MAPS THAT MUST BE AFFINE OR CONJUGATE AFFINE: A PROBLEM OF VLADIMIR ARNOLD

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Abstract. A k-flat in a vector space is a k dimensional affine subspace. Our basic result is that an injection $T\colon \mathbb{C}^n \to \mathbb{C}^n$ such that for some $k \in \{1, 2, , \ldots, n-1\}$ T maps all k-flats to flats of $\mathbb{C}^n$ and is either continuous at a point or Lebesgue measurable, is either an affine map or a conjugate affine map. An analogous result is proven for injections of the complex projective spaces. In the case of continuity at a point this is generalized in several directions, the main one being that the complex numbers can replaced by $\mathbb{F}[\sqrt{-1}]$ where $\mathbb{F}$ is an Archimedean ordered field. This extends work of Gorinov on a problem of V. I. Arnold.

1. Introduction

In his book of problems [1] V. I. Arnold asks if a homeomorphism, or more generally a bijection, $\mathbb{C}^n \to \mathbb{C}^n$ that sends affine subspaces to affine subspaces is necessarily either an affine mapping, or the complex conjugate of such a map, with a similar question being asked about homeomorphisms of the complex projective space $\mathbb{CP}^n$ and posing analogous questions about the quaternionic affine and projective spaces $\mathbb{H}^n$ and $\mathbb{HP}^n$. (Cf. [1] Problems 2000-8 (p. 134), 2002-9, 2002-10 (pp. 144–145) and the comments on these problems p. 614 and p. 627). For homeomorphisms A. G. Gorinov, [8], points out that in the case of $\mathbb{C}^n$ and $\mathbb{CP}^n$ the answer is affirmative and is a direct consequence of the fundamental theorem of projective geometry. He also shows the answer to a generalization of this question affirmative in the case of the quaternionic spaces.

In the cases of $\mathbb{C}^n$ and $\mathbb{CP}^n$ we extend Gorinov's results in several directions. In the case of $\mathbb{C}^n$ the global continuity condition can be replaced either by the condition the map is continuous at least at one point or the condition the map is Lebesgue measurable. Second the condition that all affine subspaces are mapped to affine subspaces can be weakened to the condition that affine subspaces of some fixed dimension are mapped to affine subspaces (not necessarily of the same dimension). Also in this setting the map only needs to be an injection or surjection rather than a bijection. In the case where it is assumed that the map is only continuous at a single point, the complex numbers can be replaced by a field of the form $\mathbb{F}[\sqrt{-1}]$ where $\mathbb{F}$ is a Archimedean ordered field. There are analogous results for $\mathbb{CP}^n$. Finally in Section 3.2 we prove as lemmas for our main results some extensions of the Fundamental Theorems of Affine and Projective Geometry which may be of independent interest.

2. Definitions and Statement of Main Results.

Let $\mathbb{D}$ be a division ring (all our division rings are associative). Let $\mathbb{D}^n$ denote the (left) vector space of all n-tuples over $\mathbb{D}$ and $\mathbb{DP}^n$ projective space of dimension

n over $\mathbb{D}$. (The points of $\mathbb{DP}^n$ are the one-dimensional subspaces of $\mathbb{D}^{n+1}$.) A ***k-flat in*** $\mathbb{D}^n$ is a k-dimensional affine subspace of $\mathbb{D}^n$ (that is, a translate of a k-dimensional linear subspace of $\mathbb{D}^n$). Note that in the affine setting, the empty set is taken as a -1-flat. A ***k-flat in*** $\mathbb{DP}^n$ is a k-dimensional projective subspace of $\mathbb{FP}^n$.

Let σ be an automorphism of the division ring $\mathbb{D}$. A map $T\colon \mathbb{D}^n \to \mathbb{D}^n$ is ***σ-semilinear*** if and only if $T(x+y) = T(x) + T(y)$ and $T(cx) = \sigma(c)T(x)$ for all $x, y \in \mathbb{D}^n$ and $c \in \mathbb{D}$. A map $f\colon \mathbb{D}^n \to \mathbb{D}^n$ is ***σ-semiaffine*** if and only if it is of the form $f(x) = T(x) + b$ where $b \in \mathbb{D}^n$ and T is σ-semilinear. A map is ***semilinear*** (respectively ***semiaffine***) if and only if it is σ-semilinear (respectively σ-semiaffine) for some automorphism σ of $\mathbb{D}$. When σ is the identity map these are linear and affine maps.

These notions also apply to projective spaces. For a nonzero vector $v \in \mathbb{D}^{n+1}$ let $\langle v \rangle$ be the one dimensional subspace space spanned by v. Then $\langle v \rangle \in \mathbb{DP}^n$. If $A\colon \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ is a nonsingular linear map then the map $T\colon \mathbb{DP}^n \to \mathbb{DP}^n$ defined by

$$T\langle v \rangle = \langle Av \rangle$$

will be called a ***linear map*** on $\mathbb{DP}^n$. And if σ is an automorphism of $\mathbb{D}$ and A is σ-linear, the map T just defined is ***σ-linear***. And T is ***semilinear*** if it is σ-linear for some σ.

In the case $\mathbb{D} = \mathbb{C}$ and σ is complex conjugation, we refer to ***conjugate linear*** and ***conjugate affine*** maps. In the case $\mathbb{D} = \mathbb{F} = (\mathbb{F}, +, \cdot, <)$ is an ordered field, $\mathbb{F}$ has the ***order topology***, which is the topology that has the open intervals (a, b) as a base. This defines a unique topology on any finite dimensional affine or projective space over $\mathbb{F}$, see Sections 3.1 and 4 for the definitions. In any ordered field there is the usual ***absolute value***, $|a| = \max\{a, -a\}$, and it satisfies the standard properties such as $|ab| = |a||b|$ and the triangle inequality. Also in this case, $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$ is a two dimensional vector space over $\mathbb{F}$ and thus $\mathbb{K}$ also has a natural topology as do finite dimensional affine and projective spaces over $\mathbb{K}$. The field $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$ has a natural notion of complex conjugation, $a + b\sqrt{-1} \mapsto a - b\sqrt{-1}$ and therefore for the projective and affine spaces over $\mathbb{K}$ the notations of conjugate linear and conjugate affine make sense. Finally an ordered field $\mathbb{F}$ is ***Archimedean*** if for all $a \in \mathbb{F}$ there is a natural number n such that $|a| < n$. This is equivalent to the rational numbers being dense in $\mathbb{F}$ with respect to the order topology on $\mathbb{F}$.

For the rest of this section $\mathbb{F}$ is an Archimedean ordered field and $\mathbb{K}$ is a finite dimensional division algebra over $\mathbb{F}$. Thus $\mathbb{K}$ is finite dimensional vector space over $\mathbb{F}$. We give $\mathbb{K}$ and all other finite dimensional vector and projective spaces over $\mathbb{F}$ the topologies described in Section 3.1 and Section 4. Call an automorphism $\sigma\colon \mathbb{K} \to \mathbb{K}$ that fixes $\mathbb{F}$ pointwise a ***$\mathbb{F}$-automorphism***. And in the following when we say that T ***maps k-flats into j-flats*** we mean that for any k-flat P, there is a j-flat P' with $T[P] \subseteq P'$, but $T[P]$ might be a proper subset of P'. And T ***maps k-flats to j-flats*** means that for any k-flat P, that $T[P] = P'$ for some j-flat P'.

Main Theorem 1. *Let V be a finite dimensional vector space over $\mathbb{K}$ with dimension at least 2. Let $T\colon V \to V$. If*

(i) T is surjective,
(ii) T is continuous at some point of V or $\mathbb{K} = \mathbb{C}$ and T is Lebesgue measurable, and

(iii) There is some k with $1 \le k < \dim V$, such that T maps each k-flat into a k-flat,

then T is a bijection and is σ-semiaffine for some $\mathbb{F}$-automorphism of $\mathbb{K}$. If $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$, then T either affine or conjugate affine.

Main Theorem 2 (Affine Version)**.** *Let V be a finite dimensional vector space over $\mathbb{K}$ with dimension at least 2. Let $T\colon V \to V$. If*

(i) T is injective,
(ii) T is continuous at some point of V or $\mathbb{F} = \mathbb{C}$ and T is Lebesgue measurable, and
(iii) There is some k with $1 \le k < \dim V$, such that T maps each k-flat to a flat,

then T is a bijection and is σ-affine for some $\mathbb{F}$ automorphism of $\mathbb{K}$. If $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$, it is either affine or conjugate affine.

Main Theorem 2 (Projective Version)**.** *Let V be a finite dimensional projective space over $\mathbb{K}$ with projective dimension at least 2. Let $T\colon V \to V$. If*

(i) T is injective,
(ii) T is continuous at some point of V or $\mathbb{F} = \mathbb{C}$ and T is Lebesgue measurable, and
(iii) There is some k with $1 \le k < \dim V$, such that T maps each k-flat to a flat,

then T is a bijection and is σ-linear for some $\mathbb{F}$ automorphism of $\mathbb{K}$. If $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$, it is either linear or conjugate linear.

3. Preliminary results.

3.1. **Additive Functions.** Let $S\colon V \to W$ be a map between vector spaces over the field $\mathbb{F}$. Then S is ***additive*** if and only if it satisfies

$$S(x+y) = S(x) + S(y) \quad \text{for all } x, y \in V. \tag{1}$$

This equation is often called ***Cauchy's functional equation*** after Cauchy who proved that an additive continuous map from the reals to the reals is linear.

One well known extension of Cauchy's result is

Theorem 1 (Fréchet–Banach–Sierpiński)**.** *Let V and W be finite dimensional real vector spaces and $S\colon V \to W$ an additive map. If S is Lebesgue measurable, then it is linear.* □

It is not hard to see that if this is true with $W = \mathbb{R}$, then it is true in general. (Write $S(v) = \sum_{j=1}^{n} f_j(v) w_j$ where $w_1, \dots, w_n$ is a basis of W. Then each f_j will be additive and be measurable and therefore linear.) In the case that V is the real numbers, this was originally proven by Fréchet [7]. It was later proven independently by Banach [3] and Sierpiński [13]. The proof given by Banach easily generalizes to the present case. A proof in the general case can also be found in Járai's book [9].

Another extension of Cauchy's result in the case of $S\colon \mathbb{R} \to \mathbb{R}$ is that if S is additive and continuous at a single point, then it is linear, a result due to Darboux [6] in 1875. We wish to extend this to maps between finite dimensional vector spaces over an ordered field, $\mathbb{F}$. We give $\mathbb{F}^n$ the product topology where each of the factors, $\mathbb{F}$, has the order topology. If we set

$$|v|_{\mathbb{F}^n} = |(v_1, v_2, \dots, v_n)|_{\mathbb{F}^n} := \max\{|v_j| : 1 \le j \le n\}$$

then for $a = (a_1, \ldots, a_n) \in \mathbb{F}^n$ and $r \in \mathbb{F}$ with $r > 0$ the open ball $B_{\mathbb{F}^n}(a, r) := \{v \in \mathbb{F}^n : |v - a|_{\mathbb{F}^n} < r\}$ is the Cartesian product, $\bigtimes_{j=1}^n (a_j - r, a_j + r)$, of open intervals in $\mathbb{F}$ and thus is open. In fact these balls form a basis for the topology of $\mathbb{F}^n$. An equivalent way to define the product topology that is that it is ***weak topology*** generated by the coordinate functions. (That is the smallest topology that makes the coordinates functions continuous.) With this topology it is not hard to see that all linear maps $S\colon \mathbb{F}^n \to \mathbb{F}^n$ are continuous. (Identify S with a matrix $S = [s_{ij}]$ and let $M = \max\{|s_{ij}| : 1 \leq i, j \leq n\}$, then for $v \in \mathbb{F}^n$ the inequality $|Sv|_{\mathbb{F}^n} = \max_{1\leq j \leq n} \bigl|\sum_{i=1}^n s_{ij} v_i\bigr| \leq nM|v|_{\mathbb{F}^n}$ holds. Therefore $|Sv - Sw|_{\mathbb{F}^n} = |S(v - w)|_{\mathbb{F}^n} \leq nM|v - w|_{\mathbb{F}^n}$ which implies S is continuous.) Thus if $S\colon \mathbb{F}^n \to \mathbb{F}^n$ is linear and nonsingular its inverse is linear and therefore continuous. Thus any nonsingular linear map $S\colon \mathbb{F}^n \to \mathbb{F}^n$ is a homeomorphism.

If V is a finite dimensional vector space over $\mathbb{F}$ then choose a linear isomorphism $T\colon V \to \mathbb{F}^n$ of V with $\mathbb{F}^n$. Define a topology on V by requiring T to be a homeomorphism. That is the open sets of V are the sets $T^{-1}[U]$ where U is an open set in $\mathbb{F}^n$. This topology is independent of the choice of T. This follows from the fact that if T' is another linear isomorphism $T'\colon V \to \mathbb{F}^n$, then $T' = S \circ T$ for a nonsingular linear map $S\colon \mathbb{F}^n \to \mathbb{F}^n$. The map S is a homeomorphism and therefore if $U \subseteq \mathbb{F}^n$ we have $(T')^{-1}[U] = T^{-1}\bigl[S^{-1}[U]\bigr]$ and U is open if and only if $S^{-1}[U]$ is open. Thus T and T' define the same open sets in V.

Theorem 2 (An Extended Darboux's Theorem for Ordered Fields)**.** *Let $\mathbb{F}$ be an ordered field, and let V and W be vector spaces over $\mathbb{F}$ of finite positive dimensions. Every additive map from V to W which is continuous at a some point is a linear transformation if and only if the field $\mathbb{F}$ is Archimedean.*

Proof. From the definition of the topologies on the vector spaces V and W there is no loss of generality in assuming $V = \mathbb{F}^m$ and $W = \mathbb{F}^n$. Assume S is continuous at the point x_0. Then for every $\varepsilon > 0$ in $\mathbb{F}$ there is a $\delta_\varepsilon > 0$ in $\mathbb{F}$ such that $|x - x_0|_V < \delta_\varepsilon$ implies $|S(x) - S(x_0)|_W < \varepsilon$. Let y_0 be any point of V and let $|y - y_0|_V < \delta_\varepsilon$. Then $|(y - y_0 + x_0) - x_0|_V < \delta_\varepsilon$ and therefore $|S(y) - S(y_0)|_W = |S(y - y_0 + x_0) - S(x_0)|_W < \varepsilon$. Thus S is continuous at y_0.

Now assume $\mathbb{F}$ is Archimedean. We first do the case when both V and W are one dimensional. That is we show if $f\colon \mathbb{F} \to \mathbb{F}$ is additive and continuous at a point, then f is linear. We have already shown that f is continuous on all of $\mathbb{F}$. Additivity implies for all rational numbers q that $f(qx) = qf(x)$ and in particular $f(q) = f(1)q$. Let $g\colon \mathbb{F} \to \mathbb{F}$ by $g(x) := f(1)x$. Then f and g are both continuous and agree on the set $\mathbb{Q}$ of rational numbers. As $\mathbb{F}$ is Archimedean, $\mathbb{Q}$ is a dense subset of $\mathbb{F}$. But two continuous functions that agree on a dense set are equal. Thus $f(x) = g(x) = f(1)x$ for all $x \in \mathbb{F}$ showing that f is linear.

This leaves the case of $S\colon V \to W$ with V and W finite dimensional over $\mathbb{F}$ and S continuous at some point of V. We have just seen this implies S is continuous at all points of V. As S is additive, to show that it is linear we only need to show $S(tv) = tS(v)$ for all scalars $t \in \mathbb{F}$. Let $v \in V$ and let $\lambda\colon W \to \mathbb{F}$ be any linear functional on W. Define $f_v\colon \mathbb{F} \to \mathbb{F}$ by $f_v(t) = \lambda(S(tv))$. Then f_v is additive and continuous as S and λ are both additive and continuous. Thus, by what we have just shown, f_v is linear. Therefore $f_v(t) = tf_v(1)$ for $t \in \mathbb{F}$. That is $\lambda(S(tv)) = t\lambda(S(v)) = \lambda(tS(v))$, where we have used that λ is linear. But $\lambda(S(tv)) = \lambda(tS(v))$ for all linear functionals λ on W, implies $S(tv) = tS(v)$.

For the converse, suppose that $\mathbb{F}$ is not Archimedean. Then $\mathbb{F}$ must have some infinitesimal elements other than 0. Let $\mathbb{I}$ be the set of all the m-tuples of infinitesimal elements of $\mathbb{F}$. Observe that both $\mathbb{I}$ and $V = \mathbb{F}^m$ are vector spaces over $\mathbb{Q}$. Let B_0 be a basis for $\mathbb{I}$ over $\mathbb{Q}$. Let $\mathbf{e} := \langle 1, 0, 0, \dots \rangle \in V$. Since $\mathbf{e} \notin \mathbb{I}$, the set $B_0 \cup \{\mathbf{e}\}$ is linearly independent over $\mathbb{Q}$. Extend this set to a basis B of V over $\mathbb{Q}$. Now let $\mathbf{e}^* = \langle 1, 0, 0 \dots \rangle \in W$. Let S be the linear transformation (over $\mathbb{Q}$) from V to W defined, for all $u \in V$, by $S(u) = r\mathbf{e}^*$ where r is the coefficient of $\mathbf{e}$ when u is written as linear combination over B. Then S is certainly additive. However, were S linear over $\mathbb{F}$ we would have $S(a\mathbf{e}) = aS(\mathbf{e}) = a\mathbf{e}^*$ for all $a \in \mathbb{F}$. But $S(a\mathbf{e}) = 0$ whenever a is an infinitesimal element of $\mathbb{F}$—so $S(a\mathbf{e}) \neq a\mathbf{e}^*$ when a is a nonzero infinitesimal. Therefore S is not linear over $\mathbb{F}$. It remains to show that S is continuous. As shown above, it is enough to show it is continuous at the zero vector. To this end, let $\varepsilon > 0$. We must produce a $\delta > 0$ so that for all $u \in \mathbb{F}^m$, if $|u|_{\mathbb{F}^m} < \delta$, then $|S(u)|_{\mathbb{F}^n} < \varepsilon$. Take δ to be any infinitesimal with $\delta > 0$. Then $|u|_{\mathbb{F}^m} < \delta$ entails that u is a m-tuple of infinitesimals. But then $|S(u)|_{\mathbb{F}^n} = |0|_{\mathbb{F}^n} < \varepsilon$. □

Remark 3. In addition to continuity conditions, our Main Theorems have hypotheses concerning the surjectivity or injectivity of the maps involved. The map we constructed in the proof of the theorem above has neither of these properties. Nevertheless, we are unable eliminate the Archimedean hypothesis from our Main Theorems through the use of these additional hypotheses. Indeed, over any non-Archimedean ordered field on any finite dimensional vector space there will always be continuous, additive, bijective maps that are not linear operators. Let S be the map produced in the proof of Theorem 2. Let $V = W = \mathbb{F}^n$ and let $a, b \in \mathbb{Q}$. Define $S_{a,b} \colon V \to W$ via

$$S_{a,b}(u) = aS(u) + bu \text{ for all } u \in V.$$

Evidently, each $S_{a,b}$ is continuous and additive. Recalling that S is a linear map, when $\mathbb{F}^n$ is construed as a vector space over $\mathbb{Q}$, it is easy to see that $S_{c,d}$ is the inverse of $S_{a,b}$ where $c = -a/(b(a+b))$ and $d = 1/b$, provided $b(a+b) \neq 0$. So, for example, $S_{1,1}(u) = S(u) + u$ is invertible and its inverse is $S_{-1/2,1}(u) = -1/2S(u) + u$. On the other hand, $S = 1/a(S_{a,b} - bI)$. Since we know that S is not linear over $\mathbb{F}$, we see that $S_{a,b}$ cannot be linear over $\mathbb{F}$ either.

In this way, we see that over any non-Archimedean ordered field continuous additive bijective functions need not be linear.

3.2. **Extended forms of the Fundamental Theorem of Affine and Projective Geometry.** Let $\mathbb{D}$ be a division ring. We use as our model of $\mathbb{DP}^n$, that is the n-dimensional projective space over $\mathbb{D}$, the space of one dimensional subspaces of $\mathbb{D}^{n+1}$. If $v \in \mathbb{D}^{n+1}$ with $v \neq 0$ let $\langle v \rangle$ be the one dimensional subspace spanned by v. If $A \colon \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ is semilinear and nonsingular, then it induces a projective map $\hat{A} \colon \mathbb{DP}^n \to \mathbb{DP}^n$ by

$$\hat{A}\langle v \rangle = \langle Av \rangle.$$

For each $n \geq 2$, let $\mathbb{L}(\mathbb{D}^n)$ (respectively $\mathbb{L}(\mathbb{DP}^n)$) be the lattice of all flats in $\mathbb{D}^n$ (respectively $\mathbb{DP}^n$). The following are special cases of the Fundamental Theorems of Affine and Projective Geometry.

Theorem 4 (Fundamental Theorem of Affine Geometry)**.** *For $n \geq 2$, a bijection $T \colon \mathbb{D}^n \to \mathbb{D}^n$ induces an automorphism of $\mathbb{L}(\mathbb{D}^n)$ if and only if T is semiaffine.* □

Theorem 5 (Fundamental Theorem of Projective Geometry)**.** *For $n \geq 2$, If a bijection $T\colon \mathbb{DP}^n \to \mathbb{DP}^n$ induces an automorphism of $\mathbb{L}(\mathbb{DP}^n)$, then there is a semilinear map $A\colon \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ such that T is the induced map $T = \hat{A}$.* □

This version of the Fundamental Theorem of Projective Geometry follows from [2, Thm 2.26 p. 88] and the Fundamental Theorem for Affine Geometry can be derived from the projective version. For a direct proof of the affine version see [4, pp. 201–202]

A version of the Fundamental Theorem of Affine Geometry where the assumption of the map T being bijective replaced by T being surjective was proven by Alexander Chubarev and Iosif Pinelis [5]:

Theorem 6 (Surjective Fundamental Theorem of Affine Geometry)**.** *Let $\mathbb{D}$ and $\mathbb{D}'$ be division rings such that $\mathbb{D}$ has more than two elements. Let $\mathcal{A}$ and $\mathcal{A}'$ be affine spaces of finite dimensions n and n' over $\mathbb{D}$ and $\mathbb{D}'$ respectively and let $n' \geq n \geq 2$. If T is a map from $\mathcal{A}$ to $\mathcal{A}'$ so that*

(i) T is surjective and
(ii) There is some k with $1 \leq k < n$ such that T maps each k-flat into a k-flat,

then T is bijective and semiaffine. □

The third stipulation in the affine version of our Main Theorem 2 only insists that the image of every k-flat is a flat and replaces surjectivity with injectivity.

Theorem 7 (Injective Fundamental Theorem of Affine Geometry)**.** *Let $\mathbb{D}$ be a division ring. Let $\mathcal{A}$ be the affine space of finite dimension $n > 1$ over $\mathbb{D}$. If T is a map from $\mathcal{A}$ to $\mathcal{A}$ so that*

(i) T is injective and
(ii) There is some k with $1 \leq k < n$ such that T maps each k-flat to a flat,

then T is bijective and semiaffine.

Proposition 8. *Let $\mathbb{D}$ be a division ring with more than two elements, and $n \geq 2$ be a natural number. Let V be either the affine space or the projective space of dimension n over $\mathbb{D}$. If T is a map from V to V such that*

(i) T is injective, and
(ii) there is some k with $1 \leq k < n$ such that T maps each k-flat to a flat,

then T is bijective and the map $Q \mapsto T[Q]$ is an automorphism of the lattice $\mathbb{L}(V)$.

Lemma 9. *Under the hypotheses of Proposition 8, T maps every flat to a flat. Moreover T is bijective and for every $k \leq n$, T maps every k-flat to a k-flat.*

Proof. We leave the "moreover" portion of the lemma to the end of this proof. We consider three cases.

Case: $\dim Q = k$.
It is one of our hypotheses that $T[Q]$ is a flat.

Case: $\dim Q < k$.
There is a finite set $\{P_0, P_2, \dots, P_{m-1}\}$ of k-flats so that

$$Q = \bigcap_{j<m} P_j.$$

As T is injective, $T[Q] = \bigcap_{j<m} T[P_j]$. Each $T[P_j]$ is a flat and therefore $T[Q]$ is the intersection of flats and thus is itself a flat.

Case: $\dim Q > k$.
We use the fact that a subset of V is a flat if and only if it contains the line through any two of its points. Let $y_1, y_2 \in T[Q]$ be distinct. Then there are distinct points $x_1, x_2 \in Q$ with $y_1 = T(x_1)$ and $y_2 = T(x_2)$. As $\dim Q > k \geq 1$ there is a flat $P \subset Q$ with $\dim P = k$ and $x_1, x_2 \in P$. The image $T[P]$ is a flat and contains the points $y_1 = T(x_1)$ and $y_2 = T(x_2)$ and thus contains the line through y_1 and y_2. As $T[P] \subset T[Q]$ this shows that $T[Q]$ contains the line through y_1 and y_2 and as these are arbitrary points of $T[Q]$, we have that $T[Q]$ is a flat.

Now let us consider the "moreover" portion of the Lemma. Let $\dim Q = \ell$. There is a strictly increasing chain $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n = \mathbb{D}^n$ of flats such that $\dim Q_j = j$ and $Q_\ell = Q$. Here, the flat Q_0 can be any point of Q. Then

$$T[Q_0] \subsetneq T[Q_1] \subsetneq \cdots \subsetneq T[Q_n]$$

is a strictly increasing chain of flats of $\mathbb{D}^n$, since T is injective. As $T[Q_{j+1}]$ strictly contains $T[Q_j]$ the inequality $\dim T[Q_{j+1}] \geq 1 + \dim T[Q_j]$ holds. Therefore $\dim T[Q_j] \geq j$ for all j. So that $\dim T[Q_n] \geq n$. But $\mathbb{D}^n$ has only one flat with dimension at least n and that is $\mathbb{D}^n$ itself. So $\dim T[Q_n] = n$. This is only possible if $\dim T[Q_j] = j$ for all $j \leq n$. In particular $\dim T[Q] = \dim T[Q_\ell] = \ell = \dim Q$, as required. Finally $T[\mathbb{D}^n]$ is flat of dimension n and therefore $T[\mathbb{D}^n] = \mathbb{D}^n$ which shows that T is surjective and therefore bijective. □

Proof of Proposition 8. By the lemma above and the injectivity of T the map $P \mapsto T[P]$ is an injective map from $\mathbb{L}(V)$ to itself that preserves the lattice operations. All that remains is to show that this map is surjective. As $T[T^{-1}[Q]] = Q$ it is enough to show that $T^{-1}[Q]$ is a flat whenever Q is a flat. If $\dim Q = 0$ this is clear. So assume $\dim Q \geq 1$. Let $x_1, x_2 \in Q$ be distinct and let L be the line through x_1 and x_2. Then by the lemma $T[L]$ is a line and as it contains the points $T(x_1)$ and $T(x_2)$ of the flat $T[Q]$ the line $T[L]$ is contained in $T[Q]$. Thus $L = T^{-1}[T[L]] \subseteq T^{-1}[T[Q]] = Q$. Therefore Q contains the line through any two of its points and hence it is a flat. □

Now consider the Injective Fundamental Theorem of Affine Geometry. Suppose that T fulfills the hypotheses of that theorem. Then, by Proposition 8, T must map every k-flat to a k-flat for every $k \leq n$. This implies T also fulfills the hypotheses of the Fundamental Theorem of Affine Geometry and completes the proof. □

3.3. **Extended form of the Fundamental Theorem Projective Geometry.**

Theorem 10 (Injective Fundamental Theorem of Projective Geometry)**.** *Let $\mathbb{D}$ be a division ring. Let V be the projective space of finite dimension $n > 1$ over $\mathbb{D}$. If T is a map from V to V such that*

(i) T is injective and
(ii) There is some k with $1 \leq k < n$ such that T maps each k-flat to a flat,

then T is bijective and semilinear. □

Proof. Assume that $T \colon V \to V$ satisfies the hypothesis of Theorem 10. Then by Proposition 8 the map T is bijective and the map $Q \mapsto T[Q]$ is an automorphism of the lattice $\mathbb{L}(V)$. Therefore Theorem 5 implies that T is semilinear, which completes the proof. □

We initially believed there was a projective analog of the Surjective Fundamental Theorem of Affine Geometry (Theorem 6). Rather surprisingly, it fails even for the complex projective plane.

Theorem 11 (Surjective Fundamental Theorem of Projective Geometry Fails)**.** *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero that has infinite transcendence degree over the rational numbers (i.e. the complex numbers). Then for any integer $n \geq 2$, there is a map $T \colon \mathbb{KP}^n \to \mathbb{KP}^n$ such that*

(i) T is surjective,
(ii) each point $y \in \mathbb{KP}^n$ is the image of infinitely points under T and so T is not injective,
(iii) for any $(n-1)$-flat P of $\mathbb{KP}^n$ the image $T[P]$ is a $(n-1)$-flat in $\mathbb{KP}^n$, and
(iv) every $(n-1)$-flat of $\mathbb{KP}^n$ is the image of some $(n-1)$ flat under T.

The germ of the idea for this proof (cf. Section 5) is based on ideas from Examples 1 and 2 (pages 377–378) from the paper by Rigby [12], however the algebraic details are substantially more complicated.

4. Proof of the main theorems.

Before starting the proofs, we define the topologies on the spaces we are working with. If $\mathbb{F}$ is a ordered field, then we have already defined the topology that $\mathbb{F}$ induces on any finite dimensional vector space over $\mathbb{F}$. If V is a n dimensional affine space over $\mathbb{F}$, then choosing a point $x_0 \in V$ to use as an origin makes V into a n dimensional vector space over $\mathbb{F}$, and this defines a topology on V. Using that translations are continuous, it is not hard to check this topology is independent of the choice of the origin.

Let V be a finite dimensional projective space over $\mathbb{F}$. Let H be a hyperplane in V. Then $V \setminus H$, the complement of H in V, is an affine space and thus has the topology just defined. Define a set $U \subseteq V$ to be open if and only if for every hyperplane H, the set $U \cap (V \setminus H)$ is open in $V \setminus H$. To see that this topology does not have more open sets than expected, we now argue that if H_1 and H_2 are hyperplanes in V, then the set $V \setminus (H_1 \cup H_2)$ inherits the same topology. We regard $V \setminus (H_1 \cup H_2)$ as a subset of the affine space $V \setminus H_1$ or as a subset of the affine space $V \setminus H_2$. Let $x_1, x_2, \ldots, x_n \colon (V \setminus H_1) \to \mathbb{F}$ be affine coordinates. That is the map $p \mapsto (x_1(p), x_2(p), \ldots, x_n(p))$ is an affine isomorphism between $V \setminus H_1$ and the affine space $\mathbb{F}^n$. Then our definitions imply that the topology on $V \setminus H_1$ is the weak topology generated by the functions $x_1, x_2, \ldots, x_n$. Let $y_1, y_2, \ldots, y_n \colon (V \setminus H_2) \to \mathbb{F}$ be affine coordinates on $V \setminus H_2$. Then on $V \setminus (H_1 \cup H_2)$ these two sets of coordinates are related by

$$y_i = \frac{a_{i0} + a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n}{a_{00} + a_{01}x_1 + a_{02}x_2 + \cdots + a_{0n}x_n}$$

where $[a_{ij}]_{i,j=0}^n$ is an invertible $(n+1) \times (n+1)$ matrix over $\mathbb{F}$. (This is the general form of a projective map in terms of affine coordinates, and $a_{00} + a_{01}x_1 + a_{02}x_2 + \cdots + a_{0n}x_n = 0$ is the equation for $(V \setminus H_1) \cap H_2$ in the coordinates $x_1, x_2, \ldots, x_n$.) Thus on $V \setminus (H_1 \cup H_2)$ the functions $y_1, y_2, \ldots, y_n$ are continuous functions of $x_1, x_2, \ldots, x_n$. It follows that the weak topology generated on $V \setminus (H_1 \cup H_2)$ by the functions $y_1, y_2, \ldots, y_n$ is a subset of the weak topology generated by $x_1, x_2, \ldots, x_n$. By symmetry we have that the weak topology generated on $V \setminus (H_1 \cup H_2)$ by the functions $x_1, x_2, \ldots, x_n$ is a subset of the weak topology generated by $y_1, y_2, \ldots, y_n$.

Therefore the topologies induced on $V \smallsetminus (H_1 \cup H_2)$ by the two affine sets $V \smallsetminus H_1$ and $V \smallsetminus H_2$ are equal as required.

4.1. **Proof of the affine results.** Consider a map T that fulfills the hypotheses of either of the affine versions of our Main Theorems. Using either the Surjective or Injective Fundamental Theorems of Affine Geometry, we see that T must be semiaffine. Thus there is an element $b \in \mathbb{K}^n$, an automorphism σ of $\mathbb{K}$ and a map $S \colon \mathbb{K}^n \to \mathbb{K}^n$ such that

$$\begin{aligned} S(x+y) &= S(x) + S(y) && \text{for all } x, y \in \mathbb{K}^n \\ S(cx) &= \sigma(c)S(x) && \text{for all } x \in \mathbb{K}^n \text{ and } c \in \mathbb{K} \\ T(x) &= S(x) + b && \text{for all } x \in \mathbb{K}^n \end{aligned}$$

Since T fulfills hypothesis (2) of our Main Theorems, so must S. Observe that $\mathbb{K}^n$, being a finite dimensional vector space over $\mathbb{K}$, is also a finite dimensional vector space over $\mathbb{F}$. We conclude using Theorem 2 that S is a linear operator on $\mathbb{K}^n$ considered as a vector space over $\mathbb{F}$. So for every element $r \in \mathbb{F}$ and for every $x \in \mathbb{K}^n$ we have

$$rS(x) = S(rx) = \sigma(r)S(x).$$

Because S does not map everything to the zero-vector, we find

$$r = \sigma(r) \quad \text{for all} \quad r \in \mathbb{F}.$$

Therefore σ is a $\mathbb{F}$-automorphism as required.

Finally assume $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$. Observe that $x^2 + 1$ is irreducible over $\mathbb{F}$ and $\mathbb{K}$ is a degree two extension of $\mathbb{F}$ that contains both the roots of this polynomial. Thus $\mathbb{K}$ is the splitting field of $x^2 + 1$. It follows that the Galois group of $\mathbb{K}$ over $\mathbb{F}$ is just the two element group. So there are only two automorphisms of $\mathbb{K}$ that fix each member of $\mathbb{F}$: the identity map and conjugation. In the first alternative, T will be an affine map, while in the second alternative T will be conjugate-affine. This completes the proofs.

4.2. **Proof of the projective result.** Let $T \colon V \to V$ be a map that satisfies the hypothesis of projective version of Main Theorem 2. Then by the Injective Fundamental Theorem of Projective Geometry, we see that T is semilinear. Let H be any hyperplane in V. Then $T[H]$ is also a hyperplane in V. As the group of linear automorphisms of V is transitive on the set of hyperplanes, there is a linear automorphism S of V such that such that $S[T[H]] = H$. But then $S \circ T$ maps $V \smallsetminus H$ onto itself and $V \smallsetminus H$ is an affine space. Therefore by the affine versions of our Main Theorems, the restriction $(S \circ T)\big|_{V \smallsetminus H} \colon (V \smallsetminus H) \to (V \smallsetminus H)$ is σ-semiaffine for a $\mathbb{F}$-automorphism of $\mathbb{K}$. As S is linear this implies $T\big|_{V \smallsetminus H} = S^{-1} \circ (S \circ T)\big|_{V \smallsetminus H}$ is σ-linear. From this it is not hard to check that T is σ-linear as required. If $K = \mathbb{F}[\sqrt{-1}]$, then S is either affine or conjugate affine, which implies that T is either linear or conjugate linear. □

5. Examples

5.1. **Algebraic preliminaries on Puiseux series.** Let $\mathbb{K}$ be an field of characteristic zero. For any variable x we denote by $\mathbb{K}((x))$ the ring of formal Laurent series in x. Thus if $f(x) \in \mathbb{K}((x))$ is not the zero element, there is an unique integer k such that $f(x)$ is of the form

$$f(x) = \sum_{j=k} f_j x^j$$

where $f_j \in \mathbb{K}$ and $f_k \neq 0$. The integer k is the ***order*** $f(x)$ and is denoted by $\operatorname{ord}(f(x))$. For completeness we define $\operatorname{ord}(0) = +\infty$. In analogy with complex analysis the order of $f(x)$ can be thought of as order of the zero of $f(x)$ at the origin, with the usual convention that when $\operatorname{ord}(f(x))$ is negative that the origin is a pole. If $\operatorname{ord}(f(x)) \geq 0$, then we can evaluate $f(x)$ at $x = 0$ in which case $f(0) = f_0$, the coefficient of 1 in the series $f(x) = \sum_j f_j x^j$. For a nonzero $f(x) \in \mathbb{K}((x))$ the coefficient of $x^{\operatorname{ord}(f(x))}$ is the ***lead coefficient*** of $f(x)$ and we will denote it by $\operatorname{lead}(f(x))$. Set $\operatorname{lead}(0) = 0$. With these definitions it is not hard to check for $f(x), g(x) \in \mathbb{K}((x))$ that

$$\operatorname{ord}(f(x)g(x)) = \operatorname{ord}(f(x)) + \operatorname{ord}(g(x)), \tag{2}$$

$$\operatorname{lead}(f(x)g(x)) = \operatorname{lead}(f(x)) \operatorname{lead}(g(x)). \tag{3}$$

If $\mathbb{K}[[x]]$ is the ring of formal power series over $\mathbb{K}$, then $\mathbb{K}((x))$ is the quotient ring of $\mathbb{K}[[x]]$. A fact that we will use is that $\mathbb{K}[[x]]$ is a principal ideal domain (the ideals are all of the form (x^m) for some nonnegative integer m).

Let n be a positive integer. A variant on the above is $\mathbb{K}((x^{1/n}))$, the formal Laurent series in $x^{1/n}$. In this case the order of a nonzero $f(x) \in \mathbb{K}((x^{1/n}))$ is still defined as the smallest exponent of a nonzero term in the sum $f(x) = \sum_{r \in \frac{1}{n}\mathbb{Z}} f_r x^r$ where $\frac{1}{n}\mathbb{Z} = \{k/n : k \in \mathbb{Z}\}$. Thus in this case $\operatorname{ord}(f(x)$ is a rational number of the form k/n where k is an integer. Likewise the lead coefficient, $\operatorname{lead}(f(x))$ is still defined and if we still use the convention that $\operatorname{ord}(0) = +\infty$ and $\operatorname{lead}(0) = 0$, the formulas (2) and (3) still hold. Also $\mathbb{K}((x^{1/n}))$ is the quotient ring of $\mathbb{K}[[x^{1/n}]]$ and $\mathbb{K}[[x^{1/n}]]$ is a principal ideal domain.

Finally the ***field of Puiseux series*** over $\mathbb{K}$ is the union

$$\mathbb{K}\langle\langle x \rangle\rangle = \bigcup_{n=1}^{\infty} \mathbb{K}((x^{1/n})).$$

For $f(x) \in \mathbb{K}\langle\langle x \rangle\rangle$ the $\operatorname{ord}(f(x))$ and $\operatorname{lead}(f(x))$ are defined and satisfy (2) and (3). If $\operatorname{ord}(f(x)) \geq 0$ then the evaluation, $f(0)$, is defined in the natural way. Evaluation and the lead coefficient are related as follows. If $\operatorname{ord}(f(x)) = k$ then $f(x)$ is of the form $f(x) = x^k \widetilde{f}(x)$ where $\operatorname{ord}(\widetilde{f}(x)) = 0$. Then

$$\operatorname{lead}(f(x)) = \widetilde{f}(0).$$

We will need the following result on the algebraic closure of the field, $\mathbb{K}((x))$.

Theorem 12 (The Newton-Puiseux Theorem)**.** *If $\mathbb{K}$ is an algebraically closed field of characteristic zero, then $\mathbb{K}\langle\langle x \rangle\rangle$ is an algebraic closure of $\mathbb{K}((x))$.* $\square$

A proof is in [15, pp. 98–102]. Newton and Puiseux original versions can be found in [10], [11].

The algebraic result at the heart of our examples is:

Proposition 13. *Let $\mathbb{K}$ be an algebraic closed field of characteristic zero that has infinite transcendence degree over its prime field $\mathbb{Q}$. Then the fields $\mathbb{K}$ and $\mathbb{K}\langle\langle x \rangle\rangle$ are isomorphic. In particular $\mathbb{C}$ and $\mathbb{C}\langle\langle x \rangle\rangle$ are isomorphic fields.*

Proof. A basic result in the theory of transcendental field extensions is the theorem of Steinitz that two algebraically closed fields with the same characteristic and same transcendence degree over their prime field are isomorphic (cf. [14]). The fields $\mathbb{K}$ and $\mathbb{K}\langle\langle x\rangle\rangle$ both have characteristic zero. The transcendence degree of $\mathbb{K}\langle\langle x\rangle\rangle$ over $\mathbb{Q}$ is one greater than the transcendence degree of $\mathbb{K}$. As $\mathbb{K}$ has infinite transcendence degree over $\mathbb{Q}$ this implies that $\mathbb{K}$ and $\mathbb{K}\langle\langle x\rangle\rangle$ have the same transcendence degree over $\mathbb{Q}$. □

5.2. **Construction of the examples.** Let $\mathbb{F}$ be a field. Then in this section we use the notation $\mathbb{P}^n(\mathbb{F})$ for the projective space $\mathbb{KP}^n$ realized as set of points with homogeneous coordinates $[a_0 : a_1 :, \dots, : a_n]$. If $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{F}^{n+1} \setminus \{0\}$ let $[\mathbf{a}]$ be the point in $\mathbb{P}^n(\mathbb{F})$ with homogeneous coordinates $[a_0 : a_1 :, \dots, : a_n]$. Then for $\mathbf{a}, \mathbf{b} \in \mathbb{F}^{n+1} \setminus \{0\}$ we have $[\mathbf{a}] = [\mathbf{b}]$ if and only if $\mathbf{a} = \lambda\mathbf{b}$ for some nonzero $\lambda \in \mathbb{F}$.

Let $\mathbf{a}(x) = (a_0(x), a_1(x), \dots, a_n(x))$ be a $n + 1$ tuple of elements from $\mathbb{K}\langle\langle x\rangle\rangle$. We extend the definition of ord to such tuples by

$$\operatorname{ord}(\mathbf{a}(x)) = \min_{0\le j\le n} \operatorname{ord}(a_j(x)).$$

As in the case of elements of $\mathbb{K}\langle\langle x\rangle\rangle$ if $\operatorname{ord}(\mathbf{a}(x)) \ge 0$, we can evaluate $\mathbf{a}(x)$ at zero by

$$\mathbf{a}(x) = (a_0(0), a_1(0), \dots, a_n(0)).$$

If $\mathbf{a}(x) \in \mathbb{K}\langle\langle x\rangle\rangle^{n+1} \setminus \{0\}$, write

$$\mathbf{a}(x) = x^{\operatorname{ord}(\mathbf{a}(x))}\widetilde{\mathbf{a}}(x)$$

where $\operatorname{ord}(\widetilde{\mathbf{a}}(x)) = 0$. As $\widetilde{\mathbf{a}}(x)$ has order zero, the evaluation $\widetilde{\mathbf{a}}(0) \in \mathbb{K}^{n+1}$ satisfies $\widetilde{\mathbf{a}}(0) \ne 0$. Define the ***lead coefficient*** of $\mathbf{a}(x)$ by

$$\operatorname{lead}(\mathbf{a}(x)) = \widetilde{\mathbf{a}}(0).$$

Set $\operatorname{lead}(0) = 0$. The proof of the following is left to the reader.

Lemma 14. *If $\lambda(x) \in \mathbb{K}\langle\langle x\rangle\rangle$ and $\mathbf{a}(x) = (a_0(x), a_1(x), \dots, a_n(x))$ is a $n+1$ tuple of elements from $\mathbb{K}\langle\langle x\rangle\rangle$ then the equations*

$$\operatorname{ord}(\lambda(x)\mathbf{a}(x)) = \operatorname{ord}(\lambda(x)) + \operatorname{ord}(\mathbf{a}(x))$$
$$\operatorname{lead}(\lambda(x)\mathbf{a}(x)) = \operatorname{lead}(\lambda(x))\operatorname{lead}(\mathbf{a}(x))$$

hold. □

Definition 15. If $\mathbb{K}$ is a field of characteristic zero and n a positive integer the ***lead coefficient map*** is the function $L\colon \mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle) \to \mathbb{P}^n(\mathbb{K})$ given by

$$L([\mathbf{a}(x)] = [\operatorname{lead}(\mathbf{a}(x))]$$

(This is well defined by Lemma 14). □

Proposition 16. *Let $\mathbb{K}$ be a field of characteristic zero. Then for any integer $n \ge 2$ the lead coefficient map $L\colon \mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle) \to \mathbb{P}^n(\mathbb{K})$ is surjective. Any $[\mathbf{b}] \in \mathbb{P}^n(\mathbb{K})$ is the image under L of infinitely many elements of $\mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle)$ and thus L is not injective. For for every $(n-1)$-flat, P, of $\mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle)$, the image $L[P]$ is a $(n-1)$-flat in $\mathbb{P}^n(\mathbb{K})$. Moreover every $(n-1)$-flat in $\mathbb{P}^n(\mathbb{K})$ is the image of some $(n-1)$-flat of $\mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle)$.*

Proof. Let $[\mathbf{a}] = [(a_0, a_1, \dots, a_n)] \in \mathbb{P}^n(\mathbb{K})$. Choose any elements $b_0(x), b_1(x), \dots, b_n(x) \in \mathbb{K}\langle\langle x\rangle\rangle$ such that $\operatorname{ord}(b_j(x)) > 0$ for $j \in \{0, 1, \dots, n\}$. Then

$$L([(a_0 + b_0(x), a_1 + b_1(x), \dots, a_n + b_n(x))]) = [\mathbf{a}].$$

Thus L is surjective. There are infinitely many choices for $b_0(x), b_1(x), \dots, b_n(x)$, thus any point of $\mathbb{P}^n(\mathbb{K})$ is the image of infinitely many points of $\mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle)$.

The $(n-1)$-flats of $\mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle)$ are of the form

$$H(\mathbf{c}(x)) = \Big\{[\mathbf{a}(x)] \in \mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle) : \sum_{j=0}^n c_j(x)a_j(x) = 0\Big\}$$

where $\mathbf{c}(x) \in \mathbb{K}\langle\langle x\rangle\rangle^{n+1} \smallsetminus \{0\}$. Note for any nonzero $\lambda(x) \in \mathbb{K}\langle\langle x\rangle\rangle$

$$H(\lambda(x)\mathbf{c}(x)) = H(\mathbf{c}(x)).$$

So by replacing $\mathbf{c}(x)$ by $x^{-\operatorname{ord}(\mathbf{c}(x))}\mathbf{c}(x)$ we can assume that $\operatorname{ord}(\mathbf{c}(x)) = 0$.

Likewise the $(n-1)$-flats of $\mathbb{P}^n(\mathbb{K})$ are of the form

$$H'(\mathbf{d}) = \Big\{[\mathbf{b}] \in \mathbb{P}^n(\mathbb{K}) : \sum_{j=0}^n d_j b_j = 0\Big\}$$

where $\mathbf{d} \in \mathbb{K}^{n+1} \smallsetminus \{0\}$ and for any nonzero $\lambda \in \mathbb{K}$ we have $H'(\lambda\mathbf{d}) = H'(\mathbf{d})$.

Let $H(\mathbf{c}(x))$ be a $(n-1)$-flat in $\mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle)$. Without loss of generality we assume that $\operatorname{ord}(\mathbf{c}(x)) = 0$. Then $\operatorname{lead}(\mathbf{c}(x)) = \mathbf{c}(0)$ and thus $L([\mathbf{c}(x)]) = [\mathbf{c}(0)]$.

Claim. $L[H(\mathbf{c}(x)] = H'(\mathbf{c}(0))$.

Let $[\mathbf{a}(x)] \in H(\mathbf{c}(x))$. By replacing $\mathbf{a}(x)$ with $\lambda(x)\mathbf{a}(x)$ with $\lambda(x) = x^{-\operatorname{ord}(\mathbf{a}(x))}$ we can assume that $\operatorname{ord}(\mathbf{a}(x)) = 0$. Since $[\mathbf{a}(x)] \in H(\mathbf{c})$ we have

$$\sum_{j=0}^n c_j(x)a_j(x) = 0.$$

Evaluating this at 0 gives

$$\sum_{j=0}^n c_j(0)a_j(0) = 0.$$

Since $L([\mathbf{a}_j(x)] = [\mathbf{a}(0)]$ this implies $L([\mathbf{a}(x)]) \in H'(\mathbf{c}(0))$. Therefore $L[H(\mathbf{c}(x)] \subseteq H'(\mathbf{c}(0))$.

Let $[\mathbf{u}] \in H'(\mathbf{c}(0))$. Then $\sum_{j=0}^n c_j(0)u_j = 0$. As $\operatorname{ord}(\mathbf{c}(x)) = 0$, this implies

$$\operatorname{ord}\Big(\sum_{j=0}^n c_j(x)u_j\Big) > 0.$$

Since $\mathbb{K}\langle\langle x\rangle\rangle = \bigcup_{m=1}^\infty \mathbb{K}((x^{1/m}))$ there is an m such that $c_j(x) \in \mathbb{K}((x^{1/m}))$ for $j \in \{0, 1, \dots, n\}$. As $\operatorname{ord} c_j(x) \geq 0$ for all j, we have $c_j(x) \in \mathbb{K}[[x^{1/m}]]$, the formal power series in $x^{1/m}$. Also since $\operatorname{ord}(\mathbf{c}(x)) = 0$ there is at least one index j_0 such that $\operatorname{ord}(c_{j_0}(x)) = 0$. This implies the constant term of $c_{j_0}(x)$ is nonzero and thus $c_{j_0}(x)$ is a unit in $\mathbb{K}[[x^{1/m}]]$. Therefore that greatest common divisor $\gcd(c_0(x), c_1(x), \dots, c_n(x)) = 1$ in the ring $\mathbb{K}[[x^{1/m}]]$. Let $r = \operatorname{ord}\Big(\sum_{j=0}^n c_j(x)u_j\Big)$. Then $r > 0$ and so

$$\sum_{j=0}^n c_j(x)u_j = x^r f(x)$$

where $\operatorname{ord}(f(x)) = 0$ and therefore $f(x) \in \mathbb{K}[[x^{1/m}]]$. As this ring is a principal idea domain and $\gcd(c_0(x), c_1(x), \ldots, c_n(x)) = 1$, there are $g_0(x), g_1(x), \ldots, g_n(x) \in \mathbb{K}[[x^{1/m}]]$ such that

$$\sum_{j=0}^{n} c_j(x) g_j(x) = f(x).$$

Set $a_j(x) = u_j - x^r f(x)$. Then

$$\sum_{j=0}^{n} c_j(x) a_j(x) = \sum_{j=0}^{n} c_j(x)(u_j - x^r g_j(x)) = x^r f(x) - x^r f(x) = 0.$$

Thus $[\mathbf{a}(x)] \in H(\mathbf{c}(x))$. And $L([\mathbf{a}(x)]) = [\mathbf{u}]$. As $[\mathbf{u}]$ was an arbitary element of $H'(\mathbf{c}(x))$, this implies $H'(\mathbf{c}(x)) \subseteq L[H(\mathbf{c}(x))]$. Therefore $H'(\mathbf{c}(x)) = L[H(\mathbf{c}(x))]$. This completes the proof of the claim.

All that remains is to show that every $(n-1)$-flat $H'(\mathbf{d})$ in $\mathbb{P}^n(\mathbb{K})$ is the image of some $(n-1)$-flat of $\mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle)$. Let $H'(\mathbf{d})$ be a $(n-1)$-flat in $\mathbb{P}^n(\mathbb{K})$. Choose any $\mathbf{c}(x) \in \mathbb{K}\langle\langle x\rangle\rangle^{n+1}$ of with $\operatorname{ord}(\mathbf{c}(x)) = 0$ and $\mathbf{c}(0) = \mathbf{d}$ (there are infinitely many choices for $\mathbf{c}(x)$). Then $\operatorname{lead}(\mathbf{c}(x)) = \mathbf{c}(0) = \mathbf{d}$ and thus by the claim $H'(\mathbf{d}) = L[H(\mathbf{c}(x))]$. $\square$

Proof of Theorem 11. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero with infinite transcendence degree over $\mathbb{Q}$. Then by Proposition 13 there is a field isomorphism $\varphi\colon \mathbb{K} \to \mathbb{K}\langle\langle x\rangle\rangle$. Then the map $\widehat{\varphi}\colon \mathbb{P}^n(\mathbb{K}) \to \mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle)$ given by

$$\widehat{\varphi}([(a_0, a_1, \ldots, a_n)]) = [(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_n))]$$

is an projective isomorphism of $\mathbb{P}^n(\mathbb{K})$ with $\mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle)$. In particular the map $P \mapsto \widehat{\varphi}[P]$ is a bijection of the $(n-1)$-flats of $\mathbb{P}^n(\mathbb{K})$ with the $(n-1)$-flats of $\mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle)$. Let $L\colon \mathbb{P}^n(\mathbb{K}\langle\langle x\rangle\rangle) \to \mathbb{P}^n(\mathbb{K})$ be the leading coefficient map. Then Proposition 16 composition $T = L \circ \widehat{\varphi}\colon \mathbb{P}^n(\mathbb{K}) \to \mathbb{P}^n(\mathbb{K})$ satisfies the conditions of Theorem 11. $\square$

Acknowledgements.

We owe a very large debt to the referee. He or she corrected a mistake in the first version of this paper, noted that our main results could be generalized from affine and projective spaces over $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$, to affine and projective spaces over any division ring $\mathbb{K}$ finite dimensional over $\mathbb{F}$, and also pointed out the relevance of Rigby's paper, [12], which lead to Theorem 11 and the results of Section 5. This is definitely a better paper because of her or his patience and constructive criticism.

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