Mathematics 554H/701I Homework

We now review a bit from the beginning of the term. Let $f\colon E \to E'$ be a map between sets. Recall that if $A \subseteq E$, then the ***image*** of A under f is

$$f(S) = \{f(x) : x \in A\}.$$

And if $B \subseteq E'$ the ***preimage*** of B under f is

$$f^{-1}(B) = \{x \in E : f(x) \in B\}.$$

We recall that taking preimages behaves well with respect to taking unions and intersections.

Proposition 1. *Let $f\colon E \to E'$ be a map between sets and let $\{S_\alpha\}_{\alpha\in I}$ be a collections of subsets of E'. (That is for each $\alpha \in A$ the $S_\alpha \subseteq E'$.) Then*

$$\begin{aligned} f^{-1}\Big(\bigcup_{\alpha\in I} S_\alpha\Big) &= \bigcup_{\alpha\in I} f^{-1}(S_\alpha) \qquad \textit{and} \\ f^{-1}\Big(\bigcap_{\alpha\in I} S_\alpha\Big) &= \bigcap_{\alpha\in I} f^{-1}(S_\alpha), \end{aligned}$$

Proof. To prove the first equality:

$$\begin{aligned} x \in f^{-1}\Big(\bigcup_{\alpha\in I} S_\alpha\Big) &\iff f(x) \in \bigcup_{\alpha\in I} S_\alpha \\ &\iff f(x) \in S_\alpha \quad \text{for at least one } \alpha \in I \\ &\iff x \in f^{-1}(S_\alpha) \quad \text{for at least one } \alpha \in I \\ &\iff x \in \bigcup_{\alpha\in I} f^{-1}(S_\alpha). \end{aligned}$$

This shows that $f^{-1}\Big(\bigcup_{\alpha\in I} S_\alpha\Big)$ and $\bigcup_{\alpha\in I} f^{-1}(S_\alpha)$ have the same elements and therefore are equal.

Likewise

$$\begin{aligned} x \in f^{-1}\Big(\bigcap_{\alpha\in I} S_\alpha\Big) &\iff f(x) \in \bigcap_{\alpha\in I} S_\alpha \\ &\iff f(x) \in S_\alpha \quad \text{for all } \alpha \in I \\ &\iff x \in f^{-1}(S_\alpha) \quad \text{for all } \alpha \in I \\ &\iff x \in \bigcap_{\alpha\in I} f^{-1}(S_\alpha). \end{aligned}$$

and therefore $f^{-1}\Big(\bigcap_{\alpha\in I} S_\alpha\Big)$ and $f^{-1}\Big(\bigcap_{\alpha\in I} S_\alpha\Big)$. □

Problem 1. As a review let $f\colon E \to E'$ be a function between sets and let $S_1, s_2 \subseteq E'$. Then show directly the equalities

$$f^{-1}(S_1 \cup S_2) = f^{-1}(S_1) \cup f^{-1}(S_2) \quad \text{and} \quad f^{-1}(S_1 \cap S_2) = f^{-1}(S_1) \cap f^{-1}(S_2).$$

hold. $\square$

We recall that in the books notation if S is a subset of some set E then the ***compliment*** of S in E is

$$\mathcal{C}(S) = \{x \in E : x \notin S\}.$$

That is $\mathcal{C}(S)$ is the set of points of E that are not in S. Taking compliments is also well behaved with respect to taking preimages.

Proposition 2. *Let $f\colon E \to E'$ be a map between sets and let $S \subseteq E'$. Then*

$$f^{-1}(\mathcal{C}(S)) = \mathcal{C}(f^{-1}(S)).$$

(Here $\mathcal{C}(S)$ is the compliment of S in E' and $\mathcal{C}(f^{-1}(S))$ is the compliment of $f^{-1}(S)$ in E.)

Problem 2. Prove this. $\square$

We summarize this as

Taking preimages preserves unions, intersections, and compliments.

We now relate continuity of functions to taking preimages of open sets.

Lemma 3. *Let $f\colon E \to E'$ be a map between metric spaces. Then the following are equivalent:*

(a) For every open subset $S \subseteq E'$ the preimage $f^{-1}(S)$ is open in E. (That is the preimages of open sets are open.)
(b) For every closed subset $S \subseteq E'$ the preimage $f^{-1}(S)$ is closed in E. (That is the preimages of closed sets are closed.)

Problem 3. Prove this. *Hint:* Let $S \subseteq E'$. Then we have seen that $f^{-1}(\mathcal{C}(S)) = \mathcal{C}(f^{-1}(S))$. Assume that (a) holds, that is that the preimages under f of open sets are open. Let S be closed. Then $\mathcal{C}(S)$ is open and therefore $f^{-1}(\mathcal{C}(S))$ is open. But then $\mathcal{C}(f^{-1}(\mathcal{C}(S))) = f^{-1}(\mathcal{C}(\mathcal{C}(S)))$ is closed. But what is $\mathcal{C}(\mathcal{C}(S))$? This shows that (b) holds and thus that (a) implies (b). Do a similar argument to show that (b) implies (a). $\square$

Theorem 4. *Let $f\colon E \to E'$ be a map between metric spaces. Then then the following are equivalent*

(a) f is continuous.
(b) For every open set $S \subseteq E'$ the preimage $f^{-1}(S)$ is open in E.
(c) For every closed set $S \subseteq E'$ the preimage $f^{-1}(S)$ is closed in E.

Problem 4. Prove this. *Hint:* (b) $\iff$ (c) holds is is covered by 3. So we only need to show that (a) $\iff$ (b) holds.

(a) $\implies$ (b). Assume that f is continuous and that $S \subseteq E'$ is open. We need to show that $f^{-1}(S)$ is open. That is for any $p_0 \in f^{-1}(S)$ we need to show that $f^{-1}(S)$ contains a ball about p_0. As S open in E' there is a $\varepsilon > 0$

such that $B(f(p_0),\varepsilon)\subseteq S$. Use that f is continuous at p_0 to show that there is a $\delta>0$ such that for all $p\in B(p_0,\delta)$ we have $f(p)\in B(f(p_0),\varepsilon)\subseteq S$ and use this to show $B(p_0,\delta)\subseteq f^{-1}(S)$ and therefore that $f^{-1}(S)$ contians a ball about p_0.

(b) $\implies$ (a). Assume that (b) holds, that is that the preimage of open sets by f are open and we wish to show that f is continuous at all points of E. Let $p_0\in E$ and $\varepsilon>0$. Then the ball $B(f(p_0),\varepsilon)$ is an open set in E' and therefore the preimage $f^{-1}(B(f(p_0)))$ is open. As $p_0\in f^{-1}(B(f(p_0)))$ and $f^{-1}(B(f(p_0)))$ is open we have that $f^{-1}(B(f(p_0)))$ contains an open ball about p_0, say $B(p_0,\delta)\subseteq f^{-1}(B(f(p_0)))$. Use this to show that if $d(p,p_0)<\delta$, then $d'(f(p),f(p_0))<\varepsilon$ and therefore f is continuous at p_0. $\square$

At first it may not seem that rewriting the condition of f being continuous in terms of preimages of open sets is useful, but we now show that it makes some proofs easy.

Recall that a set in a metric space is connected if and only if it is not the disjoint union of two disjoint nonempty open sets.

Theorem 5. *Let E be a connected metric space and $f\colon E\to E'$ a continuous function. Then the image $f(E)$ is connected.*

Problem 5. Prove this. *Hint:* Toward a contradiction assume that $f(E)$ is not connected. Then $f(E)$ has a disconnection. That is $f(E)=U\cup V$ where U and V are nonempty open sets in $f(E)$ and $U\cap V=\varnothing$. Now show $E=f^{-1}(U)\cup f^{-1}(V)$ is a disconnection of E, contradicting that E is connected. $\square$

Recall that we have shown that the only connected subsets of $\mathbb{R}$ are the intervals. We now combine this with Theorem 5 to prove the intermediate value theorem.

Theorem 6 (General Intermediate Value Theorem)**.** *Let E be a connected metric space and let $f\colon E\to\mathbb{R}$ be a continuous function. Let $p_0,p_1\in E$ with $f(p_0)<f(p_1)$. Then for every real number y with $f(p_0)<y<f(p_1)$ there is a $x\in E$ with $f(x)=y$.*

Problem 6. Prove this. *Hint:* By Theorem 5 the set $f(E)$ is a connected subset of $\mathbb{R}$ and therefore $f(E)$ is an interval. We have $f(p_0),f(p_1)\in f(E)$ and as $f(E)$ is an interval this implies that $f(E)$ contains every point between $f(p_0)$ and $f(p_1)$. $\square$

Theorem 7 (Intermediate Value Theorem)**.** *Let $[a,b]$ be a closed interval in $\mathbb{R}$ and $f\colon[a,b]\to\mathbb{R}$ a continuous function with $f(a)\neq f(b)$. Then for every y between $f(a)$ and $f(b)$ the equation $f(x)=y$ has a solution with $a<x<b$.*

Problem 7. (a) Prove this as a corollary of Theorem 6 and the fact that $[a,b]$ is connected.

(b) Draw some pictures illustrating why the theorem is true. $\square$

The intermediate value theorem is useful in showing that equations have solutions, even in cases where we can not solve them explicitly. Here is an example: the equation $x^7 - 3x + 1 = 0$ has at least some solution with $0 < x < 1$. To see this note that $f(x) = x^7 - 3x + 1$ is continuous on $[0, 1]$. Also $f(0) = 1$ is positive, and $f(1) = -1$ is negative. Therefore by Theorem 7 f takes on the value 0 at some point in $(0, 1)$. That is there there is x_0 with $0 < x_0 < 1$ with $f(x_0) = x_0^7 - 3x_0 + 1 = 0$.

Problem 8. Show that the following have solutions.

(a) $x^3 = \sqrt{7 + x}$ on the interval $[0, 2]$. *Hint:* This can be rewritten as $x^3 - \sqrt{1 + x} = 0$.
(b) $x^3 + 2x + 2 = 0$ on $[-2, 2]$.
(c) $x^5 - 4x^3 + x - 9 = 0$ on $[-3, 3]$.

Proposition 8. *Every polynomial of degree* 3 *has at least one real root. That is if* $f(x) = a_3x^3 + a_2x^2 + a_1x^1 + a_0$ *with* $a_3 \neq 0$ *there is a real number* x_0 *with* $f(x_0) = 0$.

Proof. We wish to solve

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0.$$

As $a_3 \neq 0$ we can divide by a_3 and get the equivalent equation

$$x^3 + b_2x^2 + b_1x + b_0 = 0 \quad \text{where} \quad b_i = \frac{a_j}{a_3} \quad \text{for } j = 0, 1, 2.$$

Let

$$f(x) = x^3 + b_2x^2 + b_1x + b_0.$$

We will now find a c_0 such that $f(c) > 0$ and $f(-c) < 0$ and therefore $f(x) = 0$ with have a solution $x = x_0$ with $-c < x_0 < c$ by the Intermediate value Theorem. We start by writing $f(x)$ as

$$f(x) = x^3\Big(1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_2}{x^3}\Big) = x^3q(x)$$

where

$$q(x) = 1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_2}{x^3}.$$

Now note if $|x| \geq 1$ that

$$\begin{aligned}
q(x) &= 1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_2}{x^3} \\
&\geq 1 - \left|\frac{b_2}{x}\right| - \left|\frac{b_1}{x^2}\right| - \left|\frac{b_2}{x^3}\right| \\
&\geq 1 - \frac{|b_2|}{|x|} - \frac{|b_1|}{|x|} - \frac{|b_0|}{|x|} && \text{(as } |x| \geq 1\text{)} \\
&= 1 - \frac{|b_2| + |b_1| + |b_0|}{|x|}
\end{aligned}$$

Therefore if $|x| \geq 2(|b_2| + |b_1| + |b_0|)$ we have

$$q(x) \geq 1 - \frac{|b_2| + |b_1| + |b_0|}{|x|} \geq 1 - \frac{|b_2| + |b_1| + |b_0|}{2(|b_2| + |b_1| + |b_0|)} = \frac{1}{2}.$$

Whence if we set $c = 2(|b_2| + |b_1| + |b_0|)$ we have that

$$|x| \geq c \quad \text{implies} \quad q(x) > \frac{1}{2} > 0$$

Thus $q(q)$ and $q(-c)$ are both positive numbers and so

$$f(c) = c^3 q(c) > 0, \quad \text{and} \quad f(-c) = (-c)^3 q(-c) = -c^3 q(-c) < 0.$$

Therefore $f(x)$ change sign on $[-c, c]$ and f is continuous so by the Intermediate Value Theorem $f(x) = 0$ has a solution on $[-c, c]$. □

Problem 9. For any even integer $n = 2k$ given an example of a polynomial $f(x)$ such that $f(x) = 0$ has no solutions for any $x \in \mathbb{R}$. *Hint:* For $n = 2$ and example is $f(x) = x^2 + 1$. □

Theorem 9. *Let $f(x)$ be a polynomial of odd degree. Then there is a real number x_0 with $f(x_0) = 0$. That is all polynomial of odd degree have at least one root.*

Problem 10. Prove this for polynomial of degree 5. *Hint:* Look at the proof of 8. □