INSTRUCTIONS:

(1) Write your solutions on only one side of your paper.
(2) Start each new problem on a separate page.
(3) Write your name (or just your initials) on the top of each page.
(4) Before handing in the exam, put the problems in order and then consecutively number your pages.
(5) Each of the 8 problems is worth 12 points. Following the instructions is worth 4 points.

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certi...

Signature / Date

Name (printed) :

Problem 1. Let (X, ρ) be a metric space. Throughout this problem, A and B are <u>*nonempty*</u>, <u>*closed*</u>, <u>*disjoint*</u> subsets of X. Define the distance $d(A, B)$ between A and B by

$$d(A,B) \;=\; \inf\{\rho(x,y) : x \in A \text{ and } y \in B\} \;. \tag{1}$$

(a) Given an example of two such subsets A and B of some metric space X such that $d(A, B) = 0$.

(b) Now assume, furthermore, that B is compact. Show that $d(A, B) > 0$.

Problem 2. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Show *Young's inequality*, i.e. show that if $x, y \geq 0$ then

$$xy \;\leq\; \frac{x^p}{p} + \frac{y^q}{q} \;. \tag{2}$$

You may use, without proving, the fact that $\varphi(x) = -\ln x$ is a convex function on $(0, \infty)$.

(b) Show *Hölder's inequality* for sequence spaces, i.e.
show that if $x = \{x_i\}_{i=1}^{\infty} \in \ell_p$ and $y = \{y_i\}_{i=1}^{\infty} \in \ell_q$ then $\{x_i\, y_i\}_{i=1}^{\infty} \in \ell_1$ and

$$\|\{x_i\, y_i\}_{i=1}^{\infty}\|_{\ell_1} \;\leq\; \|\{x_i\}_{i=1}^{\infty}\|_{\ell_p} \;\cdot\; \|\{y_i\}_{i=1}^{\infty}\|_{\ell_q} \;. \tag{3}$$

(c) Show *Hölder's inequality* for function spaces, i.e.
show that if $f \in L_p$ and $g \in L_q$ then $fg \in L_1$ and

$$\|fg\|_{L_1} \;\leq\; \|f\|_{L_p} \;\cdot\; \|g\|_{L_q} \;. \tag{4}$$

Problem 3. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Let $f \in L_\infty(\Omega, \mathcal{M}, \mu)$.

(a) Show that $f \in L_p(\Omega, \mathcal{M}, \mu)$ for each $1 \leq p < \infty$.

(b) Show that $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$.

Problem 4. Let $g\colon [a,b] \to [c,d]$ and $f\colon [c,d] \to \mathbb{R}$ be *absolutely continuous* functions.

(a) Define what it means for a function $h\colon [a,b] \to \mathbb{R}$ to be *absolutely continuous*.

(b) Assume, furthermore, that g is *monotone increasing*. Show that $f \circ g$ is absolutely continuous.

Problem 5. Let $L_1 = \{f\colon \mathbb{R} \to \mathbb{R} \mid f \text{ is Lebesgue integrable}\}$.
Establish the Riemann-Lebesgue Theorem: if $f \in L_1$ then $\lim_{n \to \infty} \int_{\mathbb{R}} f(x) \ \cos(nx)\, dx = 0$.
You may use, without proving, that step functions (i.e. functions that are finite linear combinations of characteristic functions of intervals of finite length) are dense in L_1.

Problem 6. Let $(\mathbb{R}, \mathcal{M}, m)$ be the Lebesgue measure space on $\mathbb{R}$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L_p(\mathbb{R}, \mathcal{M}, m)$ and $g \in L_q(\mathbb{R}, \mathcal{M}, m)$.

(a) Define the (convolution) function $f * g\colon \mathbb{R} \to \mathbb{R}$ by

$$(f * g)(x) \quad = \quad \int_{\mathbb{R}} f(x-y) g(y)\, dy \ . \tag{5}$$

Show that the integral in (5) exists for each $x \in \mathbb{R}$ and that

$$\sup_{x \in \mathbb{R}} |(f*g)(x)| \ \leq \ \|f\|_p \ \|g\|_q \ . \tag{6}$$

(b) Show that $f * g$ is uniformly continuous.

Problem 7. Onto Complex. Recall $\mathbb{N} = \{1, 2, 3, \ldots\}$.

(a) Fill in the blanks as to complete the statement of Cauchy's Integral Formula. Let $n \in \mathbb{N}$. If $f\colon \mathbb{C} \to \mathbb{C}$ is analytic inside and on a simple closed curve C and a is any point inside C, then $f(a) =$ _____ and $f^{(n)}(a) =$ _____ where C is traversed in the positive (counterclockwise) sense.

(b) Prove Liouville's Theorem: A bounded entire function $f\colon \mathbb{C} \to \mathbb{C}$ must be constant.

~~Problem 8.~~ The Fundamental Theorem of Algebra states that a polynomial $p\colon \mathbb{C} \to \mathbb{C}$ of degree $n \in \mathbb{N}$ has exactly n complex zeros, counting multiplicity. Prove the Fundamental Theorem of Algebra using Liouville's Theorem.