A GENERALIZED AFFINE ISOPERIMETRIC INEQUALITY

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Abstract.

A purely analytic proof is given for an inequality that has as a direct consequence the two most important affine isoperimetric inequalities of plane convex geometry: The Blaschke-Santalo inequality and the affine isoperimetric inequality of affine differential geometry.

Let $H^1(\mathbf{S})$ be the space of functions $u\colon \mathbf{S} \to \mathbf{R}$ such that u is absolutely continuous and $u' \in L^2(\mathbf{S})$. We use the norm

$$\|u\|_{H^1} = \left(\int [u^2 + (u')^2] \, d\theta \right)^{\frac{1}{2}}.$$

The space $H^1(\mathbf{S})$ can also be described as the space of functions whose first distributional derivative is in L^2. The norm is a Hilbert space norm with corresponding inner product $\langle u, v\rangle_{H^1} = \int_{\mathbf{S}} (uv + u'v') \, d\theta$.

Theorem 1 (Two Dimensional Analytic Affine Isoperimetric Inequality)**.** *Assume*

i) F and h are non-negative 2π periodic functions that do not vanish identically.

ii) F is measurable and satisfies the integrability condition

$$\int_{\mathbf{S}} F^{1/3}(\theta) \, d\theta < \infty$$

and the orthogonality conditions

$$\int_{\mathbf{S}} F^{1/2}(\theta) \cos\theta \, d\theta = 0 = \int_{\mathbf{S}} F^{1/2}(\theta) \sin\theta \, d\theta. \tag{1}$$

iii) $h \in H^1(\mathbf{S})$.

Then

$$\left(\int_{\mathbf{S}} F^{1/2}(\theta) h(\theta) \, d\theta \right)^2 \geq \frac{1}{4\pi^2} \left(\int_{\mathbf{S}} F^{1/3} \, d\theta \right)^3 \left(\int_{\mathbf{S}} [h^2 - (h')^2] \, d\theta \right). \tag{2}$$

Equality holds if and only if there exist $k_1, k_2, a > 0$, and $\alpha \in \mathbf{R}$ such that

$$h(\theta) = k_1 \sqrt{a^2 \cos^2(\theta - \alpha) + a^{-2} \sin^2(\theta - \alpha)} \tag{3}$$

and F is given almost everywhere by

$$F(\theta) = k_2 (a^2 \cos^2(\theta - \alpha) + a^{-2} \sin^2(\theta - \alpha))^{-3}. \tag{4}$$

Remark. The functions $h(\theta)$ of the form (3) are exactly support functions of the ellipses centered at the origin.

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