A ring theory problem.

Problem 1 on the January 2013 algebra exam is

Problem 1. Let d be a positive integer and $\mathbb{Q}$ is the field of rational numbers For each polynomial $f = a_0 + a_1 + a_2t^2 + \cdots + a_nt^n \in \mathbb{Q}[t]$ and integer with $0 \le i \le d-1$ let

$$N_i(f) = \sum_{j \equiv i \mod d} a_j.$$

Let I be the set of polynomials

$$I = \{f \in \mathbb{Q}[t] : N_0(f) = N_1(f) = N_2(f) = \cdots = N_{d-1}(f)\}.$$

Is I an ideal of $\mathbb{Q}[t]$? If no, give an example. If yes, then

(a) prove that I is an ideal.
(b) give a generator of the ideal, and
(c) prove your answer to (b) is correct.

I will just give a solution for $d = 2$ and $d = 3$ and leave the general case to you.

For $d = 2$ we have

$$\begin{aligned} N_0(f) &= a_0 + a_2 + a_4 + \cdots \\ N_1(f) &= a_1 + a_3 + a_5 + \cdots \end{aligned}$$

Since we this separates the coefficients of the even and odd degreed terms it suggests looking at $f(-1)$

$$\begin{aligned} f(-1) &= a_0 - a_1 + a_2 - a_n + \cdots \\ &= N_0(f) - N_1(f). \end{aligned}$$

From this we see that $N_0(f) = N_1(f)$ if and only if $f(-1) = 0$. Therefore

$$I = \{f(t) \in \mathbb{Q}[t] : f(-1) = 0\}$$

and this is just the ideal generated by $(x+1)$, that is $I = \langle x+1 \rangle$.

For $d = 3$ we have for $f = a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots$ that

$$\begin{aligned} N_0(f) &= a_0 + a_3 + a_6 + a_9 + \cdots \\ N_1(f) &= a_1 + a_4 + a_7 + a_{10} + \cdots \\ N_2(f) &= a_2 + a_5 + a_8 + a_{11} + \cdots \end{aligned}$$

we wish to separate terms by looking at their degrees $\mod 3$. So this time it makes sense to use a primitive third root of unity, ω, rather than looking at the values 1 and -1. The properties of ω we will use are

$$\begin{aligned} \omega^3 &= 1 \\ 1 + \omega + \omega^2 &= 0. \end{aligned}$$

These imply

$$\begin{aligned}
f(1) &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots \\
&= N_0(f) + N_1(f) + N_2(f) \\
f(\omega) &= a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4 + a_5\omega^5 + a_6\omega^6 + \cdots \\
&= a_0 + a_1\omega + a_2\omega^2 + a_3 + a_4\omega + a_5\omega^2 + a_6 + \cdots \\
&= N_0 + \omega N_1(f) + \omega^2 N_2(f) \\
f(\omega^2) &= a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + a_4\omega^8 + a_5\omega^10 + a_6\omega^{12} + \cdots \\
&= a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega^2 + a_5\omega + a_6 + \cdots \\
&= N_0 + \omega^2 N_1(f) + \omega N_2(f)
\end{aligned}$$

We can write this in matrix form as

$$\begin{bmatrix} f(1) \\ f(\omega) \\ f(\omega^2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix} \begin{bmatrix} N_0(f) \\ N_1(f) \\ N_2(f) \end{bmatrix}$$

If $N_0(f) = N_1(f) = N_2(f)$ and using $1 + \omega + \omega^2 = 0$ this becomes

$$\begin{bmatrix} f(1) \\ f(\omega) \\ f(\omega^2) \end{bmatrix} = \begin{bmatrix} 3N_0(f) \\ 0 \\ 0 \end{bmatrix}$$

and as the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix}$$

is non-singular (wee problem below) we see that

$$N_0(f) = N_1(f) = N_2(f) \quad \text{if and only if} \quad f(\omega) = f(\omega^2) = 0.$$

That is

$$I = \{f : f(\omega) = f(\omega^2) = 0\}$$

This is an ideal and its generator is

$$(t-\omega)(t-\omega^2) = t^2 - (\omega + \omega^2)t + \omega\omega^2 = t^2 + t + 1.$$

Thus $I = \langle t^2 + t + 1 \rangle$.

In the general case a reasonable conjecture is that is the ideal

$$I = \langle t^{d-1} + t^{d-2} + \cdots + t + 1 \rangle.$$

There is almost certainly a more direct method of doing this than what I have outlined for $d = 2, 3$. □

Problem 2. Let $\mathbb{F}$ be a field and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct members of $\mathbf{F}$. Show the matrix

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{n-1} & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{n-1}^2 & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \cdots & \lambda_{n-1}^{n-2} & \lambda_n^{n-2} \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_{n-1}^{n-1} & \lambda_n^{n-1} \end{bmatrix} = \left[\lambda_j^{i-1}\right]_{i,j=1}^{n}.$$

is non-singular. *Hint:* Let v be the column vector

$$v = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{bmatrix}.$$

Let M^t be the transpose of M and show

$$M^t v = \begin{bmatrix} f(\lambda_1) \\ f(\lambda_2) \\ \vdots \\ f(\lambda_{n-1}) \\ f(\lambda_n) \end{bmatrix}.$$

where $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}$. Thus if $Mv = 0$, the polynomial $f(t)$ has the n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. $\square$