Mathematics 739 Homework 2: Some examples of vector bundles.

1. Divisors and line bundles.

In this section M will be a compact complex manifold. By a ***hypersurface*** in M we mean a closed subset of M such that for each $p \in V$ there is $f \in \mathcal{O}_p$ such that f does not vanish identically and near p the set V is given by $\{f = 0\}$. To be more precise, since f is the germ of a function, this can be restated by saying that for all $p \in V$ there is a connected open set U with $p \in U$ and a holomorphic function, not identically zero, $f \colon U \to \mathbb{C}$ such that $V \cap U = \{x \in U : f(x) = 0\}$. The hypersurface is ***irreducible*** if and only if it is not the union of two distinct hypersurfaces. The ***divisor group*** of M is the set of formal sums

$$\sum_V a_V V$$

where the sum is over all irreducible hypersurfaces of M, the a_V'x are integers and all but finitely many of the a_V'x are zero. If V is an irreducible and $f \colon M \to \mathbb{C}$ is a meromorphic function then the ***order*** of f along V is well defined.

If V is an irreducible hypersurface and $f \colon M \to \mathbb{C}$ is meromorphic, then we can define the ***order***, $\operatorname{ord}_V(f)$ along V. To be just a little bit more explicit, if $p \in V$, then near p we can write

$$f = \frac{g}{h}$$

where $g, h \in \mathcal{O}_p$ and g and h are relatively prime in $\mathcal{O}_p$. (Recall that $\mathcal{O}_p$ is a UFD.) Then let $\operatorname{ord}_V(g)$ is the order that g vanishes along V and $\operatorname{ord}_V(h)$ the order that h vanishes along V. Set

$$\operatorname{ord}_V(f) = \operatorname{ord}_V(g) - \operatorname{ord}_V(h).$$

Problem 1. Review the definitions involved here and convince yourself this is all well defined. $\square$

Now given a meromorphic function f on M we can define a divisor

$$(f) = \sum_V \operatorname{ord}_V(f) V.$$

Proposition 1. *If f, g are meromorphic on M show that*

$$(fg) = (f) + (g)$$

and therefore $\{(f) : f$ is meromorphic on M is a subgroup of $\operatorname{Div}(M)$. $\square$

Problem 2. Prove this. $\square$

Given a divisor D on M near each point $p \in M$ there is an open neighborhood, U, of p and a meromorphic function f on U such that

$$D \cap U = (f) \cap U.$$

(If $p \notin D$, then choose U with $D \cap U = \varnothing$ and f to be nonvanishing on U.) If f_1 and f_2 are both meromorphic on U and define $D \cap U$, then f_1/f_2 is holomorphic and nonvanishing on U. Therefore given a divisor D we can cover M with open set $\{U_\alpha\}_{\alpha \in A}$ such that on each U_α there is a meromorphic function f_α that defines $D \cap U_\alpha$. On each overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$ define nonvanishing holomorphic functions $g_{\alpha\beta}$ by

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}.$$

Proposition 2. *This data, that is the cover $\{U_\alpha\}$, the meromorphic functions $\{f_\alpha\}$, and the functions $\{g_{\alpha\beta}\}$ define a holomorphic line bundle, L_D, over M and that this bundle has meromorphic section which has D as its divisor.*

Problem 3. Give a precise statement of the last proposition (this includes defining what it means for a section of a holomorphic vector bundle to be meromorphic) and prove it. □

Proposition 3. *If $D_1, D_2 \in \mathrm{Div}(M)$, then $L_{D_1+D_2} = L_{D_1} \otimes L_{D_2}$.*

Problem 4. Prove this. □

Proposition 4. *Let $D \in \mathrm{Div}(M)$. Then L_D is the trivial bundle (that is the product bundle $M \times \mathbb{C}$) if and only if $D = (f)$ for some meromorphic function f on M.*

Problem 5. Prove this. □

Let $\mathrm{Pic}(M)$ be the set of isomorphism classes of holomorphic line bundles over M. Make this into a group using tensor product for the group operation. This is the ***Picard group*** of M.

Proposition 5. *Let M be a compact complex manifold such that every holomorphic line bundle has a nontrivial meromorphic section. Then there is a group isomorphism*

$$\mathrm{Pic}(M) \approx \mathrm{Div}(M)/\{(f) : f \textit{ is meromorphic on } M.\}$$

Problem 6. Prove this. □

There are lots of examples of compact complex manifolds where every holomorphic line bundle has a nontrivial section. For example all some projective varieties have this property. In particular all compact Riemann surfaces have this property.

On the other hand there are also lots of examples of compact complex varieties that do not have this property. Here is an example. Let $n \geq 2$ and let $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ and $|\lambda| \neq 1$. On $C^n \setminus \{0\}$ define an equivalence by

$$v \sim w \qquad \Longleftrightarrow \qquad w = \lambda^k v \text{ for some } k \in \mathbb{Z}.$$

Then

$$M = \mathbb{C}^n \setminus \{0\}/\sim$$

is a compact complex manifold called a ***Hopf manifold***.

Problem 7. Prove that M is a complex manifold and that it is diffeomorphic to $S^{2n-1} \times S^1$. □

It is known that there are no nonzero divisors on the Hopf surface. That is $\operatorname{Div}(M) = \{0\}$. But in this case $\operatorname{Pic}(M) = \mathbb{C}^*$. Thus any nontrivial holomorphic vector bundle over M has no nontrivial meromorphic section.

2. Using line bundles to embed complex manifolds into projective spaces.

Let M be a compact complex manifold. Let $pE \to M$ be a holomorphic vector bundle over M. Let $\Gamma(M, E)$ be the vector space of all holomorphic sections of M. For many vector bundles this will just be the trivial vector space $\{0\}$. The following is known and follows from some facts about partial differential equations.

Proposition 6. *If $p\colon E \to M$ is a holomorphic vector bundle over a compact manifold, then the space of holomorphic sections $\Gamma(M, E)$ is finite dimensional.* □

Proposition 7. *Let $p\colon L \to M$ be a holomorphic line bundle over a compact complex manifold M. Assume that $\dim_{\mathbb{C}}(\Gamma(M, L)) > 1$ and that for each $p \in M$ there is a $s \in \Gamma(M, L)$ with $s(p) \neq 0$. Let $\mathbb{P}^*(\Gamma(M, L))$ be the projective space of all codimension one linear subspaces of $\Gamma(M, L)$. Define a map $\phi\colon M \to \mathbb{P}^*(\Gamma(M, L))$ by*

$$\phi(p) = \{s \in \mathbb{P}^*(\Gamma(M, L)) : s(p) = 0\}.$$

Then ϕ is holomorphic.

Problem 8. Prove this. □

We will see later that this method can be used to show that some complex manifolds can be embedded in $\mathbb{CP}^N$.

3. Pull back bundles.

Let $p\colon E \to B$ be a fiber bundle with fiber F. Let $f\colon X \to B$ be a continuous map. Then the ***pull back*** of E by f is the unique (up to isomorphism) bundle $\pi\colon f^*E \to X$ such that there is a continuous map $\phi\colon f^*E \to E$ such that

$$\begin{array}{ccc} f^*E & \xrightarrow{\phi} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B \end{array}$$

commutes and for each $x \in X$ we have that $\phi\big|_{f^*E_x} \to E_{f(x)}$ is a homeomorphism. If E is a vector bundle, then we require that each $\phi\big|_{f^*E_x}$ be a linear isomorphism.

Proposition 8. *Let $p\colon E \to B$ be a fiber bundle and assume that $\mathcal{U} = \{U_\alpha\}$ and $\{g_{\alpha\beta}$ transition functions that define E. Let $f\colon X \to B$ be a map and let $V_\alpha = f^{-1}[U_\alpha]$. On the overlaps $V_{\alpha\beta}$ define $g^*_{\alpha\beta}(x) = g_{\alpha\beta}(f(x))$. Then $\{g^*_{\alpha\beta}\}$ is a set of transition functions for the bundles f^*E. If $p\colon E \to B$ is a holomorphic bundle, then so is f^*E.*

Problem 9*.* Prove this. □

Proposition 9. *Let $f\colon M \to N$ be a holomorphic map between complex manifolds such that for each $x \in M$, the derivative $f'(x)\colon T(M)_x \to T(N)_{f(x)}$ is a linear isomorphism. Then $f^*(T(N)) = T(M)$.*

Problem 10*.* Prove this. □

Now let us look at the example of Riemann surfaces. That is compact complex manifolds of (complex) dimension one. If M is such a manifold, then by definition there is an open cover $\{U_\alpha\}$ and maps $z_\alpha\colon U_\alpha \to \mathbb{C}$ (the local coordinates) such that on each overlap

$$z_\alpha \circ z_\beta^{-1}\colon z_\beta[U_{\alpha\beta}] \to z_\alpha[U_{\alpha\beta}]$$

is both a diffeomorphism and holomorphic. On each U_α the vector field

$$\frac{d}{dz_\alpha}$$

is a nonvanishing section of the tangent bundle $T(M)$. On the overlaps these are related by

$$\frac{d}{dz_\beta} = \left(\frac{dz_\alpha}{dz_\beta}\right)\frac{d}{dz_\alpha}.$$

Thus the transition functions for the tangent bundle are

$$g_{\alpha\beta} = \frac{dz_\alpha}{dz_\beta}.$$

Likewise the one forms dz_α on U_α are nonvanishing sections of the cotangent bundle $T^*(M)$ and these are related by

$$dz_\beta = \left(\frac{dz_\beta}{dz_\alpha}\right) dz_\alpha$$

and so the transition functions for the cotangent bundle are

$$h_{\alpha\beta} = \frac{dz_\beta}{dz_\alpha}.$$

We now recall a bit of complex analysis.

Proposition 10. *Let U be a neighborhood of 0 in $\mathbb{C}$ and $f\colon U \to \mathbb{C}$ a holomorphic map with $f(0) = 0$. Assuming that f is not identically zero, then is a unique integer $k \geq 1$ (the order of the zero) and an analytic function defined in a neighborhood $V \subseteq U$ of 0 such that in V*

$$f(z) = g(z)^k, \quad g(0) = 0, \quad g'(0) \neq 0.$$

Problem 11. Prove this. *Hint:* If k is the order of the zero of f at 0, then we can write

$$f(z) = z^k h(z)$$

where $h(z)$ is holomorphic near 0 and $h(0) \neq 0$. There there is a neighborhood V of 0 where h has a k-th root. That is $h(z) = g_0(z)^k$ for some holomorphic function g_0. Now let $g(z) := z g_0(z)$. □

If M is a Riemann surface and $p \in M$, then a local coordinate $z\colon U \to \mathbb{C}$ (where U is an open subset of M) is ***centered at*** p if $p \in U$ and $z(p) = 0$.

Proposition 11. *Let $f\colon M \to N$ be a holomorphic map between compact Riemann surfaces and $p \in M$. Then there is a local coordinate z on M defined on a open neighborhood U of p and centered at p and a local coordinate w defined on an open neighborhood V of $f(p)$ and centered at $f(p)$ and a positive integer k such that*

$$f[U] = V, \quad \text{and} \quad w(f(x)) = z(x)^k.$$

*The integer k is uniquely determined by f and p. The number $k-1$ is the **ramification** of f at p. The ramification is 0 except at finitely many points.*

Problem 12. Prove this by reducing it to Proposition 10. □

If $f\colon M \to N$ is holomorphic map between Riemann surfaces then the ***ramification divisor*** of f is

$$R_f = \sum_{p \in M} (\text{ramification of } f \text{ at } p) p$$

In the case of a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$, then ramification of f at a point z_0 is just the order of the zero of the derivative, f', of f at z_0.

3.1. **Computing the ramification divisor of a rational function.** Let use consider the case of $M = \mathbb{CP}^1$. Use homogeneous coordinates $[z_0 : z_1]$ on M and let

$$U_0 := \{[z_0 : z_1] \in M : z_1 \neq 0\}$$
$$U_1 := \{[z_0 : z_1] \in M : z_0 \neq 0\}.$$

Then $\{U_0, U_1\}$ is an open cover of M and if we let $z\colon U_0 \to \mathbb{C}$ be

$$\phi([z_0 : z_1]) = \frac{z_0}{z_1}$$

and

$$\psi([z_0 : z_1]) = \frac{z_1}{z_0}$$

then $\{(U_0, \phi), (U_1, \psi)\}$ is an atlas for $M = \mathbb{CP}^1$.

Let us look at example of a particular function $f \colon \mathbb{CP}^1 \to \mathbb{CP}^1$. Let $f \colon \mathbb{C} \to \mathbb{C}$ be the polynomial

$$f(z) = z^2 - 3z + 2 = (z - 1)(z - 2).$$

In the set U_0 this given by

$$f([z : 1]) = [z^2 - 3z + 2 : 1].$$

The extension as a map form $\mathbb{CP}^1 \to \mathbb{CP}^1$, which we still denote by f, is given by

$$\begin{aligned} f([z : w]) &= f\left(\left[\frac{z}{w} : 1\right]\right) \\ &= \left[\frac{z^2}{w^2} - 3\frac{z}{w} + 2 : 1\right] \\ &= [z^2 - 3zw + 2w^2 : w^2]. \end{aligned}$$

In the open set U_0 we have

$$f([z, 1]) = [z^2 - 3z + 1 : 1]$$

and so in the coordinate path U_0 we can just view f as the polynomial we started with, that is $f(z) = z^2 - 3z + 2$. The derivative is

$$f'(z) = 2z - 3$$

and so in the coordinate patch U_0 we just have the one ramification point $z = 3/2$ and it has ramification 1.

In the coordinate path U_0 we have that f is given by

$$\begin{aligned} f([1 : w]) &= [1 - 3w + 2w^2 : w^2] \\ &= \left[1 : \frac{w^2}{1 - 3w + 2w^2}\right]. \end{aligned}$$

So in the coordinate patch U_1 we are looking at the rational function

$$p(w) = \frac{w^2}{1 - 3w + 2w^2}.$$

The derivative is

$$p'(w) = \frac{w(2 - 3w)}{(1 - 3w + w^2)^2}$$

which has the two zeros

$$w = 0, \frac{2}{3}.$$

Both of these are simple zeros and so have ramification 1. The point $w = 2/3$ is

$$[1 : 2/3] = [3/2 : 1]$$

and therefore corresponds to the point $z = 3/2$, a point we had already found. Therefore the ramification divisor of f is

$$R_f = [3 : 2] + [1 : 0].$$