ANALYSIS QUALIFYING EXAM AUGUST 23 1989.

Throughout this exam, unless otherwise specified, the terms measurable, a.e., refer to the Lebesgue measure λ on the real line $\mathbb{R}$, and L^p of an interval to L^p of that interval with respect to Lebesgue measure on that interval.

1. Let $\langle g_n \rangle$ be a sequence of Lebesgue measurable functions which converge a.e. to an integrable function g. Let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and $\langle f_n \rangle$ converges to f a.e. Prove that if
$$\int g d\lambda = \lim \int g_n d\lambda,$$
then
$$\int f d\lambda = \lim \int f_n d\lambda.$$

2. Let $f \in L^1(\mathbb{R}, \lambda)$. Prove that
$$\lim_{s \to 0} \int |f(x+s) - f(x)| d\lambda(x) = 0.$$

3. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete measure spaces. Let $g \in L^1(X, \mathcal{A}, \mu)$ and $h \in L^1(Y, \mathcal{B}, \nu)$, and define f on $X \times Y$ by $f(x,y) = g(x)h(y)$. Prove that
   a. $f \in L^1(X \times Y, \mu \times \nu)$.
   b. $\int_{X \times Y} f d(\mu \times \nu) = \int_X g d\mu \int_Y h d\nu$.
   Note: We do not assume that μ and ν are σ-finite!

4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Suppose $\langle \nu_n \rangle$ is sequence of measures on $(X, \mathcal{A})$ such that for some $M > 0$ we have
$$\nu_n(E) \leq M\mu(E) \text{ for all } E \in \mathcal{A}.$$
Suppose $\nu(E) = \lim_{n \to \infty} \nu_n(E)$ exists for all $E \in \mathcal{A}$. Prove that:
   a. ν is a measure
   b. There exists $f \in L^\infty(X, \mathcal{A}, \mu)$ such that $\nu(E) = \int_E f d\mu$.

5. Let $f \in L^1([0,1])$ such that $f(0) = 0$ and $f'(0)$ exists. Prove that $g \in L^1([0,1])$, where $g(x) = f(x)/x^{\frac{3}{2}}$.

6. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $f_n \in L^p(X, \mu)$ for $1 \leq p < \infty$. Suppose $f_n \to f$ in measure and that for all $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies that

$$\left(\int_E |f_n|^p d\mu\right)^{\frac{1}{p}} < \epsilon$$

for all n. Prove that $\|f_n - f\|_p \to 0$.

7. Let $\langle f_n \rangle$ be a sequence of increasing absolutely continuous functions on $[0,1]$ such that $f_n(0) = 0$ and $f_n(1) = \frac{1}{2^n}$. Prove that $f = \sum f_n$ is absolutely continuous.

8. Show $\lambda(E) = \sup\{\lambda(K) : K \subset E, K \text{ compact }\}$ for all measurable sets $E \subset \mathbb{R}$.

9. True or False. Prove or give a counterexample.
   a. If $f_n \to f$ in $L^1(X, \mu)$, then $f_n \to f$ in measure.
   b. If $f_n \to f$ in measure, then $f_n \to f$ in $L^1([0,1])$.
   c. If $f_n \to 0$ uniformly on $\mathbb{R}$ and f_n Lebesgue integrable, then $\int f_n d\lambda \to 0$.
   d. If f_n Lebesgue measurable and $f_1 \geq f_2 \geq \cdots \geq 0$, then $\lim_{n\to\infty} \int_E f_n d\lambda = \int_E \lim_{n\to\infty} f_n d\lambda$.
   e. Let $0 \leq f$ be Lebesgue integrable over $\mathbb{R}$. Then for all $\epsilon > 0$ there exists $E \subset \mathbb{R}$ with $\lambda(E) < \infty$ such that $\int f d\lambda \leq \int_E f d\lambda + \epsilon$