Absolutely continuity and bounded variation.

We have talked a bit about ***absolutely continuous*** functions and functions of ***bounded variation***. I will not write out the definitions here.

We start with some easy observations about functions of bounded variation. Let $V_a^b(f)$ be the ***total variation*** of f on $[a,b]$. That is

$$V_a^b(f) = \sup_{a=t_0\le t_1\le\cdots\le t_{n-1}\le t_n=b} \sum_{j=1}^n |f(t_{j-1}) - f(t_j)|$$

where the supremum is over all partitions of $[a,b]$. Using this definition it is not hard to show that if $a<b<c$ and f is continuous at b that

$$V_a^b(f) + V_b^c(f) = V_a^b(f).$$

That is V is additive over intervals. Also if f is monotone increasing on $[a,b]$ then then for any partition $|f(t_j) - f(t_{j-1})| = f(t_j) - f(t_{j-1})$ and therefore

$$\sum_{j=1}^n |f(t_{j-1}) - f(t_j)| = \sum_{j=1}^n (f(t_{j-1}) - f(t_j)) = f(b) - f(a)$$

as the series is telescoping. Thus in this case

$$V_a^b(f) = f(b) - f(a).$$

Note that f does not have to be continuous for this to hold. Likewise if f is monotone decreasing

$$V_a^b(f) = f(a) - f(b) = |f(b) - f(a)|.$$

Now assume that f is piecewise monotone on $[a,b]$ with the changes of monotonicity at $a < t_1 < t_2 < \cdots t_{n-1} < b$ and set $t_0 = a$, $t_n = b$, see Figure 1.

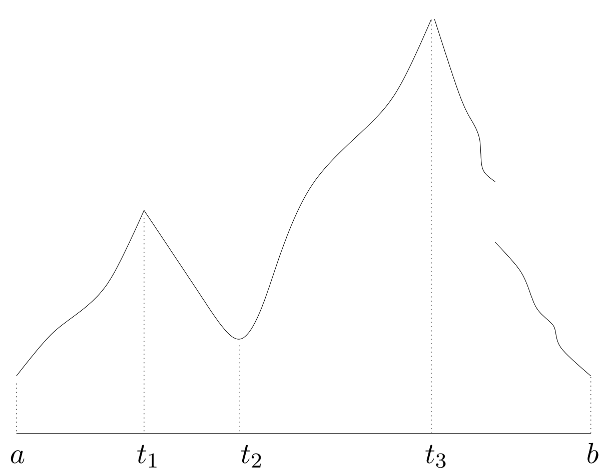

FIGURE 1.

Problem 1. For a function as in Figure 1 assume that f is continuous at $t_1, \ldots, t_{n-1}$. Show

$$V_a^b(f) = \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|.$$

What, if anything, goes wrong if f is not continuous at one of the t_j's? $\square$

The big theorem for absolutely continuous functions is that $f \in \mathrm{AC}([a,b])$ if and only if f' exists almost everywhere on $[a,b]$ with $f' \in L^1[a,b]$ and for all $x \in [a,b]$

$$f(x) - f(a) = \int_a^x f'(t)\,dt.$$

Problem 2. Assume that $f \in \mathrm{AC}([a,b])$ and that f is monotone on $[a,b]$. Show

$$\int_a^b |f'(t)|\,dt = |f(b) - f(a)| = V_a^b(f).$$

Now use that $V_a^b(f)$ is additive on intervals to show that if $f \in \mathrm{AC}([a,b])$ and is piecewise monotone in the sense of Figure 1 that

$$\int_a^b |f'(t)|\,dt = V_a^b(f). \tag{1}$$

$\square$

We would like Equation 1 to hold for all $f \in \mathrm{AC}([a,b])$. Here is a start on this.

Problem 3. Let $f \in \mathrm{AC}([a,b])$ and $\varepsilon > 0$. Then $f' \in L^1([a,b])$ and therefore there is a step function ϕ on $[a,b]$ with

$$\int_a^b |f'(t) - \phi(t)|\,dt < \varepsilon.$$

Define g on $[a,b]$ by

$$g(x) = f(a) + \int_a^x \phi(t)\,dt.$$

(a) Show that g is piecewise linear and continuous. Thus g is piecewise monotone and continuous and thus

$$V_a^b(g) = \int_a^b |g'(t)|\,dt = \int_a^b |\phi(t)|\,dt.$$

(b) Let $a = t_0 < t_1 < \cdots t_{n-1} < t_n = b$ be a partition of $[a,b]$. Then show

$$\begin{aligned}\Big| |f(t_j) - f(t_{j-1})| - |g(t_j) - g(t_{j-1})| \Big| &\le \Big| (f(t_j) - f(t_{j-1})) - (g(t_j) - g(t_{j-1})) \Big| \\ &= \left| \int_{t_{j-1}}^{t_j} (f'(t) - \phi(t))\,dt \right| \\ &\le \int_{t_{j-1}}^{t_j} |f'(t) - \phi(t)|\,dt\end{aligned}$$

and use this to show

$$|V_a^b(f) - V_a^b(g)| \le \int_a^b |f'(t) - \phi(t)|\, dt < \varepsilon.$$ □

Problem 4. Let $f \in \mathrm{AC}([a,b])$ and let ϕ_n be a step function on $[a,b]$ with

$$\int_a^b |f'(t) - \phi_n(t)|\, dt \le \frac{1}{n}.$$

Let

$$g_n(x) = f(a) + \int_a^x \phi_n(t)\, dt.$$

(a) Show for $x \in [a,b]$

$$|f(x) - g_n(x)| \le \frac{(b-a)}{n}$$

and therefore $g_n \to f$ uniformly on $[a,b]$.

(b) Show

$$\lim_{n\to\infty} V_a^b(g_n) = V_a^b(f).$$

(c) Show

$$\lim_{n\to\infty} \int_a^b |g_n'(t)|\, dt = \int_a^b |\phi_n(t)|\, dt = \int_a^b |f'(t)|\, dt.$$ □.

Theorem 1. *Let $f \in \mathrm{AC}([a,b])$ then the total variation of f is*

$$V_a^b(f) = \int_a^b |f'(t)|\, dt.$$

Problem 5. Prove the last theorem. □

The next problem is has appeared in some form or anther on at least one of the qualifying exams.

Problem 6. Let $f \in \mathrm{BV}([a,b])$ and assume that

(a) The function $x \mapsto V_x^b(f)$ is continuous at $x = a$,

(b) for each $x \in (a,b]$ the restriction $f\big|_{[x,b]}$ is in $\mathrm{AC}([x,b])$.

Then $f \in \mathrm{AC}([a,b])$. □

Now for the standard example. For $\alpha, \beta > 0$ let $f_{\alpha,\beta}$ be defined on $[0,1]$ by

$$f_{\alpha,\beta}(x) = \begin{cases} x^\alpha \sin(1/x^\beta), & x > 0; \\ 0, & x = 0. \end{cases}$$

Problem 7. Use the Theorem above to show that $f_{\alpha,\beta} \in \mathrm{AC}([0,1])$ if and only if $f_{\alpha,\beta} \in \mathrm{BV}([0,1])$. □

Problem 8. For what α, β are the functions $f_{\alpha,\beta}$ of bounded variation? □

Problem 9. Show that $f_{\alpha,\beta}$ is differentiable at $x = 0$ if and only if $\alpha > 1$. □