Some analysis problems.

Here is a fact we mentioned in class.

Theorem 1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $L^1(X, \mu)$ is a complete metric space. (Where the distance between $f, g \in L^1(X, \mu)$ is $d(f, g) = \|f - g\|_{L^1} = \int_X |f - g| \, d\mu$.)*

The proof this will use the following metric space facts.

Problem 1. Let (X, d) be a metric space and $\langle x_k \rangle_{k=1}^{\infty}$ be a Cauchy sequence in X.

(a) Show that if this has a subsequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$ that converges, then the original sequence converges.

(b) Show the sequence has a subsequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$ with

$$d(x_{n_k}, x_{n_{k-1}}) < \frac{1}{2^k} \qquad \square$$

Problem 2. Prove Theorem 1. *Hint:* We need to show every Cauchy sequence $\rangle f_n \rangle_{n=0}^{\infty}$ in $L^1(X, \mu)$ is convergent. By use of Problem we can replace this sequence with one of its subsequences and assume

$$\|f_k - f_{k-1}\|_{L^1} = \int_X |f_k - f_{k-1}| \, d\mu < \frac{1}{2^k}.$$

Note that

$$f_n = f_0 + \sum_{k=1}^{n} (f_k - f_{k-1}).$$

Now apply one (or more) of the convergence theorems we have discussed. $\square$

Problem 3. Let $f \in L^n(X, \mu)$ and let f_n be

$$f_n(x) = \begin{cases} f(x), & |f(x)| \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

Show

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

and

$$\lim_{n \to \infty} \|f - f_n\|_{L^1} = 0. \qquad \square$$

Here are some problems I got off of old exams.

Problem 4 (January 1984). Let $g \colon (0, \infty) \to \mathbb{R}$ be measurable and with

$$\int_0^{\infty} |g(t)| \, dt < \infty, \qquad \int_0^{\infty} t|g(t)) \, dt < \infty$$

and define

$$f(x) = \int_0^{\infty} \sin(xt) \, dt.$$

(a) Show that $f(x)$ is defined for all x and is a bounded function.

(b) Prove that f is differentiable and

$$f'(x) = \int_0^\infty t g(t) \cos(xt)\, dt.$$

Hint: $|\sin b - \sin a| \leq |b - a|$. $\square$

Problem 5 (January 1987). Compute

$$\lim_{n\to\infty} \int_0^\infty \frac{x^2 - n^2}{x^2 + n^2} e^{-x}\, dx.$$

Justify all the steps in your calculations. $\square$

Problem 6 (Motivated by a Problem August 1987). Let $f \in L^1([0,\infty))$ show that for almost all $x \in [0,1]$ that

$$\lim_{n\to\infty} f(x+n) = 0.$$

Hint: Define $g_n\colon [0,1] \to \mathbb{R}$ by $g_n(x) = f(x+n)$ and show

$$\int_0^\infty |f(x)|\, dx = \sum_{n=0}^\infty \int_0^1 |g_n(x)|\, dx$$ $\square$

Problem 7 (August 1988). Let $f \in L^1([0,1])$. Prove $\lim_{n\to\infty} \int_0^\infty x^n f(x)\, dx = 0.$ $\square$

Problem 8 (January 1990). (a) Prove

$$\left(1 - \frac{x}{n}\right)^n < e^{-x} \quad \text{for } 0 < x < n$$

and

$$\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = e^{-x} \quad \text{for } 0 < x < \infty.$$

(b) For $\alpha > 0$ prove

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{\alpha-1}\, dx = \int_0^\infty e^{-x} x^{\alpha-1}\, dx.$$ $\square$