Mathematics 242 Solutions to Homework 9.

Problem 1. Let x satisfy the initial value problem

$$x'' + 5x' + 6x = 5 + 3e^{-2t} - 4\sin(2t) \qquad x(0) = 4, \quad x'(0) = -3$$

(a) Find the Laplace transform $X(s) = \mathcal{L}\{x(t)\}$.
(b) Find the partial fraction decomposition of $X(s)$ and say what software or web site you used to do this. (there are partial fraction computers as Wolfram Alpha, Symbol Lab, and Voovers. The program Maple will do partial fractions instruction's here. I have been using SageMath, which can be down loaded for free here. Instructions for partial fractions in SageMath can be found here. If you know of anther program/website that you like to do partial fractions send me a link and I will pass the information along to the rest of the class.)
(c) Take the inverse Laplace transform and give the solution to the given initial value problem. □

Solution. Letting $X = X(s) = \{x(t)\}$ and using the formulas

$$\begin{aligned}
\mathcal{L}\{x''(t)\} &= s^2X(s) - sx(0) - x'(0) = s^2X(s) - 4s + 3 \\
\mathcal{L}\{x'(s)\} &= sX(s) - x(0) = sX(s) - 4 \\
\mathcal{L}\{5\} &= \frac{5}{s} \\
\mathcal{L}\{3e^{-2t}\} &= \frac{3}{s+2} \\
\mathcal{L}\{-4\sin(2t) &= \frac{-8}{s^2+4}
\end{aligned}$$

taking the Laplace transform of the initial value problem leads to

$$\big(s^2X(s) - 4s + 3\big) + 5\,(sX(s) - 4) + 6X(s) = \frac{5}{s} + \frac{3}{s+2} - \frac{8}{s^2+4}$$

which can be rewritten as

$$(s^2 + 5s + 6)X(s) = 4s + 17 + \frac{5}{s} + \frac{3}{s+2} - \frac{8}{s^2+4}.$$

Dividing by $(s^2 + 5s + 6) = (s + 2)(s + 3)$ gives

$$X(s) = \frac{4s+17}{(s+2)(s+3)} + \frac{5}{s(s+2)(s+3)} + \frac{3}{(s+2)^2(s+3)} - \frac{8}{(s^2+4)(s+2)(s+3)}$$

In used the program SageMath to get the partial fraction decomposition:

$$X(s) = \frac{5\,s-2}{13\,(s^2+4)} + \frac{11}{39\,(s+3)} + \frac{5}{2\,(s+2)} + \frac{5}{6\,s} + \frac{3}{(s+2)^2}$$

I again used SageMath to get the inverse Laplace transform and thus the solution to the value problem:

$$x(t) = \frac{1}{78}\left(30\,\cos(2\,t)\,e^{(2\,t)} - 6\,e^{(2\,t)}\sin(2\,t) + 234\,t + 65\,e^{(2\,t)} - 234\right)e^{(-2\,t)} + \frac{11}{2}\,e^{(-2\,t)} + \frac{11}{39}\,e^{(-3\,t)}$$

$\square$

Problem 2. This problem is just to show that the method works on higher order equations. Let $x(t)$ satisfy

$$x'''(t) - 8x'(t) = 4e^{2t} + 3e^{-2t}, \qquad x(0) = 1, \quad x'(0) = 2, \qquad x'''(0) = 3.$$

(a) Find the Laplace transform $X(s) = \mathcal{L}\{x(t)\}$.
(b) Find the partial fraction decomposition of $X(s)$ and say what software or web site you used to do this.
(c) Take the inverse Laplace transform and give the solution to the given initial value problem. $\square$

Solution. This time we use

$$\begin{aligned}
\mathcal{L}\{x'''(t)\} &= s^3X(s) - s^2x(0) - sx'(0) - x''(0) = s^3X(s) - s^2 - 2s - 3 \\
\mathcal{L}\{x''(t)\} &= s^2X(s) - sx(0) - x'(0) = s^2X(s) - s - 2 \\
\mathcal{L}\{x'(s)\} &= sX(s) - x(0) = sX(s) - 1 \\
\mathcal{L}\{4e^{2t}\} &= \frac{4}{s-2} \\
\mathcal{L}\{3e^{-2t}\} &= \frac{3}{s+2}
\end{aligned}$$

So the Laplace transform of the initial value problem is

$$\big(s^3X - s^2 - 2s - 3\big) - 8(sX - s - 2) = \frac{4}{s-2} + \frac{3}{s+2}$$

and thus

$$(s^3 - 8s)X = s^2 + s - 21 + \frac{4}{s-2} + \frac{3}{s+2}.$$

Divide by $s^3 - s - s(s^2 - 8)$

$$X = \frac{s^2 + s - 21}{s(s^2-8)} + \frac{4}{s(s^2-8)(s-2)} + \frac{3}{s(s^2-8)(s+2)}$$

The partial fraction decomposition is

$$X(s) = -\frac{9\,s + 68}{16\,(s^2-8)} + \frac{3}{8\,(s+2)} - \frac{1}{2\,(s-2)} + \frac{27}{16\,s}$$

Taking the inverse transform gives the solution as

$$x(t) = -\frac{9}{16}\cosh(\sqrt{2}t) - \frac{17}{4\sqrt{2}}\sinh(\sqrt{2}t) + \frac{3}{8}e^{-2t} - \frac{1}{2}e^{2t} + \frac{27}{16}. \qquad \square$$

In class today we defined the ***convolution*** of the functions f and g as

$$f * g(t) = \int_0^t f(t-u)g(u)\,dx = \int_0^t f(u)g(it-u)\,du$$

and showed that

$$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

Written in terms of the inverse transform this is

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t). \tag{1}$$

Here is an example of this in action. Let us solve

$$x''(t) - 3x'(t) + 2x(t) = 3H(t-5), \qquad x(0) = 1, \quad x'(0) = -2.$$

where $H(t)$ is the ***Heaviside function***

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t. \end{cases}$$

Letting $X(s) = \mathcal{L}\{x(t)\}$ and using the usual rules we find

$$s^2X - s + 2 - 3(sX - 1) + 2X = 3\mathcal{L}\{H(t-5)\}.$$

This can be rearranged as

$$(s^2 - 3s + 2)X = s - 5 + 3\mathcal{L}\{H(t-5)\}.$$

and therefore

$$X(s) = \frac{s-5}{s^2-3s+2} + \frac{3}{s^2-3s+2}\mathcal{L}\{H(t-5)\}. \tag{2}$$

We have the partial fraction decompositions (SageMath again)

$$\frac{s-5}{s^2-3s+2} = \frac{4}{s-1} - \frac{3}{s-2}$$

so that

$$\mathcal{L}^{-1}\left(\frac{s-5}{s^2-3s+2}\right) = 4e^t - 3e^{2t}, \tag{3}$$

and

$$\frac{3}{s^2-3s+2} = \frac{3}{s-2} - \frac{3}{s-1}$$

and therefore

$$\mathcal{L}\{3e^{2t} - 3e^t\} = \frac{3}{s^2-3s+2}$$

and therefore

$$\frac{3}{s^2-3s+2}\mathcal{L}\{H(t-5)\} = \mathcal{L}\{3e^{2t} - 3e^t\}\mathcal{L}\{H(t-5)\}$$

Therefore equation (1) above gives

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{3}{s^2-3s+2}\mathcal{L}\{H(t-5)\}\right\} &= \left(3e^{2t} - 3e^t\right) * H(t-5) \\
&= \begin{cases} 0, & t < 5; \\ \displaystyle\int_5^t \left(3e^{2(t-u)} - 3e^{t-u}\right)\, du, & 5 \le t \end{cases} \\
&= \begin{cases} 0, & t < 5; \\ \dfrac{3(e^{2(t-5)} - 1)}{2} - 3(e^{t-5} - 1), & 5 \le t \end{cases}
\end{aligned}$$

Putting this together with equations (2) and (3) gives us our final solution

$$x(t) = \begin{cases} 4e^t - 3e^{2t}, & t < 5; \\ 4e^t - 3e^{2t} + \dfrac{3(e^{2(t-5)} - 1)}{2} - 3(e^{t-5} - 1), & 5 \le t \end{cases}$$

Problem 3. Let $x(t)$ satisfy the initial value problem

$$x''(t) + 4x(t) = 3H(t-4), \qquad x(0) = 6, \quad x'(0) = 12$$

Use the method above to solve this equation. $\square$

Solution. If $X(s) = \{x(t)\}$, then the Laplace transform of the initial value problem is

$$s^2 X(s) - 6s - 12 + X(s) = 3\mathcal{L}\{H(t-4)\}.$$

Solving for $X(s)$ gives

$$X(s) = \frac{6s+12}{s^2+4} + \frac{3}{s^2+4}\mathcal{L}\{x(t)\}.$$

Therefore

$$x(t) = 6\cos(2t) + 6\sin(2t) + 3\sin(2t) * H(t-4).$$

The convolution is

$$\begin{aligned} 3\sin(2t) * H(t-4) &= 3\int_0^t H(u-4)\sin(\sin(2(t-u)))\,du \\ &= \begin{cases} 0, & t \le 4; \\ 3\displaystyle\int_4^t \sin(2(t-u))\,du, & 4 < t \end{cases} \\ &= \begin{cases} 0, & t \le 4 \\ \dfrac{3\cos(2(t-4)) - 3}{2}, & 4 < t. \end{cases} \\ &= H(t-4)\frac{3\cos(2(t-4)) - 3}{2} \end{aligned}$$

and therefore the solution is

$$x(t) = 6\cos(2t) + 6\sin(2t) + H(t-4)\frac{3\cos(2(t-4)) - 3}{2}$$ $\square$

Problem 4. Find the inverse Laplace transforms of the following functions:

(a) $F(s) = \dfrac{3s+5}{(s-7)^2}$

(b) $G(s) = \dfrac{2s-9}{2s^2+18}$

(c) $H(s) = \dfrac{3s+12}{3s^2+12s+21}$

Solution. Letting $F(s) = \{f(t)\}$, $G(s) = \{g(t)\}$, and $H(s) = \{h(t)\}$ we have

(a)

$$F(s) = \frac{3s+5}{(s-7)^2} = \frac{3(s-7+7)+5}{(s-7)^2} = \frac{3}{(s-7)} + \frac{26}{(s-7)^2}$$

and therefore

$$f(t) = e^{7t}(1+26t).$$

(b)

$$G(s) = \frac{1}{2}\left(\frac{2s-9}{s^2+9}\right)$$

which gives

$$g(t) = \cos(3t) - \frac{3}{2}\sin(3t).$$

(c)

$$H(s) = \frac{1}{3}\left(\frac{3s+12}{s^2+6s+9}\right) = \frac{1}{3}\left(\frac{3(s+3)+3}{(s+3)^2}\right) = \frac{1}{3}\left(\frac{3}{(s+3)} + \frac{3}{(s+3)^2}\right) = \frac{1}{s+3} + \frac{1}{(s+3)^2}$$

Whence

$$h(t) = e^{-3t}(1+t).$$