NAKAJIMA'S PROBLEM: CONVEX BODIES OF CONSTANT WIDTH AND CONSTANT BRIGHTNESS

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Dedicated to Rolf Schneider on the occasion of his 65th birthday

Let $\mathbb{G}(n,k)$ denote the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$. A convex body K in $\mathbb{R}^n$ is said to have *constant k-brightness*, $k \in \{1,\dots,n-1\}$, if the k-volume $V_k(K|U)$ of the orthogonal projection of K to the linear subspace $U \in \mathbb{G}(n,k)$ is independent of that subspace. The map

$$\pi_k \colon \mathbb{G}(n,k) \to \mathbb{R}, \qquad U \mapsto V_k(K|U),$$

is referred to as the *kth projection function* of K. Hence a convex body K has constant width (i.e. constant 1-brightness) if it has constant 1st projection function (width function).

1. **Theorem.** *Let K be a convex body in $\mathbb{R}^n$ having constant width and constant k-brightness with $2 \le k < (n+1)/2$, or $k = 3, n = 5$. Then K is a Euclidean ball.*

The preceding two theorems can be generalized to pairs of convex bodies K, K_0 having proportional projection functions, provided that K_0 is centrally symmetric and has a minimal amount of regularity.

2. **Theorem.** *Let K, K_0 be convex bodies in $\mathbb{R}^n$, and let K_0 be centrally symmetric with positive principal radii of curvature on some Borel subset of the unit sphere of positive measure. Let $2 \le k < (n+1)/2$, or let $k = 3, n = 5$ in which case assume the surface area measure $S_4(K_0, \cdot)$ of K_0 is absolutely continuous with positive density. Assume that there are constants $\alpha, \beta > 0$ such that*

$$\pi_1(K) = \alpha\, \pi_1(K_0) \qquad \textit{and} \qquad \pi_k(K) = \beta\, \pi_k(K_0).$$

Then K and K_0 are homothetic.

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