ANALYSIS QUALIFYING EXAMINATION

August 1995

General Instruction:

(1) PLEASE WRITE ON ONLY ONE SIDE OF YOUR PAPER.

(2) Write your solution to each problem on a separate sheet.

(3) Markings: 1-8 $\sim$ 11 pt each; 9 $\sim$ 12 pt.

General Notation/Conventions:

(1) Let $\mathrm{R} = (-\infty, \infty)$ and $\mathbb{C}$ be the complex plane.

(2) All measure spaces are assumed to be complete.

[1] Let (X_i, ρ_i) be metric spaces and $f\colon X_1 \to X_2$ be a continuous map. Let $Y_1 \subset X_1$ and $Y_2 = f(Y_1)$.

(1a) Show that a continuous image of a compact set is compact. Namely, show that if Y_1 is compact, then Y_2 is compact.

(1b) Prove or give a counterexample to the following converse: *If Y_2 is compact, the Y_1 is compact.*

[2] Let (X, ρ) be a metric space. Let X be sequentially compact and $Y \subset X$. Let $f\colon X \to Y$ be an isometry between X and Y. Recall that this means that f is a distance-preserving homeomorphism from X onto Y. Thus for each $x, \tilde{x} \in X$,

$$\rho(x, \tilde{x}) = \rho(f(x), f(\tilde{x})) .$$

Show that $Y = X$.

[3] Let (X, Σ, μ) be a finite measure space and L_0 be the collection of all μ-measurable functions from X into R.

(3a) Prove Egoroff's Theorem, namely: Let $\{f_n\}$ be a sequence of functions from L_0 that converge almost everywhere to $f_0 \in L_0$. Show that $f_n \to f_0$ in measure also.

(3b) Does the statement of (3a) remain true if (X, Σ, μ) is an arbitrary (ie., not necessarily finite) measure space? Prove or give a counterexample.

[4] Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that f^{-1} of a Borel set is again a Borel set. That is, let $\mathcal{B}$ be the Borel subsets of $\mathbb{R}$ and show that $f^{-1}[\mathcal{B}] \subset \mathcal{B}$.

[5] Let E be a subset of R that is the union of a family of quite arbitrary intervals, each being open, closed, or half open and half closed. Prove that E is Lebesgue measurable.

[6] Fix $1 \leq p < \infty$. Let $([0,1], \mathcal{M}, m)$ is the Lebesgue measure space on $[0,1]$ and

$$L_p = \left\{ f\colon [0,1] \to \mathrm{R} : f \text{ is } m\text{-measurable } \& \ \| f \|_p \equiv \left[\int | f |^p \, d\mu \right]^{\frac{1}{p}} < \infty \right\} .$$

Consider a sequence $\{f_n\}$ of L_p functions such that $\| f_n \|_p \leq 1$ for each n.

(6a) Let $1 < p < \infty$. Show that $\{f_n\}$ is uniformly integrable. That is, show that for each $\epsilon > 0$ there exists $\delta > 0$ such that if $E \in \mathcal{M}$ and $m(E) < \delta$ then

$$\int_E | f_n | \, dm < \epsilon$$

for each $n \in \mathbb{N}$.

(6b) Does the statement of (6a) hold for $p = 1$? Prove or give a counterexample.

[7] State and prove the Fundamental Theorem of Algebra.

[8] Compute

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+2x+2)} \, dx \, .$$

[9] Let

(1) $z_0 \in \mathbb{C}$

(2) $0 < r < R$

(3) f and g be analytic functions on $D(z_0, R) \equiv \{z \in \mathbb{C}\colon \ | z - z_0 | < R\}$

(4) $\gamma(t) = z_0 + re^{it}$ for $0 \leqslant t \leqslant 2\pi$

(5) $| g(z) | < | f(z) |$ for all $z \in \gamma^* \equiv \{\gamma(t) \in \mathbb{C}\colon \ 0 \leq t \leq 2\pi\}$.

Show that the number of zeros of f inside of γ is equal to the number of zeros of $f + g$ inside of γ (counting multiplicity of course).