Admission to Candidacy Examination
in
Real Analysis
January 1992

Notation: $\mathbb{R}$ denotes the Real Numbers and λ denotes Lebesgue measure on $\mathbb{R}$.

1. State and prove Hölder's inequality.

2. State and prove Egorov's theorem.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f \in L^1(X, \mathcal{A}, \mu)$. Prove each of the following:

a. Given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \int_A f \, d\mu \right| < \epsilon \quad \text{for all } A \in \mathcal{A} \text{ with } \mu(A) < \delta.$$

b. Given $\epsilon > 0$ there exists $A \in \mathcal{A}$ with $\mu(A) < \infty$ such that

$$\left| \int_{X \setminus A} f \, d\mu \right| < \epsilon.$$

4. Let $\{p_n\}$ be a sequence of 2π–periodic measurable functions on $\mathbb{R}$ satisfying

(a) $p_n(t) \geq 0$ for all n and t,
(b) $\int_{-\pi}^{\pi} p_n(t)dt = 1$,
(c) For each $\delta > 0$, $\lim_{n \to \infty} \int_{\delta \leq |t| \leq \pi} p_n(t)dt = 0$.

For f continuous and 2π–periodic on $\mathbb{R}$, set

$$f_n(x) = \int_{-\pi}^{\pi} p_n(x - t) f(t) dt.$$

Prove that $\lim_{n \to \infty} f_n(x) = f(x)$ uniformly on $\mathbb{R}$.

5. **Definition.** $f : [a, b] \to \mathbb{R}$ is in $Lip_1([a, b])$ if there exists a positive constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$.

(a) Prove that if $f \in Lip_1([a, b])$, then f is absolutely continuous on $[a, b]$.

(b) If f is absolutely continuous on $[a, b]$, prove that $f \in Lip_1([a, b])$ if and only if $f' \in L^\infty([a, b])$.

6. Let f be a bounded real valued function on $[a, b]$. Prove that if f is Riemann integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$, and the two integrals are equal.

7. Let f be a nonnegative real valued function on $\mathbb{R}$ with $f \in L^1(\mathbb{R})$. Define φ on $[0, \infty)$ by

$$\varphi(t) = \lambda(\{x : f(x) \geq t\}) \qquad t \geq 0.$$

Prove that φ is nonincreasing and that

$$\int_0^\infty \varphi(t)dt = \int_{\mathbb{R}} f\, d\lambda.$$

8. Suppose $f, f_1, f_2, \ldots$ are Lebesgue integrable functions on $[a, b]$ with $\lim_{k\to\infty} \int_a^b |f - f_n|\, d\lambda = 0$. Assume further that $g, g_1, g_2, \ldots$ are measurable, bounded in L^∞, and that $g_k \to g$ a.e. as $k \to \infty$. Prove that

$$\lim_{k\to\infty} \int_a^b |f_k g_k - fg|\, d\lambda = 0.$$

9. If $f \in L^1(\mathbb{R})$, then prove that $\lim_{h\to 0} \|f_h - f\|_{L^1(\mathbb{R})} = 0$, where $f_h(x) := f(x + h)$.