Problems on normal forms and related topics.

Let V be a finite dimensional vector space over a field $\mathbb{F}$ and $A \in \operatorname{Hom}_{\mathbb{F}}(V, V)$ (that is $A\colon V \to V$ is an $\mathbb{F}$ linear map. Then make V into a $\mathbb{F}[x]$ module by setting

$$f(x) \cdot v = f(A)v \quad \text{for } f(x) \in \mathbb{F}[x] \text{ and } v \in V.$$

We will denote this module by V_A (this is not standard notation and there is no generally accepted notation for this module that I know of).

Problem 1. Let V and W be vector spaces over $\mathbb{F}$ and $A \in \operatorname{Hom}_{\mathbb{F}}(V, V)$, $B \in \operatorname{Hom}_{\mathbb{F}}(W)$. Show that a $S \in \operatorname{Hom}_{\mathbb{F}}(V, W)$ (that is a $\mathbb{F}$ linear map from V to W) is in $\operatorname{Hom}_{\mathbb{F}[x]}(V_A, W_B)$ if and only if $SA = BS$. (Here $S \in \operatorname{Hom}_{\mathbb{F}[x]}(V_A, W_B)$ means that S is a homomorphism of $\mathbb{F}[x]$ modules, which in this case means that not only is S linear over $\mathbb{F}$, but it is linear over the ring $\mathbb{F}[x]$.) $\square$

Recall that if $A \in \operatorname{Hom}(V, V)$ and $B \in \operatorname{Hom}_{\mathbb{F}}(W, W)$, then A and B are ***similar*** if and only if there is an invertible linear map $S\colon V \to W$ such that $B = S^{-1}AS$. This equation can be rewritten as $AS = BS$ which also makes sense when S is not invertible.

Problem 2. With this notation show that A and B are similar if and only if V_A and V_B are isomorphic as $\mathbb{F}[x]$ modules. $\square$

Definition 1. Let R be a commutative ring and M a R module. Then M is cyclic if and only if there is an $e \in M$ such that $M = Re := \{re : r \in R\}$. In this case we say that e ***generates*** M. $\square$

Definition 2. In the previous definition if the ring is $\mathbb{F}[x]$ and the module is V_A then we say e is a ***cyclic vector*** for A.

Problem 3. Let V be a finite dimensional vector space over $\mathbb{F}$ and assume that e is cyclic vector for V. If $n = \dim_{\mathbb{F}} V$ show that $e, Ae, A^2e, \ldots, a^{n-1}$ is a basis of V. $\square$

Problem 4. With notation as in the last problem we have that $A^n e$ can be expressed in the basis $e, Ae, \ldots, A^{n-1}e$, say

$$A^n e = a_0 e + a_1 Ae + a_2 A^2 e + \cdots + a_{n-1} A^{n-1} e.$$

Show that the minimal polynomial of A on V is

$$\min_A(x) = x^n - a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_1 x - a_0$$

and that the matrix of A in this basis is

$$[A] = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & a_{n-1} \end{bmatrix}$$

Hint: If you have not done this before it is probably best (which is only really saying this is what I found best when learning this) is to do a representative small dimensional case such as $n = 5$ where the matrix is

$$[A] = \begin{bmatrix} 0 & 0 & 0 & 0 & a_0 \\ 1 & 0 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 & a_2 \\ 0 & 0 & 1 & 0 & a_3 \\ 0 & 0 & 0 & 1 & a_4 \end{bmatrix} \qquad \square$$

Problem 5. If $A \in \mathrm{Hom}_{\mathbb{F}}(V, V)$ and $\deg \min_A(x) < \dim_{\mathbb{F}}(V)$ show that V does not have any cyclic vectors for A. $\square$

The following two results re either a corollary of the structure theorem of finitely generated modules over $\mathbb{F}[x]$, or, in some presentations, a lemmas used in the proof of the structure theorem.

Proposition 1. *If $A \in \mathrm{Hom}_A(V, V)$ and $g(x) = \min_A(x)$ then V has a vector of order $g(x)$. That is there is a vector $e \in V$ such that*

$$\mathrm{ann}(e) := \{f(x) \in \mathbb{F}[x] : f(A)e = 0\} = \langle g(x) \rangle. \qquad \square$$

Proposition 2. *If $A \in \mathrm{Hom}_{\mathbb{F}}(V, V)$, then V contains a cyclic vector for A if and only if $\deg(\min_A(x)) = \dim(V)$.* $\square$

Problem 6. Let $A \in \mathrm{Hom}_{\mathbb{F}}(V, V)$ and assume V has a cyclic vector for A. Let $BA = AB$. Show that $B = f(A)$ for some $f(x) \in \mathbb{F}[x]$. $\square$

Problem 7 (January 2011). Classify up to similarity the 4×4 matrices over the field of complex numbers that have characteristic polynomial $(x - 1)^2(x + 1)^2$. $\square$

Problem 8. Find the invariant factors of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

and give both its rational and Jordan canonical forms. $\square$