Mathematics 739 Homework 2: Differential forms.

On $\mathbb{R}^n$ a zero form is just a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$. A one form is an expression

$$\alpha = \sum_{j=1}^{n} dx^j$$

where $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$ and the a_j's are smooth functions. We also view each dx^j as a linear functional on $\mathbb{R}^n$ by letting

$$dx^j \left(\sum_{k=1}^{n} v^k \frac{\partial}{\partial x^k} \right) = v^j$$

where $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}$ is the standard basis of $\mathbb{R}^n$. Here we are identifying vectors with point deprivations. That is if $p \in \mathbb{R}^n$ and v is a vector at p (i.e. $v \in TM_p$) then we can also view v as the directional derivative in the direction of v. That is if f is a smooth real valued function, then

$$v(f) = \frac{d}{dt} f(p + tv)\bigg|_{t=0}.$$

This operator satisfies that $f \mapsto v(f)$ is linear over $\mathbb{R}$ and than

$$v(fg) = f(p)v(g) + v(f)g(p).$$

Proposition 1. *If V is an operator on smooth real valued functions on $\mathbb{R}^n$ such that V is linear over $\mathbb{R}$ and for some point $p \in \mathbb{R}^n$*

$$V(fg) = V(f)g(p) + f(p)V(g)$$

then there is a vector v at p such that V is given by

$$V(f) = \frac{d}{dt} f(p + tv)\bigg|_{t=0}.$$

Thus V is naturally identified with a vector to $\mathbb{R}^n$ at the point p.

Problem 1. Prove this. *Hint:* First show $V(c) = 0$ for any constant c. Then show if h_1 and h_2 are smooth functions with $h_1(p) = h_2(p) = 0$, then for any smooth function g that $V(h_1 h_2 g) = 0$. Now use some form or anther of Taylor's theorem to write the smooth function f as

$$f = f(p) + \sum_{j=1}^{n} a_j (x^j - p^j) + \sum_{j,k=1}^{n} (x^j - p^j)(x^k - p^k) g_{jk}$$

where the a_j's are constants and the $g_{jk}s$'s are smooth functions. Put this all together to conclude

$$V(f) = \sum_{j=1}^{n} a_j V(x^j) = \frac{d}{dt} f(p + tv)\bigg|_{t=0}$$

where v is the vector $v = \sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}$. $\square$

One reason for viewing vectors this way is that this definition is easy to generalize to manifolds. Let M be a smooth manifold and, $C^\infty(M)$ the algebra of smooth real valued functions on M and $p \in M$ a point. Then we can define a ***point derivation*** at p to be a map $f \mapsto V(f)$ which is linear over $\mathbb{R}$ and such that for $f, g \in \mathbb{C}^\infty(M)$

$$V(fg) = V(f)g(p) + f(p)V(g).$$

Then the set of all such point derivations at p form the tangent space, TM_p, to M at p. To make this definition a bit more geometric let $c\colon (-\delta, \delta)$ be a smooth curve with $c(0) = p$. Then

$$V(f) = \frac{d}{dt} f(c(t))\bigg|_{t=0}$$

is a point derivation at p which we denote, naturally enough, at $c'(0)$. It is the tangent vector to c at $t = 0$. An easy extension of Proposition 1 shows that every $v \in TM_p$ can be realized as the tangent vector to a curve through p.

If $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$ is anther coordinate system on $\mathbb{R}^n$ and let the one form α be given in the two coordinate systems by

$$\alpha = \sum_{j=1}^n a_j\, dx^j = \sum_{j=1}^n \tilde{a}_j\, d\tilde{x}^j.$$

Then

$$\tilde{a}_j = \sum_{k=1}^n \frac{\partial x^k}{\partial \tilde{x}^j} a_k.$$

This is often written as

$$\tilde{a}_j = \frac{\partial x^k}{\partial \tilde{x}^j} a_k$$

with the convention that we sum over any repeated index.[1]

Problem 2. Prove this transformation rule. Also show that if a vector field is given in the two coordinates systems as

$$\sum_{j=1}^n a^j \frac{\partial}{\partial x^j} = \sum_{j=1}^n \tilde{a}^j \frac{\partial}{\partial \tilde{x}^j}$$

then

$$\tilde{a}^j = \sum_{k=1}^n \frac{\partial \tilde{x}^j}{\partial x^k} a^k = \frac{\partial \tilde{x}^j}{\partial x^k} a^k.$$

□

[1]This convention seems to have been introduced by Einstein in his paper *Die Grundlage der allgemeinen Relativitätstheorie* in Annalen der Physik in 1916. This is why it is often called the Einstein summation convention.

If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function we define its ***differential*** (also called its ***exterior derivative***) by

$$df = \sum_{=1}^{n} \frac{\partial f}{\partial x^j}\, dx^j.$$

The chain rule shows that this is the linear functional defined on vectors by

$$df_p(v) = \frac{d}{dt} f(p+tv)\bigg|_{t=0}.$$

Problem 3. Show that the definition of df is independent of the coordinate system used to define it. □

If $1lek \le n$ a smooth k-form is sum of the form

$$\alpha = \sum_{1\le j_1<j_2<\cdots<j_k\le n} a_{j_1 j_2\cdots j_k}\, dx^{j_1}\wedge dx^{j_2}\wedge\cdots\wedge dx^{j_k}$$

where each of the $a_{j_1 j_2\cdots j_k}$ are smooth functions. The wedge product $\wedge$ is so that

$$dx^j \wedge dx^k = -dx^k \wedge dx^j$$

which implies that for any j

$$dx^j \wedge dx^j = 0.$$

The products $dx^{j_1}\wedge dx^{j_2}\wedge\cdots\wedge dx^{j_k}$ can be view as k-linear alternating functions as follows. For $k=2$

$$dx^{j_1}\wedge dx^{j_2}(u,v) = dx^{j_1}(u)dx^{j_2}(v) - dx^{j_1}(v)dx^{j_2}(u) = \det\begin{bmatrix} dx^{j_1}(u) & dx^{j_2}(v) \\ dx^{j_1}(v) & dx^{j_2}(u)\end{bmatrix}$$

and in general

$$dx^{j_1}\wedge dx^{j_2}\wedge\cdots\wedge dx^{j_k}(v_1,v_2,\ldots,v_k) = \det\left(\left[dx^{j_s}(v_t)\right]_{s,t=1}^{k}\right).$$

In writing differential forms it is useful to use the multi-index notation. Let $J=(j_1,j_2,\ldots,j_k)$ then set

$$dx^J = dx^{j_1}\wedge dx^{j_2}\wedge\cdots dx^{j_k}.$$

Problem 4. With this notation

(a) If J has a repeated index, then $dx^J = 0$.
(b) If the elements of J' are a permutation of the elements of J, say $J' = (j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(k)})$ with σ a permutation of $\{1,2,\ldots,k\}$, then $dx^{J'} = \operatorname{sign}(\sigma) dx^J$.
(c) If J and L have an element in common, then $dx^J\wedge dx^L = 0$.
(d) If J and L have no element in common and J has degree k and L has degree ℓ, then $dx^L \wedge dx^J = (-1)^{k\ell} dx^J\wedge dx^L$.

We can now write a k form α as

$$\alpha = \sum_J a_J \, dx^J$$

where, depending on which is more useful in a given context, the sum is either over all length k multi-indices or over all increasing multi-indices.

If α and β are forms, say

$$\alpha = \sum_J a_J \, dx^J, \qquad \beta = \sum_L b_L \, dx^L$$

then the ***wedge product*** (also called the ***exterior product***) of these is

$$\alpha \wedge \beta = \sum_{J,L} a_J b_L \, dx^J \wedge dx^L.$$

Problem 5. Show this product is associative and its definition is independent of the coordinate system used. □

Problem 6. Let α be a k-form and β a ℓ-form.
(a) Show that $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$.
(b) Show that if k is odd, then $\alpha \wedge \alpha = 0$.
(c) Let $\omega = dx^1 \wedge dx^2 + dx^2 \wedge dx^4$. Show $\omega \wedge \omega = 2\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \neq 0$. Thus it is not true $\alpha \wedge \alpha = 0$ for all forms. □

We can now extend the definition of the differential, df, of a smooth function to general forms. Let

$$\alpha = \sum_J a_J \, dx^J.$$

Then its ***exterior derivative*** is

$$d\alpha = \sum_J da_J \wedge dx^J.$$

Proposition 2. *This definition is independent of the coordinate system used to define it. Also*
(a) For any form α

$$dd\alpha = 0.$$

(b) If α *is a* k *form and* β *is a* ℓ *form*

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta).$$

The following is one of the more important results about differential forms.

Theorem 3 (Poincaré lemma)**.** *Let* α *be a smooth form defined on a contractable open subset* U *of* $\mathbb{R}^n$*. If* $d\alpha = 0$*, then there is a form* β *with*

$$d\beta = \alpha.$$