Mathematics 300 Homework, October 16, 2017.

For the rest of the term you will be expected to know the formulas

$$\begin{aligned}
x^2 - y^2 &= (x-y)(x+y) \\
x^3 - y^3 &= (x-y)(x^2+xy+y^2) \\
x^4 - y^4 &= (x-y)(x^3+x^2y+xy^2+y^3) \\
x^5 - y^5 &= (x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)
\end{aligned}$$

and in general

$$x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+x^2y^{n-3}+xy^{n-1}+y^{n-3})$$

Here are a few problems for practice.

1. Show that $x^3 - 27$ factors.

Solution: Write this as $x^3 - 3^3$ and then we have

$$\begin{aligned}
x^3 - 27 = x^3 - 3^3 & \\
&= (x-3)(x^2+x(3)+(3)^2 \\
&= (x-3)(x^2+3x+9).
\end{aligned}$$

□

2. Show $a^5 + b^5$ factors.

Solution: Because of the plus sign this does not fit exactly in to the $x^n - y^n$ pattern. But just a little bit of trickery fixes this. Use that $(-1)^5 = -1$ to write $a^5 + b^5 = a^5 - (-b)^5$ and therefore

$$\begin{aligned}
a^5 + b^5 &= a^5 - (-b)^5 \\
&= (a-(-b))(a^4+a^3(-b)+a^2(-b)^2+a(-b)^3+(-b)^4 \\
&= (a+b)(a^4-a^3b+a^2b^2-ab^3+b^4).
\end{aligned}$$

□

3. Show that for all positive integers n that $9^n - 2^n$ is divisible by 7.

Solution: We use our $x^n - y^n$ identity again.

$$\begin{aligned}
9^n - 2^2 &= (9-2)(9^{n-1}+9^{n-2}2+\cdots+9^2 2^{n-2}+2^{n-1}) \\
&= 7q
\end{aligned}$$

where

$$q = 9^{n-1}+9^{n-2}2+\cdots 9^2 2^{n-2}+2^{n-1}$$

and $q \in \mathbb{Z}$. Thus $7 \big| (9^n - 2^n)$. □

Recall that the ***absolute value*** of a real number x is defined by

$$|x| = \begin{cases} x, & x \geq 0; \\ -x, & x < 0. \end{cases}$$

4. Show that for all $x \in \mathbb{R}$ that $|x| \geq 0$.

Solution: There are two cases $x \geq 0$ and $x < 0$.

Case 1. $x \geq 0$. In this case $|x|x \geq 0$ as required.

Case 2 $x < 0$. That is x is negative. Then $|x| = -x$ and the negative of a negative number is positive and so in this case $|x| = -x > 0$. $\square$

5. Show that for all $x, y \in \mathbb{R}$ that $|xy| = |x||y|$.

Solution: This is annoying in that the natural proof splits into two many cases.

We first note that $|0| = 0$. Also any number multiplied by 0 is 0. Therefore is either x or y is 0, then both of $|xy|$ and $|x||y|$ are zero and therefore the required equality holds if either x or y is zero. We therefore only need to consider case where both x and y are non-zero.

Case 1. $x > 0$ and $y > 0$. Then from basic properties of inequalities (that is positive times positive is positive) $xy > 0$. Therefore

$$|xy| = xy = |x||y|.$$

Case 2. $x > 0$ and $y < 0$. Then from properties of inequalities (this time positive times negative is negative) $xy < 0$. Therefore

$$|xy| = -xy = x(-y) = |x||y||xy| = -xy = x(-y) = |x||y|.$$

Case 3. $x < 0$ and $y > 0$. Then (this time using negative times positive is negative) we have $xy < 0$ and so

$$|xy| = -xy = (-x)(y) = |x||y|.$$

Case 3. $x < 0$ and $y < 0$. Then (negative times negative is positive) we have $xy > 0$ and thus

$$|xy| = xy = (-x)(-y) = |x||y|.$$

This covers all cases and completes the proof. $\square$

Proposition 1. *If r and s are real numbers and $n > 0$. Assume*

$$0 \leq r, s < n.$$

Then

$$|r - s| < n.$$

Proof. There are two cases.

Case 1. $r \leq s$. Then $s - r \geq 0$ (which is the same as $r - s < 00$ and therefore

$$|r - s| = -(r - s) = s - r.$$

Then

$$\begin{aligned} |r - s| &= s - r \\ &< n - r && \text{(as } s < n\text{)} \\ &\leq n && \text{(as } r \geq 0\text{)} \end{aligned}$$

And therefore $|r - s| < n$ in this case. $\square$

Case 2. $s < r$. **Doing this case is homework to be handed in.**