Some problems related to solvable groups.

Recall that we called a subgroup H of a group G ***characteristic*** if and only if $\phi[H] = H$ for all $\phi \in \operatorname{Aut}(G)$. We will use the notation $H \underset{\text{char}}{\lhd} G$ for "H is a characteristic subgroup of G".

Problem 1. Show that any characteristic subgroup of a group is also a normal subgroup. $\square$

Problem 2. $H \underset{\text{char}}{\lhd} N$ and $N \underset{\text{char}}{\lhd} G$, then $H \underset{\text{char}}{\lhd} G$. $\square$

If G is a group let G' (the ***commutator subgroup***, also called the ***derived subgroup***) of G is

$$G' = \langle aba^{-1}b^{-1} : a, b \in G \rangle$$

is the subgroup of G generated by the commutators of G.

Problem 3. Prove the G' is a characteristic subgroup of G. $\square$

Problem 4. Show that G/G' is Abelian. $\square$

Problem 5. Let $N \lhd G$ with G/N Abelian. Show that $G' \leq N$. $\square$

Problem 6. For $n \geq 3$ show that the commutator subgroup of symmetric group $G = S_n$ is $G' = A_n$ is the alternating group. *Hint:* Let a, b, c be distinct elements of $\{1, 2, \ldots, n\}$ and then the commutator of the elements $x = (ab)$ and $y = (bc)$ is $xyx^{-1}y^{-1} = (abc)$. Therefore G' contains all the three cycles and the three cycles generate A_n. $\square$

Problem 7. If $n \geq 5$ and $G = A_n$, show $G' = G$. *Hint:* If $a = (123)$ and $b = (145)$ show $aba^{-1}b^{-1} = (153)$. Generalize this calculation to show that G' contains all the three cycles and thus $G' = A_n$. This shows that for $n \geq 5$ that neither S_n or A_n are solvable. $\square$

For any group G the ***derived series*** is the sequence of subgroups defined by

$$\begin{aligned} G^{(0)} &= G \\ G^{(1)} &= G' \\ G^{(2)} &= (G^{(1)})' \\ G^{(2)} &= (G^{(1)})' \\ G^{(3)} &= (G^{(2)})' \\ \vdots &= \vdots \\ G^{(k+1)} &= (G^{(k)})'. \end{aligned}$$

The group if ***solvable*** if and only if there is an n such that $G^{(n)} = \langle 1 \rangle$. Note in an Abelian group all commutators are $aba^{-1}b^{-1} = 1$ and thus $G' = \langle 1 \rangle$. So all Abelian groups are solvable.

Problem 8. Show that if $|G| = p^n$ for some prime p, that G is solvable. $\square$

Problem 9. Let $\mathbb{F}$ be a field and G the group of 3×3 nonsingular upper triangular matrices over $\mathbb{F}$:

$$G = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} : \text{ such that } adf \neq 0 \right\}$$

Show G is solvable. Generalize this to $n \times n$ nonsingular upper triangular matrices over $\mathbb{F}$. $\square$

For the next few problems my source my source is McNulty's notes, Lecture 18.

A ***subnormal series*** for a group G is a sequence of subgroups

$$G_0 = G > G_1 > G_2 > \cdots G_{n-1} > G_n = \langle 1 \rangle.$$

where for each k we have $G_{k+1} \triangleleft G_k$. But we are not assuming that G_{k+1} is normal in G, just that it is normal in the subgroup in the series just above it.

Problem 10. Some authors define a group to be solvable if and only if it has a subnormal series with each quotient G_k/G_{k+1} Abelian. Show this definition is equivalent to the one we have given. $\square$

If $A, B \subseteq G$ with G a group, let

$$[A, B] = \langle aba^{-1}b^{-1} : a \in A, b \in B \rangle.$$

Define anther sequence of subgroups of G by

$$\begin{aligned} G^{[0]} &= G \\ G^{[1]} &= \big[G, G^{[0]}\big] \\ G^{[2]} &= \big[G, G^{[1]}\big] \\ G^{[3]} &= \big[G, G^{[2]}\big] \\ G^{[4]} &= \big[G, G^{[3]}\big] \\ \vdots &= \vdots \\ G^{[k+1]} &= [G, G^{[k]}]. \end{aligned}$$

The groups is ***nilpotent*** if and only if there is an n so that $G^{[n]} = \langle 1 \rangle$.

Problem 11. Show that any nilpotent group is solvable. *Hint:* Show $G^{[k]} \leq G^{(k)}$. $\square$

Problem 12. Show that for each k we have $G^{[k]} \underset{\text{char}}{\triangleleft} G$ and thus each $G^{[k]}$ is normal in G. $\square$

Problem 13. Show that if G is nilpotent, then its center $Z(G)$ is nontrivial. Thus any solvable group with trivial center (i.e. S_3) is an example of a solvable group that is not nilpotent. *Hint:* Consider $G^{[n-1]}$ where $G^{[n-1]} \neq \langle 1 \rangle$ and $G^{[n]} = \langle 1 \rangle$. $\square$

Problem 14. As a variant on Problem 9 show that the group G in that problem is not nilpotent, but that

$$N = \left\{ \begin{bmatrix} 1 & b & c \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \, : \, b, c, e \in \mathbb{F} \right\}$$

is nilpotent, and generalize this to $n \times n$ matrices. $\square$

Problem 15. (A challenge problem.) Show that any finite nilpotent group is the direct product of its Sylow subgroups. $\square$