Mathematics 242 Homework.

Problem 1. Use the method of undetermined coefficients to find the general solutions to the following equations.

(a) $y'' - 4y = \sin(2x)$.
(b) $y'' - 4y' + 3y = 2e^{-x} - 2$.
(c) $y'' - 4y' + 4y = 2x - 12e^{3x}$.
(d) $y'' - 3y' + 2y = 9 - 4e^{2x}$. *Hint:* The function $4e^{2x}$ is a solution to $y'' - 3y' + 2y = 0$, so this is a case where you will have to multiple by x. $\square$

Problem 2. Find the solution to each of the following initial value problems.

(a) $y'' - y' - 2y = 5\sin(x)$, $y(0) = 1$, $y'(0) = -1$.
(b) $y'' + y = \cos(x)$, $y(0) = y'(0) = 0$. $\square$

Here we derive Lagrange's method of variation of parameters. Assume that y_1 and y_2 are linearly independent solutions to the homogeneous equation

$$y'' + p(x)y' + q(x) = 0.$$

We want to find a particular solution to

$$y'' + p(x)y' + q(x)y = f(x).$$

Lagrange's idea is to let

$$y_p = u_1 y_1 + u_2 y_2 \tag{1}$$

where u_1 and u_2 are function which satisfy

$$u_1' y_1 + u_2' y_2 = 0. \tag{2}$$

Problem 3. Assuming that y_p is given by (1) and that u_1 and u_2 satisfy (2)

(a) Show

$$y_p' = u_1 y_1 u_2 y_2.$$

(b) Show

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2 y_2'$$

(c) Combine these formulas to show

$$\begin{aligned} y_p'' + py_p' + qy_p = u_1\Big(y_1'' + py_1' + qy_1\Big) + u_2\Big(y_2'' + py_2' + qy_2\Big) \\ + u_1' y_1' + u_2' y_2' \end{aligned}$$

(d) Use y_1 and y_2 are solutions to the homogeneous equation to show that the formula of part (c) reduces to

$$y_p'' + py_p' + qy_p = u_1' y_1' + u_2' y_2'$$

$\square$

A summary of what you have shown in Problem 3 is that if u_1 and u_2 are solutions to the pair of equations

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned}$$

then

$$y_p = u_1 y_1 + u_2 y_2$$

is a particular solution to the inhomogeneous equation. A little bit of algebra shows that the equations in question can be solved for u_1' and u_2' and the solutions are

$$\begin{aligned} u_1' &= \frac{-f y_2}{y_1 y_2' - y_1' y_2} \\ u_2' &= \frac{f y_1}{y_1 y_2' - y_1' y_2} \end{aligned}$$

Recalling that the Wronskian of y_1 and y_2 is $W = y_1 y_2' - y_1' y_2$ this can be rewritten as

$$\begin{aligned} u_1' &= \frac{-f y_2}{W} \\ u_2' &= \frac{f y_1}{W} \end{aligned}$$

Now u_1 and u_2 can be found by integration:

$$u_1 = \int \frac{-f(x) y_2(x)}{W}(x)\,dx, \qquad u_2 = \int \frac{f(x) y_1(x)}{W(x)}\,dx$$

Problem 4. Find the general solution to

$$y'' + y = \sec x.$$

Problem 5. Show that $y_1 = x$ and $y_2 = x^2$ are solutions to

$$x^2 y'' - 2xy' + 2y = 0$$

on the interval $(0, \infty)$. Use variation of parameters to find the general solution to

$$x^2 y'' - 2xy' + 2y = 1 + x$$

on this interval. *Hint:* Note this is not in the form $y'' + p(x)y' + q(x)y = f(x)$ so do not forget to first put the equation in this form. $\square$