Bailey Heath

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Algebra. January 2013. 1. Let $d \in \mathbb{N}$. For $f = a_0 + a_1 t + \cdots + a_n t^n \in \mathbb{Q}[t]$ and every integer $0 \leq i \leq d-1$, let

$$N_i(f) = \sum_{j \equiv i \mod (d)} a_j.$$

Let

$$I = \{f \in \mathbb{Q}[t] \mid N_0(f) = N_1(f) = \cdots = N_{d-1}(f)\}.$$

1.a. Is I an ideal of $\mathbb{Q}[t]$?

Claim: Yes, I is an ideal of $\mathbb{Q}[t]$.

Proof: It is 'clear' that $I \leq (\mathbb{Q}[t], +)$. (One needs to think about this, and on a qual I would write a line or two to prove it, but it is straightforward enough that I am not writing it here.) We now must show that I is closed under multiplication by elements of $\mathbb{Q}[t]$. Let

$$f = a_0 + a_1 t + \cdots + a_n t^n \in I$$

and

$$g = b_0 + b_1 t + \cdots + b_m t^m \in \mathbb{Q}[t].$$

Note that, since $f \in I$,

$$\sum_{j \equiv 0 \mod (d)} a_j = \sum_{j \equiv 1 \mod (d)} a_j = \cdots = \sum_{j \equiv d-1 \mod (d)} a_j \qquad (\star).$$

Define

$$h = fg = c_0 + c_1 t + \cdots + c_{m+n} t^{m+n}.$$

We want to show that $h \in I$, i.e., that $N_0(h) = N_1(h) = \cdots = N_{d-1}(h)$.

Our first important observation is that for all integers $0 \leq j \leq m+n$,

$$c_j = \sum_{k=0}^{j} a_k b_{j-k},$$

where $a_k = b_l = 0$ for all integers $k \geq n+1$ and $l \geq m+1$. Now, observe that

$$\begin{aligned} N_0(h) &= c_0 + c_d + \cdots \\ &= (a_0 b_0) + (a_0 b_d + a_1 b_{d-1} + \cdots + a_{d-1} b_1 + a_d b_0) + \cdots \\ &= b_0 \sum_{j \equiv 0 \mod (d)} a_j + b_1 \sum_{j \equiv d-1 \mod (d)} a_j + b_2 \sum_{j \equiv d-2 \mod (d)} a_j + \cdots \\ &= \sum_{j \equiv 0 \mod (d)} a_j \sum_{k=0}^{m} b_k, \end{aligned}$$

where the final equality follows from $(\star)$. From here, we may see that more generally, for $0 \leq q \leq d-1$,

$$\begin{aligned} N_q(h) &= b_0 \sum_{j \equiv q \mod (d)} a_j + b_1 \sum_{j \equiv q-1 \mod (d)} a_j + \cdots \\ &= \sum_{j \equiv 0 \mod (d)} a_j \sum_{k=0}^{m} b_k \\ &= N_0(h), \end{aligned}$$

which proves that $N_0(h) = N_1(h) = \cdots = N_{d-1}(h)$, as desired. $\square$

1.b/c. Give a generator for I and prove that your generator is correct.

Claim: I is generated by $f = t^{d-1} + t^{d-2} + \cdots + t + 1$.

Proof: Since $\mathbb{Q}$ is a field, $\mathbb{Q}[t]$ is a principal ideal domain, so I is generated by a single polynomial $g \in I$. It is clear that g must be of minimal degree in I, as otherwise $g \notin I$. We first show that $\deg(g) = d-1$. Indeed, suppose $h \in I$ has at least one nonzero coefficient.

Then, either $N_0(h) = N_1(h) = \cdots = N_{d-1}(h) > 0$, in which case $\deg(h) \geq d - 1$, or $N_0(h) = N_1(h) = \cdots = N_{d-1}(h) = 0$, in which case h must be of degree at least d to 'cancel out' any nonzero coefficients of t^i for $0 \leq i \leq d - 1$.

Now, it suffices to show that all of the coefficients of g are equal. This, too, follows immediately from the definition of I, as for a polynomial of degree $d - 1$ each of the sums in the definition of I is just a single coefficient! Hence, every polynomial of degree $d - 1$ is a constant multiple of f, which now implies that $I = \langle f \rangle$. $\square$