Some complex analysis problems.

Let $U \subseteq \mathbb{C}$ be an open set and $f \colon U \to \mathbb{C}$. Then f is ***analytic*** in U if and only if f is complex differentiable at each point of U. That is for all $a \in U$ the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists.

Theorem 1. *If $f = u_i v$ is analytic in U, then u and v satisfy the **Cauchy Riemann equations***

$$u_x = v_y \qquad u_y = -v_x.$$

Conversely if the partial derivatives u_x, u_y, v_x, v_y are continuous and $f = u_i v$ satisfies the Cauchy Riemann equations, then f is analytic in U. In this case its derivative is given by

$$f'(z) = u_x + iv_x = v_y - iu_x.$$

Problem 1. Use the Cauchy Riemann equations to show that if an analytic function on a connected open set is real valued, then it is constant. $\square$

Problem 2. Use the Cauchy Riemann equations to show that if f is analytic on a connected open set and $|f(z)|$ is constant, then f is constant. $\square$

If $z = x + iy$ define the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Problem 3. If $f = u + iv$ show that f satisfies the Cauchy Riemann equations if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0$$

and if f is analytic, then

$$f'(z) = \frac{\partial f}{\partial z}.$$

$\square$

Problem 4. Let $f = u + iv$ be analytic in an open set. Show that gradient vector fields $\nabla u = (u_x, u_y)$ and $\nabla v = (v_x, v_y)$ have have the same length at each point and are pointwise orthogonal. Interpret this in terms of the geometry of the level sets $u = c_1$ and $v = c_2$. $\square$

Problem 5. If $f = u + iv$ is analytic in an open set U then both u are v are harmonic in U. That is

$$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0.$$

(You may assume that the second partial derivative exist.) $\square$