z)$ such that $f(1/n) = (-1)^n/n^4$ for $n \geq 1$.