Admission to Candidacy Examination in Algebra
January 2011

Note! You must show sufficient work to support your answer. Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc. If some problem is incorrect, then give a counterexample.

1. Prove that every subgroup of a cyclic group is cyclic.

2. Let G be an abelian group. Suppose that a and b are elements of G of finite order and that the order of a is relatively prime to the order of b. Prove that $<a> \cap <b> = <1>$ and $<a,b> = <ab>$.

3. Let $R \subseteq S$ be commutative rings, with R a subring of S, and let $u \in S$. Define $R[u]$ and prove that $R[u]$ is isomorphic to a quotient of the polynomial ring $R[x]$.

4. Prove that there are no simple subgroups of order 56.

5. Define *solvable group*. Prove that S_4 is solvable.

6. Let F be an imperfect field of characteristic $p > 0$ and let $E = F(u)$, where u satisfies an equation of the form $x^p - a = 0$ with $a \in F \setminus F^p$. Describe the Galois group of E over F.

7. Define the splitting field of a polynomial over a field. Let F be a field and $f = x^2 + ax + b \in F[x]$. Assume that f is irreducible over F. Prove that $F[x]/(f)$ is a splitting field for f over F.

8. Classify up to similarity the 4×4 matrices over the field of complex numbers that have characteristic polynomial $(x-1)^2(x+1)^2$.

9. Let G be a non-abelian group of order 6. Prove that G is isomorphic to S_3.