Math 242 Final.

- *This is due on Tuesday, December 8 at 6:30pm, which is when the final would have ended if we were doing it on campus. You are to work alone in it.*
- *Print your name on the first page of the exam. You can look up definitions and the statements of theorems we have covered in class. And if there is an integral where you want to use a computer, your calculator, or a source such as Wolfram Alpha to compute it, that is fine, but say that this is what you did. (For example "I computed $\int x^2 e^x \, dx$ using the program Maple".) Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg for direct help on the problems.*
- *Since you have plenty of time for this and I have to have grades in shortly after the exam, I would just as soon not have any late papers.*
- *If you are using light pencil or pen it is good idea to scan a page or two and email to yourself to see how readable it is. (And remember my eyes are very likely not as good as yours.)*

Show your work for full credit.

Problem 1 (15 points). For the first order differential equation

$$y' = \frac{1.5y(y-15)(20-y)}{1+y^4}$$

(a) Find the critical points.
(b) Sketch the critical solutions and the solutions with initial conditions $y(0) = -3$, $y(0) = 10$, $y(0) = 17$, and $y(0) = 25$ all on the same axis.
(c) Which of the critical points are stable and which are unstable?
(d) For the solution with $y(0) = 17$ what is $\lim_{x\to\infty} y(x)$?
(e) For the solution with $y(10) = 3$ estimate $y(123)$.
(f) If $y(11) = 13$ compute $y'(11)$.
(g) If $y(12) = 13$ estimate $y(11.4)$. □

Problem 2 (25 points). Find the general solution to the following equations

(a) $\dfrac{dy}{dx} - ky = mx + b$ where k, m, and b are constants.
(b) $y' = \dfrac{e^{x+2y}}{y}$.
(c) $\dfrac{dy}{dx} = \dfrac{x^2 + xy + 2y^2}{x^2 + 2xy}$

Problem 3 (20 points). Solve the following initial value problems.

(a) $xy' + x^2y = 3x^3, \quad y(1) = 4.$

(b) $\dfrac{dy}{dx} = \frac{x+2}{y+4}, \quad y(2) = 0.$

Problem 4 (30 points). Solve the following equations. If no initial values are given find the general solution. If initial values are given then solve the initial value problem.

(a) $y'' - 4y' + 3y = 0,$

(b) $y'' - 4y' + 5y = 0,$

(c) $y'' - 6y' + 9y = 0, \qquad y(0) = 3, \quad y'(0) = 9,$

(d) $y'' - 5y' + 6y = 12\cos(x)$

Problem 5 (15 points). Find particular solutions to the following.

(a) $y''' + 3y'' + y' + y = 12e^x + 3x + 4$

(b) $y'' + y' = 6x^2$

(c) $y' + y = e^x \cos(x).$

Problem 6 (10 points). Find the solution to the initial value problem

$$x^2y'' + xy' - 4y = -3x, \qquad y(1) = 4, \quad y'(1) = -1.$$

Problem 7 (10 points). If $x(t)$ is a solution to the initial value problem

$$x''(t) + 5x'(t) + 4x(t) = \cos(2t), \qquad x(0) = 3, \quad x'(0) = -1$$

find the Laplace transform $\mathcal{L}\{x(t)\}$.

Problem 8 (10 points). If $x(t)$ is a function with Laplace transform

$$\{x(t)\} = \frac{3}{s} + \frac{2s+5}{s^2+4s+5}$$

what is $x(t)$?

Problem 9 (10 points). Let $x(t)$ and $y(t)$ satisfy the system of differential equations

$$\frac{dx}{dt} = .3x(1 - x - .4y)$$
$$\frac{dy}{dt} = .2y((1 - .5x - y)$$

(a) If $x(4) = 5$ and $y(4) = 6$ find $x'(4)$ and $y'(4)$.

(b) Is $y(t)$ increasing or decreasing when $t = 4$, explain how you know.

(c) Estimate $x(4.05)$ and $y(4.05)$.

Problem 10 (5 points). Make sure your name is on the first page of the test and that you the exam turned in on time.