Series.

Consider two power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$g(x) = \sum_{k=0}^{\infty} b_k x^k$$

It we assume that we can multiply these the same way we would polynomials we get

$$\begin{aligned} f(x)g(x) &= \left(a_1 + a_1 x + a_2 x^2 + a^3 x^3 + \cdots\right)\left(b_1 + b_1 x + b_2 x^2 + b^3 x^3 + \cdots\right) \\ &= a_0 b_1 + (a_0 b_1 + a_1 b_0)\, x + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, x^2 + \cdots \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_k b_{k-j} \right) x^k \end{aligned}$$

Problem 1. Here is anther way to see this. Let

$$h(x) = f(x)g(x).$$

Then the first couple of derivatives of h are

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ h''(x) &= f''(x)g(x)f'(x)g'(x) + f(x)g''(x) \\ h'''(x) &= f'''(x)g(x) + 3f''(x)g'(x) + 3f'g''(x) + g'''(x) \end{aligned}$$

which reminds use of the Binomial Theorem.

(a) Prove that k-th derivative of $h(x)$ is

$$h^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} f^{(j)}(x) g^{(k-j)}(x).$$

(b) Let

$$a_k = \frac{f^{(k)(0)}}{k!} \quad \text{and} \quad b_k = \frac{g^{(k)(0)}}{k!}$$

and use the formula for $h^{(k)}(0)$ to show

$$\frac{h^{(k)}(0)}{k!} = \sum_{j=0}^{k} a_j b_{k-j}.$$

□

If we assume that both series for $f(x)$ and $g(x)$ both converge for $x = 1$ we can let $x = 1$ the result is

$$\left(\sum_{k=0}^{\infty} a_k \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_k b_{k-j} \right).$$

This motivates:

Definition 1. Let

$$\sum_{k=0}^{\infty} a_k \qquad \sum_{k=0}^{\infty} b_k$$

be two series then the ***Cauchy product*** of these series is the series

$$\sum_{k=0}^{\infty} c_k$$

where

$$c_k = \sum_{j=0}^{k} a_j b_{k-j} = \sum_{i+j=k} a_i b_j.$$

□

Theorem 2. *Let*

$$A = \sum_{k=0}^{\infty} a_k \qquad B = \sum_{k=0}^{\infty} b_k$$

we convergent series with at least one of the two absolutely convergent. Let

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

Then the series $\sum_{k=0}^{\infty} c_k$ *converges and*

$$\sum_{k=0}^{\infty} c_k = AB.$$

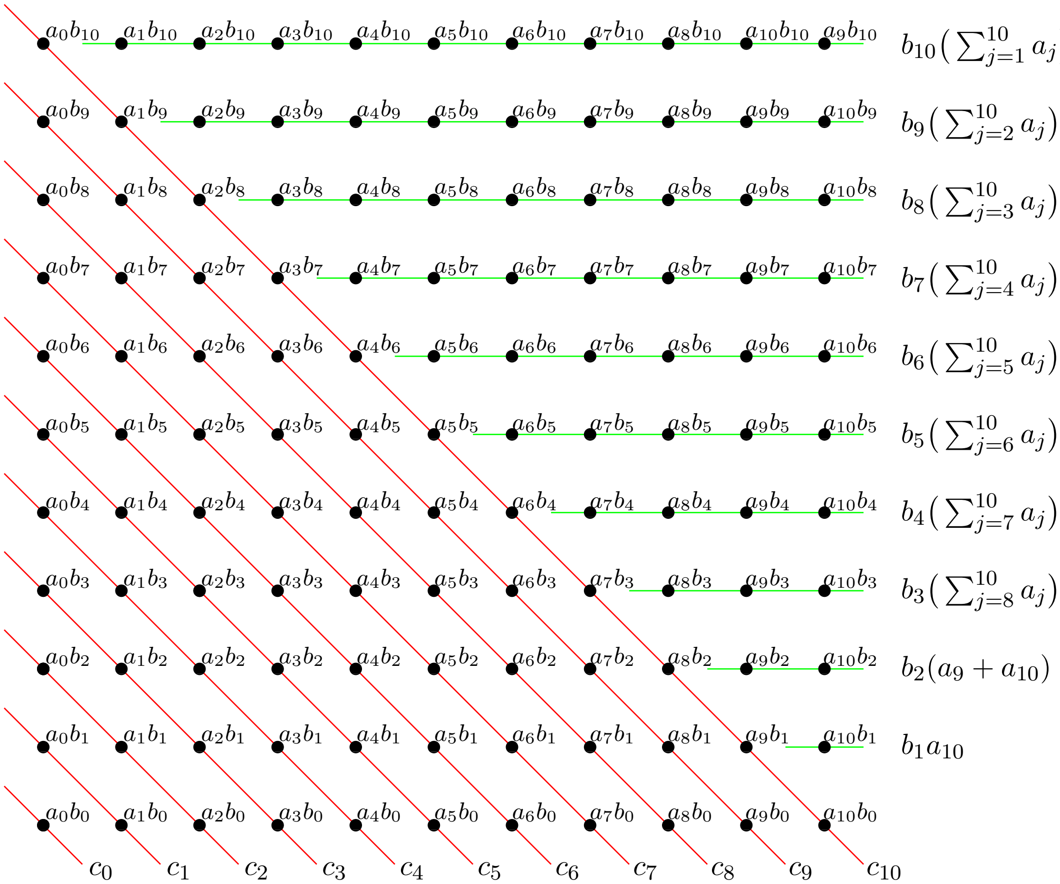
Problem 2*.* Prove this along the following lines. Let

$$A_n = \sum_{k=0}^{n} a_k \qquad B_n = \sum_{k=0}^{n} b_k \qquad C_n = \sum_{k=0}^{n} c_k$$

be the partial sums. Note that the product

$$A_n B_n = (a_0 + a_1 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)$$

is the sum of the $(n+1)^2$ products $a_j b_k$ with $0 \leq i, j \leq n$. For $n = 0$ these terms are shown in the following figure.



(a) Using the figure above as a guide show for all $n = 1, 2, \cdots$ that

$$A_n B_n - C_n = \sum_{k=1}^{n} b_k \bigg(\sum_{j=n-k}^{n} a_j \bigg).$$

(b) Let $0 < m < n$. Explain why

$$\begin{aligned} |A_n B_n - C_n| &= \left| \sum_{k=1}^{n} b_k \bigg(\sum_{j=n-k}^{n} a_j \bigg) \right| \\ &\le \sum_{k=1}^{n} |b_k| \left| \sum_{j=n-k}^{n} a_j \right| \\ &= \sum_{k=1}^{m} |b_k| \left| \sum_{j=n-k}^{n} a_j \right| + \sum_{k=m+1}^{n} |b_k| \left| \sum_{j=n-k}^{n} a_j \right| \end{aligned}$$

(c) Without loss of generality we may assume that $\sum_{k=0} b_k$ is absolutely convergent. Explain why there is constant $\beta \geq 0$ such that for all m

$$\sum_{k=1}^{m} |b_k| \leq \beta.$$

(d) The series $\sum_{j=1}^{\infty} a_j$ is convergent. That is $\lim_{n\to\infty} A_n$ exists. Show this implies there is a constant C such that $|A_n| \leq C$ for all n and then use

$$\sum_{j=n-k}^{n} a_j = A_n - A_{n-k-1}$$

to show there is a constant $\alpha \geq 0$ such that

$$\left| \sum_{j=n-k}^{n} a_j \right| \leq \alpha$$

for all n and k with $0 \leq k \leq n$.

(e) Combine parts (b), (c), and (d) to show

$$\begin{aligned} |A_n B_n - C_n| &\leq \beta \left| \sum_{j=n-k}^{n} a_j \right| + \alpha \sum_{k=m+1}^{n} |b_k| \\ &= \beta |A_n - A_{n-k-1}| + \alpha \sum_{k=m+1}^{n} |b_k| \end{aligned}$$

when $0 \leq k \leq m \leq n$.

(f) Let $\varepsilon > 0$. Explain where are are $N_1, N_2 > 0$ such that

$$m \geq N_1 \quad \text{implies} \quad \sum_{k=m+1}^{n} |b_k| < \frac{\varepsilon}{2\alpha},$$

and

$$n \geq N_2 \text{ and } n-k-1 \geq N_2 \qquad \text{implies} \qquad |A_n - A_{n-k-1}| < \frac{\varepsilon}{2\beta}.$$

(g) Let $n \geq N_1 + N_2 + 2$, set $m = N_1$, and show that for any k with $0 \leq k \leq m$ that the inequalities

$$m \geq N_1, \quad n \geq N_2, \quad n-k-1 \geq N_2$$

all hold and that this in turn yields

$$n \geq N_1 + N_2 + 2 \quad \text{implies} \quad |A_n B_n - C_n| < \varepsilon.$$

(h) Conclude from part (f) that

$$\lim_{n\to\infty} (A_n B_n - C_n) = 0.$$

(i) Complete the proof by showing

$$\lim_{n\to\infty} C_n = AB. \qquad \square$$

Problem 3. Here is an example to show that it is important that in Theorem 2 at least one of the two series is absolutely convergent. Let $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} (-1)^k/\sqrt{k+1}$. The Cauchy product is $\sum_{k=0}^{\infty} c_k$ where

$$c_k = (-1)^k \sum_{j=0}^{k} \frac{1}{\sqrt{(j+1)(k-j+1)}}.$$

Show

$$|c_k| = \sum_{j=0}^{k} \frac{1}{\sqrt{(j+1)(k-j+1)}} \geq 1$$

and therefore the series $\sum_{k=0}^{\infty} c_n$ diverges. □

Theorem 3. *Let*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad g(x) = \sum_{k=0}^{\infty} b_k x^k$$

be power series with radius convergence at least R. Let

$$c_n = \sum_{j=0}^{n} a_j b_{n-j}$$

then the power series

$$h(x) = \sum_{n=0}^{\infty} c_n x^n$$

also has radius of convergence at least R and

$$h(x) = f(x)g(x)$$

for $|x| < R$.

Proof. This follows easily form Theorem 2. □

We now give a short indication of how to divide power series. Assume that we wish to find the power series expansion of

$$f(x) = \frac{h(x)}{g(x)}$$

where

$$h(x) = \sum_{k=0}^{\infty} c_k x^k \qquad g(x) = \sum_{k=0}^{\infty} b_k x^k.$$

and we wish to find the series for $f(x)$. Assume that $f(x)$ has an expansion

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Then $h(x) = f(x)g(x)$ and so we have

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

Assume that $g(0) \neq 0$, that is $b_0 \neq 0$. Then the last equation can be rewritten as

$$a_k = \frac{1}{b_0}\left(c_k - \sum_{j=0}^{n-1} a_j b_{k-j}\right).$$

For small values of k these formulas are

$$\begin{aligned}
a_0 &= \frac{c_0}{b_0} \\
a_1 &= \frac{1}{b_0}\big(c_1 - a_0 b_1\big) \\
a_2 &= \frac{1}{b_0}\big(c_2 - a_0 b_2 - a_1 b_1\big) \\
a_3 &= \frac{1}{b_0}\big(c_3 - a_0 b_3 - a_1 b_2 - a_2 b_1\big) \\
a_4 &= \frac{1}{b_0}\big(c_4 - a_0 b_4 - a_1 b_3 - a_2 b_2 - a_3 b_1\big)
\end{aligned}$$

This allows us to find the coefficients $a_0, a_1, a_2, \ldots$ of $f(x)$ recursively. Unfortunately this method does not tell us anything about the radius of convergence of $f(x)$ in terms of the radii of convergence of $g(x)$ and $h(x)$. But if we already know that all three have positive radius of convergence, it does give us a method for finding the coefficients of $f(x)$ from the coefficients of $g(x)$ and $h(x)$.

Problem 4. Find the first three nonzero terms in the power series of

$$f(x) = \frac{e^{2x}}{\cos(x)}.$$ □

Problem 5. Find the first couple terms of the power series of the following and thus convince yourself that using series tells you more than using L'Hôspital's rule.

(a) $\dfrac{\sin(2x)}{4x}$

(b) $\dfrac{1-\cos(5x)}{x^2}$

(c) $\dfrac{e^x - 1 - x}{1-\cos(2x)}$. □

Problem 6. Find the power series of the following functions around the indicated points x_0.

(a) $f(x) = \sin(x)$ around $x_0 = \pi/4$.
(b) $f(x) = e^{2x}$ around the point $x_0 = 1$.
(c) $f(x) = \sqrt{4-x}$ around the point $x = 4$.