ADMISSION TO CANDIDACY EXAMINATION

IN

REAL ANALYSIS

JANUARY 1989

Throughout the exam m will denote the Lebesgue measure on R (the real numbers). Integrals with respect to m will be denoted by $\int f(x)dx$ or $\int f(t)dt$.

1. State and prove Fatou's Lemma.

2. Let (x,A,μ) be a measure space and f a nonnegative measurable function on x. Define ν on A by

$$\nu(E) = \int_E f d\mu \quad \text{for} \quad E \in A.$$

Prove that

(a) ν is a measure on A.

(b) For any nonnegative measurable function g on x

$$\int g\, d\nu = \int g\, f\, d\mu.$$

3. Prove that the interval $[a,b]$ is compact.

4. Let E be a measurable subset of $[0,1]$ and $m(E) > 0$. Prove that for every $\varepsilon > 0$ there exists a closed nowhere dense set K such that $K \subset E$ and $m(E\setminus K) < \varepsilon$.

5. Let $\langle k_n \rangle$ be a sequence of positive integers so that $\sum_{n=1}^{\infty} \frac{1}{k_n} < \infty$.

For $t \in [0,1)$ define

$$f_n(t) = t^{k_n}, \qquad n = 1,2\ldots$$

and let

$$f(t) = \sum_{n=1}^{\infty} f_n(t).$$

Prove that

(a) $f \in L^1([0,1])$.

(b) f is nondecreasing (with $+\infty$ as a possible value).

(c) For all $t \in [0,1)$, $f(t) < \infty$.

6. Let

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ g(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1, \end{cases}$$

where g is the Cantor ternary function.
Let μ be the measure on R such that F is its cumulative distribution function, i.e. $\mu((-\infty,x]) = F(x)$.
Prove that μ is mutually singular with respect to Lebesgue measure.

7. For $f \in L^1(R)$ and $g \in L^p(R)$, $1 \leq p < \infty$, let

$(f * g)(t) = \int f(t-x)g(x)dx.$

Prove that $(f * g)(t)$ exists a.e. and $||f * g||_p \leq ||f||_1||g||_p$.

8. Let f be a nonnegative integrable function on R, and let $\psi(t) = m(\{x : f(x) > t\})$. Prove that

(a) $m \times m(\{\langle x,y\rangle : 0 \leq y \leq f(x)\}) = m \times m(\{\langle x,y\rangle : 0 < y < f(x)\})$

$= \int f(x)dx.$

(b) ψ is a decreasing function and

$$\int_0^{\infty} \psi(t)dt = \int f(x)dx$$

9. True or false. Prove or give a counterexample.

(a) If $f_n \to f$ a.e. on $[0,1]$ then $f_n \to f$ in measure.

(b) If $f_n \to f$ in measure then $f_n \to f$ a.e.

(c) If $||f_n - f||_1 \to 0$ then $f_n \to f$ a.e.

(d) If $f_n \to f$ a.e. then $||f_n - f||_1 \to 0$.

(e) Let f be an integrable function on R. If $\langle E_n \rangle$ is a decreasing sequence of measurable sets and $E = \cap E_n$ then

$$\int_E f\, dm = \lim_{n\to\infty} \int_{E_n} f\, dm.$$

(f) If f is nondecreasing and continuous on $[0,1]$ and $f'(x) = 0$ a.e., then f is constant.