Admission to Candidacy Examination

in Real Analysis

August 1985

Notation: R – real numbers; λ – Lebesgue measure on R; M_λ – the Lebesgue measurable sets; χ_A – characteristic function of A.

1. State and prove Lebesgue's Dominated Convergence Theorem.

2. State and prove Holder's inequality.

3. Prove that for every $\varepsilon > 0$, there exists an open dense subset O of R such that $\lambda(O) < \varepsilon$.

4. Let (X, A, μ) be a finite measure space, f a real valued A-measurable function on X with range of $f \subset [-M, M]$ for some M, $0 < M < \infty$. Define ν on the Borel subsets of $[-M, M]$ by

$$\nu(B) = \mu(f^{-1}(B)).$$

Prove that
$$\int_{-M}^{M} x \, d\nu = \int_X f \, d\mu \quad ,$$

5. Let $f: R \to R$. For $x \in R$, $\delta > 0$, define

$$\omega(x, \delta) = \sup \{|f(z) - f(y)| : z, y \in N_\delta(x)\}$$

and let
$$\omega(x) = \inf_{\delta > 0} \omega(x, \delta)$$

a) Prove that f is continuous at $x \Leftrightarrow \omega(x) = 0$.

b) Prove that for each $\alpha \in R$, the set

$$O_\alpha = \{x \in R: \omega(x) < \alpha\} \text{ is open.}$$

c) Prove that the set $\{x \in R: f \text{ is continuous at } x\}$ is a G_δ set.

6. a) Let $f, g \in L^1([0,1], M_\lambda, \lambda)$ and let $F(x) = \int_0^x f \, d\lambda$. Show that F is nondecreasing if and only if $f \geq 0$ a.e. on $[0,1]$.

b) Show that if $\{F_n\}_{n=1}^\infty$ is a sequence of nondecreasing absolutely continuous functions on $[0,1]$ such that $F_n(0) = 0$ and $\sum_{n=1}^\infty F_n(1) < \infty$, then $\sum_{n=1}^\infty F_n$ is absolutely continuous.

7. Let (X, A, μ) and (Y, B, ν) be complete measure spaces. Let $g \in L^1(X, A, \mu)$ and $h \in L^1(Y, B, \nu)$. Define f on $X \times Y$ by $f(x,y) = g(x)h(y)$. Prove that $f \in L^1(X \times Y, M_{\mu\times\nu}, \mu \times \nu)$ and

$$\int_{X\times Y} f \, d(\mu \times \nu) = \int_X g \, d\mu \int_Y h \, d\nu$$

8. Let μ_1 and μ_2 be nonzero finite measures on a measurable space (X, A). Define for $E \in A$

$$(\mu_1 \vee \mu_2)(E) = \sup \{\mu_1(A) + \mu_2(E - A): A \in A, A \subset E\}.$$

It can be shown directly that $\mu_1 \vee \mu_2$ is a finite measure on (X, A). However, note that μ_1, μ_2 are absolutely continuous with respect to $\mu_1 + \mu_2$. Let

$$f_1 = \frac{d\mu_1}{d(\mu_1 + \mu_2)} \quad \text{and} \quad f_2 = \frac{d\mu_2}{d(\mu_1 + \mu_2)} .$$

Show that

$$(\mu_1 \vee \mu_2)(E) = \int_E (f_1 \vee f_2) \, d(\mu_1 + \mu_2)$$

where $(f_1 \vee f_2)(x) = \max \{f_1(x), f_2(x)\}$.

9. Let $I: L^1([0,1], M_\lambda, \lambda) \to R$ satisfy

a) I is linear.

b) If $f \geq g$, then $I(f) \geq I(g)$.

c) $I(\chi_J) =$ length of J if J is an interval.

d) If $f_n \downarrow 0$, then $I(f_n) \downarrow 0$.

Prove the following:

i) $I(\chi_O) = \lambda(O)$ for O open.

ii) $I(\chi_F) = \lambda(F)$ if F is compact.

iii) $I(\chi_E) = \lambda(E)$ if $E \in M_\lambda$.

iv) $I(f) = \int f \, d\lambda$ if f is simple.

v) $I(f) = \int f \, d\lambda$ if $f \in L^1([0,1], M_\lambda, \lambda)$.