Mathematics 552 Homework, February 1, 2020

On Monday you will have the following quiz.

Problem 1. Give the series definition of the following functions: e^z, $\sin(z)$, and $\cos(z)$.

Solution.

$$\begin{aligned}
e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} \cdots \\
\sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \frac{z^{13}}{13!} - \cdots \\
\cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \frac{z^{12}}{12!} - \cdots
\end{aligned}$$

$\square$

Problem 2. Give Euler's equation for e^{iz}.

Solution.

$$e^{iz} = \cos(z) + i\sin(z).$$

$\square$

Problem 3. Write $\sin(z)$ and $\cos(z)$ in terms of e^{iz} and e^{-iz}.

Solution.

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

$\square$

Problem 4. Use the formulas of the last problem to show that $\sin^2(z) + \cos^2(z) = 1$ for all complex numbers z.

Solution.

$$\begin{aligned}
\sin^2(z) + \cos^2(z) &= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 \\
&= \frac{e2iz - 2 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} \\
&= \frac{-e^{2iz} + 2 - e^{-2iz}}{4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

$\square$

Problem 5. Give the definitions of the following: $\log(z)$, and z^α where $z \neq 0$ and α is any complex number.

Solution.

$$\begin{aligned} \log(z) &= \ln(|z|) + i\arg(z) \\ z^\alpha &= e^{\alpha \log(z)} \end{aligned}$$

$\square$

Problem 6. Compute $\log(-4+4i)$ and $(-4+4i)^{2i}$.

Solution.

$$\begin{aligned} \log(-4+4i) &= \ln(|-4+4i|) + i\arg(-4+4i) \\ &= \ln(4\sqrt{2}) + i\left(\frac{3\pi}{4} + 2\pi n\right) \end{aligned}$$

where n can be any integer.

$$\begin{aligned} (-4+4i)^{2i} &= e^{2i\log(-4+4i)} \\ &= e^{2i\left(\ln(4\sqrt{2}) + i\left(\frac{3\pi}{4}+2\pi n\right)\right)} \\ &= e^{-2\frac{3\pi}{4} - 4\pi n + 2i\ln(4\sqrt{2})} \\ &= e^{-\frac{3\pi}{2} - 4\pi n}\left(\cos\left(2\ln(4\sqrt{2})\right) + i\sin\left(2\ln(4\sqrt{2})\right)\right) \\ &= e^{-\frac{3\pi}{2} - 4\pi n}\left(\cos\left(\ln(32)\right) + i\sin\left(\ln(32)\right)\right) \end{aligned}$$

$\square$

where n is an integer.