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ANALYSIS QUALIFYING EXAM
AUGUST 2015

Instructions: Write your name legibly on each sheet of paper. Write only on one side of each sheet of paper. The points for each question part are indicated. $L_1(\mathbb{R}^n)$ denotes the class of integrable functions with respect to n-dimensional Lebesgue measure and 'dx' denotes integration with respect to one-dimensional Lebesgue measure.

1. (a) [6] Suppose that $A \subseteq \mathbb{R}$ is closed and that $B \subseteq \mathbb{R}$ is compact with $0 \notin B$. Prove that $AB := \{ab\colon a \in A, b \in B\}$ is closed.

(b) [4] Give an example of a closed set A and a compact set B such that AB is not closed.

2. [10] Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f\colon X \to Y$ be any mapping from X to a nonempty set Y. Show that $\mathfrak{N} := \{A \subseteq Y\colon f^{-1}(A) \in \mathfrak{M}\}$ is a σ-algebra of subsets of Y and that $\nu(A) := \mu(f^{-1}(A))$ is a measure on the measurable space $(Y, \mathfrak{N})$.

$\int \sin^2(n_k x) \to 0$

$\pi \to 0$

3. (a) [3] State the Bounded Convergence Theorem.

(b) [5] Prove that there does **not** exist any increasing sequence of integers $0 < n_1 < n_2 < \ldots$ such that $\lim_{k\to\infty} \sin(n_k x) = 0$ a.e. in $[0, 2\pi]$. (Hint: Evaluate $\int_0^{2\pi} \sin^2(nx)\,dx$.) $= \pi$

Ask Ralph → ?

(c) [2] Prove that there exists a set $A \subset [0, 2\pi]$ which is **dense** in $[0, 2\pi]$ such that $\lim_{n\to\infty} \sin(2^n x) = 0$ for all $x \in A$.

4. (a) [3] State Hölder's inequality.

(b) [7] Suppose that $1 < p < \infty$ and $\int_0^1 f(x)^p\,dx = 1$, where f is a non-negative measurable function on $[0, 1]$. Prove that

$$\int_0^1 x^2 f(x)\,dx \le \left(\frac{p-1}{3p-1}\right)^{1-1/p} = \quad \frac{p-1}{p} \quad ?$$

and find an f for which equality is attained.

5. (a) [3] Define 'f is absolutely continuous on $[a, b]$'.

(b) [7] Suppose that f is absolutely continuous on $[a, b]$ and $f(x) \neq 0$ for all $a \le x \le b$. Prove that $1/f$ is absolutely continuous on [a,b] and deduce that

$$\int_a^b \frac{f'(x)}{f(x)^2}\,dx = \frac{f(b) - f(a)}{f(a)f(b)}.$$

6. (a) [3] State Fubini's theorem. ask Ralph!

(b) [7] Suppose that $f, g \in L_1(\mathbb{R})$. Prove that $f(x-y)g(y) \in L_1(\mathbb{R}^2)$ and that

$$h(x) := \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \quad \text{exists a.e. and is finite a.e.}$$

(You may assume without proof that $(x, y) \mapsto f(x-y)g(y)$ is Lebesgue-measurable on $\mathbb{R}^2$.)

7. (a) [4] Evaluate $\int_{[3\sqrt{2}, -3+3i]} \frac{1}{z}\,dz$, where $[3\sqrt{2}, -3+3i]$ denotes the directed line segment joining $3\sqrt{2}$ to $-3+3i$. ask Ralph

(b) [6] Suppose that $f(z)$ is analytic on a domain U and that $f(z) \notin (-\infty, 0]$ $(z \in U)$. Let γ be any piecewise-smooth oriented curve contained in U with initial point a and terminal point b. Prove a simple necessary and sufficient condition for $\int_\gamma \frac{f'(z)}{f(z)}\,dz$ to be purely imaginary.

8. (a) [3] State the Casorati-Weierstrass theorem about essential singularities.

(b) [7] Suppose that $f(z)$ is an entire function satisfying $|f(z)| \geq |z|^n$ if $|z| > 1$ (for some integer $n \geq 0$). Prove that $f(z)$ is a polynomial of degree at least n. (Hint: Consider the behavior of $w^n g(w)$ near $w = 0$, where $g(w) = f(1/w)$.)

9. True or False? Prove or give a counterexample in each case.

(a) [4] If f is continuous on $[0,1]$, $f'(x)$ exists a.e., and f' is integrable on $[0,1]$, then $\int_0^1 f'(x)\,dx = f(1) - f(0)$. FTC II

(b) [4] If f is continuous on $[0,1]$ and $f([0,1])$ has Lebesgue measure zero then f is constant.

(c) [4] If each f_n is a non-negative measurable function on $[0,1]$ satisfying $f_n(x) \leq 1/x$ $(n \geq 1, 0 < x \leq 1)$, and $f(x) := \lim f_n(x)$ exists a.e., then $\lim \int_0^1 \sqrt{f_n(x)}\,dx = \int_0^1 \sqrt{f(x)}\,dx$.

(d) [4] If $f(z)$ is a non-constant entire function then $f(z) = a$ has a solution for every $a \in \mathbb{C}$.

(e) [4] If $f(z)$ is analytic on the unit disk $D(0,1)$ and satisfies $|f(1/n)| \leq 2^{-n}$ for all $n \geq 2$, then $f(z)$ is identically zero on $D(0,1)$.

$|f(\frac{1}{n})| \leq \frac{1}{2^n}$