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| RSI 2016 PROJECT REPORT |
| Fascination with numbers |
| Rationality and Irrationality |
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| **Ram Srinivasa Bharathy ( PS Sr Sec School, Chennai ) & Varshini Balaji (DAV Mogappair, Chennai)** |
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| Abstract : To understand what rational, irrational and transcendental numbers are and understand their various properties |

Mentor: Professor T E Venkatabalaji – Department of Mathematics

Indian Institute of Technology Madras, Chennai, India 600 020

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References

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Definitions:

Definition of rational numbers:

Rational numbers are real numbers of the form p/q, such that p and q are integers and q not equal to 0.

Definition of irrational numbers:

Irrational numbers are real numbers that cannot be expressed in the form p/q, where p and q are integers and q not equal to 0.

I (set of irrational numbers) =

Definition of real numbers:

Real numbers are defined in two ways.

1. Through Cauchy Sequences
2. Through Dedekind cuts

The Dedekind method of defining Real Numbers has been discussed.

Dedekind cuts:

A Dedekind cut is basically a partition of an ordered field, (A,B) such that A and B are both non-empty and A is closed upwards and B is closed downwards and also A contains no greatest element. Real numbers are constructed by this way.

The general definition starts when we suppose that there is a way to divide the set of all rational numbers into 2 classes A and B such that every element *b* of class B is greater than every element *a* of class A.

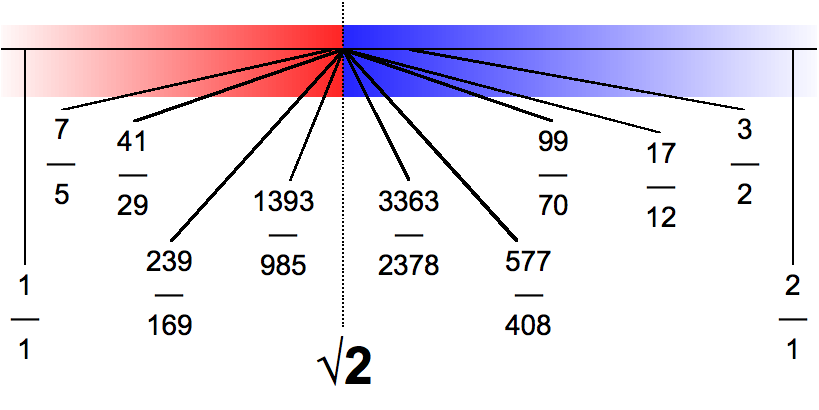
There are 2 possibilities:

* Set A has the largest element or Set B has the smallest element. This is a point where we are actually cutting at a rational number which is getting included in one of the sets.
* The other possibility is that we cut at a point where set A and set B together form the rational numbers but the point where we cut cannot be placed within any one of the sets.

This point where we have made the cut, which does not lie with the set of rational numbers is defined to be an irrational number.

For Example:

Let us say we divide the set of all rational numbers into two sets A and B such that A contains all the negative rational numbers, 0 and all positive rational numbers less than the square root of 2 and B contains all the positive rational numbers >



Now A U B, contains all the rational numbers however the point where we made the cut () is simply defined to be an irrational number.

This is the basic idea behind the Dedekind way of defining Real numbers.

A few interesting proofs were discussed such as:

* Every real number has a decimal expansion.
* Fundamental theorem of Arithmetic – Prime Factorization
* Proof that golden number is irrational and algebraic

Definitions:

Algebraic numbers:

The set of all numbers that are roots of any polynomial expression are called algebraic numbers.

The technical definition is “An Algebraic number is any complex number that is a root of a non-zero polynomial in one variable with rational coefficients. “

We will be dealing with Real Algebraic numbers for the time being.

Proofs for learning properties of the various types of numbers discussed above:

Theorem: Set of all rational numbers are countable

Proof:

To prove that all rational numbers are countable, we are going to write a series which will list out all the rational numbers

Every rational number will appear somewhere within this list (pattern), so rational numbers are countable.

Aliter:

A more illustrative proof can proceed like this.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1/1 | 1/2 | 1/3 | ¼ | 1/5 | 1/6 | 1/7 | 1/8 | 1/9 |
| 2 | 2/1 | 2/2 | 2/3 | 2/4 | 2/5 | 2/6 | 2/7 | 2/8 | 2/9 |
| 3 | 3/1 | 3/2 | 3/3 | ¾ | 3/5 | 3/6 | 3/7 | 3/8 | 3/9 |
| 4 | 4/1 | 4/2 | 4/3 | 4/4 | 4/5 | 4/6 | 4/7 | 4/8 | 4/9 |
| ….  …  …  … | 5/1  ..  ..  .. | 5/2  …  …  … | 5/3  …  …  … | 5/4  …  … | 5/5  ..  .. | 5/6  ..  .. | 5/7  ..  .. | 5/8  ..  .. | 5/9  ..  .. |

This table will clearly list all the rational numbers from 0 to infinity. So when we create a similar table for the negative rational numbers and add a 0, we would have listed all the rational numbers that exist.

Since this is a countably infinite set its cardinality is that of natural numbers which is (Aleph Null).

Theorem: Real numbers are uncountable

Proof:

Let us assume by way of contradiction that Real numbers are countable.

Now, let us make a bijection to a set containing infinite number of infinite binary sequences.

So, any sequence of infinite binary digits must correspond to a real number.

S=

{

0010010010…

10010001001…

00100100101…

…

...

}

Now, if we take 1st element in 1st series and take its complementary digit. Similarly take the 2nd element in 2nd series and take its complementary digit and so on until you create a new sequence of infinite binary digits.

New sequence is 110…

This series does not exist in S but since R contains all real numbers, this series must correspond to a real number in R implying that it exists in S.

This poses a contradiction and implies that Real numbers are uncountable.

This is known as Cantor’s diagonal argument.

Theorem: All algebraic numbers are countable

Proof:

Now, to prove that all algebraic numbers are countable, we need to show that the polynomials are countable. If polynomials are countable, then it would follow that algebraic numbers are countable as they are the roots of the polynomial.

Let us create a set

P(n) =

Let n

Now, if we take a union of these countable sets P(0), P(1), P(2), P(3)…

From the theory that the union of countable sets is countable, we will get a countable set.

This implies that all algebraic numbers are countable.

Theorem: The square root of a prime number is an irrational number

Proof: We will be using a proof by contradiction.

So, we first assume that the square root of a prime number is **rational**.

Therefore, Let us take an arbitrary prime number say **p** whose square root is assumed to be RATIONAL.

Then , where a, b ,and a, b are co-primes, b0.

Now, p= , (squaring on both sides)

p=

Therefore, p is a factor of .

Then =( - (factorization)

Then p should be one among all the factors, as it is a prime number. Then let us say p= , then it becomes obvious that p is also a factor of a.

Then a= pk,(where k is some integer)

Then, p)2, b2= pk2,

then p is a factor of b2, then p is also a factor of b,

Therefore, p is a common factor for both a, b which means that they can’t be co-primes. Therefore, it leads to a contradiction.

Hence, our assumption that p is rational is wrong, thus it is irrational.

Theorem: ‘e’ is irrational

Proof:

Let us assume by way of contradiction that e is rational. i.e. e=p/q where p and q are integers, co-primes.

We know from the Taylor series expansion that

= > e

= > <

The final expression is an infinite GP which has a sum.

= >

= > 0 < e Multiply by n!

= > 0 < n!( e) <

Now, for any n>q, certainly n>1 so

= > 0 < n!( e) < < 1

So, we can deduce that n!( e) is not an integer.

By assumption, we know that e is a rational number p/q .

= > n!( e) = n!(

Now, n > q = > q divides n! ;1

Also in the second term

n! each and every term in the expansion of this expression is an integer. ;2

From ;1 and ;2 we get that

n!( e) is an integer

But we have already proved that n!( e) is not an integer.

So, by way of contradiction, our initial assumption was wrong

= > e is irrational

Definitions:

Transcendental Numbers:

Transcendental numbers are defined as numbers that are non-algebraic.

Examples: Liouville numbers, e,

Theorem: Transcendental numbers exist

Proof:

Real numbers are uncountable.

Rational numbers are countable.

= > Transcendental numbers exist.

Now, we shall investigate a special set of numbers called as Liouville numbers.

Liouville number:

A number is defined to be a Liouville number if it is an irrational number and obeys the following condition:

Theorem:Liouville numbers are transcendental

Lemma 1:

With Then, there is a constant A = > 0 such that if a,b and b > 0 then

Proof for Lemma:

Let M be the maximum value of on [].

Let be the distinct roots of f(x) which are different from

Fix A<min{1,}

Assume that the lemma is incorrect for some integers a,b where b>0.

Then

< A < min{1,}

=>

a,b and a,b{a1 , a2 …am}

Now, according to the Mean Value theorem,

There is an x0 between a/b and such that

= >

Now, = 0 and So,

Also,

, Then multiply and divide by

= >

Substituting this in the Mean Value expression we get

Now, A = 1/M, then we get

Which is what we assumed to be false.

This gives rise to a contradiction, and thus implies that the initial assumption was incorrect

= >

Note:

What this lemma is proving is that the error between the number and the approximation has to be greater than some limit if we place a restriction on the limit on the error scale like 1/where n is fixed.

Now that the lemma has been proved, we can get to the proof

Proof:

Firstly, we need to prove that Liouville numbers are irrational as the lemma works only for irrational numbers.

So, Let us assume that is rational and is = where

c,d , Let n be a positive integer such that , then

(We can remove the mod for b and d as both are > 0,

also bc-ad always as c/ba/b)

Now

So, we have proved that

which goes against the definition of a Liouville number,

= > that is an irrational number

Note:

The reason we need to prove that is an irrational number

is that the lemma holds only for irrational numbers so in order to use the lemma, we need to prove its irrationality.

The next part is to prove that the number is non-algebraic because if we do so, then the number has to be transcendental.

To prove, let us assume that is an algebraic number and is the root of a polynomial .

Then according to the lemma there exists an A such that the lemma holds. Let r be a positive rational number such that

(because A<1)

Then there are numbers a and b such that they are integers and b>0 such that

but this goes against the lemma, which implies that is not algebraic.

If a number is not algebraic, then the number is transcendental.

**This proves that all Liouville numbers are transcendental.**

Theorem: is a Liouville number.

Let

Let a/b be equal to where n is a fixed positive integer

Now,

Here is where there is a leap in logic as we declare

This becomes obvious as the expression on the right is a GP (Geometric Progression) whereas the term on the left is not as regular.

(This follows from the sum of infinite terms of a diminishing GP)

Thus we have proved that

Which is the definition of liouville numbers, thus

**is a Liouville number. (and hence transcendental)**

Note:

* It is not necessary that 2 needs to be the base at the denominator as you can see that the proof does not depend on 2.
* We can do a similar proof for where p is any positive integer.
* We also discussed about the measure of the set of Liouville numbers.
* It turns out that the measure is 0 but the set is dense and there are an infinite number of Liouville numbers.

Theorem: e is transcendental

Proof:

Suppose there were integers with such that

Now, we will define numbers M, in the following way

The p in the expression represents a prime number

Note:

The big number within the box when expanded becomes This is a polynomial with integer coefficients which has been raised to the power ‘p’

When we focus on M, we can apply the integral to each term and add them up

=>

where the are certain integers, and = ±(n!) p.

Now, the integral is in the form of the formula

Thus, simplifying the above expression using this formula we get

When we obtain the term

So, when p>n, this term is an integer not divisible by p.

When

Which is divisible by p. (when adding all these numbers, the result will not be divisible by p)

So, M is an integer which is not divisible by p.

Now, let us take

Now, let us transform this expression into something similar to the previous simplification by taking

u=x-k and du=dx

So,

The importance of the expression in the numerator is that it has the factor u in the kth place, so the pth factor contains the term



This is a polynomial with integer coefficients, every term of which has at least a degree at least p. Thus, when we expand the numerator and simplify by taking the coefficients common and expressing in terms of summation for every term of the polynomial we get

Using the k! integral formula we get the expression in the Right Hand Side.

Now, since starts with 1, all the terms are divisible by p.

k=1,2,…n

This relation can be derived from the definitions.

Note:

In fact, the definitions were made in such a way that this relation could be defined.

Substituting this power of e into the polynomial and multiply by M, we get

[]+[] = 0

When we require p > n, and also stipulating that p > , we know that is not divisible by p. Since is divisible by p, the sum of M and all of these terms will not be divisible by p. This essentially means that it is a non-zero integer.

So in order to prove e is transcendental we can prove that

[]

Can be made as small as we want (positive or negative), which would mean that the sum could never be equal to 0 implying that e is not algebraic.

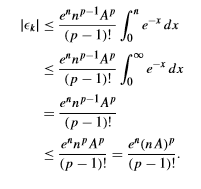
To do this, we need to prove that can be made as small as possible by choosing a p large enough.

Now, the point to remember is that n is fixed (degree of the polynomial) and that 1 then

We can see that this inequality holds and now let us maximize

A = maximum of  for x in [0,n].

Then,



Now, n and A are fixed => that can be made as small as desired by making p sufficiently large.

So, then can be as small => that the other non-zero integer

() term cannot be cancelled and the whole expression cannot be equated to 0.

**Thus, e cannot be algebraic => that e is transcendental**

Discussions of topics in Number Theory:

We discussed various topics and proofs from different parts of mathematics relating to number theory.

Complex logarithms:

We discussed how to apply logarithms to complex numbers.

This question arose when we tried to find out how to handle transcendental numbers when they are complex.

If they were powers of e, then we saw that applying logarithms would be useful.

As an exercise, we evaluated the value of and found to our surprise that it was a real number.

Using

Now, can be visualized as

From complex logarithms we got a formula which is

Log(Z) = Log(r) + i

For apply the formula to the log at the power of e

You get which is a Real number and has also been proven to be transcendental.

Harmonic Series and the Zeta function:

We investigated the Taylor series and had a look into the Zeta function.

We discussed various proofs that caught our interest such as

* diverges and tends to infinity whereas where a can be as small as you want but a> 0 is a converging series and has a limit.
* but not proved to be transcendental
* , , , , in this list, one of them is transcendental
* The list of all contains infinitely many irrational numbers as their limits.

Other than these proofs, we have absolutely no idea about the zeta function and that is what makes it so interesting. They are very closely related to the primes. That can be seen in Euler’s relation.

* =

Continuum Hypothesis and Set theory:

We understood what a Hypothesis was and also understood a few examples such as the Continuum hypothesis and the Reimann Hypothesis.

The Continuum Hypothesis states that there is no set whose cardinality lies between that of integers and the set of Real numbers.

We also had a look into Zermelo-Fraenkel set theory and tried to understand how those basic set of axioms could theoretically build up the whole of mathematics.

Formula for :

When trying to understand more about the properties of we were able to derive a formula for it from trigonometry which is actually an infinite sum.

Now, tan ()=1 = >

= > 4

Assume that the principal tan curve is divided into n parts where n is a positive integer.

= 4

=

Normal numbers:

We also had a very brief look into normal numbers, more specifically just knowing what they are and a few examples.

A normal number is defined as an irrational number for which any finite pattern of numbers occurs with the expected limiting frequency in the expansion in a given base (or all bases)

For example, for a normal decimal number, each digit 0-9 would be expected to occur 1/10 of the time, each pair of digits 00-99 would be expected to occur 1/100 of the time.

Also, a number that is normal in every base b=1,2,3… is said to be absolutely normal.

Conclusions:

* We clearly defined Rational, Irrational, Algebraic and Transcendental numbers
* We proved irrationality of a root of prime number and irrationality of e
* We defined Liouville numbers and proved them to be Transcendental
* We proved that e is transcendental
* We looked into various concepts in Number theory such as Zeta function, Complex Logarithms, and Normal numbers to name a few.

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This book contains the proof for’ e is transcendental’ along with many other important ideas in calculus.

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