More problems on metric spaces.

Problem 1. Let $A, B \subseteq \mathbb{R}^n$ be non-empty sets and let

$$A + B = \{a + b : a \in A, b \in B\}.$$

Prove or give counterexamples to the following:
(a) A and B compact implies $A + B$ compact.
(b) A closed and B compact implies $A + B$ closed.
(c) A and B both closed implies $A + B$ closed.
(d) A open and B arbitrary implies $A + B$ open.

Problem 2. From the August 2011 exam: Does the series

$$\sum_{n=1}^{\infty} \frac{x}{n + n^3 x^3}$$

converge uniformly on $[0, \infty)$?

Problem 3. Show the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^4 x)}{n^2}$$

is continuous on $\mathbb{R}$.

Problem 4. From the January 2008 exam: For each positive integers n define

$$f_n(x) = \frac{x^n e^{-x}}{(2n)!}.$$

Determine whether of not the sequence $\langle f_n \rangle_{n=1}^{\infty}$ converges uniformly on $[0, \infty)$.