QUALIFYING EXAM IN ANALYSIS

(AUGUST 2007)

Name :
S.S. # :

Throughout this examination the term measurable refers to the Lebesgue measure m on the real line. Integrals with respect to Lebesgue measure will be denoted by $\int f$. Problems are 10 points each.

1. Prove the Banach's fixed point principle.

Theorem. *If Ω is a contraction mapping on a complete metric space (X, ρ), then the equation $\Omega(x) = x$ has one and only one solution (i.e. the mapping $\Omega : X \to X$ leaves one and only one point unchanged).*

2. Let $\{x_n\}$ be a sequence of real numbers. Show that

$$\limsup_{n\to\infty}(x_1 + \cdots + x_n)/n \leq \limsup_{n\to\infty} x_n.$$

3. Let

$$M_n := \sup_{0 \le x \le 1} \frac{x^n(1-x)}{\ln(n+1)}.$$

Prove that

$$\sum_{n=1}^{\infty} \frac{x^n(1-x)}{\ln(n+1)}$$

converges uniformly on $[0,1]$, but that $\sum_{n=1}^{\infty} M_n$ diverges.

4. Does the series

$$\sum_{n=1}^{\infty} \frac{nx}{n^2 + n^4 x^3}$$

converge uniformly on $[0, \infty)$?

5. Prove Liouville's Theorem: A bounded entire function on $\mathbb{C}$ is a constant.

6. Prove Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n.$$

7. Let $f > 0$ be integrable on $[0, 1]$. Suppose that a sequence $\{E_k\}$ of measurable subsets of $[0, 1]$ has a property

$$\lim_{k\to\infty} \int_{E_k} f = \int_0^1 f.$$

Prove that

$$\lim_{k\to\infty} mE_k = 1.$$

8. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $(-\infty, \infty)$ such that $f_n \to f$ a.e., and suppose that $\int f_n \to \int f < \infty$. Prove that for each measurable set E we have $\int_E f_n \to \int_E f$.

9. Prove the Hölder Inequality:

$$\int fg \leq \|f\|_p \|g\|_{p'}, \quad 1 < p < \infty.$$

10. Let f be such that $|f|^p$ integrable on $[0,2]$, $1 \leq p < \infty$. Prove that

$$\lim_{h \to 0^+} \int_0^1 |f(x+h) - f(x)|^p dx = 0.$$