Analysis Qualifying Exam
August 2010

Instructions: Write your name legibly on each sheet of paper. Write only on one side of each sheet of paper. Each question is worth 10 points.

1. Let (X, ρ) and (Y, σ) be metric spaces.
(a) [**6**] Suppose that $f\colon X \to Y$ is *uniformly continuous*. Prove that f maps every Cauchy sequence in X onto a Cauchy sequence in Y.
(b) [**4**] Show that if X is complete then the result of (a) holds for every *continuous* function from X to Y.

2. (a) [**3**] Define: "$\mathcal{M}$ is a σ-algebra of subsets of a nonempty set X".
(b) [**3**] Let $\mathcal{C}$ be a collection of subsets of a nonempty set X. Show that there is a *smallest* σ-algebra $\mathcal{M}$ containing $\mathcal{C}$.
(c) [**4**] Suppose that $f\colon X \to X$ satisfies $f^{-1}(C) \in \mathcal{M}$ for all $C \in \mathcal{C}$. Prove carefully that $f^{-1}(A) \in \mathcal{M}$ for all $A \in \mathcal{M}$.

3. Let μ and ν be finite measures defined on a σ-algebra $\mathcal{M}$ of subsets of X.
(a) [**6**] Show that if there exists $\varepsilon > 0$ such that for all $n \geq 1$ there exists $A_n \in \mathcal{M}$ such that $\nu(A_n) \leq 2^{-n}$ and $\mu(A_n) \geq \varepsilon$, then there exists $A \in \mathcal{M}$ such that $\nu(A) = 0$ and $\mu(A) \geq \varepsilon$. (Hint: Consider $\cap_{n\geq 1} \cup_{k\geq n} A_k$.)
(b) [**4**] Suppose that for all $A \in \mathcal{M}$, if $\nu(A) = 0$ then $\mu(A) = 0$. Deduce from (a) that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\nu(A) < \delta$ then $\mu(A) < \varepsilon$ for each $A \in \mathcal{M}$.

4. Suppose that (f_n) is a sequence of nonnegative *monotone decreasing* functions on $[0, \infty)$ satisfying $f_n(0) \leq 1$ and $\int_0^\infty f_n(x)\,dx \leq 1$.
(a) [**4**] Show that $f_n(x) \leq \min(1, 1/x)$.
(b) [**6**] Suppose that $f_n(x) \to f(x)$ pointwise a.e. Deduce that for all $p > 1$,

$$\lim_{n\to\infty} \int_0^\infty |f_n - f|^p\,dx = 0.$$

5. (a) [**3**] Let $1 < p < \infty$. Defining q appropriately, state Hölder's inequality for $f \in L_p[0,1]$ and $g \in L_q[0,1]$.
(b) [**5**] Suppose that f is monotone increasing on $[0,1]$. Prove that

$$\int_0^1 x^{60} f'(x)^{1/4}\,dx \leq \frac{(f(1) - f(0))^{1/4}}{27}.$$

(c) [**2**] Find a nonconstant function f for which equality is attained.

6. (a) [**3**] Define: "f is absolutely continuous on $[a,b]$".
(b)[**3**] Show that the product of two absolutely continuous functions on $[a,b]$ is absolutely continuous.
(c) [**4**] Suppose that f is absolutely continuous on $[a,b]$. Prove that for all $a \leq x \leq b$,

$$f(x) = f(a) + \lim_{n\to\infty} (\int_a^x (f(y) - f(a))^{n-1} f'(y)\,dy)^{1/n}.$$

Hint: Power Rule!

7. Suppose that f and g are real-valued measurable functions defined on $\mathbb{R}$. The convolution $f * g$ is defined thus:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

(a) **[6]** Suppose that f and g are integrable. Show that $(f * g)(x) < \infty$ a.e. and that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. (Here $\|f\|_1 := \int_{-\infty}^{\infty} |f|\,dx$ as usual.) You may **assume without proof** the measurability of $f(x-y)g(y)$.
(b) **[4]** Suppose, in addition, that f is differentiable everywhere and that $|f'(x)| \leq 2010$ for all $x \in \mathbb{R}$. Using the Dominated Convergence Theorem, or otherwise, prove carefully that $f * g$ is differentiable everywhere and that $(f * g)' = f' * g$.

8. Let $f(z) = u(x, y) + iv(x, y)$ be analytic on an open set $U \subseteq \mathbb{C}$. (Here u and v are real-valued.)
(a)**[3]** State the Cauchy-Riemann equations for u and v.
(b)**[3]** Hence show that u is *harmonic*, i.e. $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$.
(c)**[4]** Deduce that $\ln|f|$ is harmonic provided f does not vanish on U.

9. Suppose that $f(z)$ is analytic on a convex domain $U \subseteq \mathbb{C}$.
(a) **[5]** Show that f'/f has a simple pole at every zero z_0 of f and compute $\mathrm{Res}(f'/f, z_0)$.
(b) **[3]** Hence show that $N := \int_\gamma f'/f\,dz$ is a nonnegative integer for every positively oriented simple closed piecewise-smooth curve γ in U which avoids the zeros of f.
(c)**[2]** What does N represent?

10. True or False? Prove or give a counterexample with justification.
(a) **[3]** If f is continuous and monotone increasing on $[0, 1]$ then $\int_0^1 f'(x)\,dx = f(1) - f(0)$.
(b) **[4]** Suppose that (f_n) is a sequence of nonnegative continuous functions on $[0, 1]$ which converges pointwise to zero. Then $\int_0^1 f_n\,dx \to 0$ as $n \to \infty$.
(c) **[3]** Suppose that f has an essential singularity at $z = 0$. Then there exists a sequence (z_n) such that $z_n \to 0$ and $f(z_n) \to 2010i$.