ADMISSION TO CANDIDACY EXAMINATION

REAL ANALYSIS
January 1986

Notation: λ denotes Lebesgue measure on the real line R.

1. State each of the following theorems:
   a. Lebesgue's Dominated Convergence Theorem.
   b. Egoroff's Theorem.
   c. Radon-Nikodym Theorem.
   d. Tonelli's or Fubini's Theorem (indicate which).

2. Let f be a monotone increasing real valued function on $[a,b]$. Suppose f' exists λ-a.e. on $[a,b]$. Prove that f' is measurable, nonnegative and

$$\int_a^b f' \, d\lambda \leq f(b) - f(a).$$

3. Let f_0 be continuous on $[0,1]$. For $n \geq 0$ define

$$f_{n+1}(x) = \int_0^x f_n(t)dt \, , \qquad 0 \leq x \leq 1.$$

   Prove that $\sum_{n=1}^{\infty} f_n(x)$ converges on $[0,1]$ and is continuous.

4. Suppose f is nonnegative and increasing on $[0,1]$ with $f(t)/t$ decreasing. Prove that f is absolutley continuous on $[\varepsilon,1]$ for all $\varepsilon > 0$. $\frac{f(x_1)}{x_1} - \frac{f(x_2)}{x_2} \leq 0$ iff $x_1 \geq x_2$
   then $\Rightarrow$ $f(x_1) - f(x_2) \leq \frac{f(\varepsilon)}{\varepsilon}(x_1 - y_2)$

5. a. Let $f_n \in L^2([0,1],\lambda)$ with $||f_n||_2 \leq 1$ for all $n = 1,2,..$ and $f_n \to 0$ λ-a.e. as $n \to \infty$. Prove that

$$\int_0^1 |f_n| d\lambda \to 0 \quad \text{as} \quad n \to \infty.$$