

Review for test 1

We started out with some preliminary results about bijective maps. You should know the definition of *injective*, *surjective*, and *bijective*.

We then gave the axioms for affine geometry, which are

Affine Geometry: Axiom 1 (Two points determine a unique line). *If P and Q are distinct points of \mathbb{A}^2 (that is $P \neq Q$) then there is a unique line that passes through both of them.* \square

Affine Geometry: Axiom 2 (Parallel Axiom). *If ℓ is a line of \mathbb{A}^2 and P is a point that is not on ℓ , then there is a unique line m that goes through P and is parallel to ℓ .* \square

Affine Geometry: Axiom 3. *There exists a set of four points in \mathbb{A}^2 such that no three of which are all on the same line.* \square

You should have these memorized. And you should know the definition of *parallel* (and that under our definition a line is parallel to itself).

Problem 1. If ℓ_1 and ℓ_2 are parallel lines and m is a line that intersects ℓ_1 in a single point, then it intersects ℓ_2 in a single point.

Solution. If $\ell_1 = \ell_2$, there is nothing to prove. So assume that $\ell_1 \neq \ell_2$. We first show that m intersects ℓ_2 in at least one point. For if this were not the case, then let $P = m \cap \ell_1$. Towards a contradiction assume m does not intersect ℓ_2 in any point. Then m is parallel to ℓ_1 and we are given that ℓ_2 is parallel to ℓ_1 . Thus the point P has two lines passing through it that are parallel to ℓ_2 , contradicting the parallel axiom which says that there is only one.

To see that ℓ_2 intersects m in at most one point, note if it intersected m in two points by the first axiom of affine geometry we would have $m = \ell_2$ which this line is parallel to ℓ_1 , (as $\ell_2 \parallel \ell_1$) and that it only intersects ℓ_1 in one point (as m only intersects ℓ_1 in one point). This is a contradiction. \square

Problem 2. Show that if ℓ , m and n are lines with $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.

Solution. If $\ell = m$, or $m = n$, or $\ell = n$ then the result is true. So we assume that ℓ , m and n are distinct. Towards a contradiction assume that ℓ and n are not parallel. Then they intersect at some point, call it P . Then P is not on m (as P is on ℓ and ℓ and m have no points in common as $\ell \parallel m$ and $\ell \neq m$). But then the point P has two distinct lines through it (ℓ and n) which are parallel to m which contradicts the parallel axiom. \square

Problem 3. Let ℓ be a line and m a line that intersects ℓ in exactly one point. Then there is a bijective correspondence between the points, P , on m and the lines, ℓ' , parallel to ℓ . Proof by picture, provided there is enough description in English, is fine.

Solution. The correspondence is given as follows. For each point, P , on m let $\ell'(P)$ be the line through P parallel to ℓ . For each line ℓ' parallel to ℓ let $P(\ell') = \ell' \cap m$ be the point where ℓ' intersects m . From Figure 1 we see that each point P on m determines a unique line ℓ' parallel to ℓ (by the Parallel Axiom) and each line ℓ' parallel to ℓ determines a unique point $P = m \cap \ell'$ on m (by Problem 1). This gives a bijective correspondence.

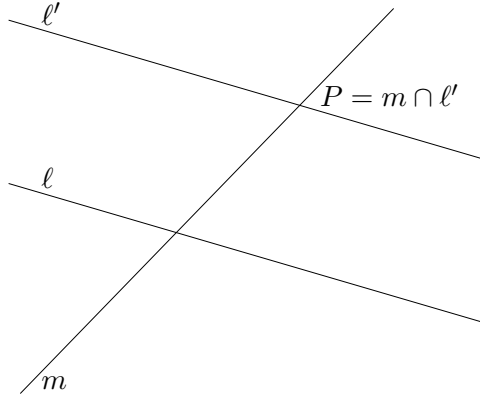


FIGURE 1

□

Problem 4. Show that any affine plane contains at least 6 lines.

Solution. You think about this one in light of Affine axiom 3. □

You should understand Theorem 14 on Homework 1.

Problem 5. In Figure 2 find the following

- (a) The line through C parallel to the line e .
- (b) All lines parallel to c .
- (c) $\overleftrightarrow{(f \cap d)(a \cap e)}$

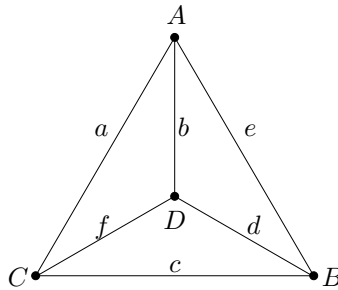


FIGURE 2

Solution. (a) f , (b) b and c , (c) $\overleftrightarrow{(f \cap d)(a \cap e)} = \overleftrightarrow{DA} = b$

Problem 6. In Figure 3

- (a) What is the line through E and parallel to b ?
- (b) What is the line through I and parallel to g ?
- (c) What are the lines parallel to b ?
- (d) What are the lines parallel to a ?

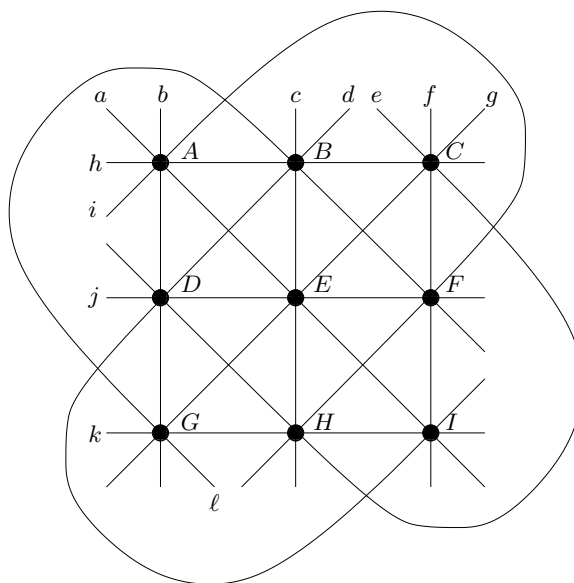


FIGURE 3

Solution. (a) c , (b) d , (c) b , c , and f , (d) a , e , and ℓ .

After giving the basic axioms, propositions, and some examples of affine geometries, we went about the project of showing that the basic coordinate plane $\mathbb{F}^2 = \mathbb{A}^2$, that is the set of ordered pairs (x, y) with $x, y \in \mathbb{F}$ we know and love from high school geometry is an example of an affine plane where we defined a line to be a set

$$L(a, b, c) = \{(x, y) : ax + by + c = 0\}$$

where a and b are not both zero. In the case that our field is $\mathbb{F} = \mathbb{R}$ you should certainly be able to graph these lines.

Problem 7. Graph the following

- (a) $L(1, 2, 3)$. What is the slope of this line? What is its slope intercept equation?
- (b) $L(1, 0, 5)$
- (c) $L(m, -1, \beta)$

Here is a particular case of showing that the first axiom of affine geometry holds.

Problem 8. Find the line that goes through $(3, -2)$ and $(4, 1)$.

Solution. A vector parallel to this line is $(4, 1) - (3, -2) = \langle 1, 3 \rangle$. Thus $\vec{n} = \langle -3, 1 \rangle$ is a perpendicular to this line. Therefore one equation for this line is

$$\vec{n} \cdot ((x, y) - (4, 1)) = 0.$$

That is

$$-3(x - 4) + 1(y - 1) = 0$$

which can be rewritten as

$$-3x + y + 11 = 0.$$

As a check note

$$-3(3) + (-2) + 11 = 0 \quad \text{and} \quad (-3)(4) + (1) + 11 = 0.$$

So the line is $L(-3, 1, 11)$. This is the same as $L(3, -1, -11)$ or $L(6, -2, -22)$. \square

More generally

Problem 9. If $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ are distinct points of \mathbb{A}^2 , show that there is a line $L(a, b, c)$ that passes through these points.

Solution. This works just like the last problem. The vector

$$Q - P = \langle x_1 - x_0, y_1 - y_0 \rangle$$

is parallel to the line. Thus

$$\vec{n} = \langle -(y_1 - y_0), (x_1 - x_0) \rangle$$

is perpendicular to the line. Thus the equation should be

$$\langle -(y_1 - y_0), (x_1 - x_0) \rangle \cdot \langle x - x_0, y - y_0 \rangle = 0.$$

This simplifies down to

$$(y_0 - y_1)x + (x_1 - x_0)y + (x_0y_1 - x_1y_0) = 0.$$

Thus $L(y_0 - y_1, x_1 - x_0, x_0y_1 - x_1y_0)$ should work. Verify by direct calculation that P and Q are on $L(y_0 - y_1, x_1 - x_0, x_0y_1 - x_1y_0)$ and explain why $(y_0 - y_1)$ and $(x_1 - x_0)$ are not both zero.

For another way to do this problem see Remark 6 on Homework 2. \square

Here is a particular case of showing that the second axiom of affine geometry holds.

Problem 10. Find the line through $(3, 7)$ that is parallel to $L(6, -2, 9)$.

Solution. The lines parallel to $L(6, -2, 9)$ are of the form $L(6, -2, c)$. We want to choose c so that $(3, 7)$ is on this line. That is we want

$$6(3) - 2(7) + c = 0$$

This gives $c = -18 + 14 = -4$. Thus the line is $L(6, -2, -4)$. \square

Problem 11. Let $L(a, b, c)$ be a line and $P = (x_0, y_0)$ any point of \mathbb{A}^2 . Show there is a line through P and parallel to $L(a, b, c)$.

Solution. The lines parallel to $L(a, b, c)$ are of the form $L(a, b, c')$. We wish to choose c' so that P is on the line. That is so that

$$ax_0 + by_0 + c' = 0.$$

This clearly works when $c' = -ax_0 - by_0$. So the required parallel to $L(a, b, c)$ through P is $L(a, b, -ax_0 - by_0)$. \square

We next started to review some linear algebra. The first topic we hit here was Cramer's rule.

Theorem 1 (Cramer's rule). *If $a, b, c, \in \mathbb{F}$ with*

$$ad - bc \neq 0$$

then the system

$$ax + by = e$$

$$cx + dy = f$$

has a unique solution. It is given by

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - fb}{ad - bc}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ce}{ad - bc}$$

*We call $ad - bc$ the **determinant of the system**.*

A special case that we have use repeatedly is

Theorem 2 (Cramer's rule for homogeneous equations). *If $a, b, c, \in \mathbb{F}$ with*

$$ad - bc \neq 0$$

then the system

$$ax + by = 0$$

$$cx + dy = 0$$

only has the solution

$$x = 0 \quad y = 0.$$

Some basic definitions are

Definition 3. The two vectors \vec{a} and \vec{b} are **linearly dependent** iff there are scalars α and β with at least one of them nonzero with

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}.$$

Thus to show two vectors \vec{a} and \vec{b} are linearly dependent, your job is to find scalars α and β , not both zero such that $\alpha\vec{a} + \beta\vec{b} = \vec{0}$.

Problem 12. Show that the vector $\vec{a} = \langle 2, 4 \rangle$ and $\vec{b} = \langle 3, 6 \rangle$ are linearly dependent.

Solution. Note

$$3\vec{a} + (-2)\vec{b} = 3\langle 2, 4 \rangle - 2\langle 3, 6 \rangle = \langle 0, 0 \rangle = \vec{0}.$$

Therefore if $\alpha = 3$ and $\beta = -2$ we have $\alpha\vec{a} + \beta\vec{b} = \vec{0}$ with $\alpha \neq 0$. Thus \vec{a} and \vec{b} are linearly dependent. (Note this is not the only possible choice of α and β that works. For example $\alpha = 15$ and $\beta = -10$ would work. But to show vectors are linearly dependent, you don't have to find all α and β that work, you only to find one pair of α and β that work.) \square

Problem 13. For any vector \vec{c} show that $4\vec{a}\vec{c}$ and $\vec{b} = 5\vec{c}$ are linearly dependent.

Solution. Just note that

$$5\vec{a} + (-4)\vec{b} = 20\vec{c} - 20\vec{c} = \vec{0}.$$

Thus for $\alpha = 5$ and $\beta = -4$ we have $\alpha\vec{a} + \beta\vec{b} = \vec{0}$ and $\beta \neq 0$. So the vectors are linearly dependent. \square

Definition 4. The vectors \vec{a} and \vec{b} are *linearly independent* iff

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}$$

for scalars α and β implies $\alpha = \beta = 0$.

Problem 14. Show the vectors $\vec{a} = \langle 1, 2 \rangle$ and $\vec{b} = \langle 3, -1 \rangle$ are linearly independent.

Solution. If $\alpha\vec{a} + \beta\vec{b} = \vec{0}$ we have

$$\alpha\langle 1, 2 \rangle + \beta\langle 3, -1 \rangle = \langle \alpha + 3\beta, 2\alpha - \beta \rangle = \langle 0, 0 \rangle.$$

This gives the system

$$\alpha + 3\beta = 0$$

$$2\alpha - \beta = 0$$

The determinant of this system is $(1)(-1) - (3)(2) = -7 \neq 0$. Thus by Cramer's rule $\alpha = \beta = 0$. Which is just what we needed to show that the vectors are linearly independent. \square

You should look at Homework 3 and understand the basic properties of the *determinant*, $\det(\vec{a}, \vec{b})$ of two vector \vec{a} and \vec{b} . It would not hurt to be able to prove the properties of \det given in Proposition 3 of Homework 3.

Problem 15. Let \vec{a} and \vec{b} be vectors with $\det(\vec{a}, \vec{b}) \neq 0$. Then show that \vec{a} and \vec{b} are linearly independent.

Solution. Let $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$. We are assuming that

$$\det(\vec{a}, \vec{b}) = a_2b_2 - a_2b_1 \neq 0.$$

If α and β are scalars such that

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}$$

then

$$\alpha\vec{a} + \beta\vec{b} = \langle a_1\alpha + b_1\beta, a_2\alpha + b_2\beta \rangle = \langle 0, 0 \rangle.$$

This gives the system

$$a_1\alpha + b_1\beta = 0$$

$$a_2\alpha + b_2\beta = 0$$

for α and β . By assumption the determinant of this system is not zero, and therefore by Cramer's rule we have that $\alpha = \beta = 0$. Which is what is required to show that \vec{a} and \vec{b} are linearly independent. \square

This is enough sample problems. Here are some other topics that you should review.

- (1) The definition of affine linear combination of two or three points. Review the quiz.
- (2) Be prepared for surprise mystery questions.