Analysis Qualifying Exam
January 2003

Instructions: Write your name legibly on each sheet of paper. Write only on one side of of each sheet of paper. Try to answer all questions. Questions 1–8 are each worth 10 points and question 9 is worth 20 points.
Terminology: Measurability and integrability on $\mathbb{R}$ or an interval will always refer to the Lebesgue measure, except if otherwise specified. Lebesgue measure will be denoted by m, dx or dy depending on the context.

1. Let $f : [a,b] \to [c,d]$ be an absolutely continuous function and let $g : [c,d] \to \mathbb{R}$ be a Lipschitz function. Prove that the composition $g \circ f : [a,b] \to \mathbb{R}$ is absolutely continuous.

2. Let $1 \le p \le 2$ and let $f \in L_p([0,\infty))$.

   a. Prove that the function f is integrable over any bounded interval $[a,b] \subset [0,\infty)$.

   b. Define $g(x) = \int_x^{x^2} f(t)\,dt$. Prove that

$$\lim_{x\to\infty} \frac{g(x)}{x} = 0.$$

3. Let $0 \le f_n : [0,1] \to \mathbb{R}$ be Lebesgue measurable such that $f_n(x) \to 0$ a.e. Prove that

$$\lim_{n\to\infty} \int_0^1 \frac{f_n}{1+f_n}\,dx = 0.$$

4. Let (X, Σ, μ) be a measure space, where Σ is the σ-algebra of all μ^*-measurable sets.

   a. Let $A_n \subset B_n$ for $n = 1,2$ where each B_n is μ^*-measurable and $B_1 \cap B_2 = \emptyset$. Prove that $\mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2)$.

   b. Let now (A_n), (B_n) be sequences of sets such that $A_n \subset B_n$ for all n, and $B_n \cap B_m = \emptyset$ for all $n \ne m$. Prove that $\mu^*(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu^*(A_n)$.

5. Let f be integrable over the bounded interval $[a,b]$ and assume that $\int_a^x f(t)\,dt = 0$ for all $x \in [a,b]$. Prove that $f(x) = 0$ a.e.

6. Let f be a non-constant entire function such that $f(\mathbb{R}) \subset \mathbb{R}^+$. Prove that all real zeros of f have even order.

7. Let $f : \{z : |z| < 1\} \to \mathbb{C}$ be a holomorphic function such that $|f(z)| < \frac{1}{|z|}$ for all $z \neq 0$. Prove that $|f(z)| \leq 1$ for all $|z| < 1$.

8. Compute
$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx.$$

9. True or False. Prove, or give a counterexample.

   a. Let $E_n \subset \mathbb{R}$ such that $\sum_{n=1}^{\infty} m^*(E_n) < \infty$. Then
   $$m^*(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k) = 0.$$

   b. Let f be integrable over $[0,1]$ such that
   $$\left| \int_0^1 f(t)g(t)\, dt \right| \leq 1$$
   for all continuous functions g on $[0,1]$ with $\|g\|_\infty \leq 1$. Then
   $$\int_0^1 |f(t)|\, dt \leq 1.$$

   c. If f is integrable over $[0,1]$, then f is bounded on $[0,1]$.

   d. If f is analytic on $|z| < 1$, then there exists a $k \geq 1$ such that $|f^{(k)}(0)| < k!4^k$.

   e. Let $|a_{nm}| \leq 1$ for all $n, m \geq 1$. Then
   $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{nm}.$$