Mathematics 552 Homework.

Here are some power series we know:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} \cdots \tag{1}$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots \tag{2}$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots \tag{3}$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + z^4 - z^5 + z^6 - \cdots \tag{4}$$

We can combine these with some easy tricks to get the series for some more complicated functions. For example to get the series $\sin(z^3)$ we replace z by z^3 in equation (2) to get

$$\begin{aligned}
\sin(z^3) &= \sum_{n=0}^{\infty} \frac{(-1)^n (z^3)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{6n+3}}{(2n+1)!} \\
&= z^3 - \frac{z^9}{3!} + \frac{z^{15}}{5!} - \frac{z^{21}}{7!} + \frac{z^{27}}{9!} - \cdots
\end{aligned}$$

Then if we wanted the series for the function

$$F(z) = \int_0^z \sin(t^3)\, dt$$

we can just integrate term at a time to get

$$\begin{aligned}
F(z) &= \int_0^z \sin(t^3)\, dt \\
&= \int_0^z \left(t^3 - \frac{t^9}{3!} + \frac{t^{15}}{5!} - \frac{t^{21}}{7!} + \frac{t^{27}}{9!} - \cdots \right) dt \\
&= \frac{z^4}{4} - \frac{z^{10}}{3!(10)} + \frac{z^{16}}{5!(16)} - \frac{z^{22}}{7!(22)} + \frac{z^{28}}{9!(28)} - \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{6n+4}}{(2n+1)!(6n+4)}.
\end{aligned}$$

Problem 1. Find the power series for the following functions:

(a) e^{-2z},

(b) $\cos(3z^2)$,

(c) $\log(1+z) - \displaystyle\int_0^z \frac{dt}{1+z}$.

(d) $\displaystyle\int_0^z e^{3t^2}\,dt.$

In class we used these methods to show

$$\begin{aligned}\arctan(z) &= \int_0^z \frac{dt}{1+t^2}\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}\\ &= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \frac{z^9}{9} - \cdots\end{aligned}$$

This has radius of convergence $R = 1$.

To compute π we can use the series for the arctan For this to be efficient we wish to use values of z that are close to zero. To get a reasonably rapidly convergent series note if

$$\alpha = \arctan(1/2), \qquad \beta = \arctan(1/3)$$

then using the addition angle for the tangent we havd

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} = \frac{1/2+1/3}{1-(1/2)(1/3)} = 1.$$

Therefore

$$\alpha + \beta = \frac{\pi}{4}$$

which implies

$$\pi = 4\arctan\left(\frac{1}{2}\right) + 4\arctan\left(\frac{1}{3}\right) = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{2^{2k+1}} + \frac{1}{3^{2k+1}}\right).$$

Stopping this series at $k = 13$ gives the value of π to 10 decimal places.

In 1796 John Machin showed that[1]

$$\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239},$$

which leads to the series

$$\pi = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{16}{5^{2k+1}} - \frac{4}{239^{2k+1}}\right)$$

which converges much faster. Using 73 terms gives π to 100 hundred decimal places.

Using the following variant on this theme, that

$$\pi = 48\arctan\frac{1}{49} + 128\arctan\frac{1}{57} - 20\arctan\frac{1}{239} + 48\arctan\frac{1}{110443},$$

was used by Yasumasa Kanada of Tokyo University in 2002 to compute π to 1,241,100,000,000 digits.

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[1] If you wish to prove this, probably the easiest way is to notice that $(5+i)^4(239-i) = 114244(1+i)$ and use the polar form of complex numbers to get the result.

For a modern method there is the formula found in 1995 by Simon Plouffe:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

which nice form the point of view of computing as powers of 16 are very easy to compute in hexadecimal. In particular using the first n terms of this series gives at least the first n-hexadecimal digits of π. $\square$