IMSc

Report

Proving W-hardness for variants of Dominating sets

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Contents

1 Aim

In this report, I will elaborate on proving W-hardness for Roman Domination as well as Weak Roman Domination for Split Graphs through Parameter Preserving Reduction. I shall also prove W-hardness for Bipartite Graphs in Roman Domination.

2 Definitions

Set Cover:

Given a set of elements $\{1, 2, ..., n\}$ (called the universe) and a collection S of m sets whose union equals the universe, the Set Cover problem is to identify the smallest sub-collection of S whose union equals the universe. It can be instanced by (U, F, k) where U is the set of all vertices in the graph, F is the set of vertices belonging to the Set Cover and k is the size of F.

2-Redundant Set Cover:

Given a set of elements $\{1, 2, ..., n\}$ (called the universe) and a collection S of m sets whose union equals the universe, the 2-Redundant Set Cover problem is to identify the smallest sub-collection of S whose union covers each element in the universe at least twice. It can be instanced by (U, F, k) where U is the set of all vertices in the graph, F is the set of vertices belonging to the 2-Redundant Set Cover and k is the size of F.

Roman Domination:

A *Roman dominating function (RDF)* on a graph $G = (V, E)$ is defined as a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The weight of a RDF is the value $f(V) = \sum_{v \in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$ is the minimum weight of all possible RDF's on a graph. A graph G is a *Roman* graph if $\gamma_R(G) = 2\gamma(G)$.

Weak Roman Domination:

Let V_0, V_1 and V_2 be the set of vertices assigned the values 0, 1 and 2 respectively, under f. There is a one to one correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, $f = (V_0, V_1, V_2)$.

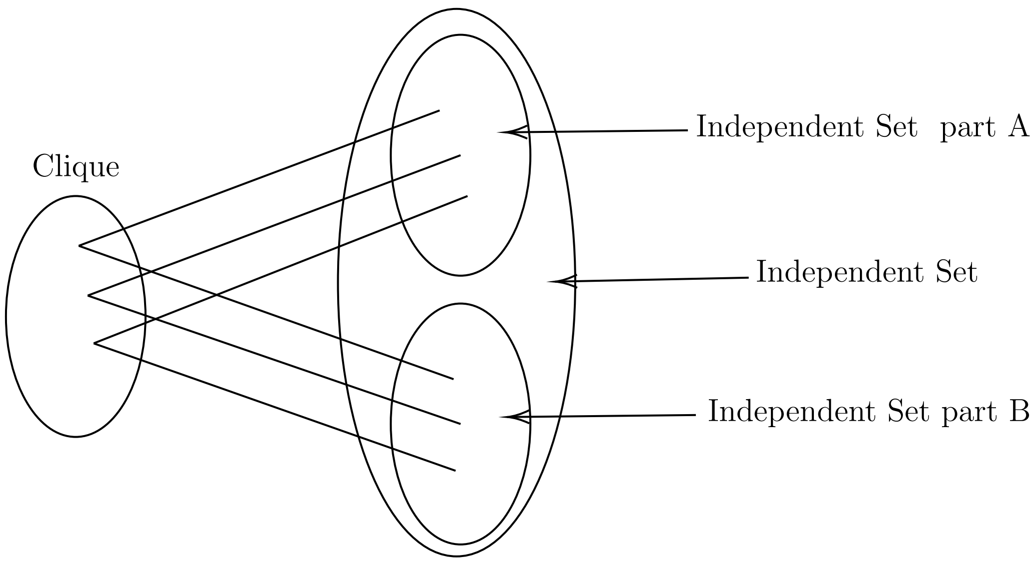
A vertex $u \in V_0$ is *undefended*, if it is not adjacent to a vertex in V_1 or V_2. The function f is a *weak Roman Dominating function* (wRDF) if each vertex $u \in V_0$ is adjacent to a vertex $v \in V_1 \cup V_2$ such that the function $f' : V \to \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. The weight $w(f)$ of f is defined to be $\|V_1\| + 2\|V_2\|$. The *Weak Roman Dominating number*, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G.

3 Parameter Preserving Reduction from Set Cover to Roman Domination on Split Graphs

Construction:
Consider the following graph constructed from the properties of Set Cover. This graph will be a split graph consisting of a clique and an independent set.

- Every vertex in the clique refers to a particular set in the collection S. The size of this clique will be m.
- There will be two vertices in the Independent Set that will correspond to the same element of the universe in the Set Cover. The will be an edge between these two vertices to all the vertices in the clique that correspond to the particular families that cover them in the collection S. Both the vertices in the Independent Set will be exact copies of each other with regards to graph structure. The size of this Independent Set will be $2n$.

Thus, the set cover problem translates into the problem of finding a subset of vertices in the clique that dominate all the elements in the independent set.



Claim 3.1 : In the given universe as well as the given collection, $\exists$ a set cover of size k if and only if in the above constructed split graph, $\exists$ a Roman Domination of size at most $2k$ wherein $\|V_1\|$ is minimized.

Proof. **Part I**:
Assume that $\exists$ a Set Cover of size k consisting of k elements in the clique. We need

to prove that $\exists$ a Roman Domination of size at most $2k$ and $\|V_1\|$ is minimized. Let us implement the following labelling:

- Let us label the vertices in the clique which correspond to the set cover as 2.
- Let us label the rest of the vertices in the clique and the entire independent set as 0.

Proposition 3.2: The vertices labelled 0 present in the clique have at least one neighbour labelled 2.

Proof. We know that at least one of the vertices is labelled 2 because $\exists$ a Set Cover of size k where all its vertices are labelled 2 by our labelling. This ensures that every vertex labelled 0 is adjacent to at least one vertex labelled 2 due to the nature of a clique. □

Proposition 3.3: The vertices labelled 0 in the Independent Set have at least one neighbour labelled 2.

Proof. By definition, the Set Cover covers all the vertices in the Independent Set, so, if every vertex in the Set Cover is labelled 2, then every element in the Independent Set must have at least one vertex labelled 2 as its neighbour. □

With the above labelling for the vertices, along with **Proposition 3.2** and **Proposition 3.3**, we have proved that this labelling is a Roman Dominating Function. The weight of this function is $2k$.

Part II:
For the given graph, assume that $\exists$ a Roman Dominating Function of size $2k$, where $\|V_1\|$ is minimized.We need to prove that $\exists$ a set cover of size k in the clique.

Claim 3.4: A Set Cover of size k exists for the above constructed graph given $\exists$ an RDF of size at most $2k$ for the graph where $\|V_1\|$ is minimized.

Proof. To prove the above, we need to prove a few more statements.

Lemma 3.5: There are no edges between vertices in V_1 and vertices in V_2 in the above graph.

Proof. If there did exist an edge between two vertices labelled 1 and 2, then I could relabel the vertex labelled as 1 to 0, which would still result in a RDF and this would also reduce $\|V_1\|$. So, there are no edges between V_1 and V_2. □

Remark: The relabelling of 1 in the above proof would not break the conditions associated with RDF because vertices labelled 1 have no such conditions or restrictions in the definition of RDF.

Proposition 3.6: If $\exists$ an RDF of size $2k$ in the above graph, then $\exists$ a RDF of size $2k$ such that all the vertices present in the Independent Set are labelled 0

Proof. Case 1: Suppose the vertices in the Independent Set(IS) are labelled 2

- If a particular vertex labelled 2 in the IS is connected to another vertex in the clique labelled 2, then the vertex labelled 2 in the IS (as well as its exact copy) can be relabelled to 0 and this would still result in an RDF wherein the weight will not increase. This is because the 0's in the IS will have that particular vertex labelled 2 in the clique adjacent to it, thus maintaining the status of the RDF as well as not increasing in weight.
- No vertex labelled 2 in the IS is connected to a vertex in the clique labelled 1 due to **Lemma 3.5**.
- If a particular vertex labelled 2 in the IS is connected to another vertex in the clique labelled 0, then we can relabel the vertex in the IS (as well as its exact copy) to 0, and relabel that particular vertex in the clique as 2. This vertex will be adjacent to both the 0's created due to this relabelling, ensuring that this labelling is an RDF. The weight of this RDF also cannot increase due to this relabelling.

Case 2: Suppose the vertices in the Independent Set are labelled 1.

- No vertex labelled 1 in the IS is connected to a vertex in the clique labelled 2 due to **Lemma 3.5**.
- If a particular vertex labelled 1 in the IS is connected to another vertex labelled 1 in the clique, then we can relabel the vertex in the IS to 0 (as well as its exact copy) and relabel that particular vertex in the clique to 2. The 0's created due to this relabelling will be adjacent to the newly relabelled 2, ensuring that this is an RDF. The weight of this RDF also will not increase.
- If a particular vertex labelled 1 in the IS is connected to another vertex labelled 0 in the clique, then we have three different situations.The exact copy of the

vertex can have three different labels 0,1, and 2.

- When the copy vertex has label 2, then both the vertex and its copy in the IS can be relabelled to 0 and the vertex labelled 0 in the clique can be relabelled as 2. The weight decreases and the resultant labelling is still an RDF as the 2 in the clique ensures that the 0's created in the IS due to the new labelling is covered.
- When the copy vertex is labelled 1, a similar argument is made as the case where it is labelled 2, except this time the weight is the same, which still doesn't break the rules of RDF.
- When the copy vertex is labelled 0, then we need to provide a different argument. Now, we are given that the current labelling is an RDF. This ensures that every 0 is adjacent to atleast one 2, which must include the 0 present in the copy vertex. This should be different from the vertex labelled 0 in the clique and must be another vertex in the clique which is labelled 2. Since the current copy vertex is covered by this 2, the considered vertex in the IS should also be adjacent to this 2. This ensures that we can relabel the 1 to 0 while ensuring that the function remains and RDF. The weight will decrease in this case.

Case 3: If the vertices in the Independent Set are labelled 0, then it adheres to the condition of the proof.

So, by considering cases 1,2 and 3, we can conclude that there exists a RDF which has weight of at most $2k$ such that the vertices in the Independent Set are all labelled 0. Proposition 3.6 is now proved.

□

We have an RDF of weight at most $2k$ where all labels > 0 are on the clique. Consider this labelling:

- By the structure of the graph, we know that $\exists k$ vertices which cover all the vertices in the IS. Label all these vertices as 2.
- Rest of the vertices in the clique and all the vertices in the IS are labelled 0.

The labelling clearly shows an RDF of weight at most $2k$ with $\|V_1\|$ minimized. The vertices labelled 2 correspond to the set cover of size k. **Claim 3.4 is now proved.** □

Due to Part I and Part II, Claim 3.1 is now proved. □

Claim 3.7: The RDF on a Split Graph is a W-hard problem
Proof: Set Cover problem is W-hard. Since we have proved Claim 3.1, we have done a parameter preserving reduction for the Roman Domination. The weight of RDFs on Split Graphs is a perfect function of k, specifically $2k$ where k is the size of the set cover.

4 Parameter Preserving Reduction from Set Cover to 2-Redundant Set Cover

Consider a Set Cover instance (U, F, k) where U is the set of all vertices in the graph, F is the set of vertices belonging to the Set Cover and k is the size of F.

Consider a 2-Redundant Set Cover instance $(U, F^{'}, k+1)$ where U is the set of all vertices in the graph, $F^{'}$ is the set of vertices belonging to the 2-Redundant Set Cover and $k+1$ is the size of $F^{'}$.

Claim 4.1: In the given universe U, $\exists$ a Set Cover of size k if and only if $\exists$ a 2-Redundant Set Cover of size $k+1$.

Part I:

Proof. Assume that $\exists$ a Set Cover (U, F, k). Let us construct a 2-Redundant Set Cover $(U, F^{'}, k+1)$ where $F^{'} = F + set\ S$ where $S = U$.
Now, size of $F^{'}$ is $k+1$ as we have the extra set S.
S will cover each element in U once and F will cover each element in U at least once. This ensures that $F^{'}$ will cover each element at least twice. □

Part II:

Proof. Assume that $\exists$ a 2-Redundant Set Cover $(U, F, k+1)$. There are two cases:

- F contains the set S.
- F does not contain the set S.

Let us prove that a Set Cover of size k exists for both cases.
Case 1: Simple remove set S from F, so the Set Cover now becomes $(U, F-S, k)$. The rest of the sets in F will ensure that each element in the universe is covered at least once.
Case 2: Replace any set in F with the set S. This replacement adheres to the rules of 2-Redundant Set Cover as F still covers each element at least twice. Now, similar to Case 1, remove set S from F.

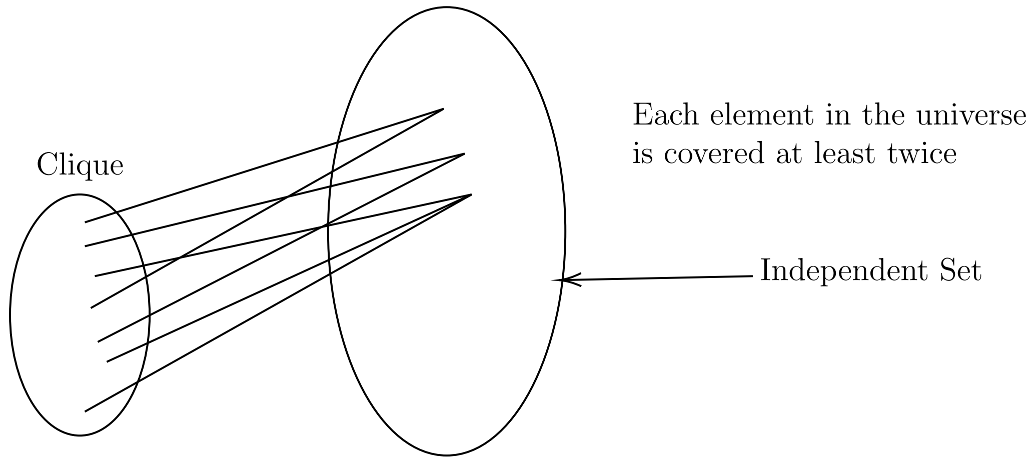
In both the above cases, we have constructed a Set Cover of size k. □

5 Parameter Preserving Reduction from 2-Redundant Set Cover to Weak Roman Domination on Split Graphs

Construction:
Consider the following graph constructed from the properties of Set Cover. This graph will be a split graph consisting of a clique and an independent set.

- Every vertex in the clique refers to a particular set in the collection S. The size of this clique will be m.
- There will be one vertex in the Independent Set that will correspond to a particular element of the universe in the Set Cover. The will be an edge between this vertex and all the vertices in the clique that correspond to the particular families that cover this vertex in the collection S. The size of this Independent Set will be n.

Thus, the 2-Redundant Set Cover problem translates into the problem of finding a subset of vertices in the clique that dominate all the elements in the independent set at least twice.



Claim 5.1: In the given universe as well as the given collection, $\exists$ a 2-Redundant Set Cover of size k if and only if in the above constructed split graph, $\exists$ a Weak Roman Domination of size at most k.

Proof. **Part I**:
Assume that $\exists$ a 2-Redundant Set Cover of size k consisting of k elements in the clique.We need to prove that $\exists$ a WRDF of size at most k.

Let us implement the following labelling:

- Let us label the vertices in the clique which correspond to the 2-R Set Cover as 1.
- Let us label the rest of the vertices in the clique and the entire independent set as 0.

The total weight of this labelling is k.

Proposition 5.2: The vertices labelled 0 present in the clique follow the rules of Weak Roman Domination.

Proof. We know that $\exists$ at least k vertices labelled 1 in the clique. So, every 0 in the clique is adjacent to at least 1 1.Also, if a 1 becomes a 0 and another 0 becomes a 1, there are still k other 1's in the clique which will ensure that they do not remain undefended. □

Proposition 5.3: The vertices labelled 0 in the Independent Set follow the rules of Weak Roman Domination.

Proof. By definition, the 2-Redundant Set Cover covers all the vertices in the Independent Set at least twice, so, if every vertex in the 2-R Set Cover is labelled 1, then every element in the Independent Set must have at least two vertex labelled 1 as its neighbour. This ensure that the labelling is a WRDF. □

With the above labelling for the vertices, along with **Proposition 5.2** and **Proposition 5.3** , we have proved that this labelling is a Weak Roman Dominating Function.
The weight of this function is k.

Part II: For the given graph, let $\exists$ a Weak Roman Domination Function of weight k. We need to prove that $\exists$ a 2-R Set Cover of size k for this graph.

Proposition 5.3: If $\exists$ a Weak Roman Domination of weight k in the given graph, then $\exists$ a Weak Roman Domination of weight at most k in the given graph such that all the elements in the Independent Set are labelled 0.

Proof. Case 1: Suppose $\exists$ vertices labelled 2 in the IS:

- If a particular vertex labelled 2 in the IS is connected to a vertex labelled 2 in the clique, then the vertex in the IS can be relabelled as 0 and this would adhere to the rules of Weak Roman Domination.
- If a particular vertex labelled 2 in the IS is connected to a vertex labelled 1 in the clique, then there are further three more cases to consider. Since we know that for each vertex in the IS, at least two vertices (let's call them c_1 and c_2) are in the clique that are adjacent to each vertex in the IS. If c_1 is labelled 1, then c_2 has 3 possibilities, $0, 1, 2$.
  - If c_2 is labelled 2, then the vertex in the IS can be labelled to 0.
  - If c_2 is labelled 1, then we have two vertices labelled 1 in the clique adjacent to the vertex in the IS.
  - If c_2 is labelled 0, then we can relabel the vertex in the IS from 2 to 0 and label c_2 as 1, resulting in two 1's adjacent to this vertex.
- If a particular vertex labelled 2 in the IS is connected to a vertex labelled 0 in the clique, then there are three more cases to consider. Since we know that for each vertex in the IS, at least two vertices in the clique(lets say c_1 and c_2) are adjacent to each vertex in the IS. If c_1 is labelled 0, then c_2 has 3 possibilities, $0, 1, 2$.
  - If c_2 is labelled 2, then the vertex in the IS can be labelled as 0.
  - If c_2 is labelled 1, then we must relabel the vertex in the IS from 2 to 0, and relabel c_1 to 1, which will ensure that two vertices labelled 1 are adjacent to the 0 in the IS.
  - If c_2 is labelled 0, We must relabel the vertex in the IS as 0 and label c_1 as 1 and c_2 as 1.

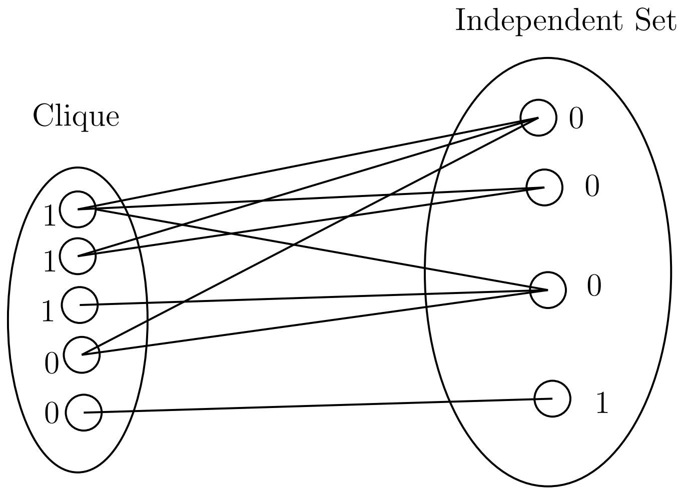
Before starting Case 2, there are few properties to be observed.

- If we want a WRDF of size k where $k¡n$, the weight of the clique is at least 1.

  Proof. Assume that the weight of the clique was 0. Then, to cover all the vertices in the clique, every element in the IS must be labelled 1. This results in $k = n$, which is a contradiction. □

Case 2: If the vertices in the Independent Set are labelled 1:

- If a particular vertex labelled 1 in the IS is connected to a vertex labelled 2 in the clique, then the vertex in the IS is labelled as 0 and the rules of WRD is followed.
- If a particular vertex labelled 1 in the IS is connected to a vertex labelled 1 in the clique, then there are more cases to consider.Since we know that for each vertex in the IS, at least two vertices in the clique(lets say c_1 and c_2) are adjacent to each vertex in the IS. If c_1 is labelled 1, then c_2 has 3 possibilities, $0, 1, 2$.
  - If c_2 is labelled 2, then the vertex in the IS is labelled 0.
  - If c_2 is labelled 1, then the vertex in the IS is labelled 0.
  - If c_2 is labelled 0, the the vertex in the IS is labelled 0 and c_2 is labelled 1.
- If a particular vertex labelled 1 in the IS is connected to a vertex labelled 0 in the clique, more cases are to be considered.Since we know that for each vertex in the IS, at least two vertices in the clique(lets say c_1 and c_2) are adjacent to each vertex in the IS. If c_1 is labelled 0, then c_2 has 3 possibilities, $0, 1, 2$.
  - If c_2 is labelled 2, then the vertex in the IS is labelled 0.
  - If c_2 is labelled 1, then the vertex in the IS is labelled 0 and c_1 is labelled 1.
  - If c_2 is labelled 0, then we have a special condition. Now, from the observation, we know that the clique contains a weight >= 1. What is done is that the vertex in the IS is labelled as 0, c_1 is labelled as 1, then c_2 is also labelled as 1. Why can't we simply relabel another vertex in the clique as 0 to preserve the weight? It is due to the following reason. Suppose you did relabel another vertex p in the clique to 0.
    Now, we need to prove that the resulting label is a WRDF of size k.Basically, the only way this relabelling could result in breaking the WRDF condition is when there is another vertex(q) in the IS which is labelled 0 and this vertex(p) in the clique is one of the two vertices labelled 1 connected to vertex p in the IS.
    Now, when p is chosen, such a p is avoided when a different p that preserves the WRDF condition exists. However, there may be a chance that all other vertices in the clique follow the above condition.



Example Scenario

Assume that such a scenario was possible. This would imply that every other vertex in the IS was labelled 0 and had at least two neighbours labelled 1 in the clique such that none of those vertices in the clique actually covered the chosen vertex labelled 1 in the IS. We know that the vertex labelled 1 in the IS has two neighbours in the clique labelled 0 which are different from the vertices labelled 1 in the clique.
In this condition, we can see that the weight of the WRDF is actually less than k. Now, by the structure of the graph, we know that both the vertices labelled 0 must be present within the 2-R Set Cover of size k as the graph structure is a modelled 2-R Set Cover. So, this relabelling, though increases the weight by 1, would still ensure that weight lies below k.

Case 3: If the vertices in the Independent Set are labelled 0, then it adheres to the condition of the proof.

So, by considering cases 1,2 and 3, we can conclude that there exists a WRDF which has weight of at most k such that the vertices in the Independent Set are all labelled 0.

Due to Part I and Part II, Claim 5.3 is now proved. □

We have an WRDF of weight at most k where all labels > 0 are on the clique. Consider this labelling:

- By the structure of the graph, we know that ∃ k vertices which cover all the vertices in the IS at least twice. Label these vertices as 1.

- Rest of the vertices in the clique and all the vertices in the IS are labelled 0.

This labelling clearly shows a WRDF of weight at most k. The vertices labelled 1 correspond to the 2-R Set Cover. Claim 5.1 is now proved. □

Claim 5.4 The WRDF on a Split Graph is a W-hard problem

Proof. Due to 2-Redundant Set Cover being W-hard, and the weight of WRDFs on Split Graphs is a perfect function of k which is parameter preserving.So, WRDF on a Split Graph is W-hard. □

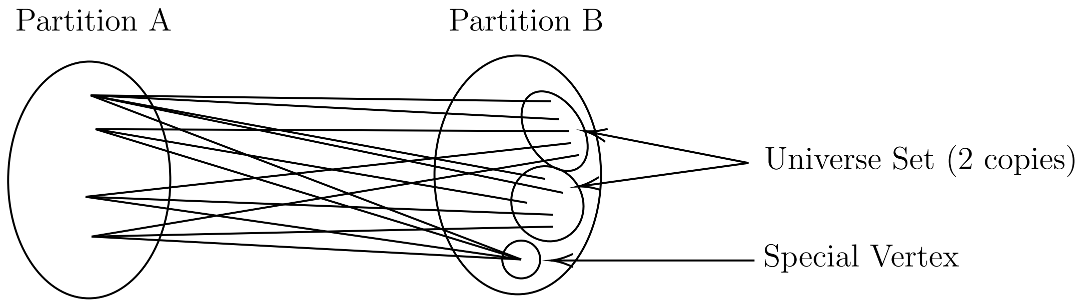
6 Parameter Preserving Reduction from Set Cover to Roman Domination on Bipartite Graphs

Construction:

Consider the following graph constructed from the properties of Set Cover. This graph will be a Bipartite Graph consisting of two partitions A and B.

- Every vertex in Partition A refers to a particular set in the collection S. The size of this partition will be m.
- There will be two vertices in Partition B that will correspond to the same element of the universe in the Set Cover. The will be an edge from these two vertices to all the vertices in Partition A that correspond to the particular families that cover them in the collection S. Both the vertices in Partition B will be exact copies of each other with regards to graph structure.
- There will be an extra special vertex in Partition B that will have an edge with all the vertices in partition A. The size of this partition will be $2n + 1$.

Thus, the set cover problem translates into the problem of finding a subset of vertices in Partition A (including the special vertex) that dominate all the elements in Partition B(excluding the special vertex).



Claim 6.1: In the given universe as well as the given collection, $\exists$ a Set Cover of size k if and only if in the above constructed Bipartite Graph, $\exists$ a Roman Domination of size at most $2(k + 1)$ wherein $\|V_1\|$ is minimized.

Proof. **Part I**:
Assume that $\exists$ a Set Cover of Partition B (excluding special element) consisting of k elements in Partition A.We need to prove that $\exists$ a RDF of size at most $2(k + 1)$ where $\|V_1\|$ is minimized.
Let us implement the following labelling:

- Let us label the vertices in Partition A which correspond to the set cover as 2.

- Let us label the special vertex in Partition B as 2.
- Let us label the rest of the vertices in both the Partitions as 0.

Proposition 6.2: The vertices labelled 0 present in Partition A have at least one neighbour labelled 2.

Proof. We know that at least one of the vertices is labelled 2 because the special vertex in Partition B is labelled 2. This special vertex is adjacent to all the vertices in Partition A which in turn is adjacent to all the vertices in Partition A that are labelled 0. □

Proposition 6.3: The vertices labelled 0 in Partition B have at least one neighbour labelled 2.

Proof. By definition, the Set Cover covers all the vertices in Partition B, so, if every vertex in the Set Cover is labelled 2, then every element in Partition B must have at least one vertex labelled 2 as its neighbour. □

With the above labelling for the vertices, along with **Proposition 6.2** and **Proposition 6.3**, we have proved that this labelling is a Roman Dominating Function. The weight of this function is $2(k+1)$.

Part II:
For the given graph, assume that $\exists$ a Roman Dominating Function of size at most $2(k+1)$, where $\|V_1\|$ is minimized. We need to prove that $\exists$ a Set Cover of size k for the above graph.

Assumption: It is assumed that $k < \frac{m}{2} - \frac{1}{2}$. This is to ensure that we find a better labelling than weight $m+1$ for the above graph.

Claim 6.4: If $\exists$ a RDF of size $2(k+1)$ in the above graph, then $\exists$ a RDF of size at most $2(k+1)$ such that all the vertices present in Partition B (excluding the special vertex) are labelled 0 and the special vertex is labelled 2.

Proof. The arguments follow a similar structure to the proof for split graphs, however in this case, we don't need to worry about vertices in Partition A not adhering to the rules of Roman Domination as the special vertex labelled 2 will ensure that vertices labelled 0 are dominated. If we label the special vertex as 2, we are able to dominate all the vertices in Partition A with a weight of 2, which is the minimum weight required to dominate them. Any other RDF will take up a weight of at least 2, ensuring that $\exists$ an RDF of weight at most $2(k+1)$ following the Claim 6.4 condition if $\exists$ an RDF for the above graph. Now, to prove the existence of such labelling for the rest of the vertices in Partition B, we need to consider 3 cases. Case 1: Suppose a vertex in Partition B is labelled 2.

- If this vertex is connected to a vertex labelled 2 in Partition A, then the vertex in Partition B can be labelled 0.
- This cannot be connected to a vertex labelled 1 due to Lemma 3.5.
- If this vertex is connected to a vertex labelled 0 in Partition A, then this vertex can be labelled 0 and the vertex in Partition A can be labelled 2.

As we can notice, the solution is similar to the case of Parameter Preserving Reduction for Split Graphs.
Case 2: Suppose a vertex in Partition B is labelled 1.

- It cannot be connected to a vertex labelled 2 due to Lemma 3.5.
- If it is connected to a vertex labelled 1 in Partition A, then this vertex can be labelled as 0 and the vertex in Partition A can be labelled as 2. This doesn't decrease the weight.
- If it is connected to a vertex labelled 0 in Partition A, then we have three situations. The vertex in Partition A must also be connected to the exact copy of this vertex and that exact copy can have 3 labels, $0, 1, 2$.
  - If the exact copy is labelled 2, then both copies in Partition B can be labelled as 0 and the vertex in Partition A can be labelled as 2. This would decrease the weight as well.
  - If the exact copy is labelled 1, then both copies in Partition B can be labelled as 0 and the vertex in Partition A can be labelled as 2. This would maintain the weight of the RDF.
  - When the exact copy is labelled 0, we need to provide a different argument. Now, we are given that the current labelling is an RDF. This ensures that every 0 is adjacent to at least one 2, which must include the 0 labelling of the copy vertex. This vertex labelled 2 should be different from the vertex labelled 0 in Partition A that we are considering and must be another vertex in Partition A. Since the copy vertex is covered by this 2, the considered vertex in this case should also be adjacent to this 2. It is essentially proven to be an instance of a different case. This ensures that we can relabel the 1 to 0 while ensuring that the function remains and RDF.

Case 3: If the vertices in Partition B (excluding the special vertex) is labelled 0, it already adheres to the condition of the proof.

So, by considering cases 1,2 and 3, we can conclude that $\exists$ a RDF which has weight at most $2(k+1)$ such that the vertices in Partition B are labelled 0 and the special vertex is labelled 2.

Now, we have an RDF where weight $2k$ worth of labels are on Partition A. Consider this labelling:

- By the structure of the graph, we know that $\exists k$ vertices which cover the vertices in Partition A. Label these vertices as 2.
- Rest of the vertices in Partition A will be labelled 0.

This labelling clearly shows an RDF of weight at most $2(k+1)$ where $\|V_1\|$ is minimized. The vertices labelled 2 in Partition A correspond to the Set Cover of size k. **Claim 6.4 is now proved.** □

Due to Part I and Part II, Claim 6.1 is now proved. □

Claim 6.5: The RDF on Bipartite Graph is a W-hard problem.

Proof. Set Cover is W-hard. Since we have proved Claim 6.1, we have a parameter preserving reduction for Roman Domination. The weight functions is a perfect function of k, specifically $2(k+1)$ where k is the size of the Set Cover. □