Qualifying Exam in Analysis
August 1993

1. Let E and F be disjoint closed subsets of a metric space (X, d). Show that there is a continuous function $f : X \to [0,1]$ such that

$$f(x) = \begin{cases} 0 & \text{for all } x \in E, \\ 1 & \text{for all } x \in F. \end{cases}$$

2. Let E be a measurable subset of $\mathbb{R}$ with $m(E) < \infty$. Prove that given $\epsilon > 0$, there exists a compact set $F \subset E$, such that

$$m(E \sim F) < \epsilon.$$

3. State and prove Egoroff's theorem.

4. Let $f \in L^1(\mathbb{R})$ and define $g : \mathbb{R} \to \mathbb{C}$ by

$$g(t) = \int_{\mathbb{R}} f(x)\, e^{itx}\, dx.$$

Prove that if $xf(x) \in L^1(\mathbb{R})$, then g is differentiable on $\mathbb{R}$ and

$$g'(t) = \int_{\mathbb{R}} ixe^{itx} f(x)\, dx.$$

5. Let $\{p_n\}$ be a sequence of $2\pi-$ periodic measurable functions on $\mathbb{R}$ satisfying
   (a) $p_n(t) \geq 0$ for all n and t,
   (b) $\int_{-\pi}^{\pi} p_n(t)dt = 1$,
   (c) For each $\delta > 0$, $\lim_{n\to\infty} \int_{\delta \leq |t| \leq \pi} p_n(t)dt = 0$.

For f continuous and $2\pi-$ periodic on $\mathbb{R}$, set

$$f_n(x) = \int_{-\pi}^{\pi} p_n(x-t) f(t) dt.$$

Prove that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly on $\mathbb{R}$.

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6. **Definition.** $f : [a,b] \to \mathbb{R}$ is in $Lip_1([a,b])$ if there exists a positive constant M such that $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in [a,b]$.
   a. Prove that if $f \in Lip_1([a,b])$, then f is absolutely continuous on $[a,b]$.
   b. If f is absolutely continuous on $[a,b]$, prove that $f \in Lip_1([a,b])$ if and only if $f' \in L^\infty([a,b])$.

7. Let f be defined by

$$f(x) = \begin{cases} x^{-1/3}, & 0 < |x| < 1, \\ 0, & x = 0, \text{ or } |x| \ge 1, \end{cases}$$

   and let $\{r_n\}$ be an enumeration of the rationals in $\mathbb{R}$.
   a. Prove that

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - r_n)$$

   converges almost everywhere and that the function F is integrable.
   b. Compute $\int_{-\infty}^{\infty} F(x)\,dx$.

8. True or False! If the result is true, prove it; if the result is false, provide a counterexample.
   a. If f is monotone increasing on $[a,b]$ and $f'(x) = 0$ a.e. on $[a,b]$, then f is constant on $[a,b]$.
   b. If f is monotone on $[a,b]$ and f' exists a.e. on $[a,b]$, then $f' \in L^1([a,b])$.
   c. If f is differentiable on (a,b), $c \in (a,b)$, then

$$\lim_{x \to c} f'(x) = f'(c).$$

   d. If f is non-constant and analytic in the open disk $D = \{z : |z - 3| < 2\}$ and continuous on the closed disk $\overline{D} = \{z : |z - 3| \le 2\}$, then the minimum value $m = \min\{|f(z)| : z \in \overline{D}\}$ cannot be attained by $|f|$ at any point inside D.

9. State and prove (using complex methods) the Fundamental Theorem of Algebra.

10. a. State the Residue Theorem.
    b. Evaluate $\int_\Gamma \frac{z}{z^4 - 1} dz$, where Γ is the ellipse (counterclockwise orientation)

$$\frac{x^2}{3} + 4y^2 = 1.$$