Math 242 Test 3.

- *This is due on Sunday, November 22 at midnight. You are to work alone in it. You can look up definitions and the statements of theorems we have covered in class. And if there is an integral where you want to use a computer, your calculator, or a source such as Wolfram Alpha to compute it, that is fine, but say that this is what you did. (For example "I computed $\int x^2 e^x\,dx$ using the program Maple".) Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg for direct help on the problems.*
- *Print your name on the first page of the exam.*
- *Use a dark pen or pencil so that the writing stands out even after being copied.*

Show your work for full credit.

Problem 1 (20 points). Let $H(t)$ be the Heaviside function

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

Use the Laplace transform to solve the initial value problem

$$x''(t) + 2x'(t) + 2x(t) = 3H(t-4), \qquad x(0) = 1, \quad x'(0) = 2.$$

Hint: Look at Homework 8 for an example of this type of problem. □

Problem 2 (20 points). We have talked about both systems of differential equations and also the Laplace transform. It is worth noting that the Laplace transforms can be used to solve constant coefficient linear systems (and in fact you have already done an example of this on Problem 4 on Test 2). Let us consider the initial value problem

$$\begin{aligned} x'(t) &= ax(t) + by(t) & \qquad x(0) &= x_0 \\ y'(t) &= cx(t) + dy(t) & \qquad y(0) &= y_0. \end{aligned}$$

where a, b, c, d, x_0, y_0 are constants. Find formulas for the Laplace transforms $\mathcal{L}\{x(t)\}$ and $\mathcal{L}\{y(t)\}$. (The answers will involve all of the constants a, b, c, d, x_0, y_0 along with the variable s.) *Hint:* The algebra will be simplified if you use Cramer's rule. *Note:* Only the Laplace transforms $\mathcal{L}\{x(t)\}$ and $\mathcal{L}\{y(t)\}$ are being ask for, you do not have to solve for $x(t)$ and $y(t)$. In principle this is "easy" as it only involves taking the inverse Laplace transform, but that is more algebra than I want to inflict on you. □

Problem 3 (20 points). For the initial value problem

$$y'' + 2y' + xy = 0 \qquad y(0) = 1, \quad y'(0)$$

assume there is a series solution

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

(a) What are a_0 and a_1?
(b) Find the recursion relation on the coefficients a_k.
(c) Explicitly what are the first five terms of the series. That is $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$. □

Problem 4 (20 points). Consider the system of differential equations

$$\frac{dx}{dt} = .1x(1 - x - .2y)$$
$$\frac{dy}{dt} = .05y(1 - .5x - y)$$

(a) If $x(3) = 4$ and $y(3) = 7$ find $x'(3)$ and $y'(3)$.
(b) If $x(3) = 4$ and $y(3) = 7$ is $x(t)$ increasing or decreasing when $t = 3$? Explain how you know.
(c) If $x(3) = 4$ and $y(3) = 7$ estimate $x(3.1)$ and $y(3.1)$. □

Recall if we have an autonomous system of equations:

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

then we can get a first order equation relating x and y along a solution by use of the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g(x, y)}{f(x, y)}.$$

For example if

$$\frac{dx}{dt} = x(2 - y)$$
$$\frac{dy}{dt} = -y(1 - x)$$

then we have

$$\frac{dy}{dx} = \frac{-y(1 - x)}{x(2 - y)}.$$

This is a separable equation and can be rewritten as

$$\frac{2 - y}{y}\, dy = -\frac{1 - x}{x}\, dx$$

Split up the fractions and integrate

$$\int \left(\frac{2}{y} - 1\right) dy = -\int \left(\frac{1}{x} - 1\right) dx$$

which leads x and y satisfying the equation

$$2\ln(|y|) - y = -\left(\ln(|x|) - x\right) + C_0$$

where C_0 is a constant of integration. Therefore the solutions move on the curves defined by

$$2\ln(|y|) + \ln(|x|) - x - y = C_0$$

If the initial conditions are $x(0) = 3$ and $y(0) = 4$ we can compute C_0

$$C_0 = 2\ln(4) + \ln(3) - 3 - 4 = -3.41648106154389\ldots$$

and thus in this case the solution $(x(t), y(t))$ with this initial condition moves on the curve

$$2\ln(|y|) + \ln(|x|) - x - y = -3.41648106154389\ldots$$

Problem 5 (20 points)*.* For the system

$$\frac{dx}{dt} = .1x(1 - .2y)$$
$$\frac{dy}{dt} = .3y(1 - .4x)$$

(a) Find the first order equation between x and y.
(b) Use this to find an relating x and y and a constant of integration.
(c) If $x_0 = 10$ and $y(0) = 20$ what is the equation relating x and y?