Take home portion of final

We start by extending our definition of $\sup(S)$ and $\inf(S)$ for nonempty subsets of $\mathbf{R}$. If S is bounded above, the $\sup(S)$ is the usual ***least upper bound*** or ***supremum***. If S is not bounded above set $\sup(S) = +\infty$. Likewise if S is bounded below, then $\inf(S)$ is the usual ***greatest lower bound*** or ***infinmum***, if S is not bounded below, then $\inf(S) = -\infty$.

We also want to extend the definition of the limit of a sequence.

Definition 1. Let $\langle a_k\rangle_{k=1}^{\infty}$ be a sequence of real numbers. Then

$$\lim_{k\to\infty} a_k = +\infty$$

if and only if for all real numbers B there is a N such that

$$k \geq N \quad \text{implies} \quad a_k > B.$$

□

Definition 2. Let $\langle a_k\rangle_{k=1}^{\infty}$ be a sequence of real numbers. Then

$$\lim_{k\to\infty} a_k = -\infty$$

if and only if for all real numbers A there is a N such that

$$k \geq N \quad \text{implies} \quad a_k < A.$$

□

Definition 3. The ***extended real numbers*** is the set $\mathbf{R} \cup \{-\infty, +\infty\}$. That is the extended real numbers is just the usual real numbers with the two valued $-\infty$ and $+\infty$ thrown in. □

Note that now every nonempty subset of $\mathbf{R}$ has an sup and inf in the extended real numbers.

One of the more useful results from last term was that any bounded monotone sequence is convergent. With the above definition we can drop the requirement that the sequence be bounded.

Proposition 4. *Let $\langle a_k\rangle_{k=1}^{\infty}$ be a monotone sequence. Then it has a limit in the extended real numbers.*

Proof. Assume that $\langle a_k\rangle_{k=1}^{\infty}$ is monotone increasing. If the sequence is bounded above, then we saw last term that it it converged to $\sup\{a_1, a_2, a_3, \ldots\}$. So assume that it is not bounded above. Then for any real number B, there is some N with $a_N > B$. But $\langle a_k\rangle_{k=1}^{\infty}$ is monotone increasing and therefore if $k \geq N$ we have $a_k \geq a_N > B$, and therefore $\lim_{k\to\infty} a_k = +\infty$. □

Definition 5. Let $\langle a_k\rangle_{k=1}^{\infty}$ be a sequence of real numbers, and set

$$S_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$
$$I_n = \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

Then $\langle S_n\rangle_{n=1}^{\infty}$ is monotone increasing and $\langle I_n\rangle_{n=1}^{\infty}$ is monotone decreasing. Therefore the limits of these two sequences exist in the extended real numbers. We set

$$\limsup_{k\to\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+1}, \dots\}$$
$$\liminf_{k\to\infty} a_k = \lim_{n\to\infty} I_n = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, a_{n+2}, \dots\}. \qquad \square$$

Proposition 6. *For any sequence* $\langle a_k\rangle_{k=1}^{\infty}$ *show that*

$$\liminf_{k\to\infty} a_k \leq \limsup_{k\to\infty} a_k.$$

Problem 1. Prove this. $\square$

Problem 2. Find the following:

(a) $\liminf_{n\to\infty}(3+(-1)^n)$.
(b) $\limsup_{n\to\infty}(3+(-1)^n)$.
(c) $\liminf_{n\to\infty}(-2)^n$.

Problem 3. Give examples of sequences $\langle a_k\rangle_{k=1}^{\infty}$ such thatg

(a) $\liminf_{k\to\infty} a_k = -1$ and $\limsup_{k\to\infty} a_k = +1$.
(b) $\liminf_{k\to\infty} a_k = 0$ and $\limsup_{k\to\infty} a_k = +\infty$.

Proposition 7. *If* $\langle a_k\rangle_{k=1}^{\infty}$ *is a sequence with such that*

$$\lim_{k\to\infty} a_k = L$$

exists, then

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L.$$

Problem 4. Prove this. $\square$

Proposition 8. *If* $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$, *then* $\lim_{k\to\infty} a_k$ *exists.*

Problem 5. Prove this. $\square$

Proposition 9. *If* $\langle a_k\rangle_{k=1}^{\infty}$ *and* $\langle b_k\rangle_{k=1}^{\infty}$ *are two sequences of real numbers, then*

$$\limsup_{k\to\infty}(a_k+b_k) \leq \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k.$$

Problem 6. Prove this and give an example where

$$\limsup_{k\to\infty}(a_k+b_k) < \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k. \qquad \square$$