Prime and Maximal Ideals.

In all that follows we assume that rings have a multiplicate identity.
The most basic results about prime and maximal ideals are

Proposition 1. *The ideal I of the commutative ring is maximal if and only if the quotient R/I is a field.* $\square$

Proposition 2. *The ideal I of the commutative ring is prime if and only if the quotient R/I is an integral domain.* $\square$

The basic existence results are usual proven using Zorn's lemma.

Proposition 3. *Every ideal in a ring R is contained in some maximal ideal.* $\square$

Recall that a set S is a ring is a ***multiplicative set*** if and only if $s_1, s_2 \in S$ implies $s_1 s_2 \in S$.

Proposition 4. *Let S be a multiplicative set with $0 \notin S$ in the commutative ring R. Then any ideal that is maximal with respect to being disjoint from S is maximal.*

Problem 1. Prove this. $\square$

Problem 2. Let R be a commutative ring and $a \in R$ an element that is not nilpotent. Then R has a prime ideal P with $a \notin P$. $\square$

Here are two results in the same spirit as this result, but which are tricker to prove.

Proposition 5. *Let R be a commutative ring and let I be an ideal which is maximal in the set of ideals of R that are not finitely generated. Then I is a prime ideal of R.* $\square$

Problem 3. Prove this. $\square$

Proposition 6. *Let R be a commutative ring and I an ideal that is maximal in the set of ideals of R that are not principal. Then I is a prime ideal.*

Problem 4. Prove this. $\square$

Proposition 5 can be used to show that if every prime ideal of a commutative ring is finitely generated, then every ideal is finitely generated. Likewise Proposition can be use to show if every prime ideal is principal, then every ideal is principal.

Problem 5. Let R be a UFD and $p \in R$ a prime. Then the principal ideal $\langle p \rangle$ is a minimal prime ideal of R. (That is $\langle p \rangle$ is prime and if P is a prime ideal of R with $\langle 0 \rangle \subseteq P \subseteq \langle p \rangle$, then $P = \langle 0 \rangle$, or $P = \langle p \rangle$.) $\square$

Problem 6 (January 2010, Problem 9)*.* Find all the prime ideals of $\mathbb{Z}[x]$ that are minimal with respect to containing the polynomial

$$f(x) = 6(x-1)(x^3+3x-2).$$

Also describe explicitly three distinct maximal ideals of $\mathbb{Z}[x]$, each containing at least two of the minimal prime ideals containing $f(x)$. □

Problem 7 (January 2012, Problem 8)*.* Let I be the ideal of $\mathbb{Z}[x]$ generated by $\{3, x^3-x^2+2x-1\}$. Is $\mathbb{Z}[x]/I$ an integral domain? Prove your answer. □

Problem 8 (January 2013, Problem 7.)*.* For each field F listed below, factor $f(x) = x^7 - 1$ into irreducible factors in $F[x]$.

(a) $F = \mathbb{Q}$ (the field of rational numbers),
(b) $F = \mathbb{Z}_7$ (the field with 7 elements),
(c) $F = \mathbb{R}$ (the field of real numbers), and
(d) $F = \mathbb{Z}_2$ (the field with 2 elements). □

Problem 9 (August 2014, Problem 6.)*.* Let R be a commutative ring.

(a) State the definition of a prime ideal of R.
(b) Let I be an ideal of R. Prove that I is a prime ideal if and only if R/I is an integral domain.
(c) Let $P, Q \subset R$ be prime ideals. Prove that

$$\mathrm{Hom}_R(R/P, R/Q) \neq \{0\} \quad \iff \quad P \subseteq Q.$$ □

Problem 10 (January 2015, Problem 7.)*.* (a) Let R be a PID. Prove every non-zero prime ideal of R is maximal.
(b) Give an example of a commutative ring R and a non-zero ideal that is not maximal.
(c) Let K be a field that is *not* algebraically closed. Give an example of a maximal ideal of the ring $R = K[X,Y]$ that is *not* of the form $\langle X-a, Y-b\rangle$ with $a, b \in K$.

Recall that one statement of the Hilbert Nullstellesatz is that that when K is an algebraically closed field, that the maximal ideals of the polynomial ring $K[X_1, X_1, \ldots, X_n]$ are all of the form $\langle X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n\rangle$ for some $a_1, a_2, \ldots, a_n \in K$. The previous problem shows that K being algebraically closed is necessary and sufficient for the Nullstellesatz to hold in $K[X_1, X_2, \ldots, X_n]$.