More problems on L^p spaces.

Problem 1. This problem is review, but we will be using it below. Let (X,μ) be a measure space and $f\colon X\to\mathbb{R}$ a measurable function. Show there is a sequence of non-negative simple functions $\langle\phi_n\rangle_{n=1}^\infty$ that increase pointwise to f. That is $0\le\phi_1\le\phi_2\le\phi_3\le\cdots$ and $\lim_{n\to\infty}\phi_n(x)=f(x$ for all x. *Hint:* For each pair of positive integers n,k let

$$E_{n,k}=\left\{x\in X:\frac{k-1}{2^n}\le f(x)<\frac{k}{2^n}\right\}.$$

and set

$$\phi_n=\sum_{k=0}^{n2^n}\frac{k}{2^n}\mathbb{1}_{E_{n,k}}.$$

Verify this does the trick. □

Recall the function $\operatorname{sgn}\colon\mathbb{R}\to\{-1,0,1\}$ is

$$\operatorname{sgn}(x)=\begin{cases}+1, & x>0\\ 0, & x=0;\\ -1, & x<0.\end{cases}$$

Problem 2. Let $X,\mu)$ be a measure space and $f\colon X\to\mathbb{R}$ a measurable function. Show there is a sequence of simple functions $\langle\psi_n\rangle_{n=1}^\infty$ such that

- $|\psi_n|\le|\psi_{n+1}|$ for all n,
- $\psi_n(x)f(x)\ge 0$ for all $x\in X$,
- $|\psi_n|\le|f|$, and
- $\lim_{n\to\infty}\psi_n(x)=f(x)$ for all x.

Hint: Using the function $|f|$ in place of f in the previous problem find a sequence of simple $0\le\phi_n\nearrow|f|$ and let $\psi_n=\operatorname{sgn}(f)\phi_n$. □

Problem 3 (August 1984). Let (X,μ) be a measure space and let $1<p<\infty$ and $1/p+1/q=1$. Let $g\in L^1(X)$ such that there is a constant M with

$$\left|\int_0^1 gs\,d\mu\right|\le M\|s\|_{L^p}$$

for all simple functions s. Prove $g\in L^q(X)$ and $\|g\|_{L^q}\le M$. *Hint:* Let $\psi_n\to g$ as in the last problem. Show

$$\begin{aligned}0\le&|\psi_n|^q\le|q|^q\\ |\psi_n|^q\le&g\operatorname{sgn}(\psi_n)|\psi_n|.\end{aligned}$$

The function $\operatorname{sgn}(\psi_n)|\psi_n|^{q-1}$ is a simple function and thus

$$\begin{aligned}\|\psi_n\|_{L^q}^q &= \int_X |\psi_n|^q \, d\mu \\ &\le \int_X g \operatorname{sgn}(\psi_n)|\psi_n|^{q-1} \, d\mu \\ &\le M \big\| \operatorname{sgn}(\psi_n)|\psi_n|^{q-1} \big\|_{L^p}.\end{aligned}$$

Show this implies

$$\|\psi_n\|_{L^q} \le M$$

and then use the monotone convergence theorem to show

$$\|g\|_{L^q} = \lim_{n\to\infty} \|\psi_n\|_{L^q} \le M$$

to complete the proof. $\square$