Mathematics 555 Homework.

Proposition 1. *Let* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be two convergent series and let* $c_1, c_2 \in \mathbf{R}$. *Then the series* $\sum_{k=1}^{\infty}(c_1 a_k + c_2 b_k)$ *is also convergent and*

$$\sum_{k=1}^{\infty}(c_1 a_k + c_2 b_k) = c_1 \sum_{k=1}^{\infty} a_k + c_2 \sum_{k=1}^{\infty} b_k.$$

Problem 1. Prove this. *Hint:* Let $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$, and $S_n = \sum_{k=1}^{n}(c_1 a_k + c_2 b_k)$ be the partial sums of the series in question. Start by showing $S_n = c_1 A_n + c_2 B_n$. $\square$

Problem 2. Determine the set of points where the following series converge absolutely and for which values they converge conditionally.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2 4^n}$,

(b) $\displaystyle\sum_{k=0}^{\infty} (1+\cos(x))^k$,

(c) $\displaystyle\sum_{j=1}^{\infty} \frac{(-1)^j x^{2j+1}}{7^j \sqrt{j}}$

Problem 3. In this problem we show that the function e^x can be expressed as a power series

(a) For each positive integer n and x any real number show

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{e^{\xi} x^{n+1}}{(n+1)!}$$

where ξ is between 0 and x.

(b) With this notation show

$$\lim_{n\to\infty} \frac{e^{\xi} x^{n+1}}{(n+1)!} = 0.$$

Hint: One way to start is by noticing

$$\left| \frac{e^{\xi} x^{n+1}}{(n+1)!} \right| \le \frac{e^{|x|}|x|^{n+1}}{(n+1)!},$$

that the series

$$\sum_{n=0}^{\infty} \frac{e^{|x|}|x|^{n+1}}{(n+1)!}$$

converges (by ratio test), and finally that the terms of a convergent series converge to zero.

(c) Show

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$$ $\square$

Problem 4. Prove that e is an irrational number. *Hint:* By the last problem

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Towards a contradiction assume that e is rational, say that

$$e = \frac{p}{q}$$

where p and q are positive integers. Then

$$\frac{p}{q} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{q!} + \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \cdots$$

Multiply this by $q!$ to get

$$\begin{aligned} p(q-1)! &= q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!} + \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \cdots \\ &= N + R \end{aligned}$$

where

$$N = q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!}$$

and

$$R = \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \cdots$$

Show N is an integer. We now estimate R by comparing it with a geometric series

$$\begin{aligned} R &= \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \frac{q!}{(q+3)!} + \frac{q!}{(q+4)!} + \cdots \\ &= \frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots \\ &< \frac{1}{(q+1)} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \frac{1}{(q+1)^4} + \frac{1}{(q+1)^5} + \cdots \end{aligned}$$

Sum this last series and use the result to show

$$0 < R < \frac{1}{q} \leq 1.$$

But we also have

$$R = p(q-1)! - N$$

which implies that R is an integer. Explain why this is a contradiction. $\square$