IMSc

Report

Proving W-hardness for variants of Dominating sets

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Contents

1 Aim

In this report, I will elaborate on proving W-hardness for Roman Domination as well as Weak Roman Domination for Split Graphs. I shall also prove W-hardness for Bipartite Graphs in Roman Domination

2 Definitions

Set Cover:

Given a set of elements $\{1, 2, ..., n\}$ (called the universe) and a collection S of m sets whose union equals the universe, the $SetCover$ problem is to identify the smallest sub-collection of S whose union equals the universe.

Roman Domination:

A *Roman dominating function (RDF)* on a graph $G = (V, E)$ is defined as a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The weight of a RDF is the value $f(V) = \sum_{v \in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$ is the minimum weight of all possible RDF's on a graph. A graph G is a *Roman* graph if $\gamma_R(G) = 2\gamma(G)$.

Weak Roman Domination:

Let V_0, V_1 and V_2 be the set of vertices assigned the values 0, 1 and 2 respectively, under f. There is a one to one correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, $f = (V_0, V_1, V_2)$.

A vertex $u \in V_0$ is *undefended*, if it is not adjacent to a vertex in V_1 or V_2. The function f is a *weak Roman Dominating function* (wRDF) if each vertex $u \in V_0$ is adjacent to a vertex $v \in V_1 \cup V_2$ such that the function $f' : V \to \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. The weight $w(f)$ of f is defined to be $\|V_1\| + 2\|V_2\|$. The *Weak Roman Dominating number*, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G.

3 Set Cover to Roman Domination on Split Graphs

Construction:

Consider the following graph constructed from the properties of Set Cover. This graph will be a split graph consisting of a clique and an independent set.

- Every vertex in the clique refers to a particular set in the collection S. The size of this clique will be m.
- There will be two vertices in the Independent Set that will correspond to the same element of the universe in the Set Cover. The will be an edge between these two vertices to all the vertices in the clique that correspond to the particular families that cover them in the collection S. Both the vertices in the Independent Set will be exact copies of each other with regards to graph structure. The size of this Independent Set will be $2n$.

Thus, the set cover problem translates into the problem of finding a subset of vertices in the clique that dominate all the elements in the independent set.

Claim 1 : In the given universe as well as the given collection, $\exists$ a set cover of size k if and only if in the above constructed split graph, $\exists$ a Roman Domination of size at most $2k$ wherein $\|V_1\|$ is minimized.

Part I:

Assume that $\exists$ a Set Cover of size k consisting of k elements in the clique.
Let us implement the following labelling:

- Let us label the vertices in the clique which correspond to the set cover as 2.
- Let us label the rest of the vertices in the clique and the entire independent set as 0.

Proposition A: The vertices labelled 0 present in the clique have at least one neighbour labelled 2.
Proof: We know that at least one of the vertices is labelled 2 because $\exists$ a Set Cover of size k where all its vertices are labelled 2 by our labelling. This ensures that every vertex labelled 0 is adjacent to at least one vertex labelled 2 due to the nature of a clique.
Proposition B: The vertices labelled 0 in the Independent Set have at least one neighbour labelled 2.
Proof: By definition, the Set Cover covers all the vertices in the Independent Set, so, if every vertex in the Set Cover is labelled 2, then every element in the Independent Set must have at least one vertex labelled 2 as its neighbour.

With the above labelling for the vertices, along with **Proposition A** and **Proposition B**, we have proved that this labelling is a Roman Dominating Function. The weight of this function is $2k$.

Part II:
For the given graph, assume that $\exists$ a Roman Dominating Function of size $2k$, where $\|V_1\|$ is minimized.
Claim 2: $\exists$ a Set Cover of size k for the above graph
Proposition C: There are no edges between vertices in V_1 and vertices in V_2 in the above graph.
Proof: If there did exist an edge between two vertices labelled 1 and 2, then I could relabel the vertex labelled in V_1 to 0, which would still result in a RDF but this would contradict the fact that $\|V_1\|$ is minimized. So, there are no edges between V_1 and V_2.
Remark: The relabelling of 1 would not break the conditions associated with RDF because vertices labelled 1 have no such conditions or restrictions in the definition of RDF.
Claim 3: If $\exists$ a RDF of size $2k$ in the above graph, then $\exists$ a RDF of size $2k$ such that all the vertices present in the Independent Set are labelled 0
Case 1: Suppose the vertices in the Independent Set are labelled 2

- If a particular vertex labelled 2 in the IS is connected to another vertex in the clique labelled 2, then the vertex labelled 2 in the IS (as well as its exact copy) can be relabelled to 0 and this would still result in an RDF wherein the weight will not increase. This is because the 0's in the IS will have that particular vertex labelled 2 in the clique adjacent to it, thus maintaining the status of the RDF as well as not increasing in weight.
- No vertex labelled 2 in the IS is connected to a vertex in the clique labelled 1 due to **Proposition C**.
- If a particular vertex labelled 2 in the IS is connected to another vertex in the clique labelled 0, then we can relabel the vertex in the IS (as well as its exact copy) to 0, and relabel that particular vertex in the clique as 2. This vertex will be adjacent to both the 0's created due to this relabelling, ensuring that this labelling is an RDF. The weight of this RDF also cannot increase due to this relabelling.

Case 2: Suppose the vertices in the Independent Set are labelled 1.

- No vertex labelled 1 in the IS is connected to a vertex in the clique labelled 2 due to **Proposition C**.

- If a particular vertex labelled 1 in the IS is connected to another vertex labelled 1 in the clique, then we can relabel the vertex in the IS to 0 (as well as its exact copy) and relabel that particular vertex in the clique to 2. The 0's created due to this relabelling will be adjacent to the newly relabelled 2, ensuring that this is an RDF. The weight of this RDF also will not increase.
- If a particular vertex labelled 1 in the IS is connected to another vertex labelled 0 in the clique, then we have three different situations.The exact copy of the vertex can have three different labels 0,1, and 2.
  - When the copy vertex has label 2, then both the vertex and its copy in the IS can be relabelled to 0 and the vertex labelled 0 in the clique can be relabelled as 2. The weight decreases and the resultant labelling is still an RDF as the 2 in the clique ensures that the 0's created in the IS due to the new labelling is covered.
  - When the copy vertex is labelled 1, a similar argument is made as the case where it is labelled 2, except this time the weight is the same, which still doesn't break the rules of RDF.
  - When the copy vertex is labelled 0, then we need to provide a different argument. Now, we are given that the current labelling is an RDF. This ensures that every 0 is adjacent to atleast one 2, which must include the 0 present in the copy vertex. This should be different from the vertex labelled 0 in the clique and must be another vertex in the clique which is labelled 2. Since the current copy vertex is covered by this 2, the considered vertex in the IS should also be adjacent to this 2. This ensures that we can relabel the 1 to 0 while ensuring that the function remains and RDF. The weight will decrease in this case.

Case 3: If the vertices in the Independent Set are labelled 0, then it adheres to the condition of the proof.

So, by considering cases 1,2 and 3, we can conclude that there exists a RDF which has weight of at most $2k$ such that the vertices in the Independent Set are all labelled 0. Claim 2 is now proved.

Claim 4: We have an RDF of weight at most $2k$ when we label the k vertices in the clique, which "cover" all the vertices in the Independent Set, as 2 and the rest of the vertices as 0

Proof: By the structure of the graph, we know that $\exists$ k vertices which "cover" all the vertices in the IS and once we label these vertices as 2 and the rest of the vertices as 0, then we have an RDF of weight $2k$ and all the 0's in the clique are adjacent to the 2's in the clique.Claim 3 is now proved.

Due to Part I and Part II, Claim 1 is now proved.

Corollary: The RDF on a Split Graph is a W-hard problem
Proof: Due to Set Cover being W-hard, and the weight of RDFs on Split Graphs is a perfect function of k, specifically $2k$, RDF on a Split Graph is W-hard.

4 2-Redundant Set Cover to Weak Roman Domination on Split Graphs

Construction:

Consider the following graph constructed from the properties of Set Cover. This graph will be a split graph consisting of a clique and an independent set.

- Every vertex in the clique refers to a particular set in the collection S. The size of this clique will be m.
- There will be one vertex in the Independent Set that will correspond to a particular element of the universe in the Set Cover. The will be an edge between this vertex and all the vertices in the clique that correspond to the particular families that cover this vertex in the collection S. The size of this Independent Set will be n.

Thus, the 2-Redundant Set Cover problem translates into the problem of finding a subset of vertices in the clique that dominate all the elements in the independent set exactly twice.

Claim 5: In the given universe as well as the given collection, $\exists$ a 2-Redundant Set Cover of size k if and only if in the above constructed split graph, $\exists$ a Weak Roman Domination of size at most k wherein $\|V_1\|$ is minimized.
Part I:
Assume that $\exists$ a 2-Redundant Set Cover of size k consisting of k elements in the clique.
Let us implement the following labelling:

- Let us label the vertices in the clique which correspond to the set cover as 1.
- Let us label the rest of the vertices in the clique and the entire independent set as 0.

The total weight of this labelling is k.
Proposition A2: The vertices labelled 0 present in the clique follow the rules present for Weak Roman Domination.
Proof: We know that $\exists$ at least k vertices labelled 1 in the clique. So, every 0 in the clique is adjacent to at least 1 1.Also, if a 1 becomes a 0 and another 0 becomes a 1, there are still k other 1's in the clique which will ensure that they do not remain undefended.
Proposition B2: The vertices labelled 0 in the Independent Set follow the rules of

Weak Roman Domination.
Proof: By definition, the 2-Redundant Set Cover covers all the vertices in the Independent Set twice, so, if every vertex in the Set Cover is labelled 1, then every element in the Independent Set must have at least one vertex labelled 1 as its neighbour. Now, for each vertex labelled 0 in the IS, there are two vertices labelled 1 as its neighbours. So, this ensures that this vertex follows the 3rd rule of Weak Roman Domination.

With the above labelling for the vertices, along with **Proposition A2** and **Proposition B2**, we have proved that this labelling is a Weak Roman Dominating Function. The weight of this function is k.

Part II: For the given graph, let $\exists$ a Weak Roman Domination Function of weight k.
Claim 6: If $\exists$ a Weak Roman Domination of weight k in the given graph, then $\exists$ a Weak Roman Domination of weight k in the given graph such that all the elements in the Independent Set are labelled 0.
Proof: Case 1: Suppose $\exists$ vertices labelled 2 in the IS:

- If a particular vertex labelled 2 in the IS is connected to a vertex labelled 2 in the clique, then the vertex in the IS can be relabelled as 0 and this would adhere to the rules of Weak Roman Domination.
- If a particular vertex labelled 2 in the IS is connected to a vertex labelled 1 in the clique, then there are further three more cases to consider. Since we know that for each vertex in the IS, at least two vertices in the clique are adjacent to each vertex in the IS. The second vertex has 3 possibilities, $0, 1, 2$.
  - If the second vertex in the clique is labelled 2, then the vertex in the IS can be labelled to 0 and would still adhere to the rules of Weak Roman Domination.
  - If the second vertex in the clique is labelled 1, then we have two vertices labelled 1 in the clique which ensures that if we relabel the vertex adjacent to them in the IS as 0, we will adhere to the rules of WRD.
  - If the second vertex in the clique is labelled 0, then we can relabel the vertex in the IS from 2 to 0 and label the second vertex in the clique as 1, resulting in two 1's adjacent to this vertex, which ensures that it adheres to WRD rules.
- If a particular vertex labelled 2 in the IS is connected to a vertex labelled 0 in the clique, then there are further three more cases to consider. Since we know

that for each vertex in the IS, at least two vertices in the clique are adjacent to each vertex in the IS. The second vertex has 3 possibilities, $0, 1, 2$.

- If the second vertex in the clique is labelled 2, then the vertex in the IS can be labelled to 0 and would still adhere to the rules of WRD.
- If the second vertex in the clique is labelled 1, then we must relabel the vertex in the IS to 0, and relabel the first vertex in the clique to 1, which will ensure that two vertices are adjacent to the 0 in the IS, which would adhere to the rules of WRD.
- If the second vertex in the clique is labelled 0, we must use the reasoning that the weight on the clique is at least 1, which would ensure that we can swap the 0 in the second vertex with this 1, resulting in the case above. Work to be done here.

Case 3: If the vertices in the Independent Set are labelled 0, then it adheres to the condition of the proof.

So, by considering cases 1,2 and 3, we can conclude that there exists a WRDF which has weight of at most k such that the vertices in the Independent Set are all labelled 0.

Due to Part I and Part II, Claim 5 is now proved.

Claim 7: We have an RDF of weight at most k when we label the k vertices in the clique, which "cover" all the vertices in the Independent Set exactly twice, as 1 and the rest of the vertices as 0

Proof: By the structure of the graph, we know that $\exists$ k vertices which "cover" all the vertices in the IS twice. We know by **Claim 5** that there $\exists$ a WRDF such that the weight is at most k and the vertices in the IS are labelled 0. So, let us label the vertices considered above with 1 and the vertices in the clique other than the 2-Redundant Set Cover, we can label them as 0. They will be adjacent to at least 2 1's in the clique, ensuring that they adhere to the rules of Weak Roman Domination.

Corollary: The WRDF on a Split Graph is a W-hard problem

Proof: Due to 2-Redundant Set Cover being W-hard, and the weight of WRDFs on Split Graphs is a perfect function of k, specifically k, WRDF on a Split Graph is W-hard.

5 Set Cover to Roman Domination on Bipartite Graphs

Construction:

Consider the following graph constructed from the properties of Set Cover. This graph will be a Bipartite Graph consisting of two partitions A and B.

- Every vertex in Partition A refers to a particular set in the collection S. The size of this partition will be m.
- There will be two vertices in Partition B that will correspond to the same element of the universe in the Set Cover. The will be an edge from these two vertices to all the vertices in Partition A that correspond to the particular families that cover them in the collection S. Both the vertices in Partition B will be exact copies of each other with regards to graph structure. The size of this partition will be $2n$.
- There will be an extra special vertex in Partition B that will have an edge with all the vertices in partition A.

Thus, the set cover problem translates into the problem of finding a subset of vertices in Partition A (including the special vertex) that dominate all the elements in Partition B(excluding the special vertex).

Claim 8: In the given universe as well as the given collection, $\exists$ a Set Cover of size k if and only if in the above constructed Bipartite Graph, $\exists$ a Roman Domination of size at most $2(k+1)$ wherein $\|V_1\|$ is minimized.
Part I:
Assume that $\exists$ a Set Cover of Partition B (excluding special element) of size k consisting of k elements in Partition A.
Let us implement the following labelling:

- Let us label the vertices in Partition A which correspond to the set cover as 2.
- Let us label the special vertex in Partition B as 2.
- Let us label the rest of the vertices in both the Partitions as 0.
- All the above labelling will also apply to the vertices' exact copies

Proposition A: The vertices labelled 0 present in Partition A have at least one neighbour labelled 2.
Proof: We know that at least one of the vertices is labelled 2 because the special vertex in Partition B is labelled 2. This special vertex is adjacent to all the vertices

in Partition A which in turn is adjacent to all the vertices in Partition A that are labelled 0.
Proposition B: The vertices labelled 0 in Partition B have at least one neighbour labelled 2.
Proof: By definition, the Set Cover covers all the vertices in Partition B, so, if every vertex in the Set Cover is labelled 2, then every element in Partition B must have at least one vertex labelled 2 as its neighbour.

With the above labelling for the vertices, along with **Proposition A** and **Proposition B**, we have proved that this labelling is a Roman Dominating Function.
The weight of this function is $2(k+1)$.

Part II:
For the given graph, assume that $\exists$ a Roman Dominating Function of size $2(k+1)$, where $\|V_1\|$ is minimized.
Claim 2: $\exists$ a Set Cover of size k for the above graph
Claim 3: If $\exists$ a RDF of size $2(k+1)$ in the above graph, then $\exists$ a RDF of size $2(k+1)$ such that all the vertices present in Partition B (excluding the special vertex) are labelled 0
Proof: The arguments follow a similar structure to the proof for split graphs.
Assumption: It is assumed that $k < \frac{m}{2} - \frac{1}{2}$ where $m+1$. This is to ensure that we find a better labelling that $m+1$
Proposition D: The special vertex in **Partition B** must be labelled 2 in the given RDF.