The Maximum Modulus Principle and Schwarz's Lemma.

Theorem 1 (Mean Value Princple)**.** *Let U be an open set in $\mathbb{C}$ and $a \in U$ so that the disk $B(a,r) := \{z : |z-a| < r\}$ has its closure contained in U. Let f be analytic in U. Then the value $f(a)$ is the average of f over the circle $\{z : |z-a| = r\}$. More explicitly*

$$f(a) = \frac{1}{2\pi} \int_{|z-a|=r} f(a + re^{i\theta})\, d\theta.$$

Problem 1. Prove this. *Hint:* From the Cauchy integral formula

$$f(a) = \frac{1}{2\pi} \int_{|z-a|=r} \frac{f(z)}{z-a}\, dz.$$

Use the parameterization $z = a + re^{i\theta}$ and simplify. $\square$

Theorem 2 (Mean value Principle second form)**.** *With the same hypothesis as in the previous theorem we can also compute the value of $f(a)$ as the average over $B(a,r)$ with respect to the area measure. That is*

$$f(a) = \frac{1}{\pi r^2} \iint_{B(a,r)} f(z)\, dx\, dy,$$

where $z = x + iy$.

Problem 2. Prove this. *Hint:* Using that if $z = a + \rho e^{i\theta}$ and we use ρ, θ as polar coordinates centered at a, then $dx\, dz = \rho\, d\theta\, d\rho$ and thus

$$\iint_{B(a,r)} f(z)\, dx\, dy = \int_0^r \int_0^{2\pi} f(a + \rho e^{i\theta}) \rho\, d\theta\, d\rho.$$

Now use $f(a) = \frac{1}{2\pi\rho} \int_0^{2\pi} f(a + \rho e^{i\theta})\, d\theta$ for $0 \le \rho \le r$. $\square$

Theorem 3 (Maximum Modulus Principle Form 1)**.** *Let f be analytic in the connected open set U. Then if $|f(z)|$ has a local maximum at some point of U, then f is constant.*

Problem 3. Prove this. *Hint:* Assume that f has a local maximum at $z = a \in U$. Choose $r > 0$ small enough that $\overline{B}(a,r) \subseteq U$, and that $|f(z)|$ achives its maximum on $\overline{B}(a,r)$ at $z = a$. Then use the Mean value principle to show

$$\begin{aligned}
|f(a)| &= \left| \frac{1}{\pi r^2} \iint_{B(a,r)} f(z)\, dx\, dx \right| \\
&\le \frac{1}{\pi r^2} \iint_{B(a,r)} |f(z)|\, dx\, dy \\
&\le \frac{1}{\pi r^2} \iint_{B(a,r)} |f(a)|\, dx\, dy \\
&= |f(a)|
\end{aligned}$$

and explain why this implies $|f(z)|$ is constant on $B(a,r)$ and why this in turn implies $f(z)$ is constant on $B(a,r)$. Finally use the uniqueness principle (or analytic continuation) to show $f(z)$ is constant on all of U. $\square$

Theorem 4 (Maximum Modulus Principle Form 2)**.** *Let U be a bounded open set and $f(z)$ a function that is analytic in U and continuous on the closure $\overline{U}$. Then the maximum of $|f(z)|$ occurs on the boundary, ∂U, of U.*

Problem 4. Prove this. $\square$

Problem 5. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant entire function and let

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

Show M_f is a strictly increasing function. $\square$

Problem 6 (January 2017, Problem 7). Let G be a bounded region and let f and g be nonvanishing continuous functions on $\overline{G}$ which are holomorphic on G. Assume $|f(z)| = |g(z)|$ for all $z \in \partial G$. Prove there is a constant λ with $|\lambda| = 1$ and $f(z) = \lambda g(z)$ for all $z \in G$. $\square$

Problem 7 (August 2002, Problem 7). Let $f, g\colon D \to \mathbb{C}$ be two holomorphic function where D is the unit disk such that $|f(z)| = |g(z)|$ for all $z \in D$. Prove every zero of g is also a zero of f and that $f = \lambda g$ for some constant λ with $|\lambda| = 1$. $\square$

Problem 8. Let f be analytic in $D = \{z : |z| < 1\}$ and continuous on $\overline{D} = \{z : |z| \leq 1\}$. Assume $|f(z)| \leq 1$ and $f(z) = 0$. Then

(a) $|f(z)| \leq |z|$ for all $z \in D$ and if equality holds at some point $z = z_0$ with $z_0 \neq 0$, then $f(z) = cz$ for some constant c with $|z| = 1$.
(b) $|f'(0)| \leq 1$ and if equality holds, then $f(z) = cz$ for some constant c with $|c| = 1$.

Hint: Let $g\colon \overline{D} \to \mathbb{C}$ be

$$g(z) = \begin{cases} \dfrac{f(z)}{z}, & z \neq 0; \\ \\ f'(0), & z = 0. \end{cases}$$

Show that $g(z)$ is analytic in D and continuous on $\overline{D}$. So by the Maximum Principle the maximum of $|g(z)|$ occurs on the boundary. Use this to show $|g(z)| \leq 1$, which implies $|f(z)| \leq |z|$. If $|g(z_0)| = 1$ at some point $z_0 \in D$, then $|g(z)|$ has a local maximum and therefore is constant. Consider the two cases $z_0 \neq 0$ and $z_0 = 0$. $\square$

Theorem 5 (Schwarz's Lemma)**.** *Let f be analytic in $D = \{z : |z| < 1\}$ with $|f(z)| \leq 1$ in D and $f(0) = 0$. Then*

(a) $|f(z)| \leq |z|$ for all $z \in D$ and if equality holds at some point $z = z_0$ with $z_0 \neq 0$, then $f(z) = cz$ for some constant c with $|z| = 1$.

(b) $|f'(0)| \le 1$ *and if equality holds, then* $f(z) = cz$ *for some constant* c *with* $|c| = 1$*.*

Problem 9*.* Prove this. *Hint:* This only differs from Problem 8 in that f is not defined on the closure $\overline{D}$. Define $g(z)$ just as before. Then we still have that $g(z)$ is analytic in D. But we have to work a little harder to show $|g(z)| \le 1$. Let $0 < r < 1$. Then $g(z)$ is a analytic and on the closed disk $\overline{B}(0,r) = \{z : |z| \le r\}$. Thus $|g(z)|$ obtains its maximum on $\overline{B}(0,r)$ on the boundary of $\overline{B}(0,r)$. Use this to show

$$|g(z)| \le \frac{1}{r}$$

on the disk $\overline{B}(0,r)$. Now let $r \nearrow 1$ to conclude $|g(z)| \le 1$ on D. The rest of the proof is as in Problem 8. $\square$

Schwarz's lemma has lots of generalizations. Here is one:

Proposition 6. *Let* $f(z)$ *be analytic in the disk* $D = \{z : |z| < 1\}$ *with* $|f(z)| \le 1$ *in* D *and* $f(0) = f'(0) = 0$*. Then*

(a) $|f(z)| \le |z|^2$ *for all* $z \in D$ *and if equality holds at some point* $z = z_0$ *with* $z_0 \ne 0$*, the* $f(z) = cz^2$ *for some constant* c *with* $|z| = 1$*.*

(b) $|f''(0)| \le 2$ *and if equality holds, then* $f(z) = cz$ *for some constant* c *with* $|c| = 1$*.*

Problem 10*.* Prove this. *Hint:* Let $g(z) = f(z)/z^2$ for $z \ne 0$ and $g(0) = f''(0)$. Show $g(z)$ is analytic and that $|g(z)| \le 1$ in D. $\square$

Problem 11*.* Let $D = \{z : |z| < 1\}$ and let $a \in D$ and set

$$\varphi_a(z) = \frac{z-a}{1-\overline{a}z}.$$

Show $\varphi_a(0) = 0$ and that $\varphi_a \colon D \to D$ is a bijection with inverse $\varphi_a^{-1} = \varphi_{-a}$. Also show

$$\varphi_a'(a) = \frac{1}{1-|a|^2}.$$

$\square$

Problem 12*.* Let f be analytic in $D = \{z : |z| < 1\}$ with $|f(z)| \le 1$ in D and $f(a) = 0$ for some $a \in D$. Then

(a) $|f(z)| \le |\varphi_a(z)|$ for all $z \in D$ and if equality holds at some point $z = z_0$ with $z_0 \ne a$, then $f(z) = c\varphi_a(z)$ for some constant c with $|z| = 1$.

(b) $|f'(a)| \le 1/(1-|a|^2)$ and if equality holds, then $f(z) = c\varphi_a(z)$ for some constant c with $|c| = 1$. $\square$

Problem 13 (January 2015, Problem 3)*.* Let $f \colon D \to \mathbb{C}$ be a bounded holomorphic function where D is the unit disk. Let $d = \sup\{|f(z) - f(w)| : z, w \in D\}$ be the diameter of the image $f[D]$. Prove $2|f'(0)| \le d$. $\square$