Constructing Approximations to Functions.

This is a somewhat edited version of a homework/notes I wrote for the undergraduate honors analysis class.

Given a function, f, if is often useful to it is often useful to approximate it by "nicer" functions. For example give a continuous function, f, it can be useful to find a sequence of differentiable functions $f_1, f_2, f_3, \ldots$ that converge to f uniformly. Here we give one of the basic methods for doing this.

Definition 1. A sequence of functions $K_1, K_2, K_3, \ldots$ defined on $\mathbb{R}$ is a ***Dirac sequence***, or an ***approximation to the identity*** iff it satisfies the following conditions.

(a) $K_n \geq 0$ for all k,

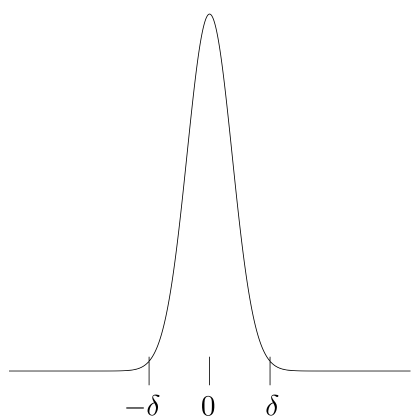
(b) For all n

$$\int_{-\infty}^{\infty} K_n(x)\,dx = 1.$$

(c) For all $\delta > 0$

$$\lim_{n\to\infty} \int_{|x|\geq\delta} K_n(x)\,dx = 0. \qquad \square$$

The condition (c) says that all most all of the mass of K_n is in $(-\delta, \delta)$.



For large n almost all of the area under the graph of $y = K_n(x)$ is between $-\delta$ and δ.

Here is a standard method of constructing Dirac sequences.

Proposition 2. *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function with*

$$\phi \geq 0, \qquad \textit{and} \qquad \int_{-\infty}^{\infty} \phi(x)\,dx = 1.$$

Then

$$K_n(x) = n\phi(nx)$$

is a Dirac sequence.

Problem 1. Prove this. □

Theorem 3. *Let f be a bounded continuous function on $\mathbb{R}$ and $\langle K_n\rangle_{n=1}^{\infty}$ be a Dirac sequence. Let*

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy$$

then

$$\lim_{n\to\infty} f_n(x) = f(x)$$

pointwise.

Problem 2. Prove this. *Hint:* The basic trick is to note that as $\int_{-\infty}^{\infty} K_n(y)\,dy = 1$ we have

$$f(x) = f(x)\cdot 1 = f(x)\int_{-\infty}^{\infty} K_n(y)\,dy = \int_{-\infty}^{\infty} f(x)K_n(y)\,dy.$$

Therefore for any $\delta > 0$ we have

$$\begin{aligned} f(x) - f_n(x) &= \int_{-\infty}^{\infty} f(x)K_n(y)\,dy - \int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy \\ &= \int_{-\infty}^{\infty} (f(x)-f(x-y))K_n(y)\,dy \\ &= \int_{|y|<\delta} (f(x)-f(x-y))K_n(y)\,dy + \int_{|y|\geq\delta} (f(x)-f(x-y))K_n(y)\,dy \\ &= I_{\delta,n}(x) + J_{\delta,n}(x). \end{aligned} \tag{1}$$

Now let $\varepsilon > 0$. Then as f is continuous at x there is a $\delta > 0$ such that

$$|y = x| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| < \frac{\varepsilon}{2}.$$

Explain why the following holds

$$|I_{\delta,n}(x)| \leq \int_{|y|<\delta} |f(x)-f(x-y)|K_n(y)\,dy < \int_{|y|<\delta} \left(\frac{\varepsilon}{2}\right) K_n(y)\,dy \leq \frac{\varepsilon}{2}.$$

Using this in the displayed sequence of equalities (1) gives

$$|f(x) - f_n(x)| \leq |I_{\delta,n}(x)| + |J_{\delta,n}(x)| < \frac{\varepsilon}{2} + |J_{\delta,n}(x)|.$$

This holds for all n. The function f is bounded thus there is a constant B such that $|f(x)| \leq B$ for all x. It follows that for all $x, y \in \mathbb{R}$ that

$$|f(x) - f(x-y)| \leq |f(x)| + |f(x-y)| \leq 2B.$$

Therefore

$$|J_{\delta,n}| \leq \int_{|y|\geq\delta} |f(x)-f(x-y)|K_n(y)\,dy \leq 2B\int_{|y|\geq\delta} K_n(y)\,dy.$$

If you now look back at the definition of a Dirac sequence you should be able to use the last inequality to show

$$\lim_{n\to\infty} |J_{\delta,n}(x)| = 0$$

and thus there is a N such that $n > N$ implies $|J_{\delta,n}(x) < \varepsilon/2$. □

We can do a bit better.

Theorem 4. *Let f function on $\mathbb{R}$ that is both bounded and uniformly continuous and let $\langle K_n \rangle_{n=1}^{\infty}$ be a Dirac sequence. Define*

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy.$$

Then

$$\lim_{n\to\infty} f_n(x) = f(x)$$

uniformly on $\mathbb{R}$.

Problem 3. Prove this. *Hint:* This is just a matter of rewriting the proof of Theorem 3 and making sure that you can make the choices of quantities such as δ and N in a way that is independent of x. □

The following gives a large number of examples of functions where Theorem 4 applies.

Proposition 5. *Let f be a continuous function such that for some interval $[\alpha, \beta]$ we have $f(x) = 0$ for all $x \notin [\alpha, \beta]$. Then f is bounded and uniformly continuous.*

Problem 4. Prove this. *Hint:* This is a good problem to review several of the results we have been working with. (Continuous on closed bounded intervals are bounded and uniformly continuous). □

Proposition 6. *Let f be bounded and continuous on $\mathbb{R}$ and let $\langle K_n \rangle_{n=1}^{\infty}$ be a Dirac sequence and*

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy.$$

Then f_n can be rewritten as

$$f_n(x) = \int_{-\infty}^{\infty} f(y)K_n(x-y)\,dy$$

Problem 5. Prove this. *Hint:* As far as y is concerned, x is a constant. So if we do the change of variable $z = x - y$ we have $dz = -dy$. □

Remark 7. In what follows we will use which ever formula for f_n given by Proposition 6 that it convenient without refering Proposition 6.

We are now in a position to prove one of the most famous theorems in analysis, the *Weierstrass Approximation Theorem*, which says that a continuous function on a closed bounded interval can be uniformly approximated by a polynomial. To start we need a Dirac sequence that is constructed from polynomials.

Lemma 8. *Let*

$$K_n(x) := \begin{cases} c_n(1-x^2)^n, & |x| \le 1; \\ 0, & |x| > 0. \end{cases}$$

where

$$c_n := \frac{1}{\displaystyle\int_{-1}^{1} (1-x^2)^n \, dx}.$$

Then $\langle K_n \rangle_{n=1}^{\infty}$ *is a Dirac sequence.*

Proof. That $K_n \ge 0$ and $\int_{-\infty}^{\infty} K_n(x)\,dx = 1$ are easy, so it remains to show that for $\delta > 0$ the limit $\lim_{n\to\infty} \int_{|x|\ge\delta} K_n(x)\,dx = 0$. We first give a bound on c_n.

$$\int_{-1}^{1} (1-x^2)^n \, dx = 2\int_0^1 (1+x)^n(1-x)^n \, dx \ge 2\int_0^1 (1-x)^n \, dx = \frac{2}{n+1}.$$

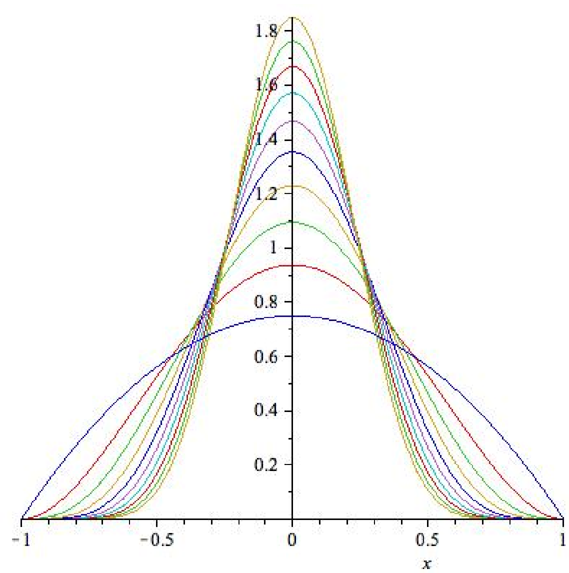
Thus

$$c_n \le \frac{n+1}{2}.$$

Let $0 < \delta < 1$. Then

$$\int_{|x|\ge\delta} K_n(x)\,dx = 2c_n \int_\delta^1 (1-x^2)^n \, dx \le 2c_n \int_\delta^1 (1-\delta)^n \, dx \le (n+1)(1-\delta^2)^n.$$

But $(1-\delta^2) < 1$ so $\lim_{n\to\infty}(n+1)(1-\delta^2)^n = 0$ which completes the proof. $\square$



The graphs of $y = K_n(x)$ for $n = 1, 2, \ldots, 10$.

Problem 6. While the exact value of $\int_{-1}^{1}(1-x^2)^n\,dx$ is not needed in the last proof, it is fun to compute it. So find $\int_{-1}^{1}(1-x^2)^n\,dx$. *Hint:* This is a case where it pays to generalize. Let

$$I(m,n) := \int_{-1}^{1}(1-x)^m(1+x)^n\,dx.$$

Then we are trying to compute $I(n,n)$. Use integration by parts to show

$$I(m,n) = \frac{m}{n+1}I(m-1,n+1)$$

when $m \geq 1$ and $n \geq 0$ and note that $I(0,k) = \int_{-1}^{1}(1+x)^k\,dx$ is easy to compute. □

Proposition 9. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function so such that $f(x) = 0$ for all $x \notin [0,1]$ and let K_n be as in Lemma 8. Set*

$$p_n(x) = \int_{-1}^{1} K_n(x-y)f(y)\,dy$$

then $p_n \to f$ uniformly and the restriction of p_n to $[0,1]$ is a polynomial.

Proof. By Proposition 5 f is bounded and uniformly continuous. Let B be a bound for f, that is $|f(x)| \leq B$ for all $x \in \mathbb{R}$. As f is uniformly continuous for $\varepsilon > 0$ here is a $\delta > 0$, such that

$$|x-y| \leq \delta \text{ and } x,y \in [0,1] \quad\implies\quad |f(x)-f(y)| \leq \varepsilon.$$

As f is bounded and uniformly continuous, Theorem 4 implies $p_n \to f$ uniformly. All that remains is to show that p_n restricted to $[0,1]$ is a polynomial. If $x,y \in [0,1]$, then $x-y \in [-1,1]$ and therefore

$$\begin{aligned} K_n(x-y) &= c_n(1-(x-y)^2)^n \\ &= g_0(y) + g_1(y)x + g_2(y)x^2 + \cdots + g_{2n}(y)x^{2n} \\ &= \sum_{k=0}^{2n} g_k(y)x^k \end{aligned}$$

where we have just expanded $c_n(1-(x-y)^2)^n$ and grouped by powers of x. (Each $g_k(y)$ is a polynomial in y, but this does not really matter for us.) As $f(y) = 0$ for $y \notin [0,1]$ if $x \in [0,1]$ we have

$$\begin{aligned} f_n(x) &:= \int_0^1 K_n(x-y)f(y)\,dy \\ &= \int_0^1 \sum_{k=0}^{2n} g_k(y)x^k f(y)\,dy \\ &= \sum_{k=0}^{2n}\left(\int_0^1 g_k(y)\,dy\right)x^k \end{aligned}$$

which is clearly a polynomial. □

Lemma 10. *Let $f\colon [\alpha,\beta] \to \mathbb{R}$ be a continuous function with $f(x) = 0$ for $x \notin [\alpha,\beta]$. Define $F\colon [0,1] \to \mathbb{R}$ to be the function*

$$F(x) := f\left(\alpha + (\beta - \alpha)x\right)$$

and let $P_n\colon [0,1] \to \mathbb{R}$ be polynomials such that $P_n \to F$ uniformly and set

$$p_n(x) = P_n\left(\frac{x-\alpha}{\beta-\alpha}\right).$$

Then each p_n is a polynomial and $p_n \to f$ uniformly.

Problem 7. Prove this. *Hint:* This is not hard, so don't be long winded.

Theorem 11 (Weierstrass Approximation Theorem). *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then there is a sequence of polynomial $p_n\colon [a,b] \to \mathbb{R}$ with $p_n \to f$ uniformly.*

Problem 8. Prove this. *Hint:* Extend f to $\mathbb{R}$ (we still denote the extended function by f) by

$$f(x) := \begin{cases} 0, & x < a-1; \\ (x-(a-1))f(a), & a-1 \le x < a; \\ f(x), & a \le x \le b; \\ ((b+1)-x)f(b), & b < x \le b+1; \\ 0, & b+1 < x. \end{cases}$$

This is continuous (don't prove this, just draw the picture and say it is clear). Let $\alpha := a-1$ and $\beta = b+1$. Then use Proposition 9 and Proposition 5 to complete the proof.

We now give some applications of these results.

Problem 9. Let $f\colon [a,b] \to \mathbb{R}$ be continuous and assume that

$$\int_a^b f(x)x^n\,dx = 0$$

for all $n = 0,1,2,3,\ldots$. Then show $f(x) = 0$ for all $x \in [a,b]$. *Hint:* Show that $\int_a^b f(x)p(x)\,dx = 0$ all polynomials. Then choose a sequence of polynomials $p_n \to f$ uniformly. Use this sequence to conclude $\int_a^b f(x)^2\,dx = 0$. □

Problem 10. Let $f,g\colon [a,b] \to \mathbb{R}$ be continuous functions such that

$$\int_a^b f(x)x^n\,dx = \int_a^b g(x)x^n\,dx$$

for all $n = 0,1,2,3,\ldots$. Show that $f(x) = g(x)$ for $x \in [a,b]$. *Hint:* Reduce this to the last problem. □

Convention. For the rest of this homework $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that for some $b > 0$ we have $f(x) = 0$ for all x with $|x| \geq b$ and f is Lebesgue integrable on $[-b, b]$ and that there is a constant B such that $|f(x)| \leq B$ for all x. □

Theorem 12. *If $\langle K_k \rangle_{n=1}^{\infty}$ is a Dirac sequence and*

$$f_n(x) = \int_{-\infty}^{\infty} K_n(y) f(x-y)\, dy = \int_{-\infty}^{\infty} K_n(x-y) f(y)\, dy$$

then at any point x where f is continuous

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Problem 11. Prove this. *Hint:* This is an easier version of an earlier theorem. □

Definition 13. A Dirac sequence $\langle K_n \rangle_{n=1}^{\infty}$ is ***differentiable*** iff for each n K_n is differentiable and

$$\lim_{h\to 0} \frac{K_n(x+h) - K_n(x)}{h} = K_n'(x)$$

uniformly. Explicitly this means that for each n and $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|h| \leq \delta \quad \Longrightarrow \quad \left| \frac{K_n(x+h) - K_n(x)}{h} - K_n'(x) \right| \leq \varepsilon \tag{2}$$

for all $x \in \mathbb{R}$ □

Proposition 14. *Let f be as in the convention and $\langle K_k \rangle_{n=1}^{\infty}$ a differentiable Dirac sequence. Then for each n*

$$f_n(x) = \int_{-\infty}^{\infty} K_n(x-y) f(y)\, dy$$

is differentiable and

$$f_n'(x) = \int_{-\infty}^{\infty} K_n'(x-y) f(y)\, dy.$$

(It is not being assumed that f is differentiable.)

Problem 12. Prove this. *Hint:* First show

$$\begin{aligned}
&\left(\frac{f_n(x+h) - f_n(x)}{h} \right) - \int_{-\infty}^{\infty} K_n'(x-y) f(y)\, dy \\
&\qquad = \int_{-\infty}^{\infty} \left(\frac{K_n(x-y+h) - K_n(x-y)}{h} - K_n'(x-y) \right) f(y)\, dy \\
&\qquad = \int_{-b}^{b} \left(\frac{K_n(x-y+h) - K_n(x-y)}{h} - K_n'(x-y) \right) f(y)\, dy
\end{aligned}$$

take absolute values and then use (2).

Lemma 15. *Let f be as in the convention and also assume that f is differentiable with f' uniformly continuous and let $\langle K_k \rangle_{n=1}^{\infty}$ be a differentiable Dirac sequence. Then the derivative of*

$$f_n(x) = \int_{-\infty}^{\infty} K_n(x-y) f(y)\, dy$$

can be written as

$$f_n'(x) = \int_{-\infty}^{\infty} K_n(x-y) f'(y)\, dy$$

Problem 13. Prove this. *Hint:* Starting with Proposition 7 show

$$\begin{aligned} f_n'(x) &= \int_{-\infty}^{\infty} K_n'(x-y) f(y)\, dy \\ &= -\int_{-\infty}^{\infty} \left(\frac{d}{dy} K_n(x-y) \right) f(y)\, dy \\ &= -\int_{-b}^{b} \left(\frac{d}{dy} K_n(x-y) \right) f(y)\, dy \end{aligned}$$

and use integration by parts along with $f(-b) = f(b) = 0$. □

Theorem 16. *Let f be as in the convention and also assume that f is differentiable with f' uniformly continuous and let $\langle K_k \rangle_{n=1}^{\infty}$ be a differentiable Dirac sequence. Then if*

$$f_n(x) = \int_{-\infty}^{\infty} K_n(x-y) f(y)\, dy$$

the limit

$$\lim_{n\to\infty} f_n' = f'$$

holds uniformly.

Problem 14. Prove this. □