THE SHARP SOBOLEV INEQUALITY AND THE BANCHOFF-POHL INEQUALITY ON SURFACES

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Abstract. Let (M, g) be a complete two dimensional simply connected Riemannian manifold with Gaussian curvature $K \leq -1$. If f is a compactly supported function of bounded variation on M then f satisfies the Sobolev inequality

$$4\pi \int_M f^2 \, dt + \left(\int_M |f| \, dA \right)^2 \leq \left(\int_M \|\nabla f\| \, dA \right)^2 .$$

Conversely letting f be the characteristic function of a domain $D \subset M$ recovers the sharp form $4\pi A(D) + A(D)^2 \leq L(\partial D)^2$ of the isoperimetric inequality for simply connected surfaces with $K \leq -1$. Therefore this is the Sobolev inequality "equivalent" to the isoperimetric inequality for this class of surfaces. This is a special case of a result that gives the equivalence of more general isoperimetric inequalities and Sobolev inequalities on surfaces.

Under the same assumptions on (M, g) if $c\colon [a, b] \to M$ is a closed curve and $w_c(x)$ is the winding number of c about x then the Sobolev inequality implies

$$4\pi \int_M w_c^2 \, dA + \left(\int_M |w_c| \, dA \right)^2 \leq L(c)^2$$

which is an extension of the Banchoff-Pohl inequality to simply connected surfaces with curvature ≤ -1.

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