Analysis Qualifying Exam
August 2009

Instructions: Write your name legibly on each sheet of paper. Write only on one side of of each sheet of paper. Try to answer all questions. Questions 1–8 are each worth 10 points and question 9 is worth 20 points.
Terminology: Measurability and integrability on $\mathbb{R}$ or a measurable subset of it will always refer to the Lebesgue measure, except if otherwise specified. Lebesgue measure will be denoted by m, dx or dy depending on the context. If A is a subset of $\mathbf{R}$ then $L^p(A)$ is considered with respect to the Lebesgue measure.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and assume that

$$\lim_{|x|\to\infty} f(x) = 0.$$

Prove f is uniformly continuous on $\mathbb{R}$.

2. Let $X \subset \mathbb{R}$ be a measurable set with $m(X) < \infty$ and assume $0 \leq f : X \to \mathbb{R}$ is integrable. Assume that

$$\int_X f\,dx = \int_X f^n\,dx$$

for all $n \geq 2$. Prove that there exists a measurable set $E \subset X$ such that $f = \chi_E$ a.e.

3. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and assume $f' \in L^2([a, b])$. Prove that there exists a constant M such that $|f(x)-f(y)| \leq M|x-y|^{\frac{1}{2}}$ for all $x, y \in [a, b]$

4. Let $f(x) = \sqrt{x}$ for all $x \in [0, 1]$. Prove the following items.

a. The function f is absolutely continuous on $[0, 1]$. State explicitly any theorem you are using.

b. The inequality $|f(x) - f(y)| \leq |x - y|^{\frac{1}{2}}$ holds for all $x, y \in [0, 1]$.

c. $f' \notin L^2([0, 1])$.

(Note; this problem shows that the converse to the previous problem is false.)

5. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Prove that its graph $\{(x, f(x)); x \in \mathbb{R}\}$ has Lebesgue measure zero.

6. Evaluate $\int_0^{\pi/2} \frac{dx}{2+\sin^2 x}$.

7. Let f be analytic except a sequence $\{p_n\}$ where $\lim |p_n| = \infty$. Assume $|f(z)| \geq C > 0$ on $\mathbb{C} \setminus \{p_n : n = 1, 2, \cdots\}$. Prove that f is constant.

8. Suppose that the function $f(z, t)$ is continuous as a function of two variables when z lies in a region Ω and $a \leq t \leq b$. Suppose also that for each fixed $t \in [a, b]$ the function $f(\cdot, t)$ is analytic in Ω. Prove that the function $F(z) = \int_a^b f(z, t)dt$ is analytic in Ω and find a formula for $F^{(n)}(z)$.

9. True or False. Prove, or give a counterexample.

- **a.** If $O \subset [0, 1]$ is an open dense set, then $m([0, 1] \setminus O) = 0$.
- **b.** Any continuous function defined on a closed and bounded subset of a metric space is uniformly continuous.
- **c.** Let (A_n) be a sequence of measurable subsets of $[0, 1]$ with $m(A_n) \to 1$. Then there exists a subsequence $(A_{n_k})_k$ of (A_n) such that

$$m(\{x \in [0, 1] : x \notin A_{n_k} \text{ for infinitely many } k\text{'s }\}) = 0.$$

- **d.** Suppose f is analytic in the region $0 < |z| < 1$ and suppose that for each r, $0 < r < 1$, the integral $\oint_{C_r} f(z)\, dz = 0$, where C_r is the circle $|z| = r$. Then f is analytic on the open unit disc.
- **e.** If $f_n \in L^2(\mathbb{R})$ and $\sum_{n=1}^{\infty} \|f_n\|_2 < \infty$, then $f_n(x) \to 0$ a.e. on $\mathbb{R}$.