Qualifying Examination in Algebra

August 2010

The field of complex numbers will be denoted by $\mathbf{C}$ and the field of rational numbers by $\mathbf{Q}$. The ring of integers is denoted by $\mathbf{Z}$.

Note! You must show sufficient work to support your answer. Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc. If some problem is incorrect, then give a counterexample.

1. Let G be a group and let $H \subseteq G$ be a normal subgroup. Verify that the operation of G/H is well-defined.

2. Let F be a field, and let $F[X]$ be the polynomial ring in one variable over F.

a. Prove that every ideal in $F[X]$ is principal.

b. Give examples of ideals that are not principal in the rings $\mathbf{Z}[X]$ and $F[X, Y]$.

3. Describe the Galois group of $\mathbf{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbf{Q}$, and find all the intermediary extensions. Prove your answers.

4. List all the isomorphism classes of abelian groups of order 360. Justify your answer.

5. If p, q are prime numbers, prove that there are no simple groups of order pq.

6. Let F denote the set of complex numbers z that are algebraic over $\mathbf{Q}$.

a. Prove that F is a field.

b. If $z \in \mathbf{C}$ is algebraic over F, prove that $z \in F$.

7. Let $F \subseteq E$ be a finite field extension.

a. Define what it means for the extension to be normal.

b. Give examples of finite extensions that are normal, and finite extensions that are not normal. Explain why your examples have the required properties.

c. If $F \subset E \subset L$ are finite field extensions, is it true or false that $F \subseteq L$ normal implies $F \subset E$ normal? Prove if true, give a counterexample if false.

8. Give an example of a field E, a subfield F, and two elements θ and φ of E such that $E = F(\theta, \varphi)$, but there does not exist ξ in E with $E = F(\xi)$. Prove that your example has the required properties.

9. State and prove the Chinese Remainder Theorem.

10. a. Let $\varphi : G \to G'$ be a group homomorphism. What quotient of G is isomorphic to the image of φ?

b. Let H and N be subgroups of a group G with N a normal subgroup of G. Prove that

$$\frac{H}{H \cap N} \cong \frac{HN}{N}.$$

You may appeal to the result stated in (a).