More analysis problems.

The first couple of problems have to do with the completeness of the L^p spaces.

Theorem 1. *Let (X,μ) and $1 \le p < \infty$ be a measure space and let $g_1, g_2, g_3, \ldots \in L^p(X,\mu)$, so that*

$$M := \sum_{k=1}^{\infty} \|g_k\|_{L^p} < \infty.$$

Then the series

$$g(x) := \sum_{k=1}^{\infty} g_k(x)$$

converges absolutely for almost all $x \in X$. Also $g \in L^p(X)$ and the series converges to g in L^p in the sense that

$$\lim_{n\to\infty} \left\| g - \sum_{k=1}^{n} g_k \right\|_{L^p} = 0.$$

Problem 1. Prove this. *Hint:* Here is an outline of one way to do this. First a bit of notation. Let

$$G_n(x) = \sum_{k=1}^{n} |g_k(x)|.$$

(a) Use the Minkowski inequality (that is the triangle inequality in L^p) to show

$$\left(\int_X G_n^p \, d\mu \right)^{\frac{1}{p}} = \|G_n\|_{L^p} \le \sum_{k=1}^{\infty} \|g_n\|_{L^p} = M.$$

and therefore

$$\int_X G_n^p \, d\mu \le M^p < \infty$$

for all n.

(b) Show the sequence $\langle G_n^p \rangle_{k=1}^{\infty}$ satisfies the hypothesis of the Monotone Convergence Theorem and use this to show the limit

$$S(x) = \lim_{n\to\infty} G_n(x)^p = \left(\sum_{k=1}^{n} |g_k(x)| \right)^p$$

exists almost everywhere, which is equivalent to $\sum_{k=1}^{\infty} g_k(x)$ being absolutely convergent almost everywhere, and

$$\int_X \left(\sum_{k=1}^{\infty} |g_k(x)| \right)^p d\mu = \int_X S(x) \, d\mu \le M^p.$$

(c) Use Part (b) to show $g \in L^p(X)$ and
$$\|g\|_{L^p} \le \sum_{k=1}^{\infty} \|g_k\|_{L^p}.$$
(d) Let $S_n = \sum_{k=1}^{n} g_k$ be the n-th partial sum for the series for g. Then $g - S_n = \sum_{k=n+1}^{\infty}$ and therefore applying where we have already proven to the sequence $g_{n+1}, g_{n+1}, \ldots$ we have
$$\|g - S_n\|_{L_p} \le \sum_{k=n+1}^{\infty} \|g_k\|_{L^p}.$$
Use this to show $\lim_{n\to\infty} \|g - S_n\|_{L^p} = 0$ and complete the proof. $\square$

Theorem 2 (Riesz–Fischer Theorem)**.** *For any measure space* (X, μ) *and* $1 \le p < \infty$ *the space* $L^p(X)$ *is a complete metric space.*

Problem 2. Prove this. *Hint:* Let $\langle f_k \rangle_{k=1}^{\infty}$ be a Cauchy sequence in $L^p(X)$. Show that by replacing this with a subsequence may assume $\|f_k - f_{k-1}\|_{L^p} < 1/2^k$ for all $k \ge 2$. Let $g_1 = f_1$ and $g_k = f_k - f_{k-1}$ for $k \ge 2$, so that the partial sums of the series $\sum_{k=1}^{\infty} g_k$ are $\sum_{k}^{n} g_k = f_n$. Now use Problem 1 to complete the proof. $\square$

Problem 3. This is a lemma for the next problem. Show that for any $a, b \ge 0$ the inequality
$$\frac{a+b}{1+a+b} \le \frac{a}{1+a} + \frac{b}{1+b}$$
holds. $\square$

Let (X, μ) be a measure space with $\mu(X) < \infty$. Let $L^0(X)$ be the set of measurable functions $f \colon X \to \mathbb{R}$. For $f \in L^0(X)$ let
$$\|f\| = \int_X \frac{|f(x)|}{1 + |f(x)|}\, d\mu(x).$$
(a) Show that for all $f \in L^0(X)$
$$0 \le \|f\| \le \mu(X)$$
and that $\|f\| = 0$ if and only if $f = 0$ almost everywhere.
(b) For $f, g \in L^0(X)$ show
$$\|f + g\| \le \|f\| + \|g\|.$$
(c) For $f, g \in L^0(X)$ define
$$d(f, g) = \|f - g\|.$$
Show this makes $L^0(X)$ into a metric space.
(d) Show
$$\lim_{n\to\infty} \|f - f_n\| = 0$$
if and only if $f_n \to f$ in measure.
(e) Show with this metric the space $L^0(X)$ is a complete metric space. $\square$

Problem 4. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. That is the set $\{x : \varphi(x) \neq 0\}$ has compact closure. Show that φ is uniformly continuous and that

$$\lim_{h\to 0} \int_{\mathbb{R}} |\varphi(x+h) - \varphi(x)|\, dx = 0. \qquad \square$$

Problem 5. (a) Let $[a,b]$ be a bounded interval in $\mathbb{R}$ and let $s = \mathbb{1}_{[a,b]}$. Show

$$\lim_{h\to 0} \int_{\mathbb{R}} |s(x+h) - s(x)|\, dx = 0$$

(b) Let $\varphi = \sum_{k=0}^{n} c_k \mathbb{1}_{[a_k,b_k]}$ be a step function with compact support. Use Part (a) and linearity to show

$$\lim_{h\to 0} \int_{\mathbb{R}} |\varphi(x+h) - \varphi(x)|\, dx = 0. \qquad \square$$

Problem 6. Let $f \in L^1(\mathbb{R})$. Show

$$\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h) - f(x)|\, dx = 0.$$

Hint: Reduce this to either Problem 4 or Problem 5. $\square$