QUALIFYING EXAM IN ANALYSIS

(August 12, 2011)

Name :
S.S. # :

Throughout this examination the term measurable refers to the Lebesgue measure m on the real line. Integrals with respect to Lebesgue measure will be denoted by $\int f$. Problems 1-8 are worth 10 points each. Each part of problem 9 is worth 4 points.

1. The function $\Omega : C[0,1] \to C[0,1]$ is defined by

$$\Omega(\phi)(x) := \int_0^x \phi(t)dt \quad (\phi \in C[0,1], x \in [0,1]).$$

Prove that Ω is not a contraction mapping, but that $\Omega^2 := \Omega \circ \Omega$ is a contraction mapping.

2. Let E be a compact set in a metric space (X, ρ). Prove that there are points $a, b \in E$ such that

$$\rho(a,b) = \sup_{x,y \in E} \rho(x,y).$$

3. Does the series

$$\sum_{n=1}^{\infty} \frac{x}{n + n^3 x^3}$$

converge uniformly on $[0, \infty)$?

4. Let f be measurable on $[0,1]$.

(i) Prove that the condition

$$\lim_{k \to \infty} km(\{x : |f(x)| > k\}) = 0 \tag{1}$$

is a necessary condition for $f \in L_1([0,1])$.

(ii) Give an example showing that (1) is not a sufficient condition for $f \in L_1([0,1])$.

5. Let f be integrable on $[0,2]$. Prove that

$$\lim_{h \to 0^+} \int_0^1 |f(x+h) - f(x)| dx = 0.$$

6. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $(-\infty, \infty)$ such that $f_n \to f$ a.e., and suppose that $\int f_n \to \int f < \infty$. Prove that for each measurable set E we have $\int_E f_n \to \int_E f$.

7. If γ is the positively oriented unit circle, compute

$$\frac{1}{2\pi\imath}\int_\gamma \frac{e^z - e^{-z}}{z^4} dz.$$

8. Prove Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a measurable set E, then

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n.$$

9. Mark each of the following statements as True or False. In order to obtain points you have to provide proofs or counterexamples to justify your answers.

(a) Let $\{x_n\}$ be a sequence of positive numbers. Show that

$$\limsup_{n\to\infty}(x_1 \cdots x_n)^{1/n} \leq \limsup_{n\to\infty} x_n.$$

(b) The trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$$

is the Fourier series of a continuous 2π-periodic function.

(c) Let f be absolutely continuous on $[0,1]$ with $f' \in L_p([0,1])$, $1 < p < \infty$. There is a constant C such that

$$|f(b) - f(a)| \leq C|b-a|^{1-1/p}$$

for all $a, b \in [0,1]$.

(d) Let f be a function which is regular (holomorphic) on a closed disc D, (i.e. regular on a region of the complex plane which contains the closed disc D). If $|f|$ is constant on the boundary of D then f is constant.

(e) There exist a non-empty compact set A, and a non-empty closed set B of the complex plane such that $A \cap B = \emptyset$ and $\inf\{|a-b| : a \in A, \quad b \in B\} = 0$.