Analysis Qualifying Exam
January 2004

Instructions: Write your name legibly on each sheet of paper. Write only on one side of of each sheet of paper. Try to answer all questions. Questions 1–8 are each worth 10 points and question 9 is worth 20 points.
Terminology: Measurability and inegrability on $\mathbb{R}$ or an interval will always refer to the Lebesgue measure, except if otherwise specified. Lebesgue measure will be denoted by m, dx or dy depending on the context.

1. Let f_n be absolutely continuous on $[0,1]$ and let $f_n(0) = 0$. Assume that
$$\int_0^1 |f_n'(x) - f_m'(x)|\, dx \to 0$$
as $m, n \to \infty$. Prove that f_n converges uniformly to a function f on $[0,1]$ and that f is absolutely continuous on $[0,1]$.

2. let f be a non-negative measurable function on $[0,1]$. Prove that $\int_0^1 f(x)^n\, dx$ exists in $\mathbb{R}$ if and only if $m\{x : f(x) > 1\} = 0$.

3. Let A be a Lebesgue measurable subset of $\mathbb{R}^2$ with $m \times m(A) > 0$ and let $\{l_i : i \in I\}$ be a collection of lines in $\mathbb{R}^2$ such that $A \subset \cup_{i\in I} l_i$. Prove that I is uncountable.

4. Let $f \in L_3([-1,1])$. Prove that $\int_{-1}^1 \frac{|f(x)|}{\sqrt{|x|}}\, dx < \infty$.

5. Let f_n be measurable functions on $[0,1]$ such that $f_n(x) \to f(x)$ a.e. Assume that $f(x) \neq 0$ a.e.. Prove that for all $\epsilon > 0$ there exists $c > 0$, a measurable set $E \subset [0,1]$ and $N \in \mathbb{N}$ such that $|f_n(x)| \geq c$ on E for all $n \geq N$ and such that $m([0,1] \setminus E) < \epsilon$.

6. Let $G \subset \mathbb{C}$ be a region and let $f : G \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = C$ for all $z \in G$. Prove that f is constant on G.

7. Let f be a holomorphic function defined in a neighborhood of the origin and let $f'(0) \neq 0$.

   (a) Show that there exists $r > 0$ such that the unique solution of the equation $f(z) = f(0)$ in the disc $|z| < r$ is $z = 0$.

(b) Prove that if $r > 0$ is suffieciently small, then

$$\frac{1}{f'(0)} = \frac{1}{2\pi i} \int_{|z|=r} \frac{1}{f(z) - f(0)} \, dz,$$

where the circle $|z| = r$ is traversed counterclockwise.

8. Let $f : \{z : |z| < 1\} \to \mathbb{C}$ be a holomorphic function such that $|f(z)| \leq 1$ for all $|z| < 1$, $f(0) = 0$, and $|f(\frac{1}{4})| = \frac{1}{4}$. Prove that there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for all $|z| < 1$.

9. True or False. Prove, or give a counterexample.

a. If $m^*(A) = 0$, then $m^*(B \setminus A) = m^*(B)$ for all subsets B of $\mathbb{R}$.

b. Let $\mathbb{Q} = \{r_n : n = 1, 2, \cdots\}$. Then

$$\mathbb{R} \setminus \bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2} \right) \neq \emptyset.$$

c. The function $f(z) = e^{\overline{z}}$ is analytic everywhere except at 0.

d. Let f be an entire function such that $\lim_{z \to \infty} |f(z)| = \infty$. Then f is a polynomial.

e. Let f_n be integrable functions on $\mathbb{R}$ such that $\lim_{n \to \infty} \int f_n \, dx = 0$. Then there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and an integrable function g such that $|f_{n_k}| \leq g$ a.e. for all k.