Math 552 Test 1.

- *This is due on Tuesday, February 16 by midnight. It should be submitted via Blackboard as a pdf document.*
- *You are to work alone on it. You can look up definitions and the statements of theorems we have covered in class. Needless to say (but I will say it anyway) no use of online help sites such as Stack Overflow or Chegg.*
- *Please print your name on the first page of the test.*
- *Since you have plenty of time on this test you should submit neat papers. By this I do not mean handwriting, but more not having crossed out work and also taking the time to write sentences explaining what you are doing. If you are writing the paper by hand, it is good idea to make a rough draft to get the details correct before making the final copy.*
- *Related to the last point, use common sense about simplification. Leaving a fraction as* $\frac{15}{20}$ *(rather than* $\frac{3}{4}$*), or worse yet* $\frac{6}{2}$ *(rather than* 3*) does not make a good impression. Likewise* $\cos(\pi/4)$ *is* $\sqrt{2}/2$ *etc.*

1. (15 points) Compute the following and put the answer in the form $x = iy$.

(a) $(a + bi)^5$ *Hint:* Binomial Theorem.

(b) $\displaystyle\sum_{k=0}^{11} 3(1+i)^k$

(c) The solution to $\dfrac{z-i}{z+i} = 2 + 3i$.

2. (5 points) Let $a = 1 - i$ and let $n = 4k$ where k is an integer. Show a^n is a real number.

3. (10 points) Let $p(z)$ be the polynomial

$$p(z) = c_4x^4 + c_3z^3 + c_2z^2 + c_1z + c_0$$

where the coefficients $c_0 \ldots c_4$ are real numbers.

(a) Show for any complex numbers z that

$$\overline{p(z)} = p(\overline{z}).$$

(b) Use this to show that if α is a root of $p(z)$, then so is $\overline{\alpha}$. (These facts are true for polynomials with real coefficients of any degree, but you only need to do the case of degree = 4.)

4. (10 points)

(a) If w is a complex number with $\overline{w} = \dfrac{1}{w}$, then $|w| = 1$.

(b) Show that for any real number x the complex number $w = \dfrac{x+i}{x-i}$ satisfies $|w| = 1$.

5. (5 points) Find both values of $\sqrt{-2+2i\sqrt{3}}$ in the form $z + iy$.

6. (10 points)

(a) Find all solutions to $e^{3z+2} = 1 - i$.

(b) Let a be a real number with $a > 1$. Find all solutions to $\tan(z) = ia$.

7. (15 points) Let $b = \dfrac{-1+i}{8}$.

(a) Find all cube roots of b in the form $x + iy$.

(b) Find all values of $\log(b)$ in the form $x + iy$.

(c) What is $\mathrm{Log}(b)$?

(d) Give all values of b^{1+i} in the form $x + iy$.

8. (10 points) Show that if α, β and γ are the interior angles of a triangle then

$$\cos(\alpha)\cos(\beta)\cos(\gamma) - \cos(\alpha)\sin(\beta)\sin(\gamma) - \sin(\alpha)\cos(\beta)\sin(\gamma) - \sin(\alpha)\sin(\beta)\cos(\gamma) = -1.$$

Hint: This is really a corollary to Euler's Theorem $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. To see why recall that the angles of a triangle satisfy $\alpha + \beta + \gamma = \pi$ and therefore $-1 = e^{i\pi} = e^{i(\alpha+\beta+\gamma)}$.

We have proven the ***Cauchy-Riemann equations*** (which we will often shorten to the "CR-equations") which are that if $f(z) = u + iv$ is analytic (that is complex differentiable) in an open U, then

$$u_x = v_y$$
$$u_y = -v_x$$

and the derivative of f is given by either of the formulas

$$f'(z) = u_x + iv_x$$
$$f'(z) = v_y - iu_y$$

9. (5 points) The function

$$f(z) = x^2 - y^2 + x - y + i(2xy + x + y)$$

is analytic. (This is given, so you do not have to prove it). Give a formula for the derivative $f'(z)$.

10. (5 points) Let $f = u + iv$ be analytic on the open set U. Use the CR-equations to show the equations

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$

hold.

11. (10 points) Let h be a real valued function on the open U that satisfies

$$h_{xx} + h_{yy} = 0$$

in U. Let $u = h_x$ and $v = -h_y$. Show that $f = u + iv$ satisfies the CR-equation.