Mathematics 574 Homework

Read sections 2.1 and 2.2 in the text.

Here is some review of Math 142 that we will be using. Let $f(x)$ be a from some open interval (a, b) containing 0. Then recall that $f(x)$ has a ***Taylor series***[1]

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

We now derive formulas for the coefficients x_n. To start with we will write $f(x)$ as

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

and first find a formula for a_0.

1. Let $x = 0$ if the formula for $f(x)$ to show that $a_0 = f(0)$. $\square$

2. We now take some derivatives of $f(x)$. Show the following

$$\begin{aligned}
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots \\
f''(x) &= 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + 6 \cdot 5a_6 x^5 + \cdots \\
f'''(x) &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + 5 \cdot 4 \cdot 3a_5 x^2 + 6 \cdot 5 \cdot 4a_6 x^3 + \cdots \\
f^{(4)}(x) &= 4 \cdot 3 \cdot 2a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5 x + 6 \cdot 5 \cdot 4 \cdot 3a_6 x^2 + \cdots \\
f^{(5)}(x) &= 5 \cdot 4 \cdot 3 \cdot 2a_5 + 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2a_6 x + \cdots
\end{aligned}$$

$\square$

3. If we let $x = 0$ in the formuala for $f'(x)$ we get $f'(x) = a_1 + 0 = a_1$. Thus $a_1 = f'(x)$. If we let $x = 0$ in the formula for $f''(x)$ we get

$$f''(x) = 2a_0 + 0 = 2a_2$$

and whence

$$a_2 = \frac{f''(0)}{2}$$

(a) Let $x = 0$ in the formula for $f'''(0)$ to get a formula for a_3.
(b) Let $x = 0$ in the formula for $f^{(4)}(x)$ to get a formula for a_4.
(c) Let $x = 0$ in the formula for $f^{(5)}(x)$ to get a formula for a_5. $\square$

4. After the last problem your can probably guess what this problem will be. Compute the n-th derivative $f^{(n)}(x)$ and let $x = 0$ in this formula to show that

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

$\square$

Putting these pieces together we have

[1] Not every function has a Taylor series. We will only get working with ones that do, so we will just assume all functions that come up do have power series.

Theorem 1. *If the function $f(x)$ has a Taylor series around $x = 0$, then*

$$f(x) = \sum_{n=0}^{\infty} a_n x \qquad \textit{where} \qquad a_n = \frac{f^{(n)}(0)}{n!}.$$

□

Here is an example. Let

$$f(x) = \frac{1}{1+x} = (1+x)^{-1}.$$

Then we have

$$\begin{aligned} f'(x) &= -(1+x)^{-2} \\ f''(x) &= (-1)(-2)(1+x)^{-3} \\ f'''(x) &= (-1)(-2)(-3)(1+x)^{-4} \\ f^{(4)}(x) &= (-1)(-2)(-3)(-4)(1+x)^{-5} \\ \vdots \quad &\vdots \\ f^{(n)} &= (-1)(-2)(-3)\cdots(-n)(1+x)^{-n-1}. \end{aligned}$$

Thus

$$f^{(n)}(0) = (-1)(-2)(-3)\cdots(-n)(1+0)^{-n-1} = (-1)^n n!$$

and therefore

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^n n!}{n!} = (-1)^n.$$

This gives

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \cdots$$

5. *Here is a generalization of this for you to do. Let α be any real number and set*

$$f(x) = (1+x)^{\alpha}.$$

Then

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1} \\ f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\ f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \\ f^{(4)}(x) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)(1+x)^{\alpha-4} \\ f^{(5)}(x) &= \alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(1+x)^{\alpha-5} \\ \vdots \quad &\vdots \end{aligned}$$

Use this start to get a formula for $f^{(n)}(x)$ and use it to find a formula for the coefficient a_n in the Taylor series of $f(x)$. □