Some group theory problems.

In George McNulty's list of group theory problems:
`http://people.math.sc.edu/mcnulty/qfers/gp.pdf`
look at the following Problems 6, 7, 8 (these three are all based on the same idea) 15, 35, 30–46.

Here are some propositions and problems related to group actions. One reference for some of the material is Lecture 16 of McNulty's notes `http://people.math.sc.edu/mcnulty/algebrafirst.pdf`.

Let G act on the set X. Then for $x \in X$, the ***stabilizer*** of x is

$$\mathbf{Stab}(x) = \{a \in G : ax = x\}.$$

Proposition 1. *The set* $\boldsymbol{Stab}(x)$ *is a subgroup of* G *and if* $g \in G$ *the stabilizers of* x *and* gx *are related by*

$$\boldsymbol{Stab}(gx) = g\,\boldsymbol{Stab}(x)g^{-1}.$$

Problem 1. Prove this. □

The action of G on X is ***transitive*** if and only if for all $x_1, x_2 \in X$ there is a $g \in G$ with $gx_1 = x_2$.

If $H \leq G$, then the coset space

$$G/H = \{\xi H : \xi \in G\}$$

is a G space by the natural actions

$$g(\xi H) = (g\xi)H.$$

As if $g = \eta(\xi)^{-1}$ then $g(\xi H) = \eta H$ and therefore this action is transitive. Up to isomorphism this gives all transitive G-spaces:

Proposition 2. *Let* G *have a transitive action on* X *and let* $x_0 \in X$. *Let* $H = \boldsymbol{Stab}(x_0)$. *Define* $f \colon G/H \to X$ *by*

$$f(\xi H) = \xi(x_0).$$

Then f *is an isomorphism of* G *spaces. (That is* f *is bijective and* $f(g\xi H) = g(f(\xi H))$*.)*

Problem 2. Prove this. (Do not forget to show f is well defined.) □

Note that if G is finite and acts transitively on X, then the last proposition implies

$$|X| = |G/H| = [G : H]$$

and in particular this implies $|X|$ divides $|G|$.

If G acts on X and $x \in X$ then the ***orbit*** of x is

$$\mathcal{O}_x = \{gx : g \in G\}.$$

Note that X is the disjoint union of the orbits of G and if $\mathcal{O}$ an orbit, then G has a transitive action on $\mathcal{O}$. Thus when G is finite $|\mathcal{O}|$ divides $|G|$.

Proposition 3. *Let the finite group G act on the finite set X and let*

$$\mathrm{Fix}_G(X) = \{x \in X : gx = x \text{ for all } g \in G\}$$

be the set of points are are fixed by every element of G. Assume that $|G| = p^n$ for some prime p. Then

$$|\,\mathrm{Fix}_G(X)| \equiv |X| \mod p.$$

Problem 3. Prove this. □

Problem 4. As an example of using this let G be a finite group and p a prime that divides $|G|$. Let H subgroup of G with $|H| = p^k$ for some k and let

$$\Omega = \{P \in \mathrm{Syl}_p(G) : H \subseteq P\}$$

be the set of p-Sylow subgroups of G that contain H. Then

$$|\Omega| \equiv 1 \mod p.$$

□

Problem 5. Let p be the smallest prime dividing the order of the finite group G. Show every subgroup of G on index p is normal. □

Proposition 4. *Let G be a finite simple group. Show that for any subgroup H of G that*

$$|G| \mid [G : H]!.$$

□

Problem 6. Here is a problem about infinite groups which is most easily done using group actions. Show that if G has a subgroup of finite index, then it also has a normal subgroup of finite index. □