

Mathematics 552 Homework, January 27, 2020

The book takes a different approach to defining the exponential function e^z . We defined it in terms of the series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

The book defines it (see Equation (2.4) in Section 2.6) as

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y),$$

which we proved as consequences of **Euler's formula**

$$e^z = \cos(z) + i \sin(z)$$

and that $e^{z+w} = e^z e^w$, where we have defined $\sin(z)$ and $\cos(z)$ by their series

$$\begin{aligned}\sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots \\ \cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots\end{aligned}$$

We showed in class that Euler's formula implies

$$\begin{aligned}\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i}.\end{aligned}$$

The book uses these for the definitions of $\cos(z)$ and $\sin(z)$.

We say that w is a **logarithm** of z if and only if

$$e^w = z.$$

Note that a complex number generally has infinitely many logarithms. For example we have seen that the general solution to

$$e^w = 1$$

is

$$w = 2n\pi i \quad \text{where } n \in \mathbb{Z}.$$

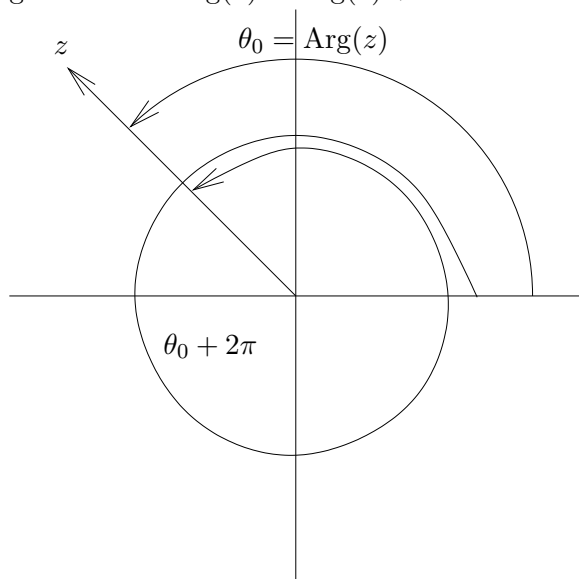
Therefore $\ln(1) = 2n\pi i$ where $n \in \mathbb{Z}$.

The logarithms of a complex number are closely related to its polar form. Let

$$z = re^{i\theta} = r(\cos \theta + i \sin \theta)$$

with $r > 0$ and $\theta \in \mathbb{R}$. The angle θ is called an **argument** of z and denoted by $\arg(z)$. (See Section 1.8 on Page 4 of the text for more about this.) There are infinitely many choices for $\theta = \arg(z)$. But any two of them differ by an integer number of whole revolutions, that is by an integer multiple of $2\pi = 360^\circ$. The **principle value** of the argument (see Section 1.8 on

Page 4 of the text) is the argument θ with $-\pi < \theta \leq \pi$. We will denote the principle argument by $\text{Arg}(z)$. The figure shows the principle value of z along another argument with $\arg(z) = \text{Arg}(z) + 2\pi$.



You should now look at Problem 2.13 on Page 56 of the text along with its solution. There it is shown that the general solution to

$$e^w = z$$

that is the general form of the logarithm of z is

$$\ln z = \ln |z| + i \arg z.$$

Put somewhat differently $w = \ln |z| + i \arg z$ is the general solution to $e^w = z$. Note that there are infinitely many solutions as there are infinitely many choices for $\arg z$. To be more explicit we can write

$$\ln z = \ln |z| + i \text{Arg } z + 2n\pi i$$

where $n \in \mathbb{Z}$.

Problem 1. Part (b) of Problem is to find the values of $\ln(1 - i)$. Use the method of that problem to find

- (a) all values of $\ln(2 - 2\sqrt{3}i)$
- (b) all solutions to $e^{2z} + 2e^z + 2 = 0$. *Hint:* If $w = e^z$ this equation becomes the quadratic equation $w^2 + 2w + 2 = 0$. Solving this gives two solutions: $w = -1 + i$ and $w = -1 - i$. So the original problem now splits into the two problems of finding all solutions to $e^z = -1 + i$ and $e^z = -1 - i$.
- (c) Find all solutions to $e^{2z} - e^z - 2 = 0$. □