Some problems related to the Residue Theorem.

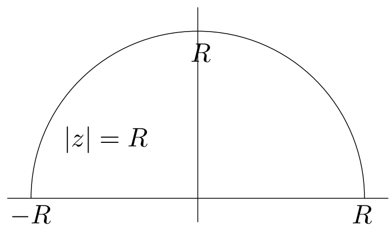
One of the most famous applications of complex analysis is to use the residue theorem to evaluate definite integrals that are hard or just about impossible to do by other means. Let us do an example (which is chosen because it illustrates several ideas): evaluate

$$\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3+9x}\,dx.$$

We will do this by evaluating the complex integral

$$\int_{C_R} \frac{e^{2iz}-1}{z^3+9z}. \tag{1}$$

over the curve:



which is the top half of the circle $|z| = R$ together with the part of the real axis between $x = -R$ and $x = R$.

Problem 1. Show that the function

$$f(z) = \frac{e^{2iz}-1}{z^3+9z}$$

has a removable singularity at $z = 0$. *Hint:* There are several ways to do this, but what I find easiest is to first write the function as

$$f(z) = \left(\frac{e^{2iz}-1}{z}\right)\left(\frac{1}{z^2+9}\right)$$

The fraction $\dfrac{1}{z^2+9}$ is clearly analytic at $z = 0$, so it is enough to show

$$g(z) = \frac{e^{2iz}-1}{z}$$

has a removable singularity at $z = 0$. Now you should be able to show that

$$\lim_{z\to 0} g(z) = L$$

exists (and you should give the value of L). This shows that $g(z)$ is bounded near $z = 0$ and we have a theorem about functions bounded near an isolated singularity. $\square$

Problem 2. Explain why the only singularities of

$$f(z) = \frac{e^{2iz}-1}{z^3+9z}$$

are at $z = 3i$ and $z = -3i$ and that why in computing the integral in (1) only the singularity at $z = 3i$ matters. $\square$

Problem 3. Compute the number

$$\rho_{3i} = \operatorname{Res}(f, 3i)$$

where $f(z)$ is as above. *Hint:* Your answer should be a positive real number. $\square$

By the residue theorem we have

$$\int_{C_R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz = 2\pi i \operatorname{Res}(f, 3i) = 2\pi i \rho_{3i}. \tag{2}$$

Split the integral over C_R into two pieces, the the part on the real axis and the part on the circle, call the circular part S_R. Then

$$\int_{C_R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz = \int_{-R}^{R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz + \int_{S_R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz \tag{3}$$

Problem 4. Use the parameterization $z = x$ on the real axis to show

$$\begin{aligned}\int_{-R}^{R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz &= \int_{-R}^{R} \frac{\cos(2x) - 1 + i\sin(2x)}{x^3 + x}\, dx \\ &= \int_{-R}^{R} \frac{\cos(2x) - 1}{x^3 + x}\, dx + i \int_{-R}^{R} \frac{\sin(2x)}{x^3 + x}\, dx.\end{aligned}$$

$\square$

Now we deal with the integral over S_R. Recall that we have proven

Proposition 1. *Let γ be a curve of length L and $f(z)$ a function defined on γ such that $|f(z)| \le M$ for some number M. Then*

$$\left| \int_{\gamma} f(z)\, dz \right| \le ML.$$

We will now apply this to the integral

$$\int_{S_R} \frac{e^{2iz} - 1}{z^3 + 9z}\, dz.$$

Since with have not done much with inequalities on class, I will do the estimate of $\frac{e^{2iz}-1}{z^3+9z}$ over the curve S_R. First observe that for $z = z + iy$ on S_R we have

$$\begin{aligned}|e^{2iz}| &= |e^{i(x+iy)}| \\ &= |e^{-y+ix}| \\ & |e^{-y}| && (\text{as } |e^{ix}| = 1) \\ &\le 1 && (\text{as on } S_R \text{ we have } y \ge 0 \text{ and so } e^{-y} \le 1).\end{aligned}$$

Therefore, when $R > 3$,

$$\begin{aligned}
\left|\frac{e^{2iz}-1}{z^3+9z}\right| &= \frac{|e^{2iz}-1|}{|z^3+9z|} \\
&\le \frac{|e^{2iz}|+1}{|z^3+9z|} && \text{(triangle inequality)} \\
&\le \frac{2}{|z^3+9z|} && \text{(using } |e^{iz}| \le 1\text{)} \\
&\le \frac{2}{|z^3|-9|z|} && \text{(using the inequality } |a+b| \ge |a|-|b|\text{)} \\
&= \frac{2}{R^3-9R} && \text{(as } |z| = R \text{ on } S_R\text{).}
\end{aligned}$$

Problem 5. Use Proposition 1 and the inequalities just given to show

$$\left|\int_{S_R} \frac{e^{2iz}-1}{z^3+9z}\,dz\right| \le \frac{2\pi R}{R^3-9R} = \frac{2\pi}{R^2-9}$$

and then use this to show

$$\lim_{R\to\infty} \int_{S_R} \frac{e^{2iz}-1}{z^3+9z}\,dz = 0.$$

Problem 6. Put together the last several problems to conclude

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\cos(2x)-1}{x^3+x}\,dx + i\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3+x}\,dx \\
&= \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{\cos(2x)-1}{x^3+x}\,dx + i\int_{-R}^{R} \frac{\sin(2x)}{x^3+x}\,dx\right) \\
&= \lim_{R\to\infty} \int_{C_R} \frac{e^{2iz}-1}{z^3+9z}\,dz \\
&= \lim_{R\to\infty} 2\pi i \rho_{3i} \\
&= 2\pi i \rho_{3i}
\end{aligned}$$

where ρ_{3i} is the residue you computed in Problem 3. Since ρ_{3i} is a positive real number we can compare the real and imaginary parts of this to get our final result:

$$\int_{-\infty}^{\infty} \frac{\sin(2x)}{x^3+x}\,dx = 2\pi\rho_{3i}$$

$$\int_{-\infty}^{\infty} \frac{\cos(2x)-1}{x^3+x}\,dx = 0. \qquad \square$$

One of the final topics we covered was

Theorem 2 (Roché's Theorem)**.** *Let D be a bounded domain in $\mathbb{C}$ with nice boundary and let $f(z)$ and $g(z)$ be functions that are analytic on the closure, $\overline{D}$, of D. Assume*

$$|g(z)| < |f(z)|$$

on the boundary ∂D. Then $f(z) + g(z) = 0$ and $f(z) = 0$ have the same number of solutions in D (where the number of solutions is counted with multiplicity). □

Note that since we are counting with multiplicity, the function $f(z) = z^n$ has n solutions inside the unit circle $|z| = 1$ as the solution $z = 0$ is counted n times.

Here is an example of using this. Let $a, b, c \in \mathbb{C}$ with

$$|a| + |b| + |c| < 1$$

then the polynomial

$$p(z) = z^3 + az^2 + bz + c$$

has three roots inside of the circle $|z| = 1$. To see this let

$$f(z) = z^3$$
$$g(z) = az^2 + bz + c.$$

Let $D = \{z : |z| < 1\}$. We wish to show that all three of the roots of $p(z) = f(z) + g(z)$ are in D and we will use Roché's Theorem to do this. Note that on ∂D we have $|z| = 1$ and thus on ∂D

$$|f(z)| = |z|^3 = 1^3 = 1.$$

And for z on ∂D we can use the triangle inequality to conclude

$$|g(z)| = |az^2 + bz + c| \leq |a||z|^2 + |b||z| + |c| = |a| + |b| + |c| < 1 = |f(z)|.$$

Thus by Roché's $p(z) = f(z) + g(z)$ and $f(z) = z^3$ have the same number of roots in D. As $f(z) = z^3$ has three roots in D so does $p(z)$.

Here is a generalization of this example.

Proposition 3. *Let $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$ with*

$$|a_0| + |a_1| + \cdots + |a_{n-1}| < 1.$$

Then the polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-1} + \cdots + a_1 z + a_0$$

has all n of its roots inside the circle $|z| = 1$.

Problem 7. Prove this. □

It is worth remarking that with only a little more work this argument can be used to give anther proof of the Fundamental Theorem of Algebra.

Here we look at anther application of the residue theorem. Recall that if we have a fraction

$$f(z) = \frac{g(z)}{h(z)}$$

then at a point $z = a$ where $h(a) = 0$ and $h'(a) \neq 0$ then $f(z)$ has a simple pole at $z = a$ and the residue of $f(z)$ at $z = a$ is

$$\operatorname{Res}(f, a) = \frac{g(a)}{h'(a)}.$$

We use this to evaluate integrals of the form

$$\int_0^{2\pi} R(\cos t, \sin t)\, dt$$

where $R(x, y)$ is a rational function of x and y. The idea is to let

$$z = e^{it}$$

with $0 \leq t \leq 2\pi$. Then

$$\begin{aligned}
\cos t &= \frac{e^{it} + e^{-it}}{2} \\
&= \frac{z + z^{-1}}{2} \\
&= \frac{z^2 + 1}{2z} \\
\sin t &= \frac{e^{-it} - e^{-it}}{2i} \\
&= \frac{z - z^{-1}}{2i} \\
&= \frac{z^2 - 1}{2iz}.
\end{aligned}$$

Also

$$dz = ie^{it}\, dt = iz\, dt$$

and therefore

$$dt = \frac{dz}{iz}.$$

Also as t goes from 0 to 2π the variable $z = e^{it}$ moves over the unit circle defined by $|z| = 1$. Therefore with this change of variable we have

$$\int_0^{2\pi} R(\cos t, \sin t)\, dt = \int_{|z|=1} R\left(\frac{z^2 + 1}{2z}, \frac{z^2 - 1}{2iz}\right) \frac{dz}{iz}.$$

Here is an example. Let $a > 1$ and let us compute

$$I(a) = \int_0^{2\pi} \frac{\cos t}{a + \cos t} dt.$$

Using the substitution $z = e^{it}$ this becomes

$$\begin{aligned} I(a) &= \int_{|z|=1} \frac{\frac{z^1+1}{2z}}{a + \frac{z^2+1}{2z}} \frac{dz}{iz} \\ &= \int_{|z|=1} \frac{z^2+1}{2z} \frac{1}{a + \frac{z^2+1}{2z}} \frac{dz}{iz} \\ &= \int_{|z|=1} \frac{z^2+1}{(2az + z^2 + 1)} \frac{dz}{iz} \\ &= \frac{1}{i} \int_{|z|=1} \frac{z^2+1}{z(z^2+2az+1)} dz \end{aligned}$$

The singularities of the integrand

$$f(z) = \frac{z^2+1}{z(z^2+2az+1)}$$

are where the denominator is zero. That is when $z = 0$ or $z^2 + 2az + 1 = 0$. The easiest way to solve the later is by completing the square. The equation is equivalent to

$$z^2 + 2az + 1 = (z+a)^2 - a^2 + 1 = 0$$

and so

$$(z+a)^2 = \sqrt{a^2-1}$$

and therefore

$$a = -a \pm \sqrt{a^2-1}.$$

Of these two roots one is $-a - \sqrt{a^2-1} < -1$ and therefore is not in the circle $|z| = 1$. The other root is

$$\beta = -a + \sqrt{a^2-1}$$

which is inside of the unit circle. We now compute the residues. Our function is

$$f(z) = \frac{z^2+1}{z(z^2+2az+1)} = \frac{g(z)}{h(z)}.$$

with $g(z) = z^2 + 1$ and $h(z) = z(z^2 + 2az + 1) = z^3 + 2az^2 + z$. We will also need the derivative of $h(z)$ which is

$$h'(z) = 3z^2 + 4az + 1.$$

So the residue at $z = 0$ is

$$\operatorname{Res}(f, 0) = \frac{g(0)}{h'(0)} = \frac{1}{1} = 1.$$

The residue at $z = \beta$ is

$$\operatorname{Res}(f, \beta) = \frac{g(\beta)}{h'(\beta)} = -2a\sqrt{a^2 - 1}.$$

(A lot of algebra was skipped in simplifying $g(\beta)/h'(\beta)$.) And now we are pretty much done:

$$\begin{aligned} I(a) &= \frac{1}{i} \int_{|z|=1} \frac{z^2 + 1}{z(z^2 + 2az + 1)} dz \\ &= \frac{1}{i} 2\pi i \Big(\operatorname{Res}(f, 0) + \operatorname{Res}(f, \beta) \Big) \\ &= 2\pi \big(1 - 2a\sqrt{a^2 - 1}\, \big) \end{aligned}$$