Analysis Qualifying Exam
August 1992

Directions. 1. Write your solution to each problem on a separate sheet.
2. Write on one side of the paper only.
3. Questions one to eight are each worth 10 points. Question nine is worth 20 points (4 points for each part).

1. PROVE: The closed interval $[a, b]$ is connected.

2. Let $\mathcal{B}$ denote the Borel $\sigma-$algebra of the real line and let $\mathcal{C}$ be a collection of Borel sets such that the $\sigma-$algebra generated by $\mathcal{C}$ is $\mathcal{B}$, i.e. $\sigma(\mathcal{C}) = \mathcal{B}$.
PROVE: Let $f : \mathbb{R} \to \mathbb{R}$. Then f is Lebesgue measurable if and only if, for every $C \in \mathcal{C}$, $f^{-1}(C)$ is Lebesgue measurable.

3. Let μ be a finite Borel measure on $[0,1]$ such that if E is a closed nowhere dense set in $[0,1]$ then $\mu(E) = 0$. PROVE: $\mu([0,1]) = 0$.
HINT: Let D denote the rational numbers in $[0,1]$. Show that, for every $\varepsilon > 0$, there exists an open set $\mathcal{O}$ such that $D \subset \mathcal{O}$ and $\mu(\mathcal{O}) < \varepsilon$.

4. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let f be a nonnegative integrable function on X, i.e. $f \in L_1^+(\mu)$.
PROVE: There exist a sequence (E_n) of measurable sets and a sequence (a_n) of nonnegative numbers such that

$$f(x) = \sum_{n=1}^{\infty} a_n \chi_{E_n}(x) \text{ a.e. } [\mu]$$

and

$$\int f \, d\mu = \sum_{n=1}^{\infty} a_n \mu(E_n).$$

5. Let (f_n) be a sequence of differentiable functions on $[a, b]$ such that
(i) $f_n(a) = 0$ for all n,
(ii) $|f_n'(x)| \leq M$ for all $x \in [a, b]$ and for all n,
(iii) $(f_n'(x))$ converges a.e. on $[a, b]$.
PROVE: (f_n) converges uniformly to an absolutely continuous function f on $[a, b]$.

6. Let f be a bounded measurable function on $[0,1]$. PROVE:

$$[\int x f(x) \, dx]^2 \leq \frac{1}{3} \int f(x)^2 \, dx.$$

7. Suppose that $f(z)$ is an analytic function on the open disk $D_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

PROVE: If $f(z)$ is not identically zero, then there exists $\delta > 0$ such that $f(z) \neq 0$ for all $z \in D_\delta(a) \backslash \{a\}$.

8. Suppose that E is an $m \times m$ measurable set in $[0,1] \times [0,1]$ (m is Lebesgue measure on $[0,1]$ here) such that $m \times m(E) \geq \frac{1}{2}$.
PROVE: $m\{x \in [0,1]; m(E_x) \geq \frac{1}{4}\} \geq \frac{1}{3}$.
Note: $E_x = \{y \in [0,1] : (x,y) \in E\}$.

9. TRUE or FALSE: Prove or give a counterexample.

T (a) If (f_n) is a sequence of nonnegative measurable functions on $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in \mathbb{R}$ and $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = 0$, then $f(x) = 0$ a.e.

(b) If $f(z)$ is an analytic function on an open set $\mathcal{O}$ and γ is a piecewise smooth closed curve in $\mathcal{O}$, then $\int_\gamma f(z)\,dz = 0$.

(c) If $f(z)$ and $g(z)$ are analytic on an open set $\mathcal{O}$,

$$\overline{D_r(a)} = \{z \in \mathbb{C} : |z-a| \leq r\} \subset \mathcal{O},$$

for some $r > 0$, and $f(z) = g(z)$ for all $z \in T_r(a) = \{z \in \mathbb{C} : |z-a| = r\}$, then $f(z) = g(z)$ on $D_r(a) = \{z \in \mathbb{C} : |z-a| < r\}$

(d) If $f(z)$ is an entire function and $|f(z)| \geq M$ for all $z \in \mathbb{C}$ (some $M > 0$) then $f(z)$ is constant.

F (e) If f is a continuous function of bounded variation on $[a,b]$ and $f'(x) = 0$ a.e. then f is constant on $[a,b]$.