Math 552, February 26, 2020.

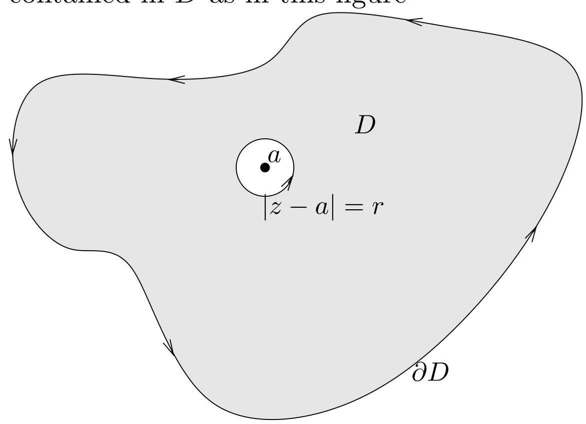
The following problems are to prepare for results that we are about to prove. We have just proven

Theorem 1 (Cauchy's Theorem). *Let D be a bounded domain with nice boundary and $f(z)$ a function that is analytic on the closure of D. Then*

$$\int_{\partial D} f(z)\,dz = 0$$

where, as usual, we orient ∂D so that we move with the inside on our left. □

Problem 1. Let $f(z)$ be analytic in a domain D and let $a \in K$. Let $r > 0$ be so small that the disk $|z-a| \le r$ is contained in D as in this figure



(a) Use the Cauchy Integral Theorem to show

$$\int_{\partial D} \frac{f(z)}{z-a}\,dz = \int_{|z-a|=r} \frac{f(z)}{z-a}\,dz.$$

Be sure to say why Cauchy Integral Formula applies.

(b) Use Part (a) and the parameterization of $|z-a| = r$ given by $z = a + re^{it}$ with $0 \le t \le 2\pi$ to show

$$\int_{\partial D} \frac{f(z)}{z-a}\,dz = \int_{|z-a|=r} \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+re^{it})\,dt.$$

Problem 2. With the same set up as in Problem 1 explain why

$$\lim_{r\to 0^+} \int_0^{2\pi} f(a+re^{it})\,dt = 2\pi i f(a)$$

and use this to show

$$\int_{\partial D} \frac{f(z)}{z-a}\,dz = 2\pi f(a).$$

□

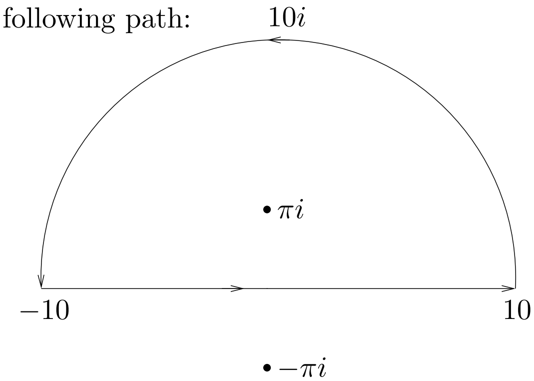
You have just proven what may be the most important result in Complex Analysis:

Theorem 2 (Cauchy Integral Formula). *Let D be a bounded domain with nice boundary and $f(z)$ be analytic on the closure of D. Then for any point $a \in D$*

$$f(a) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)\,dz}{z-a}.$$

□

Example. Consider the following path:



We now use the Cauchy Integral formula to evaluate

$$\int_\gamma \frac{e^z}{z^2+\pi^2}\,dz.$$

This function is analytic except where the denominator becomes zero. That is where $z^2 + \pi^2 = 0$. Note that $z^2+\pi^2 = (z-\pi i)(z+\pi i)$. So that the bad points are $z = \pi i$ and $z = -\pi i$. Thus our integral becomes

$$\int_\gamma \frac{e^z}{(z-\pi i)(z+\pi i)}\,dz.$$

We only need to work about the point πi as it is the only non-analytic point inside of γ. Rewrite the integral as

$$\int_\gamma \frac{e^z/(z+\pi i)}{(z-\pi i)}\,dz = \int_\gamma \frac{f(z)}{(z-\pi i)}\,dz$$

where

$$f(z) = \frac{e^z}{z+\pi i}.$$

The function $f(z)$ is analytic inside of γ. So by the Cauchy integral formula

$$\int_\gamma \frac{e^z}{z^2+\pi^2}\,dz = \int_\gamma \frac{f(z)}{(z-\pi i)}\,dz = 2\pi i f(\pi i) = 2\pi i\frac{e^{\pi i}}{\pi i+\pi i} = e^{\pi i} = -1.$$

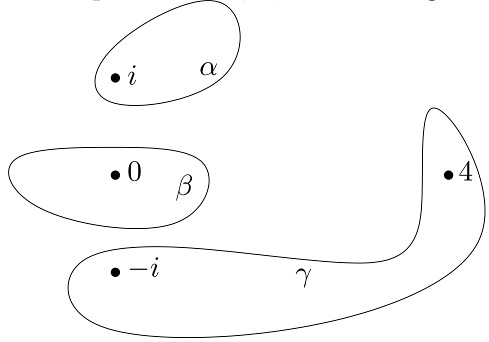
$\square$

Problem 3. Let z_1 be a complex number and γ a simple closed curve that does not pass through z_1. Show

$$\int_\gamma \frac{dz}{z-z_1} = \begin{cases} 2\pi i, & \text{if } z_1 \text{ is inside of } \gamma, \\ 0, & \text{if } z_1 \text{ is outside of } \gamma. \end{cases}$$

Hint: Use part (d) of Problem 2, or the Cauchy Integral Formula, with $f(z) = 1$, D the region inside of γ, and $z = z_1$.

Problem 4. Figure 3 shows the points i, $-i$, 0, and 4 along with three paths α, β, and γ.



Use the Cauchy integral formula to

(a) Evaluate $\displaystyle\int_\alpha \frac{2z+1}{z(z-4)(z^2+1)}\,dz$,

(b) Evaluate $\displaystyle\int_\beta \frac{2z+1}{z(z-4)(z^2+1)}\,dz$,

(c) Evaluate $\displaystyle\int_\gamma \frac{2z+1}{z(z-4)(z^2+1)}\,dz$.