Qualifying Exam in Analysis

August 1997

The real numbers will be denoted by $\mathbb{R}$, and the complex numbers by $\mathbb{C}$. Lebesgue measure, defined on the set of measurable subsets of $\mathbb{R}$, will be denoted by m. Lebesgue outer measure, defined on the set of all subsets of the real numbers, will be denoted by m^*. The integral $\int_{[a,b]} f\, dm$ will also be written as $\int_a^b f(x)\, dx$.

Instructions. Please work each problem on separate sheets of paper and only use one side of the sheet.

1. (10 POINTS) Let $E, F \subset \mathbb{R}$ such that E and F are closed and E is bounded. Prove that $E + F = \{x + y : x \in E, y \in F\}$ is closed.

2. (10 POINTS) Let $\langle f_n \rangle$ be a sequence in $L^1([0,1])$ such that $\sum_{n=1}^{\infty} \int_{[0,1]} |f_n|\, dm < \infty$. Then prove

(a) $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ for almost all $x \in [0,1]$.

(b) If $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then $f \in L^1([0,1])$.

(c) $\sum_{n=1}^{N} f_n \to f$ in $L^1([0,1])$.

3. (10 POINTS) Let $A \subset [0,1]$. Prove A is measurable if and only if

$$m^*(A) = \sup\{m(F) : F \text{ is closed and } F \subseteq A\}.$$

4. (10 POINTS) Let $F\colon [a,b] \to \mathbb{R}$ be a monotone increasing function. Then show

(a) If $F(x) - F(a) = \int_0^x F'(t)\, dt$ for all $x \in [a,b]$ then F is absolutely continuous.

(b) If $F(b) - F(a) = \int_a^b F'(t)\, dt$ then F is absolutely continuous on $[a,b]$.

5. (10 POINTS) Let $f : \mathbb{R} \to [0,\infty)$ be measurable and set $F(t) = m\{x : f(x) > t\}$. Then show

$$\int_0^{\infty} F(t)\, dt = \int_{\mathbb{R}} f\, dm$$

6. (10 POINTS) Let $C([a,b])$ be the set of continuous real valued functions on $[a,b]$. Then show $C([a,b])$ is dense in $L^1([a,b])$, that is for every $f \in L^1(10,1])$ and $\varepsilon > 0$ there is an $h \in C([a,b])$ so that $\int_{[a,b]} |f - h|\, dm < \varepsilon$.

7. (10 POINTS) Prove that a bounded entire function is constant and use this to prove the fundamental theorem of algebra.

8. (20 POINTS) True or False. For the following state if they are true or false and give a short reason why.

(a) If $\langle f_n \rangle$ is a sequence of measurable functions on $[0,1]$ and $\int_{[0,1]} |f_n|\, dm \to 0$ then $f_n \to 0$ in measure.

(b) If $\langle f_n \rangle$ is a sequence of measurable real valued functions in $\mathbb{R}$ and $f_n \searrow 0$ then $\int f_n\, dm \searrow 0$.

(c) Let F be a function of bounded variation on $[a, b]$ such that $F'(x) = 0$ almost everywhere. Then F is constant.

(d) If f is an analytic function the annulus $U = \{z \in \mathbb{C} : 1 < |z| < 4\}$, then f has an antiderivative in U

(e) If $f \in L^2(\mathbb{R})$ then for $a \geq 0$

$$\left|\int_0^a tf(t)\,dt\right| \leq \frac{a^{\frac{3}{2}}}{\sqrt{3}}\|f\|_2.$$

9. (10 POINTS) Let γ be the curve in Figure 1. Then compute

$$\int_\gamma \frac{1+z+z^3}{z(z-1)(z-i)}\,dz.$$

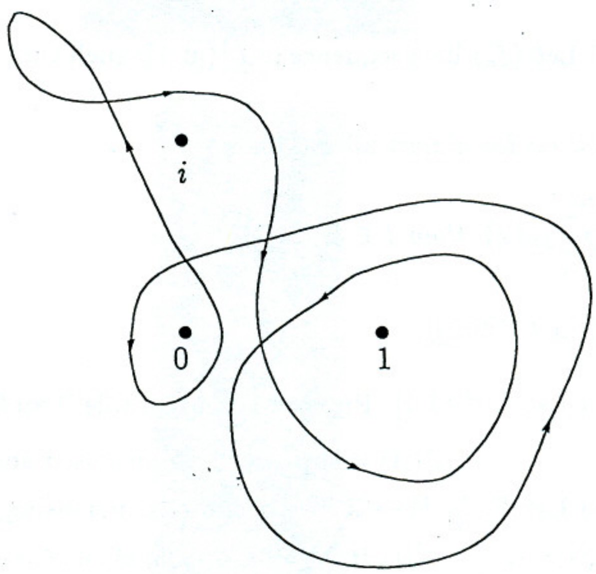

FIGURE 1