Qualifying Exam in Analysis

January 1993

Lebesgue measure, defined on the set of measurable subsets of the real numbers $\mathbb{R}$, will be denoted by m. Lebesgue outer measure, defined on the set of all subsets of the real numbers, will be denoted by m^*. The integral $\int_{[a,b]} f\,dm$ will also be written as $\int_a^b f(x)\,dx$.

1 (10 points) (a) Give an example of an infinite subset of $\mathbb{R}$ that has exactly 3 accumulation points.
(b) Show that every uncountable subset of $\mathbb{R}$ has an accumulation point.

2 (10 points) Let $\langle r_i \rangle_{i=1}^{\infty}$ be an enumeration of the rational numbers in $[0,1]$. Define a function f on $\mathbb{R}$ by

$$f(x) = \sum_{i=1}^{\infty} \chi_{[r_i, r_i + 1/2^i]}(x).$$

Show that the series defining f converges almost everywhere.

3 (10 points) Find all entire functions f so that $f(2z) = -if(z)$ for all $z \in \mathbb{C}$.

4 (10 points) Let $\langle h_k \rangle_{k=1}^{\infty}$ be a sequence of real valued functions defined on $\mathbb{R}$ so that

$$|h_k(x)| \le 10 \quad \text{for all } x \text{ and } k,$$

and

$$\lim_{k\to\infty} \int_a^b h_k(x)\,dx = 3(b-a)$$

for all $a < b$ in $\mathbb{R}$. Show for all $f \in L^1(\mathbb{R})$ that

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} h_k(x) f(x)\,dx = 3\int_{-\infty}^{\infty} f(x)\,dx.$$

5 (10 points) Let μ be a finite measure on a compact set $K \subset \mathbb{C}$. Define F by

$$F(z) = \int \frac{1}{w-z}\,d\mu(w).$$

Prove F is analytic with derivative

$$F'(z) = \int \frac{1}{(w-z)^2}\,d\mu(w)$$

on the complement of K in $\mathbb{C}$.

6 (10 points) Let f be a continuously differentiable function on $\mathbb{R}$ with

$$\int_{\mathbb{R}} |f'|^2 \, dm \le 16.$$

Show if $x, y \in \mathbb{R}$ then

$$|f(x) - f(y)| \le 4\sqrt{|x-y|}.$$

7 (10 points) Prove or give a counterexample:

(a) If K is a compact subset of $\mathbb{R}$ and there is a continuous onto function $f : K \to [0,1]$, then $m(K) > 0$.

(b) If f is analytic on an open subset U of $\mathbb{C}$ and γ is a piecewise smooth closed curve in U, then

$$\int_\gamma f(z)\, dz = 0.$$

(c) If $\langle f_n \rangle_{n=1}^\infty$ is a sequence of increasing on $\mathbb{R}$ and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx = 0,$$

then $\langle f_n \rangle_{n=1}^\infty$ converges to 0 in measure.

(d) If f is an integrable measurable function on $[0,1]$ then there is a subinterval $(a,b) \subset [0,1]$ so that f is continuous on (a,b).

8 (10 points) Let $A \subset \mathbb{R}$ be a non-measurable subset of $\mathbb{R}$. Show for any $\rho \in (0,1)$ there is an interval $I \subset \mathbb{R}$ so that $m^*(A \cap I) \ge \rho m^*(I)$.

9 (10 points) Let $\langle f_n \rangle_{n=1}^\infty$ be a sequence of monotone increasing functions on $[0,1]$. Assume that

$$\int_0^1 f_n'(x)\, dx = 0 \quad \text{and} \quad f_n(0) = 0$$

for all n, and that

$$\sum_{n=1}^\infty f_n(1) \le 12.$$

Show that f defined by

$$f = \sum_{n=1}^\infty f_n$$

converges uniformly on all of $[0,1]$ and that

$$\int_0^1 f'(x)\, dx = 0$$

10 (10 points) Let $f \in L^1(m \times m) = L^1([0,1]\times[0,1])$. Prove for every $\varepsilon > 0$ there is a finite set of intervals $[a_i, b_i], [c_i, d_i] \subseteq [0,1]$ and numbers A_i ($1 \le i \le n$) so that

$$\int_0^1 \int_0^1 \left| f(x,y) - \sum_{i=1}^n A_i \chi_{[a_i,b_i]\times[c_i,d_i]}(x,y) \right| dx dy < \varepsilon$$