Mathematics 700 Final Name:______________________________

Show your work to get credit. An answer with no work will not get credit.

(1) Define the following:

(a) eigenvalue.

(b) eigenvector.

(c) a linear operator $S\colon V \to V$ is diagonalizable.

(d) the three elementary row operations.

(e) the Smith Normal Form of a matrix over a Euclidean domain.

(f) The quotient vector space V/W where W is a subspace of the vector space V.

(2) State the following

(a) The uniqueness theorem for n-linear alternating functions on $M_{n\times n}(R)$ where R is a communitive ring.

(b) The rank plus nullity theorem.

(c) The formula for $\dim(U \cap W)$ where U and W are subspaces of a vector space V.

(d) The Cayley-Hamilton theorem.

(e) The fundemential theorem of arithmetic in a Euclidean domain.

(3) Let V be a vector space and let $v_1, \ldots, v_n \in V$. Let $w \in V$ be a vector so that $w = a_1 v_1 + a_2 v_2 + \ldots a_n v_n$ for scalars $a_1, \ldots, a_n$ with $a_n \neq 0$. Show $\text{Span}\{v_1, \ldots, v_{n-1}, v_n\} = \text{Span}\{v_1, \ldots, v_{n-1}, w\}$

(4) Let V and W be vector spaces and $S\colon V \to W$ a linear map. Let $v_1, v_2, v_3 \in V$.

(a) If Sv_1, Sv_2, Sv_3 are linearly independent show that v_1, v_2, v_3 are linearly independent.

(b) If v_1, v_2, v_3 are linearly independent and $\operatorname{Span}\{v_1, v_2, v_3\} \cap \ker S = \{0\}$ show that Sv_1, Sv_2, Sv_3 are linearly independent.

(c) Given an example where v_1, v_2, v_3 are linearly independent, but Sv_1, Sv_2, Sv_3 are linearly dependent.

(5) Let A be an $n \times n$ matrix with $A^2 = 0$. Show that $\mathrm{rank}(A) \leq n/2$.

(6) Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Then find a basis of $\{v_1\}^\perp \subset \mathbf{R}^{3*}$.

(7) What is the Jordan canonical form, over the complex numbers, of a matrix that has elementary divisors $x - 2$, $(x - 2)^2$, $x^2 - 8x + 25$, $(x^2 - 8x + 25)^2$?

(8) Let

$$A = \begin{bmatrix} 0 & 2 & -4 \\ 0 & 1 & 0 \\ 1 & -2 & 5 \end{bmatrix}$$

Then for A find

(a) The elementary divisors.

(b) The minimal polynomial.

(c) The rational canonical form.

(9) Prove Cramer's rule for solving the 3×3 linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

using nothing but elementary properties of the determinant.

(10) Let V be a finite dimensional vector space and $S\colon V \to V$ a linear operator on V. Let W be a non-trival subspace of V that is invarant under S and let $S_1 := S\big|_W$ be the restriction of S to W.

(a) Show that the minimal polynomial $\min_{S_1}(x)$ of S_1 divides the minimil polynomial $\min_S(x)$ of S.

(b) Show that if S is diagonalizable, then so is S_1. (You may use the fact that a linear operator is diagonalizable if and only if its minimal polynomial factors into linear factors.)

(11) Let $\mathcal{P}_2 = \mathrm{Span}\{1, x, x^2\}$ be the real polynomials of degree ≤ 2. Define $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ by

$$T(p)(x) = e^{-x}\frac{d}{dx}\left(e^{x}p(x)\right).$$

Let $\mathcal{P}_2^*$ be the dual space to $\mathcal{P}_2$ and let $\Lambda \in \mathcal{P}_2^*$ be the functional

$$\Lambda(p) = p(2).$$

Then compute $\langle x^2, T^*\Lambda\rangle$.

Have a nice holiday!