Mathematics 554H/701I Homework

The topic we have started since the last test is the convergence of sequences.

Definition 1. Let E be a metric space and $\langle p_n \rangle_{n=1}^{\infty} = \langle p_1, p_2, p_3, \ldots \rangle$ a sequence in E. Then

$$\lim_{n\to\infty} p_n = p$$

if and only if for all $\varepsilon > 0$ there is a $N > 0$ such that

$$n > N \qquad \Longrightarrow \qquad d(p_n, p) < \varepsilon.$$

In the case we say that the sequence $\langle p_n \rangle_{n=1}^{\infty}$ ***converges*** to p. □

Problem 1. Let $\lim_{n\to\infty} p_n = p$ in the metric space E. Let $a_n = p_{2n}$. Show that $\lim_{n\to\infty} a_n = p$ also holds. □

Problem 2. Write out the proof from the definition that if $\lim_{n\to\infty} x_n = x$ in $\mathbb{R}$, that $\lim_{n\to\infty} -5x_n = -5x$. □

Problem 3. Write out the proof that if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ in $\mathbb{R}$ that

$$\lim_{n\to\infty} (10x_n - 12y_n) = 10x - 12y.$$

□

We did a proof of the following in class.

Proposition 2. *If $\langle x_n \rangle$ be a convergent sequence in $\mathbb{R}$. Then there is a constant M such that $|x_n| < M$ for all M.* □

Theorem 3. *Let*

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y$$

in $\mathbb{R}$. Then

$$\lim_{n\to\infty} x_n y_n = xy.$$

Problem 4. Prove this. *Hint:* Start with

> **Scratch work that the no one else needs to see:** Our goal is to make $|x_n y_n - xy|$ small. We compute

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \qquad \text{(Adding and subtracting trick.)} \\ &\le |x_n y_n - x y_n| + |x y_n - xy| \\ &= |x_n - x||y_n| + |x||y_n - y| \end{aligned}$$

> The factors $|x_n - x|$ and $|y_n - y|$ are both good in that we can make them small. The factor $|x|$ is independent of n and thus is not a problem. The sequence $\langle y_n \rangle_{n=1}^{\infty}$ is convergent and thus bounded, so we bound the factor $|y_n|$. We now return to our regularly scheduled proof.

Let $\varepsilon > 0$. The sequence $\langle y_n \rangle_{n=1}^{\infty}$ is convergent thus it is bounded. Therefore there is an M so that

$$|y_n| \leq M \quad forall \quad n.$$

As $\lim_{n\to\infty} x_n = x$ there There is a $N_n > 0$ such that

$$n > N_2 \quad \text{implies} \quad |x_n - x| < \frac{\varepsilon}{2(M+1)}$$

and as $\lim_{n\to\infty} y_n = y$ there is a $N_2 > 0$ such that

$$n > N_2 \quad \text{implies} \quad |y - y_n| < \frac{\varepsilon}{2(|x|+1)}.$$

Now let $N = \max\{N_1, N_2\}$ and use the calculation from our scratch work to show

$$n > N \quad \text{implies} \quad |x_n y_n - xy| < \varepsilon$$

which completes the proof. □

Proposition 4. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in the metric space E. Let $\langle p_{n_k} \rangle_{k=1}^{\infty}$ be a subsequence of this sequence. Then $\langle p_{n_k} \rangle_{k=1}^{\infty}$ is also convergent and has the same limit at the original sequence.*

Problem 5. Prove this. *Hint:* For all k we have $n_k \geq k$. □

Definition 5. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers. Then this sequence is ***monotone increasing*** if and only if $x_n \leq x_{n+1}$ for all n. It is ***monotone decreasing*** if and only if $n_n \geq x_{n_1}$ for all n. It is ***monotone*** if it is either monotone increasing or monotone decreasing. □

Theorem 6. *A bounded monotone sequence in $\mathbb{R}$ is convergent.*

Proof. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a bounded monotone sequence. We first assume that it is monotone increasing. Let

$$S = \{x_n : n = 1, 2, \ldots\}$$

be the set of values of the sequence. As the sequence is bounded, this set is bounded. Therefore, by Least Upper bound Axiom, this set has a least upper bound $b = \sup(S)$. We now show that the sequence converges to b.

Let $\varepsilon > 0$. Then $b - \varepsilon < b$ and b is the least upper bound of S, therefore $b - \varepsilon$ is not an upper bound for S. Therefore there is positive integer N such that $b - \varepsilon < x_N$. Then for any $n > N$ we have

$$\begin{aligned} b - \varepsilon &< x_N \\ &\leq x_n \qquad (x_N \leq x_n \text{ as the sequence is monotone increasing.}) \\ &\leq b \qquad (\text{as } b \text{ is an upper bound for } S \text{ and } x_n \in S.) \end{aligned}$$

Therefore we have $b - \varepsilon < x_n \leq b$ for all $n > N$. Therefore $n > N$ implies $|x_n - b| < \varepsilon$ and thus $\lim_{n\to\infty} x_n = b$. □ □

Problem 6. Modify the last proof so show that if $\langle x_n \rangle_{n=1}^{\infty}$ is bounded and monotone decreasing that it converges to $\inf\{x_n : n = 1, 2, 3, \ldots\}$. □

The following is a very important idea.

Definition 7. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in a metric space E. Then the sequence is a ***Cauchy sequence*** if and only if for all $\varepsilon > 0$, there is a $N > 0$ such that $m, n > N$ implies $d(p_m, p_n) < \varepsilon$. $\square$

A brief version would be that $\langle p_n \rangle_{n=1}^{\infty}$ is Cauchy if and only if

$$\forall \varepsilon > 0 \; \exists N > 0 \big[\, m, n > n \quad \implies \quad d(p_m, p_n) < \varepsilon \,\big].$$

Proposition 8. *Every convergent sequence is a Cauchy sequence.*

Problem 7. Prove this. *Hint:* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in the metric space and let p be its limit. Let N be so that

$$n > N \quad \text{implies} \quad d(p_n, p) < \frac{\varepsilon}{2}.$$

Then show that

$$m, n > n \text{implies} \quad d(p_m, p_n) < \varepsilon.$$

The converse is not true. There are Cauchy sequences that are not convergent.

Problem 8. Let $E = (0, 1)$ be the open unit interval with metric $d(x, y) = |x - y|$. Then show that the sequence $\langle 1/n \rangle_{n=1}^{\infty}$ is a Cauchy sequence that is not convergent to any point of E. $\square$

You may feel that the example of the last problem is a bit of a cheat as the sequence does converge in the larger space of all real numbers. And is some sense this is true, given a metric space, E, there is a natural way to expand it to a somewhat larger space that contains the limits of all Cauchy sequences from E.

Proposition 9. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in the metric space E, such that some subsequence of $\langle p_{n_k} \rangle_{k=1}^{\infty}$ converges. Then the original sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges.*

Problem 9. Prove this. *Hint:* Let $\varepsilon > 0$. As the sequence is Cauchy, there is a N such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \frac{\varepsilon}{2}.$$

Let $n > N$, then for any k we have by the triangle inequality that

$$d(p_n, p) \le d(p_n, p_{n_k}) + d(p_{n_k}, p).$$

Now show that it is possible to choose k such that both $d(p_n, p_{n_k})$ and $d(p_{n_k}, p)$ are less than $\varepsilon/2$. $\square$

Theorem 10. *Every sequence of real numbers has a monotone subsequence.*

Problem 10. Prove this. *Hint:* Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of real numbers. Call x_n a ***peak point*** if $x_n \ge x_m$ for all $m > n$. (That is x_n is greater than or equal to all the values that follow it.)

Case 1: *There are infinitely many peak points.* In this case there is an infinite subsequence of the sequence consisting of peak points. Show this subsequence is monotone decreasing.

Case 2: *There are only finitely many peak points.* Let N be the largest n such that x_n is a peak point. Thus if $n > N$ the point x_n is not a peak point and therefore there is $m > n$ with $x_n > x_n$. Let $n_1 = N_1$. Then $n_1 > N$ and so there is a $n_2 > n_1$ with $x_{n_2} > x_{n_1}$. But then $n_2 > N$ and thus there is $n_3 > n_2$ with $x_{n_3} > x_{n_2}$. Continue in this manner to show that there is an infinite increasing subsequence. $\square$

Proposition 11. *Let E be a metric space. Then every Cauchy sequence in E is bounded. (That is the sequence is contained in some ball.)*

Problem 11. Prove this. *Hint:* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in E. Let $\varepsilon = 1$ (or any other positive number that you like). Then there is $N > 0$ such that

$$m, n > N \quad \text{implies} \quad d(p_m, p_n) < \varepsilon = 1.$$

Let $a = N + 1$ and set

$$r = 1 + \max\{1, d(a, x_1), d(a, x_2), \ldots, d(a, x_N)\}.$$

Then show that $p_n \in B(a, r)$ for all n. $\square$

Theorem 12. *Every Cauchy sequence in $\mathbb{R}$ converges.*

Problem 12. Prove this. *Hint:* Let $\langle x_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}$. Then by Proposition 11 this sequence is bounded. By Theorem 10 this sequence has a monotone subsequence. By Theorem 6 this monotone subsequence converges. Put these facts together with Proposition 9 to prove that the sequence $\langle x_n \rangle_{n=1}^{\infty}$ converges. $\square$

This property of a metric space, that Cauchy implies convergent, is important enough to give a name.

Definition 13. The metric space E is ***complete*** if and only if every Cauchy sequence in E converges. $\square$

So we can restate Theorem 12 as

Proposition 14. *The real numbers, $\mathbb{R}$, with their usual metric is a complete metric space.* $\square$

We can not get more examples by looking at closed subsets of complete metric spaces.

Proposition 15. *Let E be a metric space and F a closed subset of E. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of points of F that converges in E to some point p. Then $p \in F$. (A nice restatement of this is that a closed set contains all its limit points.)*

Problem 13. Prove this. *Hint:* Towards a contradiction assume that $p \notin F$. Then as F is closed, the compliment $\mathcal{C}(F)$ is open. As $p \in \mathcal{C}F$ by the definition an open set, there is a $r > 0$ such that $B(p, r) \subseteq \mathcal{C}(F)$. But $\lim_{n\to\infty} p_n = p$ and therefore if we let $\varepsilon = r$ there is a $N > 0$ such that $n > 0$ implies $d(p_n, p) < \varepsilon = r$. This this leads to a contradiction. $\square$

Proposition 16. *Let E be a complete metric space and F a closed subset of E. Then F, considered as a metric space in its own right, is complete.*

Problem 14. Prove this. *Hint:* Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence from F. As E is complete this sequence converges to some point, p, of E. To finish the proof it is enough to show that $p \in F$. $\square$