NOTES ON ANALYSIS

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Contents

1. Metric Spaces.

Definition 1. A ***metric space*** is a nonempty set E with a function $d\colon E\times E\to[0,\infty)$ such that for all $p,q,r\in E$ the following hold

(a) $d(p,q)\geq 0$,
(b) $d(p,q)=0$ if and only if $p=q$,
(c) $d(p,q)=d(q,p)$, and
(d) $d(p,r)\leq d(p,q)+d(q,r)$. □

The function d is called the ***distance function*** on E. The condition $d(p,q)=d(q,p)$ is that the distance between points is ***symmetric***. The inequality $d(p,r)\leq d(p,q)+d(q,r)$ is the ***triangle inequality***.

The most basic example if a metric space is when $E\subseteq\mathbb{R}$ is a nonempty subset of the real numbers, $\mathbb{R}$, and the distance is defined by

$$d(p,q)=|p-q|.$$

Problem 1. Show that this makes E into a metric space. □

We have seen that if $p=(p_1,\dots,p_n)$ and $q=(q_1,\dots,q_n)$ are points in $\mathbb{R}^n$ and we define the ***length*** or ***norm*** of p to be

$$\|p\|=\sqrt{p_1^2+\cdots+p_n^2}$$

then the inequality

$$\|p+q\|\leq\|p\|+\|q\|$$

holds.

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Proposition 2. *Let E be a nonempty subset of $\mathbb{R}^n$ and for $p, q \in E$ let*

$$d(p,q) = \|p-q\|.$$

Then E with the distance function d is a metric space.

Problem 2. Prove this. $\square$

Here are some inequalities that we will be using later.

Proposition 3 (Reverse triangle inequality)**.** *Let E be a metric space with distance function d and let $x, y, z \in E$. Then*

$$|d(x,y) - d(x,z)| \le d(y,z).$$

Problem 3. Prove this. Then draw a picture in the case of $\mathbb{R}^2$ with its standard distance function showing why this inequality is reasonable. $\square$

Proposition 4. *Let E be a metric space with distance function d and $x_1, \ldots, x_n \in E$. Then*

$$d(x_1, x_n) \le d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n).$$

Problem 4. Prove this. Then draw a picture in the plane showing why this is reasonable. *Hint:* Induction. $\square$

Definition 5. Let E be a metric space with distance function d. Let $a \in E$, and $r > 0$.

(a) The ***open ball*** of radius r centered at x is

$$B(a,r) := \{x : d(a,x) < r\}.$$

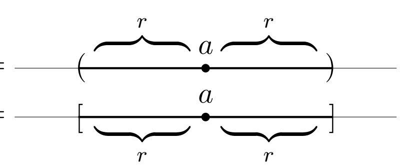
(b) The ***closed ball*** or radius r centered at a is

$$\overline{B}(a,r) := \{x : d(a,x) \le r\}.$$

$\square$

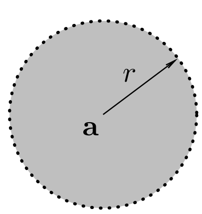
In the real numbers with their usual metric $d(x,y) = |x-y|$ the open and closed balls about a are intervals with center a:

$B(a,r) = (a-r, a+r) =$
$\overline{B}(a,r) = [a-r, a+r] =$

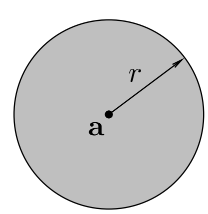


In the plane, $\mathbb{R}^2$, with its usual metric $d(\mathbf{p},\mathbf{q}) = \|\mathbf{p}-\mathbf{q}\|$, the open and closed balls about $\mathbf{a}$ are disks centered at $\mathbf{a}$.

$B(\mathbf{a}, r) =$



$\overline{B}(\mathbf{a}, r) =$



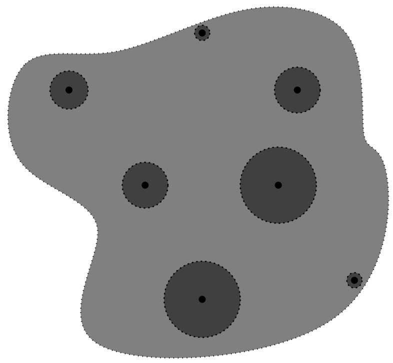


FIGURE 1. A set is open if and only if each of its points is the center of an open ball contained in the set.

Definition 6. Let E be a metric space with distance function d. Then $S \subseteq E$ is an ***open set*** if and only if for all $x \in S$ there is an $r > 0$ such that $B(x, r) \subseteq S$. □

This can be restated by saying that S is open if and only if each of its points is the center of an open ball contained in S. See Figure 1.

Proposition 7. *In any metric space E, the sets E and $\varnothing$ are open.* □

Proof. Let $p \in E$, then for any $r > 0$ we have $B(p, r) = \{x \in E : d(x, p) < r\} \subseteq E$. Thus E contains not only some open ball about p, it contains every open ball about p. Therefore E is open.

That $\varnothing$ is open is a case of a vicious implication. To see this consider the statement

$$p \in \varnothing \text{ and } r > 0 \implies B(p, r) \subseteq \varnothing.$$

If this statement is true, then $\varnothing$ satisfies the definition of beiing open. But this is a true statement as an implication $P \implies Q$ is true whenever the hypotheses, P, is false. And the hypothesis "$p \in \varnothing$ and $r > 0$" is false as "$p \in \varnothing$" is false. □

Proposition 8. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the open ball $B(x, r)$ is an open set.*

Problem 5. Prove this. *Hint:* Let $x \in B(a, r)$. Then $d(a, x) < r$. Set $\rho := r - d(a, x) > 0$ and show $B(x, \rho) \subseteq B(a, r)$ □

Proposition 9. *In the real numbers, $\mathbb{R}$, with their standard metric, the open intervals (a, b) are open.*

Problem 6. Prove this. □

Proposition 10. *Let E be a metric space. Then for any $a \in E$ and $r > 0$ the compliment, $\mathcal{C}(\overline{B}(a, r))$, of the closed ball $\overline{B}(a, r)$ is open.*

Proposition 11. *Prove this.* Hint: *If $x \in \mathcal{C}(B(a, r))$, then $d(x, a) > r$. Let $\rho := d(a, x) - r > 0$ and show $B(a, \rho) \subseteq \mathcal{C}(B(a, r))$.* □

Proposition 12. *If U and V are open subsets of E, then so are $U \cup V$ and $U \cap V$.*

Proof. Let $x \in U \cup V$. Then $x \in U$ or $x \in V$. By symmetry we can assume that $x \in U$. Then, as U is open, there is an $r > 0$ such $B(x,r) \subseteq U$. But then $B(x,r) \subseteq U \subseteq U \cup V$. As x was any point of $U \cup V$ this shows that $U \cup V$ is open.

Let $x \in U \cap V$. Then $x \in U$ and $x \in V$. As $x \in U$ there is an $r_1 > 0$ such that $B(x,r_1) \subseteq U$. Likewise there is an $r_2 > 0$ such that $B(x,r_2) \subseteq V$. Let $r = \min\{r_1, r_2\}$. Then

$$B(x,r) \subseteq B(x,r_1) \subseteq U \quad \text{and} \quad B(x,r) \subseteq B(x,r_2) \subseteq V$$

and therefore $B(x,r) \subseteq U \cap V$. As x was any point of $U \cap V$ this shows that $U \cap V$ is open. □

Proposition 13. *Let E be a metric space.*

(a) Let $\{U_i : i \in I\}$ be a (possibly infinite) collection of open subsets of E. Then the union $\bigcup_{i \in I} U_i$ is open.
(b) Let $U_1, \ldots, U_n$ be a finite collection of open subsets of E. Then the intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is open.

Problem 7. Prove this. □

Problem 8. In $\mathbb{R}$ let U_n be the open set $U_n := (-1/n, 1/n)$. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not open.

Definition 14. Let E be a metric space. Then a subset S of E is ***closed*** if and only if its compliment, $\mathcal{C}(S)$ is open. □

Because the compliment of the compliment is the original set this implies that a set, S, is open if and only if its compliment $\mathcal{C}(S)$ is closed. Likewise a set, S, is closed if and only if its compliment $\mathcal{C}(S)$ is open.

Proposition 15. *In any metric space E the sets $\varnothing$ and E are both closed.*

Proof. We have seen the sets E and $\varnothing$ are open, thus their compliments $\mathcal{C}(E) = \varnothing$ and $\mathcal{C}(\varnothing) = E$ are closed. □

Proposition 16. *If E is a metric space, $a \in E$, and $r > 0$, then the closed ball $\overline{B}(a,r)$ is closed.* □

Problem 9. Show that in $\mathbb{R}$ with its usual metric the closed intervals are closed. □

Proposition 17. *If E is a metric space, then every finite subset of E is closed.*

Problem 10. Prove this. □

Problem 11. In the real numbers show that the half open interval $[0,1)$ is neither open or closed. □

Problem 12. The integers, $\mathbb{Z}$, are a metric space with the metric $d(m,n) = |m-n|$. Note that for this metric space if $m \neq n$ that $d(m,n)$ is a nonzero positive integer and thus $d(m,n) \geq 1$. Assuming these facts prove the following

(a) Let $r = 1/2$, then for each $n \in \mathbb{Z}$ the open ball $B(n,r)$ is the one element set $B(n,r) = \{n\}$ and therefore $\{n\}$ is open.
(b) Every subset of $\mathbb{Z}$ is open. *Hint:* Let $S \subseteq \mathbb{Z}$, then $S = \bigcup_{n\in S}\{n\}$ and use Proposition 13 to conclude that S is open.
(c) Every subset of $\mathbb{Z}$ is closed. $\square$

Proposition 18. *Let E be a metric space.*

(a) Let $\{F_i : i \in I\}$ be a (possibly infinite) collection of closed subsets of E. Then the intersection $\bigcap_{i\in I} F_i$ is closed.
(b) Let $F_1,\dots,F_n$ be a finite collection of closed subsets of E, then the union $U_1 \cup \cdots \cup U_n$ is closed.

Problem 13. Prove this. *Hint:* The correct way to do this is to deduce it directly from Proposition 13. For example to show that the union of two closed sets is closed, let F_1 and F_2 be closed. Then the compliments $\mathcal{C}(F_1)$ and $\mathcal{C}(F_1)$ are open and the intersection of two open sets is open. Therefore $\mathcal{C}(F_1) \cap \mathcal{C}(F_2)$ is open and thus the compliment of this set is closed. That is

$$F_1 \cup F_2 = \mathcal{C}(\mathcal{C}(F_1) \cap \mathcal{C}(F_2))$$

is closed. $\square$

Let E be a metric space. Then a function $f\colon E \to \mathbb{R}$ is ***Lipschitz*** if and only if there is a constant $M \geq 0$ such that

$$|f(p) - f(q)| \leq Md(p,q) \quad \text{for all } p, q \in E.$$

Proposition 19. *Let E be a metric space and $f\colon E \to \mathbb{R}$ a Lipschitz function. Then for all $c \in \mathbb{R}$ the sets*

$$f^{-1}\big[(c,\infty)\big] = \{p \in E : f(p) < c\}$$
$$f^{-1}\big[(-\infty,c)\big] = \{p \in E : f(p) > c\}$$

are open and the sets

$$f^{-1}\big[[c,\infty)\big] = \{p \in E : f(p) \geq c\}$$
$$f^{-1}\big[(-\infty,c]\big] = \{p \in E : f(p) \leq c\}$$

are closed.

Half of the proof. Assume that f satisfies $|f(p) - f(q)| \leq Md(p,q)$ for $p, q \in E$. We will show that $f^{-1}\big[(-\infty,c)\big]$ is open. We need to show that for any

$q \in f^{-1}\big[(-\infty, c)\big]$ the set $f^{-1}\big[(-\infty, c)\big]$ contains an open ball about q. As $q \in f^{-1}\big[(-\infty, c)\big]$ we have $f(q) < c$. Therefore

$$r = \frac{c - f(q)}{M}$$

is positive. Let $pe \in B(q, r)$. Then

$$\begin{aligned}
f(p) &= f(q) + (f(p) - f(q)) \\
&\leq f(q) + |f(p) - f(q)| && (\text{as } (f(p) - f(q)) \leq |f(p) - f(q)|) \\
&\leq f(q) + Md(p, q) && (\text{as } f \text{ is Lipschitz}) \\
&< f(q) + Mr && (\text{as } p \in B(q, r), \text{ so } d(p, q) < r) \\
&= f(q) + M\left(\frac{c - f(q)}{M}\right) && (\text{from our definition of } r) \\
&= c.
\end{aligned}$$

Therefore if $p \in B(q, r)$ we have $f(p) < c$ and thus $B(q, r) \subseteq f^{-1}\big[(-\infty, c)\big]$. Whence $f^{-1}\big[(-\infty, c)\big]$ contains an open ball about any of its points q and therefore $f^{-1}\big[(-\infty, c)\big]$ is open.

We now show $f^{-1}\big[[c, \infty]\big] = \{p \in E : f(p) \geq c\}$ is closed. We know $f^{-1}\big[(-\infty, c)\big] = \{p \in E : f(p) < c\}$ is open. Its compliment is

$$\mathcal{C}\left(f^{-1}\big[(-\infty, c)\big]\right) = f^{-1}\big[[c, \infty)\big].$$

Therefore $f^{-1}\big[[c, \infty)\big]$ is the compliment of an open set, which means that $f^{-1}\big[[c, \infty)\big]$ is closed. $\square$

Problem 14. Prove the other half of Proposition 19, that is show $f^{-1}\big[(c, \infty)\big]$ is open and $f^{-1}\big[(-\infty, c]\big]$ is closed. $\square$

Proposition 20. *Let E be a metric space and $f \colon E \to \mathbb{R}$ a Lipschitz function. Then for any $c \in \mathbb{R}$ the set*

$$f^{-1}[c] = \{p \in E : f(p) = c\}$$

is a closed set.

Problem 15. Prove this. *Hint:* Write $f^{-1}[c]$ as the intersection of two closed sets. $\square$

We can now give some more examples of open and closed sets in the plane, $\mathbb{R}^2$ where we are using the distance function $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$. Let $\mathbf{a} = (a_1, a_2)$ be a vector in $\mathbb{R}^2$ define $f \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2 + b$$

where b is a real number. If $\mathbf{x} = (x_1, x_2)$ this can be written in vector form as

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b.$$

If $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$ then

$$\begin{aligned} |f(\mathbf{p}) - f(\mathbf{q})| &= |\mathbf{a} \cdot \mathbf{p} + b - (\mathbf{a} \cdot \mathbf{q} + b)| \\ &= |\mathbf{a} \cdot (\mathbf{p} - \mathbf{q})| \\ &\leq \|\mathbf{a}\| \|\mathbf{p} - \mathbf{q}\| && \text{(Cauchy-Schwartz)} \\ &= M d(\mathbf{p}, \mathbf{q}) \end{aligned}$$

where $M = \|\mathbf{a}\|$. Thus f is a Lipschitz function.

Consider the case where $\mathbf{a} = (1, 0)$ and $b = 0$. Then for any $c \in \mathbb{R}$. In this case $f(x_1, x_2) = x_1$, or in slightly different notation $f(x, y) = x$. Therefore Proposition 19 implies the sets

$$\{(x, y) : x > c\}, \quad \{(x, y) : x < c\}$$

are open and that

$$\{(x, y) : x \geq c\}, \quad \{(x, y) : x \leq c\}$$

are closed.

Problem 16. Let $(a, b) \in \mathbb{R}^2$ be a nonzero vector and $c \in \mathbb{R}$.

(a) Show that the line

$$\{(x, y) \in \mathbb{R}^2 : ax + by = c\}$$

is closed.

(b) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by > c\}$$

is open (call such a half plane an ***open half plane***).

(c) Show that the half plane

$$\{(x, y) \in \mathbb{R}^2 : ax + by \geq c\}$$

is closed (call such a half plane a ***closed half plane***).

(d) Show that the triangle

$$T = \{(x, y) \in \mathbb{R}^2 : x, y > 0, x + y < 0\}$$

is an open set. *Hint:* Write this as the intersection of three open half planes.

(e) Show that the triangle

$$S = \{(x, y) : x, y \geq 0, x + y \leq 0\}$$

is a closed subset of the plane. *Hint:* Write this as the interestion of three closed half planes. □

1.1. Definition of limit in a metric space and some special limits in $\mathbb{R}$.

Definition 21. Let E be a metric space and $\langle p_n \rangle_{n=1}^{\infty} = \langle p_1, p_2, p_3, \ldots \rangle$ a sequence in E. Then
$$\lim_{n\to\infty} p_n = p$$
if and only if for all $\varepsilon > 0$ there is a $N > 0$ such that
$$n > N \qquad \Longrightarrow \qquad d(p_n, p) < \varepsilon.$$
In the case we say that the sequence $\langle p_n \rangle_{n=1}^{\infty}$ ***converges*** to p. $\square$

Problem 17. Let $\lim_{n\to\infty} p_n = p$ in the metric space E. Let $a_n = p_{2n}$. Show that $\lim_{n\to\infty} a_n = p$ also holds. $\square$

Here are some examples of working with limits in $\mathbb{R}$.

Example 22. If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} 5x_n = 5x$.

Proof. Let $\varepsilon > 0$. Note that
$$|5x_n - 5x| = 5|x_n - x|.$$
From the definition of $\lim_{n\to\infty} x_n = x$ there is a $N > 0$ such that
$$n > N \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{5}.$$
But then (multiply by 5)
$$n > N \quad \text{implies} \quad |5x_n - 5x| < \varepsilon.$$
But this is just the definition of $\lim_{n\to\infty} 5x_n = 5x$. $\square$

Proposition 23. *Let $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ be sequences in $\mathbb{R}$ with*
$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y.$$
Then
$$\lim_{n\to\infty} (x_n + y_n) = x + y.$$

Proof. Let $\varepsilon > 0$. Then from the definition of $\lim_{n\to\infty} x_n = x$, there is a $N_1 > 0$ such that
$$n > N_1 \quad \text{implies} \quad |x - x_n| < \frac{\varepsilon}{2}.$$
Likewise $\lim_{n\to\infty} y_n = y$ implies there is a $N_2 > 0$ such that
$$n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{2}.$$
Set
$$N = \max\{N_1, N_2\}.$$
If $n > N$, then $n > N_1$ and $n > N_2$ and thus
$$|(x+y) - (x_n + y_n)| \leq |x - x_n| + |y - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is

$$n > N \quad \text{implies} \quad |(x+y) - (x_n + y_n)| < \varepsilon$$

which is exactly the definition of $\lim_{n\to\infty}(x_n + y_n) = x + y$. $\square$

Proposition 24. *Let $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ be sequences in $\mathbb{R}$ with*

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y.$$

Then for any real numbers a and b

$$\lim_{n\to\infty}(ax_n + by_n) = ax + by.$$

Problem 18. Prove this. $\square$

Proposition 25. *If $\langle x_n \rangle$ be a convergent sequence in $\mathbb{R}$. Then $\langle x_n \rangle$ is bounded. That is a constant M such that $|x_n| \le M$ for all M.* $\square$

Problem 19. Prove this. *Hint:* Let $\varepsilon = 1$. Then there is a N such that

$$n > N \quad \text{implies} \quad |x - x_n| < 1.$$

Therefore is $n > N$ we have

$$|x_n| = |x + (x_n - x)| \le |x| + |x - x_n| < |x| + 1.$$

This bounds all the terms with $n > N$. Let

$$M = \max\left\{|x| + 1, |x_1|, |x_2|, \ldots, |x_N|\right\}.$$

Then $|x_n| \le M$ for all n, which shows that the sequence is bounded. $\square$

Theorem 26. *Let*

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y$$

in $\mathbb{R}$. Then

$$\lim_{n\to\infty} x_n y_n = xy.$$

Problem 20. Prove this. *Hint:* Start with

> **Scratch work that the no one else needs to see:** Our goal is to make $|x_n y_n - xy|$ small. We compute

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \qquad \text{(Adding and subtracting trick.)} \\ &\le |x_n y_n - x y_n| + |x y_n - xy| \\ &= |x_n - x||y_n| + |x||y_n - y| \end{aligned}$$

> The factors $|x_n - x|$ and $|y_n - y|$ are both good in that we can make them small. The factor $|x|$ is independent of n and thus is not a problem. The sequence $\langle y_n \rangle_{n=1}^{\infty}$ is convergent and thus bounded, so we bound the factor $|y_n|$. We now return to our regularly scheduled proof.

Let $\varepsilon > 0$. The sequence $\langle y_n \rangle_{n=1}^{\infty}$ is convergent thus it is bounded. Therefore there is an M so that

$$|y_n| \leq M \quad \text{for all} \quad n.$$

As $\lim_{n\to\infty} x_n = x$ there There is a $N_2 > 0$ such that

$$n > N_2 \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{2(M+1)}$$

and as $\lim_{n\to\infty} y_n = y$ there is a $N_2 > 0$ such that

$$n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{2(|x|+1)}.$$

Now let $N = \max\{N_1, N_2\}$ and use the calculation from our scratch work to show

$$n > N \quad \text{implies} \quad |x_n y_n - xy| < \varepsilon$$

which completes the proof. □

Corollary 27. *If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = x$, then*

$$\lim_{n\to\infty} x_n^2 = x^2.$$

Proof. Use $\langle x_n \rangle = \langle y_n \rangle$ in Theorem 26. □

Proposition 28. *Let k be a positive integer and $\langle p_n \rangle$ a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} p_n = p$. Then*

$$\lim_{n\to\infty} p_n^k = p^k$$

Problem 21. Prove this. *Hint:* What is probably the easiest why is to use induction. □

Problem 22. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the quadratic polynomial $f(x) = ax^2+bx+c$ where a, b, c are constants. Let $\langle p_n \rangle$ be a convergent sequence, $\lim_{n\to\infty} p_n = p$. Then

$$\lim_{n\to\infty} f(p_n) = f(p).$$

□

Lemma 29. *Let $a \in \mathbb{R}$ with $a \neq 0$. Let $|x| > \dfrac{|a|}{2}$. Then*

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2},$$

$$\frac{1}{|x|} < \frac{2}{|a|},$$

and

$$\left| \frac{1}{x} - \frac{1}{a} \right| \leq \frac{2|x-a|}{|a|^2}.$$

Problem 23. Prove this. □

Proposition 30. *Let $\langle x_n \rangle$ be a sequence with $\lim_{n\to\infty} = a$ and $a \neq 0$. Then*

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{a}.$$

Problem 24. Prove this. *Hint:* First note there is a N_1 such that

$$n > N_1 \quad \text{implies} \quad |x_n - a| < \frac{|a|}{2}.$$

Now let $\varepsilon > 0$ There is also a N_2 such that

$$n > N_2 \quad \text{implies} \quad |x_n - a| < \frac{|a|^2}{2}\varepsilon.$$

Now let $N = \max\{N_1, N_2\}$ and use the last lemma to show that

$$n > N \quad \text{implies} \quad \left|\frac{1}{x_n} - \frac{1}{a}\right| < \varepsilon.$$

□

Proposition 31. *Let E be a metric space and $f\colon E \to \mathbb{R}$ be a Lipschitz map. (That is there is a constant M such that for all $p, q \in E$ the inequality $|f(p) - f(q)| \le Md(p, q)$ holds.) Let $\langle p_n \rangle$ be a sequence in E with $\lim_{n\to\infty} p_n = p$ where $p \in E$. Then*

$$\lim_{n\to\infty} f(p_n) = f(p).$$

Problem 25. Prove this. □

1.1.1. *Limits and rational functions.* We show that limits play will with polynomials and rational functions. Recall a polynomial, $f(x)$, is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_0, a_1, \dots, a_n$ are real numbers. If $a_n \ne 0$, then the ***degree*** of $f(x)$ is $\deg f(x) = n$. The following is trivial, but useful in doing induction proofs involving polynomials.

Proposition 32. *Let $f(x)$ be a polynomial of degree $n \ge 1$. There there is a polynomial $g(x)$ of degree $(n-1)$ and a constant c such that*

$$f(x) = xg(x) + c.$$

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ Then factor out x from all the non-constant terms:

$$\begin{aligned} f(x) &= x\big(a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1\big) + a_0 \\ &= xg(x) + c \end{aligned}$$

where $g(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_2 x + a_1$ and $c = a_0$. □

Theorem 33. *Let $\langle p_k \rangle_{k=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$, say*

$$\lim_{k\to\infty} p_k = p.$$

Then for any polynomial $f(x)$

$$\lim_{k\to\infty} f(p_k) = f(p).$$

Problem 26. Prove this. *Hint:* Use induction on $\deg f(x)$. The base case is $\deg f(x) = 0$, that is $f(x) = a_0$ is a constant. This case $f(p_k) = a_0$ for all n and thus $\langle f(p_k)\rangle_{k=1}^{\infty}$ is a sequence of constants and so the result it true in this case. The case of $\deg f(x) = 1$ is also easy. In this case $f(x) = a_1x + a_0$. And from some some of our earlier results we have

$$\begin{aligned} \lim_{k\to\infty} f(p_k) &= \lim_{k\to\infty} (a_1 p_k + a_0) \\ &= a_1 p + a_0 \\ &= f(p). \end{aligned}$$

Now do the induction step. Assume we know the result is true for polynomials of degree $(n-1)$ and let $f(x)$ be a polynomial of degree n. By Proposition 32 write

$$f(x) = xg(x) + c$$

where $g(x)$ is a polynomial of degree $(n-1)$ and c is a constant. Then by the induction hypothesis we have

$$\lim_{k\to\infty} g(p_k) = g(p).$$

Now use our earlier results about limits of products and sums to finish the induction step and complete the proof. $\square$

Lemma 34. *Let $g(x)$ be a polynomial and $p \in \mathbb{R}$ a point with $g(p) \neq 0$. Let $\langle p_k\rangle_{k=1}^{\infty}$ a sequence with*

$$\lim_{k\to\infty} p_k = p.$$

Then $g(p_k) \neq 0$ for all but at most finitely many k's, and this the sequence

$$\left\langle \frac{1}{g(p_k)} \right\rangle_{k=1}^{\infty}$$

is defined for all but finitely many values of k and

$$\lim_{k\to\infty} \frac{1}{g(p_k)} = \frac{1}{g(p)}.$$

Problem 27. Prove this. *Hint:* First show that $g(p_k) \neq 0$ for all but finitely may k. One way to do this is to let $\varepsilon = |g(p)|/2$ in the definition of a limit. We know from Theorem 33 that

$$\lim_{k\to\infty} g(p_k) = g(p).$$

Let $\varepsilon = |g(p)|/2$ in the definition of $\lim_{k\to\infty} g(p_k) = g(p)$. to find a $N > 0$ such that

$$n > N \quad \text{implies} \quad |g(p_k) - g(p)| < |g(p)|/2.$$

Use this to show

$$n > N \quad \text{implies} \quad |g(k)| > |g(p)|/2$$

and therefore

$$n > N \quad \text{implies} \quad g(p_k) \neq 0.$$

You should now be able to use Proposition 30 to finish the proof.

A ***rational function*** is a function

$$h(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomials, $g(x)$ is not identically zero and the domain of $h(x)$ is the set of points where $g(x) \neq 0$.

Theorem 35. *Let $\langle p_k \rangle_{k=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$,*

$$\lim_{k\to\infty} p_k = p.$$

Let

$$h(x) = \frac{f(x)}{g(x)}$$

be a rational function with $g(p) \neq 0$. Then

$$\lim_{k\to\infty} h(p_k) = h(p).$$

Problem 28. Prove this by putting together Lemma 34 and Proposition 30. □

We can now do some limits you recall from calculus. For example let us compute

$$\lim_{n\to\infty} \frac{3n^2 - 3n + 7}{4n^2 + 6}.$$

Divide the numerator and denominator of the fraction in the limit to get

$$\frac{3n^2 - 3n + 7}{4n^2 + 6} = \frac{3 - 3(1/n) + 7(1/n)^2}{4 + 6/(1/n)^2} = \frac{f(1/n)}{g(1/n)}$$

where $f(x)$ and $g(x)$ are the polynomials

$$f(x) = 3 - 3x + 7x^2 \quad g(x) = 4 + 6x^2.$$

And have seen that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore Theorem 35 gives

$$\lim_{n\to\infty} \frac{3n^2 - 3n + 7}{4n^2 + 6} = \lim_{n\to\infty} \frac{f(1/n)}{g(1/n)} = \frac{f(0)}{g(0)} = \frac{3}{4}.$$

Problem 29. Find the following limits and give a justification (which can just be quoting the right proposition or theorem) for your answer.

(a) $\displaystyle\lim_{n\to\infty} \frac{4n^3 + 5n_6}{7n^3 - 8n + 7}$

(b) $\displaystyle\lim_{n\to\infty} \frac{-3n^2 + 1}{7n^5 - 19}$ □

Problem 30. Let $0 < a < 1$. Give a ε, N proof that

$$\lim_{n\to\infty} a^k = 0.$$

Hint: Let $\varepsilon > 0$. We have proven that for $a \in (0,1)$ there is a natural number N with $a^N < \varepsilon$. □

Problem 31. Let $a > 0$ and $x \geq a/4$. Show

$$|\sqrt{x} - \sqrt{a}| \leq \frac{2|x-a|}{3\sqrt{a}}.$$

□

Proposition 36. *If $\langle a_n \rangle_{n=1}^{\infty}$ is a convergent sequence in $\mathbb{R}$, say*

$$\lim_{n\to\infty} a_n = a$$

with $a > 0$. Then

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}.$$

Problem 32. Give a N, ε proof of this. □

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