Hampton

Ph.D. Qualifying Examination
In Algebra
January 2012

Problem 0.

"If G is a group and $H$and K are normal subgroups of G such that $H \cong K$, then $G/H \cong G/K$." Decide if the statement above is true or false. If true, prove it. If false, give a counterexample.

Problem 1.

a. Let G be a finite Abelian group and d a positive integer that divides the order of G. Prove that G contains a subgroup of order d.
b. Let G be a finite Abelian group, and let d be a square-free positive integer that divides the order of G. Prove that G contains an element of order d. Give an example to show that the statement is no longer true if d is not assumed square-free.
c. Give an example of a group of order 24 that does not contain any element of order 6. Justify your example.

Problem 2.

Identify a familiar group that is isomorphic to the Galois group of $x^3 - 2$ over $\mathbb{Q}$ and prove the group you identified is indeed isomorphic to this Galois group.

Problem 3.

Prove that each subring of the ring of rational numbers is a principal ideal domain.

Problem 4.

Prove that $\mathbb{Q}(2^{\frac{1}{4}}, i)$ is a normal extension of $\mathbb{Q}$, and find the degree $[\mathbb{Q}(2^{\frac{1}{4}}, i) : \mathbb{Q}]$ of this extension.

Problem 5.

Let $f(x) \in \mathbb{Q}[x]$ and let E be the splitting field of $f(x)$ over $\mathbb{Q}$. Prove that if $[E : \mathbb{Q}] = 1323$, then $f(x)$ is solvable be radicals. [Help: $1323 = 3^3 \cdot 7^2$.]

Problem 6.

Prove that $\csc u$ is transcendental whenever u is an algebraic nonzero real number.

Problem 7.

Let H and K be simple groups, and let $G = H \times K$. Let N be a nontrivial proper normal subgroup of G. Prove that N is isomorphic to H or to K.

Problem 8.

Let I be the ideal of $\mathbb{Z}[x]$ generated by $\{3, x^3 - x^2 + 2x - 1\}$. Is $\mathbb{Z}[x]/I$ an integral domain? Prove your answer.

Problem 9.

Prove that every algebraically closed field has infinitely many subfields.