

Mathematics 551 Homework, January 15, 2020

We start by reviewing some vector algebra. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be vectors in \mathbb{R}^2 and c a scalar. Then the sum of \mathbf{a} and \mathbf{b} is

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2)$$

and the product of \mathbf{a} by the c is

$$c\mathbf{a} = (ca_1, cb_1).$$

It is common to use the notation

$$\mathbf{i} = (1, 0), \quad \mathbf{j} = (0, 1).$$

With this notation we can write \mathbf{a} as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$

The *inner product* of \mathbf{a} and \mathbf{b} is

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2.$$

Then

$$\mathbf{a} \cdot \mathbf{a} = (a_1)^2 + (a_2)^2$$

which is the square of the length of \mathbf{a} . We use the notation

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

for the *length* of \mathbf{a} .

If c_1 and c_2 are scalars and \mathbf{u} , \mathbf{v} , and \mathbf{w} are vectors are scalars then the inner product has the following properties:

- $\mathbf{b} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{b}$.
- $(c_1\mathbf{u} + c_2\mathbf{v}) \cdot \mathbf{w} = c_1\mathbf{u} \cdot \mathbf{w} + c_2\mathbf{v} \cdot \mathbf{w}$.

Consequences of these that will come up are

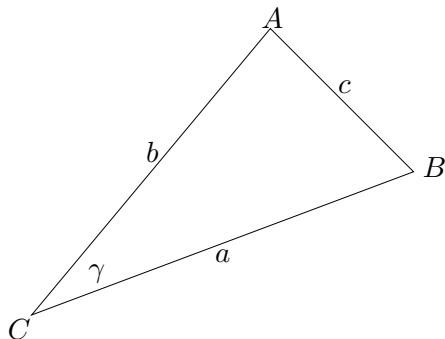
$$\begin{aligned}\|\mathbf{a} + \mathbf{b}\|^2 &= \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2 \\ \|\mathbf{a} - \mathbf{b}\|^2 &= \|\mathbf{a}\|^2 - 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2.\end{aligned}$$

Problem 1. Prove these formulas. □

A very important property of the inner product is given by

Theorem 1. *If \mathbf{a} and \mathbf{b} are nonzero vectors and θ is the angle between them, then*

$$(1) \quad \mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta).$$



Problem 2. In the triangle shown a , b , and c are the side lengths of $\triangle ABC$ and γ is the angle between \overrightarrow{CB} and \overrightarrow{CA} . Use Theorem 1 to show

$$c^2 = a^2 + b^2 - 2ab \cos(\gamma).$$

Hint: Let $\mathbf{u} = \overrightarrow{CB}$, $\mathbf{v} = \overrightarrow{CA}$ and $\mathbf{w} = \overrightarrow{AB}$. Then $\mathbf{w} = \mathbf{u} - \mathbf{v}$ and therefore $\|\mathbf{w}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$. \square

A Corollary of Theorem 1 is that non-zero vectors \mathbf{a} and \mathbf{b} are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Problem 3. Show that for any vectors \mathbf{a} and \mathbf{b} that $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular if and only if $\|\mathbf{a}\| = \|\mathbf{b}\|$. \square

Problem 4. Define a map $J: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$J(x, y) = (-y, x).$$

Show for all non-zero vectors \mathbf{v} that

- (a) $\|J\mathbf{v}\| = \|\mathbf{v}\|$
- (b) \mathbf{v} and $J\mathbf{v}$ are always perpendicular. \square

We now recall some calculus for functions $\mathbf{c}: [a, b] \rightarrow \mathbb{R}^2$ or more generally functions $\mathbf{c}: [a, b] \rightarrow \mathbb{R}^n$ where \mathbb{R}^n is the space of n -tuples (x_1, x_2, \dots, x_n) . We give the formulas for $n = 2$, and generalizing to higher dimensions is easy. First if

$$\mathbf{c}(t) = (x(t), y(t))$$

then the **derivative** of $\mathbf{c}(t)$ is

$$\mathbf{c}'(t) = (x'(t), y'(t)).$$

That is computing the derivative of the function $\mathbf{c}(t) = (x(t), y(t))$ is the same as computing the derivative of each component. The official definition is in terms of a limit;

$$\mathbf{c}'(t) = \lim_{h \rightarrow 0} \frac{1}{h} (\mathbf{c}(t+h) - \mathbf{c}(t)).$$

With this definition some of the results of single variable calculus carry over without much change.

Theorem 2. Let $\mathbf{f}, \mathbf{g}: [a, b] \rightarrow \mathbb{R}^2$ be differentiable vector valued functions and define a scalar valued function by

$$h(t) = \mathbf{f}(t) \cdot \mathbf{g}(t).$$

Then the product rule

$$h'(t) = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t)$$

holds. That is

$$(\mathbf{f}(t) \cdot \mathbf{g}(t))' = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t).$$

Problem 5. Prove this. *Hint:* Reduce this to the one variable product as follows. Let $\mathbf{f}(t) = (f_1(t), f_2(t))$ and $\mathbf{g}(t) = (g_1(t), g_2(t))$. Then

$$\mathbf{f}(t)\mathbf{g}(t) = f_1(t)g_1(t) + f_2(t)g_2(t).$$

Then

$$(\mathbf{f}(t)\mathbf{g}(t))' = (f_1(t)g_1(t))' + (f_2(t)g_2(t))'$$

You can now use the one variable product rule on the terms $(f_1(t)g_1(t))'$ and $(f_2(t)g_2(t))'$ and rearrange the results to get the desired formula. \square

Corollary 3. Let $\mathbf{f}: [a, b] \rightarrow \mathbb{R}^2$ be differentiable. Then

$$\frac{d}{dt} \|\mathbf{f}(t)\|^2 = 2\mathbf{f}(t) \cdot \mathbf{f}'(t).$$

and at points where $\mathbf{f}(t) \neq \mathbf{0}$

$$\frac{d}{dt} \|\mathbf{f}(t)\| = \frac{\mathbf{f}(t)}{\|\mathbf{f}(t)\|} \cdot \mathbf{f}'(t).$$

Problem 6. Prove this. *Hint:* For the first formula let $\mathbf{g} = \mathbf{f}$ in Theorem 2. For the second use $\|\mathbf{f}(t)\| = (\|\mathbf{f}(t)\|^2)^{\frac{1}{2}}$. \square

Here is another product rule.

Proposition 4. Let $\mathbf{f}: [a, b] \rightarrow \mathbb{R}^2$ be a differentiable vector valued function and let $h: [a, b] \rightarrow \mathbb{R}$ be a differentiable scalar valued function. Then

$$\frac{d}{dt} (h(t)\mathbf{f}(t)) = h'(t)\mathbf{f}(t) + h(t)\mathbf{f}'(t).$$

Problem 7. Prove this. \square

We now integrate vector valued functions. Let $\mathbf{f}: [a, b] \rightarrow \mathbb{R}^2$ be continuous and write it as

$$\mathbf{f}(t) = (f_1(t), f_2(t)).$$

Then

$$\int_a^b \mathbf{f}(t) dt = \left(\int_a^b f_1(t) dt, \int_a^b f_2(t) dt \right).$$

That is integrating a vector valued function is the same as integrating each of its component functions.

Proposition 5. Let $\mathbf{f}: [a, b] \rightarrow \mathbb{R}^2$ be a vector valued function and \mathbf{a} a constant vector. Then

$$\mathbf{a} \cdot \int_a^b \mathbf{f}(t) dt = \int_a^b \mathbf{a} \cdot \mathbf{f}(t) dt.$$

Problem 8. Prove this. *Hint:* Write $\mathbf{f}(t) = (f_1(t), f_2(t))$ and $\mathbf{a} = (a_1, a_2)$ and expand each side of the equation to be proven in terms of the definitions. \square

Using that $\cos(\theta) \leq 1$ for all θ we see that Equation (1) implies

$$\mathbf{a} \cdot \mathbf{b} \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

for all vectors \mathbf{a} and \mathbf{b} . This is the **Cauchy-Schwartz inequality**.

Theorem 6. Let $\mathbf{f}: [a, b] \rightarrow \mathbb{R}^2$ be a continuous vector valued function. Then the inequality

$$\left\| \int_a^b \mathbf{f}(t) dt \right\| \leq \int_a^b \|\mathbf{f}(t)\| dt$$

holds.

Problem 9. If $\int_a^b \mathbf{f}(t) dt = \mathbf{0}$, then the result holds. So assume that $\int_a^b \mathbf{f}(t) dt \neq \mathbf{0}$. Now prove the result along the following lines.

(a) Let \mathbf{a} be the vector

$$\mathbf{a} = \left\| \int_a^b \mathbf{f}(t) dt \right\|^{-1} \int_a^b \mathbf{f}(t) dt.$$

and show that \mathbf{a} is a unit vector, that is

$$\|\mathbf{a}\| = 1$$

and that

$$\left\| \int_a^b \mathbf{f}(t) dt \right\| = \mathbf{a} \cdot \int_a^b \mathbf{f}(t) dt$$

(b) Now use Proposition 5 to show

$$\left\| \int_a^b \mathbf{f}(t) dt \right\| = \int_a^b \mathbf{a} \cdot \mathbf{f}(t) dt.$$

(c) Use that \mathbf{a} is a unit vector and the Cauchy-Schwartz inequality to show

$$\mathbf{a} \cdot \mathbf{f}(t) \leq \|\mathbf{f}(t)\|.$$

(d) Put all these pieces together to complete the proof. \square