Riemann Integration.

1. Step functions and their integrals

If $A \subseteq \mathbf{R}$ is a subset of $\mathbf{R}$ then the ***characteristic function***, also called the ***indicator function*** of A is defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A. \end{cases}$$

In these notes by a bounded interval we mean a set of the form

$$[a,b], \quad [a,b), \quad (a,b], \quad (a,b)$$

where $a \le b$. Note that we are ruling our infinite intervals such as $[a,\infty)$ or $(-\infty,b)$. But we are not ruling out the "degenerate" case of $[a,a] = \{a\}$. So for us a one element set is a bounded interval. When we wish to distinguish a one element interval from an bounded interval with non-empty interior we will call the one element interval ***degenerate*** and the interval with non-empty interior a ***proper*** interval.

Definition 1. A ***step function*** is a function of the form

$$\phi = \sum_{j=1}^{n} a_j \chi_{I_j}$$

where $I_1, \ldots, I_n$ is any finite collection of bounded intervals and a_j are any real numbers. $\square$

Note that we do not require the intervals $I_1, \ldots, I_n$ to be disjoint. This means that a step function can be written in many ways as a sum of characteristic functions of intervals.

Problem 1. Let ϕ be the step function

$$\phi = 2\chi_{[0,3]} + 4\chi_{[1,5]} - 3\chi_{[2,4]}.$$

(a) Graph ϕ.
(b) Show ϕ also has the representation

$$\phi = 2\chi_{[0,1)} + 6\chi_{[1,2)} + 3\chi_{[2,3]} + 1\chi_{(3,4]} + 4\chi_{(4,5]}.$$

(c) Find anther representation of ϕ step function.