Mathematics 242 Homework.

The main topic we covered in class today was homogeneous second order with constant coefficients. That is equations of the form

$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$. The ***characteristic equation*** of this differential equation is the algebraic equation

$$ar^2 + br + c = 0.$$

This is quadratic equation in r and can be solved by factoring, completing the square, or the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

What we saw today was

Theorem 1. *If the roots r_1 and r_2 of the characteristic equation are real and distinct (that is $r_1 \neq r_2$) then the general solution to*

$$ay'' + by' + cy = 0$$

is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$ □

Example 2. Find the general solution to the equation

$$y'' + 3y' - 10y.$$

In this case the characteristic equation is

$$r^2 + 3r - 10 = (r + 5)(r - 2)$$

so that the characteristic roots are $r_1, r_2 = -5, 2$ and thus the general solution is

$$y = c_1 e^{-5x} + c_2 e^{2x}.$$ □

Example 3. Find the general solution to

$$y'' + 2y' - 2y = 0.$$

In principle this is not any harder than the previous example, other than the characteristic equation

$$r^2 + 2r - 2 = 0$$

does not factor nicely. So we use the quadratic formula to get

$$r_1, r_2 = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2} = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}.$$

Therefore the general solution is

$$y = c_1 e^{(-1-\sqrt{3})x} + c_2 e^{(-1+\sqrt{3})x}.$$ □

Problem 1. Find the general solutions to the following:
(a) $y'' - 9y' + 20y = 0$
(b) $3y'' - 8y' + 4y = 0$
(c) $y'' - ky = 0$ where $k > 0$ is a constant.
(d) $y'' + 2ky' - y = 0$ where k is a constant. □

We also saw that if the values of y and y' are given at some point, then we can solve for c_1 and c_2 in the general solution.

Example 4. For the equation

$$r^2 + 3r - 10 = (r+5)(r-2)$$

find the solution with $y(0) = 5$ and $y'(0) = 4$. We have already seen that the general solution to this equation is

$$y = c_1 e^{-5x} + c_2 e^{2x}.$$

Then the derivative is

$$y' - 5c_1 e^{-5x} + c_2 e^{2x}.$$

Then we want to choose c_1 and c_2 so that

$$y(0) = c_1 + c_2 = 5$$
$$y'(0) = -5c_1 + 2c_2 = 4$$

(where we have used $e^0 = 1$). Solving (I leave the algebra to you) we get

$$c_1 = \frac{6}{7}, \qquad c_2 = \frac{29}{7}$$

and therefore the solution we are after is

$$y = \frac{6}{7}e^{-5x} + \frac{29}{7}e^{2x}.$$ □

Example 5. Find the solution to

$$y'' + 3y' + 2 = 0$$

with $y(1) = 2$ and $y'(1) = -7$. The characteristic equation is

$$r^2 + 3r + 2 = (r+2)(r+1) = 0$$

so the characteristic roots are -1 and -2, the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

and its derivative is

$$y = -c_1 e^{-x} - 2c_2 e^{-2x}.$$

We need to solve for c_1 and c_2 in

$$y(1) = c_1 e^{-1} + c_2 e^{-2} = 2$$
$$y'(1) = -c_1 e^{-1} - 2e^{-2} = -7.$$

I again leave the algebra to you in showing

$$c_1 = -3e, \qquad c_2 = 5e^2$$

and therefore the solution we are after is

$$y = -3ee^{-x} + 5e^2e^{-2} = -3e^{-(x-1)} + 5e^{-2(x-1)} \qquad \square$$

Problem 2. Solve the following initial value problems:
(a) $y'' + 7y' + 12y = 0$, with $y(0) = 4$, and $y'(0) = -3$.
(b) $y'' + 7y' + 12y = 0$, with $y(3) = 4$, and $y'(3) = -3$.
(c) $y'' - 4y' - 2y = 0$, $y(0) = 1$, and $y'(0) = -3$.
(d) $y'' - ky = 0$ with $y(0) = y_0$, and $y'(0) = y_1$ and $k > 0$ is a constant. $\square$