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Analysis Qualifying Exam
January 2017

Instructions: Write your name legibly on each sheet of paper. Write only on one side of of each sheet of paper. Try to answer all questions. Questions 1–8 are each worth 10 points and question 9 is worth 20 points.
Terminology: Measurability and integrability on $\mathbb{R}$ or a measurable subset of it will always refer to the Lebesgue measure, except if otherwise specified. Lebesgue measure will be denoted by m, dx or dy depending on the context. If A is a subset of $\mathbb{R}$ then $L_p(A)$ is considered with respect to the Lebesgue measure. You can quote without proof any of the standard theorems covered in Math 703-704, but do indicate why the relevant hypotheses hold.

10 1. Let (X, d), (Y, ρ) be metric spaces and let $f : X \to Y$. Prove that f is continuous if and only if f restricted to any compact subset of X is continuous.

10 2. Let $f : \mathbb{R} \to [0, 1]$ be Lebesgue measurable. Prove that either $f = \chi_A$ a.e. for some measurable set A or there exists $\epsilon > 0$ such that

$$m(\{x \in \mathbb{R} : \epsilon < f(x) < 1 - \epsilon\}) > 0.$$

10 3. Let f_n, f be Lebesgue integrable functions on $\mathbb{R}$. Assume $f_n(x) \to f(x)$ a.e. on $\mathbb{R}$ as $n \to \infty$ and

$$\int |f_n|\, dx \to \int |f|\, dx < \infty$$

as $n \to \infty$. Prove that $\int |f_n - f|\, dx \to 0$ as $n \to \infty$.

Fatou? mimic pf of OCT, maybe even use OCT? make new sequence

6 4. Let E_n be Lebesgue measurable subsets of $[0, 1]$ and assume

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{\text{already } \leq}{=} \sum_{n=1}^{\infty} m(E_n).$$

know $\leq$, & $<$ + find contradiction?

10 Prove that $m(E_i \cap E_j) = 0$ for all $i \neq j$.

contrapositive seems promising?

10 5. Let $a < b$ in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$. Assume that there exitst $M > 0$ such that the total variations $T^b_{a+\epsilon}(f) \leq M$ for all $\epsilon > 0$. Prove that f is of bounded variation on $[a, b]$, but that not necessarily $T^b_a(f) \leq M$.

OCT? MCT?

**** think of pf for F abs cts on $[\varepsilon, 1]$ + cts on $[0,1] \Rightarrow$ F abs cts on $[0,1]$

does increasing $\Rightarrow$ cts? Nope...

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maybe can approximate or something? Hölder? honestly no idea

6. Let $1 \le p < \infty$ and $f_n \in L_p(\mathbb{R})$ such that

   (a) $f_n(x) \to 0$ a.e.

   (b) For all $\epsilon > 0$ there exists a measurable set E with $m(E) < \infty$ such that $\int_{E^c} |f_n|^p \, dx < \epsilon$ for all $n \ge 1$.

   (c) For all $\epsilon > 0$ there exists a $\delta > 0$ such that for all measurable sets E with $m(E) < \delta$ we have $\int_E |f_n|^p \, dx < \epsilon$ for all $n \ge 1$.

   Prove that $\int |f_n|^p \, dx \to 0$.

10 ~~7~~. Let G be bounded region and let f and g be continuous nowhere zero functions on $\overline{G}$, which are holomorphic on G. Assume that $|f(z)| = |g(z)|$ for all $z \in \partial G$. Prove that there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that for all $z \in G$ we have $f(z) = \lambda g(z)$. $f(z) \neq 0 \ \forall z \in G$

8 ~~8~~. Compute

$$\int_{-\infty}^{\infty} \frac{2x^2 + x + 1}{x^4 + 5x^2 + 4} \, dx.$$

9. True or False. Prove, or give a counterexample.

   ~~a.~~ If $A \subset \mathbb{R}$ is such that int (A) and $\partial(A)$ are compact, then A is compact.

   4 ~~b.~~ Let $E \subset \mathbb{R}$ be a measurable set of finite measure. Then for $\epsilon > 0$ there exists a closed set $F \supset E$ such that $m(F \setminus E) < \epsilon$.

   ~~c.~~ There exists a function f holomorphic in an neighborhood of 0 such that $f(\frac{1}{n}) = \frac{1}{n^2-1}$ for all $n \ge 2$. uniqueness thm?

   d. If f_n are measurable functions on $\mathbb{R}$ for which there exists an integrable function f such that $f_n \le f$ for all n, then

$\int \underline{\lim} f_n \le \underline{\lim} \int f_n$

$\Rightarrow -\lim \int f_n \le -\int \lim f_n$

$\Rightarrow \overline{\lim} \int f_n dx \le \int \overline{\lim} f_n dx$

but may not be unsigned.

$$\limsup \int f_n \, dx \le \int \limsup f_n \, dx.$$

$\overline{\lim} \int f_n dx = -\underline{\lim} \int f_n dx$.

$\int \overline{\lim} f_n dx = -\int \underline{\lim} f_n dx$

   e. $\lim_{n\to\infty} \int_0^1 \frac{\sin(x^n)}{x^n} \, dx = 1$.

as $n \to \infty$, $x^n \to 0$ a.e. since $x \in [0,1]$.

$\therefore \frac{\sin(x^n)}{x^n} \to 1$ as $n \to \infty$. is $\frac{\sin(x^n)}{x^n}$ bdd