Mathematics 551 Homework, January 12, 2020

We start by reviewing some vector algebra. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be vectors in $\mathbb{R}^2$ and c a scalar. Then the sum of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2)$$

and the product of $\mathbf{a}$ by the c is

$$c\mathbf{a} = (ca_1, cb_1).$$

It is common to use the notation

$$\mathbf{i} = (1, 0), \qquad \mathbf{j} = (0, 1).$$

With this notation we can write $\mathbf{a}$ as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$

The ***inner product*** of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

Then

$$\mathbf{a} \cdot \mathbf{a} = (a_1)^2 + (a_2)^2$$

which is the square of the length of $\mathbf{a}$. We use the notation

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

for the ***length*** of $\mathbf{a}$.

If c_1 and c_2 are scalars and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors are scalars then the inner product has the following properties:

- $\mathbf{b} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
- $(c_1\mathbf{u} + c_2\mathbf{v}) \cdot \mathbf{w} = c_1\mathbf{u} \cdot \mathbf{w} + c_2\mathbf{v} \cdot \mathbf{w}$.

Consequences of these that will come up are

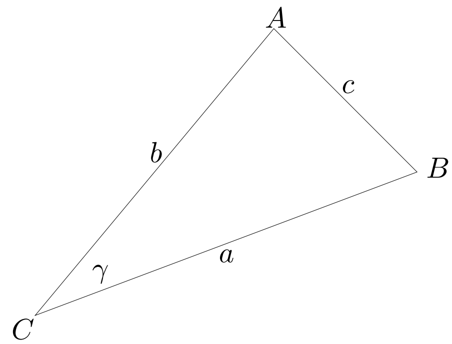
$$\begin{aligned}
\|\mathbf{a} + \mathbf{b}\| &= \|\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{a}\|^2 \\
\|\mathbf{a} - \mathbf{b}\| &= \|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} - \|\mathbf{a}\|^2.
\end{aligned}$$

Problem 1. Prove these formulas. □

A very important property of the inner product is given by

Theorem 1. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are nonzero vectors and* θ *is the angle between them, then*

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos(\theta).$$



Problem 2. In the triangle shown a, b, and c are the side lengths of $\triangle ABC$ and γ is the angle between $\overrightarrow{CB}$ and and $\overrightarrow{CA}$. Use Theorem 1 to show

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

Hint: Let $\mathbf{u} = \overrightarrow{CB}$, $\mathbf{v} = \overrightarrow{CA}$ and $\mathbf{w} = \overrightarrow{AB}$. Then $\mathbf{w} = \mathbf{u} - \mathbf{v}$ and therefore $\|\mathbf{w}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$. □

A Corollary of Theorem 1 is that non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Problem 3. Show that for any vectors $\mathbf{a}$ and $\mathbf{b}$ that $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular if and only if $\|\mathbf{a}\| = \|\mathbf{b}\|$. □

Problem 4. Define a map $J \colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$J(x, y) = (-y, x).$$

Show for all non-zero vectors $\mathbf{v}$ that

(a) $\|J\mathbf{v}\| = \|\mathbf{v}\|$
(b) $\mathbf{v}$ and $J\mathbf{v}$ are always perpendicular. □