Mathematics 555 Test 2, Take Home Portion.

The problems are 10 points each. You can choose any four to use as the take home part of the test. If you turn in more than four, I will use the four best four the grade. You are allowed to use any result in the homework notes

1. Let

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Show

$$\lim_{n\to\infty} (H_n - \ln(n))$$

exists. $\square$

2. Let $f\colon [a,b] \to \mathbf{R}$ be a continuous function. Assume that $f \geq 0$ on $[a,b]$ and that

$$\int_a^b f(x)\,dx = 0.$$

Prove $f(x) = 0$ for all $x \in [a,b]$. $\square$

3. Let $\langle a_k \rangle_{k=0}^{\infty}$ be a sequence of real numbers such that $L = \lim_{k\to\infty} a_k$ exists. For $k \geq 1$ set $b_k = (a_{k-1} - a_k)$. Prove (i.e. there should be some ε's and N's)

$$\sum_{k=1}^{\infty} b_k = a_0 - L.$$

$\square$

4. Let $p\colon \mathbf{R} \to \mathbf{R}$ be a continuous function and let

$$P(x) = \int_0^x p(t)\,dt.$$

Define a function

$$u(x) = e^{P(x)}.$$

Show that u satisfies $u'(x) = p(x)u(x)$ and $u(0) = 1$. $\square$

5. Let $f_1, f_2, f_3, \ldots : [a,b] \to \mathbf{R}$ be a sequence of continuous functions and assume that $\lim_{n\to\infty} f_n(x) = f(x)$ and the limit converges uniformly.

(a) Quote a theorem that tells us that f is continuous.

(b) Prove $\displaystyle\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$ $\square$

6. (A generalization of the usual integral test) Let $f\colon [1,\infty) \to \mathbf{R}$ be a twice differentiable function such that

$$\int_1^{\infty} |f''(x)|\,dx := \lim_{n\to\infty} \int_1^{\infty} |f''(x)|\,dx < \infty.$$

Show that $\displaystyle\sum_{k=1}^{\infty} f(n)$ converges if and only if $\displaystyle\int_1^{\infty} f(x)\,dx$ converges. (Where, by definition, $\displaystyle\int_1^{\infty} f(x)\,dx$ converging means $\displaystyle\lim_{n\to\infty} \int_0^n f(x)\,dx$ exists.) $\square$