Analysis Qualifying Exam
August 2003

Instructions: Write your name legibly on each sheet of paper. Write only on one side of each sheet of paper. Try to answer all questions. Prove all your claims. Questions 1-8 are worth 10 points each and question 9 is worth 20 points.
Terminology: Measurability and integrability on $\mathbb{R}^n$ ($n \in \mathbb{N}$) or interval, or product of intervals will always refer to the Lebesgue measure except if otherwise specified. Lebesgue measure will be denoted by λ, dx or dy depending on the context.

1. **a.** Prove that if (X, Σ, μ) is a measure space and $A_n \subseteq X$ for all $n \in \mathbb{N}$ with $\sum_n \mu(A_n) < \infty$ then $\mu(\limsup_n A_n) = 0$.
b. For every $x \in (0, 1]$ write the dyadic expression of x,

$$x = \sum_{n=1}^{\infty} \frac{d_n(x)}{2^n} = .d_1(x)d_2(x)\dots,$$

each $d_n(x)$ being 0 or 1; if a number has two dyadic expressions we choose the one that terminates with ones (e.g. we write $\frac{1}{2^2} + \frac{1}{2^3} + \cdots$ rather than $\frac{1}{2}$). For every $n \in \mathbb{N}$ we define the function $\ell_n : (0, 1] \to \mathbb{N}$ by $\ell_n(x) = 0$ if and only if $d_n(x) = 1$; $\ell_n(x) = k \geq 1$ if and only if $d_n(x) = d_{n+1}(x) = \cdots = d_{n+k-1}(x) = 0$ and $d_{n+k}(x) = 1$ (i.e. $\ell_n(x)$ is a finite number and is equal to the number of consequtive zeros in the dyadic expression of x starting counting from the n^{th} decimal). Compute

$$\lambda(\limsup_n \{x \in (0, 1] : \ell_n(x) \geq 2\log_2 n\}).$$

(Recall that if X is a set and $A_n \subseteq X$ for all $n \in \mathbb{N}$ then we define $\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$).

2. Prove that every Borel subset of $\mathbb{R}$ is Lebesgue measurable.

3. State Lebesgue' Dominated Convergence Theorem (LDCT) and Egoroff's theorem. Prove LDCT for spaces of finite measure using Egoroff's theorem.

4. Suppose that f is a non-negative measurable function on a σ-finite measure space $X, \Sigma, \mu)$. Prove that $\{(x, y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\}$ is $\mu \times \lambda$ measurable and that $\int_X f d\mu = (\mu \times \lambda)\{(x, y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\}$.

5. Let $f : [0, 1] \to \mathbb{R}$ be a function of bounded variation and let $v : [0, 1] \to \mathbb{R}$ defined by $v(x) = T_a^x(f)$ (the total variation of f from 0 to x).
a. Prove that if v is absolutely continuous then f is absolutely continuous.
b. Prove that if $T_0^1(f) = \int_0^1 |f'| d\lambda$ then f is absolutely continuous. (The fact that if $g : [a, b] \to \mathbb{R}$ is a function of bounded variation then $T_a^b(g) \geq \int_a^b |g'| d\lambda$ can be used without proof if needed).

6. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$. Compute

$$\int_\gamma \left(\frac{z}{z-1}\right)^n dz$$

for all positive integers n.

7. Let Ω be a region in $\mathbb{C}$ and $f, g : \Omega \to \mathbb{C}$ be analytic functions such that $f(z)g(z) = 0$ for all $z \in \Omega$. Then prove that $f \equiv 0$ or $g \equiv 0$ (i.e. $f(z) = 0$ for all $z \in \Omega$ or $g(z) = 0$ for all $z \in \Omega$).

8. Evaluate

$$\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx.$$

9. True or False. Prove or disprove, whichever is appropriate, in order to obtain credit.

a. If (X, Σ, μ) is a finite measure space, $1 \le r \le s < \infty$ and $f : X \to \mathbb{R}$ is a measurable function then

$$\|f\|_r \le \|f\|_s \mu(X)^{\frac{1}{r}-\frac{1}{s}}.$$

(Recall $\|f\|_r = \left(\int_X |f|^r d\mu\right)^{\frac{1}{r}}$).

b. There exists a measurable set $A \subseteq [0,1] \times [0,1]$ such that $\lambda(A_x) = 0$ λ-a.e. and $\lambda(A^y) > 0$ λ-a.e. (Recall that we denote $A_x = \{y \in [0,1] : (x,y) \in A\}$ and $A^y = \{x \in [0,1] : (x,y) \in A\}$).

c. The function $f(z) = z \sin \frac{1}{z}$ has a pole at 0.

d. There exists a region Ω of $\mathbb{C}$ which is mapped in a one-to-one way onto $\{z \in \mathbb{C}\backslash\{0\} : |z| < 1\}$ through the exponential map.

e. There exists an analytic function $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$ such that $f(\{z \in \mathbb{C} : |z| < 1\})$ is exactly a line segment.