Mathematics 172 Homework

The problems cover some of the material that would have been covered in the lectures we missed from the storm days. Recall that the ***absolute value*** of a real number is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

In what follows you can use the fact that $x < 0$ implies that $-x > 0$.

1. Show that for all real numbers x that $|x| \geq 0$. *Hint:* There are two cases (1) $x \geq 0$ and (2) $x < 0$.

2. Show that for $x, y \in \mathbb{R}$ that $|xy| = |x||y|$. *Hint:* There are four cases (1) $x, y \geq 0$, (2) $x \geq 0$ and $y < 0$, (3) $x < 0$ $y \geq 0$, and (4) $x, y < 0$.

3. Show that if a is a positive number and $-a < x < a$, then $|x| < a$. *Hint:* There are two cases, $x \geq 0$ and $x < 0$.

4. Show that if $0 \leq r_1 < n$ and $0 \leq r_2 < n$, then $|r_1 - r_2| < n$. *Hint:* We are given the inequality

$$0 \leq r_1 < n \tag{1}$$

$$0 \leq r_2 < n. \tag{2}$$

Multiply the inequality (2) by -1 (and remember that multiplying and inequality by a negative number reverses the inequality) to get

$$-n < -r_2 \leq 0.$$

Add this to the inequality (1) to get

$$-n < r_1 - r_2 < n.$$

Now you can use Problem 3 (with $a = n$ and $x = r_1 - r_2$).

Recall the ***division algorithm*** with says that if n is a positive integer and a is any integer, then there exist unique integers q (the ***quotient***) and r (the ***remainder***) such that

$$a = nq + r \quad \text{and} \quad 0 \leq < r.$$

Also recall that $a \equiv b \mod n$ means that $n \mid (a - b)$

5. Show that if a and b have the same remainder when divided by n that $a \equiv b \mod n$. *Hint:* We have done this at least once in class.

6. If $a \equiv b \mod n$, then a and b have the same remainder when divided by n. *Hint:* Assume that $a \equiv b \mod n$. That is $n|(a-b)$. Now the remainders when a and b are divided by n are defined by

$$a = q_1 n + r_1$$

$$b = q_2 n + r_2$$

where q_1, q_2, r_1, r_2 are ingeters and $0 \le r_1, r_2 < n$.

Goal: to show $\quad r_1 = r_2.$

Do this in steps:

(a) Use that $n \mid (a-b)$ to show there is an integer k such $(a-b) = kn$.

(b) Show
$$(r_1 - r_2) = (a-b) + (q_2 - q_1)n.$$

(c) Combine parts (a) and (b) so show there is an integer ℓ such that
$$r_1 - r_2 = \ell n$$
for some integer ℓ.

(d) Use Problem 3 to show that $|r_1 - r_2| < n$. (Use $a = n$ and $x = r_1 - r_2$.)

(e) Use Parts (c) and (d) to show that $|\ell| < 1$.

(f) Finish the proof by noting that the only integer ℓ with $|\ell| < 1$ is $\ell = 0$. Use this to show $r_1 - r_2 = 0$ thereby finish the proof. $\square$

Note that Problems 5 and 6 together give

Theorem 1. *Let n be a positive integer and a and b any integers. Then $a \equiv b \mod n$ if and only if a and b have the same remainder when divided by n.* $\square$

7. Show the number α is irrational if and only if $\dfrac{1+2\alpha}{2\alpha}$ is irrational.

8. Show the following are equivalent for the real number β:

(a) β is irrational.

(b) $1 + 2\beta$ is irrational.

(c) $\dfrac{1+\beta}{1-\beta}$ is irrational.