Mathematics 552 Test 1 Name: Answer key

1. Compute the following and put the answers in the form $a+bi$:

(a) $(3-2i)(5+3i) = 21-i$

(b) $\dfrac{1+2i}{3+4i} = \dfrac{11}{25} + \dfrac{2}{25}i$

(c) $(1-i)^8$

Solution: There a several ways to do this. Maybe the most natural is to write in polar form:

$$(1-i)^8 = \left(\sqrt{2}e^{-\pi i/4}\right)^8 = 2^{\frac{8}{2}}e^{-2\pi i} = 2^4(1) = 16.$$

A couple of you did the clever trick of repeated squaring:

$$(1-i)^8 = \left((1-i)^2\right)^4 = (-2i)^4 = \left((-2i)^2\right)^2 = (-4)^2 = 16.$$

And for yet anther method use the general definition of $z^\alpha = e^{\alpha \log(z)}$:

$$(1-i)^8 = e^{8\log(1-i)} = e^{8(\ln(\sqrt{2})+(-\pi/4+2\pi n)} = e^{8(\frac{1}{2}\ln(2))}e^{(-2\pi+16\pi n)} = 16(1) = 16.$$

And finally a few of you used Pascal's triangle to expand $(1-i)^8$. This works, but involves way too much work. $\square$

(d) $\sqrt{-2+2i\sqrt{3}}$

Solution: Anther one where the polar form is most natural:

$$\sqrt{-2+i2\sqrt{3}} = \left(4e^{i(2\pi/3+2\pi n)}\right)^{\frac{1}{2}} = 2e^{i\pi/3+n\pi i} = \pm 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right) = \pm\left(1+i\sqrt{3}\right)$$

$\square$

(e) $\log(-1-i)$

Solution:

$$\log(-1-i) = \ln(|-1-i|) + i\arg(-1-i) = \ln(\sqrt{2}) + i\left(\frac{5\pi}{4}+2\pi n\right).$$

$\square$

2. State Euler's formula for e^{iz}.

Solution:

$$e^{iz} = \cos(z) + i\sin(z).$$

$\square$

3. Find all solutions to $z^3 = -8$.

Solution: The solutions are $z = (-8)^{\frac{1}{3}}$.

$$\begin{aligned}(-8)^{\frac{1}{3}} &= \left(8e^{\pi i + 2n\pi i}\right)^{\frac{1}{3}} \\ &= 2e^{\frac{(2n+1)\pi i}{3}} \\ &= 2e^{\frac{pii}{3}}, 2e^{\pi i}, 2e^{\frac{5\pi i}{3}} \\ &= 1 + \sqrt{3}\, i,\ -2,\ 1 - \sqrt{3}\, i.\end{aligned}$$

□

4. Find the sum of the series

$$f(z) = \sum_{n=0}^{\infty} 5(-1)^n z^{3n} = 5 - 5z^3 + 5z^6 - 5z^9 + 5z^{12} - \cdots$$

Solution: This is a geometric series therefore the sum is

$$f(z) = \frac{\mathsf{first}}{1 - \mathsf{ratio}} = \frac{5}{1 - (-z^3)} = \frac{5}{1 + z^3}$$

This converges when $|\mathsf{ratio}| = |-z^3| < 1$. That is when $|z| < 1$. □

5. Give the series definitions of the following functions:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots$$

6. (a) Let $f(z)$ be defined on the open set U and let $z_0 \in U$. Give the limit definition of what it means for $f(z)$ to be differentiable at z_0.

Solution: f is differentiable at z_0 if the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists.

It is equally acceptable to write the limit in the forms:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

□

(b) What is the definition of $f(z)$ being ***analytic*** in the open set U.

Solution: f is analytic in U if and only if $f'(z)$ exists at all points $z \in U$. □

7. (a) Let $f(z) = u + iv$ in the open set U. State what it means for $f(z)$ to satisfy the ***Cauchy-Riemann equations***.

Solution: The Cauchy-Riemann equations are

$$u_x = v_y, \qquad u_y = -v_x$$

or if you prefer not using the subscript notation for partial derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

□

(b) Let $g(z) = (x^2 + y^2) + i(3xy + 1)$. Explain why $g(z)$ is not analytic.

Solution: It is a theorem that a function $u + iv$ defined on an open subset of $\mathbb{C}$ is analytic if and only if it satisfies the CR-equations on that set. In this case $u = x^2 + y^2$ and $v = 3xy + 1$. Therefore

$$\begin{aligned} u_x &= 2x & \qquad v_x &= 3y \\ u_y &= 2y & \qquad v_y &= 3x \end{aligned}$$

and we see that $u_x \neq v_y$ and $u_y \neq v_x$. Therefore neither CR-equation holds. Thus g is not analytic. □

(c) Let $f(z) = (x^2 - y^2 - 2x) + i(2xy - 2y)$. Explain why $f(z)$ is analytic on $\mathbb{C}$ and give a formula for its derivative.

Solution: If $f = u + iv$ satisfies the CR-equations it is analytic and its derivative is $f' = u_x + iv_x$. Here $u = x^2 - y^2 - 2x$ and $v = 2xy - 2y$ and

$$\begin{aligned} u_x &= v_y = 2x - 2 \\ u_y &= -v_x = -2y \end{aligned}$$

Thus CR-equations hold and therefore f is analytic on all of $\mathbb{C}$. Its derivative is

$$f'(z) = u_x + v_x i = 2x - 2 + 2y\, i.$$

Anther way to have done this is note that if $z = x + iy$, then

$$z^2 - 2z = (x + iy)^2 - 2(x + iy) = (x^2 - y^2 - 2x) + (2xy - 2y)i = f(z).$$

Therefore $f(z) = z^2 - 2z$ is a polynomial and therefore analytic and

$$f'(z) = 2z + 2 = (2x + 2) + 2y\, i$$

as before. □

(d) If $f(z) = u + iv$ is analytic in the open set U show that v satisfies the equation

$$v_{xx} + v_{yy} = 0.$$

Solution: This uses the CR-equations:

$$v_{xx} + v_{yy} = (v_x)_x + (v_y)_y = (-u_y)_x + (u_x)_y = -u_{xy} + u_{xy} = 0.$$

□

8. Find the center and radius of the circle of convergence of the following series

(a) $g(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)^3 2^k}$.

Solution: We use the ratio test:

$$\begin{aligned}
\mathsf{ratio} &= \lim_{k\to\infty} \left| \frac{(k+1)\text{-st term}}{k\text{-th term}} \right| \\
&\lim_{k\to\infty} \left| \frac{z^{k+1}}{((k+1)+2)^3 2^{k+1}} \, \frac{(k+2)^3 2^k}{z^k} \right| \\
&= \lim_{k\to\infty} \frac{(k+2)^3 z^k}{(k+3)^3 2} \\
&= \frac{|z|}{2}
\end{aligned}$$

The series converges when

$$\mathsf{ratio} = \frac{|z|}{2} < 1$$

which is equivalent to $|z| < 2$. Thus

$$\text{Center} = 0, \qquad \text{Radius} = 2.$$

□

(b) $f(z) = \sum_{n=0}^{\infty} \frac{n(z+2i)^{2n+1}}{9^n}$.

Solution: This time for convergence we want

$$\begin{aligned}
\mathsf{ratio} &= \lim_{n\to\infty} \left| \frac{(n+1)(z+2i)^{2(n+1)+1}}{9^{n+1}} \, \frac{9^n}{n(z+2i)^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)(z+2i)^2}{9n} \right| \\
&= \frac{|x+2i|^2}{9} \\
&< 1.
\end{aligned}$$

This is equivalent to $|z+2i|^2 < 9$ which in turn is equivalent to $|z+2i| < 3$. Thus

$$\text{Center} = -2i \qquad \text{Radius} = 3.$$

□

9. Find all solutions to $\cos(z) = -\frac{5}{4}$.

Proof. Solution: Using formula for $\cos(z)$ in terms of the exponential we have

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = -\frac{5}{4}.$$

Multiply by $4e^{-iz}$ to get

$$2(e^{iz})^2 + 2 = -5e^{iz}.$$

which can be rearranged as an equation that is quadratic in e^{-z}:

$$2(e^{iz})^2 + 5e^{iz} + 2 = 0.$$

By the quadratic formula

$$e^{iz} = \frac{-5 \pm \sqrt{25 - 4 \cdot 4}}{4} = \frac{-5 \pm \sqrt{9}}{4} = \frac{-5 \pm 3}{4} = -2, -\frac{1}{2}.$$

Solving $e^{iz} = -2$ gives

$$z = \frac{1}{i} \left(\log(-2) = \ln(2) + i \arg(-2)\right) = -i \ln(2) + \pi + 2n\pi.$$

Solving $e^{iz} = -\frac{1}{2}$ gives

$$z = \frac{1}{i} \log\left(\frac{-1}{2}\right) = i \ln(2) + \pi + 2n\pi.$$

Therefore the solutions are

$$z = (2n+1)\pi \pm i \ln 2.$$

$\square$