Math 554 Test 2.

- *This is due on Tuesday, March 23 by midnight. It should be submitted via Blackboard as a pdf document.*
- *You are to work alone on it. You can look up definitions and the statements of theorems we have covered in class. Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg.*
- *Please print your name on the first page of the test.*
- *As I said in class, this will be graded in part on the writing. It does not have to be long winded, but work with just formulas and no English will lose points. Think you work being read by someone who does no know how to solve the problem and write in such a way that they will not have any understanding your solution.*

Problem 1. (10 points) Let E be a metric space and let $a \in E$. Define a function $f_a \colon E \to \mathbb{R}$ by

$$f(p) = d(p, a).$$

Prove that f satisfies

$$|f(p) - f(q)| \leq d(p, q).$$

$\square$

Recall that a function $f \colon E \to \mathbb{R}$ is ***Lipschitz*** if and only if there is a constant M such that

$$|f(p) - f(q)| \leq M d(p, q)$$

for all $p, q \in E$. Thus Problem 1 tells us that the function $f(p) = d(a, p)$ is Lipschitz with $M = 1$.

We have also proven

Proposition 1. *If $f \colon E \to \mathbb{R}$ is Lipschitz and $\langle p_n \rangle_{n=1}^{\infty}$ is a convergent sequence, say*

$$\lim_{n\to\infty} p_n = p$$

then

$$\lim_{n\to\infty} f(p_n) = f(p).$$

Problem 2. (10 points) As a bit of review give the proof of this. That is show for all $\varepsilon > 0$ there is a N such that

$$n \geq N \qquad \text{implies} \qquad |f(p) - f(p_n)| < \varepsilon. \qquad \square$$

Definition 2. Let E be a metric space. Then a subset S of E is ***sequentially compact*** if and only if for every sequence of points $\langle p_n \rangle_{n=1}^{\infty} \subseteq S$ from S there is a subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$ such that for some point $p \in S$

$$p = \lim_{k\to\infty} p_{n_k}. \qquad \square$$

Problem 3. (25 points) Let E be a metric space and $S \subseteq E$ a sequentially compact subset of E. Let $a \in E$. Show there is a point $p \in S$ such that

$$d(a,p) \leq d(a,q) \qquad \text{for all} \qquad q \in S.$$

(That is there is a point of S that is closest to a.) *Hint:* One way to do this is by using the following steps:

(a) Let D be the set of distances

$$D = \{d(a,q) : q \in S\}.$$

Show that this is a subset of $\mathbb{R}$ that is bounded below and thus D has an infimum (greatest lower bound). Let

$$\beta = \inf(D).$$

Also give a sentence saying why

$$\beta \leq d(a,q) \qquad \text{for all} \qquad q \in S.$$

(b) Explain why for each positive integer n there is a point $p_n \in S$ with

$$\beta \leq d(a,p_n) < \beta + \frac{1}{n}.$$

(c) Explain why $\langle p_n \rangle_{n=1}^{\infty}$ has a subsequence that converges to a point $p \in S$, that is

$$\lim_{k\to\infty} p_{n_k} = p \in S.$$

(d) Show that

$$\lim_{k\to\infty} d(a, p_{n_k}) = d(a,p).$$

(*Subhint:* Use Problem 1 and Proposition 1.)

(e) Use Part (b) of the problem to also show that

$$\lim_{k\to\infty} d(x, p_{n_k}) = \beta.$$

(f) Now finish the proof by showing $d(a,p) = \beta$ and explaining why this shows $d(a,p) \leq d(a,q)$ for all $q \in S$. $\square$

Problem 4. (10 points) Let E be a metric space and let $\langle p_n\rangle_{n=1}^{\infty}$ be a convergent sequence in E, say

$$\lim_{n\to\infty} p_n = p.$$

Let

$$S = \{p\} \cup \{p_n : n = 1, 2, 3, \ldots\}.$$

Show S is closed and bounded. □

Here is anther of our recent results:

Theorem 3. *Let $\langle p_n\rangle_{n=1}^{\infty}$ be a Cauchy sequence in a metric space such that it has a convergent subsequence $\langle p_{n_k}\rangle_{k=1}^{\infty}$. Then the original series $\langle p_n\rangle_{n=1}^{\infty}$ converges.* □

Problem 5. (15 points) Let E be a metric space were every closed bounded set is sequentially compact. Show E is complete. *Hint:* To show E is complete we need to show that every Cauchy sequence in E converges. So let $\langle p_n\rangle_{n=1}^{\infty}$ be a Cauchy sequence in E and let S be defined as in Problem 4. By that problem S is closed and bounded. Now explain why the assumption on sequential compactness and Theorem 4 implies the sequence converges. □

The following may be useful in the next problem.

Theorem 4. *A bounded monotone sequence in $\mathbb{R}$ is convergent.* □

Problem 6. (20 points) Define a sequence in $\mathbb{R}$ by

$$\begin{aligned}
x_1 &= 1\\
x_2 &= \frac{3}{4}x_1 + 12\\
x_3 &= \frac{3}{4}x_2 + 12\\
x_4 &= \frac{3}{4}x_3 + 12
\end{aligned}$$

and in general

$$x_n = \frac{3}{4}x_{n-1} + 12.$$

(a) Compute x_2, x_3, and x_4.
(b) Use induction to show the sequence $\langle x_n\rangle_{n=1}^{\infty}$ is increasing. That is show $x_n > x_{n-1}$ for all $n \geq 2$.
(c) Use induction to show $x_n \leq 100$ for all n.
(d) Show $\langle x_n\rangle_{n=1}^{\infty}$ converges and find its limit. □

Problem 7. (10 points) Show that the subset S be the subset of $\mathbb{R}^2$ defined by

$$S = \{(x, y) : 0 < x^2 + y^2 \leq 1\}.$$

(a) Draw a picture of S.

(b) Show that S is not sequentially compact. $\square$