# The Elements of the Calculus of Finite Difference

## 1. The fundamental theorem of summation theory.

Let $f\colon \mathbf{Z} \to \mathbf{R}$ be a function from the integers, $\mathbf{Z}$, to the real numbers, $\mathbf{R}$. We wish to find methods to evaluate sums of the form

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

and in particular the special case

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n).$$

**Definition 1.** Let $f\colon \mathbf{Z} \to \mathbf{R}$. Then the ***diffenence***, $\Delta f$, of $f$ is the function

$$\Delta f(x) = f(x+1) - f(x).$$

The operator $\Delta$ is called the ***difference operator***. □

For example if $f(x) = 3x + 2$, then

$$\Delta f(x) = f(x+1) - f(x) = (3(x+1)+2) - (3x+2) = 3.$$

If $f(x) = x^2$, then

$$\Delta f(x) = f(x+1) - f(x) = (x+1)^2 - x^2 = 2x + 1$$

In the following table $a, b, c, r$ are constants.

| $f(x)$ | $\Delta f(x)$ |
|---|---|
| $c$ | $0$ |
| $ax+b$ | $a$ |
| $cr^x$ | $c(r-1)r^x$ |

**Problem 1.** Verify these. □

**Theorem 1 (Fundamental Theorem of Summation Theory).** *Let $f\colon \mathbf{Z} \to \mathbf{R}$ and let $F$ be an* ***anti-difference*** *of $f$. That is $\Delta F = f$. Then for $a, b \in \mathbf{Z}$ with $a < b$*

$$\sum_{k=a}^{b} f(k) = F(b+1) - F(a).$$

*In particular*

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1).$$

*Proof.* This uses the basic trick about telescoping sums:

$$\begin{aligned}
\sum_{k=a}^{b} f(k) &= \sum_{k=a}^{b} (F(k+1) - F(k)) \\
&= \sum_{k=a}^{b} F(k+1) - \sum_{k=a}^{b} F(k) \\
&= \big(F(a+1) + F(a+2) + \cdots + F(b) + F(b+1)\big) \\
&\qquad - \big(F(a) + F(a+1) + \cdots + F(b-1) + F(b)\big) \\
&= F(b+1) - F(a)
\end{aligned}$$

as required. □

Theorem 1 makes it interesting to find anti-differences of functions. Here are some basic examples of functions $f(x)$ defined on the integers and their anti-differences ($a$, $r$ and $b$ are constants).

| $f(x)$ | $F(x)$ |
|---|---|
| $ax + b$ | $a\dfrac{x(x-1)}{2} + bx$ |
| $ar^x$ | $\dfrac{ar^x}{1-r}$ |

**Problem 2.** Verify these. (You just need to check $F(x+1) - F(x) = f(x)$). □

**Problem 3** (Sum of finite geometric series)**.** Use that $\dfrac{ar^x}{1-r}$ as the anti-difference of $ar^x$ and Theorem 1 to show

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r} = \frac{\text{first} - \text{next}}{1 - \text{ratio}}.$$

□

## 2. Falling factorial powers and sums of powers.

For Theorem 1 to be useful we need more functions $f(x)$ where we know the anti-difference $F(x)$. As a start we give

**Definition 2.** For natural number $p$ define the ***falling factorial power*** of $x \in \mathbf{R}$ as $x^{\underline{0}} = 1$ and for $p \geq 1$

$$x^{\underline{p}} = x(x-1)(x-2)\cdots(x-(p-1)).$$

(This product has $p$ terms.) □

For small values of $p$ this becomes

$$\begin{aligned}
x^{\underline{0}} &= 1 \\
x^{\underline{1}} &= x \\
x^{\underline{2}} &= x(x-1) \\
x^{\underline{3}} &= x(x-1)(x-2) \\
x^{\underline{4}} &= x(x-1)(x-2)(x-3) \\
x^{\underline{5}} &= x(x-1)(x-2)(x-3)(x-4).
\end{aligned}$$

**Proposition 1.** *If $f(x) = x^{\underline{p}}$ where $p$ is a natural number, then $\Delta f(x) = px^{\underline{p-1}}$. That is*

$$\Delta x^{\underline{p}} = px^{\underline{p-1}}.$$

**Problem 4.** Prove this. □

*Remark* 1. The formula should remind you of the c formula $\frac{d}{dx}x^p = px^{p-1}$ for derivatives. □

**Proposition 2.** *If $f(x) = x^{\underline{p}}$ where $p$ is a non-negative integer, then $F(x) = \dfrac{1}{p+1}x^{\underline{p+1}}$ is an anti-difference of $f$.*

**Problem 5.** Prove this as a corollary of Proposition 1 by noting (by replacing $p$ by $p+1$), that $\Delta x^{\underline{p+1}} = (p+1)x^{\underline{p}}$ and dividing by $(p+1)$. □

**Problem 6.** Show that if $p \geq 2$ that $1^{\underline{p}} = 0$. (For example $1^{\underline{3}} = 1(1-1)(1-2) = 0$.) □

**Proposition 3.** *If $p$ is a positive integer, then*

$$\sum_{k=1}^{n} k^{\underline{p}} = \frac{(n+1)^{\underline{p+1}}}{p+1}.$$

*Remark* 2. This should remind you of the formula $\displaystyle\int_0^x t^p\,dt = \frac{x^{p+1}}{p+1}$.

**Problem 7.** Prove this. Hint: Let $f(x) = x^{\underline{p}}$. Then $F(x) = \dfrac{x^{\underline{p+1}}}{p+1}$ is an anti-difference of $f(x)$ and thus by Theorem 1

$$\sum_{k=1}^{n} f(k) = F(n+1) - F(1)$$

and use Problem 6 to see that $F(1) = 0$. □

**Proposition 4.** *The equalities*

$$\begin{aligned}
x &= x^{\underline{1}} \\
x^2 &= x^{\underline{2}} + x^{\underline{1}} \\
x^3 &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \\
x^4 &= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} \\
x^5 &= x^{\underline{5}} + 10x^{\underline{4}} + 25x^{\underline{3}} + 15x^{\underline{2}} + x^{\underline{1}}
\end{aligned}$$

*hold.*

**Problem 8.** Verify the first three of these. □

**Problem 9.** Find formulas for

$$\sum_{k=1}^{n} k^2, \qquad \sum_{k=1}^{n} k^3.$$

HINT: Here is the idea for $\sum_{k=1}^{n} k^2$. Using the last problem and Proposition 3

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} (k^{\underline{2}} + k^{\underline{1}}) \\
&= \sum_{k=1}^{n} k^{\underline{2}} + \sum_{k=1}^{n} k^{\underline{1}} \\
&= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2} \\
&= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}.
\end{aligned}$$

We can leave the answer like this, or expand and factor to get

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Do similar calculations for $\sum_{k=1}^{n} k^3$. □

## 3. A couple of trigonometric sums.

For your convenience we recall some trig identities:

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\
\cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\
\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)
\end{aligned}$$

**Problem 10.** Let $\theta$ be a constant with $\sin(\frac{\theta}{2}) \neq 0$. Use the identities above to show

$$\sin\left(\theta\left(x+\frac{1}{2}\right)\right) - \sin\left(\theta\left(x-\frac{1}{2}\right)\right) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\theta x\right)$$

and therefore

$$F(x) = \frac{\sin\left(\theta\left(x-\frac{1}{2}\right)\right)}{2\sin(\frac{\theta}{2})}.$$

is an anti-difference of

$$f(x) = \cos(\theta x).$$

□

**Proposition 5.** *If* $\sin\left(\frac{\theta}{2}\right) \neq 0$, *then*

$$\sum_{k=1}^{n} \cos(k\theta) = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} = \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)} - \frac{1}{2}.$$

**Problem 11.** Use Problem 10 and Theorem 1 to prove this. □

There is a similar formula for sums for the sine function.

**Proposition 6.** *If* $\sin(\frac{\theta}{2}) \neq 0$, *then*

$$\sum_{k=1}^{n} \sin(k\theta) = \frac{\cos(\frac{\theta}{2}) - \cos((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}.$$

**Problem 12.** Prove this. □