Mathematics 546 Homework.

Let us review what we should all know about polynomials. Let F be a field, which for the time being we can assume is one of the following:

$$\begin{aligned} \mathbb{Q} &= \text{The rational numbers,} \\ \mathbb{R} &= \text{The real numbers,} \\ \mathbb{C} &= \text{The complex numbers, or} \\ \mathbb{Z}_p &= \text{for } p \text{ a prime number.} \end{aligned}$$

You can find a formal definition of a field in Definition 4.1.1 on Page 191 of the text, but for the time being the above examples are plenty. Let $F[x]$ be the polynomials with coefficients from F. That is (See Definition 4.1.4 on Page 194 of the text) ***polynomials*** are expressions of the form

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$

where the ***coefficients*** $a_0, a_1, \ldots, a_m$ are elements of the field F. In summation notation this is

$$f(x) = \sum_{j=0}^{m} a_j x^j$$

with the understanding that $x^0 = 1$. If $a_m \neq 0$, then

$$\deg(f(x)) = m.$$

For example

$$\begin{aligned} \deg(4x^3 - 9x^2 + 17x - 42) &= 3 \\ \deg(x^n - x) &= 1 \qquad \text{When } n \text{ is an integer} \geq 2. \\ \deg(5) &= 0. \end{aligned}$$

In general if $a_0 \neq 0$ is a nonzero constant, then the constant polynomial $f(x) = a_0 = a_0 x^0$ has $\deg(f(x)) = 0$. The zero polynomial $f(x) = 0$ is not given a degree (or some people give it the degree $\deg(0) = -\infty$).

The basic rule for exponents

$$x^j x^k = x^{j+k}$$

and the distributive law tells us how to multiply polynomials. For example using the distributive law on the product $(a_2x^2 + a_1x + a_0)(b_3x^3 + b_2x^2 + b_1x + b_0)$ leads to $3 \times 4 = 12$ terms which can then be grouped by powers of

x:

$$\begin{aligned}
&(a_2x^2 + a_1x + a_0)(b_3x^3 + b_2x^2 + b_1x + b_0)\\
&\quad = a_2x^2(b_3x^3 + b_2x^2 + b_1x + b_0)\\
&\qquad + a_1x(b_3x^3 + b_2x^2 + b_1x + b_0)\\
&\qquad + a_0(b_3x^3 + b_2x^2 + b_1x + b_0)\\
&\quad = a_2b_3x^5 + a_2b_2x^4 + a_2b_13 + a_2b_0^2\\
&\qquad + a_1b_3x^4 + a_1b_2x^3 + a_1b_1x^2 + a_1b_0x\\
&\qquad + a_0b_3x^3 + a_0b_2x^2 + a_0b_1x + a_0b_0\\
&\quad = a_2a_3x^5 + (a_2b_2 + a_1b_3)x^4 + (a_2b_1 + a_1b_2 + a_0b_3)x^3\\
&\qquad + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + (a_1b_0 + a_0b_1)x + a_0b_0.
\end{aligned}$$

In general if

$$\begin{aligned}
f(x) &= a_mx^m + a_{m-1}x^{m-1} + \cdots + a_1x + a_0\\
g(x) &= b_nx^n + b_{n-1}x^{m-1} + \cdots + a_1x + b_0
\end{aligned}$$

then the product

$$\begin{aligned}
f(x)g(x) &= c_{m+n}x^{m+n} + c_{n+m-1}x^{m+n-1} + c_{m+n-2}x^{m_n-2} + \cdots + c_1x + c_0\\
&\sum_{k=0}^{m+n} c_kx^k
\end{aligned}$$

where

$$\begin{aligned}
c_{m+n} &= a_mb_n\\
c_{m+n-1} &= a_mb_{m-1} + a_{m-1}b_n\\
c_{n+m-2} &= a_nb_{n-2} + a_{m-1}b_{n-1} + a_{n-2}b_n\\
\vdots &\quad \vdots\\
c_k &= \sum_{\substack{i+j=k\\ 0\le i\le m\\ 0\le j\le n}} a_ib_j\\
\vdots &\quad \vdots\\
c_2 &= a_2b_0 + a_1b_1 + a_0b_2\\
c_1 &= a_1b_0 + a_0b_1\\
c_0 &= a_0b_0.
\end{aligned}$$

The formula for c_k can be simplified if we set $a_i = 0$ for $i > m$ and $b_j = 0$ for $j > n$. Then

$$c_k = \sum_{i+j=k} a_jb_j = \sum_{i=0}^{k} a_ib_{k-i} = \sum_{j=0}^{k} a_{k-j}b_j.$$

Proposition 1. *If $f(x), g(x) \in F[x]$ are not the zero polynomial, then*

$$\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x)).$$

Proof. Let $\deg(f(x)) = m$ and $\deg(g(x)) = n$ then

$$\begin{aligned} f(x) &= a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \\ g(x) &= b_n x^n + b_{n-1}x^{m-1} + \cdots + a_1 x + b_0 \end{aligned}$$

where $a_m \neq 0$ and $b_n \neq 0$. Then

$$f(x)g(x) = c_{m+n}x^{m+n} + c_{m+n-1}x^{m+n-1}x^{m+n-1} + \cdots + c_1 x + c_0.$$

where $c_{m+n} = a_m b_n \neq 0$. Thus $\deg(f(x)g(x)) = m + n = \deg(f(x)) + \deg(g(x))$ as required. ☐

Problem 1. This problem is just a bit of practice (or review) in basic operations with polynomials. Let

$$\begin{aligned} f(x) &= 3x^2 - 4x + 1 \\ g(x) &= x^3 + 2x^2 - x + 5. \end{aligned}$$

Compute the following

(a) $f(x) + g(x)$ (or just write "Oh come on, you know we can all add polynoials".)
(b) $f(x)^2$
(c) $f(x)g(x)$. ☐

Problem 2. Let $a \in F$ and compute the following

(a) $(x - a)(x + a)$
(b) $(x - a)(x^2 + ax + a^2)$
(c) $(x - a)(x^3 + ax^2 + a^2x + a^3)$
(d) $(x - a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4)$
(e) At this point you should have seen a pattern. What is it? ☐

We will also want to do long division with polynomials. For example if we divide $f(x) = x^4 + 4x^3 + 3x^2 + 2x - 1$ by $g(x) = x^2 + 2x - 2$:

$$\begin{array}{rr} & x^2 + 2x + 2 \\ x^2 + 2x - 3 & \overline{)\, x^4 + 4x^3 + 3x^2 + 2x - 1} \\ & \underline{x^4 + 2x^3 - 3x^2 \qquad\qquad} \\ & 2x^3 + 6x^2 + 2x - 1 \\ & \underline{2x^3 + 4x^2 - 6x \qquad} \\ & 2x^2 + 8x - 1 \\ & \underline{2x^2 + 4x - 6} \\ & 4x + 5 \end{array}$$

we get a quotient of $q(x) = x^2 + 2x + 2$ and a remainder of $r(x) = 4x + 5$. This means

$$f(x) = q(x)g(x) + r(x).$$

Problem 3. Find the quotient and remainder when $g(x)$ is divided into $f(x)$ in the following cases.

(a) $g(x) = x - 5$ and $f(x) = 4x^2 - 3x + 7$.
(b) $g(x) = x^2 + 2x + 3$ and $f(x) = 3x^4 - 2x^3 + x^2 - 5x + 1$.
(c) $g(x) = x - s$ and $f(x) = ax^2 + bx + c$ where s, a, b, c are constants (that is elements of the field F.)