Mathematics 554H/701I Homework

Before going on to connectedness, let us practice a bit with compactness. Recall that a metric space is compact if every open over has a finite subcover.

The following was on the test

Problem 1. Show that every compact metric space can be covered by a finite number of open balls of radius one.

Solution: Let E be a compact metric space. Then the collection $\mathcal{U} = \{B(p,1) : p \in E\}$ is an open cover of E. (For if $p \in E$, then $p \in B(p,1) \in \mathcal{U}$.) Because E is compact this open cover will have a finite subcover $\mathcal{U}_0 = \{B(x_1,1), B(x_2,1), \ldots, B(x_n,1)\}$. Then $\mathcal{U}_0$ is the required finite cover of E by open balls of radius 1. $\square$

We say that a metric space, E, is ***totally bounded*** if and only if for every positive real number r the space E can be covered by a finite number of balls of radius r.

Proposition 1. *Every compact metric space is totally bounded.*

Problem 2. Prove this. *Hint:* Look at the solution to Problem 1. $\square$

Problem 3. Let E be a compact metric space. Show that there is a finite subset $F \subseteq E$ such that every point of E is within a distanct of $1/100$ of a point of F. $\square$

We now get back to connectedness.

Definition 2. Let E be a metric space. Then a subset $A \subseteq E$ is ***clopen*** if and only if it is both open and closed.

Proposition 3. *Let E be a metric space. Then the following are equivalent.*

(a) E has a clopen subset A with $A \neq \varnothing$ and $A \neq E$.
(b) There are non-empty disjoint open subsets U and V of E with $E = U \cup V$.
(c) There are non-empty disjoint closed subsets U and V of E with $E = U \cup V$.

Problem 4. Prove this. *Hint:* See class notes. $\square$

Definition 4. A metric space E is connected if and only if the only clopen sets in E are E and $\varnothing$. $\square$

Definition 5. The sets U and V are a ***disconnection*** of the metric space E if and only if U and V are open, non-empty, disjoint, and $E = U \cup V$. (In light of Proposition 3 we could also assume that U and V are both closed.) $\square$

Thus a metric space is connected if and only if it does not have a disconnection. This makes the strategy for showing something is disconnected straightforward: find a disconnection.

Problem 5. Show the following sets are disconnected.

(a) The rational numbers.
(b) The set $E = \{(x, y) : x, y \in \mathbb{R} \text{ and } |x| \le 1, 10 < |y| < 11\}$ *Hint:* Draw a picture. $\square$

Problem 6. If E is a subset of the real numbers and there are points $a, b \in E$ and a point between a and b that is not in E, then E is not connected. *Hint:* See class notes. $\square$

Problem 7. Let $r > 0$ and p, q points in a metric space with $d(p, q) > 2r$. Show that the set $E = B(p, r) \cup B(q, r)$ is not connected. $\square$

Showing sets are connected is more work. Here is a way to reduce showing that a space is connected by showing that it is the union of connected subsets all of which contain the same point.

Problem 8. Let $E = S_1 \cup S_2$ where S_1 and S_2 are both connected and there is a point x_0 that is in both S_1 and S_2. Then E is connected.

Solution: We need to show that the only clopen subsets of E are E and $\varnothing$. That is we need to show that if A is a non-empty clopen subset of E, then $A = E$. As $A \subseteq E = S_1 \cup S_2$, then will intersect at least one of S_1 or S_2. Assume that $A \cap S_1 \neq \varnothing$. Then $A \cap S_1$ is a non-empty clopen subset of S_1 and S_1 is connected, therefore $A \cap S_1 = S_1$. That is $A \subseteq S_1$. As $x_0 \in S_1 \subseteq A$ we have $x_0 \in A$. Therefore $x_0 \in A \cap S_2$ and thus $A \cap S_2 \neq \varnothing$. Then $A \cap S_2$ is a non-empty clopen subset of S_2 and S_2 is connected and therefore $A \cap S_2 = S_2$. Thus $S_2 \subseteq A$. Therefore we have $A \subseteq E = S_1 \cup S_2 \subseteq E$. This shows that $A = E$ which is just what we needed to complete the proof. $\square$

Proposition 6. *Let E be a metric space and $\{S_\alpha\}_{\alpha \in I}$ a collection of connected subsets of E such that for some point $x_0 \in E$ with*

$$x_0 \in \bigcap_{\alpha \in I} S_\alpha$$

and

$$E = \bigcup_{\alpha \in I} S_\alpha.$$

Then E is connected.

Problem 9. Prove this. *Hint:* Look at the solution of Problem 8.

Theorem 7. *Every interval in $\mathbb{R}$ is connected.*

Problem 10. Prove this. *Hint:* Let E be in interval in $\mathbb{R}$. The property of E that we will use is that if $a, b \in E$ then any real number between a and b is in E. Towards a contradiction assume that E is not connected. Then E is a disjoint union $E = U \cup V$ where U and V are both open and closed in E. Let $a \in U$ and $b \in V$. By possibly changing the names we can assume that $a < b$. Let

$$A = [a, b] \cap U \quad \text{and} \quad B = [a, b] \cap V.$$

and set

$$c = \sup(A)$$

Now show

(a) A and B are clopen in $[a, b]$.

(b) $a < c < b$ and $c \in E$ (to start note that as A is open in $[a, b]$ there is a ball $B(a, r)$ with $B(a, r) \cap [a, b] = [a, r) \subseteq A$ and likewise there is a $r' > 0$ such that $(b - r', b] \subseteq B$).

(c) c is an adherent point of A (this just uses that $c = \sup(A)$)

(d) c is an adherent point of B. (if not explain why there is $r > 0$ with $(c - r, c + r) \subseteq A$ and why this contradicts that $c = \sup(A)$.)

(e) As A and B are closed they contain their adherent points and so c is in both A and B.

(f) Explain why we have a contradiction. $\square$