Math 554 Test 3.

- *This is due on Tuesday, April 20 by midnight. It should be submitted via Blackboard as a pdf document and should have your name on the first page.*
- *You are to work alone on it. You can look up definitions and the statements of theorems we have covered in class. Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg.*
- *You will be graded in part on writing proofs up correctly. In particular you can lose points with answers that are all formulas and equations without any English.*

A note on notation. Let E be a metric space with metric d. Then it is wrong to write the distance between the points p and q of E as $|p-q|$ for a couple of reasons. First in a general metric space subtraction makes no sense and therefore $p-q$ is not defined. And for a metric space the absolute value $|x|$ of a point x in E is not defined. Therefore $|p-q|$ is doubly not defined. The distance between the points p and q of E is just $d(p,q)$. The only examples where $d(p,q)=|p-q|$ are when E is a subset of $\mathbb{R}$.

Recall that a function $: E \to E'$ between metric spaces is ***continuous*** if and only if it is continuous at all points of E. A good deal of our recent energy has gone into showing that this can be described in several ways:

Theorem 1. *Let $f\colon E \to E'$ be a function. Then the following are equivalent:*

(a) f is continuous.
(b) f does the right thing to convergent sequences. That is if $\lim_{n\to\infty} p_n = p$, then $\lim_{n\to\infty} f(p_n) = f(p)$.
(c) Preimages by f of open sets are open. That is if $U \subseteq E'$ is open, then $f^{-1}[U]$ is open in E.
(d) Preimages by f of closed sets are closed. That is if $C \subseteq E'$ is closed, then $f^{-1}[C]$ is closed in E. $\square$

Let us put this to work. To start a bit of set theory.

Proposition 2. *Let $f\colon E \to E'$ and $g\colon E' \to E''$ be functions and let $g \circ f\colon E \to E''$ be the composition $(g \circ f)(x) = g(f(x))$. Then for any subset $S \subseteq E''$*

$$(g \circ f)^{-1}[S] = f^{-1}\left[g^{-1}[S]\right]$$

Problem 1. (5 points) Prove this. □

Theorem 3. *Let E, E' and E'' be metric spaces and $f\colon E \to E'$ and $g\colon E' \to E''$ continuous functions. Then the composition $g \circ f$ is also continuous.*

Problem 2. (10 points) Give a short proof of this this using Condition (c) of Theorem 1 and Proposition 2. □

Back before Test 2 we proved

Theorem 4. *A metric space E and $S \subseteq E$. Then S is compact if and only if it is sequentially compact.* □

Before going on you should make sure that you know the definitions of compact (open covers have finite subcovers) and sequentially compact (sequences from the set have subsequences that converge to a point of the set).

Theorem 5. *Let $f\colon E \to E'$ be a continuous function between metric spaces and $K \subseteq E$ a compact subset of E. Then the image $f[K]$ is compact.*

Problem 3. (15 points) Prove this by use of Condition (b) of Theorem 1. *Hint:* In light of Theorem 4 it is enough to show that if K is sequentially compact, so is the image $f[K]$. Let $\langle y_n \rangle_{n=1}^{\infty}$ be a sequence in $f[K]$. We need to find a subsequence of this sequence that converges to a point of $f[K]$. So explain why for each n that $y_n = f(x_n)$ for some $x_n \in K$. Now use the sequential compactness of K to get a subsequence $\langle x_{n_k} \rangle_{k=1}^{\infty}$ that converges to a point of K. □

Now some more set theory.

Proposition 6. *Let $f\colon E \to E'$ be a function between sets and $U \subseteq E'$. Then*

$$f\left[f^{-1}[U]\right] = U$$

Problem 4. (5 points) Prove this. □

Problem 5. (15 points) Give anther proof of Theorem 4 using Condition (c) of Theorem 1 and Proposition 6. *Hint:* This time we use the open cover definition of compactness. So let $\mathcal{C}$ be an open cover of $f[K]$ and we need to show it has a finite subcover. Let

$$f^{-1}[\mathcal{C}] = \{f^{-1}[U] : U \in \mathcal{C}\}.$$

Show that $f^{-1}[\mathcal{C}]$ an open cover of K and therefore $f^{-1}[\mathcal{C}]$ has a finite subcover of K as K is compact. $\square$

Proposition 7. *Let E be a metric space and K a compact subset of E. Then K is bounded in E. More explicitly let p_0 be any point of E. Then there is a $r > 0$ so that $K \subseteq B(p_0, r)$.*

Problem 6. (10 points) Prove this. *Hint:* Start by showing that

$$\mathcal{C} = \{B(p_0, n) : n = 1, 2, 3, \ldots\}$$

is an open over of K. $\square$

The next result is generally considered on of the big theorems for this course.

Theorem 8. *Let E be a compact metric space and $f\colon E \to \mathbb{R}$ a continuous function. Then f is bounded on E. Explicitly this means that there is a positive number r so that $-r \leq f(p) \leq r$ for all $p \in E$.*

Problem 7. (10 points) Prove this. *Hint:* Explain by the set $f[E]$ is compact in $\mathbb{R}$. Now Proposition 7 may be relevant. $\square$

Problem 8. (20 points) Give examples of the following:

(a) A function $f\colon \mathbb{R} \to \mathbb{R}$ which is continuous at all point other than -4 and 7 but discontinuous at -4 and 7.
(b) A closed subset of $\mathbb{R}$ that is not compact.
(c) A bounded subset of $\mathbb{R}$ that is not compact.
(d) A subset of $\mathbb{R}$ that is neither open or closed.

Our most basic form of the Intermediate Value Theorem is

Theorem 9. *Let $f\colon [a, b] \to \mathbb{R}$ be continuous and assume that $f(a)$ and $f(b)$ have opposite signs (that is either $f(a) > 0$ and $f(b) < 0$ or $f(a) < 0$ and $f(b) > 0$). Then $f(x) = 0$ has a solution for some x between a and b.* $\square$

Problem 9. (10 points) Use the Intermediate Value Theorem to prove the equation

$$x^4 - 8x^2 + x + 2 = 0$$

has four solutions. *Hint:* I would start by graphing it to see if this is reasonable. $\square$