DIGITAL SIGNAL PROCESSING

Second Edition

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Chennai

Tata McGraw Hill Education Private Limited
NEW DELHI

McGraw-Hill Offices
New Delhi New York St Louis San Francisco Auckland Bogotá Caracas
Kuala Lumpur Lisbon London Madrid Mexico City Milan Montreal
San Juan Santiago Singapore Sydney Tokyo Toronto

Tata McGraw-Hill

Published by the Tata McGraw Hill Education Private Limited,
7 West Patel Nagar, New Delhi 110 008.

ISBN-13: 9780070086654
ISBN-10: 0070086656

Vice President and Managing Director— *Ajay Shukla*

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Production Manager: *Reji Kumar*

Typeset at Tej Composers, WZ-391, Madipur, New Delhi 110063, and printed at Rajkamal Electric Press, Plot No. 2, Phase IV, HSIIDC, Kundli, Sonepat, Haryana 131 028

Cover printed at: Rajkamal

The McGraw·Hill Companies

Dedicated to

My sister	**:**	*Mrs. A. Rajina Bivi, MA, B.Ed*
Brother-in-law	**:**	*Dr. A. Kalilur Rahman, MBBS, MS(ORTHO)*
Their daughter	**:**	*Er. K. Shajina, BE*
Their son	**:**	*K. Shafiq, (MBBS)*

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Preface

The main objective of this book is to explore the basic concepts of digital signal processing in a simple and easy-to-understand manner.

This text on digital signal processing has been crafted and designed to meet student's requirements. Considering the highly mathematical nature of this subject, more emphasis has been given on the problem- solving methodology. Considerable effort has been made to elucidate mathematical derivations in a step-by-step manner. Exercise problems with varied difficulty levels are given in the text to help students get an intuitive grasp on the subject.

This book with its lucid writing style and germane pedagogical features will prove to be a master text for engineering students and practitioners.

Salient Features

The salient features of this book on Digital Signal Processing are,

- proof of properties of transforms are clearly highlighted by shaded boxes
- wherever required, problems are solved in multiple methods
- additional explanations for solutions and proofs are provided in separate boxes
- different types of fonts are used for text, proof and solved problems for better clarity
- keywords are highlighted by bold, italic fonts

Organization

In this book, the concepts of discrete time signals and their transforms are organized in four chapters and two chapters are devoted for digital filter design. One chapter is devoted to each topic in digital signal processing like finite word length effects, mutirate DSP, spectrum analysis, digital signal processors and applications of DSP. Each chapter provides the foundations and practical implications with a large number of solved numerical examples for better understanding.

The important concepts are summarized at the end of each chapter which can help in quick reference. Another significant aspect of this book is MATLAB based computer exercises with complete explanations given in each chapter. This will be of great assistance to both instructors and students.

Chapter 1 deals with a general introduction about various aspects of digital signal processing and its importance in real life. Basic definitions of discrete time signals and systems, mathematical representation of discrete time systems and significance of time and frequency domain analysis are presented in brief. Introduction to various topics of digital signal processing like FIR filters, IIR filters, finite word length effects, multirate DSP, power spectrum, digital signal processors, applications of digital signal processing and usage of MATLAB in this course are also presented in a brief manner.

Chapter 2 is devoted to concepts of discrete time signals and systems and is more concerned with generation, representation, classification, mathematical operations of discrete time signals and systems, block diagram and signal flow graph notations.

The chapter also presents the methods of obtaining responses of LTI discrete time systems and various convolution methods. The deconvolution, correlation techniques and the inverse systems are clearly explained with solved numericals. In addition, the concept of sampling and its importance are dealt with briefly.

Chapter 3 explains $\mathcal{Z}$-transform and its application to discrete time signals and systems. All the important properties of $\mathcal{Z}$-transform are presented explicitly. Inverse $\mathcal{Z}$-transform and solution of difference equations describing the discrete time systems are demonstrated with numerical examples. Also, the structures for realization of IIR and FIR systems are provided.

Chapter 4 is dedicated to discrete time Fourier series and Fourier transform which forms the basics for frequency domain analysis of discrete time signals and systems. In the first half of this chapter, the discrete time Fourier series and the frequency spectrum using discrete time Fourier series are discussed with relevant examples.

The second half of the chapter details the development of discrete time Fourier transform from discrete time Fourier series, frequency spectrum, various properties of Fourier transform, and Fourier transform of some standard discrete time signals. In addition, the computation of frequency responses of LTI discrete time systems using Fourier transform are also explained with examples. The relation between Fourier transform and $\mathcal{Z}$-transform of discrete time signals is also discussed in the chapter.

Chapter 5 extends the understanding of the concepts of Discrete time Fourier transform(DTFT) to DFT (Discrete Fourier transform) and FFT (Fast Fourier Transform). Development of DFT from DTFT, properties of DFT, relation between DFT and $\mathcal{Z}$-transform, analysis of the LTI systems using DFT and FFT are extensively discussed.

Chapter 6 focuses on frequency response of FIR filters and characteristics various windows used for FIR filter design. Also, design of linear phase FIR filters by windowing and frequency sampling techniques are presented with suitable examples.

Chapter 7 explains the techniques for transforming analog filter to digital filter and the characteristics of analog Butterworth and Chebyshev filters. Also, design of Butterworth and Chebyshev digital IIR filters are presented with examples.

Chapter 8 discusses the quantization and representation of digital/binary number systems. The effects due to finite precision of filter coefficients and products, and various types of overflow in recursive computations are also discussed with appropriate examples.

Chapter 9 focuses on sampling rate conversion by decimation and interpolation and their effects on frequency spectrum. Implementation of sampling rate conversion in filters and application of multirate digital signal processing are also discussed in the chapter.

Chapter 10 is concerned with the estimation of energy spectrum of discrete time signals and power spectrum of random process. The various nonparametric methods power spectrum estimation and their performance characteristics are presented.

Chapter 11 focuses on architecture and programming of special purpose processors for digital signal processing with particular concentration to Texas Instruments digital signal processors TMS320C5x and TMS320C54x processors.

Chapter 12 provide a brief discussion on some applications of digital signal processing in speech, musical sound, audio/video, communication and biomedical signals.

The author has taken care to present the concepts of Digital Signal Processing in a simple manner and hope that the teaching and student community will welcome the book. The readers can feel free to convey their criticism and suggestions to **kani@vsnl.com** for further improvement of the book.

A.Nagoor Kani

Acknowledgements

I express my heartful thanks to my wife Ms.C. Gnanaparanjothi Nagoor Kani and my sons N. Bharath Raj alias Chandrakani Allaudeen and N.Vikram Raj for the support, encouragement and cooperation they have extended to me throughout my career.

It is my pleasure to acknowledge the contributions to our technical editors Ms.K.Jayashree, Ms. B.Hemavathy, Ms. S. Pavithra for editing and proofreading of the manuscript, and Ms. A. Selvi, Ms. M. Faritha for type setting and preparing the layout of the book.

My sincere thanks to all reviewers for their valuable suggestions and comments which helps me to explore the subject to greater depth.

Prateek Kumar	Maharana Pratap Engineering College Kanpur, Uttar Pradesh.
Ashish Suri	Shri Mata Vaishno Devi University Jammu, Jammu and Kashmir.
S.S Prasad	National Institute of Technology (NIT) Jamshedpur, Jharkhand.
Harpal Theti	Kalinga Institute of Industrial Technology, Bhubaneswar, Orissa.
Kishor Kinage	D J Sanghvi Engineering College, Mumbai.
S. Moorthi	National Insitute of Technology (NIT), Tiruchirapalli, Tamil Nadu
S. Anand	MEPCO SCHLENK Engineering Colledge, Sivakasi, Tamil Nadu.
R. Prakash	School of Electronics Sciences, Vellore Institute of Technology, Vellore, Tamil Nadu.
J. Vijayraghavan	Rajalakshmi Engineering College, Chennai.
Jagadeshwar Reddy	Sri Venkateswara Institute of Science and Technology, Kadapa, Andhra Pradesh.
P. Biswagar	R V College of Engineering, Bangalore, Karnataka.

I am also grateful to Ms.Vibha Mahajan, Mr.Ebi John, Ms. Koyel Ghosh, Mr. P.L.Pandita and Ms. Sohini Mukherjee of Tata McGraw Hill Education for their concern and care in publishing this work.

My special thanks to Ms. Koyel Ghosh of McGraw Hill Education for her care in bringing out this work at the right time.

I thank all my office staff for their cooperation in carrying out my day-to-day activities.

Finally, a special note of appreciation is due to my sisters, brothers, relatives, friends, students and the entire teaching community for their overwhelming support and encouragement to my writing.

A. Nagoor Kani

List of Symbols and Abbreviations

Symbols

A	-	Number of integer digit
A_s	-	Gain at stopband edge frequency
A_p	-	Gain at passband edge frequency
B	-	Bandwidth in Hz
b	-	Size of binary excluding sign bit
c_k	-	Fourier coefficients of exponential form of Fourier series of $x(t)$
D	-	Sampling rate reduction factor
E	-	Energy of a signal
E_r	-	Relative error due to rounding
E_t	-	Relative error due to truncation
e_r	-	Rounding error
f	-	Frequency of discrete time signal (or digital frequency) in cycles/sample
F	-	Frequency of continuous time signal (or analog frequency) in Hz
f_o	-	Fundamental frequency of discrete time signal in cycles/sample
F_o	-	Fundamental frequency of continuous time signal in Hz
F_m	-	Maximum frequency of continuous time signal in Hz
F_s	-	Sampling frequency of continuous time signal in Hz
I	-	Sampling rate mulitplication factor
j	-	complex operator, $\sqrt{-1}$
L	-	Number of segments

$\mathcal{M}$	-	Figure of merit
M	-	Mantissa
N	-	Fundamental period
N	-	Order of the filter
N_f	-	Floating point binary number
N_{tf}	-	Truncated floating point number
P	-	Power of a signal
p	-	Pole
$P_{xx}(f)$	-	Power spectrum
$P_{xx}^{B}(f)$	-	Bartlett power spectrum estimate
$P_{xx}^{BT}(f)$	-	Blackman-Tukey power spectrum estimate
$P_{xx}^{per}(f)$	-	Periodogram power spectrum estimate
$P_{xx}^{W}(f)$	-	Welch power spectrum estimate
q	-	Quantization step size
Q	-	Quality factor
R	-	Range of decimal number
r	-	Radix or base
S	-	Sign bit
$S_{xx}(f)$	-	Energy spectrum
t	-	Time in seconds
T	-	Time period in seconds
$\mathcal{V}$	-	Variabiltiy
W	-	Phase factor or Twiddle factor
x(n)	-	Discrete time signal or Ergodic random process

X(n)	-	Random process
z	-	Complex variable (z = u + jv)
z	-	Unit advance operator or Zero
z^{-1}	-	Unit delay operator
Î	-	Attenuation costant
W	-	Angular frequency of continuous time signal in rad/sec
W_o	-	Center frequency
W_s	-	Stop band edge analog frequency in rad/sec
W_p	-	Pass band edge analog frequency in rad/sec
w	-	Angular frequency of discrete time signal in rad/sample
w_k	-	Sampling frequency point
w_p	-	Pass band edge digital frequency in rad/sample
w_s	-	Stop band edge digital frequency in rad/sample
s^2	-	Variance
s^2_{eoi}	-	Steady state output noise power due to input quantization error
a_p	-	Attenuation at a pass band frequency
a_s	-	Attenuation at a stop band frequency
$*$	-	Convolution operator
$\circledast$	-	Circular convolution operator
$\int$	-	Integration operator
$\frac{d}{dt}$	-	Differentiation operator

Standard/Input/Output Signals

$\lvert A(w) \rvert$	-	Magnitude function
$h(n)$	-	Impulse response of discrete time system
$h'(n)$	-	Impulse response of inverse system
$h_d(n)$	-	Desired impulse response
$r_{xy}(m)$	-	Crosscorrelation sequence of x(n) and y(n)
$r_{xx}(m)$	-	Autocorrelation sequence of discrete time signal
$r_{xx}(m)$	-	Autocorrelation sequence of random process with finite data
$g_{xx}(m)$	-	Autocorrelation sequence of random process with infinite data
$\bar{r}_{xx}(m)$	-	Circular autocorrelation sequence of x(n)
$\bar{r}_{xy}(m)$	-	Circular crosscorrelation sequence of x(n) and y(n)
$u(n)$	-	Discrete time unit step signal
$w_R(n)$	-	Rectangular window sequence
$w_r(n)$	-	Bartlett or triangular window sequence
$w_C(n)$	-	Hanning window sequence
$w_H(n)$	-	Hamming window sequence
$w_B(n)$	-	Blackman window sequence
$w_K(n)$	-	Kaiser window sequence
$x(n)$	-	Discrete time signal
$x(n)$	-	Input of discrete time system
$x_o(n)$	-	Odd part of discrete time signal x(n)
$x_e(n)$	-	Even part of discrete time signal x(n)
$x(n-m)$	-	Delayed or linearly shifted x(n) by m units
$x((n-m))_N$	-	Circularly shifted x(n) by m units, where N is period

$x(Dn)$	-	Down sampled version of $x(n)$
$x(n/I)$	-	Upsampled version of $x(n)$
$x_p(n)$	-	Periodic extension of $x(n)$
$y(n)$	-	Output / Response of discrete time system
$y(n-m)$	-	Delayed output / Response of discrete time system
$y_p(n)$	-	Particular soultion of discrete time system
$y_n(n)$	-	Homogenous solution of discrete time system
$y_{zs}(n)$	-	Zero state response of discrete time system
$y_{zi}(n)$	-	Zero input response of discrete time time system
$d(n)$	-	Discrete time impulse signal
$d(n-m)$	-	Delayed impulse signal
t_p	-	Phase delay
t_g	-	Group delay
$q(w)$	-	Phase function

Transform Operators and Functions

$\mathcal{DFT}$	-	Discrete Fourier transform (DFT)
$\mathcal{DFT}^{-1}$	-	Inverse DFT
$E\{X\}$	-	Expected value of random variable
$\mathcal{F}$	-	Fourier transform
$\mathcal{F}^{-1}$	-	Inverse Fourier transform
$\mathcal{H}$	-	System operator
$\mathcal{H}^{-1}$	-	Inverse system operator
$H(z)$	-	Transfer function

$H(e^{jw})$	-	Frequency response of the digital filter
$H_N(z)$	-	Normalized transfer function
$H_d(e^{jw})$	-	Desired or ideal frequency response
Q[ ]	-	Quantization operations
$X(e^{jw})$	-	Discrete time Fourier transform of x(n)
$X_r(e^{jw})$	-	Real part of $X(e^{jw})$
$X_i(e^{jw})$	-	Imaginary part of $X(e^{jw})$
$X(j\Omega)$	-	Fourier transform of x(t)
X(k)	-	Discrete Fourier transform of x(n)
$X_r(k)$	-	Real part of X(k)
$X_i(k)$	-	Imaginary part of X(k)
X(z)	-	$\mathcal{Z}$-transform of x(n)
$\mathcal{Z}$	-	$\mathcal{Z}$-transform
$\mathcal{Z}^{-1}$	-	Inverse $\mathcal{Z}$-transform

Abbreviations

BIBO	-	**B**ounded **I**nput **B**ounded **O**utput
DFT	-	**D**iscrete **F**ourier **T**ransform
DIF	-	**D**ecimation **I**n **F**requency
DIT	-	**D**ecimation **I**n **T**ime
DT	-	**D**iscrete **T**ime
DTFS	-	**D**iscrete **T**ime **F**ourier **S**eries
DTFT	-	**D**iscrete **T**ime **F**ourier **T**ransform
FFT	-	**F**ast **F**ourier **T**ransform
FIR	-	**F**inite **I**mpulse **R**esponse

IIR	-	**I**nfinite **I**mpulse **R**esponse
LSD	-	**L**east **S**ignificant **D**igit
LHP	-	**L**eft **H**alf **P**lane
LTI	-	**L**inear **T**ime **I**nvariant
MSD	-	**M**ost **S**ignificant **D**igit
NTF	-	**N**oise **T**ransfer **F**unction
RHP	-	**R**ight **H**alf **P**lane
ROC	-	**R**egion **O**f **C**onvergence
Var	-	**V**ariance
QMF	-	**Q**uardrature **M**irror **F**ilter
LPF	-	**L**ow **P**ass **F**ilter

Chapter 1

Introduction to Digital Signal Processing

1.1 Introduction

Digital **S**ignal **P**rocessing (DSP) refers to processing of signals by digital systems like **P**ersonal **C**omputers (PC) and systems designed using digital **I**ntegrated **C**ircuit**s** (ICs), microprocessors and microcontrollers. DSP gained popularity in the 1960s. Earlier, DSP systems were limited to general purpose non-real-time scientific and business applications. The rapid advancement in computers and IC fabrication technology leads to complete domination of DSP systems in both real-time and non-real-time applications in all fields of engineering and technology.

The basic components of a DSP system are shown in fig 1.1. The ***DSP system*** involves conversion of analog signal to digital signal, then processing of the digital signal by a digital system and then conversion of the processed digital signal back to analog signal.

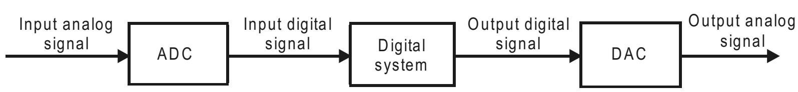


Fig 1.1 : *Basic components of a DSP system.*

The real-world signals are analog, and only for processing by digital systems, the signals are converted to digital. For conversion of signals from analog to digital, an ADC (**A**nalog to **D**igital **C**onverter) is employed. The various steps in analog to digital conversion process are sampling and quantization of analog signals, and then converting the quantized samples to suitable binary codes. The digital signals in the form of binary codes are fed to digital system for processing, and after processing, it generates an output digital signal in the form of binary codes. The output analog signal is constructed from the output binary codes using a DAC (**D**igital to **A**nalog **C**onverter).

The processing of signals are basically spectrum analysis to determine the various frequency components of a signal and filtering the signal to extract the required frequency component of the signal.

The ***digital system*** can be a specially designed programmable hardware for DSP or an algorithm/ software running on a general purpose digital system like **P**ersonal **C**omputer (PC).

Advantages of Digital Signal Processing

Some of the advantages of digital processing of signals are,

1. The digital hardware are compact, reliable, less expensive, and programmable.
2. Since the DSP systems are programmable, the performance of the system can be easily upgraded/ modified.
3. By employing high speed, sophisticated digital hardware higher precision can be achieved in processing of signals.
4. The digital signals can be permanently stored in magnetic media so that they are transportable and can be processed in non-real-time or off-line.

1.2 Signal

Any physical phenomenon that conveys or carries some information can be called a ***signal***. The music, speech, motion pictures, still photos, heart beat, etc., are examples of signals that we normally encounter in day-to-day life.

When a signal is defined continuously for any value of an independent variable, it is called an ***analog*** or ***continuous signal***. Most of the signals encountered in science and engineering are analog in nature. When the dependent variable of an analog signal is time, it is called a ***continuous time signal*** and it is denoted as "x(t)".

When a signal is defined for discrete intervals of an independent variable, it is called a ***discrete signal***. When the dependent variable of a discrete signal is time, it is called ***discrete time signal*** and it is denoted by "x(n)". Most of the discrete signals are either sampled versions of analog signals for processing by digital systems or output of digital systems.

The quantized and coded version of the discrete time signals are called ***digital signals***. In digital signals the value of the signal for every discrete time "n" is represented in binary codes. The process of conversion of a discrete time signal to digital signal involves quantization and coding.

Normally, for binary representation, a standard size of binary is chosen. In m-bit binary representation, we can have 2^m binary codes. The possible range of values of the discrete time signals are usually divided into 2^m steps called ***quantization levels***, and a binary code is attached to each quantization level. The values of the discrete time signals are approximated by rounding or truncation in order to match the nearest quantization level.

1.3 Discrete Time System

Any process that exhibits cause and effect relation can be called a ***system***. A system will have an input signal and an output signal. The output signal will be a processed version of the input signal. A system is either interconnection of hardware devices or software / algorithm.

A system which can process a discrete time signal is called a ***discrete time system***, and so the input and output signals of a discrete time system are discrete time signals.

A discrete time system is denoted by the letter $\mathcal{H}$. The input of discrete time system is denoted as "x(n)" and the output of discrete time system is denoted as "y(n)". The diagrammatic representation of a discrete time system is shown in fig 1.2.

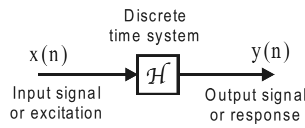


Fig 1.2 : *Representation of discrete time system.*

The operation performed by a discrete time system on input to produce output or response can be expressed as,

Response, $y(n) = \mathcal{H}\{x(n)\}$

where, $\mathcal{H}$ denotes the system operation (also called system operator).

When a discrete time system satisfies the properties of linearity and time invariance then it is called ***LTI (Linear Time Invariant) discrete time system*** .

The input-output relation of an LTI discrete time system is represented by constant coefficient difference equation shown below.

$$y(n) = -\sum_{m=1}^{N} a_m \, y(n-m) + \sum_{m=0}^{M} b_m \, x(n-m)$$

where, N = Order of the system, and $M \leq N$.

The solution of the above difference equation is the response y(n) of the discrete time system, for the input x(n).

1.4 Analysis of Discrete Time System

Mostly, the discrete time systems are designed for analysis of discrete time signals. Physically, the discrete time systems are realized in time domain. In time domain, the discrete time systems are governed by difference equations. The analysis of discrete time signals and systems in time domain involves solution of difference equations. The solution of difference equations are difficult due to assumption of a solution and then solving the constants using initial conditions.

In order to simplify the task of analysis, the discrete time signals can be transformed to some other domain, where the analysis may be easier. One such transform exists for discrete time signals is $\mathcal{Z}$-transform.The $\mathcal{Z}$-transform, will transform a function of discrete time "n" into a function of complex variable "z", where $z = re^{jw}$ .Therefore, ***$\mathcal{Z}$-transform*** of a discrete time signal will transform the time domain signal into z-domain signal.

On taking $\mathcal{Z}$ -transform of the difference equation governing the discrete time system, it becomes algebraic equation in "z" and the solution of algebraic equation will give the response of the system as a function of "z" and it is called z-domain response. The inverse $\mathcal{Z}$ -transform of the z-domain response, will give the time domain response of the discrete time system. Also, the stability analysis of the discrete systems are much easier in z-domain.

The ratio of $\mathcal{Z}$ -transform of output and input is called ***transfer function*** of the discrete time system. The inverse $\mathcal{Z}$ -transform of the system gives the ***impulse response*** of the system, which is used to study the characteristics of a system.

Another important characteristic of any signal is frequency, and for most of the applications the frequency content of the signal is an important criteria. The frequency range of some of the signals are listed in table 1.1 and 1.2.

Table 1.1 : Frequency Range of Some Electromagnetic Signals

Type of signal	Wavelength (m)	Frequency range (Hz)
Radio broadcast	10^{4} to 10^{2}	3×10^{4} to 3×10^{6}
Shortwave radio signals	10^{2} to 10^{-2}	3×10^{6} to 3×10^{10}
Radar / Space communications	1 to 10^{-2}	3×10^{8} to 3×10^{10}
Common-carrier microwave	1 to 10^{-2}	3×10^{8} to 3×10^{10}
Infrared	10^{-3} to 10^{-6}	3×10^{11} to 3×10^{14}
Visible light	3.9×10^{-7} to 8.1×10^{-7}	3.7×10^{14} to 7.7×10^{14}
Ultraviolet	10^{-7} to 10^{-8}	3×10^{15} to 3×10^{16}
Gamma rays and x-rays	10^{-9} to 10^{-10}	3×10^{17} to 3×10^{18}

Table 1.2 : Frequency Range of Some Biological and Seismic Signals

Type of Signal	Frequency Range (Hz)
Electroretinogram	0 to 20
Electronystagmogram	0 to 20
Pneumogram	0 to 40
Electrocardiogram (ECG)	0 to 100
Electroencephalogram (EEG)	0 to 100
Electromyogram	10 to 200
Sphygmomanogram	0 to 200
Speech	100 to 4000
Wind noise	100 to 1000
Seismic exploration signals	10 to 100
Earthquake and nuclear explosion signals	0.01 to 10
Seismic noise	0.1 to 1

The frequency contents of a discrete time signal can be studied by taking Fourier transform of the discrete time signal. The Fourier transform of discrete time signal is a particular class of $\mathcal{Z}$-transform in which $z = e^{jw}$, where "w" is the frequency of the discrete time signals.

The ***Fourier transform***, will transform a function of discrete time "n" into a function of frequency "w". Therefore, Fourier transform of a discrete time signal will transform the discrete time signal into frequency domain signal. The Fourier transform of the discrete time signal, is also called ***frequency spectrum*** of the discrete time signal. The Fourier transform of the impulse response of a system is called ***frequency response*** of the system. The frequency spectrum is a complex function of "w" and so can be expressed as magnitude spectrum and phase spectrum. The magnitude spectrum is used to study the various frequency components of the discrete time signal.

The frequency spectrum obtained via Fourier transform will be a continuous spectrum and so cannot be computed by digital systems, Therefore, the samples of Fourier transform can be computed at sufficient number of points by digital systems. The samples of Fourier transform can also be directly computed using DFT (**D**iscrete **F**ourier **T**ransform). The computation of DFT involves a large number of calculations. In order to reduce the computational task of DFT, a number of methods/algorithms are developed which are collectively called ***FFT*** (**F**ast **F**ourier **T**ransform). The DFT of discrete time signal will give the ***discrete frequency spectrum*** of the signal.

1.5 Filters

The filters are frequency selective devices. The two major types of digital filters are FIR (**F**inite **I**mpulse **R**esponse) and IIR (**I**nfinite **I**mpulse **R**esponse) filters.

Generally, the filter specification will be a desired frequency response. The inverse Fourier transform of the frequency response will be the impulse response of the filter, and it will be an infinite duration signal. The digital filters designed by choosing finite samples of impulse rseponse are called ***FIR filters***, and the filters designed by considering all the infinite samples are called ***IIR filters***.

Since, an FIR filter is designed from the finite samples of impluse response, the direct design of FIR filter is possible in which the transfer function of the filter is obtained by taking $\mathcal{Z}$ -transform of impulse response.

Note : Mathematically, the filter design is design of transfer function of the filter.

Since, an IIR filter is designed by considering / preserving the infinite samples of impulse response, the direct design of IIR filter is not possible. Therefore, the IIR filter is designed via analog filter. For designing IIR filter, first the specifications of IIR filter is transformed to specifications of analog filter using bilinear or impulse-invariant transformation, then an analog filter transfer function is designed using Butterworth or Chebychev approximation. Finally the analog filter transfer function is transfered to digital filter transfer function using the transformation chosen for transforming the specifications.

1.6 Finite Word Length Effects

In digital representation the signals are represented as an array of binary numbers, and the digital system employ a fixed size of binary called "word size or word length" for number representation. This finite word size for number representation leads to errors in input signals, intermediate signals in computations and in the final output signals. In general, the various effects due to finite precision representation of numbers in digital systems are called ***finite word length effects***.

Some of the finite word length effects in digital systems are given below.

- Errors due to quantization of input data.
- Errors due to quantization of filter coefficients.

- Errors due to rounding the product in multiplication.
- Errors due to overflow in addition.
- Limit cycles in recursive computations.

1.7 Multirate DSP

In many communication systems, the sampling rate conversion is a vital requirement. Some of the systems that employ sampling rate conversion are video receivers that receive both NTSC and PAL signals, audio systems that can play CDs recorded in different standards, etc.

The processing of discrete time signals at different sampling rates in different parts of a system is called ***multirate DSP***. In digital systems, the sampling rate conversion is achieved by either decimation or interpolation. In decimation, the sampling rate is reduced, whereas in interpolation the sampling rate is increased. The multirate DSP systems leads to reduction in computations, memory requirement and errors due to finite word length effects.

1.8 Energy and Power Spectrum

There are many situations where the signals are corrupted by noise like sonar signals corrupted by ambient ocean noise, speech signal from cockpit of an airplane corrupted by engine noise, etc. When the signals are corrupted by noise, then the energy or power spectrum will be useful to identify the signal from noise.

The ***energy spectrum*** can be computed for deterministic signals, and it is given by square of magnitude of Fourier transform of the signal. Alternatively, the energy spectrum is given by Fourier transform of the autocorrelation sequence of the signal.

The power spectrum can be estimated for nondeterministic signals or random process/signals. The power spectrum estimation methods can be broadly classified into two groups, namely, nonparametric methods and parametric methods.

In ***nonparametric methods***, first an estimate of autocorrelation of the random process is determined which represents the average behaviour of the signal, then the Fourier transform of estimated autocorrelation is determined, which is the power spectrum estimate of the random process.

In ***parametric methods***, first an appropriate model is selected for the given random process, then the parameters of the model are computed using the available data of the random process. Finally, the power spectrum is estimated from the constructed model.

1.9 Digital Signal Processors

The ***digital signal processors*** are specially designed microprocessors/microcontrollers for DSP applications.

The importance of special purpose processors for signal processing applications were realised in 1980s, and many companies started releasing special processors for DSP applications. The pioneers among them are Texas Instruments and Analog Devices. The Texas Instruments has released a large variety of processors in the family name TMS320Cxx and Analog Devices has released processors in the family name ADSPxx.

Some of the special features of digital signal processors are given below.

- Modified Harvard architecture with two or more internal buses for simultaneous access of code and one or two data.
- Specialized addressing modes like circular addressing and bit reversed addressing suitable for computations like convolution, correlation and FFT.
- MAC unit for performing multiply-accumulate computations involved in convolution, correlation and FFT in single clock cycle.
- Larger size ALU and accumulators with guard bits to accommodate the overflow in computation.
- Pipelining of instructions to execute different phases of four or six instructions in parallel.
- VLIW architecture to fetch and execute multiple instructions in parallel.
- Multiprocessor architecture by integrating multiple processors on a single piece of silicon for parallel processing.

1.10 Importance of Digital Signal Processing

The technology advancement in programmable digital signal processors, helps to implement more and more real time applications in digital systems.

The digital processing of signal plays a vital role in almost every field of Science and Engineering. Some of the applications of digital processing of signals in various field of Science and Engineering are listed here.

1. Biomedical

- ECG is used to predict heart diseases.
- EEG is used to study normal and abnormal behaviour of the brain.
- EMG is used to study the condition of muscles.
- X-ray images are used to predict the bone fractures and tuberculosis.
- Ultrasonic scan images of kidney and gall bladder is used to predict stones.
- . Ultrasonic scan images of foetus is used to predict abnormalities in a baby.
- MRI scan is used to study minute inner details of any part of the human body.

2. Speech Processing

- Speech compression and decompression to reduce memory requirement of storage systems.
- Speech compression and decompression for effective use of transmission channels.
- Speech recognization for voice operated systems and voice based security systems.
- Speech recognization for conversion of voice to text.
- Speech synthesis for various voice based warnings or annoucements.

3. Audio and Video Equipments

- The analysis of audio signals will be useful to design systems for special effects in audio systems like stereo, woofer, karoke, equalizer, attenuator, etc.
- Music synthesis and composing using music keyboards.
- Audio and video compression for storage in DVDs.

4. Communication

- The spectrum analysis of modulated signals helps to identify the information bearing frequency component that can be used for transmission.
- The analysis of signals received from radars are used to detect flying objects and thier velocity.
- Generation and detection of DTMF signals in telephones.
- Echo and noise cancellation in transmission channels.

5. Power electronics

- The spectrum analysis of the output of coverters and inverters will reveal the harmonics present in the output, which in turn helps to design suitable filter to eliminate the harmonics.
- The analysis of switching currents and voltages in power devices will help to reduce losses.

6. Image processing

- Image compression and decompression to reduce memory requirement of storage systems.
- Image compression and decompression for effective use of transmission channels.
- Image recognition for security systems.
- Filtering operations on images to extract the features or hidden information.

7. Geology

- The seismic signals are used to determine the magnitude of earthquakes and volcanic eruptions.
- The seismic signals are also used to predict nuclear explosions.
- The seismic noises are also used to predict the movement of earth layers (tectonic plates).

8. Astronomy

- The analysis of light received from a star is used to determine the condition of the star.
- The analysis of images of various celestial bodies gives vital information about them.

1.11 Use of MATLAB in Digital Signal Processing

MATLAB (**MAT**rix **LAB**oratory) is a software developed by The MathWork Inc, USA, which can run on any windows platform in a PC (**P**ersonal **C**omputer). This software has a number of tools for the study of various engineering subjects. It includes various tools for digital signal processing also. Using these tools, a wide variety of studies can be made on discrete time signals and systems. Some of the analysis that is relevant to this particular textbook are given below.

- Sketch or plot of discrete time signals as a function of independent variable.
- Spectrum analysis of discrete time signals.
- Solution of LTI discrete time systems.
- Perform convolution and deconvolution operations on discrete time signals.
- Perform various transforms on discrete time signals like Fourier transform, **Z**-transform, **F**ast **F**ourier **T**ransform (FFT), etc.
- Design and frequency response analysis of FIR and IIR filters.
- Decimation and interpolation of discrete time signals.
- Estimation of energy and power spectrum of discrete time signals.

Chapter 2

Discrete Time Signals and Systems

2.1 Introduction

In today's world, digital systems are employed for almost every application. The digital systems can process only discrete signals. This chapter deals with time domain analysis of discrete time signals and systems. In the first part of this chapter, the generation, representation, classification and mathematical operations on discrete time signals are discussed in detail. In the second part of this chapter, the representation, classification and response of discrete time systems are discussed in detail. The concept of LTI systems are highlighted wherever necessary.

Discrete Signal and Discrete Time Signal

The ***discrete signal*** is a function of a discrete independent variable. The independent variable is divided into uniform intervals and each interval is represented by an integer. The letter "n" is used to denote the independent variable. The discrete or digital signal is denoted by x(n).

The discrete signal is defined for every integer value of the independent variable "n". The magnitude (or value) of discrete signal can take any discrete value in the specified range. Here both the value of the signal and the independent variable are discrete. The discrete signal can be represented by a one-dimensional array as shown in the following example.

Example :

$x(n) = \{ 2, 4, -1, 3, 3, 4 \}$

Here the discrete signal x(n) is defined for, $n = 0, 1, 2, 3, 4, 5$

\ $x(0) = 2$; $x(1) = 4$; $x(2) = -1$; $x(3) = 3$; $x(4) = 3$; $x(5) = 4$.

When the independent variable is time t, the discrete signal is called ***discrete time signal***. In discrete time signal, the time is divided uniformly using the relation t = nT, where T is the sampling time period. (The sampling time period is the inverse of sampling frequency). The discrete time signal is denoted by x(n) or x(nT).

Since the discrete signals have a sequence of numbers (or values) defined for integer values of the independent variable, the discrete signals are also known as ***discrete sequence***. In this book, the term sequence and signal are used synonymously. Also in this book, the discrete signal is referred as discrete time signal.

Digital Signal

The ***digital signal*** is same as discrete signal except that the magnitude of the signal is quantized. The magnitude of the signal can take one of the values in a set of quantized values. Here quantization is necessary to represent the signal in binary codes.

The generation of a discrete time signal by sampling a continuous time signal and then quantizing the samples in order to convert the signal to digital signal is shown in the following example.

Let, x(t) = Continuous time signal

T = Sampling time

A typical continuous time signal and the sampling of this continuous time signal at uniform interval are shown in fig 2.1a and fig 2.1b respectively. The samples of the continuous time signal as a function of sampling time instants are shown in fig 2.1c. (In fig 2.1c, 1T, 2T, 3T, ...etc., represents sampling time instants and the value of the samples are functions of this sampling time instants).

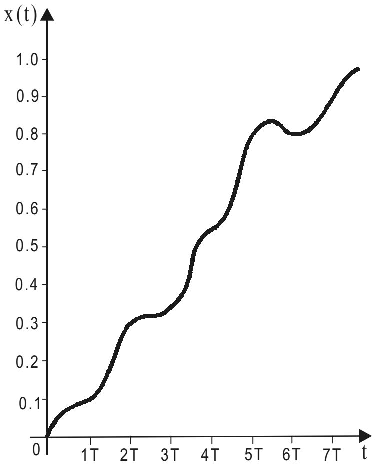


Fig 2.1a.

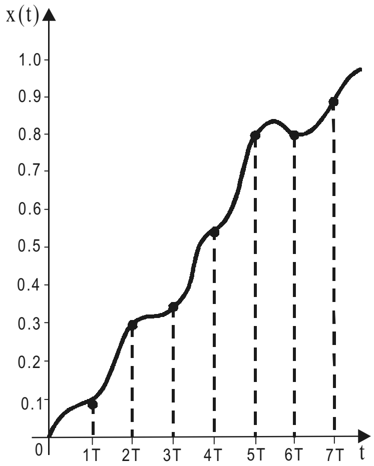


Fig 2.1b.

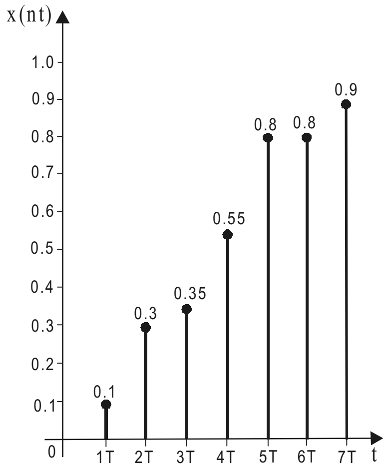


Fig 2.1c.

Fig 2.1 : *Sampling a continuous time signal to generate discrete time signal.*

When t = 0 ;	x(t) = 0	When t = 4T ;	x(t) = 0.55
When t = 1T ;	x(t) = 0.1	When t = 5T ;	x(t) = 0.8
When t = 2T ;	x(t) = 0.3	When t = 6T ;	x(t) = 0.8
When t = 3T ;	x(t) = 0.35	When t = 7T ;	x(t) = 0.9

In general, the sampling time instants can be represented as, "nT", where "n" is an integer. When we drop the sampling time "T" , then the samples are functions of the integer variable "n" alone. Therefore, the samples of the continuous time signal will be a discrete time signal, denoted as x(n), which is a function of an integer variable "n" as shown below.

$$x(n) = \{ 0,\ 0.1,\ 0.3,\ 0.35,\ 0.55,\ 0.8,\ 0.8,\ 0.9 \}$$

Here the discrete signal x(n) is defined for, n = 0, 1, 2, 3, 4, 5, 6, 7

\ $x(0) = 0$; $x(1) = 0.1$; $x(2) = 0.3$; $x(3) = 0.35$;

$x(4) = 0.55$; $x(5) = 0.8$; $x(6) = 0.8$; $x(7) = 0.9$.

The sample value lies the range of 0 to 1.

Let us choose 3-bit binary to represent the samples in binary code. Now, the possible binary codes are $2^3 = 8$, and so the range can be divided into eight quantization levels, and each sample is assigned, one of the quantization level as shown in the following table.

Quantization level (R = Range = 1)	Binary code	Range represented by quantization level for quantization by truncation
$0 \times \frac{R}{2^3} = 0 \times \frac{1}{8} = 0$	000	$0.000 \le x(n) < 0.125 \Rightarrow 0.000$
$1 \times \frac{R}{2^3} = 1 \times \frac{1}{8} = 0.125$	001	$0.125 \le x(n) < 0.250 \Rightarrow 0.125$
$2 \times \frac{R}{2^3} = 2 \times \frac{1}{8} = 0.25$	010	$0.250 \le x(n) < 0.375 \Rightarrow 0.250$
$3 \times \frac{R}{2^3} = 3 \times \frac{1}{8} = 0.375$	011	$0.375 \le x(n) < 0.500 \Rightarrow 0.375$
$4 \times \frac{R}{2^3} = 4 \times \frac{1}{8} = 0.5$	100	$0.500 \le x(n) < 0.625 \Rightarrow 0.500$
$5 \times \frac{R}{2^3} = 5 \times \frac{1}{8} = 0.625$	101	$0.625 \le x(n) < 0.75 \Rightarrow 0.625$
$6 \times \frac{R}{2^3} = 6 \times \frac{1}{8} = 0.75$	110	$0.750 \le x(n) < 0.875 \Rightarrow 0.750$
$7 \times \frac{R}{2^3} = 7 \times \frac{1}{8} = 0.875$	111	$0.875 \le x(n) \le 1.000 \Rightarrow 0.875$

Let, $x_q(n)$ = Quantized discrete time signal.

$x_c(n)$ = Quantized and coded discrete time signal.

Now, $x_q(n) = \{ 0, 0, 0.25, 0.25, 0.5, 0.75, 0.75, 0.875 \}$

$x_c(n) = \{ 000, 000, 010, 010, 100, 110, 110, 111 \}$

The quantized and coded discrete time signal $x_c(n)$ is called digital signal.

2.2 Discrete Time Signals

2.2.1 Generation of Discrete Time Signals

A discrete time signal can be generated by the following three methods.

The methods 1 and 2 are independent of any time frame but Method 3 depends critically on time.

1. Generate a set of numbers and arrange them as a sequence.

Example :

The numbers 0, 2, 4, ..., 2N form a sequence of even numbers and can be expressed as,

x(n) = 2n ; 0 £ n £ N

2. Evaluation of a numerical recursion relation will generate a discrete signal.

Example :

$x(n) = 0.2\,x(n-1)$ with initial condition $x(0) = 1$, gives the sequence, $x(n) = 0.2^n \;;\; 0 \le n < \infty$

When $n = 0$; $x(0) = 1$ (initial condition) $= 0.2^0$

When $n = 1$; $x(1) = 0.2\,x(1-1) = 0.2\,x(0) = 0.2 = 0.2^1$

When $n = 2$; $x(2) = 0.2\,x(2-1) = 0.2\,x(1) = 0.2 \times 0.2 = 0.2^2$

When $n = 3$; $x(3) = 0.2\,x(3-1) = 0.2\,x(2) = 0.2 \times 0.2^2 = 0.2^3$ and so on

$\therefore\; x(n) = 0.2^n \;;\; 0 \le n < \infty$

3. A third method is by uniformly sampling a continuous time signal and using the amplitudes of the samples to form a sequence.

Let, $x(t)$ = Continuous time signal

Now, Discrete signal, $x(nT) = x(t)\big|_{t=nT} \;;\; -\infty < n < \infty$

where, T is the sampling interval

The generation of discrete signal by sampling a continuous time signal is shown in fig 2.1.

2.2.2 Representation of Discrete Time Signals

The discrete time signal can be represented by the following methods.

1. Functional representation

In functional representation, the signal is represented as a mathematical equation, as shown in the following example.

$$x(n) = \begin{cases} -0.5 & ; \; n = -2 \\ 1.0 & ; \; n = -1 \\ -1.0 & ; \; n = 0 \\ 0.6 & ; \; n = 1 \\ 1.2 & ; \; n = 2 \\ 1.5 & ; \; n = 3 \\ 0 & ; \; \text{other } n \end{cases}$$

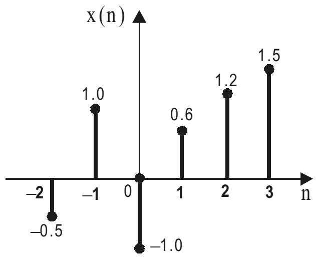


Fig 2.2 : Graphical representation of a discrete time signal.

2. Graphical representation

In graphical representation, the signal is represented in a two-dimensional plane. The independent variable is represented in the horizontal axis and the value of the signal is represented in the vertical axis as shown in fig 2.2.

3. Tabular representation

In tabular representation, two rows of a table are used to represent a discrete time signal. In the first row, the independent variable "n" is tabulated and in the second row the value of the signal for each value of "n" are tabulated as shown in the following table.

n	....	–2	-1	0	1	2	3	....
x(n)	....	–0.5	1.0	–1.0	0.6	1.2	1.5	....

4. Sequence representation

In sequence representation, the discrete time signal is represented as a one-dimensional array as shown in the following examples.

An infinite duration discrete time signal with the time origin, n = 0, indicated by the symbol - is represented as,

x(n) = { ..... – 0.5, 1.0, $\underset{-}{-1.0}$, 0.6, 1.2, 1.5, ... }

An infinite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,

x(n) = { $\underset{-}{-1.0}$, 0.6, 1.2, 1.5, ... } or x(n) = {–1.0, 0.6, 1.2, 1.5, ... }

A finite duration discrete time signal with the time origin, n = 0, indicated by the symbol - is represented as,

x(n) = { – 0.5, 1.0, $\underset{-}{-1.0}$, 0.6, 1.2, 1.5 }

A finite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,

x(n) = { $\underset{-}{-1.0}$, –0.6, 1.2, 1.5 } or x(n) = { –1.0, 0.6, 1.2, 1.5}

2.2.3 Standard Discrete Time Signals

1. Digital impulse signal or unit sample sequence

$$\text{Impulse signal, } \delta(n) = 1 \;;\; n = 0$$
$$= 0 \;;\; n \neq 0$$

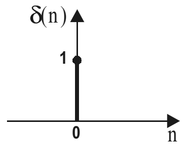


Fig 2.3 : Digital impulse signal.

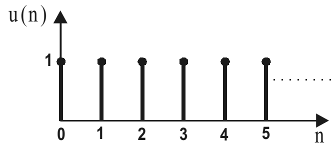


Fig 2.4 : Unit step signal.

2. Unit step signal

$$\text{Unit step signal, } u(n) = 1 \;;\; n \geq 0$$
$$= 0 \;;\; n < 0$$

3. Ramp signal

$$\text{Ramp signal, } u_r(n) = n \;;\; n \geq 0$$
$$= 0 \;;\; n < 0$$

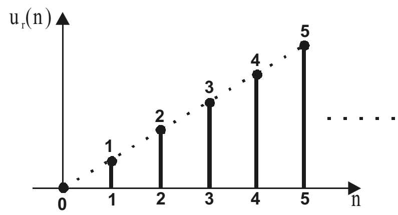


Fig 2.5 : Ramp signal.

4. Exponential signal

$$\text{Exponential signal, } g(n) = a^n \;;\; n \geq 0$$
$$= 0 \;;\; n < 0$$

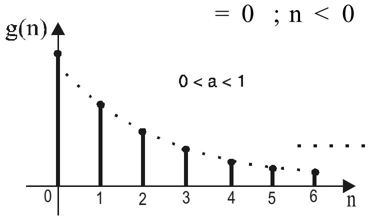


Fig 2.6a : Decreasing exponential signal.

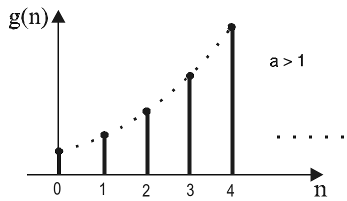


Fig 2.6b : Increasing exponential signal.

Fig 2.6 : Exponential signal.

5. Discrete time sinusoidal signal

The discrete time sinusoidal signal may be expressed as,

$$x(n) = A\cos(\omega_0 n + \theta) \ ; \text{for n in the range } -\infty < n < +\infty$$

$$x(n) = A\sin(\omega_0 n + \theta) \ ; \text{for n in the range } -\infty < n < +\infty$$

where, ω_0 = Frequency in radians/sample ; θ = Phase in radians

$$f_0 = \frac{\omega_0}{2\pi} = \text{Frequency in cycles/sample}$$

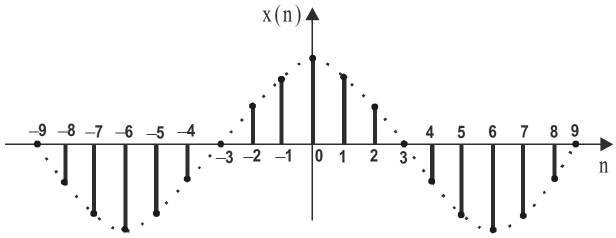


Fig 2.7a : *Discrete time sinusoidal signal represented by equation $x(n) = A\cos(\omega_0 n)$.*

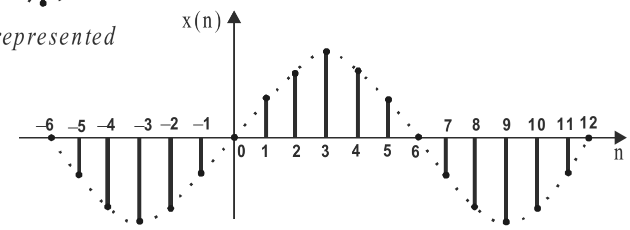


Fig 2.7b : *Discrete time sinusoidal signal represented by equation $x(n) = A\sin(\omega_0 n)$.*

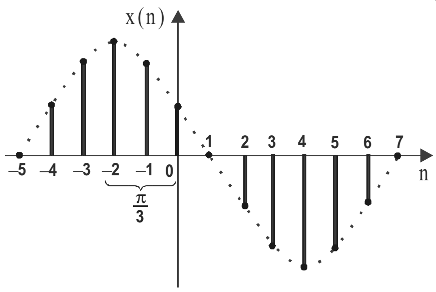


Fig 2.7c : *Discrete time sinusoidal signal represented by equation,*

$$x(n) = A\cos\left(\frac{\pi}{6}n + \frac{\pi}{3}\right); \ \omega_0 = \frac{\pi}{6} \ ; \ \theta = \frac{\pi}{3}$$

Fig 2.7 : *Discrete time sinusoidal signals.*

Properties of Discrete Time Sinusoid

1. A discrete time sinusoid is periodic only if its frequency f_0 is a rational number, (i.e., ratio of two integers).
2. Discrete time sinusoids whose frequencies are separated by integer multiples of 2π are identical.

$\therefore x(n) = A\cos[(\omega_0 + 2\pi k)\, n + \theta]$, for $k = 0,1,2$...........are identical in the interval $-\pi \le \omega_0 \le \pi$ and so they are indistinguishable.

> ***Proof :***
>
> $$\cos[(\omega_0 + 2\pi k)\, n + \theta] = \cos(\omega_0 n + 2\pi nk + \theta) = \cos[(\omega_0 n + \theta) + 2\pi nk]$$
>
> $$= \cos(\omega_0 n + \theta)\cos 2\pi nk - \sin(\omega_0 n + \theta)\sin 2\pi nk$$
>
> Since n and k are integers, $\cos 2\pi nk = 1$ and $\sin 2\pi nk = 0$
>
> $$\therefore \cos[(\omega_0 + 2\pi k)\, n + \theta] = \cos(\omega_0 n + \theta), \quad \text{for } k = 0, 1, 2, 3, ...$$

Conclusion

1. The sequences of any two sinusoids with frequencies in the range, $-p$ £ w_o £ p (or $-1/2$ £ f_0 £ $1/2$), are distinct.

$$[-p \text{ £ } w \text{ £ } p \xrightarrow{\text{divide by } 2\pi} -1/2 \text{ £ } f \text{ £ } 1/2]$$

2. Any discrete time sinusoid with frequency $w_0 > |p|$ (or $f_0 > |1/2|$) will be identical to another discrete time sinusoid with frequency $w_0 < |p|$ (or $f_0 < |1/2|$).

6. Discrete time complex exponential signal

The discrete time complex exponential signal is defined as,

$$x(n) = a^n e^{j(\omega_0 n+\theta)} = a^n [\cos(w_0 n + q) + j \sin(w_0 n + q)]$$

$$= a^n \cos(w_0 n + q) + j\, a^n \sin(w_0 n + q) = x_r(n) + j\, x_i(n)$$

where, $x_r(n)$ = Real part of $x(n) = a^n \cos(w_0 n + q)$

$x_i(n)$ = Imaginary part of $x(n) = a^n \sin(w_0 n + q)$

The real part of x(n) will give an exponentially increasing cosinusoid sequence for a > 1 and exponentially decreasing cosinusoid sequence for 0 < a < 1.

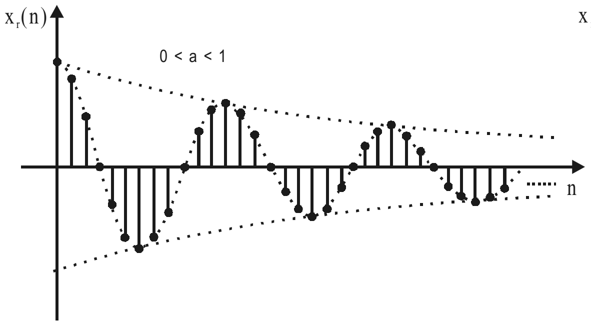


Fig 2.8a : *The discrete time sequence represented by the equation, $x_r(n) = a^n \cos \omega_0 n$ for $0 < a < 1$.*

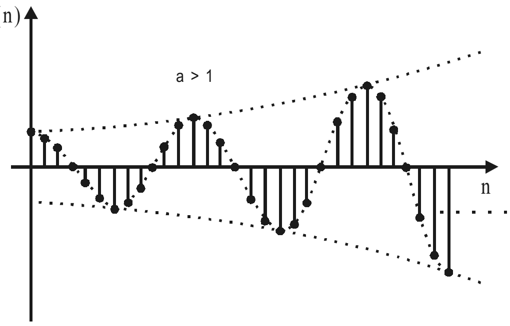


Fig 2.8b : *The discrete time sequence represented by the equation, $x_r(n) = a^n \cos \omega_0 n$ for $a > 1$.*

Fig 2.8 : *Real part of complex exponential signal.*

The imaginary part of x(n) will give rise to an exponentially increasing sinusoid sequence for a > 1 and exponentially decreasing sinusoid sequence for 0 < a < 1.

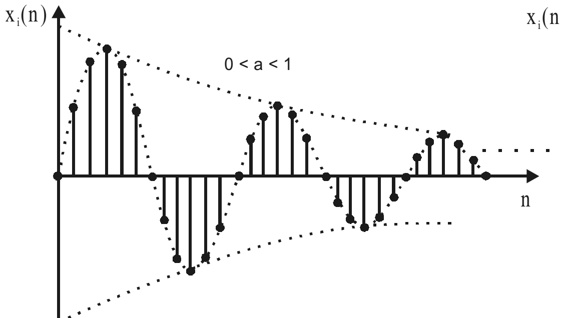


Fig 2.9a : *The discrete time sequence represented by the equation, $x_i(n) = a^n \sin \omega_0 n$ for $0 < a < 1$.*

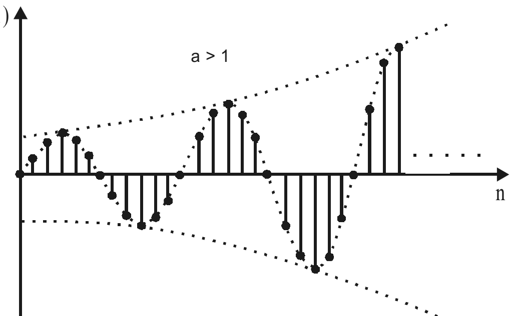


Fig 2.9b : *The discrete time sequence represented by the equation, $x_i(n) = a^n \sin \omega_0 n$ for $a > 1$.*

Fig 2.9 : *Imaginary part of complex exponential signal.*

2.3 Sampling of Continuous Time (Analog) Signals

The ***sampling*** is the process of conversion of a continuous time signal into a discrete time signal. The sampling is performed by taking samples of continuous time signal at definite intervals of time. Usually, the time interval between two successive samples will be same and such type of sampling is called ***periodic or uniform sampling***.

The time interval between successive samples is called ***sampling time*** (or sampling period or sampling interval), and it is denoted by "T". The unit of sampling period is second (s). [The lower units are millisecond (ms) and microsecond (μs)].

The inverse of sampling period is called ***sampling frequency*** (or sampling rate), and it is denoted by F_s. The unit of sampling frequency is hertz (Hz). (The higher units are kHz and MHz).

Let, $x_a(t)$ = Analog / Continuous time signal.

$x(n)$ = Discrete time signal obtained by sampling $x_a(t)$.

Mathematically, the relation between $x(n)$ and $x_a(t)$ can be expressed as,

$$x(n) = x_a(t)\big|_{t=nT} = x_a(nT) = x_a\left(\frac{n}{F_s}\right) ; \qquad \text{for n in the range } -\infty < n < \infty$$

where, T = Sampling period or interval in seconds

$F_s = \dfrac{1}{T}$ = Sampling rate or sampling frequency in hertz

> **Example :** Let, $x_a(t) = A\cos(\Omega_0 t + \theta) = A\cos(2\pi F_0 t + \theta)$
>
> where, Ω_0 = Frequency of analog signal in rad/s
>
> $F_0 = \dfrac{\Omega_0}{2\pi}$ = Frequency of analog signal in Hz
>
> Let $x_a(t)$ be sampled at intervals of T seconds to get x(n), where $T = \dfrac{1}{F_s}$
>
> $$\therefore\ x(n) = x_a(t)\big|_{t=nT} = A\cos(\Omega_0 t + \theta)\big|_{t=nT}$$
>
> $$= A\cos(\Omega_0 nT + \theta) = A\cos\left(\frac{2\pi F_0}{F_s}n + \theta\right) = A\cos(2\pi f_0 n + \theta) = A\cos(\omega_0 n + \theta)$$
>
> where, $f_0 = \dfrac{F_0}{F_s}$ = Frequency of discrete sinusoid in cycles/sample
>
> $\omega_0 = 2\pi f_0$ = Frequency of discrete sinusoid in radians/sample

2.3.1 Sampling and Aliasing

In Section 2.2, it is observed that any two sinusoid signals with frequencies in the range $-1/2 \le f \le +1/2$ are distinct and a discrete sinusoid with frequency, $f > |\pm 1/2|$ will be identical to another discrete sinusoid with frequency, $f < |\pm 1/2|$. Therefore, we can conclude that range of frequency of discrete time signal is $-1/2$ to $+1/2$. But the range of frequency of analog signal is $-\infty$ to $+\infty$. While sampling analog signals, the infinite frequency range continuous time signals are mapped (or converted) to finite frequency range discrete time signals.

The relation between frequency of analog and discrete time signal is,

$$f = \frac{F}{F_s} \qquad .....(2.1)$$

The range of frequency of discrete time signal is,

$$-\frac{1}{2} \leq f \leq \frac{1}{2} \qquad .....(2.2)$$

On substituting for f from equation (2.1) in equation (2.2) we get,

$$-\frac{1}{2} \leq \frac{F}{F_s} \leq \frac{1}{2} \qquad .....(2.3)$$

On multiplying equation (2.3) by F_s we get,

$$-\frac{F_s}{2} \leq F \leq \frac{F_s}{2} \qquad .....(2.4)$$

From equation (2.4) we can say that when an analog signal is sampled at a frequency F_s, the highest analog frequency that can be uniquely represented by a discrete time signal will be $F_s/2$. The continuous time signal with frequency above $F_s/2$ will be represented as a signal within the range $+ F_s/2$ to $- F_s/2$. Hence the signal with frequency above $F_s/2$ will have an identical signal with frequency below $F_s/2$ in the discrete form.

Hence infinite number of high frequency continuous time signals will be represented by a single discrete time signal. Such signals are called ***alias.***

The phenomenon of high-frequency component getting the identity of low-frequency component during sampling is called ***aliasing.***

Sampling an analog signal with frequency F by choosing a sampling frequency F_s such that $F_s/2 > F$ will not result in alias. But sampling frequency is selected such that $F_s/2 < F$ that the frequency above $F_s/2$ will have alias with frequency below $F_s/2$. Hence the point of reflection is $F_s/2$, and the frequency $F_s/2$ is called ***folding frequency***.

The discrete time sinusoids, A sin [$(2pf_0 + 2pk)n$], will be alias for integer values of k. It is also observed that, a sinusoidal signal with frequency F_1 will be an alias of sinusoidal signal with frequency F_2 if it is sampled at a frequency $F_s = F_1 - F_2$. In general, if the sampling frequency is any multiple of $F_1 - F_2$, [i.e., $F_s = k(F_1 - F_2)$ where k = 1, 2, 3, ...] the signal with frequency F_2 will be an alias of the signal with frequency F_1.

Let, F_{max} be maximum frequency of analog signal that can be uniquely represented as discrete time signal when sampled at a frequency F_s.

$$\text{Now, } F_{max} = \frac{F_s}{2} \qquad .....(2.5)$$

$$\therefore F_s = 2F_{max} \qquad .....(2.6)$$

The equation (2.6) gives a choice for selecting sampling frequency. From equation (2.6) we can say that for unique representation of analog signal with maximum frequency F_{max}, the sampling frequency should be greater than $2F_{max}$.

i.e., to avoid aliasing F_s ³ $2F_{max}$ ..... (2.7)

When sampling frequency F_s is equal to $2F_{max}$, the sampling rate is called ***Nyquist rate***.

It is observed that a nonshifted sinusoidal signal when sampled at Nyquist rate, will produce zero sample sequence (i.e., discrete sequence with all zeros), (because the sinusoidal signal is sampled at its zero crossings, Refer example 2.3). Hence to avoid zero sampling of sinewave, the sampling frequency F_s should be greater than $2F_{max}$, where F_{max} is the maximum frequency in the analog signal.

A discrete signal obtained by sampling can be reconstructed to an analog signal, only when it is sampled without aliasing. The above concepts of sampling analog signals are summarized as the sampling theorem, given below.

Sampling Theorem :	*A band limited continuous time signal with highest frequency (bandwidth) F_m hertz can be uniquely recovered from its samples provided that the sampling rate F_s is greater than or equal to $2F_m$ samples per second.*

Note : The effects of aliasing in frequency spectrum are discussed in Chapter-4.

Example 2.1

Consider the analog signals, $x_1(t) = 3\cos 2\pi(20t)$ and $x_2(t) = 3\cos 2\pi(70t)$.
Find a sampling frequency so that 70 Hz signal is an alias of the 20 Hz signal?

Solution

Let, the sampling frequency, $F_s = 70 - 20 = 50\,Hz$.

$$\therefore x_1(n) = x_1(t)\Big|_{t = nT = \frac{n}{F_s}} = 3\cos 2\pi(20t)\Big|_{t = \frac{n}{F_s}} = 3\cos 2\pi\left(\frac{20 \times n}{50}\right) = 3\cos\frac{4\pi}{5}n$$

$$x_2(n) = x_2(t)\Big|_{t = nT = \frac{n}{F_s}} = 3\cos 2\pi(70t)\Big|_{t = \frac{n}{F_s}} = 3\cos 2\pi\left(\frac{70 \times n}{50}\right)$$

$$= 3\cos\frac{14\pi}{5}n = 3\cos\left(2\pi n + \frac{4\pi}{5}n\right) = 3\cos\frac{4\pi}{5}n$$

For integer values of n
$\cos(2\pi n + \theta) = \cos\theta$

From the above analysis, we observe that $x_1(n)$ and $x_2(n)$ are identical , and so $x_2(t)$ is an alias of $x_1(t)$ when sampled at a frequency of 50 Hz.

Example 2.2

Let an analog signal, $x_a(t) = 10\cos 200\pi t$. If the sampling frequency is 150 Hz, find the discrete time signal x(n). Also find an alias frequency corresponding to $F_s = 150\,Hz$.

Solution

$$x(n) = x_a(t)\Big|_{t = nT = \frac{n}{F_s}} = 10\cos 200\pi t\Big|_{t = \frac{n}{F_s}} = 10\cos 200\pi \times \frac{n}{F_s}$$

$$= 10\cos\frac{200\pi \times n}{150} = 10\cos\frac{4\pi}{3}n = 10\cos\left(2\pi - \frac{2\pi}{3}\right)n = 10\cos\frac{2\pi}{3}n = 10\cos 2\pi\frac{1}{3}n$$

We know that the discrete time sinusoids whose frequencies are separated by integer multiples of 2π are identical.

$$\therefore 10\cos\frac{2\pi}{3}n = 10\cos\left(\frac{2\pi}{3} + 2\pi\right)n = 10\cos\frac{8\pi}{3}n = 10\cos 2\pi\frac{4}{3}n$$

Now, $10\cos 2\pi\frac{4}{3}n$ is an alias of $10\cos\frac{2\pi}{3}n$.

Here the frequency of the signal, $10\cos 2\pi\frac{4}{3}n$ is,

$$f = \frac{4}{3} \text{ cycles / sample}$$

We know that, $f = \frac{F}{F_s} \quad \Rightarrow \quad F = f\,F_s = \frac{4}{3} \times 150 = 200\,Hz$

$\therefore$ when, $F_s = 150\,Hz$, $F = 200\,Hz$ is an alias frequency.

Example 2.3

Consider the analog signal, $x_a(t) = 6\cos 50\pi t + 3\sin 200\pi t - 3\cos 100\pi t$.

Determine the minimum sampling frequency and the sampled version of analog signal at this frequency. Sketch the waveform and show the sampling points. Comment on the result.

Solution

The given analog signal can be written as shown below.

$$x_a(t) = 6\cos 50\pi t + 3\sin 200\pi t - 3\cos 100\pi t = 6\cos 2\pi F_1 t + 3\sin 2\pi F_2 t - 3\cos 2\pi F_3 t$$

$$\text{Where, } 2\pi F_1 = 50\pi \quad ; \quad F_1 = 25\,\text{Hz}$$

$$2\pi F_2 = 200\pi \quad ; \quad F_2 = 100\,\text{Hz}$$

$$2\pi F_3 = 100\pi \quad ; \quad F_3 = 50\,\text{Hz}$$

The maximum analog frequency in the signal is 100Hz. The sampling frequency should be twice that of this maximum analog frequency.

i.e., $F_s \geq 2F_{max} \quad \Rightarrow \quad F_s \geq 2 \times 100$

Let, sampling frequency, $F_s = 200\,\text{Hz}$

$$\therefore \quad x_a(nT) = x_a(t)\big|_{t = nT} = x_a(t)\Big|_{t = \frac{n}{F_s}}$$

$$= 6\cos\frac{50\pi n}{200} + 3\sin\frac{200\pi n}{200} - 3\cos\frac{100\pi n}{200} = 6\cos\frac{\pi n}{4} + 3\sin\pi n - 3\cos\frac{\pi n}{2}$$

For integer values of n, $\sin\pi n = 0$.

$$\therefore \quad x_a(nT) = 6\cos\frac{\pi n}{4} - 3\cos\frac{\pi n}{2}$$

The components of analog waveform and the sampling points are shown in fig1.

__Comment__ : In the sampled version of analog signal $x_a(nT)$, the component $3\sin 200\pi t$ will give always zero samples when sampled at 200Hz for any value of n. This is the drawback in sampling at Nyquist rate (i.e., sampling at $F_s = 2F_{max}$).

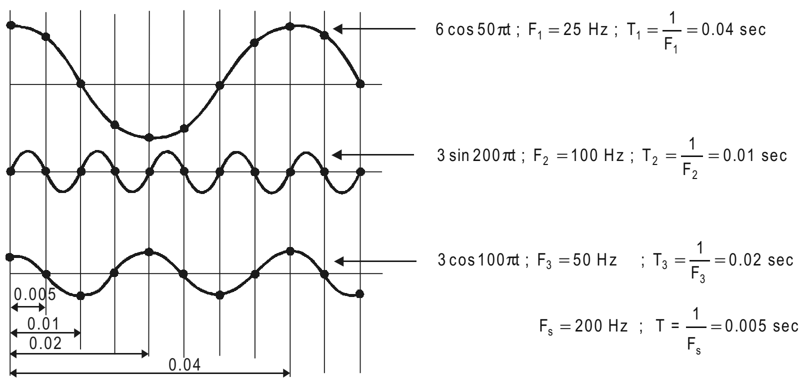


Fig 1 : Sampling points of the components of the signal $x_a(t)$.

2.4 Classification of Discrete Time Signals

The discrete time signals are classified depending on their characteristics. Some ways of classifying discrete time signals are,

1. Deterministic and nondeterministic signals
2. Periodic and aperiodic signals
3. Symmetric and antisymmetric signals
4. Energy and power signals
5. Causal and noncausal signals

2.4.1 Deterministic and Nondeterministic Signals

The signals that can be completely specified by mathematical equations are called ***deterministic signals***.The step, ramp, exponential and sinusoidal signals are examples of deterministic signals.

The signals whose characteristics are random in nature are called ***nondeterministic signals.*** The noise signals from various sources are best examples of nondeterministic signals.

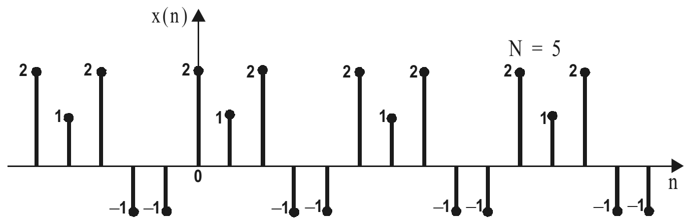
2.4.2 Periodic and Aperiodic Signals

When a discrete time signal x(n), satisfies the condition $\mathbf{x(n+N) = x(n)}$ for integer values of N, then the discrete time signal x(n) is called ***periodic signal***. Here N is the number of samples of a period.

$$\boxed{\text{i.e, if, } \mathbf{x(n+N) = x(n)}, \text{ for all n, then x(n) is periodic.}}$$

The smallest value of N for which the above equation is true is called ***fundamental period***. If there is no value of N that satisfies the above equation, then x(n) is called ***aperiodic*** or ***nonperiodic*** signal.

When N is the fundamental period, the periodic signals will also satisfy the condition x(n + kN) = x(n), where k is an integer. The periodic signals are power signals. The discrete time sinusoidal and complex exponential signals are periodic signals when their fundamental frequency, f_0 is a rational number.



$$x(n) = \{.......2, 1, 2, -1, -1, \underset{\uparrow}{2}, 1, 2, -1, -1, 2, 1, 2, -1, -1, ...\}$$

Fig 2.10 : *Periodic discrete time signal.*

When a discrete time signal is a sum or product of two periodic signals with fundamental periods N_1 and N_2, then the discrete time signal will be periodic with period given by LCM of N_1 and N_2.

Example 2.4

Determine whether following signals are periodic or not. If periodic find the fundamental period.

a) $x(n) = \cos\left(\frac{5\pi}{9}n + 1\right)$ **b)** $x(n) = \sin\left(\frac{n}{9} - \pi\right)$ **c)** $x(n) = \sin\frac{\pi}{8}n^2$

d) $x(n) = e^{\frac{j7\pi n}{4}}$ **e)** $x(n) = 2\cos\frac{5\pi n}{3} + 3e^{j\frac{3\pi n}{4}}$

Solution

a) Given that, $x(n) = \cos\left(\dfrac{5\pi}{9}n + 1\right)$

Let N and M be two integers.

Now, $x(n+N) = \cos\left(\dfrac{5\pi}{9}(n+N)+1\right) = \cos\left(\dfrac{5\pi n}{9} + 1 + \dfrac{5\pi}{9}N\right)$

Since, $\cos(q + 2pM) = \cos q$, for periodicity $\dfrac{5\pi}{9}N$ should be integral multiple of 2π.

Let, $\dfrac{5\pi}{9}N = M \times 2\pi$

$\therefore\ N = M \times 2\pi \times \dfrac{9}{5\pi} = \dfrac{18M}{5}$

Here N is an integer if, M = 5, 10, 15, 20, ...

Let, M = 5 ; $\backslash$ N = 18

When $N = 18$; $x(n+N) = \cos\left(\dfrac{5\pi n}{9} + 1 + \dfrac{5\pi}{9}\times 18\right) = \cos\left(\dfrac{5\pi n}{9} + 1 + 10\pi\right) = \cos\left(\dfrac{5\pi n}{9} + 1\right) = x(n)$

Hence x(n) is periodic with fundamental period of 18 samples.

---

b) Given that, $x(n) = \sin\left(\dfrac{n}{9} - \pi\right)$

Let N and M be two integers.

Now, $x(n+N) = \sin\left(\dfrac{n+N}{9} - \pi\right) = \sin\left(\dfrac{n}{9} + \dfrac{N}{9} - \pi\right) = \sin\left(\dfrac{n}{9} - \pi + \dfrac{N}{9}\right)$

Since, $\sin(q + 2pM) = \sin q$, for periodicity $\dfrac{N}{9}$ should be equal to integral multiple of $2p$.

Let, $\dfrac{N}{9} = M \times 2\pi$

$\backslash$ N = 18 pM

Here N cannot be an integer for any integer value of M and so x(n) will not be periodic.

---

c) Given that, $x(n) = \sin\left(\dfrac{\pi}{8}n^2\right)$

$\therefore\ x(n+N) = \sin\dfrac{\pi}{8}(n+N)^2 = \sin\dfrac{\pi}{8}(n^2 + N^2 + 2nN) = \sin\left(\dfrac{\pi}{8}n^2 + \dfrac{\pi N^2}{8} + \dfrac{\pi N}{4}n\right)$

Let, $\dfrac{\pi N^2}{8} = 2\pi M_1$

$\therefore\ N = 4\sqrt{M_1}$

Now, N is integer for $M_1 = 1^2, 2^2, 3^2, 4^2$ .....

Let, $\dfrac{\pi N}{4} = 2\pi M_2$

$\backslash$ $N = 8\,M_2$

Now, N is integer for $M_2 = 1, 2, 3, 4$ .....

When $M_1 = 2^2$ and $M_2 = 1$, we get a common value for N as, N = 8.

When $N = 8$; $x(n + N) = \sin\left(\dfrac{\pi}{8}n^2 + \dfrac{\pi 8^2}{8} + \dfrac{\pi 8}{4}n\right)$

For interger M,
sin (q + 2p M) = sinq

$= \sin\left(\left(\dfrac{\pi}{8}n^2 + 2\pi n\right) + 4\times 2\pi\right) = \sin\left(\dfrac{\pi}{8}n^2 + 2\pi n\right)$

$= \sin\dfrac{\pi}{8}n^2 = x(n)$

$\backslash$ x(n) is periodic with fundamental period, N = 18 samples.

d) Given that, $x(n) = e^{\frac{j7\pi n}{4}}$

Let N and M be two integers.

Now, $x(n+N) = e^{\frac{j7\pi(n+N)}{4}} = e^{\frac{j7\pi n}{4}} e^{\frac{j7\pi N}{4}}$

Since, $e^{j2\pi M} = 1$, for periodicity $\frac{7\pi N}{4}$ should be an integral multiple of 2π.

Let, $\frac{7\pi N}{4} = M \times 2\pi,$

$$\therefore \; N = M \times 2\pi \times \frac{4}{7\pi} = \frac{8M}{7}$$

Here, N is integer, when M = 7, 14, 21, ...

When M = 7 ; N = 8

$\therefore$ x(n) is periodic with fundamental period of 8 samples.

---

e) Given that, $x(n) = 2\cos\frac{5\pi n}{3} + 3e^{j\frac{3\pi n}{4}}$

Let, $x(n) = x_1(n) + x_2(n)$

where, $x_1(n) = 2\cos\frac{5\pi n}{3}$

$x_2(n) = 3e^{j\frac{3\pi n}{4}}$

Consider, $x_1(n) = 2\cos\frac{5\pi n}{3}$

$$\therefore \; x_1(n+N_1) = 2\cos\frac{5\pi(n+N_1)}{3}$$

$$= 2\cos\left(\frac{5\pi n}{3} + \frac{5\pi N_1}{3}\right) \quad .....(1)$$

Let, $\frac{5\pi N_1}{3} = 2\pi M_1 \;\Rightarrow\; N_1 = \frac{6}{5}M_1$

Let, $M_1 = 5$; $\therefore$ $N_1 = 6$

Substitute $N_1 = 6$ in equation (1),

$$\therefore \; x_1(n+N_1) = 2\cos\left(\frac{5\pi n}{3} + \frac{5\pi}{3} \times 6\right)$$

$$= 2\cos\left(\frac{5\pi n}{3} + 5 \times 2\pi\right)$$

$$= 2\cos\frac{5\pi n}{3} = x_1(n)$$

For integer M, $\cos(\theta + 2\pi M) = \cos\theta$

$\therefore$ $x_1(n)$ is periodic with fundamental period, $N_1 = 6$ samples.

Consider, $x_2(n) = 3e^{j\frac{3\pi n}{4}}$

$$\therefore \; x_2(n+N_2) = 3e^{j\frac{3\pi(n+N_2)}{4}}$$

$$= 3e^{j\left(\frac{3\pi n}{4} + \frac{3\pi N_2}{4}\right)} \quad .....(2)$$

Let, $\frac{3\pi N_2}{4} = 2\pi M_2 \;\Rightarrow\; N_2 = \frac{8}{3}M_2$

Let, $M_2 = 3$; $\therefore$ $N_2 = 8$

Substitute $N_2 = 8$ in equation (2),

$$\therefore \; x_2(n+N_2) = 3e^{j\left(\frac{3\pi n}{4} + \frac{3\pi \times 8}{4}\right)}$$

$$= 3e^{j\left(\frac{3\pi n}{4} + 3\times 2\pi\right)}$$

$$= 3e^{j\frac{3\pi n}{4}} = x_2(n)$$

For integer M, $e^{j(\theta + 2\pi M)} = e^{j\theta}$

$\therefore$ $x_2(n)$ is periodic with fundamental period, $N_2 = 8$ samples.

Here, $x(n) = x_1(n) + x_2(n)$, and $x_1(n)$ is periodic with period $N_1 = 6$, and $x_2(n)$ is periodic with period $N_2 = 8$. Therefore, x(n) is periodic with period N, where N is LCM of N_1 and N_2.

The LCM of 6 and 8 is 24.

$\therefore$ N = 24

$\therefore$ x(n) is periodic with fundamental period, N = 24.

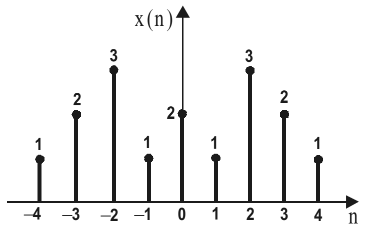
2.4.3 Symmetric (Even) and Antisymmetric (Odd) Signals

The discrete time signals may exhibit symmetry or antisymmetry with respect to n = 0. When a discrete time signal exhibits symmetry with respect to n = 0 then it is called an ***even signal.*** Therefore, the even signal satisfies the condition,

$$\boxed{x(-n) = x(n)}$$

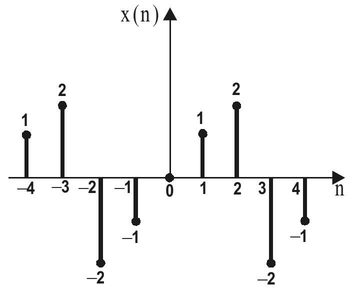
When a discrete time signal exhibits antisymmetry with respect to n = 0, then it is called an ***odd signal***. Therefore the odd signal satisfies the condition,

$$\boxed{x(-n) = -x(n)}$$



$$x(n) = \{1, 2, 3, 1, \underset{\uparrow}{2}, 1, 3, 2, 1\}$$

Fig 2.11a : *Symmetric (or even) signal.*



$$x(n) = \{1, 2, -2, -1, \underset{\uparrow}{0}, 1, 2, -2, -1\}$$

Fig 2.11b : *Antisymmetric (or odd) signal.*

Fig 2.11 : *Symmetric and antisymmetric discrete time signal.*

A discrete time signal x(n) which is neither even nor odd can be expressed as a sum of even and odd signal.

Let, $x(n) = x_e(n) + x_o(n)$

where, $x_e(n)$ = Even part of x(n)

$x_o(n)$ = Odd part of x(n)

Note : If x(n) is even then its odd part will be zero. If x(n) is odd then its even part will be zero.

Now, it can be proved that,

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

Odd part, $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

Proof :

Let, $x(n) = x_e(n) + x_o(n)$ ......(2.8)

On replacing n by –n in equation (2.8) we get,

$x(-n) = x_e(-n) + x_o(-n)$ ......(2.9)

Since $x_e(n)$ is even, $x_e(-n) = x_e(n)$

Since $x_o(n)$ is odd, $x_o(-n) = -x_o(n)$

Hence the equation (2.9) can be written as,

$x(-n) = x_e(n) - x_o(n)$ ......(2.10)

On adding equation (2.8) and (2.10) we get,

$$x(n) + x(-n) = 2\,x_e(n)$$

$$\therefore\ x_e(n) = \frac{1}{2}\big[x(n) + x(-n)\big]$$

On subtracting equation (2.10) from equation (2.8) we get,

$$x(n) - x(-n) = 2\,x_o(n)$$

$$\therefore\ x_o(n) = \frac{1}{2}\big[x(n) - x(-n)\big]$$

Example 2.5

Determine the even and odd parts of the signals.

a) $x(n) = 3^n$ **b)** $x(n) = 3\,e^{j\frac{\pi}{5}n}$ **c)** $x(n) = \{2,\ -2,\ 6,\ -2\}$

Solution

a) Given that, $x(n) = 3^n$

$$\therefore\ x(-n) = 3^{-n}$$

$$\text{Even part, } x_e(n) = \frac{1}{2}[x(n) + x(-n)] = \frac{1}{2}[3^n + 3^{-n}]$$

$$\text{Odd part, } x_o(n) = \frac{1}{2}[x(n) - x(-n)] = \frac{1}{2}[3^n - 3^{-n}]$$

b) Given that, $x(n) = 3\,e^{i\frac{\pi}{5}n}$

$$x(n) = 3\,e^{j\frac{\pi}{5}n} = 3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n$$

$$\therefore x(-n) = 3\,e^{-j\frac{\pi}{5}n} = 3\cos\frac{\pi}{5}n - j3\sin\frac{\pi}{5}n$$

$$\text{Even part, } x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}\left[3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n + 3\cos\frac{\pi}{5}n - j3\sin\frac{\pi}{5}n\right] = \frac{1}{2}\left[6\cos\frac{\pi}{5}n\right] = 3\cos\frac{\pi}{5}n$$

$$\text{Odd part, } x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= \frac{1}{2}\left[3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n - 3\cos\frac{\pi}{5}n + j3\sin\frac{\pi}{5}n\right]$$

$$= \frac{1}{2}\left[j6\sin\frac{\pi}{5}n\right] = j3\sin\frac{\pi}{5}n$$

c) Given that, $x(n) = \{\underset{\uparrow}{2}, -2, 6, -2\}$

Given that, $x(n) = \{2, -2, 6, -2\}$, \ $x(0) = 2$; $x(1) = -2$; $x(2) = 6$; $x(3) = -2$

$x(-n) = \{-2, 6, -2, 2\}$, \ $x(-3) = -2$; $x(-2) = 6$; $x(-1) = -2$; $x(0) = 2$

Even part, $x_e(n) = \frac{1}{2}\,[x(n) + x(-n)]$

At $n = -3$; $x(n) + x(-n) = 0 + (-2) = -2$

At $n = -2$; $x(n) + x(-n) = 0 + 6 = 6$

At $n = -1$; $x(n) + x(-n) = 0 + (-2) = -2$

At $n = 0$; $x(n) + x(-n) = 2 + 2 = 4$

At $n = 1$; $x(n) + x(-n) = -2 + 0 = -2$

At $n = 2$; $x(n) + x(-n) = 6 + 0 = 6$

At $n = 3$; $x(n) + x(-n) = -2 + 0 = -2$

$\therefore\ x(n) + x(-n) = \{-2, 6, -2, \underset{\uparrow}{4}, -2, 6, -2\}$

$\therefore\ x_e(n) = \frac{1}{2}\,[x(n) + x(-n)]$

$= \{-1,\ 3,\ -1,\ \underset{\uparrow}{2},\ -1,\ 3,\ -1\}$

Odd part, $x_o(n) = \frac{1}{2}\,[x(n) - x(-n)]$

At $n = -3$; $x(n) - x(-n) = 0 - (-2) = 2$

At $n = -2$; $x(n) - x(-n) = 0 - 6 = -6$

At $n = -1$; $x(n) - x(-n) = 0 - (-2) = 2$

At $n = 0$; $x(n) - x(-n) = 2 - 2 = 0$

At $n = 1$; $x(n) - x(-n) = -2 - 0 = -2$

At $n = 2$; $x(n) - x(-n) = 6 - 0 = 6$

At $n = 3$; $x(n) - x(-n) = -2 - 0 = -2$

$\therefore\ x(n) - x(-n) = \{2, -6, 2, \underset{\uparrow}{0}, -2, 6, -2\}$

$\therefore\ x_o(n) = \frac{1}{2}\,[x(n) - x(-n)]$

$= \{1,\ -3,\ 1,\ \underset{\uparrow}{0},\ -1,\ 3,\ -1\}$

2.4.4 Energy and Power Signals

The ***energy*** E of a discrete time signal x(n) is defined as,

$$\boxed{\text{Energy, } E = \sum_{n=-\infty}^{\infty} |x(n)|^2} \qquad .....(2.11)$$

The energy of a signal may be finite or infinite, and can be applied to complex valued and real valued signals.

If energy E of a discrete time signal is finite and nonzero, then the discrete time signal is called an ***energy signal.*** The exponential signals are examples of energy signals.

The average ***power*** of a discrete time signal x(n) is defined as,

$$\boxed{\text{Power, } P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2} \qquad .....(2.12)$$

If power P of a discrete time signal is finite and nonzero, then the discrete time signal is called a ***power signal***. The periodic signals are examples of power signals.

For energy signals, the energy will be finite and average power will be zero. For power signals the average power is finite and energy will be infinite.

$\therefore$ For energy signal, $0 < E < \infty$ and $P = 0$

For power signal, $0 < P < \infty$ and $E = \infty$

Example 2.6

Determine whether the following signals are energy or power signals.

a) $x(n) = \left(\frac{1}{4}\right)^n u(n)$ **b)** $x(n) = \sin\left(\frac{\pi}{3}n\right)$ **c)** $x(n) = u(n)$

Solution

a) Given that, $x(n) = \left(\frac{1}{4}\right)^n u(n)$

Here, $x(n) = \left(\frac{1}{4}\right)^n u(n)$ for all n.

$$\therefore\ x(n) = \left(\frac{1}{4}\right)^n = 0.25^n \quad ; \quad n \ge 0$$

Infinite geometric series sum formula.
$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=0}^{\infty} \left|(0.25)^n\right|^2 = \sum_{n=0}^{\infty} (0.25^2)^n$$

$$= \sum_{n=0}^{\infty} (0.0625)^n = \frac{1}{1-0.0625} = 1.067 \text{ joules}$$

Using infinite geometric series sum formula.

$$\text{Power, } P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} \left|(0.25)^n\right|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.25^2)^n = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.0625)^n$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{(0.0625)^{N+1}-1}{0.0625-1}$$

Using finite geometric series sum formula.

$$= \frac{1}{\infty} \times \frac{0.0625^{\infty}-1}{0.0625-1} = 0$$

Finite geometric series sum formula.
$$\sum_{n=0}^{N} C^n = \frac{C^{N+1}-1}{C-1}$$

Here E is finite and P is zero and so x(n) is an energy signal.

---

b) Given that, $x(n) = \sin\left(\frac{\pi}{3}n\right)$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=-\infty}^{+\infty} \sin^2\left(\frac{\pi}{3}n\right) = \sum_{n=-\infty}^{+\infty} \frac{1-\cos\frac{2\pi}{3}n}{2}$$

$$= \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty}\left(1-\cos\frac{2\pi}{3}n\right)\right) = \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty} 1^n - \sum_{n=-\infty}^{+\infty} \cos\frac{2\pi}{3}n\right) = \frac{1}{2}(\infty - 0) = \infty$$

Note : Sum of infinite 1's is infinity. Sum of samples of one period of cosinusoidal signal is zero.

$$\text{Power, } P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \sin^2\frac{\pi n}{3}$$

$$\therefore\ P = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=-N}^{N}\frac{\left(1-\cos\frac{2\pi}{3}n\right)}{2}$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\ \frac{1}{2}\left[\sum_{n=-N}^{N}1^n - \sum_{n=-N}^{N}\cos\frac{2\pi}{3}n\right]$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\ \frac{1}{2}\left[\underbrace{1+1+\ldots\ldots+1}_{N\ terms}+1+\underbrace{1+\ldots\ldots+1+1}_{N\ terms}-0\right]$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\ \frac{1}{2}\left[2N+1\right] = \underset{N\to\infty}{Lt}\ \frac{1}{2} = \frac{1}{2}\ \text{watts}$$

Since P is finite and E is infinite, x(n) is a power signal.

> *Note: The term $\cos\frac{2\pi}{3}n$ is periodic with periodicity of 3 samples. Samples of $\cos\frac{2\pi}{3}n$ for two periods are given below. It can be observed that sum of samples of a period is zero.*
>
> *When n = 0 ; $\cos\frac{2\pi}{3}n = 1$, When n = 1 ; $\cos\frac{2\pi}{3}n = -0.5$, When n = 2 ; $\cos\frac{2\pi}{3}n = -0.5$*
>
> *When n = 3 ; $\cos\frac{2\pi}{3}n = 1$, When n = 4 ; $\cos\frac{2\pi}{3}n = -0.5$, When n = 5 ; $\cos\frac{2\pi}{3}n = -0.5$*

c) Given that, x(n) = u(n)

$$E = \sum_{n=-\infty}^{+\infty}|x(n)|^2 = \sum_{n=0}^{+\infty}(u(n))^2$$

$$= \sum_{n=0}^{+\infty}u(n) = 1+1+1\ \ldots\ldots\ldots\infty = \infty$$

$$P = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2 = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=0}^{N}u(n) = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\left(\underbrace{1+1+1+\ldots\ldots\ldots+1}_{N+1\ terms}\right)$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\ (N+1) = \underset{N\to\infty}{Lt}\ \frac{N\left(1+\frac{1}{N}\right)}{N\left(2+\frac{1}{N}\right)} = \frac{1+\frac{1}{\infty}}{2+\frac{1}{\infty}} = \frac{1+0}{2+0} = \frac{1}{2}\ \text{watts}$$

Since P is finite and E is infinite, x(n) is a power signal.

2.4.5 Causal, Noncausal and Anticausal signals

A discrete time signal is said to be ***causal***, if it is defined for n ³ 0. Therefore if x(n) is causal, then x(n) = 0 for n < 0.

A discrete time signal is said to be ***noncausal***, if it is defined for either n ≤ 0, or for both n ≤ 0 and n > 0. Therefore if x(n) is noncausal, then x(n) ≠ 0 for n < 0. A noncausal signal can be converted to causal signal by multiplying the noncausal signal by a unit step signal, u(n).

When a noncausal discrete time signal is defined only for n ≤ 0, it is called an ***anticausal signal***.

Examples of Causal and Noncausal Signals

$$\left.\begin{array}{l} x(n)=\{\underline{1},-1,2,-2,3,-3\} \\ x(n)=\{\underline{2},2,3,3,\ldots\ldots\ldots\ldots\} \end{array}\right\} \text{Causal signals}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} x(n)=\{1,-1,2,-2,3,\underline{-3}\} \\ x(n)=\{\ldots\ldots\ldots\ldots,2,2,3,\underline{3}\} \end{array}\right\} \text{Anticausal signals} \\ x(n)=\{2,3,4,\underline{5},4,3,2\} \\ x(n)=\{\ldots\ldots,2,3,4,\underline{5},4,3,2,\ldots\ldots\} \end{array}\right\} \text{Noncausal signals}$$

2.5 Mathematical Operations on Discrete Time Signals

Some of the mathematical operations that can be performed on discrete time signals are,

1. Scaling : Amplitude scaling and time scaling
2. Folding
3. Shifting : Right shift (or advance) and left shift (or delay)
4. Addition
5. Multiplication

2.5.1. Scaling of Discrete Time Signals

Amplitude Scaling (or Scalar Multiplication)

Amplitude scaling of a discrete time signal by a constant A is accomplished by multiplying the value of every signal sample by the constant A.

Example :

Let y(n) be amplitude scaled signal of x(n), then y(n) = A x(n)

Let, $x(n) = 10$; $n = 0$	and $A = 0.2$,	When $n = 0$;	$y(0) = A\,x(0) = 0.2 \times 10$	$= 2.0$
$= 16$; $n = 1$		When $n = 1$;	$y(1) = A\,x(1) = 0.2 \times 16$	$= 3.2$
$= 20$; $n = 2$		When $n = 2$;	$y(2) = A\,x(2) = 0.2 \times 20$	$= 4.0$

Time Scaling (or Downsampling and Upsampling)

There are two ways of time scaling a discrete time signal. They are downsampling and upsampling.

In a signal x(n), if n is replaced by Dn, where D is an integer, then it is called ***downsampling***.

In a signal x(n), if n is replaced by $\frac{n}{I}$, where I is an integer, then it is called ***upsampling***.

Example :

If $x(n) = b^n$; $n \geq 0$; $0 < b < 1$, then

$x_1(n) = x(2n)$ will be a down sampled version of x(n) and

$x_2(n) = x\left(\frac{n}{2}\right)$ will be an up sampled version on x(n).

When $n = 0$; $x_1(0) = x(0) = b^0$

When $n = 1$; $x_1(1) = x(2) = b^2$

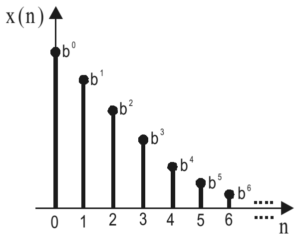
When $n = 2$; $x_1(2) = x(4) = b^4$ and so on.

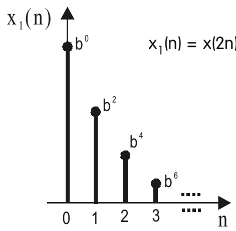
When $n = 0$; $x_2(0) = x\left(\frac{0}{2}\right) = x(0) = b^0$

When $n = 1$; $x_2(1) = x\left(\frac{1}{2}\right) = 0$

When $n = 2$; $x_2(2) = x\left(\frac{2}{2}\right) = x(1) = b^1$

When $n = 3$; $x_2(3) = x\left(\frac{3}{2}\right) = 0$ and so on.





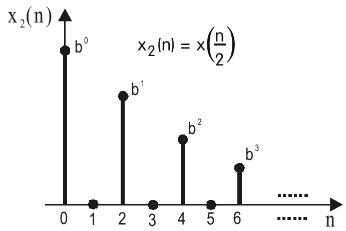


Fig 2.12a : A discrete time signal x(n). *Fig 2.12b : Down sampled signal of x(n).* *Fig 2.12c : Up sampled signal x(n).*

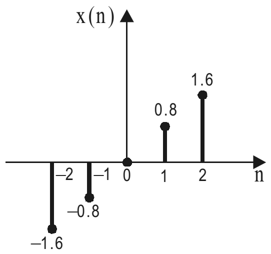
Fig 2.12 : A discrete time signal and its time scaled version.

2.5.2. Folding (or Reflection or Transpose) of Discrete Time Signals

The ***folding*** of a discrete time signal x(n) is performed by changing the sign of the time base n in x(n). The folding operation produces a signal x(–n) which is a mirror image of the signal x(n) with respect to time origin n = 0.

Example :

Let $x(n) = 0.8n$; $-2 \leq n \leq 2$. Now the folded signal, $x_1(n) = x(-n) = -0.8n$; $-2 \leq n \leq 2$



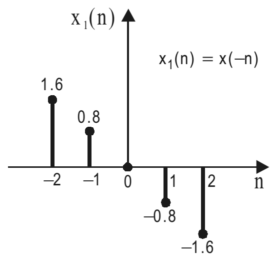


Fig 2.13a : A discrete time signal x(n). *Fig 2.13b : Folded signal of x(n).*

Fig 2.13 : A discrete time signal and its folded version.

2.5.3. Time Shifting of Discrete Time Signals

A signal x(n) may be shifted in time by replacing the independent variable n by n – m, where m is an integer. [i.e, x(n–m) is shifted version of x(n)]. If m is a positive integer, the time shift results in a delay by m units of time. If m is a negative integer, the time shift results in an advance of the signal by |m| units in time. The ***delay*** results in shifting each sample of x(n) to the right. The ***advance*** results in shifting each sample of x(n) to the left.

Example :

Let, $x(n) = 3 \; ; \; n = 2$

$= 2 \; ; \; n = 3$

$= 1 \; ; \; n = 4$

$= 0 \; ;$ for other n

Let, $x_1(n) = x(n-2)$, where $x_1(n)$ is delayed signal of x(n)

When n = 4 ; $x_1(4) = x(4-2) = x(2) = 3$

When n = 5 ; $x_1(5) = x(5-2) = x(3) = 2$

When n = 6 ; $x_1(6) = x(6-2) = x(4) = 1$

The sample x(2) is available at n = 2 in the original sequence x(n), but the same sample is available at n = 4 in $x_1(n)$. Similarly every sample of x(n) is delayed by two sampling times.

Let, $x_2(n) = x(n+2)$, where $x_2(n)$ is an advanced signal of x(n)

When n = 0 ; $x_2(0) = x(0+2) = x(2) = 3$

When n = 1 ; $x_2(1) = x(1+2) = x(3) = 2$

When n = 2 ; $x_2(2) = x(2+2) = x(4) = 1$

The sample x(2) is available at n = 2 in the original sequence x(n), but the same sample is available at n = 0 in $x_2(n)$. Similarly every sample of x(n) is advanced by two sampling times. Hence the signal $x_2(n)$ is an advanced version of x(n).

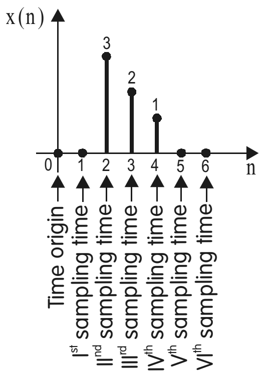


Fig 2.14a : A discrete time signal x(n) .

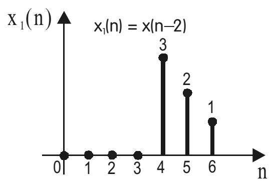


Fig 2.14b : Delayed signal of x(n).

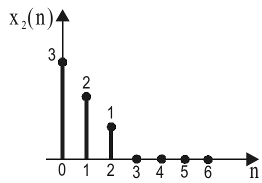


Fig 2.14c : Advanced signal of x(n).

Fig 2.14 : A discrete time signal and its shifted version.

Delayed Unit Impulse Signal

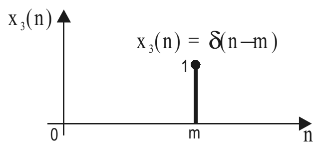


Fig 2.15 : Delayed unit impulse.

The unit impulse signal is defined as,

$\delta(n) = 1 \; ;$ for $n = 0$

$= 0 \; ;$ for $n \neq 0$

The unit impulse signal delayed by m units of time is denoted as $\delta(n-m)$.

Now, $\delta(n-m) = 1 \; ; \; n = m$

$= 0 \; ; \; n \neq m$

Delayed Unit Step Signal

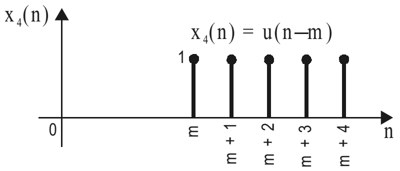


***Fig 2.16 :** Delayed unit step signal.*

The unit step signal is defined as,

$$u(n) = 1 \;;\; \text{for } n \geq 0$$
$$= 0 \;;\; \text{for } n < 0$$

The unit step signal delayed by m units of time is denoted as $u(n - m)$.

Now, $u(n - m) = 1 \;;\; n \geq m$
$= 0 \;;\; n < m$

2.5.4. Addition of Discrete Time Signals

The ***addition*** of two discrete time signals is performed on a sample-by-sample basis.

The sum of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the sum of the samples of these two signals at that instant.

i.e., $y(n) = x_1(n) + x_2(n) \;;\; -\infty < n < \infty$.

Example :

Let, $x_1(n) = \{2, 2, -1\}$ and $x_2(n) = \{-1, 1, 2\}$

When $n = 0$; $y(0) = x_1(0) + x_2(0) = 2 + (-1) = 1$

When $n = 1$; $y(1) = x_1(1) + x_2(1) = 2 + 1 = 3$

When $n = 2$; $y(2) = x_1(2) + x_2(2) = -1 + 2 = 1$

$\therefore\; y(n) = x_1(n) + x_2(n) = \{1, 3, 1\}$

2.5.5. Multiplication of Discrete Time Signals

The ***multiplication*** of two discrete time signals is performed on a sample-by-sample basis.The product of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the product of the samples of these two signals at that instant. The product is also called ***modulation***.

Example :

Let, $x_1(n) = \{ 2, 2, -1 \}$ and $x_2(n) = \{ -1, 1, 2 \}$

When $n = 0$; $y(0) = x_1(0) \times x_2(0) = 2 \times (-1) = -2$

When $n = 1$; $y(1) = x_1(1) \times x_2(1) = 2 \times 1 = 2$

When $n = 2$; $y(2) = x_1(2) \times x_2(2) = -1 \times 2 = -2$

$\therefore\; y(n) = x_1(n) \times x_2(n) = \{-2, 2, -2\}$

2.6 Discrete Time System

A ***discrete time system*** is a device or algorithm that operates on a discrete time signal, called the input or excitation, according to some well-defined rule, to produce another discrete time signal called the output or the response of the system. We can say that the input signal $x(n)$ is transformed by the system into a signal $y(n)$, and the transformation can be expressed mathematically as shown in equation (2.13).The diagrammatic representation of discrete time system is shown in fig 2.17.

Response, $y(n) = \mathcal{H}\{x(n)\}$ .....(2.13)

where, $\mathcal{H}$ denotes the transformation (also called an operator).

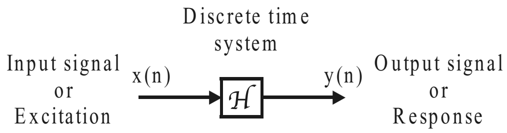


***Fig 2.17 :** Representation of discrete time system.*

LTI System

A discrete time system is linear if it obeys the principle of superposition and it is time invariant if its input-output relationship does not change with time. When a discrete time system satisfies the properties of linearity and time invariance then it is called an ***LTI system*** (Linear Time Invariant system).

Impulse Response

When the input to a discrete time system is a unit impulse $\delta(n)$ then the output is called an ***impulse response*** of the system and is denoted by h(n).

∴ Impulse Response, $h(n) = \mathcal{H}\{\delta(n)\}$ .....(2.14)

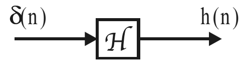


***Fig 2.18 :** Discrete time system with impulse input.*

2.6.1 Mathematical Equation Governing Discrete Time System

The mathematical equation governing the discrete time system can be developed as shown below.

The response of a discrete time system at any time instant depends on the present input, past inputs and past outputs.

Let us consider the response at n = 0. Let us assume a relaxed system and so at n = 0, there is no past input or output. Therefore the response at n = 0, is a function of present input alone.

i.e., $y(0) = F[x(0)]$

Let us consider the response at n =1. Now the present input is x(1), the past input is x(0) and past output is y(0). Therefore the response at n = 1, is a function of x(1), x(0), y(0).

i.e., $y(1) = F[y(0), x(1), x(0)]$

Let us consider the response at n = 2. Now the present input is x(2), the past inputs are x(1) and x(0), and past outputs are y(1) and y(0). Therefore the response at n = 2, is a function of x(2), x(1), x(0), y(1), y(0).

i.e., $y(2) = F[y(1), y(0), x(2), x(1), x(0)]$

Similarly, at n = 3, $y(3) = F[y(2), y(1), y(0), x(3), x(2), x(1), x(0)]$

at n = 4, $y(4) = F[y(3), y(2), y(1), y(0), x(4), x(3), x(2), x(1), x(0)]$ and so on.

In general, at any time instant n,

$$y(n) = F[y(n-1), y(n-2), y(n-3), ...y(1), y(0), x(n), x(n-1), x(n-2), x(n-3) ..... x(1), x(0)] \quad .....(2.15)$$

For an LTI system, the response y(n) can be expressed as a weighted summation of dependent terms. Therefore the equation (2.15) can be written as,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \dots\dots\dots\dots$$
$$+ b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3) + \dots\dots\dots \qquad \text{.....(2.16)}$$

where, $a_1, a_2, a_3, \dots$ and $b_0, b_1, b_2, b_3, \dots$ are constants.

Note : Negative constants are inserted for output signals, because output signals are feedback from output to input. Positive constants are inserted for input signals, because input signals are feed forward from input to output.

Practically, the response y(n) at any time instant n, may depend on N number of past outputs, present input and M number of past inputs where M £ N. Hence the equation (2.16) can be written as,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \dots\dots\dots - a_N\, y(n-N)$$
$$+ b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3) + \dots\dots\dots + b_M\, x(n-M)$$

$$\boxed{\therefore \quad y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)} \qquad \text{.....(2.17)}$$

The equation (2.17) is a constant coefficient ***difference equation***, governing the input-output relation of an LTI discrete time system.

In equation (2.17) the value of "N" gives the ***order*** of the system.

If N = 1, the discrete time system is called 1^{st} order system

If N = 2, the discrete time system is called 2^{nd} order system

If N = 3, the discrete time system is called 3^{rd} order system , and so on.

The general difference equation governing 1^{st} order discrete time LTI system is,

$$y(n) = -a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1)$$

The general difference equation governing 2^{nd} order discrete time LTI system is,

$$y(n) = -a_2\, y(n-2) - a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2)$$

2.6.2 Block Diagram and Signal Flow Graph Representation of Discrete Time System

The discrete time system can be represented diagrammatically by ***block diagram*** or ***signal flow graph***. These diagrammatic representations are useful for physical implementation of discrete time system in hardware or software.

The basic elements employed in block diagram or signal flow graph are adder, constant multiplier, unit delay element and unit advance element.

Adder : An adder is used to represent addition of two discrete time signals.

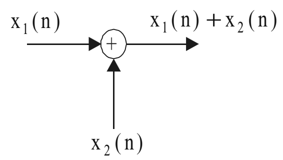
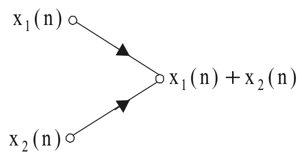
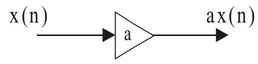
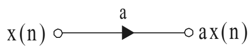
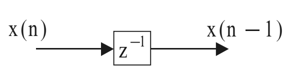
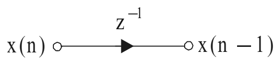
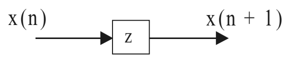
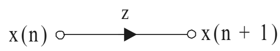
Constant Multiplier : A constant multiplier is used to represent multiplication of a scaling factor (constant) to a discrete time signal.

Unit Delay Element : A unit delay element is used to represent the delay of samples of a discrete time signal by one sampling time.

Unit Advance Element : A unit advance element is used to represent the advance of samples of a discrete time signal by one sampling time.

The symbolic representation of the basic elements of block diagram and signal flow graph are listed in table 2.1.

Table 2.1 : Basic Elements of Block Diagram and Signal Flow Graph

Element	Block diagram representation	Signal flow graph representation
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

Example 2.7

Construct the block diagram and signal flow graph of the discrete time systems whose input-output relations are described by the following difference equations.

a) $y(n) = 0.7\,x(n) + 0.7\,x(n-1)$

b) $y(n) = 0.4\,y(n-1) + x(n) - 3\,x(n-2)$

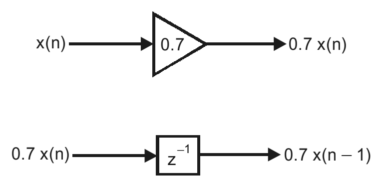
c) $y(n) = 0.2\,y(n-1) + 0.7\,x(n) + 0.9\,x(n-1)$

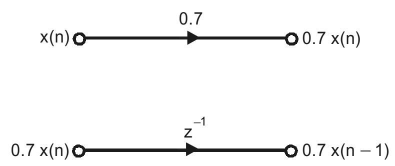
Solution

a) Given that, $y(n) = 0.7\,x(n) + 0.7\,x(n-1)$

The individual terms of the given equation are $0.7\,x(n)$ and $0.7\,x(n-1)$. They are represented by basic elements as shown below.

Block diagram representation | **Signal flow graph representation**





The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

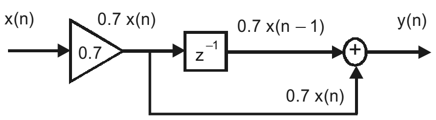


Fig 1 : Block diagram of the system $y(n) = 0.7\,x(n) + 0.7\,x(n-1)$.

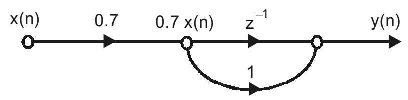
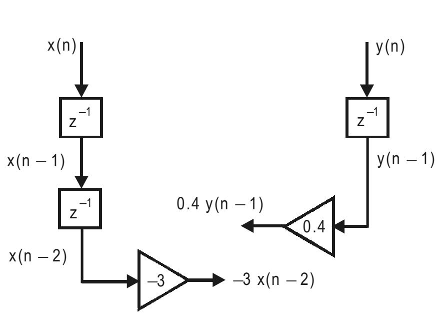


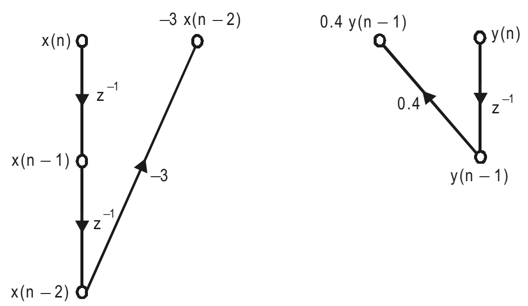
Fig 2 : Signal flow graph of the system $y(n) = 0.7\,x(n) + 0.7\,x(n-1)$.

b) Given that, y(n) = 0.4 y(n – 1) + x(n) – 3 x(n – 2)

The individual terms of the given equation are 0.4 y(n – 1) and – 3 x(n – 2). They are represented by basic elements as shown below.

Block diagram representation | **Signal flow graph representation**





The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

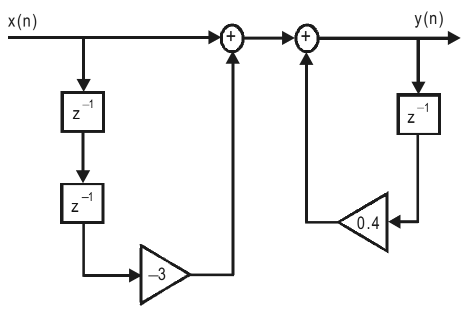


Fig 3 : Block diagram of the system described by the equation $y(n) = 0.4\,y(n-1) + x(n) - 3\,x(n-2)$.

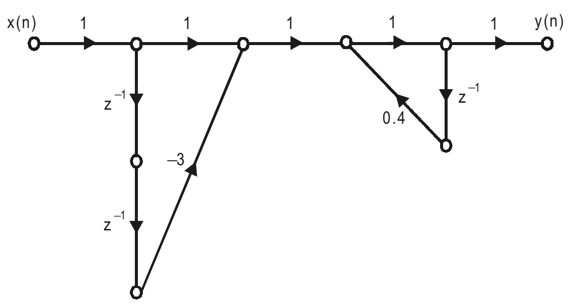
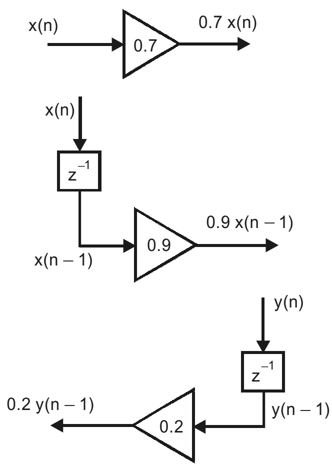


Fig 4 : Signal flow graph of the system described by the equation $y(n) = 0.4\,y(n-1) + x(n) - 3\,x(n-2)$.

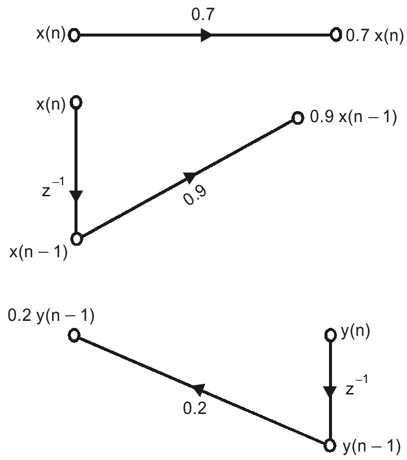
c) Given that, y(n) = 0.2 y(n – 1) + 0.7 x(n) + 0.9 x(n – 1)

The individual terms of the given equation are 0.2 y(n – 1), 0.7 x(n) and 0.9 x(n – 1). They are represented by basic elements as shown below.

Block diagram representation



Signal flow graph representation



The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

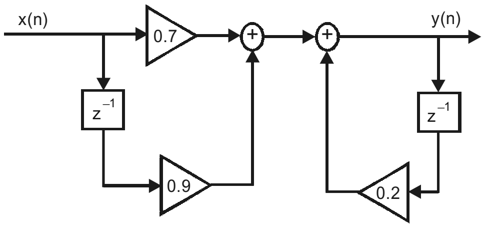


Fig 5 : Block diagram of the system described by the equation y(n) = 0.2 y(n – 1) + 0.7 x(n)+ 0.9 x(n – 1).

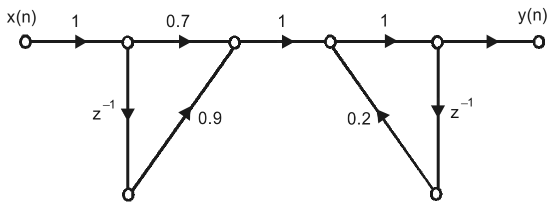


Fig 6 : Signal flow graph of the system described by the equation y(n) = 0.2 y(n – 1) + 0.7 x(n) + 0.9 x(n – 1).

2.7 Response of LTI Discrete Time System in Time Domain

The general equation governing an LTI discrete time system is,

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\therefore \quad y(n) + \sum_{m=1}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\text{(or)} \quad \sum_{m=0}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m) \quad \text{with } a_o = 1 \qquad .....(2.18)$$

The solution of the difference equation (2.18) is the ***response*** y(n) of LTI system, which consists of two parts. In mathematics, the two parts of the solution y(n) are homogeneous solution $y_h(n)$ and particular solution $y_p(n)$.

$$\boxed{\therefore \text{ Response, } y(n) = y_h(n) + y_p(n)}$$ .....(2.19)

The ***homogeneous solution*** is the response of the system when there is no input.The ***particular solution*** $y_p(n)$ is the solution of difference equation for specific input signal x(n) for $n \geq 0$.

In signals and systems, the two parts of the solution y(n) are called zero-input response $y_{zi}(n)$ and zero-state response $y_{zs}(n)$.

$$\boxed{\therefore \text{ Response, } y(n) = y_{zi}(n) + y_{zs}(n)}$$ .....(2.20)

The ***zero-input response*** is mainly due to initial conditions (or initial stored energy) in the system. Hence zero-input response is also called ***free response*** or ***natural response***. The ***zero-input response*** is given by homogeneous solution with constants evaluated using initial conditions.

The ***zero-state response*** is the response of the system due to input signal and with zero initial condition. Hence the zero-state response is called forced response.The ***zero-state response*** or ***forced response*** is given by the sum of homogeneous solution and particular solution with zero initial conditions.

2.7.1 Zero-Input Response or Homogeneous Solution

The ***zero-input response*** is obtained from homogeneous solution $y_h(n)$ with constants evaluated using initial condition.

$$\therefore \text{ Zero - input response, } y_{zi}(n) = y_h(n)\Big|_{\text{with constants evaluated using initial conditions}}$$

The ***homogeneous solution*** is obtained when x(n) = 0. Therefore the homogeneous solution is the solution of the equation,

$$\sum_{m=0}^{N} a_m \, y(n-m) = 0$$ .....(2.21)

Let us assume that the solution of equation (2.21) is in the form of an exponential.

i.e., $y(n) = \lambda^n$

On substituting $y(n) = \lambda^n$ in equation (2.21) we get,

$$\sum_{m=0}^{N} a_m \, \lambda^{n-m} = 0$$

On expanding the above equation (by taking $a_0 = 1$), we get,

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + ... + a_{N-1} \lambda^{n-(N-1)} + a_N \lambda^{n-N} = 0$$

$$\lambda^{n-N} (\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + ... + a_{N-1} \lambda + a_N) = 0$$

Now, the ***characteristic polynomial*** of the system is given by,

$$\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + ... + a_{N-1} \lambda + a_N = 0$$

The characteristic polynomial has N roots, which are denoted as $\lambda_1, \lambda_2, ... \lambda_N$.

The roots of the characteristic polynomial may be distinct real roots, repeated real roots or complex. The assumed solutions for various types of roots are given below.

Distinct Real Roots

Let the roots $\lambda_1, \lambda_2, \lambda_3, ... \lambda_N$ be distinct real roots. Now the homogeneous solution will be in the form,

$$y_h(n) = C_1 \lambda_1^n + C_2 \lambda_2^n + C_3 \lambda_3^n + ... + C_N \lambda_N^n$$

where, $C_1, C_2, C_3, ...C_N$ are constants that can be evaluated using initial conditions.

Repeated Real Roots

Let one of the real roots λ_1 repeats p times and the remaining (N – p) roots are distinct real roots. Now, the homogeneous solution is in the form,

$$y_h(n) = (C_1 + C_2 n + C_3 n^2 + ... + C_p n^{p-1})\lambda_1^n + C_{p+1}\lambda_{p+1}^n + ... + C_N \lambda_N^n$$

where, $C_1, C_2, C_3, ...C_N$ are constants that can be evaluated using initial conditions.

Complex Roots

Let the characteristic polynomial has a pair of complex roots λ and λ^* and the remaining (N – 2) roots be distinct real roots. Now, the homogeneous solution will be in the form,

$$y_h(n) = r^n [C_1 \cos n\theta + C_2 \sin n\theta] + C_3 \lambda_3^n + C_4 \lambda_4^n + ... + C_N \lambda_N^n$$

where, $\lambda = a + jb, \quad \lambda^* = a - jb, \quad r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}\frac{b}{a}$

$C_1, C_2, C_3 ... C_N$ are constants that can be evaluated using initial conditions.

2.7.2 Particular Solution

The ***particular solution***, $y_p(n)$ is the solution of the difference equation for specific input signal x(n) for $n \geq 0$. Since the input signal may have different form, the particular solution depends on the form or type of the input signal x(n).

If x(n) is constant, then $y_p(n)$ is also a constant.

> **Example :**
>
> Let, x(n) = u(n) ; now, $y_p(n)$ = K u(n)

If x(n) is exponential, then $y_p(n)$ is also an exponential.

> **Example :**
>
> Let, x(n) = a^n u(n) ; now, $y_p(n) = K a^n u(n)$

If x(n) is sinusoid, then $y_p(n)$ is also a sinusoid.

> **Example :**
>
> Let, x(n) = A $\cos \omega_0 n$; now, $y_p(n) = K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$

The general form of particular solution for various types of inputs are listed in table 2.2.

Table 2.2 : Particular Solution

Input signal, x(n)	Particular solution, $y_p(n)$
A	K
$A B^n$	$K B^n$
$A n^B$	$K_0 n^B + K_1 n^{(B-1)} + ... + K_B$
$A^n n^B$	$A^n (K_0 n^B + K_1 n^{(B-1)} + ... + K_B)$
$A \cos \omega_0 n$	$K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$
$A \sin \omega_0 n$	$K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$

2.7.3 Zero-State Response

The ***zero-state response*** or ***forced response*** is obtained from the sum of homogeneous solution and particular solution and evaluating the constants with zero initial conditions.

$$\therefore \text{ Zero - state response, } y_{zs}(n) = y_h(n) + y_p(n)\Big|_{\text{with constants } C_1,\, C_2, \ldots\, C_N \text{ evaluated with zero initial conditions}}$$

2.7.4 Total Response

The total response of discrete time system can be obtained by the following two methods.

Method-1

The ***total response*** is given by sum of homogeneous solution and particular solution.

$\therefore$ Total response, $y(n) = y_h(n) + y_p(n)$

Procedure to Determine Total Response by Method-1

1. Determine the homogeneous solution $y_h(n)$ with constants $C_1, C_2, \ldots..C_N$.
2. Determine the particular solution $y_p(n)$ and evaluate the constants K for any value of $n \geq 1$ so that no term of y(n) vanishes.
3. Now the total response is given by the sum of $y_h(n)$ and $y_p(n)$.

   $\therefore$ Total response, $y(n) = y_h(n) + y_p(n)$
4. The total response will have N number of constants $C_1, C_2, \ldots..C_N$. Evaluate the given difference equation for $n = 0, 1, 2, \ldots.N - 1$ and form one set of N number of equations. Then evaluate the total response for $n = 0, 1, 2, \ldots..\ N - 1$ and form another set of N number of equations. Now solve the constants $C_1, C_2, \ldots..C_N$ using the two sets of N number of equations.

Method-2

The ***total response*** is given by sum of zero-input response and zero-state response.

$\therefore$ Total response, $y(n) = y_{zi}(n) + y_{zs}(n)$

Procedure to Determine Total Response by Method-2

1. Determine the homogeneous solution $y_h(n)$ with constants $C_1, C_2, \ldots.C_N$.
2. Determine the zero-input response, which is obtained from the homogeneous solution $y_h(n)$ and evaluating the constants $C_1, C_2, \ldots.C_N$ using the initial conditions.
3. Determine the particular solution $y_p(n)$ and evaluate the constants K for any value of $n \geq 1$ so that no term of y(n) vanishes.
4. Determine the zero-state response, $y_{zs}(n)$ which is given by sum of homogeneous solution and particular solution and evaluating the constants $C_1, C_2, \ldots.C_N$ with zero initial conditions.
5. Now, the total response is given by sum of zero input response and zero state response.

   $\therefore$ Total response, $y(n) = y_{zi}(n) + y_{zs}(n)$

Example 2.8

Determine the response of first order discrete time system governed by the difference equation,

$$y(n) = -0.8\, y(n-1) + x(n)$$

When the input is unit step, and with initial condition **a)** $y(-1) = 0$ **b)** $y(-1) = 2/9$.

Solution

Given that, $y(n) = -0.8\, y(n-1) + x(n)$

$$\therefore \quad y(n) + 0.8\, y(n-1) = x(n) \qquad .....(1)$$

Homogeneous Solution

The homogeneous equation is the solution of equation (1) when $x(n) = 0$.

$$\therefore \quad y(n) + 0.8\, y(n-1) = 0 \qquad .....(2)$$

Put, $y(n) = \lambda^n$ in equation (2).

$$\therefore \quad \lambda^n + 0.8\, \lambda^{(n-1)} = 0$$

$$\lambda^{(n-1)}(\lambda + 0.8) = 0 \quad \Rightarrow \quad \lambda = -0.8$$

The homogeneous solution $y_h(n)$ is given by,

$$y_h(n) = C\, \lambda^n = C\,(-0.8)^n \ ; \quad \text{for } n \ge 0 \qquad .....(3)$$

Particular Solution

Given that the input is unit step and so the particular solution will be in the form,

$$y(n) = K\, u(n) \qquad .....(4)$$

On substituting for $y(n)$ from equation (4) in equation (1) we get,

$$y(n) + 0.8\, y(n-1) = x(n) \quad \Rightarrow \quad K\, u(n) + 0.8\, K\, u(n-1) = u(n) \qquad .....(5)$$

In order to determine the value of K, let us evaluate equation (5) for $n = 1$, ($\because$ we have to evaluate equation (5) for any $n \ge 1$, such that none of the term vanishes).

From equation (5) when $n = 1$, we get,

$$K + 0.8\, K = 1 \quad \Rightarrow \quad 1.8\, K = 1 \quad \Rightarrow \quad K = \frac{1}{1.8} = \frac{10}{18} = \frac{5}{9}$$

The particular solution $y_p(n)$ is given by,

$$y_p(n) = K\, u(n) = \frac{5}{9}\, u(n) \ ; \quad \text{for all } n$$

$$= \frac{5}{9} \ ; \quad \text{for } n \ge 0$$

Total Response

The total response $y(n)$ of the system is given by sum of homogeneous and particular solution.

$\therefore$ Response, $y(n) = y_h(n) + y_p(n)$

$$\therefore \ y(n) = C(-0.8)^n + \frac{5}{9} \ ; \quad \text{for } n \ge 0 \qquad .....(6)$$

When $n = 0$, from equation (1), we get, $y(0) + 0.8\, y(-1) = 1$

$$\therefore \quad y(0) = 1 - 0.8\, y(-1) \qquad .....(7)$$

When $n = 0$, from equation (6), we get, $y(0) = C + \frac{5}{9}$ .....(8)

On equating (7) and (8) we get, $C + \frac{5}{9} = 1 - 0.8\, y(-1)$

$$\therefore \ C = 1 - 0.8\, y(-1) - \frac{5}{9}$$

$$= \frac{4}{9} - 0.8\, y(-1) \qquad .....(9)$$

On substituting for C from equation (9) in equation (6) we get,

$$y(n) = \left(\frac{4}{9} - 0.8\ y(-1)\right)(-0.8)^n + \frac{5}{9}$$

a) When y(–1) = 0

$$\therefore\ y(n) = \frac{4}{9}(-0.8)^n + \frac{5}{9}\ ;\quad \text{for } n \geq 0$$

$$= \left[\frac{4}{9}(-0.8)^n + \frac{5}{9}\right] u(n)$$

b) When y(–1) = 2/9

$$\therefore\quad y(n) = \left(\frac{4}{9} - 0.8 \times \frac{2}{9}\right)(-0.8)^n + \frac{5}{9} = \frac{2.4}{9}(-0.8)^n + \frac{5}{9} = \frac{24}{90}(-0.8)^n + \frac{5}{9}$$

$$\therefore\quad y(n) = \frac{5}{9} + \frac{12}{45}(-0.8)^n;\quad \text{for } n \geq 0$$

$$= \left[\frac{5}{9} + \frac{12}{45}(-0.8)^n\right] u(n)$$

Example 2.9

Determine the response y(n), $n \geq 0$ of the system described by the second order difference equation,

$$y(n) - 0.2\ y(n-1) - 0.03\ y(n-2) = x(n) + 0.4\ x(n-1),$$

when the input signal is, $x(n) = 0.2^n\ u(n)$ and with initial conditions $y(-2) = 0$, $y(-1) = 0.5$.

Solution

Given that, y(n) – 0.2 y(n – 1) – 0.03 y(n – 2) = x(n) + 0.4 x(n – 1) .....(1)

Homogeneous Solution

The homogeneous equation is the solution of equation (1) when x(n) = 0.

$$\therefore\ y(n) - 0.2\ y(n-1) - 0.03\ y(n-2) = 0 \qquad .....(2)$$

Put $y(n) = \lambda^n$ in equation (2).

$$\therefore\ \lambda^n - 0.2\ \lambda^{n-1} - 0.03\ \lambda^{n-2} = 0$$

$$\lambda^{n-2}(\lambda^2 - 0.2\lambda - 0.03) = 0$$

The characteristic equation is,

$$\lambda^2 - 0.2\lambda - 0.03 = 0 \quad \Rightarrow \quad (\lambda - 0.3)(\lambda + 0.1) = 0$$

$\therefore$ The roots are, $\lambda = 0.3, -0.1$

The roots of quadratic,

$$\lambda^2 - 0.2\lambda - 0.03 = 0 \text{ are,}$$

$$\lambda = \frac{0.2 \pm \sqrt{0.2^2 + 4 \times 0.03}}{2}$$

$$= \frac{0.2 \pm 0.4}{2} = 0.3,\ -0.1$$

The homogeneous solution, $y_h(n)$ is given by,

$$y_h(n) = C_1\ \lambda_1^n + C_2\ \lambda_2^n$$

$$= C_1(0.3)^n + C_2(-0.1)^n\quad ;\quad \text{for } n \geq 0 \qquad .....(3)$$

Particular Solution

Given that the input is an exponential signal, $0.2^n\ u(n)$ and so the particular solution will be in the form,

$$y(n) = K\ 0.2^n\ u(n) \qquad .....(4)$$

On substituting for y(n) from equation (4) in equation (1) we get,

$$K 0.2^n u(n) - 0.2\ K\ 0.2^{(n-1)}\ u(n-1) - 0.03\ K\ 0.2^{(n-2)}\ u(n-2) = 0.2^n\ u(n) + 0.4 \times 0.2^{(n-1)}\ u(n) \qquad .....(5)$$

In order to determine the value of K, let us evaluate equation (5) for n = 2, ($\because$ we have to evaluate equation (5) for any $n \geq 1$, such that none of the term vanishes).

From equation (5) when n = 2, we get,

$$K\,0.2^2 - 0.2K \times 0.2^1 - 0.03K \times 0.2^0 = 0.2^2 + 0.4 \times 0.2^1$$

$$0.04K - 0.04K - 0.03K = 0.04 + 0.08$$

$$-0.03K = 0.12$$

$$\therefore\ K = -\frac{0.12}{0.03} = -4$$

The particular solution $y_p(n)$ is given by,

$$y_p(n) = K\,0.2^n\,u(n) = (-4)\,0.2^n\,u(n)$$

Total Response

The total response y(n) of the system is given by sum of homogeneous and particular solution.

$$\therefore\ \text{Response, } y(n) = y_h(n) + y_p(n)$$

$$= C_1\,0.3^n + C_2\,(-0.1)^n + (-4)\,0.2^n\ ; \quad \text{for } n \geq 0 \qquad .....(6)$$

To find y(0) and y(1)

When n = 0,

From equation (1) we get,

$$y(0) - 0.2\,y(-1) - 0.03\,y(-2) = x(0) + 0.4\,x(-1) \qquad .....(7)$$

Given that, $y(-1) = 0.5$, $y(-2) = 0$

$$x(n) = 0.2^n\,u(n), \qquad \therefore\ x(0) = 0.2^0 = 1$$

$$x(-1) = 0$$

On substituting the above conditions in equation (7) we get,

$$y(0) - 0.2 \times 0.5 - 0.03 \times 0 = 1 + 0$$

$$\therefore\ y(0) = 1.1 \qquad .....(8)$$

When n = 1,

From equation (1) we get,

$$y(1) - 0.2\,y(0) - 0.03\,y(-1) = x(1) + 0.4\,x(0) \qquad .....(9)$$

We know that, $y(0) = 1.1$, $y(-1) = 0.5$, $y(-2) = 0$

Given that, $x(n) = 0.2^n\,u(n)$, $\therefore\ x(0) = 0.2^0 = 1$

$$x(1) = 0.2^1 = 0.2$$

On substituting the above conditions in equation (9) we get,

$$y(1) - 0.2 \times 1.1 - 0.03 \times 0.5 = 0.2 + 0.4 \times 1$$

$$\therefore\ y(1) = 0.6 + 0.235 = 0.835 \qquad .....(10)$$

To solve constants C_1 and C_2

When n = 0,

From equation (6) we get,

$$y(0) = C_1\,0.3^0 + C_2\,(-0.1)^0 + (-4)\,0.2^0 = C_1 + C_2 - 4 \qquad .....(11)$$

From equations (8) and (11) we can write,

$$C_1 + C_2 - 4 = 1.1$$

$$\therefore\ C_1 + C_2 = 5.1 \qquad .....(12)$$

When n = 1,

From equation (6) we get,

$$y(1) = C_1 \times 0.3 + C_2\,(-0.1) + (-4)\,0.2 = 0.3\,C_1 - 0.1C_2 - 0.8 \qquad .....(13)$$

From equations (10) and (13) we can write,

$$0.3\,C_1 - 0.1C_2 - 0.8 = 0.835$$

$$\therefore\quad 0.3\,C_1 - 0.1C_2 = 1.635 \qquad .....(14)$$

$$\begin{array}{lcl}
\text{Equation (12)} \times 0.1 & \Rightarrow & 0.1C_1 + 0.1C_2 = 0.51 \\
\text{Equation (13)} & \Rightarrow & 0.3C_1 - 0.1C_2 = 1.635 \\
 & \text{Add} & 0.4C_1 \qquad\quad = 2.145
\end{array}$$

$$\therefore\quad C_1 = \frac{2.145}{0.4} = 5.3625$$

From equation(12),

$$C_2 = 5.1 - C_1 = 5.1 - 5.3625$$

$$= -\,0.2625$$

Total Response

$$y(n) = [5.3625(0.3)^n - 0.2625(-0.1)^n + (-4)\,0.2^n]\,u(n) \quad ; \quad \text{for all n}$$

2.8 Classification of Discrete Time Systems

The discrete time systems are classified based on their characteristics. Some of the classifications of discrete time systems are,

1. Static and dynamic systems
2. Time invariant and time variant systems
3. Linear and nonlinear systems
4. Causal and noncausal systems
5. Stable and unstable systems
6. FIR and IIR systems
7. Recursive and nonrecursive systems

2.8.1 Static and Dynamic Systems

A discrete time system is called ***static*** or ***memoryless*** system if its output at any instant n depends at most on the input sample at the same time but not on the past or future samples of the input. In any other case, the system is said to be ***dynamic*** or to have memory.

Example :

$$\left.\begin{array}{l} y(n) = a\,x(n) \\ y(n) = n\,x(n) + 6\,x^3(n) \end{array}\right\}\ \text{Static systems}$$

$$\left.\begin{array}{l}
\left.\begin{array}{l} y(n) = x(n) + 3\,x(n-1) \\ y(n) = \displaystyle\sum_{m=0}^{N} x(n-m) \end{array}\right\}\ \text{Finite memory is required} \\
\left.\begin{array}{l} y(n) = \displaystyle\sum_{m=0}^{\infty} x(n-m) \end{array}\right\}\ \text{Infinite memory is required}
\end{array}\right\}\ \text{Dynamic systems}$$

2.8.2 Time Invariant and Time Variant Systems

A system is said to be ***time invariant*** if its input-output characteristics do not change with time.

Definition : A relaxed system $\mathcal{H}$ is ***time invariant*** or ***shift invariant*** if and only if

$$\mathcal{H}\{x(n)\} = y(n) \text{ implies that, } \mathcal{H}\{x(n-m)\} = y(n-m)$$

for every input signal x(n) and every time shift m.

i.e., in time invariant systems, if $\mathbf{y(n) = \mathcal{H}\{x(n)\}}$ then $\mathbf{y(n-m) = \mathcal{H}\{x(n-m)\}}$.

Alternative Definition for Time Invariance

A system $\mathcal{H}$ is ***time invariant*** if the response to a shifted (or delayed) version of the input is identical to a shifted (or delayed) version of the response based on the unshifted (or undelayed) input.

i.e., In a time invariant system, $\mathbf{\mathcal{H}\{x(n-m)\} = z^{-m}\, \mathcal{H}\{x(n)\}}$; for all values of m .....(2.22)

The operator z^{-m} represents a signal delay of m samples.

The diagrammatic explanation of the above definition of time invariance is shown in fig 2.19.

Procedure to Test for Time Invariance

1. Delay the input signal by m units of time and determine the response of the system for this delayed input signal. Let this response be y(n – m).
2. Delay the response of the system for undelayed input by m units of time. Let this delayed response be $y_d(n)$.
3. Check whether y (n – m) = $y_d(n)$. If they are equal then the system is time invariant.

   Otherwise the system is ***time variant***.

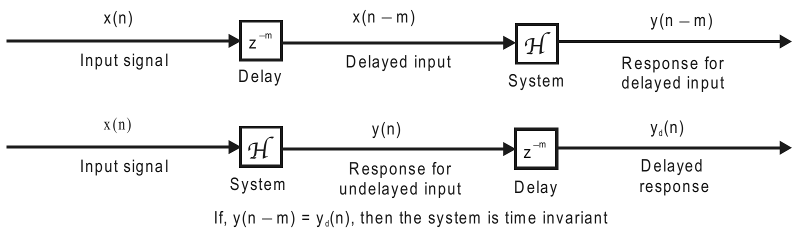


Fig 2.19 : Diagrammatic explanation of time invariance.

Example 2.10

Test the following systems for time invariance.

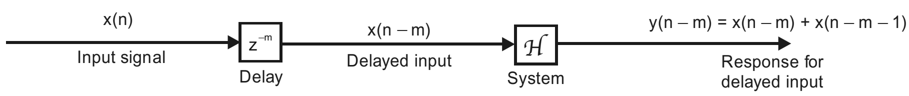
a) y(n) = x(n) + x(n – 1) **b)** y(n) = 2n x(n) **c)** y(n) = x(–n) **d)** y(n) = x(n) – b x(n – 1)

Solution

a) Given that, y(n) = x(n) + x(n – 1)

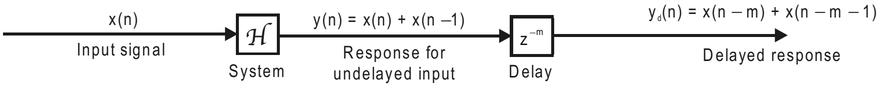
<u>Test 1 : Response for delayed input</u>

Let, y(n – m) = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

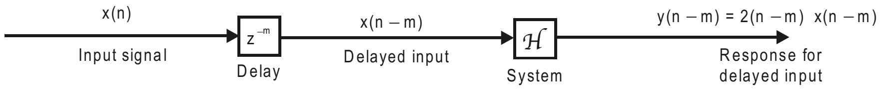


Conclusion : Here, $y(n - m) = y_d(n)$, therefore the system is time invariant.

b) Given that, y(n) = 2n x(n)

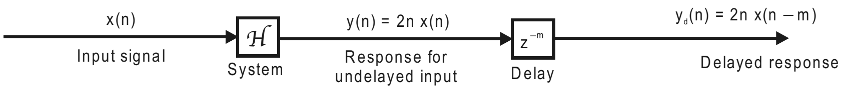
Test 1 : Response for delayed input

Let, $y(n - m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

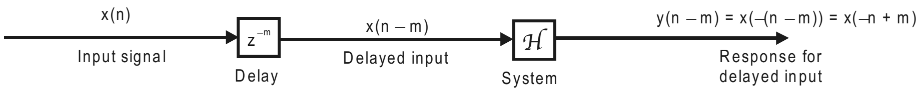


Conclusion : Here, $y(n - m) \neq y_d(n)$, therefore the system is time variant.

c) Given that, y(n) = x(–n)

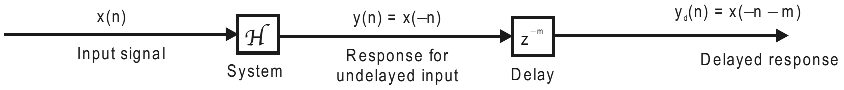
Test 1 : Response for delayed input

Let, $y(n - m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

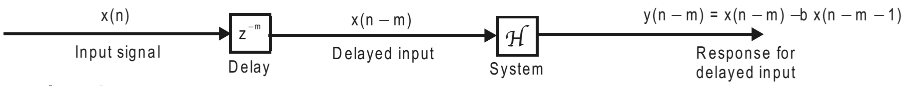


Conclusion : Here, $y(n - m) \neq y_d(n)$, therefore the system is time variant.

d) Given that, y(n) = x(n) – b x(n – 1)

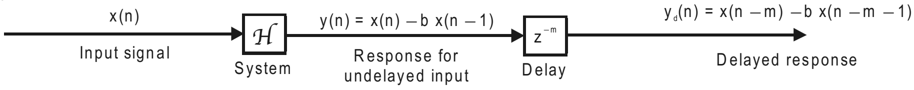
Test 1 : Response for delayed input

Let, $y(n - m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.



Conclusion : Here, $y(n - m) = y_d(n)$, therefore the system is time invariant.

Example 2.11

Test the following systems for time invariance.

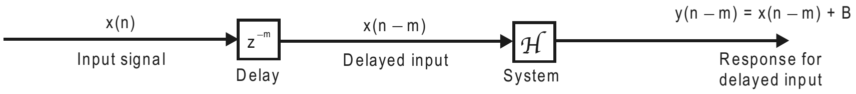
a) $y(n) = x(n) + B$ **b)** $y(n) = n\,x^3(n)$ **c)** $y(n) = b^{x(n)}$ **d)** $y(n) = \sum_{k=0}^{M} b_k\,x(n-k) - \sum_{k=1}^{N} a_k\,y(n-k)$

Solution

a) Given that, $y(n) = x(n) + B$

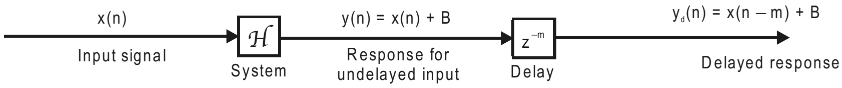
Test 1 : Response for delayed input

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

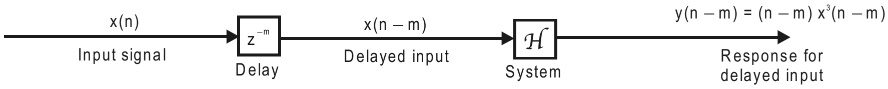


Conclusion : Here, $y(n-m) = y_d(n)$, therefore the system is time invariant.

b) Given that, $y(n) = n\,x^3(n)$

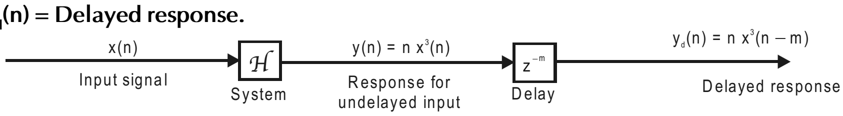
Test 1 : Response for delayed input

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

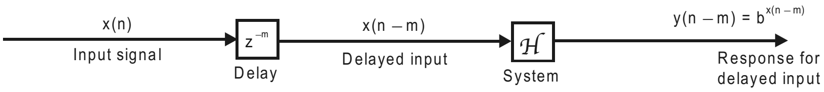


Conclusion : Here, $y(n-m) \neq y_d(n)$, therefore the system is time variant.

c) Given that, $y(n) = b^{x(n)}$

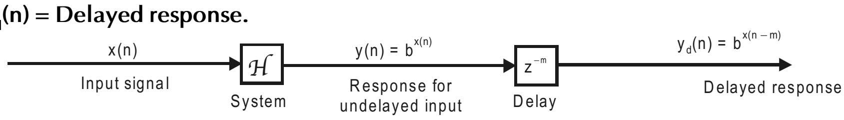
Test 1 : Response for delayed input

Let, $y(n-m)$ = Response for delayed input.



Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.

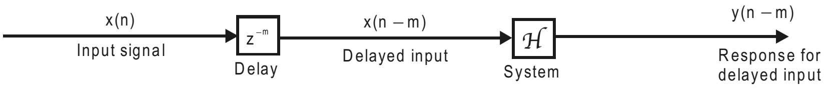


Conclusion : Here, $y(n-m) = y_d(n)$, therefore the system is time invariant.

d) Given that, $y(n) = \sum\limits_{k=0}^{M} b_k\, x(n-k) - \sum\limits_{k=1}^{N} a_k\, y(n-k)$

Test 1 : Response for delayed input

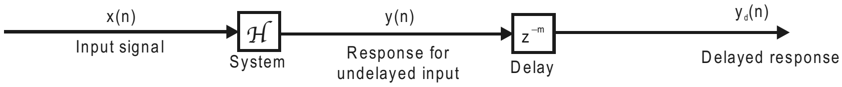
Let, $y(n-m)$ = Response for delayed input.



$$\text{Response for delayed input, } y(n-m) = H\{x(n-m)\} = \sum_{k=0}^{M} b_k\, x(n-m-k) - \sum_{k=1}^{N} a_k\, y(n-m-k)$$

Test 2 : Delayed response

Let, $y_d(n)$ = Delayed response.



$$\text{Response for undelayed input} = H\{x(n)\} = y(n) = \sum_{k=0}^{M} b_k\ x(n-k) - \sum_{k=1}^{N} a_k\ y(n-k)$$

$$\text{Delayed response, } y_d(n) = z^{-m}\ H\{x(n)\}$$

$$= z^{-m}\left[\sum_{k=0}^{M} b_k\ x(n-k) - \sum_{k=1}^{N} a_k\ y(n-k)\right]$$

$$= \sum_{k=0}^{M} b_k\ x(n-m-k) - \sum_{k=1}^{N} a_k\ \ y(n-m-k)$$

Conclusion : Here, $y(n-m) = y_d(n)$, therefore the system is time invariant.

2.8.3 Linear and Nonlinear Systems

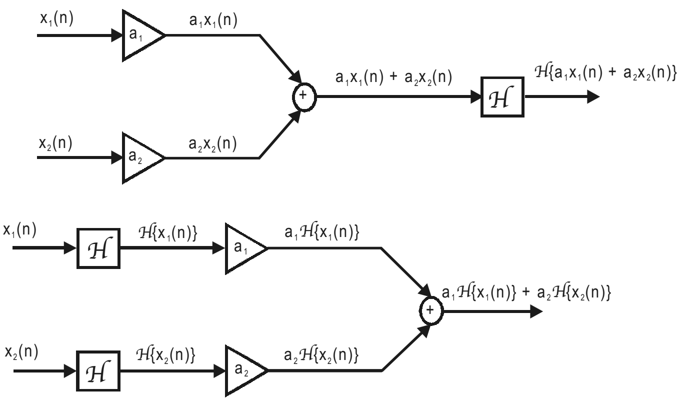
A ***linear system*** is one that satisfies the superposition principle. The ***principle of superposition*** requires that the response of the system to a weighted sum of the signals is equal to the corresponding weighted sum of the responses of the system to each of the individual input signals.

Definition : A relaxed system $\mathcal{H}$ is ***linear*** if

$$\mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = a_1\,\mathcal{H}\{x_1(n)\} + a_2\,\mathcal{H}\{x_2(n)\} \qquad .....(2.23)$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$ and for any arbitrary constants a_1 and a_2.

If a relaxed system does not satisfy the superposition principle as given by the above definition, then the system is ***nonlinear***.The diagrammatic explanation of linearity is shown in fig 2.20.



The system, $\mathcal{H}$ is linear if and only if, $\mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = a_1\,\mathcal{H}\{x_1(n)\} + a_2\,\mathcal{H}\{x_2(n)\}$

Fig 2.20 : Diagrammatic explanation of linearity.

Procedure to test for linearity

1. Let $x_1(n)$ and $x_2(n)$ be two inputs to system $\mathcal{H}$, and $y_1(n)$ and $y_2(n)$ be corresponding responses.
2. Consider a signal, $x_3(n) = a_1\,x_1(n) + a_2\,x_2(n)$ which is a weighed sum of $x_1(n)$ and $x_2(n)$.
3. Let $y_3(n)$ be the response for $x_3(n)$.
4. Check whether $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$. If they are equal then the system is linear, otherwise it is nonlinear.

Example 2.12

Test the following systems for linearity.

a) $y(n) = n\,x(n)$ **b)** $y(n) = x(n^2)$ **c)** $y(n) = x^2(n)$ **d)** $y(n) = B\,x(n) + C$

Solution

a) Given that, $y(n) = n\,x(n)$

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = n\,x(n)$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = n\,x(n)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = n\,x_1(n)$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = n\,x_2(n)$$

\ $a_1\,y_1(n) + a_2\,y_2(n) = a_1\,n\,x_1(n) + a_2\,n\,x_2(n)$ .....(1)

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

\ $y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = n[a_1\,x_1(n) + a_2\,x_2(n)] = a_1\,n\,x_1(n) + a_2\,n\,x_2(n)$ .....(2)

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is linear.

---

b) Given that, $y(n) = x(n^2)$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = x(n^2)$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = x(n^2)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = x_1(n^2)$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = x_2(n^2)$$

\ $a_1\,y_1(n) + a_2\,y_2(n) = a_1\,x_1(n^2) + a_2\,x_2(n^2)$ .....(1)

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore \; y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = a_1\, x_1(n^2) + a_2\, x_2(n^2) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is linear.

c) Given that, $y(n) = x^2(n)$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = x^2(n)$.

The system $\mathcal{H}$ operates on x(n) to produce, y(n).

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = x^2(n)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = x_1^2(n)$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = x_2^2(n)$$

$$\therefore \; a_1\, y_1(n) + a_2\, y_2(n) = a_1\, x_1^2(n) + a_2\, x_2^2(n) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore \; y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = [a_1\, x_1(n) + a_2\, x_2(n)]^2$$
$$= a_1^2\, x_1^2\,(n) + a_2^2\, x_2^2(n) + 2\; a_1\, a_2\, x_1(n) x_2(n) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is nonlinear.

d) Given that, $y(n) = B\, x(n) + C$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = B\, x(n) + C$.

The system $\mathcal{H}$ operates on x(n) to produce, y(n).

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = B\; x(n) + C$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = B\; x_1(n) + C$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = B\; x_2(n) + C$$

$$\therefore \; a_1\, y_1(n) + a_2\, y_2(n) = a_1 [B\, x_1(n) + C] + a_2 [B\, x_2(n) + C]$$
$$= B\; a_1\, x_1(n) + C\; a_1 + B\; a_2\, x_2(n) + C\; a_2 \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore \; y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = B\,[a_1\, x_1(n) + a_2\, x_2(n)] + C = B\, a_1\, x_1(n) + B\; a_2\, x_2(n) + C \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is nonlinear.

Example 2.13

Test the following systems for linearity.

a) $y(n) = e^{x(n)}$ **b)** $y(n) = b^{x(n)}$ **c)** $y(n) = n\, x^2(n)$

Solution

a) Given that, $y(n) = e^{x(n)}$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = e^{x(n)}$.

The system $\mathcal{H}$ operates on x(n) to produce, y(n).

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = e^{x(n)}$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = e^{x_1(n)}$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = e^{x_2(n)}$$

$$\therefore \quad a_1\, y_1(n) + a_2\, y_2(n) = a_1\, e^{x_1(n)} + a_2\, e^{x_2(n)} \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore \quad y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = e^{[a_1 x_1(n) + a_2 x_2(n)]} = e^{a_1 x_1(n)}\, e^{a_2 x_2(n)} \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is nonlinear.

---

b) Given that, $y(n) = b^{x(n)}$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = b^{x(n)}$.

The system $\mathcal{H}$ operates on x(n) to produce, y(n).

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = b^{x(n)}$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = b^{x_1(n)}$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = b^{x_2(n)}$$

$$\therefore \quad a_1\, y_1(n) + a_2\, y_2(n) = a_1\, b^{x_1(n)} + a_2\, b^{x_2(n)} \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore \; y_3(n) = \mathcal{H}\{a_1x_1(n) + a_2x_2(n)\} = b^{[a_1x_1(n)+a_2x_2(n)]} = b^{a_1x_1(n)} \; b^{a_2x_2(n)} \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is nonlinear.

---

c) Given that, $y(n) = n\,x^2(n)$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = n\,x^2(n)$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = n\,x^2(n)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = n\,x_1^2(n)$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = n\,x_2^2(n)$$

$$\backslash \;\; a_1\,y_1(n) + a_2\,y_2(n) = a_1\,n\,x_1^2(n) + a_2\,n\,x_2^2(n) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\backslash \;\; y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = n[a_1\,x_1(n) + a_2\,x_2(n)]^2$$

$$= n\,a_1^2\,x_1^2(n) + n\,a_2^2\,x_2^2(n) + 2\,n\,a_1\,a_2\,x_1(n)\,x_2(n) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n)\ ^1\ a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is nonlinear.

Example 2.14

Test the following systems for linearity.

a) $y(n) = 3\,x(n) + \dfrac{1}{x(n-2)}$ **b**) $y(n) = x(n) - 2\,x(n-1)$ **c**) $y(n) = \sum\limits_{m=0}^{M} b_m\,x(n-m) - \sum\limits_{m=1}^{N} c_m\,y(n-m)$

Solution

a) Given that, $y(n) = 3\,x(n) + \dfrac{1}{x(n-2)}$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = 3\,x(n) + \dfrac{1}{x(n-2)}$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = 3x(n) + \frac{1}{x(n-2)}$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = 3x_1(n) + \frac{1}{x_1(n-2)}$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = 3x_2(n) + \frac{1}{x_2(n-2)}$$

$$\therefore\ a_1\, y_1(n) + a_2\, y_2(n) = a_1 \left(3\,x_1(n) + \frac{1}{x_1(n-2)} \right) + a_2 \left(3\,x_2(n) + \frac{1}{x_2(n-2)} \right) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = 3\,[a_1\, x_1(n) + a_2\, x_2(n)] + \frac{1}{a_1\, x_1(n-2) + a_2\, x_2(n-2)} \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is nonlinear.

---

b) Given that, $y(n) = x(n) - 2\,x(n-1)$

Let, $\mathcal{H}$ be the system represented by the equation, $y(n) = x(n) - 2\,x(n-1)$.

The system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n)$.

$$x(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y(n) = \mathcal{H}\{x(n)\} = x(n) - 2x(n-1)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$x_1(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1(n) = \mathcal{H}\{x_1(n)\} = x_1(n) - 2\,x_1(n-1)$$

$$x_2(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2(n) = \mathcal{H}\{x_2(n)\} = x_2(n) - 2\,x_2(n-1)$$

$$\therefore\ a_1\, y_1(n) + a_2\, y_2(n) = a_1\, x_1(n) - a_1\, 2\, x_1(n-1) + a_2\, x_2(n) - a_2\, 2\, x_2(n-1) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for $x_3(n)$.

$$x_3(n) \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_3(n) = \mathcal{H}\{x_3(n)\}$$

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = a_1\, x_1(n) + a_2\, x_2(n) - 2[a_1\, x_1(n-1) + a_2\, x_2(n-1)]$$

$$= a_1\, x_1(n) - a_1\, 2\, x_1(n-1) + a_2\, x_2(n) - a_2\, 2\, x_2(n-1) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$. Hence the system is linear.

---

c) Given that, $y(n) = \sum\limits_{m=0}^{M} b_m\, x(n-m) - \sum\limits_{m=1}^{N} c_m\, y(n-m)$

Let, $\mathcal{H}$ be the system represented by the equation,

$$y(n) = \sum_{m=0}^{M} b_m\, x(n-m) - \sum_{m=1}^{N} c_m\, y(n-m)$$

$$\left.\begin{array}{l}\text{The response of the system}\\ \mathcal{H}\text{ for the input } x(n)\end{array}\right\} = \mathcal{H}\{x(n)\} = y(n) = \sum_{m=0}^{M} b_m\, x(n-m) - \sum_{m=1}^{N} c_m\, y(n-m)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let, $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = \sum_{m=0}^{M} b_m\, x_1(n-m) - \sum_{m=1}^{N} c_m\, y_1(n-m)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = \sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, y_2(n-m)$$

$$\therefore\ a_1\, y_1(n) + a_2\, y_2(n) = a_1 \left(\sum_{m=0}^{M} b_m\, x_1(n-m) - \sum_{m=1}^{N} c_m\, y_1(n-m) \right)$$

$$+ a_2 \left(\sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, y_2(n-m) \right) \quad .....(1)$$

Consider a linear combination of inputs, $a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$.

Let, $y_3(n)$ be the response for the input $x_3(n)$.

$$\therefore\ y_3(n) = \mathcal{H}\{x_3(n)\} = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\}$$

$$= \sum_{m=0}^{M} b_m\, (a_1\, x_1(n-m) + a_2\, x_2(n-m)) - \sum_{m=1}^{N} c_m\, y_3(n-m)$$

$$= a_1 \sum_{m=0}^{M} b_m\, x_1(n-m) + a_2 \sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, y_3(n-m) \quad .....(2)$$

By time invariant property,

If $y_3(n) = \mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\}$ then $y_3(n-m) = \mathcal{H}\{a_1\, x_1(n-m) + a_2\, x_2(n-m)\}$

If $y_2(n) = \mathcal{H}\{x_2(n)\}$ then $y_2(n-m) = \mathcal{H}\{x_2(n-m)\}$

If $y_1(n) = \mathcal{H}\{x_1(n)\}$ then $y_1(n-m) = \mathcal{H}\{x_1(n-m)\}$

$$\therefore\ y_3(n-m) = \mathcal{H}\{a_1\, x_1(n-m) + a_2\, x_2(n-m)\} = a_1\, \mathcal{H}\{x_1(n-m)\} + a_2\, \mathcal{H}\{x_2(n-m)\}$$

$$= a_1\, y_1(n-m) + a_2\, y_2(n-m)] \quad .....(3)$$

Using equation (3), the equation (2) can be written as,

$$y_3(n) = a_1 \sum_{m=0}^{M} b_m\, x_1(n-m) + a_2 \sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, [a_1\, y_1(n-m) + a_2\, y_2(n-m)]$$

$$= a_1 \sum_{m=0}^{M} b_m\, x_1(n-m) + a_2 \sum_{m=0}^{M} b_m\, x_2(n-m) - a_1 \sum_{m=1}^{N} c_m\, y_1(n-m) - a_2 \sum_{m=1}^{N} c_m\, y_2(n-m)$$

$$= a_1 \left(\sum_{m=0}^{M} b_m\, x_1(n-m) - \sum_{m=1}^{N} c_m\, y_1(n-m) \right) + a_2 \left(\sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, y_2(n-m) \right) \quad .....(4)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (4) we can say that the condition for linearity is satisfied. Therefore the system is linear.

2.8.4 Causal and Noncausal Systems

Definition : A system is said to be ***causal*** if the output of the system at any time n depends only on the present input, past inputs and past outputs but does not depend on the future inputs and outputs.

If the system output at any time n depends on future inputs or outputs then the system is called ***noncausal*** system.

The causality refers to a system that is realizable in real time. It can be shown that an LTI system is causal if and only if the impulse response is zero for $n < 0$, (i.e., $h(n) = 0$ for $n < 0$).

Let, $x(n)$ = Present input and $y(n)$ = Present output

$\therefore\ x(n-1), x(n-2), ...,$ are past inputs

$y(n-1), y(n-2), ...,$ are past outputs

In mathematical terms the output of a causal system satisfies the equation of the form,

$$y(n) = F\,[x(n), x(n-1), x(n-2), ... , y(n-1), y(n-2) ....]$$

where, $F[\cdot]$ is some arbitrary function.

Example 2.15

Test the causality of the following systems.

a) $y(n) = x(n) - x(n-2)$ **b)** $y(n) = \sum_{k=-\infty}^{n} x(k)$ **c)** $y(n) = b\,x(n)$ **d)** $y(n) = n\,x(n)$

Solution

a) Given that, $y(n) = x(n) - x(n-2)$

When $n = 0$, $y(0) = x(0) - x(-2)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past input $x(-2)$

When $n = 1$, $y(1) = x(1) - x(-1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past input $x(-1)$.

From the above analysis we can say that for any value of n, the system output depends on present and past inputs. Hence the system is causal.

---

b) Given that, $y(n) = \sum_{k=-\infty}^{n} x(k)$

When $n = 0$, $y(0) = \sum_{k=-\infty}^{0} x(k)$

$= \dots x(-2) + x(-1) + x(0)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past inputs $x(-1)$, $x(-2)$,.....

When $n = 1$, $y(1) = \sum_{k=-\infty}^{1} x(k)$

$= \dots x(-2) + x(-1) + x(0) + x(1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past inputs $x(0)$, $x(-1)$, $x(-2)$,..

From the above analysis we can say that for any value of n, the system output depends on present and past inputs. Hence the system is causal.

---

c) Given that, $y(n) = b\,x(n)$

When $n = 0$, $y(0) = b\,x(0)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$.

When $n = 1$, $y(1) = b\,x(1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$.

From the above analysis we can say that the response for any value of n depends on the present input. Hence the system is causal.

---

d) Given that, $y(n) = n\,x(n)$

When $n = 0$, $y(0) = 0 \times x(0)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$.

When $n = 1$, $y(1) = 1 \times x(1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$.

When $n = 2$, $y(2) = 2 \times x(2)$ $\Rightarrow$ The response at $n = 2$, i.e., $y(2)$ depends on the present input $x(2)$.

From the above analysis we can say that the response for any value of n depends on the present input. Hence the system is causal.

Example 2.16

Test the causality of the following systems.

a) $y(n) = x(n) + 2\,x(n+3)$ **b)** $y(n) = x(n^2)$ **c)** $y(n) = x(3n)$ **d)** $y(n) = x(-n)$

Solution

a) Given that, $y(n) = x(n) + 2\,x(n+3)$

When $n = 0$, $y(0) = x(0) + 2\,x(3)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and future input $x(3)$.

When n = 1, y(1) = x(1) + 2 x(4) $\Rightarrow$ The response at n = 1, i.e., y(1) depends on the present input x(1) and future input x(4).

From the above analysis we can say that the response for any value of n depends on present and future inputs. Hence the system is noncausal.

---

b) Given that, y(n) = x(n²)

When n = –1 ; y(–1) = x(1) $\Rightarrow$ The response at n = –1, depends on the future input x(1).
When n = 0 ; y(0) = x(0) $\Rightarrow$ The response at n = 0, depends on the present input x(0).
When n = 1 ; y(1) = x(1) $\Rightarrow$ The response at n = 1, depends on the present input x(1).
When n = 2 ; y(2) = x(4) $\Rightarrow$ The response at n = 2, depends on the future input x(4).

From the above analysis we can say that the response for any value of n (except n = 0 and n = 1) depends on future inputs. Hence the system is noncausal.

---

c) Given that, y(n) = x(3n)

When n = –1 ; y(–1) = x(–3) $\Rightarrow$ The response at n = –1, depends on the past input x(–3).
When n = 0 ; y(0) = x(0) $\Rightarrow$ The response at n = 0, depends on the present input x(0).
When n = 1 ; y(1) = x(3) $\Rightarrow$ The response at n = 1, depends on the future input x(3).

From the above analysis we can say that the response of the system for n > 0, depends on future inputs. Hence the system is noncausal.

---

d) Given that, y(n) = x(–n)

When n = –2 ; y(–2) = x(2) $\Rightarrow$ The response at n = –2, depends on the future input x(2).
When n = –1 ; y(–1) = x(1) $\Rightarrow$ The response at n = –1, depends on the future input x(1).
When n = 0 ; y(0) = x(0) $\Rightarrow$ The response at n = 0, depends on the present input x(0).
When n = 1 ; y(1) = x(–1) $\Rightarrow$ The response at n = 1, depends on the past input x(–1).

From the above analysis we can say that the response of the system for n < 0 depends on future inputs. Hence the system is noncausal.

2.8.5 Stable and Unstable Systems

Definition* :** An arbitrary relaxed system is said to be ***BIBO stable (Bounded Input-Bounded Output stable) if and only if every bounded input produces a bounded output.

Let x(n) be the input of discrete time system and y(n) be the response or output for x(n).The term ***bounded input*** refers to finite value of the input signal x(n) for any value of n. Hence if input x(n) is bounded then there exits a constant M_x such that $|x(n)| \le M_x$ and $M_x < \infty$, for all n.

Examples of bounded input signal are step signal, decaying exponential signal and impulse signal.
Examples of unbounded input signal are ramp signal and increasing exponential signal.

The term ***bounded output*** refers to finite and predictable output for any value of n. Hence if output y(n) is bounded then there exists a constant M_y such that $|y(n)| \le M_y$ and $M_y < \infty$, for all n.

In general, the test for stability of the system is performed by applying specific input. On applying a bounded input to a system if the output is bounded then the system is said to be BIBO stable.For LTI (Linear Time Invariant) systems the condition for BIBO stability can be transformed to a condition on impulse response as shown below.

Condition for Stability of LTI System

The condition for stability of an LTI system is,

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty \qquad .....(2.24)$$

i.e., an LTI system is ***stable*** if the impulse response is absolutely summable.

Proof

Let, x(n) = Input to LTI system.

y(n) = Response of LTI system for the input x(n).

Now, by convolution sum formula,

$$y(n) = x(n) * h(n) = h(n) * x(n)$$ Convolution satisfy commutative property.

$$= \sum_{m=-\infty}^{+\infty} h(m)\, x(n-m)$$

$$\therefore \quad |y(n)| = \left| \sum_{m=-\infty}^{+\infty} h(m)\, x(n-m) \right|$$ Taking absolute value on both sides.

$$= \sum_{m=-\infty}^{+\infty} |h(m)\, x(n-m)|$$ For linear system the order summation and absolute value can be interchanged.

$$= \sum_{m=-\infty}^{+\infty} |h(m)|\, |x(n-m)|$$ For linear system the order of multiplication and absolute value can be interchanged.

$$= \sum_{m=-\infty}^{+\infty} |h(m)|\, M_X$$ If input is bounded, then $|x(n-m)|$ = constant = M_x

$$= M_X \sum_{m=-\infty}^{+\infty} |h(m)|$$ M_x is indepentent of summation index m.

$$= M_X \sum_{n=-\infty}^{+\infty} |h(n)|$$ Change index m to n.

In the above equation, if

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty \qquad .....(2.25)$$

then the response y(n) is bounded.

Example 2.17

Test the stability of the following systems.

a) $y(n) = \cos[x(n)]$ **b)** $y(n) = x(-n-3)$ **c)** $y(n) = n\, x(n)$

Solution

a) Given that, $y(n) = \cos[x(n)]$

The given system is a nonlinear system, and so the test for stability should be performed for specific inputs.

The value of cos θ lies between –1 to +1 for any value of θ. Therefore the output y(n) is bounded for any value of input x(n). Hence the given system is stable.

b) Given that, $y(n) = x(-n-3)$

The given system is a time variant system, and so the test for stability should be performed for specific inputs.

The operations performed by the system on the input signal are folding and shifting. A bounded input signal will remain bounded even after folding and shifting. Therefore in the given system, the output will be bounded as long as input is bounded. Hence the given system is BIBO stable.

c) Given that, $y(n) = n\, x(n)$

The given system is a time variant system, and so the test for stability should be performed for specific inputs.

Case i : If x(n) tends to infinity or constant, as "n" tends to infinity, then y(n) = n x(n) will be infinite as "n" tends to infinity. So the system is unstable.

Case ii : If x(n) tends to zero as "n" tends to infinity, then y(n) = n x(n) will be zero as "n" tends to infinity. So the system is stable.

Example 2.18

Determine the range of values of "p" and "q" for the stability of LTI system with impulse response,

$$h(n) = p^n \quad ; \quad n < 0$$
$$= q^n \quad ; \quad n \ge 0$$

Solution

The condition to be satisfied for the stability of the system is, $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

Given that, $h(n) = p^n \quad ; \quad n < 0$

$= q^n \quad ; \quad n \ge 0$

$$\therefore \sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{-1} |p^n| + \sum_{n=0}^{\infty} |q^n| = \sum_{n=1}^{\infty} |p^{-n}| + \sum_{n=0}^{\infty} |q^n|$$

$$= \sum_{n=1}^{\infty} \left|\frac{1}{p^n}\right| + \sum_{n=0}^{\infty} |q^n| = \sum_{n=1}^{\infty} \frac{1}{|p|^n} + \sum_{n=0}^{\infty} |q|^n$$

n is always positive.

$$= \sum_{n=0}^{\infty} \left(\frac{1}{|p|}\right)^n - 1 + \sum_{n=0}^{\infty} |q|^n$$

$|p|^0 = 1$

The summation of infinite terms in the above equation converges if, $0 < 1/|p| < 1$ and $0 < |q| < 1$. Hence by using infinite geometric series formula,

$$\sum_{n=-\infty}^{+\infty} |h(n)| = \frac{1}{1 - \frac{1}{|p|}} - 1 + \frac{1}{1 - |q|}$$
$$= \text{constant}$$

Therefore, the system is stable if $|p| > 1$ and $|q| < 1$.

Infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$
if $0 < |C| < 1$

Example 2.19

Test the stability of LTI systems, whose impulse responses are,

a) $h(n) = 0.2^n\, u(n)$

b) $h(n) = 0.3^n\, u(n) + 2^n\, u(n)$

c) $h(n) = 4^n\, u(-n)$

d) $h(n) = 0.2^n\, u(-n) + 3^n\, u(-n)$

Solution

a) $h(n) = 0.2^n\, u(n)$

$$\therefore \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} |0.2^n\, u(n)| = \sum_{n=0}^{\infty} 0.2^n$$
$$= \frac{1}{1-0.2} = 1.25$$

Since, $\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$, system is stable.

Infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$
if $0 < |C| < 1$

b) $h(n) = 0.3^n u(n) + 2^n u(n)$

$$\therefore \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} \left|0.3^n u(n) + 2^n u(n)\right|$$

$$= \sum_{n=0}^{\infty} 0.3^n + \sum_{n=0}^{\infty} 2^n u(n) = \frac{1}{1-0.3} + \infty = \infty$$

$$\boxed{\sum_{n=0}^{\infty} C^n = \infty \quad \text{if } C > 1}$$

Since, $\sum_{n=-\infty}^{+\infty} |h(n)| = \infty$, system is unstable.

c) $h(n) = 4^n u(-n)$

$$\therefore \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} \left|4^n u(-n)\right| = \sum_{n=-\infty}^{0} 4^n = \sum_{n=0}^{+\infty} 4^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4^n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \sum_{n=0}^{\infty} 0.25^n = \frac{1}{1-0.25} = 1.3333$$

Since, $\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$, system is stable.

d) $h(n) = 0.2^n u(-n) + 3^n u(-n)$

$$\therefore \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} \left|0.2^n u(-n) + 3^n u(-n)\right|$$

$$= \sum_{n=-\infty}^{0} 0.2^n + \sum_{n=-\infty}^{0} 3^n = \sum_{n=0}^{+\infty} 0.2^{-n} + \sum_{n=0}^{+\infty} 3^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{0.2^n} + \sum_{n=0}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{1}{0.2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} 5^n + \sum_{n=0}^{\infty} 0.333^n = \infty + \frac{1}{1-0.333} = \infty$$

Since, $\sum_{n=-\infty}^{+\infty} |h(n)| = \infty$, system is unstable.

2.8.6 FIR and IIR Systems

In ***FIR system*** (**F**inite duration **I**mpulse **R**esponse system), the impulse response consists of finite number of samples. The convolution formula for FIR system is given by,

$$y(n) = \sum_{m=0}^{N-1} h(m)\, x(n-m) \qquad .....(2.26)$$

where, $h(n) = 0$; for $n < 0$ and $n ³ N$

From equation (2.26) it can be concluded that the impulse response selects only N samples of the input signal.In effect, the system acts as a window that views only the most recent N input signal samples in forming the ouput. It neglects or simply forgets all prior input samples. Thus a FIR system requires memory of length N. In general, a FIR system is described by the difference equation,

$$y(n) = \sum_{m=0}^{N-1} b_m\, x(n-m) \qquad .....(2.27)$$

where, $b_m = h(m)$; for $m = 0$ to $N - 1$

In ***IIR system*** (**I**nfinite duration **I**mpulse **R**esponse system), the impulse response has infinite number of samples. The convolution formula for IIR systems is given by,

$$y(n) = \sum_{m=0}^{\infty} h(m)\, x(n-m) \qquad .....(2.28)$$

Since this weighted sum involves the present and all the past input sample, we can say that the IIR system requires infinite memory. In general, an IIR system is described by the difference equation,

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

2.8.7 Recursive and Nonrecursive Systems

A system whose output y(n) at time n depends on any number of past output values as well as present and past inputs is called a ***recursive system.*** The past outputs are y(n – 1), y(n – 2), y(n – 3), etc.,.

Hence for recursive system, the output y(n) is given by,

$$y(n) = F\,[y(n-1), y(n-2), ...y(n-N), x(n), x(n-1), ...x(n-M)]$$

A system whose output does not depend on past output but depends only on the present and past input is called a ***nonrecursive system.***

Hence for nonrecursive system, the output y(n) is given by,

$$y(n) = F\,[x(n), x(n-1), ..., x(n-M)]$$

In a recursive system, in order to compute $y(n_0)$, we need to compute all the previous values y(0), y(1) ,......., $y(n_0 - 1)$ before calculating $y(n_0)$. Hence the output samples of a recursive system has to be computed in order [i.e., y(0), y(1), y(2), ...]. The IIR systems are recursive systems.

In nonrecursive system, $y(n_0)$ can be computed immediately without having $y(n_0 - 1)$, $y(n_0 - 2)$..... Hence the output samples of nonrecursive system can be computed in any order [i.e. y(50), y(5), y(2), y(100),....]. The FIR systems are nonrecursive systems.

2.9 Discrete or Linear Convolution

The ***Discrete*** or ***Linear convolution*** of two discrete time sequences $x_1(n)$ and $x_2(n)$ is defined as,

$$\boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, x_1(n-m)} \qquad .....(2.29)$$

where, $x_3(n)$ is the sequence obtained by convolving $x_1(n)$ and $x_2(n)$

m is a dummy variable

The convolution relation of equation (2.29) can be symbolically expressed as,

$$x_3(n) = x_1(n) * x_2(n) = x_2(n) * x_1(n) \qquad .....(2.30)$$

where, the symbol $*$ indicates convolution operation.

In linear convolution, the sequences $x_1(n)$ and $x_2(n)$ are nonperiodic sequences and the sequence $x_3(n)$ obtained by convolution is also nonperiodic. Hence this convolution is also called ***aperiodic convolution.***

Procedure For Evaluating Linear Convolution

Let, $x_1(n)$ = Discrete time sequence with N_1 samples

$x_2(n)$ = Discrete time sequence with N_2 samples

Now, the convolution of $x_1(n)$ and $x_2(n)$ will produce a sequence $x_3(n)$ consisting of N_1+N_2-1 samples. Each sample of $x_3(n)$ can be computed using the equation (2.29). The value of $x_3(n)$ at n = q is obtained by replacing n by q, in equation (2.29).

$$\therefore \; x_3(q) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(q-m) \qquad .....(2.31)$$

The evaluation of equation (2.31) to determine the value of $x_3(n)$ at $n = q$, involves the following five steps.

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, to get the sequences $x_1(m)$ and $x_2(m)$.

2. Folding : Fold $x_2(m)$ about $m = 0$, to obtain $x_2(-m)$.

3. Shifting : Shift $x_2(-m)$ by q to the right if q is positive, shift $x_2(-m)$ by q to the left if q is negative to obtain $x_2(q - m)$.

4. Multiplication : Multiply $x_1(m)$ by $x_2(q - m)$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2(q - m)$.

5. Summation : Sum all the values of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at $n = q$. [i.e., $x_3(q)$].

The above procedure will give the value of $x_3(n)$ at a single time instant say $n = q$. In general, we are interested in evaluating the values of the sequence $x_3(n)$ over all the time instants in the range $-\infty < n < \infty$. Hence the steps 3, 4 and 5 given above must be repeated, for all possible time shifts in the range $-\infty < n < \infty$.

Convolution of finite duration sequences

In convolution of finite duration sequences it is possible to predict the length of resultant sequence.

If the sequence $x_1(n)$ has N_1 samples and sequence $x_2(n)$ has N_2 samples then the output sequence $x_3(n)$ will be a finite duration sequence consisting of "N_1+N_2-1" samples.

i.e., if, Length of $x_1(n) = N_1$

Length of $x_2(n) = N_2$

then, Length of $x_3(n) = N_1 + N_2 - 1$

In the convolution of finite duration sequences it is possible to predict the start and end of the resultant sequence. If $x_1(n)$ starts at $n = n_1$ and $x_2(n)$ starts at $n = n_2$ then, the initial value of n for $x_3(n)$ is "$n = n_1 + n_2$". The value of $x_1(n)$ for $n < n_1$ and the value of $x_2(n)$ for $n < n_2$ are then assumed to be zero.The final value of n for $x_3(n)$ is "$n = (n_1 + n_2) + (N_1 + N_2 - 2)$".

i.e., if, $x_1(n)$ start at $n = n_1$

$x_2(n)$ start at $n = n_2$

then, $x_3(n)$ start at $n = n_1 + n_2$

and $x_3(n)$ end at $n = (n_1 + n_2) + (N_1 + N_2 - 1) - 1$

$= (n_1 + n_2) + (N_1 + N_2 - 2)$

2.9.1 Representation of Discrete Time Signal as Summation of Impulses

A discrete time signal can be expressed as summation of impulses and this concept will be useful to prove that the response of discrete time LTI system can be determined using discrete convolution.

Let, $x(n)$ = Discrete time signal

$\delta(n)$ = Unit impulse signal

$\delta(n - m)$ = Delayed impulse signal

We know that, $\delta(n) = 1 \; ; \text{ at } n = 0$

$= 0 \; ; \text{ when } n \neq 0$

and, $\delta(n-m) = 1 \; ; \text{ at } n = m$

$= 0 \; ; \text{ when } n \neq m$

If we multiply the signal x(n) with the delayed impulse $\delta(n - m)$ then the product is nonzero only at $n = m$ and zero for all other values of n. Also at $n = m$, the value of product signal is m^{th} sample x(m) of the signal x(n).

$$\therefore \; x(n)\,\delta(n-m) = x(m)$$

Each multiplication of the signal x(n) by an unit impulse at some delay m, in essence picks out the single value x(m) of the signal x(n) at n = m, where the unit impulse is nonzero. Consequently if we repeat this multiplication for all possible delays in the range $-\infty < m < \infty$ and add all the product sequences, the result will be a sequence that is equal to the sequence x(n).

For example,
$$x(n)\delta(n-(-2)) = x(-2)$$
$$x(n)\delta(n-(-1)) = x(-1)$$
$$x(n)\delta(n) = x(0)$$
$$x(n)\delta(n-1) = x(1)$$
$$x(n)\delta(n-2) = x(2)$$

From the above products we can say that each sample of x(n) can be expressed as a product of the sample and delayed impulse, as shown below.

$$\therefore \; x(-2) = x(-2)\delta(n-(-2))$$
$$x(-1) = x(-1)\delta(n-(-1))$$
$$x(0) = x(0)\delta(n)$$
$$x(1) = x(1)\delta(n-1)$$
$$x(2) = x(2)\delta(n-2)$$

$$\therefore \; x(n) = \cdots\cdots + x(-2) + x(-1) + x(0) + x(1) + x(2) + \cdots\cdots$$

$$= \cdots\cdots + x(-2)\,\delta(n-(-2)) + x(-1)\,\delta(n-(-1)) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\delta(n-2) + \cdots\cdots$$

$$= \sum_{m=-\infty}^{+\infty} x(m)\,\delta(n-m) \qquad \text{.....(2.32)}$$

In equation (2.32) each product $x(m)\,\delta(n-m)$ is an impulse and the summation of impulses gives the sequence x(n).

2.9.2 Response of LTI Discrete Time System Using Discrete Convolution

In an LTI system, the response y(n) of the system for an arbitrary input x(n) is given by convolution of input x(n) with impulse response h(n) of the system. It is expressed as,

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\,h(n-m) \qquad \text{.....(2.33)}$$

where, the symbol $*$ represents convolution operation.

Proof :

Let y(n) be the response of system $\mathcal{H}$ for an input x(n)

\ $y(n) = \mathcal{H}\{x(n)\}$ .....(2.34)

From equation (2.32) we know that the signal x(n) can be expressed as a summation of impulses,

$$\text{i.e., } x(n) = \sum_{m=-\infty}^{+\infty} x(m)\ \delta(n-m) \qquad .....(2.35)$$

where, d(n – m) is the delayed unit impulse signal.

From equations (2.34) and (2.35) we get,

$$y(n) = \mathcal{H}\left\{\sum_{m=-\infty}^{+\infty} x(m)\ \delta(n-m)\right\} \qquad .....(2.36)$$

The system $\mathcal{H}$ is a function of n and not a function of m. Hence by linearity property the equation (2.36) can be written as,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\ \mathcal{H}\{\delta(n-m)\} \qquad .....(2.37)$$

Let the response of the LTI system to the unit impulse input d(n) be denoted by h(n),

\ $h(n) = \mathcal{H}\{d(n)\}$

Then by time invariance property the response of the system to the delayed unit impulse input d(n – m) is given by,

$h(n-m) = \mathcal{H}\{d(n-m)\}$ .....(2.38)

Using equation (2.38), the equation (2.37) can be expressed as,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\ h(n-m)$$

The above equation represents the convolution of input x(n) with the impulse response h(n) to yield the output y(n). Hence it is proved that the response y(n) of LTI discrete time system for an arbitrary input x(n) is given by convolution of input x(n) with impulse response h(n) of the system.

2.9.3 Properties of Linear Convolution

The Discrete or Linear convolution will satisfy the following properties.

Commutative property : $x_1(n) * x_2(n) = x_2(n) * x_1(n)$

Associative property : $[x_1(n) * x_2(n)] * x_3(n) = x_1(n) * [x_2(n) * x_3(n)]$

Distributive property : $x_1(n) * [x_2(n) + x_3(n)] = [x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]$

Proof of Commutative Property :

Consider convolution of $x_1(n)$ and $x_2(n)$.

By commutative property we can write,

$$\underset{\text{(LHS)}}{x_1(n) * x_2(n)} = \underset{\text{(RHS)}}{x_2(n) * x_1(n)}$$

$$\text{LHS} = x_1(n) * x_2(n)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\ x_2(n-m) \qquad .....(2.39)$$

where, m is a dummy variable used for convolution operation.

Let, $n - m = p$ | when $m = -\infty$, $p = n - m = n + \infty = +\infty$

$\therefore$ $m = n - p$ | when $m = +\infty$, $p = n - m = n - \infty = -\infty$

On replacing m by (n – p) and (n – m) by p in equation (2.39) we get,

$$\text{LHS} = \sum_{p=-\infty}^{+\infty} x_1(n-p)\, x_2(p) = \sum_{p=-\infty}^{+\infty} x_2(p)\, x_1(n-p)$$

$$= x_2(n) * x_1(n)$$

p is a dummy variable used for convolution operation.

$$= \text{RHS}$$

Proof of Associative Property :

Consider the discrete time signals $x_1(n), x_2(n)$ and $x_3(n)$.

Let, $y_1(n) = x_1(n) * x_2(n)$ .....(2.40)

Let us replace n by p

$$\therefore\ y_1(p) = x_1(p) * x_2(p)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m) \qquad .....(2.41)$$

Let, $y_2(n) = x_2(n) * x_3(n)$ .....(2.42)

$$\therefore\ y_2(n) = \sum_{q=-\infty}^{+\infty} x_1(q)\, x_2(n-q)$$

$$\therefore\ y_2(n-m) = \sum_{q=-\infty}^{+\infty} x_1(q)\, x_2(n-q-m) \qquad .....(2.43)$$

where p, m and q are dummy variables used for convolution operation.

By associative property we can write,

$$\underbrace{[x_1(n) * x_2(n)] * x_3(n)}_{\text{LHS}} = \underbrace{x_1(n) * [x_2(n) * x_3(n)]}_{\text{RHS}}$$

$$\text{LHS} = [x_1(n) * x_2(n)] * x_3(n)$$

$$= y_1(n) * x_3(n)$$ Using equation (2.40)

$$= \sum_{p=-\infty}^{+\infty} y_1(p)\, x_3(n-p)$$

$$= \sum_{p=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m)\, x_3(n-p)$$ Using equation (2.41)

$$= \sum_{m=-\infty}^{+\infty} x_1(m) \sum_{p=-\infty}^{+\infty} x_2(p-m)\, x_3(n-p) \qquad .....(2.44)$$

Let, $p - m = q$ | when $p = -\infty$, $q = p - m = -\infty - m = -\infty$

$\therefore$ $p = q + m$ | when $p = +\infty$, $q = p - m = +\infty - m = +\infty$

On replacing (p – m) by q, and p by (q + m) in the equation (2.44) we get,

$$\text{LHS} = \sum_{m=-\infty}^{+\infty} x_1(m) \sum_{q=-\infty}^{+\infty} x_2(q)\, x_3(n-q-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, y_2(n-m)$$ Using equation (2.43)

$$= x_1(n) * y_2(n)$$

$$= x_1(n) * [x_2(n) * x_3(n)]$$ Using equation (2.42)

$$= \text{RHS}$$

<u>*Proof of Distributive Property :*</u>

Consider the discrete time signals $x_1(n), x_2(n)$ and $x_3(n)$. By distributive property we can write,

$$\underbrace{x_1(n) * [x_2(n) + x_3(n)]}_{\text{LHS}} = \underbrace{[x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]}_{\text{RHS}}$$

$$\text{LHS} = x_1(n) * [x_2(n) + x_3(n)]$$

$$= x_1(n) * x_4(n) \qquad \boxed{x_4(n) = x_2(n) + x_3(n)}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_4(n-m) \qquad \boxed{\text{m is a dummy variable used for convolution operation.}}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, [x_2(n-m) + x_3(n-m)] \qquad \boxed{x_4(n-m) = x_2(n-m) + x_3(n-m)}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) + \sum_{m=-\infty}^{+\infty} x_1(m)\, x_3(n-m)$$

$$= [x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]$$

$$= \text{RHS}$$

2.9.4 Interconnections of Discrete Time Systems

Smaller discrete time systems may be interconnected to form larger systems. Two possible basic ways of interconnection are ***cascade connection*** and ***parallel connection***. The cascade and parallel connections of two discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ are shown in fig 2.21.

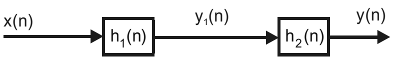


Fig 2.21a : Cascade connection.

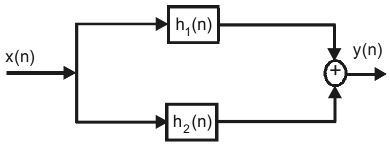


Fig 2.21b : Parallel connection.

Fig 2.21 : Interconnection of discrete time systems.

Cascade Connected Discrete Time Systems

Two cascade connected discrete time systems with impulse response $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system whose impulse response is given by convolution of individual impulse responses.

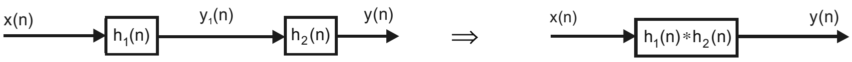


Fig 2.22 : Cascade connected discrete time system and their equivalent.

<u>*Proof:*</u>

With reference to fig 2.22 we can write,

$$y_1(n) = x(n) * h_1(n) \qquad .....(2.45)$$

$$y(n) = y_1(n) * h_2(n) \qquad .....(2.46)$$

Using equation (2.45), the equation (2.46) can be written as,

$$y(n) = x(n) * h_1(n) * h_2(n)$$

$$= x(n) * [h_1(n) * h_2(n)]$$

$$= x(n) * h(n) \qquad .....(2.47)$$

where, $h(n) = h_1(n) * h_2(n)$

From equation (2.47) we can say that the overall impulse response of two cascaded discrete time systems is given by convolution of individual impulse responses.

Parallel Connected Discrete Time Systems

Two parallel connected discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system whose impulse response is given by sum of individual impulse responses.

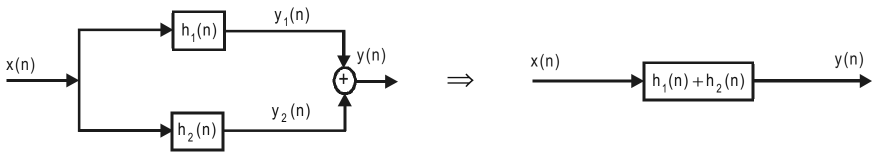


Fig 2.23 : *Parallel connected discrete time systems and their equivalent.*

Proof:

With reference to fig 2.23 we can write,

$y_1(n) = x(n) * h_1(n)$ .....(2.48)

$y_2(n) = x(n) * h_2(n)$ .....(2.49)

$y(n) = y_1(n) + y_2(n)$ .....(2.50)

On substituting for $y_1(n)$ and $y_2(n)$ from equations (2.48) and (2.49) in equation (2.50) we get,

$y(n) = [\,x(n) * h_1(n)] + [\,x(n) * h_2(n)]$ .....(2.51)

By using distributive property of convolution, the equation (2.51) can be written as shown below,

$$y(n) = x(n) * [h_1(n) + h_2(n)]$$

$$= x(n) * h(n) \qquad .....(2.52)$$

where, $h(n) = h_1(n) + h_2(n)$

From equation (2.52) we can say that the overall impulse response of two parallel connected discrete time systems is given by sum of individual impulse responses.

Example 2.20

Determine the impulse response for the cascade of two LTI systems having impulse responses,

$$h_1(n) = \left(\frac{2}{5}\right)^n u(n) \text{ and } h_2(n) = \left(\frac{1}{5}\right)^n u(n).$$

Solution

Let h(n) be the impulse response of cascade system. Now h(n) is given by convolution of $h_1(n)$ and $h_2(n)$.

$$\therefore\ h(n) = h_1(n) * h_2(n) = \sum_{m=-\infty}^{+\infty} h_1(m)\, h_2(n-m)$$

where, m is a dummy variable used for convolution operation

$$h_1(m) = \left(\frac{2}{5}\right)^m \ ; \quad h_2(m) = \left(\frac{1}{5}\right)^m \ ; \quad h_2(n-m) = \left(\frac{1}{5}\right)^{n-m}$$

The product $h_1(m)\, h_2(n-m)$ will be nonzero in the range $0 \le m \le n$. Therefore the summation index in the above equation is changed to m = 0 to n.

$$\therefore\ h(n) = \sum_{m=0}^{n} h_1(m)\, h_2(n-m) = \sum_{m=0}^{n} \left(\frac{2}{5}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{2}{5}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m} = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{2}{5}\right)^m 5^m$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{2 \times 5}{5}\right)^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} 2^m$$

$$= \left(\frac{1}{5}\right)^n \left(\frac{2^{n+1}-1}{2-1}\right) = \left(\frac{1}{5}\right)^n (2^{n+1}-1) \ ; \quad \text{for } n \ge 0$$

$$= \left(\frac{1}{5}\right)^n (2^{n+1}-1)\, u(n) \ ; \quad \text{for all } n$$

Finite geometric series sum formula

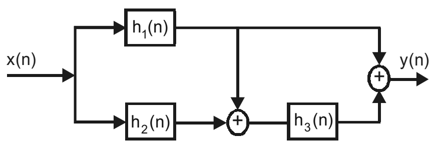
$$\sum_{n=0}^{N} C^n = \frac{C^{N+1}-1}{C-1}$$

Using finite geometric series sum formula.

Example 2.21

Determine the overall impulse response of the interconnected discrete time systems shown below,

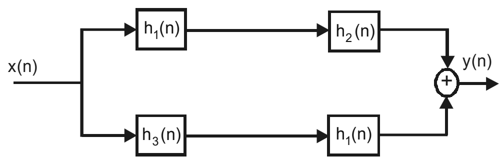
a)



$$h_1(n) = \left(\frac{1}{3}\right)^n u(n); \quad h_2(n) = \left(\frac{1}{2}\right)^n u(n);$$

$$h_3(n) = \left(\frac{1}{5}\right)^n u(n)$$

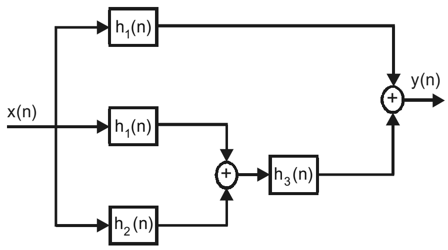
b)



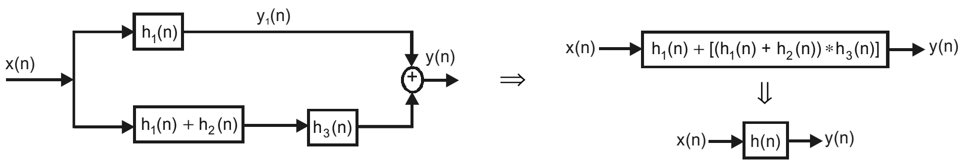
$$h_1(n) = a^n\, u(n) \quad ; \quad h_2(n) = \delta(n-1) \quad ; \quad h_3(n) = \delta(n-2)$$

Solution

a) The given system can be redrawn as shown below.



The above system can be reduced to single equivalent system as shown below.



Here, $h(n) = h_1(n) + [(h_1(n) + h_2(n)) * h_3(n)]$

$= h_1(n) + [h_1(n) * h_3(n)] + [h_2(n) * h_3(n)]$

Using distributive property.

Let us evaluate the convolution of $h_1(n)$ and $h_3(n)$.

$$h_1(n) * h_3(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_3(n-m)$$

The product of $h_1(m)\, h_3(n-m)$ will be nonzero in the range $0 \le m \le n$. Therefore the summation index in the above equation can be changed to m = 0 to n.

$$\therefore\ h_1(n) * h_3(n) = \sum_{m=0}^{n} h_1(m)\, h_3(n-m)$$

$$= \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m}$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m 5^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{5}{3}\right)^m$$

$$\therefore\ h_1(n) * h_3(n) = \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^{n+1} - 1}{\frac{5}{3} - 1}$$

Using finite geometric series sum formula.

Finite geometric series sum formula

$$\sum_{m=0}^{N} C^m = \frac{C^{N+1}-1}{C-1}$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^n \frac{5}{3} - 1}{\frac{5-3}{3}} = \left(\frac{1}{5}\right)^n \left[\frac{3}{2}\left(\frac{5}{3}\right)^n \frac{5}{3} - \frac{3}{2}\right]$$

$$= \frac{5}{2}\left(\frac{1}{5}\right)^n\left(\frac{5}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n = \frac{5}{2}\left(\frac{1}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n \ ; \text{ for } n \ge 0$$

$$= \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n)\ ; \text{ for all } n$$

Let us evaluate the convolution of $h_2(n)$ and $h_3(n)$.

$$h_2(n) * h_3(n) = \sum_{m=-\infty}^{+\infty} h_2(m)\, h_3(n-m)$$

The product of $h_2(m)$ and $h_3(n–m)$ will be nonzero in the range 0 £ m £ n. Therefore the summation index in the above equation can be change to m = 0 to n.

$$\therefore\ h_2(n) * h_3(n) = \sum_{m=0}^{n} h_2(m)\, h_3(n-m)$$

$$= \sum_{m=0}^{n}\left(\frac{1}{2}\right)^m\left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n}\left(\frac{1}{2}\right)^m\left(\frac{1}{5}\right)^n\left(\frac{1}{5}\right)^{-m}$$

Finite geometric series sum formula

$$\sum_{m=0}^{N} C^m = \frac{C^{N+1}-1}{C-1}$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n}\left(\frac{1}{2}\right)^m 5^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n}\left(\frac{5}{2}\right)^m$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{2}\right)^{n+1} - 1}{\frac{5}{2} - 1}$$

Using finite geometric series sum formula.

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{2}\right)^n \frac{5}{2} - 1}{\frac{5-2}{2}} = \left(\frac{1}{5}\right)^n \left[\frac{2}{3}\left(\frac{5}{2}\right)^n \frac{5}{2} - \frac{2}{3}\right]$$

$$= \frac{5}{3}\left(\frac{1}{5}\right)^n\left(\frac{5}{2}\right)^n - \frac{2}{3}\left(\frac{1}{5}\right)^n = \frac{5}{3}\left(\frac{1}{2}\right)^n - \frac{2}{3}\left(\frac{1}{5}\right)^n \text{ for } n \ge 0$$

$$= \frac{5}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{5}\right)^n u(n) \text{ for all } n$$

Now, the overall impulse response h(n) is given by,

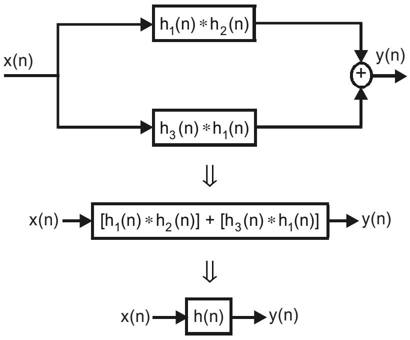
$$h(n) = h_1(n) + \left[h_1(n) * h_3(n)\right] + \left[h_2(n) * h_3(n)\right]$$

$$= \left(\frac{1}{3}\right)^n u(n) + \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{5}\right)^n u(n)$$

$$= \left(1 + \frac{5}{2}\right)\left(\frac{1}{3}\right)^n u(n) - \left(\frac{3}{2} + \frac{2}{3}\right)\left(\frac{1}{5}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n)$$

$$= \left[\frac{7}{2}\left(\frac{1}{3}\right)^n - \frac{13}{6}\left(\frac{1}{5}\right)^n + \frac{5}{3}\left(\frac{1}{2}\right)^n\right] u(n)$$

b) The given system can be reduced to single equivalent system as shown below.



Here, $h(n) = [h_1(n) * h_2(n)] + [h_3(n) * h_1(n)]$

Let us evaluate the convolution of $h_1(n)$ and $h_2(n)$.

$$h_1(n) * h_2(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_2(n-m)$$

$$= \sum_{m=-\infty}^{\infty} h_2(m)\, h_1(n-m)$$ Using commutative property.

$$= \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{(n-m)} = \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^n\, a^{-m}$$

$$= a^n \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{-m}$$

The product of $\delta(m-1)$ and a^{-m} in the above equation will be nonzero only when m = 1.

$\therefore\ h_1(n) * h_2(n) = a^n\, a^{-1} = a^{n-1}$; for $n \geq 1$

$= a^{n-1}\, u(n-1)$; for all n.

Let us evaluate the convolution of $h_3(n)$ and $h_1(n)$.

$$h_3(n) * h_1(n) = \sum_{m=-\infty}^{\infty} h_3(m)\, h_1(n-m)$$

$$= \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^{(n-m)} = \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^n\, a^{-m}$$

$$= a^n \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^{-m}$$

The product of $\delta(m-2)$ and a^{-m} in the above equation will be nonzero only when m = 2.

$\therefore\ h_1(n) * h_2(n) = a^n\, a^{-2} = a^{n-2}$; for $n \geq 2$

$= a^{n-2}\, u(n-2)$; for all n

Now, the overall impulse response h(n) is given by,

$h(n) = [h_1(n) * h_2(n)] + [h_3(n) * h_1(n)]$

$= a^{(n-1)}\, u(n-1) + a^{(n-2)}\, u(n-2)$

2.9.5 Methods of Performing Linear Convolution

Method 1: Graphical Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence.

1. Change the index "n" of input sequences to "m" to get $x_1(m)$ and $x_2(m)$.
2. Sketch the graphical representation of the input sequences $x_1(m)$ and $x_2(m)$.
3. Let us fold $x_2(m)$ to get $x_2(-m)$. Sketch the graphical representation of the folded sequence $x_2(-m)$.
4. Shift the folded sequence $x_2(-m)$ to the left graphically so that the product of $x_1(m)$ and shifted $x_2(-m)$ gives only one nonzero sample. Now multiply $x_1(m)$ and shifted $x_2(-m)$ to get a product sequence, and then sum up the samples of product sequence, which is the first sample of output sequence.
5. To get the next sample of output sequence, shift $x_2(-m)$ of previous step to one position right and multiply the shifted sequence with $x_1(m)$ to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.

2. To get subsequent samples of output sequence, the step 5 is repeated until we get a nonzero product sequence.

Method 2: Tabular Method

The tabular method is same as that of graphical method, except that the tabular representation of the sequences are employed instead of graphical representation. In tabular method, every input sequence, folded and shifted sequence is represented by a row in a table.

Method 3: Matrix Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence. In matrix method one of the sequences is represented as a row and the other as a column as shown below.

Multiply each column element with row elements and fill up the matrix array.

Now the sum of the diagonal elements gives the samples of output sequence $x_3(n)$. (The sum of the diagonal elements are shown below for reference).

	....	$x_2(0)$	$x_2(1)$	$x_2(2)$	$x_2(3)$	....
$x_1(0)$	....	$x_1(0)\,x_2(0)$	$x_1(0)\,x_2(1)$	$x_1(0)\,x_2(2)$	$x_1(0)\,x_2(3)$	....
$x_1(1)$	....	$x_1(1)\,x_2(0)$	$x_1(1)\,x_2(1)$	$x_1(1)\,x_2(2)$	$x_1(1)\,x_2(3)$	....
$x_1(2)$	....	$x_1(2)\,x_2(0)$	$x_1(2)\,x_2(1)$	$x_1(2)\,x_2(2)$	$x_1(2)\,x_2(3)$	....
$x_1(3)$	....	$x_1(3)\,x_2(0)$	$x_1(3)\,x_2(1)$	$x_1(3)\,x_2(2)$	$x_1(3)\,x_2(3)$	....
⋮		⋮	⋮	⋮	⋮	

$$\vdots$$

$$x_3(0) = \cdots\cdots + x_1(0)\,x_2(0) + \cdots\cdots$$

$$x_3(1) = \cdots\cdots + x_1(1)\,x_2(0) + x_1(0)\,x_2(1) + \cdots\cdots$$

$$x_3(2) = \cdots\cdots + x_1(2)\,x_2(0) + x_1(1)\,x_2(1) + x_1(0)\,x_2(2) + \cdots\cdots$$

$$x_3(3) = \cdots\cdots + x_1(3)\,x_2(0) + x_1(2)\,x_2(1) + x_1(1)\,x_2(2) + x_1(0)\,x_2(3) + \cdots\cdots$$

$$\vdots$$

$$\backslash \;\; x_3(n) = \{\ldots\ldots x_3(0), x_3(1), x_3(2), x_3(3), \ldots\ldots\}$$

Example 2.22

Determine the response of the LTI system whose input x(n) and impulse response h(n) are given by,

$x(n) = \{\underline{1}, 2, 0.5, 1\}$ and $h(n) = \{1, \underline{2}, 1, -1\}$

Solution

The response y(n) of the system is given by convolution of x(n) and h(n).

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\; h(n-m)$$

In this example the convolution operation is performed by three methods.

The Input sequence starts at n = 0 and the impulse response sequence starts at n = –1. Therefore the output sequence starts at n = 0 + (–1) = –1.

The input and impulse response consists of 4 samples, so the output consists of 4 + 4 – 1 = 7 samples.

Method 1 : Graphical Method

The graphical representation of x(n) and h(n) after replacing n by m are shown below. The sequence h(m) is folded with respect to m = 0 to obtain h(–m).

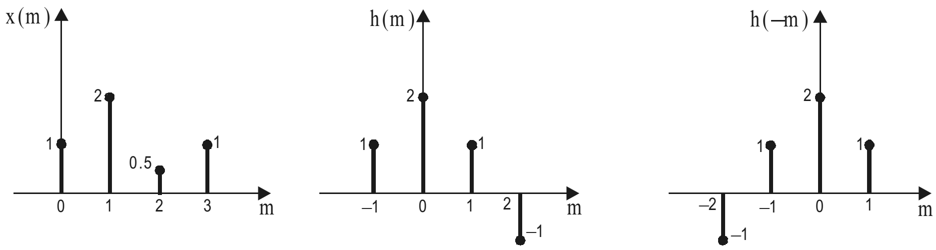


Fig 1 : Input sequence. *Fig 2 : Impulse response.* *Fig 3 : Folded impulse response.*

The samples of y(n) are computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\; h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\; h_n(m)\;;\; \text{ where } h_n(m) = h(n-m)$$

The computation of each sample using the above equation are graphically shown in fig 4 to fig 10. The graphical representation of output sequence is shown in fig 11.

When n = –1 ; $y(-1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(-1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_{-1}(m) = \sum_{m=-\infty}^{+\infty} v_{-1}(m)$

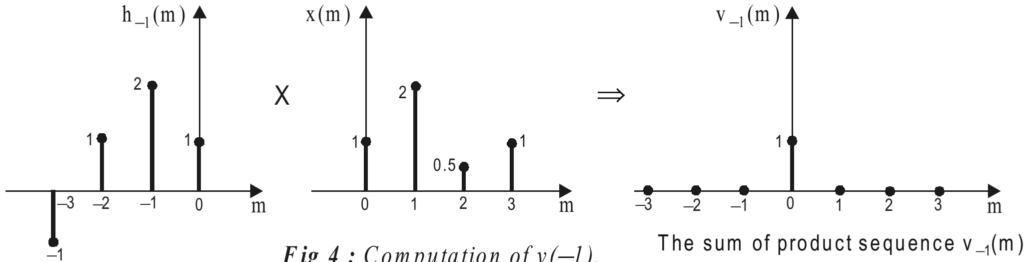


Fig 4 : Computation of y(–1).

The sum of product sequence $v_{-1}(m)$ gives $y(-1)$. $\therefore$ $y(-1) = 1$

When n = 0 ; $y(0) = \sum_{m=-\infty}^{+\infty} x(m)\, h(0-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_0(m) = \sum_{m=-\infty}^{+\infty} v_0(m)$

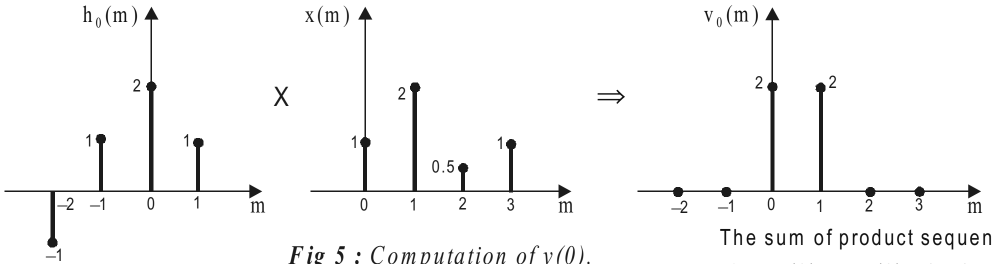


Fig 5 : Computation of y(0).

The sum of product sequence $v_0(m)$ gives $y(0)$. $\therefore$ $y(0) = 2 + 2 = 4$

When n = 1 ; $y(1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_1(m) = \sum_{m=-\infty}^{+\infty} v_1(m)$

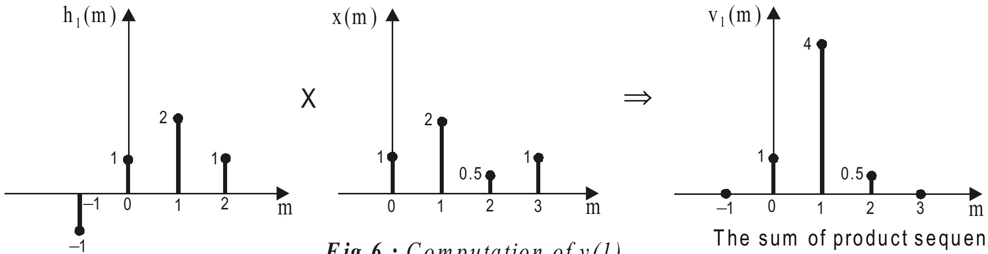


Fig 6 : Computation of y(1).

The sum of product sequence $v_1(m)$ gives $y(1)$. $\therefore$ $y(1) = 1 + 4 + 0.5 = 5.5$

When n = 2 ; $y(2) = \sum_{m=-\infty}^{+\infty} x(m)\, h(2-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_2(m) = \sum_{m=-\infty}^{+\infty} v_2(m)$

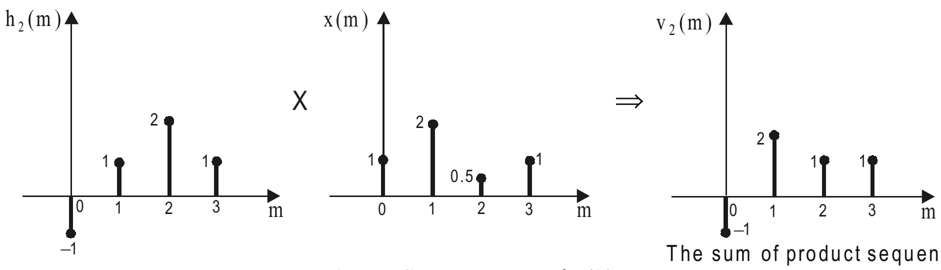


Fig 7 : Computation of y(2).

The sum of product sequence $v_2(m)$ gives $y(2)$. $\therefore$ $y(2) = -1 + 2 + 1 + 1 = 3$

When n = 3 ; $y(3) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h(3-m) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h_3(m) = \sum\limits_{m=-\infty}^{+\infty} v_3(m)$

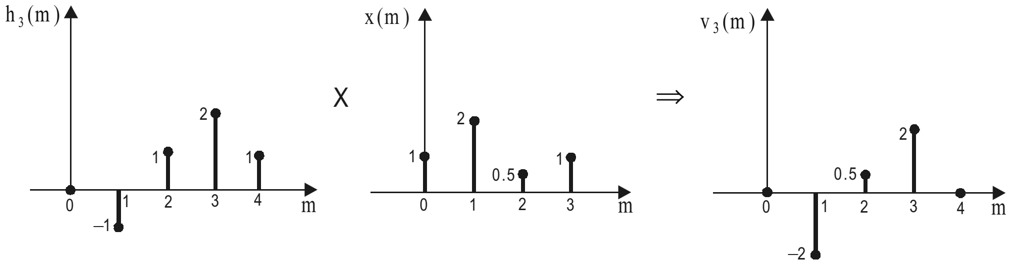


Fig 8 : Computation of y(3).

The sum of product sequence $v_3(m)$ gives y(3). $\therefore$ $y(3) = -2 + 0.5 + 2 = 0.5$

When n = 4 ; $y(4) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h(4-m) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h_4(m) = \sum\limits_{m=-\infty}^{+\infty} v_4(m)$

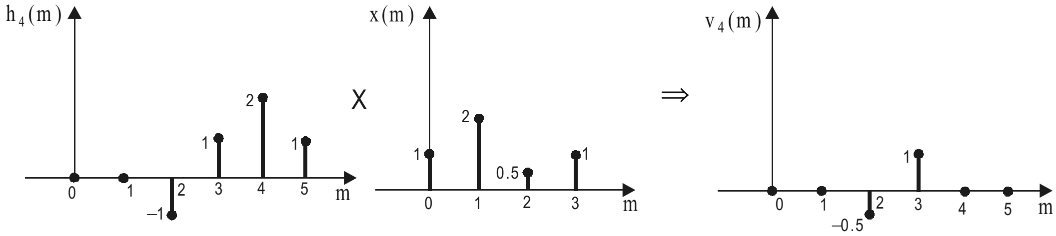


Fig 9 : Computation of y(4).

The sum of product sequence $v_4(m)$ gives y(4). $\therefore$ $y(4) = -0.5 + 1 = 0.5$

When n = 5 ; $y(5) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h(5-m) = \sum\limits_{m=-\infty}^{+\infty} x(m)\, h_5(m) = \sum\limits_{m=-\infty}^{+\infty} v_5(m)$

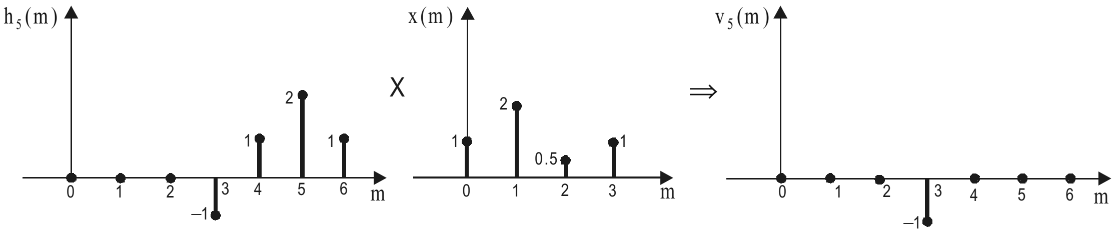


Fig 10 : Computation of y(5).

The sum of product sequence $v_5(m)$ gives y(5). $\therefore$ $y(5) = -1$

The output sequence, $y(n) = \{1,\ \underset{\uparrow}{4},\ 5.5,\ 3,\ 0.5,\ 0.5,\ -1\}$

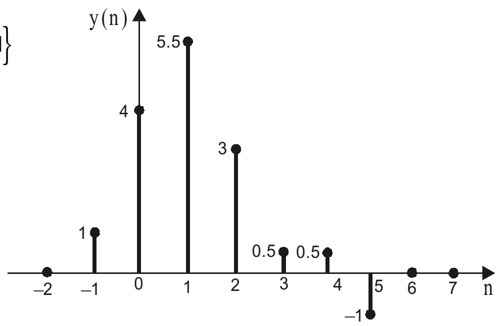


Fig 11 : Graphical representation of y(n).

Method 2 : Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

Note : The unfilled boxes in the table are considered as zeros.

m	–3	–2	–1	0	1	2	3	4	5	6
x(m)				1	2	0.5	1			
h(m)			1	2	1	–1				
h(–m)		–1	1	2	1					
$h(-1-m) = h_{-1}(m)$	–1	1	2	1						
$h(0-m) = h_0(m)$		–1	1	2	1					
$h(1-m) = h_1(m)$			–1	1	2	1				
$h(2-m) = h_2(m)$				–1	1	2	1			
$h(3-m) = h_3(m)$					–1	1	2	1		
$h(4-m) = h_4(m)$						–1	1	2	1	
$h(5-m) = h_5(m)$							–1	1	2	1

Each sample of y(n) is computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m), \quad \text{where } h_n(m) = h(n-m)$$

To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence (i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$). The sum of all the samples of the product sequence gives y(q).

When n = –1 ; $y(-1) = \sum_{m=-3}^{3} x(m)\, h_{-1}(m)$ $\boxed{\because \text{ The product is valid only for } m = -3 \text{ to } +3.}$

$$= x(-3)\, h_{-1}(-3) + x(-2)h_{-1}(-2) + x(-1)h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2) + x(3)\, h_{-1}(3)$$

$$= 0 + 0 + 0 + 1 + 0 + 0 + 0 = 1$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

When n = 0 ; $y(0) = \sum_{m=-2}^{3} x(m)\, h_0(m) = 0 + 0 + 2 + 2 + 0 + 0 = 4$

When n = 1 ; $y(1) = \sum_{m=-1}^{3} x(m)\, h_1(m) = 0 + 1 + 4 + 0.5 + 0 = 5.5$

When n = 2 ; $y(2) = \sum_{m=0}^{3} x(m)\, h_2(m) = -1 + 2 + 1 + 1 = 3$

When n = 3 ; $y(3) = \sum_{m=0}^{4} x(m)\, h_3(m) = 0 - 2 + 0.5 + 2 + 0 = 0.5$

When n = 4 ; $y(4) = \sum_{m=0}^{5} x(m)\, h_4(m) = 0 + 0 - 0.5 + 1 + 0 + 0 = 0.5$

When n = 5 ; $y(5) = \sum_{m=0}^{6} x(m)\, h_5(m) = 0 + 0 + 0 - 1 + 0 + 0 + 0 = -1$

The output sequence, $y(n) = \{1, \underset{\uparrow}{4}, 5.5, 3, 0.5, 0.5, -1\}$

Method 3 : Matrix Method

The input sequence x(n) is arranged as a column and the impulse response is arranged as a row as shown below. The elements of the two-dimensional array are obtained by multiplying the corresponding row element with the column element. The sum of the diagonal elements gives the samples of y(n).

x(n) ↓ \ h(n) →	1	2	1	–1
1	1×1	1×2	1×1	$1 \times (-1)$
2	2×1	2×2	2×1	$2 \times (-1)$
0.5	0.5×1	0.5×2	0.5×1	$0.5 \times (-1)$
1	1×1	1×2	1×1	$1 \times (-1)$

$\Rightarrow$

x(n) ↓ \ h(n) →	1	2	1	–1
1	1	2	1	–1
2	2	4	2	–2
0.5	0.5	1	0.5	–0.5
1	1	2	1	–1

$y(-1) = 1$

$y(0) = 2 + 2 = 4$

$y(1) = 0.5 + 4 + 1 = 5.5$

$y(2) = 1 + 1 + 2 + (-1) = 3$

$y(3) = 2 + 0.5 + (-2) = 0.5$

$y(4) = 1 + (-0.5) = 0.5$

$y(5) = -1$

$\therefore y(n) = \{1, 4, 5.5, 3, 0.5, 0.5, -1\}$

Example 2.23

Determine the output y(n) of a relaxed LTI system with impulse response,

$h(n) = a^n\, u(n)$; where $|a| < 1$ and

When input is a unit step sequence, i.e., x(n) = u(n).

Solution

The graphical representation of x(n) and h(n) after replacing n by m are shown below. Also the sequence x(m) is folded to get x(–m).

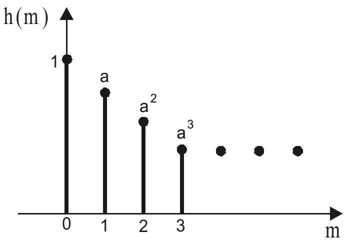


Fig 1 : Impulse response.

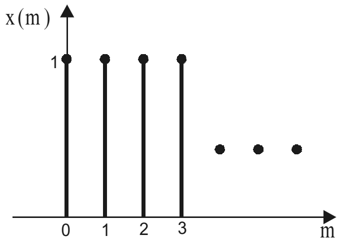


Fig 2 : Impulse sequence.

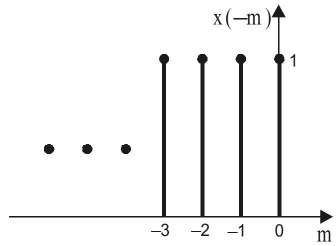


Fig 3 : Folded input sequence.

Here both h(m) and x(m) are infinite duration sequences starting at n = 0. Hence the output sequence y(n) will also be an infinite duration sequence starting at n = 0.

By convolution formula,

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m) = \sum_{m=0}^{\infty} h(m)\, x_n(m) \; ; \quad \text{where } x_n(m) = x(n-m)$$

The computation of some samples of y(n) using the above equation are graphically shown below.

When n = 0 ; $y(0) = \sum_{m=0}^{\infty} h(m)\, x(0-m) = \sum_{m=0}^{\infty} h(m)\, x_0(m) = \sum_{m=0}^{\infty} v_0(m)$

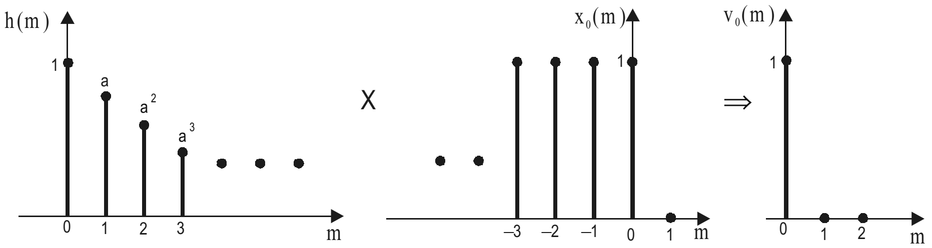


Fig 4 : Computation of y(0).

When n = 1 ; $y(1) = \sum_{m=0}^{\infty} h(m)\, x(1-m) = \sum_{m=0}^{\infty} h(m)\, x_1(m) = \sum_{m=0}^{\infty} v_1(m)$

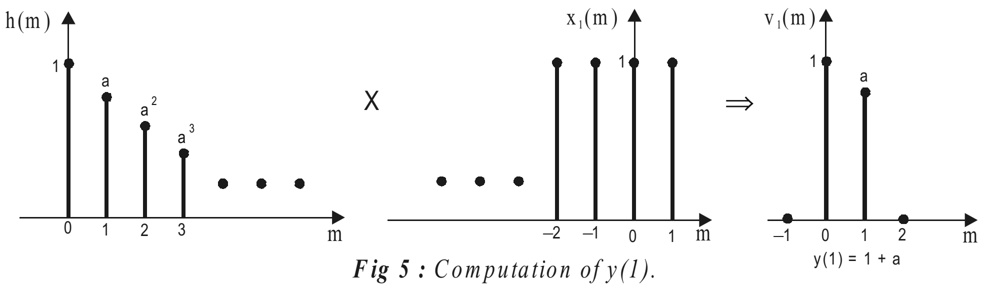


Fig 5 : Computation of y(1).

When n = 2 ; $y(2) = \sum_{m=0}^{\infty} h(m)\, x(2-m) = \sum_{m=0}^{\infty} h(m)\, x_2(m) = \sum_{m=0}^{\infty} v_2(m)$

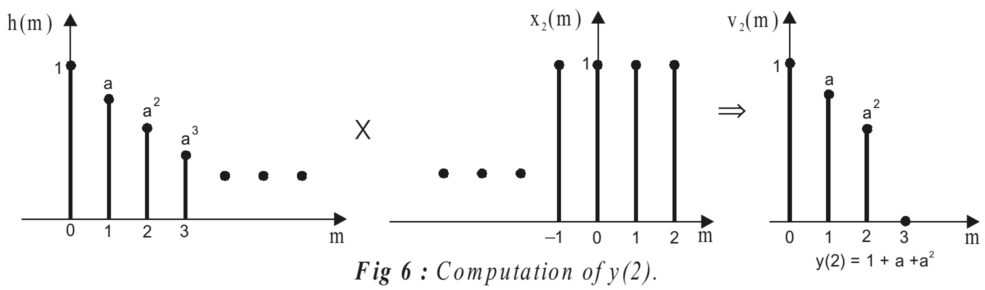


Fig 6 : Computation of y(2).

Solving similarly for other values of n, we can write y(n) for any value of n as shown below.

$$y(n) = 1 + a + a^2 + \ldots\ldots + a^n = \sum_{p=0}^{n} a^p \quad ; \quad \text{for } n \geq 0$$

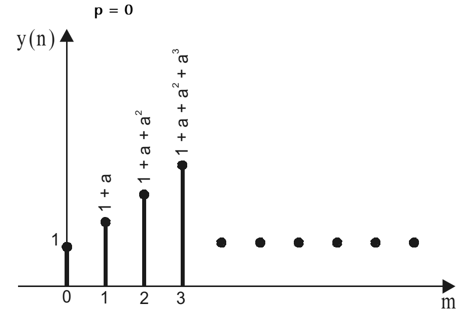


Fig 7 : Graphical representation of y(n).

2.10 Circular Convolution

2.10.1 Circular Representation and Circular Shift of Discrete Time Signal

Consider a finite duration sequence x(n) and its periodic extension $x_p(n)$. The periodic extension of x(n) can be expressed as $x_p(n) = x(n + N)$, where N is the periodicity. Let N = 4. The sequence x(n) and its periodic extension are shown in fig 2.24.

$$\text{Let, } x(n) = 1\ ;\quad n = 0$$
$$= 2\ ;\quad n = 1$$
$$= 3\ ;\quad n = 2$$
$$= 4\ ;\quad n = 3$$

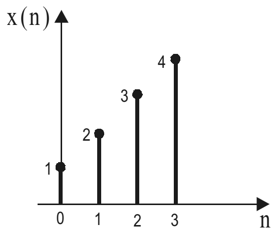


Fig 2.24a : *Finite duration sequence x(n).*

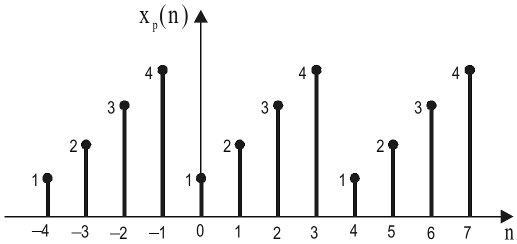


Fig 2.24b : *Periodic extension of x(n).*

Fig 2.24 : *A finite duration sequence and its periodic extension.*

Let us delay the periodic sequence $x_p(n)$ by two units of time as shown in fig 2.25(a). (For delay the sequence is shifted right). Let us denote one period of this delayed sequence by $x_1(n)$. One period of the delayed sequence is shown in fig 2.25(b).

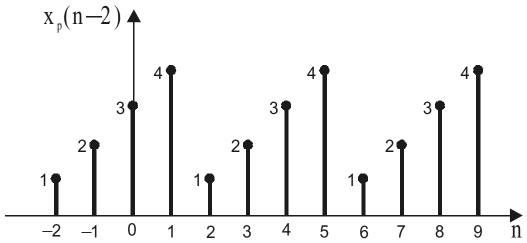


Fig 2.25a: *$x_p(n)$ delayed by two units of time.*

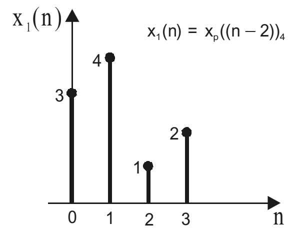


Fig 2.25b: *One period of $x_p(n-2)$.*

Fig 2.25: *Delayed version of $x_p(n)$.*

The sequence $x_1(n)$ can be represented by $x_p(n-2, (\text{mod } 4))$, or $x_p((n-2))_4$, where mod 4 indicates that the sequence repeats after 4 samples. The relation between the original sequence x(n) and one period of the delayed sequence $x_1(n)$ are shown below.

$$x_1(n) = x_p(n - 2, (\text{mod } 4)) = x_p((n-2))_4$$

\ When n = 0; $x_1(0) = x_p((0-2))_4 = x_p((-2))_4 = x(2) = 3$

When n = 1; $x_1(1) = x_p((1-2))_4 = x_p((-1))_4 = x(3) = 4$

When n = 2; $x_1(2) = x_p((2-2))_4 = x_p((0))_4 = x(0) = 1$

When n = 3; $x_1(3) = x_p((3-2))_4 = x_p((1))_4 = x(1) = 2$

The periodic sequences $x_p(n)$ and $x_1(n)$ can be represented as points on a circle as shown in fig 2.26. From fig 2.26 we can say that, $x_1(n)$ is simply $x_p(n)$ shifted circularly by two units in time, where the counter clockwise (anticlockwise) direction has been arbitrarily selected for right shift or delay.

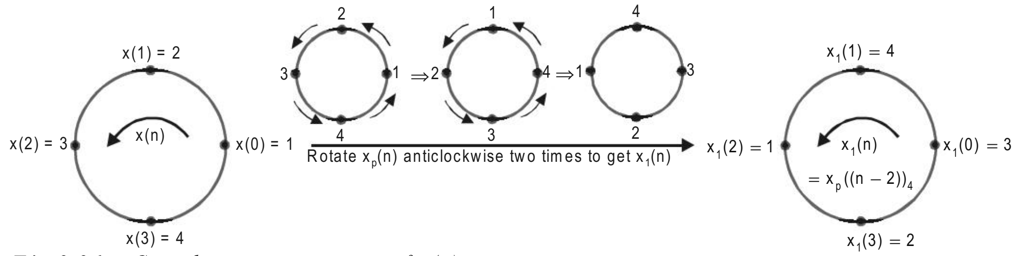


Fig 2.26a: *Circular representation of x(n).* ***Fig 2.26b:*** *Circular representation of $x_1(n)$.*

Fig 2.26: *Circular representation of a signal and its delayed version.*

Let us advance the periodic sequence $x_p(n)$ by three units of time as shown in fig 2.27(a). Let us denote one period of this advanced sequence by $x_2(n)$. One period of the advanced sequence is shown in fig 2.27(b).

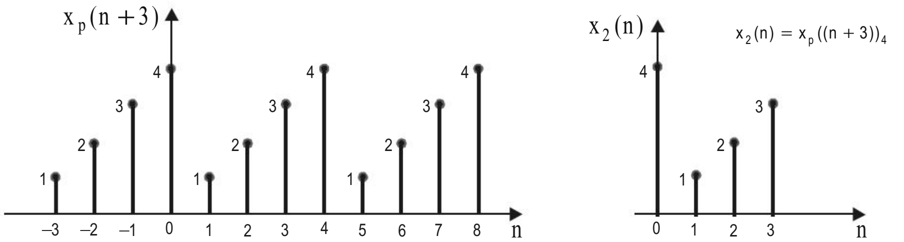


Fig 2.27a: *$x_p(n)$ advanced by three units of time.* ***Fig 2.27b:*** *One period of $x_p(n+3)$.*

Fig 2.27: *Advanced version of $x_p(n)$.*

The sequence $x_2(n)$ can be represented by $x_p(n+3, (\text{mod } 4))$ or $x_p((n+3))_4$, where mod 4 indicates that the sequence repeats after 4 samples. The relation between the original sequence x(n) and one period of the advanced sequence $x_2(n)$ are shown below.

$$x_2(n) = x_p(n+3, (\text{mod } 4)) = x_p((n+3))_4$$

\ When n = 0; $x_2(0) = x_p((0+3))_4 = x_p((3))_4 = x(3) = 4$

When n = 1; $x_2(1) = x_p((1+3))_4 = x_p((4))_4 = x(0) = 1$

When n = 2; $x_2(2) = x_p((2+3))_4 = x_p((5))_4 = x(1) = 2$

When n = 3; $x_2(3) = x_p((3+3))_4 = x_p((6))_4 = x(2) = 3$

The periodic sequences $x_p(n)$ and $x_2(n)$ can be represented as points on a circle as shown in fig 2.28. From fig 2.28 we can say that $x_2(n)$ is simply $x_p(n)$ shifted circularly by three units in time where clockwise direction has been selected for left shift or advance.

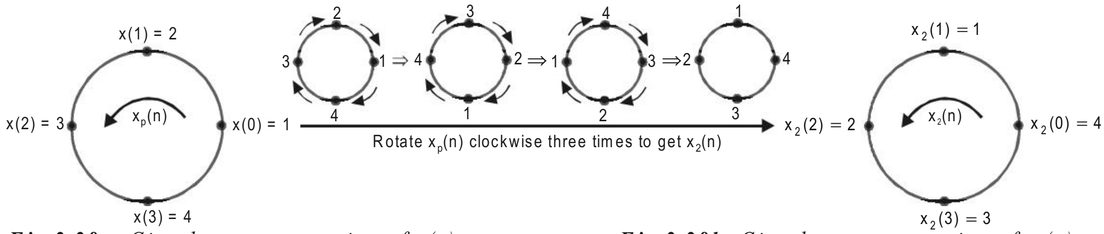


Fig 2.28a: *Circular representation of x(n).* ***Fig 2.28b:*** *Circular representation of $x_2(n)$.*

Fig 2.28: *Circular representation of a signal and its advanced version.*

Thus we conclude that a circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension and viceversa. If a nonperiodic N-point sequence is represented on the circumference of a circle then it becomes a periodic sequence of periodicity N. When the sequence is shifted circularly, the samples repeat after N shifts. This is similar to modulo-N operation. Hence, in general, the circular shift may be represented by the index mod-N. Let x(n) be an N-point sequence represented on a circle and $x'(n)$ be its ***circularly shifted sequence*** by m units of time.

Now, $x'(n) = x(n-m, \text{mod } N) \equiv x((n-m))_N$ ..... (2.53)

When m is positive, the equation (2.53) represents delayed sequence and when m is negative, the equation (2.53) represents advanced sequence.

2.10.2 Circular Symmetries of Discrete Time Signal

The circular representation of a sequence and the resulting periodicity gives rise to new definitions for even symmetry, odd symmetry and the time reversal of the sequence.

An N-point sequence is called even if it is symmetric about the point zero on the circle. This implies that,

$$x(N-n) = x(n) \quad ; \text{ for } \; 0 \le n \le N-1 \qquad \text{...... (2.54)}$$

An N-point sequence is called odd if it is antisymmetric about the point zero on the circle. This implies that,

$$x(N-n) = -x(n) \quad ; \text{ for } \; 0 \le n \le N-1 \qquad \text{...... (2.55)}$$

The time reversal of a N-point sequence is obtained by reversing its sample about the point zero on the circle. Thus the sequence x(–n, (mod N)) is simply written as,

$$x(-n, (\text{mod } N)) = x(N-n) \quad ; \text{ for } \; 0 \le n \le N-1 \qquad \text{...... (2.56)}$$

This time reversal is equivalent to plotting x(n) in a clockwise direction on a circle, as shown in fig 2.29.

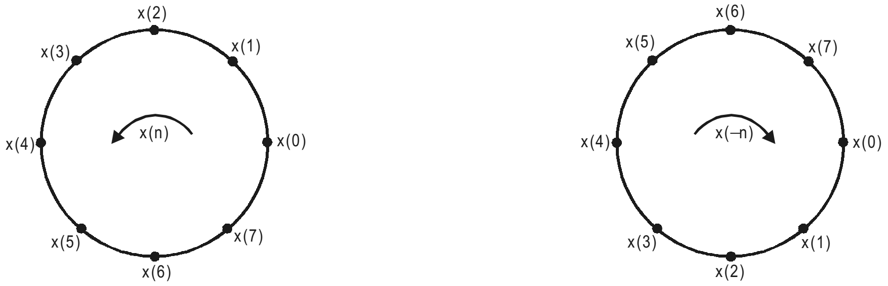


Fig 2.29: *Circular representation of an 8-point sequence and its folded sequence.*

2.10.3 Definition of Circular Convolution

The ***circular convolution*** of two periodic discrete time sequences $x_1(n)$ and $x_2(n)$ with periodicity of N samples is defined as,

$$\boxed{x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=0}^{N-1} x_2(m)\, x_1((n-m))_N} \qquad \text{.....(2.57)}$$

where, $x_3(n)$ is the sequence obtained by circular convolution,

$x_1((n-m))_N$ represents circular shift of $x_1(n)$

$x_2((n-m))_N$ represents circular shift of $x_2(n)$

m is a dummy variable.

The output sequence $x_3(n)$ obtained by circular convolution is also a periodic sequence with periodicity of N samples. Hence this convolution is also called ***periodic convolution***.

The convolution relation of equation (2.57) can be symbolically expressed as

$$x_3(n) = x_1(n) \circledast x_2(n) = x_2(n) \circledast x_1(n) \qquad .....(2.58)$$

where, the symbol $\circledast$ indicates circular convolution operation.

The circular convolution is defined for periodic sequences. But circular convolution can be performed with nonperiodic sequences by periodically extending them.The circular convolution of two sequences requires that, at least one of the sequences should be periodic. Hence it is sufficient if one of the sequences is periodically extended in order to perform circular convolution.

The circular convolution of finite duration sequences can be performed only if both the sequences consist of the same number of samples. If the sequences have different number of samples, then convert the smaller size sequence to the length of larger size sequence by appending zeros.

Circular convolution basically involves the same four steps as that for linear convolution, namely, folding one sequence, shifting the folded sequence, multiplying the two sequences and finally summing the values of the product sequence. Like linear convolution, any one of the sequence is folded and rotated in circular convolution.

The difference between the two is that in circular convolution the folding and shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences by modulo-N operation. In linear convolution there is no modulo-N operation.

2.10.4 Procedure for Evaluating Circular Convolution

Let, $x_1(n)$ and $x_2(n)$ be periodic discrete time sequences with periodicity of N-samples. If $x_1(n)$ and $x_2(n)$ are non-periodic then convert the sequences to N-sample sequences and periodically extend the sequence $x_2(n)$ with periodicity of N-samples.

Now the circular convolution of $x_1(n)$ and $x_2(n)$ will produce a periodic sequence $x_3(n)$ with periodicity of N-samples. The samples of one period of $x_3(n)$ can be computed using the equation (2.57). The value of $x_3(n)$ at $n = q$ is obtained by replacing n by q, in equation (2.57).

$$\therefore\ x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N \qquad .....(2.59)$$

The evaluation of equation (2.59) to determine the value of $x_3(n)$ at $n = q$ involves the following five steps.

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, in order to get the sequences $x_1(m)$ and $x_2(m)$. Represent the samples of one period of the sequences on circles.

2. Folding : Fold $x_2(m)$ about $m = 0$, to obtain $x_2(-m)$.

3. Rotation : Rotate $x_2(-m)$ by q times in anti-clockwise if q is positive, rotate $x_2(-m)$ by q times in clockwise if q is negative to obtain $x_2((q-m))_N$.

4. Multiplication : Multiply $x_1(m)$ by $x_2((q-m))_N$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2((q-m))_N$.

5. Summation : Sum up the samples of one period of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at $n = q$. [i.e., $x_3(q)$].

The above procedure will give the value of $x_3(n)$ at a single time instant say $n = q$. In general we are interested in evaluating the values of the sequence $x_3(n)$ in the range $0 < n < N-1$ Hence the steps 3 , 4 and 5 given above must be repeated, for all possible time shifts in the range $0 < n < N-1$

2.10.5 Linear Convolution via Circular Convolution

When two numbers of N-point sequences are circularly convolved, it produces another N-point sequence. For circular convolution, one of the sequence should be periodically extended. Also the resultant sequence is periodic with period N.

The linear convolution of two sequences of length N_1 and N_2 produces an output sequence of length $N_1 + N_2 - 1$. To perform linear convolution via circular convolution both the sequences should be converted to $N_1 + N_2 - 1$ point sequences by padding with zeros. Then perform circular convolution of $N_1 + N_2 - 1$ point sequences. The resultant sequence will be same as that of linear convolution of N_1 and N_2 point sequences.

2.10.6 Methods of Computing Circular Convolution

Method 1 : Graphical Method

In graphical method, the given sequences are converted to same size and represented on circles. In case of periodic sequences, the samples of one period are represented on circles. One of the sequence is folded and shifted circularly. Let $x_1(n)$ and $x_2(n)$ be the given sequences. Let $x_3(n)$ be the sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to get a sample of $x_3(n)$ at $n = q$.

1. Change the index n in the sequences $x_1(n)$ and $x_2(n)$ to get $x_1(m)$ and $x_2(m)$ and then represent the sequences on circles.
2. Fold one of the sequence. Let us fold $x_2(m)$ to get $x_2(-m)$.
3. Rotate (or shift) the sequence $x_2(-m)$, q times to get the sequence $x_2((q-m))_N$. If q is positive then rotate (or shift) the sequence in anticlockwise direction and if q is negative then rotate (or shift) the sequence in clockwise direction.
4. The sample of $x_3(q)$ at $n = q$ is given by,

$$x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,q}(m)$$

where, $x_{2,q}(m) = x_2((q-m))_N$

Determine the product sequence $x_1(m)\, x_{2,q}(m)$ for one period.

5. The sum of all the samples of the product sequence gives the sample $x_3(q)$ [i.e., $x_3(n)$ at $n = q$].

The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

Method 2 : Tabular Method

Let $x_1(n)$ and $x_2(n)$ be the given N-point sequences. Let $x_3(n)$ be the N-point sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to obtain one sample of $x_3(n)$ at $n = q$.

1. Change the index n in the sequences $x_1(n)$ and $x_2(n)$ to get $x_1(m)$ and $x_2(m)$ and then represent the sequences as two rows of tabular array.
2. Fold one of the sequence. Let us fold $x_2(m)$ to get $x_2(-m)$.
3. Periodically extend $x_2(-m)$. Here the periodicity is N, where N is the length of the given sequences.
4. Shift the sequence $x_2(-m)$, q times to get the sequence $x_2((q-m))_N$. If q is positive then shift the sequence to the right and if q is negative then shift the sequence to the left.

5. The sample of $x_3(q)$ at n = q is given by, $x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,q}(m)$

   where $x_{2,q}(m) = x_2((q-m))_N$

   Determine the product sequence $x_1(m)\, x_{2,q}(m)$ for one period.

6. The sum of the samples of the product sequence gives the sample $x_3(q)$ [i.e., $x_3(n)$ at n = q]. The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

Method 3: Matrix Method

Let $x_1(n)$ and $x_2(n)$ be the given N-point sequences. The circular convolution of $x_1(n)$ and $x_2(n)$ yields another N-point sequence $x_3(n)$.

In this method an (N ´ N) matrix is formed using one of the sequence as shown below. Another sequence is arranged as a column vector (column matrix) of order (N ´ 1). The product of the two matrices gives the resultant sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \ldots\ldots & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & \ldots\ldots & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & \ldots\ldots & x_2(4) & x_2(3) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_2(N-2) & x_2(N-3) & x_2(N-4) & \ldots\ldots & x_2(0) & x_2(N-1) \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \ldots\ldots & x_2(1) & x_2(0) \end{bmatrix} \times \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ \vdots \\ x_1(N-2) \\ x_1(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ \vdots \\ \vdots \\ x_3(N-2) \\ x_3(N-1) \end{bmatrix}$$

Example 2.24

Perform circular convolution of the two sequences, $x_1(n) = \{\underline{2}, 1, 2, -1\}$ and $x_2(n) = \{\underline{1}, 2, 3, 4\}$

Solution

Method 1: Graphical Method of Computing Circular Convolution

Let $x_3(n)$ be the sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$.

The circular convolution of $x_1(n)$ and $x_2(n)$ is given by,

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,n}(m)$$

where $x_{2,n}(m) = x_2((n-m))_N$ and m is the dummy variable used for convolution.

The index n in the given sequences are changed to m and each sequence is represented as points on a circle as shown below. The folded sequence $x_2(-m)$ and circularly shifted sequences $x_2(n-m)$ are also represented on the circle.

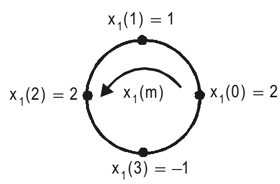


Fig 1.

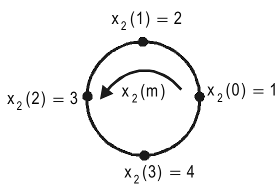


Fig 2.

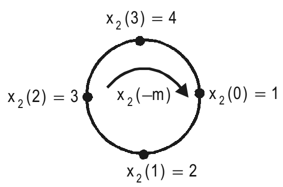


Fig 3.

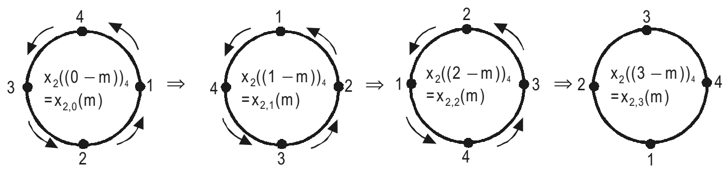


Fig 4: Circularly shifted sequences $x_2(-m)$ for n = 0, 1, 2, 3.

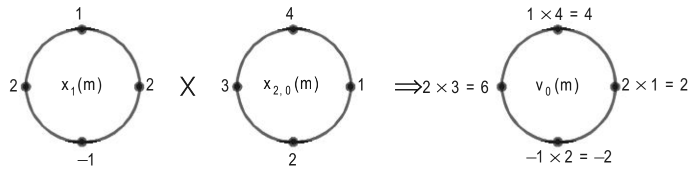
The given sequences are 4-point sequences . ∴ N = 4.

Each sample of $x_3(n)$ is given by sum of the samples of product sequence defined by the equation,

$$x_3(n) = \sum_{m=0}^{3} x_1(m)\ x_{2,n}(m) = \sum_{m=0}^{3} v_n(m) \quad ; \quad \text{where } v_n(m) = x_1(m)\ x_{2,n}(m) \qquad .....(1)$$

Using the above equation (1), graphical method of computing each sample of $x_3(n)$ are shown in fig 5 to fig 8.

When n = 0 ; $x_3(0) = \sum_{m=0}^{3} x_1(m)\ x_2((0-m))_4 = \sum_{m=0}^{3} x_1(m)\ x_{2,0}(m) = \sum_{m=0}^{3} v_0(m)$

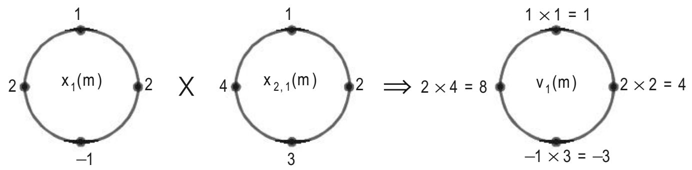


The sum of samples of $v_0(m)$ gives $x_3(0)$

Fig 5: Computation of $x_3(0)$.

$\therefore\ x_3(0) = 2 + 4 + 6 - 2 = 10$

When n = 1 ; $x_3(1) = \sum_{m=0}^{3} x_1(m)\ x_2((1-m))_4 = \sum_{m=0}^{3} x_1(m)\ x_{2,1}(m) = \sum_{m=0}^{3} v_1(m)$

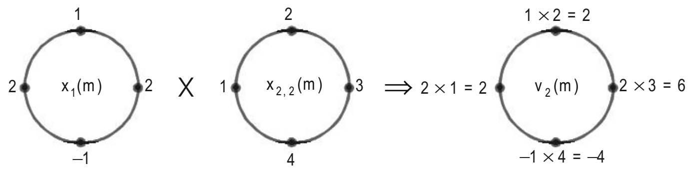


The sum of samples of $v_1(m)$ gives $x_3(1)$

Fig 6: Computation of $x_3(1)$.

$\therefore\ x_3(1) = 4 + 1 + 8 - 3 = 10$

When n = 2 ; $x_3(2) = \sum_{m=0}^{3} x_1(m)\ x_2((2-m))_4 = \sum_{m=0}^{3} x_1(m)\ x_{2,2}(m) = \sum_{m=0}^{3} v_2(m)$



The sum of samples of $v_2(m)$ gives $x_3(2)$

Fig 7: Computation of $x_3(2)$.

$\therefore\ x_3(2) = 6 + 2 + 2 - 4 = 6$

When n = 3 ; $x_3(3) = \sum_{m=0}^{3} x_1(m)\ x_2((3-m))_4 = \sum_{m=0}^{3} x_1(m)\ x_{2,3}(m) = \sum_{m=0}^{3} v_3(m)$

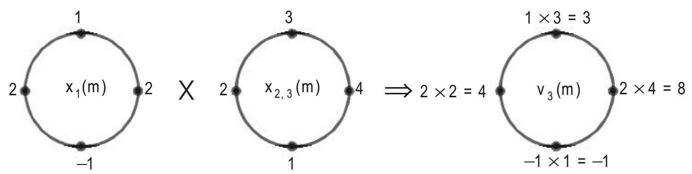


Fig 8: Computation of $x_3(3)$.

The sum of samples of $v_3(m)$ gives $x_3(3)$

$\therefore\ x_3(3) = 8 + 3 + 4 - 1 = 14$

∴ $x_3(n) = \{10, 10, 6, 14\}$

Method 2 : Circular Convolution Using Tabular Array

The index n in the given sequences are changed to m and then, the given sequences can be represented in the tabular array as shown below. Here the shifted sequences $x_{2,n}(m)$ are periodically extended with a periodicity of N = 4. Let $x_3(n)$ be the sequence obtained by convolution of $x_1(n)$ and $x_2(n)$. Each sample of $x_3(n)$ is given by the equation,

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,n}(m), \text{ where } x_{2,n}(m) = x_2((n-m))_N$$

Note : The boldfaced numbers are samples obtained by periodic extension.

m	–3	–2	–1	0	1	2	3
$x_1(m)$				2	1	2	–1
$x_2(m)$				1	2	3	4
$x_2((-m))_4 = x_{2,0}(m)$	4	3	2	1	**4**	**3**	**2**
$x_2((1-m))_4 = x_{2,1}(m)$		4	3	2	1	**4**	**3**
$x_2((2-m))_4 = x_{2,2}(m)$			4	3	2	1	**4**
$x_2((3-m))_4 = x_{2,3}(m)$				4	3	2	1

To determine a sample of $x_3(n)$ at n = q, multiply the sequence, $x_1(m)$ and $x_{2,q}(m)$, to get a product sequence $x_1(m)\, x_{2,q}(m)$. [i.e., multiply the corresponding elements of the row $x_1(m)$ and $x_{2,q}(m)$]. The sum of all the samples of the product sequence gives $x_3(q)$.

$$\text{When } n = 0\ ;\ x_3(0) = \sum_{m=0}^{3} x_1(m)\, x_{2,0}(m)$$
$$= x_1(0)\, x_{2,0}(0) + x_1(1)\, x_{2,0}(1) + x_1(2)\, x_{2,0}(2) + x_1(3)\, x_{2,0}(3)$$
$$= 2 \times 1 + 1 \times 4 + 2 \times 3 + (-1) \times 2 = 2 + 4 + 6 - 2 = 10$$

The samples of $x_3(n)$ for other values of n are calculated as shown for n = 0.

$$\text{When } n = 1;\ x_3(1) = \sum_{m=0}^{3} x_1(m)\, x_{2,1}(m) = 4 + 1 + 8 - 3 = 10$$

$$\text{When } n = 2;\ x_3(2) = \sum_{m=0}^{3} x_1(m)\, x_{2,2}(m) = 6 + 2 + 2 - 4 = 6$$

$$\text{When } n = 3;\ x_3(3) = \sum_{m=0}^{3} x_1(m)\, x_{2,3}(m) = 8 + 3 + 4 - 1 = 14$$

$$\therefore\ x_3(n) = \{\underset{\uparrow}{10},\ 10,\ 6,\ 14\}$$

Method 3 : Circular Convolution Using Matrices

The sequence $x_1(n)$ can be arranged as a column vector of order N ´ 1 and using the samples of $x_2(n)$ the N ´ N matrix is formed as shown below. The product of the two matrices gives the sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1\times 2+4\times 1+3\times 2+2\times -1 \\ 2\times 2+1\times 1+4\times 2+3\times -1 \\ 3\times 2+2\times 1+1\times 2+4\times -1 \\ 4\times 2+3\times 1+2\times 2+1\times -1 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 6 \\ 14 \end{bmatrix}$$

$\therefore\ x_3(n) = \{10, 10, 6, 14\}$

Example 2.25

Perform the circular convolution of the two sequences $x_1(n)$ and $x_2(n)$, where,

$$x_1(n) = \{0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.2,\ 1.4,\ 1.6\}$$

$$x_2(n) = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1.1,\ 1.3,\ 1.5\}$$

Solution

Let $x_3(n)$ be the result of the circular convolution of $x_1(n)$ and $x_2(n)$. The given sequences consists of eight samples. Then $x_3(n)$ will also have 8 samples.

The sequences are represented in the tabular array as shown below after replacing n by m. The sequence $x_2(m)$ is folded and shifted.

The shifted sequences $x_{2,n}(m)$ are periodically extended with a periodicity of N = 8.

Note : The boldfaced numbers are samples obtained by periodic extension

m	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
$x_1(m)$								0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$x_2(m)$								0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5
$x_2((-m))_8 = x_{2,0}(m)$	1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**	**0.7**	**0.5**	**0.3**
$x_2((1-m))_8 = x_{2,1}(m)$		1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**	**0.7**	**0.5**
$x_2((2-m))_8 = x_{2,2}(m)$			1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**	**0.7**
$x_2((3-m))_8 = x_{2,3}(m)$				1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**	**0.9**
$x_2((4-m))_8 = x_{2,4}(m)$					1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**	**1.1**
$x_2((5-m))_8 = x_{2,5}(m)$						1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5	**1.3**
$x_2((6-m))_8 = x_{2,6}(m)$							1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	1.5
$x_2((7-m))_8 = x_{2,7}(m)$								1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1

Each sample of $x_3(n)$ is given by the equation,

$$x_3(n) = \sum_{m=0}^{7} x_1(m)\ x_2((n-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,n}(m)\ ; \quad \text{where } x_{2,n}(m) = x_2((n-m))_8$$

The samples of $x_3(0)$ are calculated as shown below.

$$\text{When } n = 0\ ;\ x_3(n) = \sum_{m=0}^{7} x_1(m)\ x_2((0-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,0}(m)$$

$$= x_1(0)\ x_{2,0}(0) + x_1(1)\ x_{2,0}(1) + x_1(2)\ x_{2,0}(2) + x_1(3)\ x_{2,0}(3)$$

$$+ x_1(4)\ x_{2,0}(4) + x_1(5)\ x_{2,0}(5) + x_1(6)\ x_{2,0}(6) + x_1(7)\ x_{2,0}(7)$$

$$= 0.02 + 0.6 + 0.78 + 0.88 + 0.9 + 0.84 + 0.7 + 0.48 = 5.20$$

The samples of $x_3(n)$ for other values of n are calculated as shown for n = 0.

$$\text{When } n = 1;\ x_3(1) = \sum_{m=0}^{7} x_1(m)\ x_2((1-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,1}(m) = 6.00$$

$$\text{When } n = 2;\ x_3(2) = \sum_{m=0}^{7} x_1(m)\ x_2((2-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,2}(m) = 6.48$$

$$\text{When } n = 3;\ x_3(3) = \sum_{m=0}^{7} x_1(m)\ x_2((3-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,3}(m) = 6.64$$

$$\text{When } n = 4;\ x_3(4) = \sum_{m=0}^{7} x_1(m)\ x_2((4-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,4}(m) = 6.48$$

$$\text{When } n = 5;\ x_3(5) = \sum_{m=0}^{7} x_1(m)\ x_2((5-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,5}(m) = 6.00$$

$$\text{When } n = 6;\ x_3(6) = \sum_{m=0}^{7} x_1(m)\ x_2((6-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,6}(m) = 5.20$$

$$\text{When } n = 7;\ x_3(7) = \sum_{m=0}^{7} x_1(m)\ x_2((7-m))_8 = \sum_{m=0}^{7} x_1(m)\ x_{2,7}(m) = 4.08$$

$$\therefore x_3(n) = \{\underset{\uparrow}{5.20},\ 6.00,\ 6.48,\ 6.64,\ 6.48,\ 6.00,\ 5.20,\ 4.08\}$$

Example 2.26

Find the linear and circular convolution of the sequences, $x(n) = \{\underset{\uparrow}{1},\ 0.5\}$ and $h(n) = \{\underset{\uparrow}{0.5},\ 1\}$.

Solution

Linear Convolution by Tabular Array

$$\text{Let, } y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m)\ h(n-m) \quad ; \text{ where m is a dummy variable for convolution.}$$

Since both x(n) and h(n) starts at n = 0, the output sequence y(n) will also start at n = 0.

Since the length of x(n) and h(n) is 2, the length of y(n) is 2 + 2 – 1 = 3.

Let us change the index n to m in x(n) and h(n). The sequences x(m) and h(m) are represented in the tabular array as shown below.

Note : The unfilled boxes in the table are considered as zeros.

m	–1	0	1	2
x(m)		1	0.5	
h(m)		0.5	1	
$h(-m) = h_0(m)$	1	0.5		
$h(1-m) = h_1(m)$		1	0.5	
$h(2-m) = h_2(m)$			1	0.5

Each sample of y(n) is given by the relation,

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\ h(n-m) = \sum_{m=-\infty}^{\infty} x(m)\ h_n(m) \quad ; \quad \text{where } h_n(m) = h(n-m)$$

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=-\infty}^{\infty} x(m)\ h(-m) = \sum_{m=-1}^{1} x(m)\ h_0(m) = x(-1)\ h_0(-1) + x(0)\ h_0(0) + x(1)\ h_0(1)$$

$$= 0 \times 1 + 1 \times 0.5 + 0.5 \times 0 = 0 + 0.5 + 0 = 0.5$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=-\infty}^{\infty} x(m)\ h(1-m) = \sum_{m=0}^{1} x(m)\ h_1(m) = 1 + 0.25 = 1.25$$

$$\text{When } n = 2\ ;\ y(2) = \sum_{m=-\infty}^{\infty} x(m)\ h(2-m) = \sum_{m=0}^{2} x(m)\ h_2(m) = 0 + 0.5 + 0 = 0.5$$

$$\therefore y(n) = \{\underset{\uparrow}{0.5},\ 1.25,\ 0.5\}$$

Circular Convolution by Tabular Array

Let, $y(n) = x(n) \circledast h(n) = \sum_{m=0}^{N-1} x(m)\ h((n-m))_N$; where m is a dummy variable for convolution.

The index n in the sequences are changed to m and the sequences are represented in the tabular array as shown below. The shifted sequence $h_n(m)$ is periodically extended with periodicity N = 2.

Note : The boldfaced number is the sample obtained by periodic extension.

m	–1	0	1
x(m)		1	0.5
h(m)		0.5	1
$h((-m))_2 = h_0(m)$	1	0.5	**1**
$h((1-m))_2 = h_1(m)$		1	0.5

Each sample of y(n) is given by the equation,

$$y(n) = \sum_{m=0}^{N-1} x(m)\ h((n-m))_N = \sum_{m=0}^{N-1} x(m)\ h_n(m); \ \text{where } h_n(m) = h((n-m))_N$$

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=0}^{N-1} x(m)\ h((0-m))_2 = \sum_{m=0}^{1} x(m)\ h_0(m)$$

$$= x(0)\ h_0(0) + x(1)\ h_0(1) = 1 \times 0.5 + 0.5 \times 1 = 0.5 + 0.5 = 1.0$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=0}^{N-1} x(m)\ h((1-m))_2 = \sum_{m=0}^{1} x(m)\ h_1(m)$$

$$= x(0)\ h_1(0) + x(1)\ h_1(1) = 1 \times 1 + 0.5 \times 0.5 = 1 + 0.25 = 1.25$$

$$\therefore y(n) = \{\underset{\uparrow}{1.0},\ 1.25\}$$

Example 2.27

The input x(n) and impulse response h(n) of a LTI system are given by,

$$x(n) = \{-1, \underset{\uparrow}{1}, 2, -2) \ ;\ h(n) = \{0.5, \underset{\uparrow}{1}, -1, 2, 0.75\}$$

Determine the response of the system **a)** using linear convolution and **b)** using circular convolution.

Solution

a) Response of LTI system using linear convolution

Let y(n) be the response of LTI system. By convolution sum formula,

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) \quad ; \quad \text{where m is a dummy variable used for convolution.}$$

The sequence x(n) starts at n = 0 and h(n) starts at n = –1. Hence y(n) will start at n = 0 + (–1) = –1. The length of x(n) is 4 and the length of h(n) is 5. Hence the length of y(n) is (4 + 5 – 1) = 8. Also y(n) ends at n = 0 + (–1) + (4 + 5 –2) = 6.

Let us change the index n to m in x(n) and h(n). The sequences x(m) and h(m) are represented on the tabular array as shown below. Let us fold h(m) to get h(–m) and shift h(–m) to perform convolution operation.

Note : The unfilled boxes in the table are considered as zeros.

m	–4	–3	–2	–1	0	1	2	3	4	5	6	7
x(m)					–1	1	2	–2				
h(m)				0.5	1	–1	2	0.75				
h(–m)		0.75	2	–1	1	0.5						
$h(-1-m) = h_{-1}(m)$	0.75	2	–1	1	0.5							
$h(0-m) = h_0(m)$		0.75	2	–1	1	0.5						
$h(1-m) = h_1(m)$			0.75	2	–1	1	0.5					
$h(2-m) = h_2(m)$				0.75	2	–1	1	0.5				
$h(3-m) = h_3(m)$					0.75	2	–1	1	0.5			
$h(4-m) = h_4(m)$						0.75	2	–1	1	0.5		
$h(5-m) = h_5(m)$							0.75	2	–1	1	0.5	
$h(6-m) = h_6(m)$								0.75	2	–1	1	0.5

Each sample of y(n) is given by summation of the product sequence, x(m) h(n – m). To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence [i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$]. The sum of all the samples of the product sequence gives y(q).

$$\text{i.e., } y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m)$$

$$\text{When } n = -1 \; ; \; y(-1) = \sum_{m=-4}^{3} x(m)\, h_{-1}(m)$$

$$= x(-4)\, h_{-1}(-4) + x(-3)\, h_{-1}(-3) + x(-2)\, h_{-1}(-2) + x(-1)\, h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2) + x(3)\, h_{-1}(3)$$

$$= 0 + 0 + 0 + 0 + (-0.5) + 0 + 0 + 0 = -0.5$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

$$\text{When } n = 0 \; ; \; y(0) = \sum_{m=-3}^{3} x(m)\, h_0(m) = 0 + 0 + 0 + (-1) + 0.5 + 0 + 0 = -0.5$$

$$\text{When } n = 1 \; ; \; y(1) = \sum_{m=-2}^{3} x(m)\, h_1(m) = 0 + 0 + 1 + 1 + 1 + 0 = 3$$

$$\text{When } n = 2 \; ; \; y(2) = \sum_{m=-1}^{3} x(m)\, h_2(m) = 0 + (-2) + (-1) + 2 + (-1) = -2$$

When n = 3 ; $y(3) = \sum_{m=0}^{4} x(m)\ h_3(m) = -0.75 + 2 + (-2) + (-2) + 0 = -2.75$

When n = 4 ; $y(4) = \sum_{m=0}^{5} x(m)\ h_4(m) = 0 + 0.75 + 4 + 2 + 0 + 0 = 6.75$

When n = 5 ; $y(5) = \sum_{m=0}^{6} x(m)\ h_5(m) = 0 + 0 + 1.5 + (-4) + 0 + 0 + 0 = -2.5$

When n = 6 ; $y(6) = \sum_{m=0}^{7} x(m)\ h_6(m) = 0 + 0 + 0 + (-1.5) + 0 + 0 + 0 + 0 = -1.5$

The response of LTI system y(n) is,

$$y(n) = \{\underset{\uparrow}{-0.5}, -0.5, 3, -2, -2.75, 6.75, -2.5, -1.5\}$$

b) Response of LTI System Using Circular Convolution

The response of LTI system is given by linear convolution of x(n) and h(n). Let y(n) be the response sequence of LTI system. To get the result of linear convolution from circular convolution, both the sequences should be converted to the size of y(n) and perform circular convolution of the converted sequences. Also the converted sequences should start and end at the same value of n as that of y(n).

The length of x(n) is 4 and the length of h(n) is 5. Hence the length of y(n) is (4 + 5 – 1) = 8. Therefore both the sequences should be converted to 8-point sequences.

The x(n) starts at n = 0 and h(n) starts at n = –1. Hence y(n) will start at n = 0 + (–1) = –1. The y(n) will end at n = [0 +(–1)] + (4 + 5 – 2) = 6. Therefore the converted sequences should start at n = –1 and end at n = 6.

$$\therefore\ x(n) = \{\underset{\uparrow}{0}, -1, 1, 2, -2, 0, 0, 0\} \text{ and } h(n) = \{\underset{\uparrow}{0.5}, 1, -1, 2, 0.75, 0, 0, 0\}$$

The converted sequences x(n) and h(n) are represented on the tabular array after replacing the index n by m as shown below. The sequence h(m) is folded and shifted.

The shifted sequences $h_n(m)$ are periodically extended with a periodicity of N = 8.

Note : The boldfaced numbers are samples obtained by periodic extension of the sequences.

m	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
x(m)							0	–1	1	2	–2	0	0	0	
h(m)							0.5	1	–1	2	0.75	0	0	0	
h(–m)		0	0	0	0.75	2	–1	1	0.5						
$h((-1-m))_8 = h_{-1}(m)$	0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**	**2**	**–1**	**1**
$h((0-m))_8 = h_0(m)$		0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**	**2**	**–1**
$h((1-m))_8 = h_1(m)$			0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**	**2**
$h((2-m))_8 = h_2(m)$				0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**
$h((3-m))_8 = h_3(m)$					0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**
$h((4-m))_8 = h_4(m)$						0	0	0	0.75	2	–1	1	0.5	**0**	**0**
$h((5-m))_8 = h_5(m)$							0	0	0	0.75	2	–1	1	0.5	**0**
$h((6-m))_8 = h_6(m)$	**0**	**0**	**0.75**	**2**	**–1**	**1**	**0.5**	0	0	0	0.75	2	–1	1	0.5

Let y(n) be the sequence obtained by circular convolution of x(n) and h(n).

Now, each sample of y(n) is given by,

$$y(n) = \sum_{m=-1}^{6} x(m)\ h((n-m))_8 = \sum_{m=-1}^{6} x(m)\ h_n(m)\ ;\quad \text{where } h_n(m) = h((n-m))_8$$

To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence x(m) $h_q(m)$, [i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$]. The sum of all the samples of the product sequence gives y(q).

$$\text{When } n = -1\ ;\ y(-1) = \sum_{m=-1}^{6} x(m)\, h_{-1}(n) = x(-1)\, h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2)$$

$$+\ x(3)\, h_{-1}(3) + x(4)\, h_{-1}(4) + x(5)\, h_{-1}(5) + x(6)\, h_{-1}(6)$$

$$= 0 + (-0.5) + 0 + 0 + 0 + 0 + 0 + 0 = -0.5$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=-1}^{6} x(m)\, h_0 m = 0 + (-1) + 0.5 + 0 + 0 + 0 + 0 + 0 = -0.5$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=-1}^{6} x(m)\, h_1 m = 0 + 1 + 1 + 1 + 0 + 0 + 0 + 0 = 3$$

$$\text{When } n = 2\ ;\ y(2) = \sum_{m=-1}^{6} x(m)\, h_2 m = 0 + (-2) + (-1) + 2 + (-1) + 0 + 0 + 0 = -2$$

$$\text{When } n = 3\ ;\ y(3) = \sum_{m=-1}^{6} x(m)\, h_3 m = 0 + (-0.75) + 2 + (-2) + (-2) + 0 + 0 + 0 = -2.75$$

$$\text{When } n = 4\ ;\ y(4) = \sum_{m=-1}^{6} x(m)\, h_4 m = 0 + 0 + 0.75 + 4 + 2 + 0 + 0 + 0 = 6.75$$

$$\text{When } n = 5\ ;\ y(5) = \sum_{m=-1}^{6} x(m)\, h_5 m = 0 + 0 + 0 + 1.5 + (-4) + 0 + 0 + 0 = -2.5$$

$$\text{When } n = 6\ ;\ y(6) = \sum_{m=-1}^{6} x(m)\, h_6 m = 0 + 0 + 0 + 0 + (-1.5) + 0 + 0 + 0 = -1.5$$

The response of LTI system y(n) is,

$$y(n) = \{-0.5, \underset{\uparrow}{-0.5}, 3, -2, -2.75, 6.75, -2.5, -1.5\}$$

> *Note : 1. Since circular convolution is periodic, the convolution is performed for any one period.*
> *2. It can be observed that the results of both the methods are same.*

2.11 Sectioned Convolution

The response of an LTI system for any arbitrary input is given by linear convolution of the input and the impulse response of the system. If one of the sequences (either the input sequence or impulse response sequence) is very much larger than the other, then it is very difficult to compute the linear convolution for the following reasons.

1. The entire sequence should be available before convolution can be carried out. This makes long delay in getting the output.

2. Large amounts of memory is required to store the sequences.

The above problems can be overcome in the sectioned convolutions. In this technique the larger sequence is sectioned (or splitted) into the size of smaller sequence. Then the linear convolution of each section of longer sequence and the smaller sequence is performed. The output sequences obtained from the convolutions of all the sections are combined to get the overall output sequence. There are two methods of sectioned convolutions. They are overlap add method and overlap save method.

2.11.1 Overlap Add Method

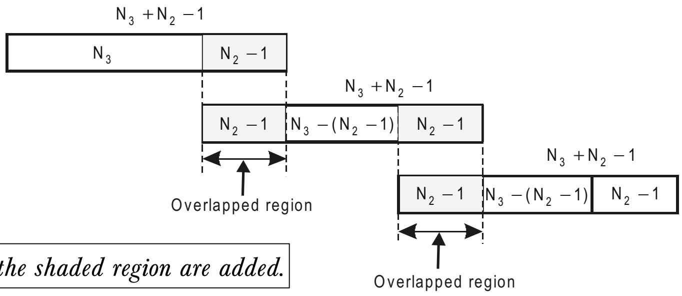
In the ***overlap add method,*** the longer sequence is divided into smaller sequences. Then linear convolution of each section of longer sequence and smaller sequence is performed. The overall output sequence is obtained by combining the output of the sectioned convolution.

Let, N_1 = Length of longer sequence

N_2 = Length of smaller sequence

Let the longer sequence be divided into sections of size N_3 samples.

Note : Normally the longer sequence is divided into sections of size same as that of smaller sequence.



Note : Samples in the shaded region are added.

Fig 2.30: *Overlapping of output sequence of sectioned convolution by overlap add method.*

The linear convolution of each section with smaller sequence will produce an output sequence of size $N_3 + N_2 - 1$ samples. In this method the last $N_2 - 1$ samples of each output sequence overlaps with the first $N_2 - 1$ samples of the next section. [i.e., there will be a region of $N_2 - 1$ samples over which the output sequence of q^{th} convolution overlaps the output sequence of $(q + 1)^{th}$ convolution]. While combining the output sequences of the various sectioned convolutions, the corresponding samples of overlapped regions are added and the samples of non-overlapped regions are retained as such.

2.11.2 Overlap Save Method

In the ***overlap save method,*** the results of linear convolution of the various sections are obtained using circular convolution. In this method, the longer sequence is divided into smaller sequences. Each section of the longer sequence and the smaller sequence are converted to the size of the output sequence of sectioned convolution. The circular convolution of each section of the longer sequence and the smaller sequence is performed. The overall output sequence is obtained by combining the outputs of the sectioned convolution.

Let, N_1 = Length of longer sequence

N_2 = Length of smaller sequence

Let the longer sequence be divided into sections of size N_3 samples.

Note : Normally the longer sequence is divided into sections of size same as that of smaller sequence.

In the ***overlap save method,*** the results of linear convolution are obtained by circular convolution. Hence each section of longer sequence and the smaller sequence are converted to the size of output sequence of size $N_3 + N_2 - 1$ samples.The smaller sequence is converted to size of $N_3 + N_2 - 1$ samples, by appending with zeros.The convertion of each section of longer sequence to the size $N_3 + N_2 - 1$ samples can be performed in two different methods.

Method-1

In this method, the first N_2-1 samples of a section is appended as last N_2-1 samples of the previous section (i.e., the overlapping samples are placed at the beginning of the section). The circular convolution of each section will produce an output sequence of size $N_3 + N_2-1$ samples. In this output the first N_2-1 samples are discarded and the remaining samples of the output of sectioned convolutions are saved as the overall output sequence.

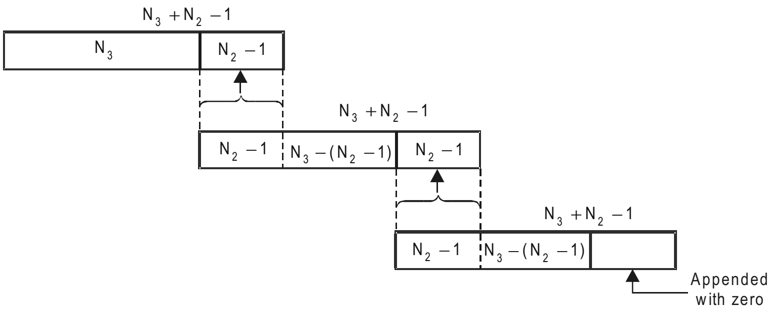
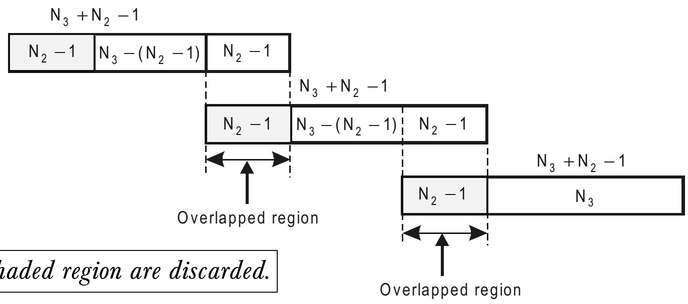


Fig 2.31: Appending of sections of input sequence in method 1 of overlap save method.



Note : Samples in the shaded region are discarded.

Fig 2.32: Overlapping of output sequence of sectioned convolution by method 1 of overlap save method.

Method-2

In this method, the last N_2-1 samples of a section is appended as last N_2-1 samples of the next section (i.e, the overlapping samples are placed at the end of the sections). The circular convolution of each section will produce an output sequence of size N_3+N_2-1 samples. In this output the last N_2-1 samples are discarded and the remaining samples of the output of sectioned convolutions are saved as the overall output sequence.

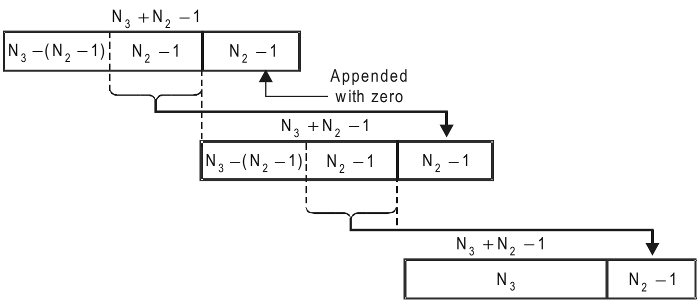


Fig 2.33: Appending of sections of input sequence in method 2 of overlap save method.

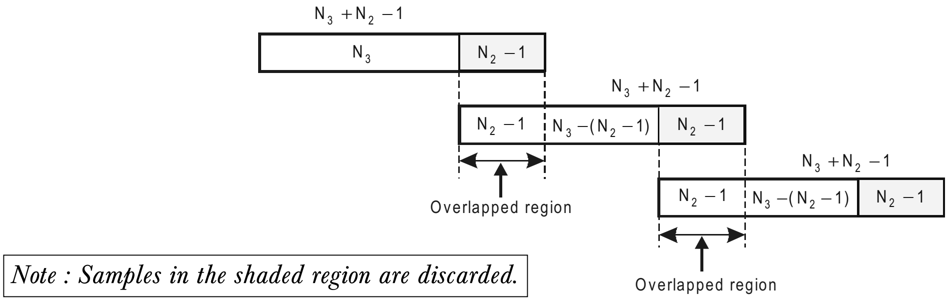


Fig 2.34: Overlapping of output sequence of sectioned convolution by method 2 of overlap save method.

Example 2.28

Perform the linear convolution of the following sequences by **a)** Overlap add method, and **b)** Overlap save method.

$$x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\} \;;\; h(n) = \{-1, 1\}$$

Solution

a) Overlap Add Method

In this method the longer sequence is sectioned into sequences of size equal to smaller sequence. Here x(n) is a longer sequence when compared to h(n). Hence x(n) is sectioned into sequences of size equal to h(n).

Given that, $x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\}$

Let x(n) can be sectioned into four sequences, each consisting of two samples of x(n) as shown below.

$x_1(n) = 1 \;;\; n = 0$	$x_2(n) = 2 \;;\; n = 2$	$x_3(n) = 3 \;;\; n = 4$	$x_4(n) = 4 \;;\; n = 6$
$= -1 \;;\; n = 1$	$= -2 \;;\; n = 3$	$= -3 \;;\; n = 5$	$= -4 \;;\; n = 7$

Let $y_1(n)$, $y_2(n)$, $y_3(n)$ and $y_4(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_4(n)$ with h(n) respectively.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 2$, $\therefore$ $y_2(n)$ will start at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 4$, $\therefore$ $y_3(n)$ will start at $n = n_3 + n_h = 4 + 0 = 4$

$x_4(n)$ starts at $n = n_4 = 6$, $\therefore$ $y_4(n)$ will start at $n = n_4 + n_h = 6 + 0 = 6$

Here linear convolution of each section is performed between two sequences each consisting of 2 samples. Hence each convolution output will consists of 2 + 2 – 1 = 3 samples. The convolution of each section is performed by tabular method as shown below.

> *Note :*
>
> *1. Here $N_1 = 8$, $N_2 = 2$, $N_3 = 2$. $\therefore$ $(N_2 - 1) = 2 - 1 = 1$ and $(N_2 + N_3 - 1) = 2 + 2 - 1 = 3$*
>
> *2. The unfilled boxes in the tables are considered as zero.*
>
> *3. For convenience of convolution operation the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$ and $h(n)$.*

Convolution of Section 1

m	–1	0	1	2
$x_1(m)$		1	–1	
h(m)		–1	1	
$h(-m) = h_o(m)$	1	–1		
$h(1-m) = h_1(m)$		1	–1	
$h(2-m) = h_2(m)$			1	–1

$$y_1(n) = x_1(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, h(n-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, h_n(m)\ ;\quad n = 0, 1, 2$$

where $h_n(m) = h(n-m)$

When $n = 0$; $y_1(0) = \sum x_1(m)\, h_0(m) = 0 - 1 + 0 = -1$

When $n = 1$; $y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 = 2$

When $n = 2$; $y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 + 0 = -1$

Convolution of Section 2

m	–1	0	1	2	3	4
$x_2(m)$				2	–2	
h(m)		–1	1			
h(–m)	1	–1				
$h(2-m) = h_2(m)$			1	–1		
$h(3-m) = h_3(m)$				1	–1	
$h(4-m) = h_4(m)$					1	–1

$$y_2(n) = x_2(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, h(n-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_2(m)\, h_n(m)\ ;\quad n = 2, 3, 4$$

where $h_n(m) = h(n-m)$

When $n = 2$; $y_2(2) = \sum x_2(m)\, h_2(m) = 0 - 2 + 0 = -2$

When $n = 3$; $y_2(3) = \sum x_2(m)\, h_3(m) = 2 + 2 = 4$

When $n = 4$; $y_2(4) = \sum x_2(m)\, h_4(m) = 0 - 2 + 0 = -2$

Convolution of Section 3

m	–1	0	1	2	3	4	5	6
$x_3(m)$						3	–3	
h(m)		–1	1					
h(–m)	1	–1						
$h(4-m) = h_4(m)$					1	–1		
$h(5-m) = h_5(m)$						1	–1	
$h(6-m) = h_6(m)$							1	–1

$$y_3(n) = x_3(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_3(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x_3(m)\, h_n(m)\ ;\quad n = 4, 5, 6$$

where $h_n(m) = h(n-m)$

When $n = 4$; $y_3(4) = \sum x_3(m)\, h_4(m) = 0 - 3 + 0 = -3$

When $n = 5$; $y_3(5) = \sum x_3(m)\, h_5(m) = 3 + 3 = 6$

When $n = 6$; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 - 3 + 0 = -3$

Convolution of Section 4

m	–1	0	1	2	3	4	5	6	7	8
$x_4(m)$								4	–4	
h(m)		–1	1							
h(–m)	1	–1								
$h(6-m) = h_6(m)$							1	–1		
$h(7-m) = h_7(m)$								1	–1	
$h(8-m) = h_8(m)$									1	–1

$$y_4(n) = x_4(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_4(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x_4(m)\, h_n(m) \quad ; \; n = 6,\ 7,\ 8$$

where $h_n(m) = h(n-m)$

When $n = 6$; $y_4(6) = \sum x_4(m)\, h_6(m) = 0 - 4 + 0 = -4$

When $n = 7$; $y_4(7) = \sum x_4(m)\, h_7(m) = 4 + 4 = 8$

When $n = 8$; $y_4(8) = \sum x_4(m)\, h_8(m) = 0 - 4 + 0 = -4$

To Combine the Output of Convolution of Each Section

It can be observed that the last sample in an output sequence overlaps with the first sample of next output sequence. In this method the overall output is obtained by combining the outputs of the convolution of all sections. The overlapped portions (or samples) are added while combining the output. The output of all sections can be represented in a table as shown below. Then the samples corresponding to same value of n are added to get the overall output.

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	–1	2	–1						
$y_2(n)$			–2	4	–2				
$y_3(n)$					–3	6	–3		
$y_4(n)$							–4	8	–4
$y(n)$	–1	2	–3	4	–5	6	–7	8	–4

$\therefore$ $y(n) = x(n) * h(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, -4\}$

b) Overlap Save Method

In this method, the longer sequence is sectioned into sequences of size equal to smaller sequence. The number of samples that will be obtained in the output of linear convolution of each section is determined. Then each section of longer sequence is converted to the size of output sequence using the samples of original longer sequence. The smaller sequence is also converted to the size of output sequence by appending with zeros. Then the circular convolution of each section is performed.

Here x(n) is a longer sequence when compared to h(n). Hence x(n) is sectioned into sequences of size equal to h(n). Given that, $x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\}$

Let x(n) be sectioned into four sequences, each consisting of two samples of x(n) as shown below.

$x_1(n) = 1$; $n = 0$	$x_2(n) = 2$; $n = 2$	$x_3(n) = 3$; $n = 4$	$x_4(n) = 4$; $n = 6$
$= -1$; $n = 1$	$= -2$; $n = 3$	$= -3$; $n = 5$	$= -4$; $n = 7$

Let $y_1(n)$, $y_2(n)$, $y_3(n)$ and $y_4(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_4(n)$ with h(n) respectively. Here linear convolution of each section will result in an output sequence consisting of $2 + 2 - 1 = 3$ samples.

The sequence h(n) is converted to 3-sample sequence by appending with zero. $\therefore$ $h(n) = \{-1, 1, 0\}$

Method - 1

In method 1, the overlapping samples are placed at the beginning of the sections. Each section of longer sequence is converted to 3-sample sequences, using the samples of original longer sequence as shown below. It can be observed that the first sample of $x_2(n)$ is placed as overlapping sample at the end of $x_1(n)$. The first sample of $x_3(n)$ is placed as overlapping sample at the end of $x_2(n)$. The first sample of $x_4(n)$ is placed as overlapping sample at the end of $x_3(n)$. Since there is no fifth section, the overlapping sample of $x_4(n)$ is taken as zero.

$x_1(n) = 1$; $n = 0$	$x_2(n) = 2$; $n = 2$	$x_3(n) = 3$; $n = 4$	$x_4(n) = 4$; $n = 6$
$= -1$; $n = 1$	$= -2$; $n = 3$	$= -3$; $n = 5$	$= -4$; $n = 7$
$= 2$; $n = 2$	$= 3$; $n = 4$	$= 4$; $n = 6$	$= 0$; $n = 8$

Now perform circular convolution of each section with h(n). The output sequence obtained from circular convolution will have three samples. The circular convolution of each section is performed by tabular method as shown below.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 2$, $\therefore$ $y_2(n)$ will start at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 4$, $\therefore$ $y_3(n)$ will start at $n = n_3 + n_h = 4 + 0 = 4$

$x_4(n)$ starts at $n = n_4 = 6$, $\therefore$ $y_4(n)$ will start at $n = n_4 + n_h = 6 + 0 = 6$

Note : 1. Here $N_1 = 8$, $N_2 = 2$, $N_3 = 2$. $\therefore$ $(N_2 - 1) = 2 - 1 = 1$ and $(N_2 + N_3 - 1) = 2 + 2 - 1 = 3$

2. The boldfaced numbers in the tables are obtained by periodic extension.

3. For convenience of convolution operation, the index n in $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$ and $h(n)$ are replaced by m.

Convolution of Section 1

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_1(m)\, h((n-m))_N$$

$$= \sum_{m=0}^{2} x_1(m)\, h_n(m)\ ; \quad n = 0,\ 1,\ 2,$$

where $h_n(m) = h((n-m))_N$

m	–2	–1	0	1	2
$x_1(m)$			1	–1	2
h(m)			–1	1	0
$h((-m))_3 = h_0(m)$	0	1	–1	**0**	**1**
$h((1-m))_3 = h_1(m)$		0	1	–1	**0**
$h((2-m))_3 = h_2(m)$			0	1	–1

When n = 0 ; $y_1(0) = \sum x_1(m)\, h_0(m) = -1 + 0 + 2 = 1$

When n = 1 ; $y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 + 0 = 2$

When n = 2 ; $y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 - 2 = -3$

Convolution of Section 2

m	–2	–1	0	1	2	3	4
$x_2(m)$					2	–2	3
h(m)			–1	1	0		
h(–m)	0	1	–1				
$h((2-m))_3 = h_2(m)$			0	1	–1	**0**	**1**
$h((3-m))_3 = h_3(m)$				0	1	–1	**0**
$h((4-m))_3 = h_4(m)$					0	1	–1

$$y_2(n) = x_2(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_2(m)\, h((n-m))_N = \sum_{m=2}^{4} x_2(m)\, h_n(m)\ ; \quad n = 2,\ 3,\ 4$$

where $h_n(m) = h((n-m))_N$

When n = 2 ; $y_2(2) = \sum x_2(m)\, h_2(m) = -2 + 0 + 3 = 1$

When n = 3 ; $y_2(3) = \sum x_2(m)\, h_3(m) = 2 + 2 + 0 = 4$

When n = 4 ; $y_2(4) = \sum x_2(m)\, h_4(m) = 0 + -2 - 3 = -5$

Convolution of Section 3

$$y_3(n) = x_3(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_3(m)\, h((n-m))_N = \sum_{m=4}^{6} x_3(m)\, h_n(m)\ ; \quad n = 4,\ 5,\ 6$$

where $h_n(m) = h((n-m))_N$

m	–2	–1	0	1	2	3	4	5	6
$x_3(m)$							3	–3	4
h(m)			–1	1	0				
h(–m)	0	1	–1						
$h((4-m))_3 = h_4(m)$					0	1	–1	**0**	**1**
$h((5-m))_3 = h_5(m)$						0	1	–1	**0**
$h((6-m))_3 = h_6(m)$							0	1	–1

When n = 4 ; $y_3(4) = \sum x_3(m)\ h_4(m) = -3 + 0 + 4 = 1$

When n = 5 ; $y_3(5) = \sum x_3(m)\ h_5(m) = 3 + 3 + 0 = 6$

When n = 6 ; $y_3(6) = \sum x_3(m)\ h_6(m) = 0 - 3 - 4 = -7$

Convolution of section 4

m	–2	–1	0	1	2	3	4	5	6	7	8
$x_4(m)$									4	–4	0
h(m)			–1	1	0						
h(–m)	0	1	–1								
$h((6-m))_3 = h_6(m)$							0	1	–1	**0**	**1**
$h((7-m))_3 = h_7(m)$								0	1	–1	**0**
$h((8-m))_3 = h_8(m)$									0	1	–1

$$y_4(n) = x_4(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_4(m)\ h((n-m))_N = \sum_{m=6}^{8} x_4(m)\ h_n(m)\ ;\ n = 6,\ 7,\ 8$$

where $h_n(m) = h((n-m))_N$

When n = 6 ; $y_4(6) = \sum x_4(m)\ h_6(m) = -4 + 0 + 0 = -4$

When n = 7 ; $y_4(7) = \sum x_4(m)\ h_7(m) = 4 + 4 + 0 = 8$

When n = 8 ; $y_4(8) = \sum x_4(m)\ h_8(m) = 0 - 4 + 0 = -4$

To Combine the Output of the Convolution of Each Section

It can be observed that the last sample in an output sequence overlaps with the first sample of next output sequence. In overlap save method the overall output is obtained by combining the outputs of the convolution of all sections. While combining the outputs, the overlapped first sample of every output sequence is discarded and the remaining samples are simply saved as samples of y(n) as shown in the following table.

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	$\cancel{1}$	2	–3						
$y_2(n)$			$\cancel{1}$	4	–5				
$y_3(n)$					$\cancel{1}$	6	–7		
$y_4(n)$							$\cancel{-4}$	8	–4
y(n)	*	2	–3	4	–5	6	–7	8	–4

$y(n) = x(n) * h(n) = \{*, 2, -3, -4, -5, 6, -7, 8, -4\}$

Note : Here y(n) is linear convolution of x(n) and h(n). It can be observed that the results of both the methods are same, except the first $N_2 - 1$ samples.

Method 2

In method 2, the overlapping samples are placed at the end of the section.Each section of longer sequence is converted to 3-sample sequence, using the samples of original longer sequence as shown below. It can be observed that the last sample of $x_1(n)$ is placed as overlapping sample at the end of $x_2(n)$. The last sample of $x_2(n)$ is placed as overlapping sample at the end of $x_3(n)$. The last sample of $x_3(n)$ is placed as overlapping sample at the end of $x_4(n)$. Since there is no previous section for $x_1(n)$, the overlapping sample of $x_1(n)$ is taken as zero.

$x_1(n) = 1\ ;\ n = 0$	$x_2(n) = 2\ ;\ n = 2$	$x_3(n) = 3\ ;\ n = 4$	$x_4(n) = 4\ ;\ n = 6$
$= -1\ ;\ n = 1$	$= -2\ ;\ n = 3$	$= -3\ ;\ n = 5$	$= -4\ ;\ n = 7$
$= 0\ ;\ n = 2$	$= -1\ ;\ n = 4$	$= -2\ ;\ n = 6$	$= -3\ ;\ n = 8$

Now perform circular convolution of each section with h(n). The output sequence obtained from circular convolution will have three samples. The circular convolution of each section is performed by tabular method as shown below.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 2$, $\therefore$ $y_2(n)$ will start at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 4$, $\therefore$ $y_3(n)$ will start at $n = n_3 + n_h = 4 + 0 = 4$

$x_4(n)$ starts at $n = n_4 = 6$, $\therefore$ $y_4(n)$ will start at $n = n_4 + n_h = 6 + 0 = 6$

> *Note :* 1. Here $N_1 = 8$, $N_2 = 2$, $N_3 = 2$. $\therefore$ $(N_2 - 1) = 2 - 1 = 1$ and $(N_2 + N_3 - 1) = 2 + 2 - 1 = 3$
> 2. *The boldfaced numbers in the tables are obtained by periodic extension.*
> 3. *For convenience of convolution the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$ and $h(n)$.*

Convolution of Section 1

m	–2	–1	0	1	2
$x_1(m)$			1	–1	0
$h(m)$			–1	1	0
$h((-m))_3 = h_0(m)$	0	1	–1	**0**	**1**
$h((1-m))_3 = h_1(m)$		0	1	–1	**0**
$h((2-m))_3 = h_2(m)$			0	1	–1

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_1(m)\, h((n-m))_N$$

$$= \sum_{m=0}^{2} x_1(m)\, h_n(m)\ ; \quad n = 0,\ 1,\ 2$$

where $h_n(m) = h((n-m))_N$

When $n = 0$; $y_1(0) = \sum x_1(m)\, h_0(m) = -1 + 0 + 0 = -1$

When $n = 1$; $y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 + 0 = 2$

When $n = 2$; $y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 + 0 = -1$

Convolution of Section 2

m	–2	–1	0	1	2	3	4
$x_2(m)$					2	–2	–1
$h(m)$			–1	1	0		
$h(-m)$	0	1	–1				
$h((2-m))_3 = h_2(m)$			0	1	–1	**0**	**1**
$h((3-m))_3 = h_3(m)$				0	1	–1	**0**
$h((4-m))_3 = h_4(m)$					0	1	–1

$$y_2(n) = x_2(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_2(m)\, h((n-m))_N = \sum_{m=2}^{4} x_2(m)\, h_n(m); \; n = 2,\ 3,\ 4,$$

where $h_n(m) \;=\; h((n-m))_N$

When $n = 2$; $y_2(2) = \sum\ x_2(m)\, h_2(m) = -2 + 0 - 1 = -3$

When $n = 3$; $y_2(3) = \sum\ x_2(m)\, h_3(m) = \ 2 + 2 + 0 = \ 4$

When $n = 4$; $y_2(4) = \sum\ x_2(m)\, h_4(m) = \ 0 - 2 + 1 \ = -1$

Convolution of Section 3

m	–2	–1	0	1	2	3	4	5	6
$x_3(m)$							3	–3	–2
h(m)			–1	1	0				
h(–m)	0	1	–1						
$h((4-m))_3 = h_4(m)$					0	1	–1	**0**	**1**
$h((5-m))_3 = h_5(m)$						0	1	–1	**0**
$h((6-m))_3 = h_6(m)$							0	1	–1

$$y_3(n) = x_3(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_3(m)\, h((n-m))_N = \sum_{m=4}^{6} x_3(m)\, h_n(m)\ ; \quad n = 4,\ 5,\ 6$$

where $h_n(m) = h((n-m))_N$

When $n = 4$; $y_3(4) = \sum\ x_3(m)\, h_4(m) = -3 + 0 - 2 \ = -5$

When $n = 5$; $y_3(5) = \sum\ x_3(m)\, h_5(m) = \ 3 + 3 + 0 \ = \ 6$

When $n = 6$; $y_3(6) = \sum\ x_3(m)\, h_6(m) = \ 0 - 3 + 2 \ = -1$

Convolution of Section 4

m	–2	–1	0	1	2	3	4	5	6	7	8
$x_4(m)$									4	–4	–3
h(m)			–1	1	0						
h(–m)	0	1	–1								
$h((6-m))_3 = h_6(m)$							0	1	–1	**0**	**1**
$h((7-m))_3 = h_7(m)$								0	1	–1	**0**
$h((8-m))_3 = h_8(m)$									0	1	–1

$$y_4(n) = x_4(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_4(m)\, h((n-m))_N = \sum_{m=6}^{8} x_4(m)\, h_n(m)\ ; \; n = 6,\ 7,\ 8$$

where $h_n(m) \;=\; h((n-m))_N$

When $n = 6$; $y_4(6) = \sum\ x_4(m)\, h_6(m) = -4 + 0 - 3 = -7$

When $n = 7$; $y_4(7) = \sum\ x_4(m)\, h_7(m) = \ 4 + 4 + 0 = \ 8$

When $n = 8$; $y_4(8) = \sum\ x_4(m)\, h_8(m) = \ 0 - 4 + 3 = -1$

To Combine the Output of the Convolution of Each Section

It can be observed that the last sample in an output sequence overlaps with the first sample of next output sequence. In overlap save method the overall output is obtained by combining the outputs of the convolution of all sections. While combining the outputs, the overlapped last sample of every output sequence is discarded and the remaining samples are simply saved as samples of y(n) as shown in the following table.

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	–1	2	~~–1~~						
$y_2(n)$			–3	4	~~–1~~				
$y_3(n)$					–5	6	~~–1~~		
$y_4(n)$							–7	8	~~–1~~
y(n)	–1	2	–3	4	–5	6	–7	8	*

Note :

Here y(n) is linear convolution of x(n) and h(n). It can be observed that the results of both the methods are same except the last N_2– 1 samples.

\ y(n) = x(n) * h(n) = {–1, 2, –3, 4, –5, 6, –7, 8, *}

Example 2.29

Perform the linear convolution of the following sequences by **a)** Overlap add method and **b)** Overlap save method.

x(n) = {1, 2, 3, –1, –2, –3, 4, 5, 6} and h(n) = {2, 1, –1}

Solution

a) Overlap Add Method

In this method the longer sequence is sectioned into sequences of size equal to smaller sequence. Here x(n) is a longer sequence when compared to h(n). Hence x(n) is sectioned into sequences of size equal to h(n).

Given that x(n) = {1, 2, 3, –1, –2, –3, 4, 5, 6}. Let x(n) can be sectioned into three sequences, each consisting of three samples of x(n) as shown below.

$x_1(n) = 1\ ;\ n = 0$	$x_2(n) = -1\ ;\ n = 3$	$x_3(n) = 4\ ;\ n = 6$
$= 2\ ;\ n = 1$	$= -2\ ;\ n = 4$	$= 5\ ;\ n = 7$
$= 3\ ;\ n = 2$	$= -3\ ;\ n = 5$	$= 6\ ;\ n = 8$

Let $y_1(n)$, $y_2(n)$ and $y_3(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$ and $x_3(n)$ with h(n) respectively.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, \ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 3$, \ $y_2(n)$ will start at $n = n_2 + n_h = 3 + 0 = 3$

$x_3(n)$ starts at $n = n_3 = 6$, \ $y_3(n)$ will start at $n = n_3 + n_h = 6 + 0 = 6$

Here linear convolution of each section is performed between two sequences each consisting of three samples. Hence each convolution output will consists of 3 + 3 – 1 = 5 samples. The convolution of each section is performed by tabular method as shown below.

Note : 1. Here $N_1 = 9$, $N_2 = 3$, $N_3 = 3$, \ $(N_2 - 1) = 3 - 1 = 2$ and $(N_2 + N_3 - 1) = 3 + 3 - 1 = 5$.

2. The unfilled boxes in the table are considered as zero.

3. For convenience of convolution operation, the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$ and h(n).

Convolution of Section 1

m	–2	–1	0	1	2	3	4
$x_1(m)$			1	2	3		
h(m)			2	1	–1		
$h(-m) = h_0(m)$	–1	1	2				
$h(1-m) = h_1(m)$		–1	1	2			
$h(2-m) = h_2(m)$			–1	1	2		
$h(3-m) = h_3(m)$				–1	1	2	
$h(4-m) = h_4(m)$					–1	1	2

$$y_1(n) = x_1(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\ h(n-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\ h_n(m)$$

for $n = 0,\ 1,\ 2,\ 3,\ 4$

where $h_n(m) = h(n-m)$

When $n = 0$; $y_1(0) = \sum x_1(m)\ h_o(m) = 0 + 0 + 2 + 0 + 0 = 2$

When $n = 1$; $y_1(1) = \sum x_1(m)\ h_1(m) = 0 + 1 + 4 + 0 = 5$

When $n = 2$; $y_1(2) = \sum x_1(m)\ h_2(m) = -1 + 2 + 6 = 7$

When $n = 3$; $y_1(3) = \sum x_1(m)\ h_3(m) = 0 - 2 + 3 + 0 = 1$

When $n = 4$; $y_1(4) = \sum x_1(m)\ h_4(m) = 0 + 0 - 3 + 0 + 0 = -3$

Convolution of Section 2

m	–2	–1	0	1	2	3	4	5	6	7
$x_2(m)$						–1	–2	–3		
h(m)			2	1	–1					
$h(-m) = h_0(m)$	–1	1	2							
$h(3-m) = h_3(m)$				–1	1	2				
$h(4-m) = h_4(m)$					–1	1	2			
$h(5-m) = h_5(m)$						–1	1	2		
$h(6-m) = h_6(m)$							–1	1	2	
$h(7-m) = h_7(m)$								–1	1	2

$$y_2(n) = x_2(n) * h(n) = \sum_{m=-\infty}^{\infty} x_2(m)\ h(n-m) = \sum_{m=-\infty}^{\infty} x_2(m)\ h_n(m);\ n = 3,\ 4,\ 5,\ 6,\ 7$$

where $h_n(m) = h(n-m)$

When $n = 3$; $y_2(3) = \sum x_2(m)\ h_3(m) = 0 + 0 - 2 + 0 + 0 = -2$

When $n = 4$; $y_2(4) = \sum x_2(m)\ h_4(m) = 0 - 1 - 4 + 0 = -5$

When $n = 5$; $y_2(5) = \sum x_2(m)\ h_5(m) = 1 - 2 - 6 = -7$

When $n = 6$; $y_2(6) = \sum x_2(m)\ h_6(m) = 0 + 2 - 3 + 0 = -1$

When $n = 7$; $y_2(7) = \sum x_2(m)\ h_7(m) = 0 + 0 + 3 + 0 + 0 = 3$

Convolution of Section 3

m	–2	–1	0	1	2	3	4	5	6	7	8	9	10
$x_3(m)$									4	5	6		
$h(m)$			2	1	–1								
$h(-m) = h_0(m)$	–1	1	2										
$h(6-m) = h_6(m)$							–1	1	2				
$h(7-m) = h_7(m)$								–1	1	2			
$h(8-m) = h_8(m)$									–1	1	2		
$h(9-m) = h_9(m)$										–1	1	2	
$h(10-m) = h_{10}(m)$											–1	1	2

$$y_3(n) = x_3(n) * h(n) = \sum_{m=-\infty}^{\infty} x_3(m)\, h(n-m) = \sum_{m=-\infty}^{\infty} x_3(m)\, h_n(m)\ ; \quad n = 6, 7, 8, 9, 10$$

where $h_n(m) = h(n-m)$

When $n = 6$; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 + 0 + 8 + 0 + 0 = 8$

When $n = 7$; $y_3(7) = \sum x_3(m)\, h_7(m) = 0 + 4 + 10 + 0 = 14$

When $n = 8$; $y_3(8) = \sum x_3(m)\, h_8(m) = -4 + 5 + 12 = 13$

When $n = 9$; $y_3(9) = \sum x_3(m)\, h_9(m) = 0 - 5 + 6 + 0 = 1$

When $n = 10$; $y_3(10) = \sum x_3(m)\, h_{10}(m) = 0 + 0 - 6 + 0 + 0 = -6$

To Combine the Output of the Convolution of Each Section

It can be observed that the last $N_2 - 1$ sample in an output sequence overlaps with the first $N_2 - 1$ sample of next output sequence. In this method, the overall output is obtained by combining the outputs of the convolution of all sections. The overlapped portions (or samples) are added while combining the output.

The output of all sections can be represented in a table as shown below. Then the samples corresponding to same value of n are added to get the overall output.

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	2	5	7	1	–3						
$y_2(n)$				–2	–5	–7	–1	3			
$y_3(n)$							8	14	13	1	– 6
$y(n)$	2	5	7	–1	–8	–7	7	17	13	1	– 6

$\therefore$ $y(n) = x(n) * h(n) = \{2, 5, 7, -1, -8, -7, 7, 17, 13, 1, -6\}$

b) Overlap Save Method

In this method the longer sequence is sectioned into sequences of size equal to smaller sequence. The number of samples that will be obtained in the output of linear convolution of each section is determined. Then each section of longer sequence is converted to the size of output sequence using the samples of original longer sequences. The smaller sequence is also converted to the size of output sequence by appending with zeros. Then the circular convolution of each section is performed.

Here x(n) is a longer sequence when compared to h(n). Hence x(n) is sectioned into sequences of size equal to h(n). Given that $x(n) = \{1, 2, 3, -1, -2, -3, 4, 5, 6\}$.

Let x(n) be sectioned into three sequences each consisting of three samples as shown below.

Let, N_1 = Length of longer sequence

N_2 = Length of smaller sequence

$N_3 = N_2$ = Length of each section of longer sequence.

$x_1(n) = 1 \; ; n = 0$	$x_2(n) = -1 \; ; n = 3$	$x_3(n) = 4 \; ; n = 6$
$= 2 \; ; n = 1$	$= -2 \; ; n = 4$	$= 5 \; ; n = 7$
$= 3 \; ; n = 2$	$= -3 \; ; n = 5$	$= 6 \; ; n = 8$

Let $y_1(n)$, $y_2(n)$ and $y_3(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$ and $x_3(n)$ with $h(n)$ respectively. Here linear convolution of each section will result in an output sequence consisting of $3 + 3 - 1 = 5$ samples.

Hence each section of longer sequence is converted to five sample sequence, using the samples of original longer sequence as shown below. It can be observed that the first $N_2 - 1$ samples of $x_2(n)$ is placed as overlapping sample at the end of $x_1(n)$. The first $N_2 - 1$ samples of $x_3(n)$ is placed as overlapping sample at the end of $x_2(n)$. Since there is no fourth section, the overlapping samples of $x_3(n)$ are considered as zeros.

$x_1(n) = \;\; 1 \; ; n = 0$	$x_2(n) = -1 \; ; n = 3$	$x_3(n) = 4 \; ; n = 6$
$= \;\; 2 \; ; n = 1$	$= -2 \; ; n = 4$	$= 5 \; ; n = 7$
$= \;\; 3 \; ; n = 2$	$= -3 \; ; n = 5$	$= 6 \; ; n = 8$
$= -1 \; ; n = 3$	$= \;\; 4 \; ; \; n = 6$	$= 0 \; ; n = 9$
$= -2 \; ; n = 4$	$= \;\; 5 \; ; \; n = 7$	$= 0 \; ; n = 10$

The sequence $h(n)$ is also converted to five sample sequence by appending with zeros.

\ $h(n) = \{2, 1, -1, 0, 0\}$

Now perform circular convolution of each section with $h(n)$. The output sequence obtained from circular convolution will have five samples. The circular convolution of each section is performed by tabular method as shown below.

Here $h(n)$ starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, \ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 3$, \ $y_2(n)$ will start at $n = n_2 + n_h = 3 + 0 = 3$

$x_3(n)$ starts at $n = n_3 = 6$, \ $y_3(n)$ will start at $n = n_3 + n_h = 6 + 0 = 6$

> *Note : 1. Here $N_1 = 9, N_2 = 3, N_3 = 3$ \ $(N_2 - 1) = 3 - 1 = 2$ and $[N_2 + N_3 - 1] = 3 + 3 - 1 = 5$ samples.*
>
> *2. The boldfaced numbers in the table are obtained by periodic extension.*
>
> *3. For convenience of convolution operation the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$ and $h(n)$.*

Convolution of Section 1

m	–4	–3	–2	–1	0	1	2	3	4
$x_1(m)$					1	2	3	–1	–2
$h(m)$					2	1	–1	0	0
$h((-m))_5 = h_0(m)$	0	0	–1	1	2	**0**	**0**	**–1**	**1**
$h((1-m))_5 = h_1(m)$		0	0	–1	1	2	**0**	**0**	**–1**
$h((2-m))_5 = h_2(m)$			0	0	–1	1	2	**0**	**0**
$h((3-m))_5 = h_3(m)$				0	0	–1	1	2	**0**
$h((4-m))_5 = h_4(m)$					0	0	–1	1	2

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m = m_i}^{m_f} x_1(m)\, h((n-m))_N = \sum_{m=0}^{4} x_1(m)\, h_n(m); \;\; n = 0, \; 1, \; 2, \; 3, \; 4$$

where $h_n(m) = h((n-m))_N$

When n = 0 ; $y_1(0) = \sum x_1(m)h_o(m) = 2 + 0 + 0 + 1 - 2 = 1$

When n = 1 ; $y_1(0) = \sum x_1(m)h_1(m) = 1 + 4 + 0 + 0 + 2 = 7$

When n = 2 ; $y_1(2) = \sum x_1(m)h_2(m) = -1 + 2 + 6 + 0 + 0 = 7$

When n = 3 ; $y_1(3) = \sum x_1(m)h_3(m) = 0 - 2 + 3 - 2 + 0 = -1$

When n = 4 ; $y_1(4) = \sum x_1(m)h_4(m) = 0 + 0 - 3 - 1 - 4 = -8$

Convolution of Section 2

m	–4	–3	–2	–1	0	1	2	3	4	5	6	7
$x_2(m)$								–1	–2	–3	4	5
h(m)					2	1	–1	0	0			
$h(-m) = h_0(m)$	0	0	–1	1	2							
$h((3-m))_5 = h_3(m)$				0	0	–1	1	2	**0**	**0**	**–1**	**1**
$h((4-m))_5 = h_4(m)$					0	0	–1	1	2	**0**	**0**	**–1**
$h((5-m))_5 = h_5(m)$						0	0	–1	1	2	**0**	**0**
$h((6-m))_5 = h_6(m)$							0	0	–1	1	2	**0**
$h((7-m))_5 = h_7(m)$								0	0	–1	1	2

$$y_2(n) = x_2(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_2(m)\ h((n-m))_N = \sum_{m=3}^{7} x_2(m)\ h_n(m);\ n = 3, 4, 5, 6, 7$$

where $h_n(m) = h((n-m))_N$

When n = 3 ; $y_2(3) = \sum x_2(m)\ h_3(m) = -2 + 0 + 0 - 4 + 5 = -1$

When n = 4 ; $y_2(4) = \sum x_2(m)\ h_4(m) = -1 - 4 + 0 + 0 - 5 = -10$

When n = 5 ; $y_2(5) = \sum x_2(m)\ h_5(m) = 1 - 2 - 6 + 0 + 0 = -7$

When n = 6 ; $y_2(6) = \sum x_2(m)\ h_6(m) = 0 + 2 - 3 + 8 + 0 = 7$

When n = 7 ; $y_2(7) = \sum x_2(m)\ h_7(m) = 0 + 0 + 3 + 4 + 10 = 17$

Convolution of Section 3

m	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8	9	10
$x_3(m)$											4	5	6	0	0
h(m)					2	1	–1	0	0						
$h(-m) = h_0(m)$	0	0	–1	1	2										
$h((6-m))_5 = h_6(m)$							0	0	–1	1	2	**0**	**0**	**–1**	**1**
$h((7-m))_5 = h_7(m)$								0	0	–1	1	2	**0**	**0**	**–1**
$h((8-m))_5 = h_8(m)$									0	0	–1	1	2	**0**	**0**
$h((9-m))_5 = h_9(m)$										0	0	–1	1	2	**0**
$h((10-m))_5 = h_{10}(m)$											0	0	–1	1	2

$$y_3(n) = x_3(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_3(m)\ h((n-m))_N = \sum_{m=6}^{10} x_3(m)\ h_n(m)\ ;\ n = 6, 7, 8, 9, 10$$

where $h_n(m) = h((n-m))_N$

When n = 6 ; $y_3(6) = \sum x_3(m)\, h_6(m) = 8 + 0 + 0 + 0 + 0 = 8$

When n = 7 ; $y_3(7) = \sum x_3(m)\, h_7(m) = 4 + 10 + 0 + 0 + 0 = 14$

When n = 8 ; $y_3(8) = \sum x_3(m)\, h_8(m) = -4 + 5 + 12 + 0 + 0 = 13$

When n = 9 ; $y_3(9) = \sum x_3(m)\, h_9(m) = 0 - 5 + 6 + 0 + 0 = 1$

When n = 10 ; $y_3(10) = \sum x_3(m)\, h_{10}(m) = 0 + 0 - 6 + 0 + 0 = -6$

To Combine the Output of Convolution of Each Section

It can be observed that the last N_2–1 samples in an output sequence overlaps with the first N_2–1 samples of next output sequence. In overlap save method the overall output is obtained by combining the outputs of the convolution of all sections. While combining the outputs, the overlapped first N_2–1 samples of every output sequence is discarded and the remaining samples are simply saved as samples of y(n) as shown in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	~~1~~	~~7~~	7	–1	–8						
$y_2(n)$				~~–1~~	~~10~~	–7	7	17			
$y_3(n)$							~~8~~	~~14~~	13	1	–6
y(n)	*	*	7	–1	–8	–7	7	17	13	1	–6

$\therefore\ y(n) = x(n) \circledast h(n) = \{*, *, 7, -1, -8, -7, 7, 17, 13, 1, -6\}$

Note : Here y(n) is linear convolution of x(n) and h(n). It can be observed that the results of both the methods are same except the first N_2 – 1 samples.

2.12 Inverse System and Deconvolution

2.12.1 Inverse System

The ***inverse system*** is used to recover the input from the response of a system. For a given system, the inverse system exists, if distinct inputs to a system leads to distinct outputs. The inverse systems exists for all LTI systems.

The inverse system is denoted by $\mathcal{H}^{-1}$. If x(n) is input and y(n) is the output of a system, then y(n) is the input and x(n) is the output of its inverse system.

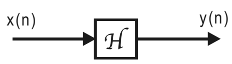


Fig 2.35a : System.

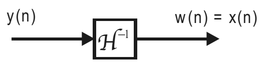


Fig 2.35b : Inverse system.

Fig 2.35 : A system and its inverse system.

Let h(n) be the impulse response of a system and h′(n) be the impulse response of inverse system. Let us connect the system and its inverse in cascade as shown in fig 2.36.

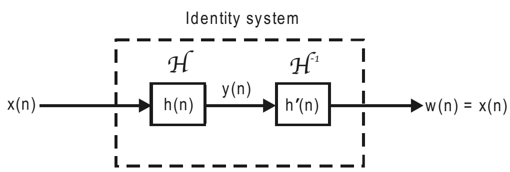


Fig 2.36 : Cascade connection of a system and its inverse.

Now it can be proved that,

$$h(n) * h'(n) = \delta(n) \qquad .....(2.60)$$

Therefore the cascade of a system and its inverse is identity system.

Proof :

With reference to fig 2.36 we can write,

$$y(n) = x(n) * h(n) \qquad .....(2.61)$$

$$w(n) = y(n) * h'(n) \qquad .....(2.62)$$

On substituting for y(n) from equation (2.61) in equation (2.62) we get,

$$w(n) = x(n) * h(n) * h'(n) \qquad .....(2.63)$$

In equation (2.63),

if, $h(n) * h'(n) = \delta(n)$, then, $x(n) * \delta(n) = x(n)$

In a inverse system, w(n) = x(n), and so,

$h(n) * h'(n) = \delta(n)$. Hence proved.

2.12.2 Deconvolution

In an LTI system the response y(n) is given by convolution of input x(n) and impulse response h(n).

i.e., $y(n) = x(n) * h(n)$

The process of recovering the input from the response of a system is called ***deconvolution***. (or the process of recovering x(n) from x(n) * h(n) is called ***deconvolution***).

When the response y(n) and impulse response h(n) are available, then the input x(n) can be computed using the equation (2.64).

$$x(n) = \frac{1}{h(0)}\left[y(n) - \sum_{m=0}^{n-1} x(m)\ h(n-m)\right] \qquad .....(2.64)$$

Proof :

Let x(n) and h(n) be finite duration sequences starting from n = 0. Consider the matrix method of convolution of x(n) and h(n) shown below.

	h(0)	h(1)	h(2)	h(3)	......
x(0)	x(0)h(0)	x(0)h(1)	x(0)h(2)	x(0)h(3)	......
x(1)	x(1)h(0)	x(1)h(1)	x(1)h(2)	x(1)h(3)	......
x(2)	x(2)h(0)	x(2)h(1)	x(2)h(2)	x(2)h(3)	......
x(3)	x(3)h(0)	x(3)h(1)	x(3)h(2)	x(3)h(3)	......
⋮	⋮	⋮	⋮	⋮	⋮

From the above two-dimensional array we can write,

$$y(n) = x(0)\,h(0) \quad \Rightarrow \quad x(0) = \frac{y(0)}{h(0)}$$

$$y(1) = x(1)\,h(0) + x(0)\,h(1) \quad \Rightarrow \quad x(1) = \frac{y(1) - x(0)\,h(1)}{h(0)}$$

$$y(2) = x(2)\,h(0) + x(1)\,h(1) + x(0)\,h(2) \quad \Rightarrow \quad x(2) = \frac{y(2) - x(0)\,h(2) - x(1)\,h(1)}{h(0)}$$

$$y(3) = x(3)\,h(0) + x(2)\,h(1) + x(1)\,h(2) + x(0)\,h(3) \;\Rightarrow\; x(3) = \frac{y(3) - x(0)\,h(3) - x(1)\,h(2) - x(2)\,h(1)}{h(0)}$$

and so on.

From the above analysis, in general for any value of n, the x(n) is given by,

$$x(n) = \frac{y(n) - x(0)\,h(n) - x(1)\,h(n-1) \; - \; ...... \; - \; x(n-1)\,h(1)}{h(0)}$$

$$\therefore \; x(n) = \frac{1}{h(0)}\left[y(n) \; - \; \sum_{m=0}^{n-1} x(m)\,h(n-m)\right]$$

Example 2.30

A discrete time system is defined by the equation, $y(n) = \sum_{m=0}^{n} x(m)$; for $n \geq 0$. Find the inverse system.

Solution

Given that, $y(n) = \sum_{m=0}^{n} x(m)$

When $n = 0$; $y(0) = \sum_{m=0}^{0} x(m) = x(0)$

When $n = 1$; $y(1) = \sum_{m=0}^{1} x(m) = x(0) + x(1) = y(0) + x(1)$

When $n = 2$; $y(2) = \sum_{m=0}^{2} x(m) = x(0) + x(1) + x(2) = y(1) + x(2)$

When $n = 3$; $y(3) = \sum_{m=0}^{3} x(m) = x(0) + x(1) + x(2) + x(3) = y(2) + x(3)$

and so on,

From the above analysis we can write,

$x(0) = y(0)$; $x(1) = y(1) - y(0)$; $x(2) = y(2) - y(1)$; $x(3) = y(3) - y(2)$ and so on,

In general for any value of n, the signal x(n) can be written as,

$$x(n) = y(n) - y(n-1)$$

Therefore the inverse system is defined by the equation,

$$x(n) = y(n) - y(n-1)$$

Example 2.31

When a discrete time system is excited by an input x(n), the response is $y(n) = \{ \underset{-}{2}, 5, 11, 17, 13, 12 \}$

If the impulse response of the system is $h(n) = \{ \underset{-}{2}, 1, 3 \}$, then what will be the input to the system?

Solution

Let N_1 be number of samples in x(n) and N_2 be number of samples in h(n), then the number of samples N_3 in y(n) is given by,

$$N_3 = N_1 + N_2 - 1$$

$$\therefore \; N_1 = N_3 - N_2 + 1 = 6 - 3 + 1 = 4 \text{ samples}$$

Therefore x(n) is 4 sample sequence.

Each sample of x(n) is given by,

$$x(n) = \frac{1}{h(0)} \left[y(n) - \sum_{m=0}^{n-1} x(m)\, h(n-m) \right]$$

$$\text{When } n = 0\ ;\ x(0) = \frac{y(0)}{h(0)} = \frac{2}{2} = 1$$

$$\text{When } n = 1\ ;\ x(1) = \frac{1}{h(0)} \left[y(1) - \sum_{m=0} x(m)\ h(1-m) \right]$$

$$= \frac{1}{h(0)} \left[y(1) - x(0)\, h(1) \right] = \frac{1}{2} \left[5 - 1 \times 1 \right] = 2$$

$$\text{When } n = 2\ ;\ x(2) = \frac{1}{h(0)} \left[y(2) - \sum_{m=0}^{1} x(m)\, h(2-m) \right]$$

$$= \frac{1}{h(0)} \left[y(2) - x(0)\, h(2) - x(1)\, h(1) \right] = \frac{1}{2} \left[11 - 1 \times 3 - 2 \times 1 \right] = 3$$

$$\text{When } n = 3\ ;\ x(3) = \frac{1}{h(0)} \left[y(3) - \sum_{m=0}^{2} x(m)\, h(3-m) \right]$$

$$= \frac{1}{h(0)} \left[y(3) - x(0)\, h(3) - x(1)\, h(2) - x(2)\, h(1) \right] = \frac{1}{2} \left[17 - 1 \times 0 - 2 \times 3 - 3 \times 1 \right] = 4$$

$$\therefore \quad x(n) = \{x(0),\ x(1),\ x(2),\ x(3)\} = \{\underset{\uparrow}{1},\ 2,\ 3,\ 4\}$$

2.13 Correlation, Crosscorrelation and Autocorrelation

The ***correlation*** of two discrete time sequences x(n) and y(n) is defined as,

$$\boxed{r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)} \qquad .....(2.65)$$

where $r_{xy}(m)$ is the correlation sequence obtained by correlation of x(n) and y(n) and m is the variable used for time shift. The correlation of two different sequences is called ***crosscorrelation*** and the correlation of a sequence with itself is called ***autocorrelation***. Hence autocorrelation of a discrete time sequence is defined as,

$$\boxed{r_{xx}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, x(n-m)} \qquad .....(2.66)$$

If the sequence x(n) has N_1 samples and sequence y(n) has N_2 samples then the crosscorrelation sequence $r_{xy}(m)$ will be a finite duration sequence consisting of $N_1 + N_2 - 1$ samples.If the sequence x(n) has N samples, then and the autocorrelation sequence $r_{xx}(m)$ will be a finite duration sequence consisting of $2N - 1$ samples.

In the equation (2.65), the sequence x(n) is unshifted and the sequence y(n) is shifted by m units of time for correlation operation. The same results can be obtained if the sequence y(n) is unshifted and the sequence x(n) is shifted opposite to that of earlier case by m units of time, hence the crosscorrelation operation can also be expressed as,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n+m)\, y(n) \qquad .....(2.67)$$

2.13.1 Procedure for Evaluating Correlation

Let, $x(n)$ = Discrete time sequence with N_1 samples

$y(n)$ = Discrete time sequence with N_2 samples

Now the correlation of x(n) and y(n) will produce a sequence $r_{xy}(m)$ consisting of N_1+N_2-1 samples. Each sample of $r_{xy}(m)$ can be computed using the equation (2.65). The value of $r_{xy}(m)$ at $m = q$ is obtained by replacing m by q, in equation (2.65).

$$\therefore\ r_{xy}(q) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-q) \qquad .....(2.68)$$

The evaluation of equation (2.68) to determine the value of $r_{xy}(m)$ at m = q involves the following three steps.

1. Shifting	:	Shift y(n) by q times to the right if q is positive, shift y(n) by q times to the left if q is negative to obtain y(n - q).
2. Multiplication	:	Multiply x(n) by y(n - q) to get a product sequence. Let the product sequence be $v_q(n)$. Now, $v_q(n) = x(n) \times y(n-q)$.
3. Summation	:	Sum all the values of the product sequence $v_q(n)$ to obtain the value of $r_{xy}(m)$ at m = q. [i.e., $r_{xy}(q)$].

The above procedure will give the value $r_{xy}(m)$ at a single time instant say m = q. In general we are interested in evaluating the values of the sequence $r_{xy}(m)$ over all the time instants in the range $-\infty < m < \infty$. Hence the steps 1, 2 and 3 given above must be repeated, for all possible time shifts in the range $-\infty < m < \infty$.

In the correlation of finite duration sequences it is possible to predict the start and end of the resultant sequence. If x(n) is N-point sequence and starts at $n = n_1$ and if y(n) is N_2-point sequence and starts at $n = n_2$ then, the initial value of $m = m_i$ for $r_{xy}(m)$ is $\mathbf{m_i = n_1 - (n_2 + N_2 - 1)}$. The value of x(n) for $n < n_1$ and the value of y(n) for $n < n_2$ are then assumed to be zero. The final value of $m = m_f$ for $r_{xy}(m)$ is $\mathbf{m_f = m_i + (N_1+N_2-2)}$.

The correlation operation involves all the steps in convolution operation except the folding. Hence it can be proved that the convolution of x(n) and folded sequence y(-n) will generate the crosscorrelation sequence $r_{xy}(m)$.

$$\text{i.e., } r_{xy}(m) = x(n) * y(-n) \qquad .....(2.69)$$

The procedure given above can be used for computing autocorrelation of x(n). For computing autocorrelation using equation (2.68) replace y(n – q) by x(n – q). Similarly when equation (2.69) is used, replace y(–n) by x(–n).

The autocorrelation of N-point sequence x(n) will give 2N – 1 point autocorrelation sequence. If x(n) starts at $n = n_x$ then initial value of $m = m_i$ for $r_{xx}(m)$ is $\mathbf{m_i = -(N-1)}$. The final value of $m = m_f$ for $r_{xx}(m)$ is $\mathbf{m_f = m_i + (2N-2)}$.

Properties of Correlation

1. The crosscorrelation sequence $r_{xy}(m)$ is simply a folded version of $r_{yx}(m)$,

   i.e., $r_{xy}(m) = r_{yx}(-m)$

   Similarly for autocorrelation sequence,

   $r_{xx}(m) = r_{xx}(-m)$

   Hence autocorrelation is an even function.

2. The crosscorrelation sequence satisfies the condition,

   $$\left|r_{xy}(m)\right| \le \sqrt{r_{xx}(0)\, r_{yy}(0)} = \sqrt{E_x\, E_y}$$

   where, E_x and E_y are energy of x(n) and y(n) respectively.

   On applying the above condition to autocorrelation sequence we get,

   $$\left|r_{xx}(m)\right| \le r_{xx}(0) = E_x$$

   From the above equations we infer that the crosscorrelation sequence and autocorrelation sequences attain their respective maximum values at zero shift/lag.

3. Using the maximum value of crosscorrelation sequence, the normalized crosscorrelation sequence is defined as,

   $$\rho_{xy}(m) \le \frac{r_{xy}(m)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}}$$

   Using the maximum value of autocorrelation sequence, the normalized autocorrelation sequence is defined as,

   $$\rho_{xx}(m) \le \frac{r_{xx}(m)}{r_{xx}(0)}$$

Methods of Computing Correlation

Method 1: Graphical Method

Let x(n) and y(n) be the input sequences and $r_{xy}(m)$ be the output sequence.

1. Sketch the graphical representation of the input sequences x(n) and y(n).
2. Shift the sequence y(n) to the left graphically so that the product of x(n) and shifted y(n) gives only one nonzero sample. Now multiply x(n) and shifted y(n) to get a product sequence, and then sum up the samples of product sequence, which is the first sample of output sequence.
3. To get the next sample of output sequence, shift y(n) of previous step to one position right and multiply the shifted sequence with x(n) to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.
4. To get subsequent samples of output sequence, the step 3 is repeated until we get a nonzero product sequence.

Method 2: Tabular Method

The tabular method is same as that of graphical method, except that the tabular representation of the sequences are employed instead of graphical representation. In tabular method, every input sequence and shifted sequence is represented on a row in a table.

Method 3: Matrix Method

Let x(n) and y(n) be the input sequences and $r_{xy}(m)$ be the output sequence. We know that the convolution of x(n) and folded sequence y(−n) will generate the crosscorrelation sequence $r_{xy}(m)$. Hence fold y(n) to get y(−n), and compute convolution of x(n) and y(−n) by matrix method.

In matrix method one of the sequence is represented as a row and the other as a column as shown below.

	....	y(0)	y(−1)	y(−2)	y(−3)	....
x(0)	....	x(0)y(0)	x(0)y(−1)	x(0)y(−2)	x(0)y(−3)	....
x(1)	....	x(1)y(0)	x(1)y(−1)	x(1)y(−2)	x(1)y(−3)	....
x(2)	....	x(2)y(0)	x(2)y(−1)	x(2)y(−2)	x(2)y(−3)	....
x(3)	....	x(3)y(0)	x(3)y(−1)	x(3)y(−2)	x(3)y(−3)	....

Multiply each column element with row elements and fill up the matrix array.

Now the sum of the diagonal elements gives the samples of output sequence $r_{xy}(m)$. (The sum of the diagonal elements are shown below for reference).

$$\vdots$$

$$r_{xy}(0) = ... + x(0)\,y(0) + ...$$

$$r_{xy}(1) = ... + x(1)\,y(0) + x(0)\,y(-1) + ...$$

$$r_{xy}(2) = ... + x(2)\,y(0) + x(1)\,y(-1) + x(0)\,y(-2) + ...$$

$$r_{xy}(3) = ... + x(3)\,y(0) + x(2)\,y(-1) + x(1)\,y(-2) + x(0)\,y(-3) + ...$$

$$\vdots$$

Example 2.32

Perform crosscorrelation of the sequences, x(n) = {1, 1, 2, 2} and y(n) = {1, 0.5, 1}.

Solution

Let $r_{xy}(m)$ be the crosscorrelation sequence obtained by crosscorrelation of x(n) and y(n).

The crosscorrelation sequence $r_{xy}(m)$ is given by,

$$r_{xy} = \sum_{n=-\infty}^{+\infty} x(n)\; y(n-m)$$

The x(n) starts at n = 0 and has 4 samples.

\ $n_1 = 0, N_1 = 4$

The y(n) starts at n = 0 and has 3 samples.

\ $n_2 = 0, N_2 = 3$

Now, $r_{xy}(m)$ will have $N_1 + N_2 - 1 = 4 + 3 - 1 = 6$ samples.

The initial value of $m = m_i = n_1 - (n_2 + N_2 - 1)$
$= 0 - (0 + 3 - 1) = -2$

The final value of $m = m_f = m_i + (N_1 + N_2 - 2)$
$= -2 + (4 + 3 - 2) = 3$

In this example the correlation operation is performed by three methods.

Method 1 : Graphical Method

The graphical representation of x(n) and y(n) are shown below.

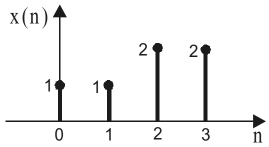


Fig 1.

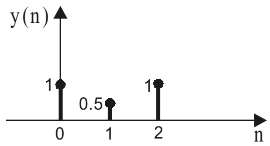


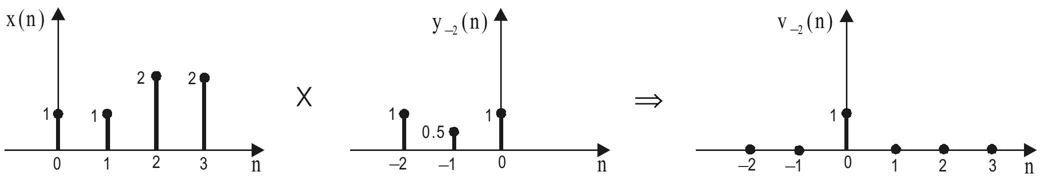
Fig 2.

The 6 samples of $r_{xy}(m)$ are computed using the equation,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\, y_m(n) \;;\; \text{where } y_m(n) = y(n-m)$$

The computation of each sample of $r_{xy}(n)$ using the above equation are graphically shown in fig 3 to fig 8. The graphical representation of output sequence is shown in fig 9.

When $m = -2$; $r_{xy}(-2) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-(-2)) = \sum_{n=-\infty}^{+\infty} x(n)\, y_{-2}(n) = \sum_{n=-\infty}^{+\infty} v_{-2}(n)$

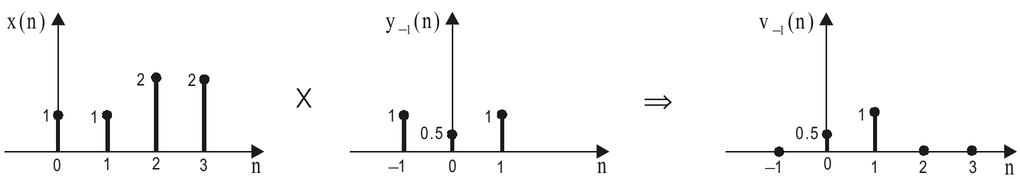


The sum of product sequence $v_{-2}(n)$ gives $r_{xy}(-2)$

Fig 3: Computation of $r_{xy}(-2)$.

$\therefore\ r_{xy}(-2) = 0 + 0 + 1 + 0 + 0 + 0 = 1$

When $m = -1$; $r_{xy}(-1) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-(-1)) = \sum_{n=-\infty}^{+\infty} x(n)\, y_{-1}(n) = \sum_{n=-\infty}^{+\infty} v_{-1}(n)$



The sum of product sequence $v_{-1}(n)$ gives $r_{xy}(-1)$

Fig 4 : Computation of $r_{xy}(-1)$.

$\therefore\ r_{xy}(-1) = 0 + 0.5 + 1 + 0 + 0 = 1.5$

When m = 0 ; $r_{xy}(0) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y(n) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y_0(n) = \sum\limits_{n=-\infty}^{+\infty} v_0(n)$

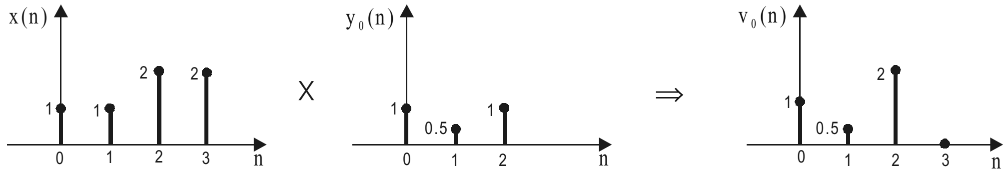


Fig 5 : Computation of $r_{xy}(0)$.

The sum of product sequence $v_0(n)$ gives $r_{xy}(0)$

$\therefore\ r_{xy}(0) = 1 + 0.5 + 2 + 0 = 3.5$

When m = 1 ; $r_{xy}(1) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y(n-1)) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y_1(n) = \sum\limits_{n=-\infty}^{+\infty} v_1(n)$

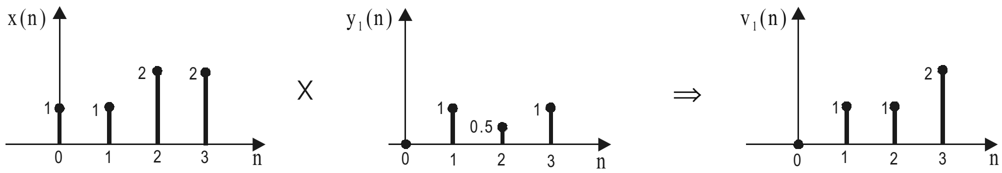


Fig 6 : Computation of $r_{xy}(1)$.

The sum of product sequence $v_1(n)$ gives $r_{xy}(1)$

$\therefore\ r_{xy}(1) = 0 + 1 + 1 + 2 = 4$

When m = 2 ; $r_{xy}(2) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y(n-2) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y_2(n) = \sum\limits_{n=-\infty}^{+\infty} v_2(n)$

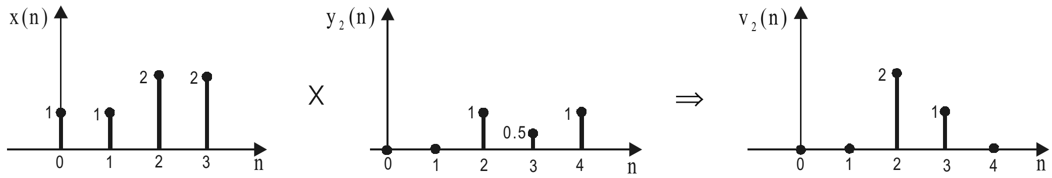


Fig 7 : Computation of $r_{xy}(2)$.

The sum of product sequence $v_2(n)$ gives $r_{xy}(2)$

$\therefore\ r_{xy}(2) = 0 + 0 + 2 + 1 + 0 = 3$

When m = 3 ; $r_{xy}(3) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y(n-3) = \sum\limits_{n=-\infty}^{+\infty} x(n)\, y_3(n) = \sum\limits_{n=-\infty}^{+\infty} v_3(n)$

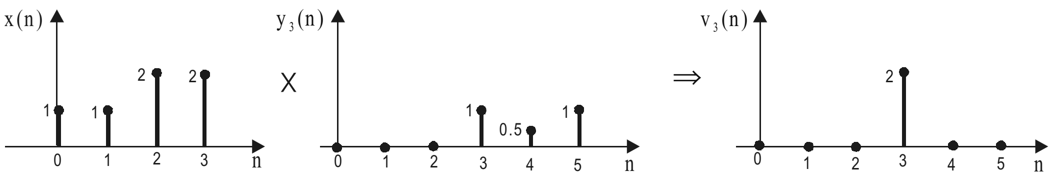


Fig 8 : Computation of $r_{xy}(3)$.

The sum of product sequence $v_3(n)$ gives $r_{xy}(3)$

$\therefore\ r_{xy}(3) = 0 + 0 + 0 + 2 + 0 + 0 = 2$

The crosscorrelation sequence, $r_{xy}(m) = \{1, 1.5, 3.5, 4, 3, 2\}$

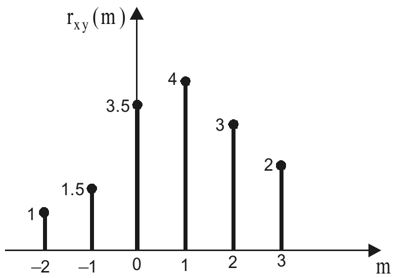


Fig 9 : Graphical representation of $r_{xy}(m)$.

Method 2: Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

n	–2	–1	0	1	2	3	4	5
x(n)			1	1	2	2		
y(n)			1	0.5	1			
$y(n-(-2)) = y_{-2}(n)$	1	0.5	1					
$y(n-(-1)) = y_{-1}(n)$		1	0.5	1				
$y(n) = y_0(n)$			1	0.5	1			
$y(n-1) = y_1(n)$				1	0.5	1		
$y(n-2) = y_2(n)$					1	0.5	1	
$y(n-3) = y_3(n)$						1	0.5	1

Note: The unfilled boxes in the table are considered as zeros.

Each sample of $r_{xy}(m)$ is given by,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\, y_m(n)\ ; \text{ where } y_m(n) = y(n-m)$$

To determine a sample of $r_{xy}(m)$ at $m = q$, multiply the sequence x(n) and $y_q(n)$ to get a product sequence [i.e., multiply the corresponding elements of the row x(n) and $y_q(n)$]. The sum of all the samples of the product sequence gives $r_{xy}(q)$.

When $m = -2$; $r_{xy}(-2) = \sum_{n=-2}^{3} x(n)\, y_{-2}(n) = 0+0+1+0+0+0 = 1$

When $m = -1$; $r_{xy}(-1) = \sum_{n=-1}^{3} x(n)\, y_{-1}(n) = 0+0.5+1+0+0 = 1.5$

When $m = 0$; $r_{xy}(0) = \sum_{n=0}^{3} x(n)\, y_0(n) = 1+0.5+2+0 = 3.5$

When $m = 1$; $r_{xy}(1) = \sum_{n=0}^{3} x(n)\, y_1(n) = 0+1+1+2 = 4$

When $m = 2$; $r_{xy}(2) = \sum_{n=0}^{4} x(n)\, y_2(n) = 0+0+2+1+0 = 3$

When $m = 3$; $r_{xy}(3) = \sum_{n=0}^{5} x(n)\, y_3(n) = 0+0+0+2+0+0 = 2$

$\therefore$ Crosscorrelation sequence, $r_{xy}(m) = \{1,\ 1.5,\ \underset{\uparrow}{3.5},\ 4,\ 3,\ 2\}$

Method 3: Matrix Method

Given that, $x(n) = \{1, 1, 2, 2\}$; $y(n) = \{1, 0.5, 1\}$; $\therefore$ $y(-n) = \{1, 0.5, 1\}$

The sequence x(n) is arranged as a column and the folded sequence y(–n) is arranged as a row as shown below. The elements of the two-dimensional array are obtained by multiplying the corresponding row element with column element. The sum of the diagonal elements gives the samples of the crosscorrelation sequence, $r_{xy}(m)$.

x(n) ↓ \ y(–n) →	1	0.5	1
1	1×1	1×0.5	1×1
1	1×1	1×0.5	1×1
2	2×1	2×0.5	2×1
2	2×1	2×0.5	2×1

$\Rightarrow$

x(n) ↓ \ y(–n) →	1	0.5	1
1	1	0.5	1
1	1	0.5	1
2	2	1	2
2	2	1	2

$r_{xy}(-2) = 1$; $r_{xy}(-1) = 1 + 0.5 = 1.5$; $r_{xy}(0) = 2 + 0.5 + 1 = 3.5$

$r_{xy}(1) = 2 + 1 + 1 = 4$; $r_{xy}(2) = 1 + 2 = 3$; $r_{xy}(3) = 2$

$\therefore$ $r_{xy}(m) = \{1, 1.5, 3.5, 4, 3, 2\}$

Example 2.33

Determine the autocorrelation sequence for $x(n) = \{1, 2, 3, 4\}$.

Solution

Let, $r_{xx}(m)$ be the autocorrelation sequence.

The autocorrelation sequence $r_{xx}(m)$ is given by,

$$r_{xx}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, x(n-m)$$

The x(n) starts at n = 0 and has 4 samples.

$\therefore$ $n_x = 0$ and $N = 4$

Now, $r_{xx}(m)$ will have, $2N - 1 = 2 \times 4 - 1 = 7$ samples.

The initial value of $m = m_i = -(N - 1) = -(4 - 1) = -3$

The final value of $m = m_f = m_i + (2N - 2) = -3 + (2 \times 4 - 2) = 3$

The autocorrelation is computed by tabular method. Hence the sequence x(n) and the shifted sequences of x(n) are tabulated in the following table.

n	–3	–2	–1	0	1	2	3	4	5	6
x(n)				1	2	3	4			
$x(n-(-3)) = x_{-3}(n)$	1	2	3	4						
$x(n-(-2)) = x_{-2}(n)$		1	2	3	4					
$x(n-(-1)) = x_{-1}(n)$			1	2	3	4				
$x(n) = x_0(n)$				1	2	3	4			
$x(n-1) = x_1(n)$					1	2	3	4		
$x(n-2) = x_2(n)$						1	2	3	4	
$x(n-3) = x_3(n)$							1	2	3	4

Each sample of $r_{xx}(m)$ is given by,

$$r_{xx}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, x(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\, x_m(n) \quad ; \quad \text{where } x_m(n) = x(n-m)$$

To determine a sample of $r_{xx}(m)$ at m = q, multiply the sequence x(n) and $x_q(n)$ to get a product sequence [i.e., multiply the corresponding elements of the row x(n) and $x_q(n)$]. The sum of all the samples of the product sequence gives $r_{xx}(q)$.

When m = −3 ; $r_{xx}(-3) = \sum_{n=-3}^{3} x(n)\, x_{-3}(n) = 0+0+0+4+0+0+0 = 4$

When m = −2 ; $r_{xx}(-2) = \sum_{n=-2}^{3} x(n)\, x_{-2}(n) = 0+0+3+8+0+0 = 11$

When m = −1 ; $r_{xx}(-1) = \sum_{n=-1}^{3} x(n)\, x_{-1}(n) = 0+2+6+12+0 = 20$

When m = 0 ; $r_{xx}(0) = \sum_{n=0}^{3} x(n)\, x_0(n) = 1+4+9+16 = 30$

When m = 1 ; $r_{xx}(1) = \sum_{n=0}^{4} x(n)\, x_1(n) = 0+2+6+12+0 = 20$

When m = 2 ; $r_{xx}(2) = \sum_{n=0}^{5} x(n)\, x_2(n) = 0+0+3+8+0+0 = 11$

When m = 3 ; $r_{xx}(3) = \sum_{n=0}^{6} x(n)\, x_3(n) = 0+0+0+4+0+0+0 = 4$

$\therefore$ Autocorrelation sequence, $r_{xx}(m) = \{4,\ 11,\ 20,\ \underset{\uparrow}{30},\ 20,\ 11,\ 4\}$

2.14 Circular Correlation

The ***circular correlation*** of two periodic discrete time sequences x(n) and y(n) with periodicity of N samples is defined as,

$$\boxed{\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N} \qquad .....(2.70)$$

where, $\bar{r}_{xy}(m)$ is the sequence obtained by circular correlation

$y^*((n-m))_N$ represents circular shift of $y^*(n)$

m is a variable used for circular time shift

The circular correlation of two different sequences is called ***circular crosscorrelation*** and the circular correlation of a sequence with itself is called ***circular autocorrelation***. Hence circular autocorrelation of a discrete time sequence is defined as,

$$\boxed{\bar{r}_{xx}(m) = \sum_{n=0}^{N-1} x(n)\, x^*((n-m))_N} \qquad .....(2.71)$$

The output sequence obtained by circular correlation is also periodic sequence with periodicity of N samples. Hence this correlation is also called ***periodic correlation***. The circular correlation is defined for periodic sequences. But circular correlation can be performed with non-periodic sequences by periodically extending them.The circular correlation of two sequences requires that, at least one of the sequences should be periodic. Hence it is sufficient if one of the sequences is periodically extended in order to perform circular correlation.

The circular correlation of finite duration sequences can be performed only if both the sequences consists of same number of samples. If the sequences have different number of samples, then convert the smaller size sequence to the size of larger size sequence by appending zeros.

In the equation (2.70), the sequence x(n) is unshifted and the sequence y*(n) is circularly shifted by m units of time for correlation operation. The same results can be obtained if the sequence y*(n) is unshifted and the sequence x(n) is circularly shifted opposite to that of earlier case by m units of time, hence the circular correlation operation can also be expressed as,

$$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x((n+m))_N \; y^*(n) \qquad .....(2.72)$$

Circular correlation basically involves the same three steps as that for correlation, namely shifting one of the sequence, multiplying the two sequences and finally summing the values of product sequence. The difference between the two is that in circular correlation the shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences by modulo-N operation. In correlation, there is no modulo-N operation.

2.14.1 Procedure for Evaluating Circular Correlation

Let, x(n) and y(n) be periodic discrete time sequences with periodicity of N-samples. If x(n) and y(n) are non-periodic then convert the sequences to N-sample sequence and periodically extend the sequence y(n) with periodicity of N-samples.

Now the circular correlation of x(n) and y(n) will produce a periodic sequence $\bar{r}_{xy}(m)$ with periodicity of N-samples. The samples of one period of $\bar{r}_{xy}(m)$ can be computed using the equation (2.70).

The value of $\bar{r}_{xy}(m)$ at m = q is obtained by replacing m by q, in equation (2.70), as shown below.

$$\bar{r}_{xy}(q) = \sum_{n=0}^{N-1} x(n) \; y^*((n-q))_N \qquad .....(2.73)$$

The evaluation of equation (2.73) to determine the value of $\bar{r}_{xy}(m)$ at m = q involves the following four steps.

1. Conjugation : Take conjugate of y(n) to get y*(n). If y(n) is a real sequence then y*(n) will be same as y(n). Represent the samples of one period of the sequences x(n) and y*(n) on circles.

2. Rotation : Rotate y*(n) by q times in anticlockwise if q is positive, rotate y*(n) by q times in clockwise if q is negative to obtain $y^*((n-q))_N$.

3. Multiplication : Multiply x(n) by $y^*((n-q))_N$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x(n) \times y^*((n-q))_N$.

4. Summation : Sum up the samples of one period of the product sequence $v_q(m)$ to obtain the value of $\bar{r}_{xy}(m)$ at m = q. [i.e., $\bar{r}_{xy}(q)$].

The above procedure will give the value of $\bar{r}_{xy}(m)$ at a single time instant say m = q. In general, we are interested in evaluating the values of the sequence $\bar{r}_{xy}(m)$ in the range $0 < m < N - 1$. Hence the steps 2 , 3 and 4 given above must be repeated, for all possible time shifts in the range $0 < m < N - 1$.

2.14.2 Methods of Computing Circular Correlation

Method 1 : Graphical Method

In graphical method the given sequences are converted to same size and represented on circles. In case of periodic sequences, the samples of one period are represented on circles. Let x(n) and y(n) be the given real sequences. Let $\bar{r}_{xy}(m)$ be the sequence obtained by circular correlation of x(n) and y(n). The following procedure can be used to get a sample of $\bar{r}_{xy}(m)$ at $m = q$.

1. Represent the sequences x(n) and y(n) on circles.
2. Rotate (or shift) the sequence y(n), q times to get the sequence $y((n-q))_N$. If q is positive then rotate (or shift) the sequence in anticlockwise direction and if q is negative then rotate (or shift) the sequence in clockwise direction.
3. The sample of $\bar{r}_{xy}(q)$ at m = q is given by,

$$\bar{r}_{xy}(q) = \sum_{n=0}^{N-1} x(n)\, y((n-q))_N = \sum_{n=0}^{N-1} x(n)\, y_q(n)$$

where, $y_q(n) = y((n-q))_N$

Determine the product sequence $x(n)y_q(n)$ for one period.

4. The sum of all the samples of the product sequence gives the sample $\bar{r}_{xy}(q)$ [i.e., $\bar{r}_{xy}(m)$ at m = q].

The above procedure is repeated for all possible values of m to get the sequence $\bar{r}_{xy}(m)$.

Method 2 : Using Tabular Array

Let x(n) and y(n) be the given real sequences. Let $\bar{r}_{xy}(m)$ be the sequence obtained by circular correlation of x(n) and y(n). The following procedure can be used to get a sample of $\bar{r}_{xy}(m)$ at $m = q$.

1. Represent the sequences x(n) and y(n) as two rows of tabular array.
2. Periodically extend y(n). Here the periodicity is N, where N is the length of the given sequences.
3. Shift the sequence y(n), q times to get the sequence $y((n-q))_N$. If q is positive then shift the sequence to the right and if q is negative then shift the sequence to the left.
4. The sample of $\bar{r}_{xy}(q)$ at m = q is given by,

$$\bar{r}_{xy}(q) = \sum_{n=0}^{N-1} x(n)\, y((n-q))_N = \sum_{n=0}^{N-1} x(n)\, y_q(n)$$

where, $y_q(n) = y((n-q))_N$

Determine the product sequence $x(n)y_q(n)$ for one period.

5. The sum of all the samples of the product sequence gives the sample $\bar{r}_{xy}(q)$ [i.e., $\bar{r}_{xy}(m)$ at m = q].

The above procedure is repeated for all possible values of m to get the sequence $\bar{r}_{xy}(m)$.

Method 3: Using Matrices

Let x(n) and y(n) be the given N-point sequences. The circular correlation of x(n) and y(n) yields another N-point sequence $\bar{r}_{xy}(m)$.

In this method an N ´ N matrix is formed using the sequence y(n) as shown below. The sequence x(n) is arranged as a column vector (column matrix) of order N ´ 1. The product of the two matrices gives the resultant sequence $\bar{r}_{xy}(m)$.

$$\begin{bmatrix} y(0) & y(1) & y(2) & ..... & y(N-1) & y(N) \\ y(N) & y(0) & y(1) & ..... & y(N-2) & y(N-1) \\ y(N-1) & y(N) & y(0) & ..... & y(N-3) & y(N-2) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ y(2) & y(3) & y(4) & ..... & y(0) & y(1) \\ y(1) & y(2) & y(3) & ..... & y(N) & y(0) \end{bmatrix} \times \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ \vdots \\ x(N-2) \\ x(N-1) \end{bmatrix} = \begin{bmatrix} \bar{r}_{xy}(0) \\ \bar{r}_{xy}(1) \\ \bar{r}_{xy}(2) \\ \vdots \\ \vdots \\ \bar{r}_{xy}(N-2) \\ \bar{r}_{xy}(N-1) \end{bmatrix}$$

Example 2.34

Perform circular correlation of the two sequences, $x(n) = \{\underset{\uparrow}{1}, 1, 2, 1\}$ and $y(n) = \{\underset{\uparrow}{2}, 3, 1, 1\}$

Solution

Method 1:Graphical Method of Computing Circular Correlation

The given sequences are represented as points on circles as shown in fig 1 and 2.

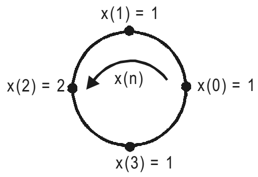


Fig 1.

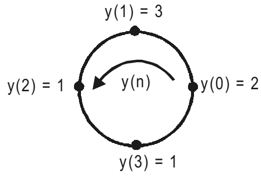


Fig 2.

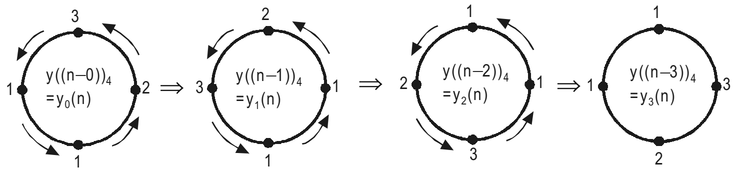


Fig 3 : Circularly shifted sequences y(n-m), for m = 0, 1, 2, 3.

Let $\bar{r}_{xy}(m)$ be the sequence obtained by circular correlation of x(n) and y(n). The given sequences are 4 sample sequences and so N = 4. Each sample of $\bar{r}_{xy}(m)$ is given by the equation,

$$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\, y((n-m))_N = \sum_{n=0}^{N-1} x(n)\, y_m(n), \quad \text{where } y_m(n) = y((n-m))_N$$

Using the above equation, graphical method of computing each sample of $\bar{r}_{xy}(m)$ are shown in fig 4 to fig 7.

When $m = 0$; $\bar{r}_{xy}(0) = \sum_{n=0}^{3} x(n)\, y((n-0))_4 = \sum_{n=0}^{3} x(n)\, y_0(n) = \sum_{n=0}^{3} v_0(n)$

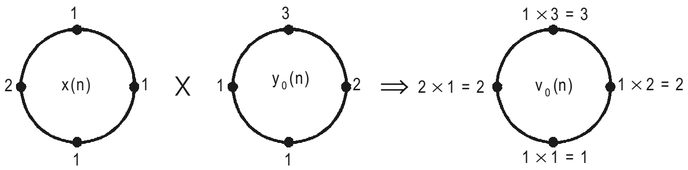


Fig 4 : Computation of $\bar{r}_{xy}(0)$.

The sum of samples of $v_0(n)$ gives $\bar{r}_{xy}(0)$

$\therefore\ \bar{r}_{xy}(0) = 2 + 3 + 2 + 1 = 8$

When $m = 1$; $\bar{r}_{xy}(1) = \sum\limits_{n=0}^{3} x(n)\ y((n-1))_4 = \sum\limits_{n=0}^{3} x(n)\ y_1(n) = \sum\limits_{n=0}^{3} v_1(n)$

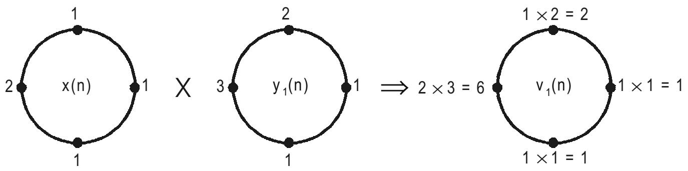


Fig 5 : Computation of $\bar{r}_{xy}(1)$.

The sum of samples of $v_1(n)$ gives $\bar{r}_{xy}(1)$

$\therefore\ \bar{r}_{xy}(1) = 1 + 2 + 6 + 1 = 10$

When $m = 2$; $\bar{r}_{xy}(2) = \sum\limits_{n=0}^{3} x(n)\ y((n-2))_4 = \sum\limits_{n=0}^{3} x(n)\ y_2(n) = \sum\limits_{n=0}^{3} v_2(n)$

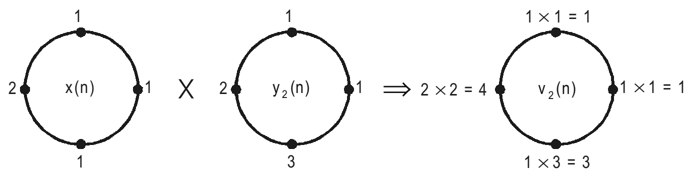


Fig 6 : Computation of $\bar{r}_{xy}(2)$.

The sum of samples of $v_2(n)$ gives $\bar{r}_{xy}(2)$

$\therefore\ \bar{r}_{xy}(2) = 1 + 1 + 4 + 3 = 9$

When $m = 3$; $\bar{r}_{xy}(3) = \sum\limits_{n=0}^{3} x(n)\ y((n-3))_4 = \sum\limits_{n=0}^{3} x(n)\ y_3(n) = \sum\limits_{n=0}^{3} v_3(n)$

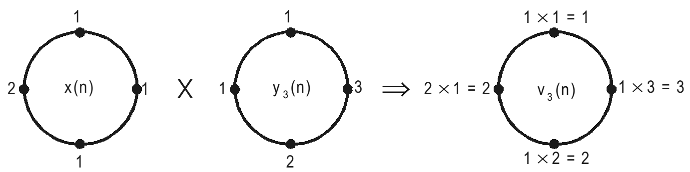


Fig 7 : Computation of $\bar{r}_{xy}(3)$.

The sum of samples of $v_3(n)$ gives $\bar{r}_{xy}(3)$

$\therefore\ \bar{r}_{xy}(3) = 3 + 1 + 2 + 2 = 8$

\ $\bar{r}_{xy}(m) = \{8, 10, 9, 8\}$

Method 2 : Circular Correlation Using Tabular Array

The given sequences are represented in the tabular array as shown below. Here the shifted sequences $y_m(n)$ are periodically extended with a periodicity of $N = 4$. Let $\bar{r}_{xy}(m)$ be the sequence obtained by circular correlation of x(n) and y(n). Each sample of $\bar{r}_{xy}(m)$ is given by the equation,

$$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\ y((n-m))_N = \sum_{n=0}^{N-1} x(n)\ y_m(n), \quad \text{where } y_m(n) = y((n-m))_N$$

Note : The boldfaced numbers are samples obtained by periodic extension.

n	0	1	2	3	4	5	6
$x(n)$	1	1	2	1			
$y(n)$	2	3	1	1			
$y_0((n-0))_4 = y_0(n)$	2	3	1	1	**2**	**3**	**1**
$y_1((n-1))_4 = y_1(n)$	**1**	2	3	1	1	**2**	**3**
$y_2((n-2))_4 = y_2(n)$	**1**	**1**	2	3	1	1	**2**
$y_3((n-3))_4 = y_3(n)$	**3**	**1**	**1**	2	3	1	1

To determine a sample of $\bar{r}_{xy}(m)$ at m = q, multiply the sequence, x(n) and $y_q(n)$, to get a product sequence x(n) $x_q(n)$ [i.e., multiply the corresponding elements of the row x(n) and $y_q(n)$]. The sum of all the samples of the product sequence gives $\bar{r}_{xy}(m)$.

$$\text{When } m = 0\ ;\ \bar{r}_{xy}(0) = \sum_{n=0}^{3} x(n)\ y_0(n)$$
$$= x(0)\ y_0(0) + x(1)\ y_0(1) + x(2)\ y_0(2) + x(3)\ y_0(3)$$
$$= 1\times 2 + 1\times 3 + 2\times 1 + 1\times 1 = 2 + 3 + 2 + 1 = 8$$

The samples of $\bar{r}_{xy}(m)$ for other values of m are calculated as shown for m = 0.

$$\text{When } m = 1;\quad \bar{r}_{xy}(1) = \sum_{n=0}^{3} x(n)\ \ y_1(n) = 1 + 2 + 6 + 1 = 10$$

$$\text{When } m = 2;\quad \bar{r}_{xy}(2) = \sum_{n=0}^{3} x(n)\ \ y_2(n) = 1 + 1 + 4 + 3 = 9$$

$$\text{When } m = 3;\quad \bar{r}_{xy}(3) = \sum_{n=0}^{3} x(n)\ \ y_3(n) = 3 + 1 + 2 + 2 = 8$$

$$\therefore\quad \bar{r}_{xy}(m) = \{\underset{\uparrow}{8},\ 10,\ 9,\ 8\}$$

Method 3 : Circular Correlation Using Matrices

The sequence $\bar{r}_{xy}(m)$ can be arranged as a column vector of order N ´ 1 and using the samples of y(n) the N ´ N matrix is formed as shown below. The product of the two matrices gives the sequence $\bar{r}_{xy}(m)$.

$$\begin{bmatrix} y(0) & y(1) & y(2) & y(3) \\ y(3) & y(0) & y(1) & y(2) \\ y(2) & y(3) & y(0) & y(1) \\ y(1) & y(2) & y(3) & y(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} \bar{r}_{xy}(0) \\ \bar{r}_{xy}(1) \\ \bar{r}_{xy}(2) \\ \bar{r}_{xy}(3) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 9 \\ 8 \end{bmatrix}$$

$$\backslash\quad \bar{r}_{xy}(m) = \{8, 10, 9, 8\}$$

2.15. Summary of Important Concepts

1. The discrete signal is a function of a discrete independent variable.
2. In a discrete time signal, the value of discrete time signal and the independent variable time are discrete.
3. The digital signal is same as discrete signal except that the magnitude of the signal is quantized.
4. A discrete time sinusoid is periodic only if its frequency is a rational number.
5. Discrete time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.
6. The sampling is the process of conversion of continuous time signal into discrete time signal.
7. The time interval between successive samples is called sampling time or sampling period.
8. The inverse of sampling period is called sampling frequency.
9. The phenomenon of high frequency component getting the identity of low-frequency component during sampling is called aliasing.
10. For analog signal with maximum frequency F_{max}, the sampling frequency should be greater than $2F_{max}$.
11. When sampling frequency F_s is equal to $2F_{max}$, the sampling rate is called Nyquist rate.
12. The signals that can be completely specified by mathematical equations are called deterministic signals.
13. The signals whose characteristics are random in nature are called nondeterministic signals.
14. A signal x(n) is periodic with periodicity of N samples if $x(n + N) = x(n)$.
15. When a signal exhibits symmetry with respect to $n = 0$ then it is called an even signal.
16. When a signal exhibits antisymmetry with respect to $n = 0$, then it is called an odd signal.
17. When the energy E of a signal is finite and nonzero, the signal is called energy signal.
18. When the power P of a signal is finite and nonzero, the signal is called power signal.
19. For energy signals, the energy will be finite and average power will be zero.
20. For power signals the average power is finite and energy will be infinite.
21. A signal is said to be causal, if it is defined for $n \geq 0$.
22. A signal is said to be noncausal, if it is defined for both $n \leq 0$ and $n > 0$.
23. A signal is said to be anticausal, if it is defined for $n \leq 0$.
24. A discrete time system is a device or algorithm that operates on a discrete time signal.
25. When a system satisfies the properties of linearity and time invariance, it is called an LTI system.
26. When the input to a discrete time system is unit impulse $\delta(n)$, the output is called impulse response, h(n).
27. In a static or memoryless system, the output at any instant n depends on input at the same time.
28. A system is said to be time invariant if its input-output characteristics do not change with time.
29. A linear system is one that satisfies the superposition principle.
30. A system is said to be causal if the output does not depends on future inputs/outputs.
31. When a system output at any time n depends on future inputs/outputs, it is called a noncausal system.
32. System is said to be BIBO stable if and only if every bounded input produces a bounded output.
33. When a system output at any time n depends on past outputs, it is called a recursive system.
34. A system whose output does not depends on past outputs is called a nonrecursive system.
35. The convolution of N_1 and N_2 sample sequences produce a sequence consisting of N_1+N_2-1 samples.
36. In an LTI system, response for an arbitrary input is given by convolution of input with impulse response.
37. The output sequence of circular convolution is also periodic sequence with periodicity of N samples.
38. The inverse system is used to recover the input from the response of a system.
39. The process of recovering the input from the response of a system is called deconvolution.
40. The correlation of two different sequences is called crosscorrelation.
41. The correlation of a sequence with itself is called autocorrelation.

2.16. Short Questions and Answers

Q2.1 ***Perform addition of the discrete time signals, $x_1(n)$ = {2, 2, 1, 2} and $x_2(n)$ = {–2, –1, 3, 2}.***

Solution

In addition operation, the samples corresponding to same value of n are added.

When n = 0, $x_1(0) + x_2(0) = 2 + (-2) = 0$	When n = 2, $x_1(2) + x_2(2) = 1 + 3 = 4$
When n = 1, $x_1(1) + x_2(1) = 2 + (-1) = 1$	When n = 3, $x_1(3) + x_2(3) = 2 + 2 = 4$

$\therefore\ x_1(n) + x_2(n) = \{0, 1, 4, 4\}$

Q2.2 ***Perform multiplication of discrete time signals, $x_1(n)$ = {2, 2, 1, 2} and $x_2(n)$ = {–2, –1, 3, 2}.***

Solution

In multiplication operation, the samples corresponding to same value of n are multiplied.

When n = 0, $x_1(0) \times x_2(0) = 2\times(-2) = -4$	When n = 2, $x_1(2) \times x_2(2) = 1 \times 3 = 3$
When n = 1, $x_1(1) \times x_2(1) = 2\times(-1) = -2$	When n = 3, $x_1(3) \times x_2(3) = 2 \times 2 = 4$

$\therefore\ x_1(n) \times x_2(n) = \{-4, -2, 3, 4\}$

Q2.3 ***Express the discrete time signal x(n) as a summation of impulses.***

If we multiply a signal x(n) by a delayed unit impulse $\delta(n - m)$, then the product is x(m), where x(m) is the signal sample at n = m [because $\delta(n - m)$ is 1 only at n = m and zero for other values of n]. Therefore, if we repeat this multiplication over all possible delays in the range $-\infty < m < \infty$ and sum all the product sequences, then the result will be a sequence that is equal to the sequence x(n).

$$\therefore\ x(n) = \ldots\ x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + \ldots$$

$$= \sum_{m=-\infty}^{+\infty} x(m)\,\delta(n-m)$$

Q2.4 ***What are the basic elements used to construct the block diagram of discrete time system?***

The basic elements used to construct the block diagram of discrete time system are adder, constant multiplier and unit delay element.

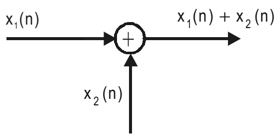


Fig a : Adder.

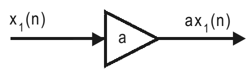


Fig b : Constant multiplier.

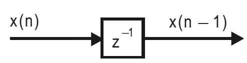


Fig c : Unit delay element.

Q2.5 ***Let, x(n) = {1, 2, 3, 4}, be one period of a periodic sequence. What is x(n – 2, mod4)?***

The x(n) can be represented on the circle as shown in fig Q2.5a. The x(n – 2, mod4) is circularly shifted sequence of x(n) by two units of time as shown in fig Q2.5b.(Here, mod 4 stands for periodicity of 4).

$\therefore\ x(n - 2, \text{mod}4) = \{3, 4, 1, 2\}$

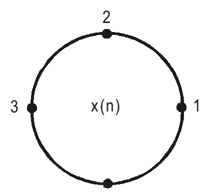


Fig Q2.5a.

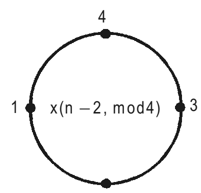


Fig Q2.5b.

Q2.6 ***Why linear convolution is important in digital signal processing ?***

The response or output of an LTI discrete time system for any input x(n) is given by linear convolution of the input x(n) and the impulse response h(n) of the system. (This means that if the impulse response of a system is known, then the response of the system for any input can be determined by convolution operation.)

Q2.7 ***In y(n) = x(n) * h(n), how will you determine the start and end point of y(n)? What will be the length of y(n)?***

Let, length of x(n) be N_1 and starts at $n = n_x$. Let, length of h(n) be N_2 and starts at $n = n_h$.

Now, y(n) will start at $n = n_x + n_h$

y(n) will end at $n = (n_x + n_h) + (N_1 + N_2 - 2)$

The length of y(n) is $N_1 + N_2 - 1$.

Q2.8 ***What is zero padding? Why is it needed?***

Appending zeros to a sequence in order to increase the size or length of the sequence is called zero padding.

In circular convolution, when the two input sequences are of different size, then they are converted to equal size by zero padding.

Q2.9 ***List the differences between linear convolution and circular convolution.***

Linear convolution	*Circular convolution*
1. The length of the input sequence can be different	1. The length of the input sequences should be same.
2. Zero padding is not required.	2. If the length of the input sequences are different, then zero padding is required.
3. The input sequences need not be periodic.	3. Atleast one of the input sequence should be periodic or should be periodically extended.
4. The output sequence is nonperiodic.	4. The output sequence is periodic. The periodicity is same as that of input sequence.
5. The length of output sequence will be greater than the length of input sequences.	5. The length of the input and output sequences are same.

Q2.10 ***Perform the circular convolution of the two sequences $x_1(n) = \{1, 2, 3\}$ and $x_2(n) = \{4, 5, 6\}$.***

Solution

Let $x_3(n)$ be the sequence obtained from circular convolution of $x_1(n)$ and $x_2(n)$. The sequence $x_1(n)$ can be arranged as a column vector of order 3 ´1 and using the samples of $x_2(n)$ a 3 ´ 3 matrix is formed as shown below. The product of two matrices gives the sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 4 & 6 & 5 \\ 5 & 4 & 6 \\ 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 31 \\ 31 \\ 28 \end{bmatrix}$$

$$\backslash \;\; x_3(n) = x_1(n) \circledast x_2(n) = \{31, 31, 28\}.$$

Q2.11 Perform the linear convolution of the two sequences $x_1(n)$ = {1, 2} and $x_2(n)$ = {3, 4} via circular convolution.

Solution

Let $x_3(n)$ be the sequence obtained from linear convolution of $x_1(n)$ and $x_2(n)$. The length of $x_3(n)$ will be 2 + 2 – 1 = 3. Let us convert $x_1(n)$ and $x_2(n)$ into three sample sequences by padding with zeros as shown below.

$$x_1(n) = \{1, 2, 0\} \text{ and } x_2(n) = \{3, 4, 0\}$$

Now the circular convolution of $x_1(n)$ and $x_2(n)$ will give $x_3(n)$. The sequence $x_1(n)$ is arranged as a column vector and using the sequence $x_2(n)$, a 3 ´3 matrix is formed as shown below. The product of the two matrices gives the sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 3 & 0 & 4 \\ 4 & 3 & 0 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 8 \end{bmatrix}$$

\ $x_3(n) = x_1(n) * x_2(n) = \{3, 10, 8\}$

Q2.12 Compare the overlap add and overlap save method of sectioned convolutions.

Overlap add method	Overlap save method
1. Linear convolution of each section of longer sequence with smaller sequence is performed.	1. Circular convolution of each section of longer sequence with smaller sequence is performed. (after converting them to the size of output sequence).
2. Zero padding is not required.	2. Zero padding is required to convert the input sequences to the size of output sequence.
3. Overlapping of samples of input sections are not required.	3. The N_2–1 samples of an input section of longer sequence is overlapped with next input section.
4. The overlapped samples in the output of sectioned convolutions are added to get the overall output.	4. Depending on method of overlapping input samples, either last N_2–1 samples or first N_2–1 samples of output sequence of each sectioned convolution are discarded.

Q2.13 In what way zero padding is implemented in overlap save method?

In overlap save method, the zero padding is employed to convert the smaller input sequence to the size of the output sequence of each sectioned convolution. The zero padding is also employed to convert either the last section or the first section of the longer input sequence to the size of the output sequence of each sectioned convolution. (This depends on the method of overlapping input samples).

Q2.14 List the similarities and differences in convolution and correlation of two sequences.

Similarities

1. Both convolution and correlation operation involves shifting, multiplication and summation of product sequence.
2. Both convolution and correlation operation produce same size of output sequence.

Differences

1. Correlation operation does not involve change of index and folding of one of the input sequence.
2. The convolution operation is commutative, [i.e., x(n) ∗ y(n) = y(n) ∗ x(n)], whereas in correlation operation in order to satisfy commutative property, while performing correlation of y(n) and x(n), the shifting has to performed in opposite direction to that of performing correlation of x(n) and y(n).

Q2.15 ***Let $r_{xy}(m)$ be the correlation sequence obtained by correlation of x(n) and y(n), how will you determine the start and end point of $r_{xy}(m)$? What will be the length of $r_{xy}(m)$?***

Let, length of x(n) be N_1 and starts at $n = n_1$. Let length of y(n) be N_2 and starts at $n = n_2$.

Now, $r_{xy}(m)$ will start at $m_i = n_1 - (n_2 + N_2 - 1)$

$r_{xy}(m)$ will end at $m_f = m_i + (N_1 + N_2 - 2)$

The length of $r_{xy}(m)$ is $N_1 + N_2 - 1$.

Q2.16 ***What are the differences between crosscorrelation and autocorrelation?***

1. Crosscorrelation operation is correlation of two different sequences, whereas autocorrelation is correlation of a sequence with itself.
2. Autocorrelation operation is an even function, whereas crosscorrelation is not an even function.

Q2.17 ***Perform the correlation of the two sequences, x(n) = {1, 2, 3} and y(n) = {2, 4, 1}.***

Solution

Given that, $x(n) = \{\underset{\uparrow}{1}, 2, 3\}$ and $y(n) = \{\underset{\uparrow}{2}, 4, 1\}$. ∴ $y(-n) = \{1, 4, \underset{\uparrow}{2}\}$

The sequence x(n) is arranged as a column and the folded sequence y(–n) is arranged as a row as shown below. The elements of the two dimensional array are obtained by multiplying the corresponding row element with column element. The sum of the diagonal elements gives the samples of the crosscorrelation sequence, $r_{xy}(m)$.

x(n) ↓ \ y(–n) →	1	4	2
1	1 × 1	1 × 4	1 × 2
2	2 × 1	2 × 4	2 × 2
3	3 × 1	3 × 4	3 × 2

⇒

x(n) ↓ \ y(–n) →	1	4	2
1	1	4	2
2	2	8	4
3	3	12	6

$r_{xy}(-2) = 1$; $r_{xy}(-1) = 2 + 4 = 6$; $r_{xy}(0) = 3 + 8 + 2 = 13$; $r_{xy}(1) = 12 + 4 = 16$; $r_{xy}(2) = 6$;

∴ $r_{xy}(m) = \{1, 6, \underset{\uparrow}{13}, 16, 6\}$

Q2.18 ***Perform the circular correlation of the two sequences, x(n) = {1, 2, 3} and y(n) = {2, 4, 1}.***

Solution

Let $\bar{r}_{xy}(m)$ be the sequence obtained from circular correlation of x(n) and y(n). The sequence x(n) can be arranged as a column vector of order 3 × 1 and using the samples of y(n) a 3 × 3 matrix is formed as shown below. The product of two matrices gives the sequence $\bar{r}_{xy}(m)$.

$$\begin{bmatrix} y(0) & y(1) & y(2) \\ y(2) & y(0) & y(1) \\ y(1) & y(2) & y(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix} = \begin{bmatrix} \bar{r}_{xy}(0) \\ \bar{r}_{xy}(1) \\ \bar{r}_{xy}(2) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 4 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 17 \\ 12 \end{bmatrix}$$

∴ $\bar{r}_{xy}(m) = \{13, 17, 12\}$

Q2.19 Perform circular autocorrelation of the sequence, x(n) = {1, 2, 3, 4}.

Solution

Let $\bar{r}_{xx}(m)$ be the sequence obtained from circular autocorrelation of x(n). The sequence x(n) can be arranged as a column vector of order 4×1 and again by using the samples of x(n) a 4×4 matrix is formed as shown below. The product of two matrices gives the sequence $\bar{r}_{xx}(m)$.

$$\begin{bmatrix} x(0) & x(1) & x(2) & x(3) \\ x(3) & x(0) & x(1) & x(2) \\ x(2) & x(3) & x(0) & x(1) \\ x(1) & x(2) & x(3) & x(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} \bar{r}_{xx}(0) \\ \bar{r}_{xx}(1) \\ \bar{r}_{xx}(2) \\ \bar{r}_{xx}(3) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 24 \\ 22 \\ 24 \end{bmatrix}$$

$$\therefore \ \bar{r}_{xx}(m) = \{30,\ 24,\ 22,\ 24\}$$

Q2.20 What is the difference between circular crosscorrelation and circular autocorrelation?

Circular crosscorrelation operation is circular correlation of two different sequences, whereas circular autocorrelation is circular correlation of a sequence with itself.

2.17 MATLAB Programs

Program 2.1

Write a MATLAB program to generate the standard discrete time signals unit impulse, unit step and unit ramp signals.

```
%****************** program to plot some standard signals

n=-20 : 1 : 20;                  %specify the range of n

%****************** unit impulse signal
x1=1;
x2=0;
x=x1.*(n==0)+x2.*(n~=0);         %generate unit impulse signal
subplot(3,1,1);stem(n,x);        %plot the generated unit impulse signal
xlabel('n');ylabel('x(n)');title('unit impulse signal');

%****************** unit step signal
x1=1;
x2=0;
x=x1.*(n>=0)+x2.*(n<0);          %generate unit step signal
subplot(3,1,2);stem(n,x);        %plot the generated unit step signal
xlabel('n');ylabel('x(n)');title('unit step signal');

%****************** unit ramp signal
x1=n;
x2=0;
x=x1.*(n>=0)+x2.*(n<0);          %generate unit ramp signal
subplot(3,1,3);stem(n,x);        %plot the generated unit ramp signal
xlabel('n');ylabel('x(n)');title('unit ramp signal');
```

OUTPUT

The output waveforms of program 2.1 are shown in fig P2.1.

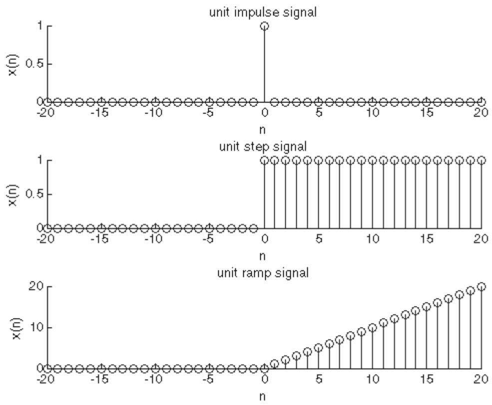


Fig P2.1 : Output waveforms of program 2.1.

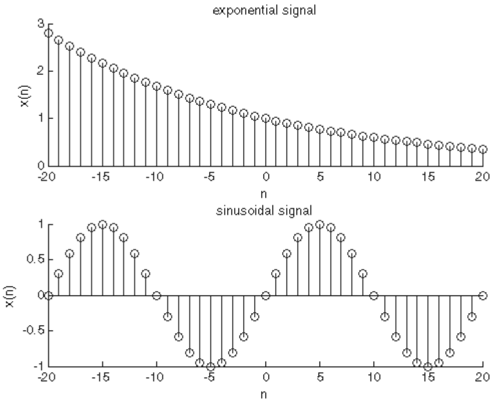


Fig P2.2 : Output waveforms of program 2.2.

Program 2.2

Write a MATLAB program to generate the standard discrete time signals exponential and sinusoidal signals.

```
%****************** program to plot some standard signals

n=-20 : 1 : 20;                %specify the range of n

%****************** exponential signal
A=0.95;
x=A.^n;                        %generate exponential signal
subplot(2,1,1);stem(n,x);      %plot the generated exponential signal
xlabel('n');ylabel('x(n)');title('exponential signal');

%****************** sinusoidal signal
N=20;                          %declare periodicity
f=1/20;                        %compute frequency
x=sin(2*pi*f*n);               %generate sinusoidal signal
subplot(2,1,2);stem(n,x);      %plot the generated sinusoidal signal
xlabel('n');ylabel('x(n)');title('sinusoidal signal');
```

OUTPUT

The output waveforms of program 2.2 are shown in fig P2.2.

Program 2.3

Write a MATLAB program to find the even and odd parts of the signal $x(n)=0.8^n$.

```
%To find the even and odd parts of the signal, x(n)= 0.8^n

n= -5 :1 :5;         %specify the range of n
A=0.8;
x1=A.^n;             %generate the given signal
x2=A.^(-n);          %generate the folded signal
```

```
if(x2==x1)
     disp('"The given signal is even signal"');
else if (x2==(-x1))
     disp('"The given signal is odd signal"');
else
     disp('"The given signal is neither even nor odd signal"');
end
end

xe=(x1+x2)/2;       %compute even part
xo=(x1-x2)/2;       %compute odd part

subplot(2,2,1);stem(n,x1);
xlabel('n');ylabel('x1(n)');title('signal x(n)');

subplot(2,2,2);stem(n,x2);
xlabel('n');ylabel('x2(n)');title('signal x(-n)');

subplot(2,2,3);stem(n,xe);
xlabel('n');ylabel('xe(n)');title('even part of x(n)');

subplot(2,2,4);stem(n,xo);
xlabel('n');ylabel('xo(n)');title('odd part of x(n)');
```

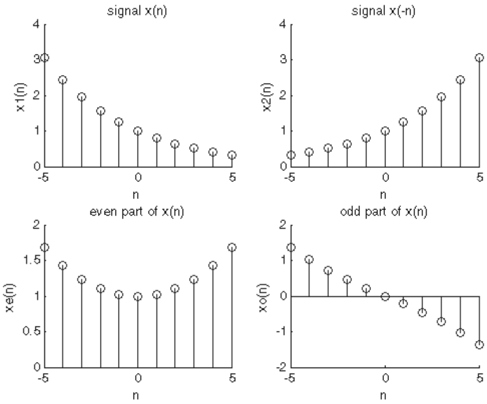


***Fig P2.3** : Output waveforms of program 2.3.*

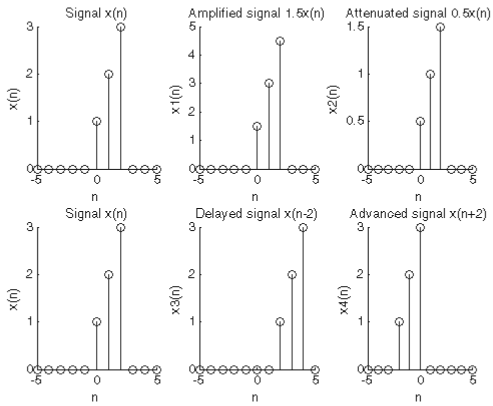


***Fig P2.4** : Output waveforms of program 2.4.*

OUTPUT

"The given signal is neither even nor odd signal"

The output waveforms of program 2.3 are shown in fig P2.3.

Program 2.4

Write a MATLAB program to perform amplitude scaling and time shift on the signal x(n) = 1+n; for n = 0 to 2.

Program to declare the given signal as function y(n)

```
% declare the given signal as function y(n)

function x = y(n)
x=(1.0 + n).*(n>=0 & n<=2);
```

Note: The above program should be stored as a separate file in the current working directory

Program to perform amplitude scaling and time shift on y(n)

```
%To Perform Amplitude scaling and Time shift on signal x(n)=1+n;
%for n= 0 to 2
%include y.m file in current work directory which declare given signal as
%function y(n)

n=-5:1:5;            %specify range of n

y0 =y(n);            %assign the given signal as y0
y1 =1.5*y(n);        %compute the amplified version of x(n)
y2 =0.5*y(n);        %compute the attenuated version of x(n)
y3 =y(n-2);          %compute the delayed version of x(n)
y4 =y(n+2);          %compute the advanced version of x(n)

%plot the given signal and amplitude scaled signal
subplot(2,3,1);stem(n,y0);
xlabel('n');ylabel('x(n)');title('Signal x(n)');
subplot(2,3,2);stem(n,y1);
xlabel('n');ylabel('x1(n)');title('Amplified signal 1.5x(n)');
subplot(2,3,3);stem(n,y2);
xlabel('n');ylabel('x2(n)');title('Attenuated signal 0.5x(n)');

%plot the given signal and time shifted signal
subplot(2,3,4);stem(n,y0);
xlabel('n');ylabel('x(n)');title('Signal x(n)');
subplot(2,3,5);stem(n,y3);
xlabel('n');ylabel('x3(n)');title('Delayed signal x(n-2)');
subplot(2,3,6);stem(n,y4);
xlabel('n');ylabel('x4(n)');title('Advanced signal x(n+2)');
```

OUTPUT

The input and output waveforms of program 2.4 are shown in fig P2.4.

Program 2.5

Write a MATLAB program to perform convolution of the following two discrete time signals.

x1(n)=1; 1<n<10   x2(n)=1;   2<n<10

```
%******************Program to perform convolution of two signals
%*****************x1(N)=1; n= 1 to 10 and x2(n)=1; n= 2 to 10

n = 0 : 1 : 15;             %specify range of n

x1=1.*(n>=1 & n<=10);       %generate signal x1(n)
x2=1.*(n>=2 & n<=10);       %generate signal x2(n)
N1=length(x1);
N2=length(x2);
x3=conv(x1,x2);             %perform convolution of signals x1(n) and x2(n)
n1=0 : 1 : N1+N2-2;         %specify range of n for x3(n)
```

```
subplot(3,1,1);stem(n,x1);
xlabel('n');ylabel('x1(n)');
title('signal  x1(n)');

subplot(3,1,2);stem(n,x2);
xlabel('n');ylabel('x2(n)');
title('signal  x2(n)');

subplot(3,1,3);stem(n1,x3);
xlabel('n');ylabel('x3(n)');
title('signal, x3(n)  =
x1(n)*x2(n)');
```

OUTPUT

The input and output waveforms of program 2.5 are shown in fig P2.5.

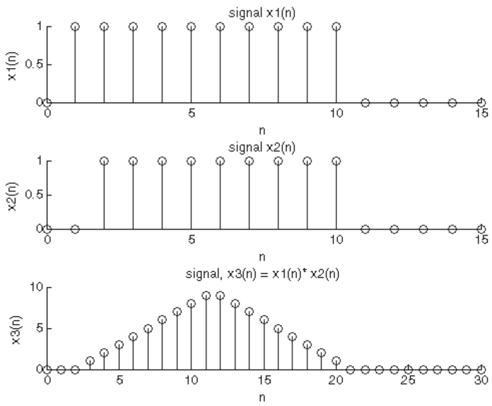


***Fig P2.5** : Output waveforms of program 2.5.*

2.18 Exercises

I. Fill in the blanks with appropriate words

1. A signal x(n) may be shifted in time by m units by replacing the independent variable n by _______.
2. The _______ of a signal x(n) is performed by changing the sign of the time base n.
3. If the average power of a signal is finite then it is called _______.
4. The smallest value of N for which $x(n + N) = x(n)$ is true is called _______.
5. In a discrete time signal x(n), if $x(n) = x(-n)$ then it is called _______ signal.
6. In a discrete time signal x(n), if $x(-n) = -x(n)$ then it is called _______ signal.
7. The output of the system with zero input is called _______.
8. A discrete time system is _______ if it obeys the principle of superposition.
9. A discrete time system is _______ if its input-output relationship do not change with time.
10. The response of an LTI system is given by _______ of input and impulse response.
11. If the output of a system depends only on present input then it is called _______.
12. A system is said to be _______ if the output does not depends on future inputs and outputs.
13. An LTI system is causal if and only if its impulse response is _______ for negative values of n.
14. When a system output at any time n depends on past output values, it is called _______ system.
15. An N-point sequence is called _______ if it is symmetric about point zero on the circle.
16. An N-point sequence is called _______ if it is antisymmetric about point zero on the circle.
17. The _______ is called aperiodic convolution.
18. The _______ is called periodic convolution.
19. Appending zeros to a sequence in order to increase its length is called _______.
20. The two methods of sectioned convolutions are _______ and _______method.

21. In _______ method of sectioned convolution, overlapped samples of output sequences are _______.
22. In _______ method, the overlapped samples in one of the output sequences are discarded.
23. The correlation of two different discrete time sequences is called _______ .
24. The cascade of a system and its inverse is _______.
25. The process of recovering the input from the response of a system is called ______ .

Answers

1. n – m	8. linear	15. even	22. overlap save
2. folding	9. time invariant	16. odd	23. cross correlation
3. power signal	10. convolution	17. linear convolution	24. identity system
4. fundamental period	11. memoryless or static	18. circular convolution	25. deconvolution
5. symmetric	12. causal	19. zero padding	
6. antisymmetric	13. zero	20. overlap add, overlap save	
7. natural response	14. recursive	21. overlap add, added	

II. State whether the following statements are True/False

1. The discrete signals are continuous function of an independent variable.
2. In digital signal the magnitudes of the signal are unquantized.
3. A discrete time signal x(n) is defined for noninteger values of n.
4. An impulse signal has a nonzero sample only for one value of n.
5. When we multiply a discrete time signal by unit step signal, the signal is converted to one-sided signal.
6. Shifting a signal to left is called delay and shifting to right is called advance.
7. Any discrete time signal can be expressed as a summation of impulses.
8. Periodic signals are power signals.
9. When the energy of a signal is infinite, it is called energy signal.
10. The output of a system for impulse input is called impulse response.
11. A system can be realized in real time only if it is noncausal and stable.
12. Dynamic systems does not require memory but static systems require memory.
13. A system is time invariant if the response to a shifted version of the input is identical to a shifted version of the response based on the unshifted input.
14. An LTI system is unstable if the impulse response is absolutely summable.
15. A system whose output depends only on the present and past input is called a recursive system.
16. The circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension.
17. For an N-point sequence represented on a circle, the time reversal is obtained by reversing its sample about the point zero on the circle.
18. When a nonperiodic N-point sequence is represented on a circle then it becomes periodic with periodicity N.
19. In linear convolution the length of the input sequences should be same.
20. In circular convolution the length of the input sequences need not be same.
21. In circular correlation the length of the input and output sequences are same.

22. The correlation operation, $r_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-m)$ is not same as $r_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n+m)\, y(n)$.
23. The cross correlation sequence $r_{xy}(m)$ is folded version of $r_{yx}(m)$.
24. The inverse systems exist for all LTI systems.
25. The final value of m in autocorrelation sequence of N-point sequence is, $m_f = m_i + (2N - 1)$.

Answers

1.	False	6.	False	11.	False	16.	True	21.	True
2.	False	7.	True	12.	False	17.	True	22.	False
3.	False	8.	True	13.	True	18.	True	23.	True
4.	True	9.	False	14.	False	19.	False	24.	True
5.	True	10.	True	15.	False	20.	False	25.	False

III. Choose the right answer for the following questions

1. ***$x(n) = \frac{x(n-1)}{4}$ with initial condition x(0) = –1, gives the sequence,***

   a) $x(n) = \left(\frac{1}{4}\right)^n$ **b)** $x(n) = -\left(\frac{1}{4}\right)^n$ **c)** $x(n) = \left(\frac{1}{4}\right)^{-n}$ **d)** $x(n) = \left(\frac{-1}{4}\right)^{-n}$

2. ***The process of conversion of continuous time signal into discrete time signal is known as,***

   a) aliasing **b)** sampling **c)** convolution **d)** none of the above

3. ***If F_s is sampling frequency then the relation between analog frequency F and digital frequency f is,***

   a) $f = \frac{F}{2F_s}$ **b)** $f = \frac{F_s}{F}$ **c)** $f = \frac{F}{F_s}$ **d)** $f = \frac{2F}{F_s}$

4. ***If F_s is sampling frequency then the highest analog frequency that can be uniquely represented in its sampled version of discrete time signal is,***

   a) $\frac{F_s}{2}$ **b)** $2F_s$ **c)** F_s **d)** $\frac{1}{F_s}$

5. ***The sampling frequency of the following analog signal, x(t) = 4 sin150πt + 2 cos50πt should be,***

   a) greater than 75 Hz **b)** greater than 150 Hz **c)** less than 150 Hz **d)** greater than 50 Hz

6. ***Which of the following signal is the example for deterministic signal?***

   a) step **b)** ramp **c)** exponential **d)** all of the above

7. ***For energy signals, the energy will be finite and the average power will be,***

   a) infinite **b)** finite **c)** zero **d)** cannot be defined

8. ***In a signal x(n), if 'n' is replaced by $\frac{n}{3}$, then it is called,***

   a) upsampling **b)** folded version **c)** downsampling **d)** shifted version

9. ***The unit step signal u(n) delayed by 3 units of time is denoted as,***

   a) $u(n+3) = 1;\ n \geq 3$; $= 0;\ n < 3$ **b)** $u(3-n) = 1;\ n \geq 3$; $= 0;\ n < 3$ **c)** $u(n-3) = 1;\ n \geq 3$; $= 0;\ n < 3$ **d)** $u(3n) = 1;\ n > 3$; $= 0;\ n < 3$

10. The zero input response (or) natural response is mainly due to,

a) Initial stored energy in the system **b)** Initial conditions in the system
c) Specific input signal **d)** both a and b

11. If $x(n) = a^n\, u(n)$ is the input signal, then the particular solution $y_p(n)$ will be,

a) $K^n\, a^n\, u(n)$ **b)** $K\, a^n\, u(n)$
c) $K_1\, a^n\, u(n) + K_2\, a^n\, u(n)$ **d)** $K\, a^{-n}\, u(n)$

12. The discrete time system, $y(n) = x(n–3) – 4x(n–10)$ is a,

a) dynamic system **b)** memoryless system **c)** time varying system **d)** none of the above

13. An LTI discrete time system is causal if and only if,

a) h(n) ¹ 0 for n < 0 **b)** $h(n) = 0$ for $n < 0$ **c)** h(n) ¹ ¥ for n < 0 **d)** h(n) ¹ 0 for n > 0

14. Which of the following system is causal?

a) $h(n) = n\left(\frac{1}{2}\right)^n u(n+1)$ **b)** $y(n) = x^2(n) - x(n+1)$ **c)** $y(n) = x(-n) + x(2n-1)$ **d)** $h(n) = n\left(\frac{1}{2}\right)^n u(n)$

15. An LTI system is stable, if the impulse response is,

a) $\sum_{n=-\infty}^{\infty}|h(n)| = 0$ **b)** $\sum_{n=-\infty}^{\infty}|h(n)| < \infty$ **c)** $\sum_{n=-\infty}^{\infty}|h(n)| \neq 0$ **d)** either a or b

16. The system $y(n) = \sin[x(n)]$ is,

a) stable **b)** BIBO stable **c)** unstable **d)** none of the above

17. Two parallel connected discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system with impulse response,

a) $h_1(n) * h_2(n)$ **b)** $h_1(n) + h_2(n)$ **c)** $h_1(n) - h_2(n)$ **d)** $h_1(n) * [h_1(n) + h_2(n)]$

18. Sectioned convolution is performed if one of the sequence is very much larger than the other in order to overcome,

a) long delay in getting output **b)** larger memory space requirment
c) both a and b **d)** none of the above

19. In overlap save method, the convolution of various sections are performed by,

a) zero padding **b)** linear convolution **c)** circular convolution **d)** both b and c

20. If $x(n)$ is N_1-point sequence,if $y(n)$ is N_2-point sequence, if $r_{xy}(m)$ is the correlation sequence starts at $m = m_i$, then the value of m corresponding to last sample of $r_{xy}(m)$ is,

a) $m_f = m_i + (N_1 + N_2 - 2)$ **b)** $m_f = m_i + (2N - 2)$ **c)** $m_f = m_i + (N_1 + N_2 - 1)$ **d)** $m_f = m_i + (2N + 1)$

21. For a system, $y(n) = nx(n)$, the inverse system will be,

a) $y\left(\frac{1}{n}\right)$ **b)** $\frac{1}{n}\, y(n)$ **c)** $ny(n)$ **d)** $n^{-1}y(n)$

22. For a system $y(n) = x(n–3)$ the impulse response of the system and the inverse system will be ——— and ——— respectively.

a) h(n) = d(n + 3), x(n) = y(n – 3) **b)** $h(n) = \delta(3n),\ x(n) = y\left(\frac{n}{3}\right)$
c) h(n) = d(n – 3), x(n) = y(n + 3) **d)** h(n) = d(n + 3), x(n) = y(3n)

23. The circular correlation $\bar{r}_{x_1x_2}(q)$ of the sequence $x_1(n)$ and $x_2(n)$ of length 'N' can be defined by the equation,

a) $\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-q)$ **b)** $\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n-q)$

c) $\sum_{n=0}^{N-1} x_1(n)\, x_2^*((n-q))_N$ **d)** $\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*((n-q))_N$

24. The evaluation of correlation involves,

a) shifting, rotating and summation **b)** shifting, multiplication and summation

c) change of index, folding and summation **d)** change of index, folding, shifting & multiplication

25. The circular correlation of N-point sequences is evaluated in the range,

a) $-N < m < N$ **b)** $-N < m < 0$ **c)** $0 < m < N$ **d)** $0 < m < N-1$

Answers

1. b	6. d	11. b	16. a	21. b
2. b	7. c	12. a	17. b	22. c
3. c	8. a	13. b	18. c	23. c
4. a	9. c	14. d	19. c	24. b
5. b	10. a	15. d	20. a	25. d

IV. Answer the following questions

1. Define discrete and digital signal.
2. Explain briefly, the various methods of representing discrete time signal with examples.
3. Define sampling and aliasing.
4. What is Nyquist rate?
5. State sampling theorem.
6. Define the impulse and unit step signal.
7. Express the discrete time signal x(n) as a summation of impulses.
8. How will you classify the discrete time signals?
9. What are energy and power signals?
10. When a discrete time signal is called periodic?
11. What is discrete time system?
12. What is impulse response? Explain its significance.
13. Write the difference equation governing the N^{th} order LTI system.
14. Write the expression for discrete convolution.
15. List the various methods of classifying discrete time systems.
16. Define time invariant system.
17. What is linear and nonlinear systems?
18. What is the importance of causality?
19. What is BIBO stability? What is the condition to be satisfied for stability?
20. What are FIR and IIR systems?
21. Write the convolution sum formula for FIR and IIR systems.
22. What are recursive and nonrecursive systems? Give examples.

23. Write the properties of linear convolution.
24. Prove the distributive property of linear convolution.
25. What are the two ways of interconnecting LTI systems?
26. Define circular convolution.
27. What is the importance of linear and circular convolution in signals and systems?
28. How will you perform linear convolution via circular convolution?
29. What is sectioned convolution? Why is it performed?
30. What are the two methods of sectioned convolution?
31. What is inverse system? What is its importance?
32. Define deconvolution.
33. Define cross correlation and autocorrelation?
34. What are the properties of correlation?
35. What is circular correlation?

V. Solve the following problems

E2.1 Determine whether the following signals are periodic or not. If periodic, find the fundamental period.

a) $x(n) = \sin\left(\frac{5\pi}{8}n + 6\right)$ **b)** $x(n) = \sin\left(\frac{7n}{3} + \pi\right)$ **c)** $x(n) = \cos\left(\frac{4\pi n}{12}\right)$

d) $x(n) = \cos\left(\frac{\pi}{32}n^2\right)$ **e)** $x(n) = e^{j9n}$ **f)** $x(n) = 4\sin\frac{3\pi n}{2} + 5\cos\frac{3\pi n}{4}$

E2.2 Determine the even and odd parts of the signals.

a) $x(n) = \frac{1}{a^{2n}}$ **b)** $x(n) = 8e^{-j\frac{\pi}{6}n}$ **c)** $x(n) = \{6, \quad \underset{\uparrow}{4}, \quad 2, \quad 2\}$

E2.3 a) Consider the analog signal $x(t) = 2 \sin 80pt$. If the sampling frequency is 60 Hz, find the sampled version of discrete time signal x(n). Also find an alias frequency corresponding to F_s = 60 Hz.

b) Consider the analog signals, $x_1(t) = 4\cos 2\pi(30t)$ and $x_2(t) = 4\cos 2\pi(5t)$. Find a sampling frequency so that 30 Hz signal is an alias of 5 Hz signal.

c) Consider the analog signal, $x(t) = 3\sin 40\pi t - \sin 100\pi t + 2\cos 50\pi t$. Determine the minimum sampling frequency and the sampled version of analog signal at this frequency. Sketch the waveform and show the sampling points. Comment on the result.

E2.4 Determine whether the following signals are energy or power signals.

a) $x(n) = \left(\frac{5}{9}\right)^n u(n)$ **b)** $x(n) = \left(\cos\frac{3\pi}{4}n\right)$ **c)** $x(n) = u(2n)$ **d)** $x(n) = 2\,u(3 - n)$

E2.5 Construct the block diagram and signal flow graph of the discrete time systems whose input-output relations are described by the following difference equations.

a) $y(n) = 2y(n-1) + 2.1\,x(n-1) + 0.5\,x(n-2)$

b) $y(n) = 1.6\,x(n-2) + 0.7\,x(n) + 3y(n-1) + 0.3y(n-2)$

E2.6 Determine the response of the discrete time systems governed by the following difference equations.

a) $y(n) = 0.1y(n-1) + x(n-1) + 0.7x(n)$; $x(n) = 2^{-n}u(n)$; $y(-1) = -1$

b) $y(n) + 2.1y(n-1) + 0.2y(n-2) = x(n) + 0.56x(n-1)$; $x(n) = u(n)$; $y(-2) = 1$; $y(-1) = -3$

E2.7 ***Test the following systems for time invariance.***

a) $y(n) = x(n+1) + x(n+2)$ **b)** $y(n) = na^{x(n)}$ **c)** $y(n) = x^2(n+2) + C$ **d)** $y(n) = (n-1)\,x^2(n) + C$

E2.8 ***Test the following systems for linearity.***

a) $y(n) = x^2(n) + x^3(n-1)$ **b)** $y(n) = bx(n+2) + ne^{x(n)}$ **c)** $y(n) = a\sqrt{x(n)} + b\,x(n)$

d) $y(n) = \sqrt{x(n)} + \dfrac{1}{\sqrt{x(n)}}$ **e)** $y(n) = \sum\limits_{m=-1}^{N} b_m\, x(n+m) + \sum\limits_{m=0}^{M} c_m\, y(n+m)$

E2.9 ***Test the causality of the following systems.***

a) $y(n) = x(n) - x(-n-2) + x(n-1)$ **b)** $y(n) = a\,x(2n) + x(n^2)$

c) $y(n) = \sum\limits_{m=-1}^{n} x(m) + \sum\limits_{m=-\infty}^{n} x(2m)$ **d)** $y(n) = (0.3)^n\, u(n+2)$ **e)** $y(n) = \sum\limits_{k=-4}^{4} x(n-k)$

E2.10 ***Test the stability of the following discrete time systems.***

a) $y(n) = x^2(n) + x(n+1)$ **b)** $y(n) = nx(n-1)$ **c)** $h(n) = (0.4)^n\, u(n+3)$

d) $h(n) = (8)^n\, u(4-n)$ **e)** $y(n) = x(n-3)$

E2.11 ***Determine the range of values of 'a' and 'b' for the stability of an LTI system with impulse response,***

$$h(n) = \begin{cases} (-4a)^n & ; \quad n \ge 0 \\ 2b^{-n} & ; \quad n < 0 \end{cases}$$

E2.12 ***a) Determine the impulse response for the cascade of two LTI systems having impulse responses,***

$$h_1(n) = \left(\frac{1}{7}\right)^n u(n) \quad \text{and} \quad h_2(n) = d(n-3)$$

b) Determine the overall impulse response of the interconnected discrete time system shown in fig E2.12.

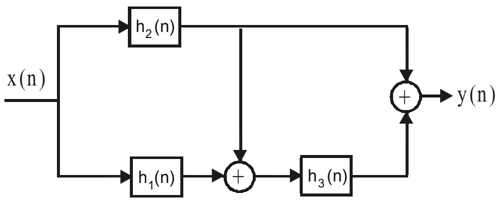


Fig E2.12.

Take, $h_1(n) = \left(\frac{1}{3}\right)^n u(n)\ ;\ h_2(n) = \left(\frac{1}{6}\right)^n u(n)\ ;\ h_3(n) = \left(\frac{1}{9}\right)^n u(n)$

E2.13 ***Determine the response of an LTI system whose impulse response h(n) and input x(n) are given by,***

a) $h(n) = \{1,\ 4,\ \underset{\uparrow}{1},\ -2,\ 1\}$, $x(n) = \{1,\ \underset{\uparrow}{3},\ 5,\ -1,\ -2\}$

b) $h(n) = \begin{cases} 1 & ; \quad 0 \le n \le 2 \\ 0 & ; \quad n \ge 3 \end{cases}$, $x(n) = a^n u(n);\quad |a| < 1$

E2.14 ***Perform circular convolution of the two sequences,***

a) $x_1(n) = \{1,\ 2,\ -1,\ 1\}$; $x_2(n) = \{2,\ 4,\ 6,\ 8\}$

b) $x_1(n) = \{0,\ 0.6,\ -1,\ 1.5,\ 2\}$; $x_2(n) = \{-2,\ 3,\ 0.2,\ 0.7,\ 0.8\}$

E2.15 ***The input x(n) and impulse response h(n) of an LTI system are given by,***

$x(n) = \{-1,\ \underset{\uparrow}{1},\ -1,\ 1,\ -1,\ 1\}$; $h(n) = \{-0.5,\ \underset{\uparrow}{0.5},\ -1,\ 0.5,\ -1,\ -2\}$

Find the response of the system using **a)** Linear convolution, **b)** Circular convolution.

E2.16 ***Perform linear convolution of the following sequences by,***

a) Overlap add method **b)** Overlap save method

$x(n) = \{1,\ -1,\ 2,\ 1,\ -1,\ 2,\ +1,\ -1,\ +2\}$; $h(n) = \{2,\ 3,\ -1\}$

E2.17 Perform crosscorrelation of the sequences,

$$x(n) = \{-1, \ \underset{\uparrow}{2} \ \ 3, \ -4,\} \ ; \qquad h(n) = \{2, \ \underset{\uparrow}{-1}, \ -3,\}$$

E2.18 Determine the autocorrelation sequence for $x(n) = \{1, \ \underset{\uparrow}{4}, \ 3, \ -5, \ 2\}$.

E2.19 Find the inverse system for the following discrete time system,

$$y(n) = \sum_{p=0}^{n} c^p \, x(p-2) \ ; \quad for \quad n \geq 0$$

E2.20 A discrete time system is excited by an input x(n), and the response is, $y(n) = \{\underset{\uparrow}{4}, \ 3, \ 6, \ 7.5, \ 3, \ 30, \ -8\}$.

If the impulse response of the system is $h(n) = \{2, \ \underset{\uparrow}{4}, \ -2\}$***, then what will be the input to the system?***

E2.21 Perform circular correlation of the sequence, $x(n) = \{-1, \ 1, \ 2, \ 6\}$ ***and*** $y(n) = \{4, \ -2, \ -1, \ 2\}$.

Answers

E2.1 **a)** periodic; N=16 **b)** nonperiodic **c)** periodic; N=6 **d)** periodic; N=32 **e)** nonperiodic. **f)** periodic; N=8

E2.2 **a)** $x_e(n) = \frac{1}{2}\left[a^{-2n} + a^{2n}\right]$ **b)** $x_e(n) = 8\cos\frac{\pi}{6}n$ **c)** $x_e(n) = \{1, \ 1, \ 2, \ \underset{\uparrow}{6}, \ 2, \ 1, \ 1\}$

$x_o(n) = \frac{1}{2}\left[a^{-2n} - a^{2n}\right]$ $\qquad x_o(n) = -j8\sin\frac{\pi}{6}n$ $\qquad x_o(n) = \{-1, \ -1, \ -2, \ \underset{\uparrow}{0}, \ 2, \ 1, \ 1\}$

E2.3 **a)** $x(n) = 2\sin\frac{4\pi n}{3}$; Alias frequency $= 100\,\text{Hz}$ **b)** $F_s = 25\,\text{Hz}$

c) $F_{s,min} = 100\,\text{Hz}$; $x(nT) = 3\sin\frac{2\pi n}{5} + 2\cos\frac{\pi n}{2}$ $(\sin \pi n = 0, \text{for integer n})$

The component sin100pt will give always zero samples when sampled at 100Hz for any value of n (Refer fig E2.3c).

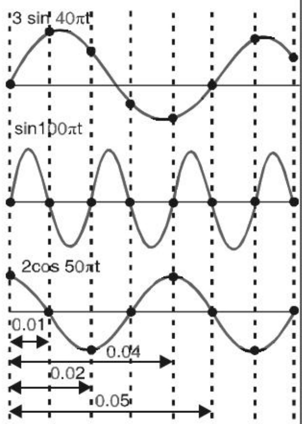


Fig E2.3c : Sampling points.

E2.4 **a)** E = 1.435J ; P = 0 ; Energy signal.

b) E = ¥ ; P = 0.5W ; Power signal.

c) E = ¥ ; P = 0.25W ; Power signal.

d) E = ¥ ; P = 2 W ; Power signal.

E2.5 **a)**

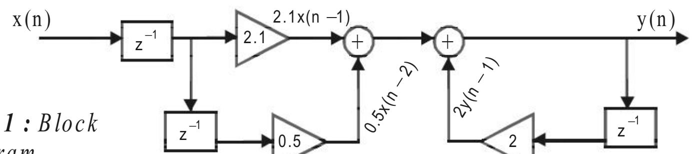


Fig E2.5a.1 : Block diagram.

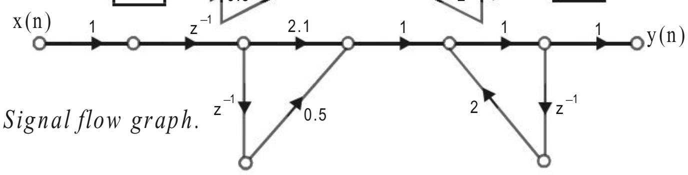


Fig E2.5a.2 : Signal flow graph.

E2.5 b)

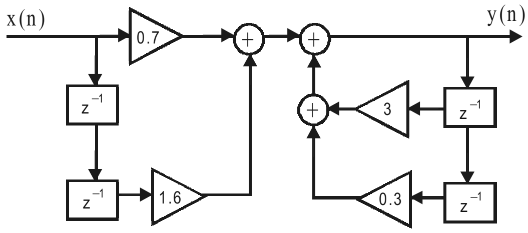


Fig E2.5b.1 : Block diagram.

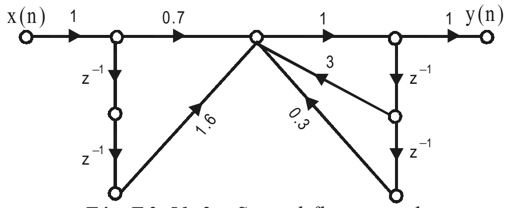


Fig E2.5b.2 : Signal flow graph.

E2.6 **a)** $y(n) = \left[-2.775(0.1)^n + 3.375\left(\frac{1}{2}\right)^n\right]u(n)$ **b)** $y(n) = \left[0.47 - 0.02\,(-0.1)^n + 6.65\,(-2)^n\right]u(n)$

E2.7 **a)** **c)** Time invariant **b)** **d)** Time variant

E2.8 **a)** **e)** Linear **b)** **c)** **d)** Nonlinear

E2.9 **a)** **b)** **c)** **d)** **e)** Noncausal

E2.10 **a)** **c)** **d)** **e)** Stable system **b)** Unstable system

E2.11 For stability, $0 < |a| < \frac{1}{4}$ and $0 < |b| < \frac{1}{2}$

E2.12 **a)** $h(n) = \left(\frac{1}{7}\right)^{(n-3)} u(n-3)$ **b)** $h(n) = \left[4\left(\frac{1}{6}\right)^n - \frac{3}{2}\left(\frac{1}{9}\right)^n + \left(\frac{3}{2}\right)\left(\frac{1}{3}\right)^n\right]u(n)$

E2.13 **a)** $y(n) = \{1,\ 7,\ 18,\ \underset{\uparrow}{20},\ -6,\ -16,\ 5,\ 3,\ -2\}$ **b)** $y(n) = \sum_{k=0}^{n} a^k$; for n = 0, 1, 2

$= \sum_{k=n-2}^{n} a^k$; for n > 2

E2.14 **a)** $x_3(n) = \{\underset{\uparrow}{8},\ -6,\ 4,\ 14\}$ **b)** $x_3(n) = \{6.08,\ \underset{\uparrow}{-0.55},\ 6.4,\ -4.28,\ 0.72\}$

E2.15 $y(n) = \{0.5,\ \underset{\uparrow}{-1},\ 2,\ -2.5,\ 3.5,\ -1.5,\ 1,\ -0.5,\ -0.5,\ 1,\ -2\}$

E2.16 **a)** Overlap add method : $y(n) = \{2,\ 1,\ 0,\ 9,\ -1,\ 0,\ 9,\ -1,\ 0,\ 7,\ -2\}$

b) Overlap save method : $y(n) = \{*,\ *,\ 0,\ 9,\ -1,\ 0,\ 9,\ -1,\ 0,\ 7,\ -2\}$

E2.17 $r_{xy}(m) = \{3,\ \underset{\uparrow}{-5},\ -13,\ 13,\ 10,\ -8\}$

E2.18 $r_{xx}(m) = \{2,\ 3,\ -11,\ -9,\ \underset{\uparrow}{55},\ -9,\ -11,\ 3,\ 2\}$

E2.19 $x(n) = \frac{1}{c^{n+2}}\,[y(n+2) - y(n+1)]$; for $n \ge -1$

with initial condition $x(-2) = y(0)$

E2.20 $x(n) = \{\underset{\uparrow}{2},\ -2.5,\ 10,\ -18.75,\ 49\}$

E2.21 $r_{xy}(m) = \{4,\ -8,\ -1,\ 29\}$

Solution for Exercise Problems

E2.1. *Determine whether the following signals are periodic or not. If periodic, find the fundamental period.*

a) $x(n) = \sin\left(\frac{5\pi}{8}n + 6\right)$

Solution

Now, $x(n+N) = \sin\left(\frac{5\pi}{8}(n+N)+6\right) = \sin\left(\frac{5\pi n}{8} + 6 + \frac{5\pi N}{8}\right)$

Since sin (q + 2pM) = sin q, for periodicity, $\frac{5\pi N}{8}$ should be integral multiple of 2p.

Let, $\frac{5\pi N}{8} = M \times 2\pi$, M and N are integers.

$$\therefore \quad N = M \times 2\pi \times \frac{8}{5\pi} = \frac{16}{5}M$$

$$N = \frac{16}{5}M, \quad \text{if } M = 5, 10, 15, 20 \text{ ..... N will be a integer.}$$

When $M = 5$, $N = 16$.

$$x(n+N) = \sin\left(\frac{5\pi}{8}n + 6 + \frac{5\pi}{8} \times 16\right) = \sin\left(\frac{5\pi}{8}n + 6 + 10\pi\right) = \sin\left(\frac{5\pi}{8}n + 6\right) = x(n).$$

\ x(n) is periodic.

Fundamental period is 16 samples.

---

b) $x(n) = \sin\left(\frac{7n}{3} + \pi\right)$

Solution

$$x(n+N) = \sin\left(\frac{7(n+N)}{3} + \pi\right) = \cos\left(\frac{7n}{3} + \pi + \frac{7N}{3}\right)$$

Since cos (q + 2pM) = cos q, for periodicity, $\frac{7N}{3}$ should be equal to integral multiple of 2p.

Let, $\frac{7N}{3} = M \times 2\pi \Rightarrow N = \frac{2\pi \times 3}{7} M = \frac{6\pi}{7}M$

Here, N cannot be an integer for any integer value of M, and so, x(n) will not be periodic.

---

c) $x(n) = \cos\left(\frac{4\pi n}{12}\right)$

Solution

$$x(n) = \cos\left(\frac{\pi}{3}n\right)$$

$$x(n+N) = \cos\left(\frac{\pi}{3}(n+N)\right) = \cos\left(\frac{n\pi}{3} + \frac{N\pi}{3}\right)$$

$$\frac{N\pi}{3} = 2\pi M \Rightarrow N = 6M$$

For M = 1, 2, 3, ... N will be integer.

For M = 1, N = 6.

\ x(n) is periodic.

Fundamental period is 6 samples.

d) x(n) = cos $\left(\frac{\pi}{32} n^2\right)$

Solution

Given that, $x(n) = \cos\left(\frac{\pi}{32} n^2\right)$

$$\therefore\; x(n+N) = \cos\frac{\pi}{32}(n+N)^2$$

$$= \cos\frac{\pi}{32}(n^2 + N^2 + 2nN)$$

$$= \cos\left(\frac{\pi}{32}n^2 + \frac{\pi N^2}{32} + \frac{\pi N}{16}n\right)$$

Let, $\frac{\pi N^2}{32} = 2\pi M_1$

$$\therefore\; N = 8\sqrt{M_1}$$

Now, N is integer for $M_1 = 1^2, 2^2, 3^2, 4^2$ .....

Let, $\frac{\pi N}{16} = 2\pi M_2$

\ $N = 32\, M_2$

Now, N is integer for $M_2 = 1, 2, 3, 4$ .....

When $M_1 = 4^2$ and $M_2 = 1$, we get a common value for N as, N = 32.

When N = 32 ; $x(n + N) = \cos\left(\frac{\pi}{32}n^2 + \frac{\pi 32^2}{32} + \frac{\pi 32}{16}n\right)$

$$= \cos\left(\left(\frac{\pi}{32}n^2 + 2\pi n\right) + 16 \times 2\pi\right)$$

$$= \cos\left(\frac{\pi}{32}n^2 + 2\pi n\right)$$

$$= \cos\frac{\pi}{32}n^2 = x(n)$$

For integer M, cos(q + 2pM) = cosq

\ x(n) is periodic with fundamental period, N = 32 samples.

---

e) x(n) = e^{j9n}

Solution

$$x(n+N) = e^{j9(n+N)} = e^{j9n}\,.\,e^{j9N}$$

Since, $e^{j2\pi M} = 1$

Let, $(9N) = M \times 2\pi \;\Rightarrow\; N = \frac{2\pi}{9}M$

For any integer value of M, N will not be an integer.

Hence x(n) is non periodic.

---

f) Given that, x(n) = 4 sin $\frac{3\pi n}{2}$ + 5 cos $\frac{3\pi n}{4}$

Solution

Let, $x_1(n) = 4\sin\frac{3\pi n}{2}$

$$\therefore\; x_1(n+N_1) = 4\sin\frac{3\pi(n+N_1)}{2}$$

$$= 4\sin\left(\frac{3\pi n}{2} + \frac{3\pi N_1}{2}\right) \qquad .....(1)$$

Let, $\frac{3\pi N_1}{2} = 2\pi M_1 \;\Rightarrow\; N_1 = \frac{4}{3}M_1$

Let, $M_1 = 3$; \ $N_1 = 4$

Let, $x_2(n) = 5\cos\frac{3\pi n}{4}$

$$\therefore\; x_2(n+N_2) = 5\cos\frac{3\pi(n+N_2)}{4}$$

$$= 5\cos\left(\frac{3\pi n}{4} + \frac{3\pi N_2}{4}\right) \qquad .....(2)$$

Let, $\frac{3\pi N_2}{4} = 2\pi M_2 \;\Rightarrow\; N_2 = \frac{8}{3}M_2$

Let, $M_2 = 3$; \ $N_2 = 8$

substitute $N_1 = 4$ in equation (1),

$$\therefore\ x_1(n+N_1) = 4\sin\left(\frac{3\pi n}{2}+\frac{3\pi}{2}\times 4\right)$$

$$= 4\sin\left(\frac{3\pi n}{2}+3\times 2\pi\right)$$

For integer M, $\sin(\theta + 2\pi M) = \sin\theta$

$$= 4\sin\frac{3\pi n}{2} = x_1(n)$$

\ $x_1(n)$ is periodic with fundamental period, $N_1 = 4$ samples.

substitute $N_2 = 8$ in equation (2),

$$\therefore\ x_2(n+N_2) = 5\cos\left(\frac{3\pi n}{4}+\frac{3\pi}{4}\times 8\right)$$

$$= 5\cos\left(\frac{3\pi n}{4}+3\times 2\pi\right)$$

For integer M, $\cos(\theta + 2\pi M) = \cos\theta$

$$= 5\cos\frac{3\pi n}{4} = x_2(n)$$

\ $x_2(n)$ is periodic with fundamental period, $N_2 = 8$ samples.

Here, $x(n) = x_1(n) + x_2(n)$, and x(n) is periodic with period $N_1 = 4$, and $x_2(n)$ is periodic with period $N_2 = 8$.

Therefore, x(n) is periodic with period N, where N is LCM of N_1 and N_2.

The LCM of 4 and 8 is 8.

\ x(n) is periodic with fundamental period, N = 8.

E2.2. Determine the even and odd parts of the signals.

a) $x(n) = \dfrac{1}{a^{2n}} \quad \Rightarrow \quad x(n) = a^{-2n}$

Solution

$$x(-n) = \frac{1}{a^{-2n}} \quad \Rightarrow \quad x(-n) = a^{2n}$$

Even part of the signal,

$$x_e(n) = \frac{1}{2}\left[x(n)+x(-n)\right] = \frac{1}{2}\left[a^{-2n}+a^{2n}\right]$$

Odd part of the signal is,

$$x_0(n) = \frac{1}{2}\left[x(n)-x(-n)\right] = \frac{1}{2}\left[a^{-2n}-a^{2n}\right]$$

b) $x(n) = 8\, e^{-j\frac{\pi}{6}n}$

Solution

$$x(n) = 8\, e^{-j\frac{\pi}{6}n} = 8\left[\cos\frac{\pi}{6}n - j\sin\frac{\pi}{6}n\right]$$

$$x(-n) = 8\, e^{-j\frac{\pi}{6}(-n)} = 8\left[\cos\frac{\pi}{6}n + j\sin\frac{\pi}{6}n\right]$$

$$x_e(n) = \frac{1}{2}\times 8\left[\cos\frac{\pi}{6}n - j\sin\frac{\pi}{6}n + \cos\frac{\pi}{6}n + j\sin\frac{\pi}{6}n\right] = 8\cos\frac{\pi}{6}n$$

$$x_0(n) = \frac{1}{2}\times 8\left[\cos\frac{\pi}{6}n - j\sin\frac{\pi}{6}n - \cos\frac{\pi}{6}n - j\sin\frac{\pi}{6}n\right] = -j8\sin\frac{\pi}{6}n$$

c) $x(n) = \{\underset{\uparrow}{6},\ 4,\ 2,\ 2\}$

Solution

Given that, $x(n) = \{\underset{\uparrow}{6},\ 4,\ 2,\ 2\}$

$$x(0) = 6,\ \ x(1) = 4,\ \ x(2) = 2,\ \ x(3) = 2$$

$$x(-n) = \{2,\ 2,\ 4,\ \underset{\uparrow}{6}\}$$

$$x(0) = 6;\ \ x(-1) = 4;\ \ x(-2) = 2;\ \ x(-3) = 2$$

Even part, $x_e(n) = \frac{1}{2}(x(n) + x(-n))$	**Odd part,** $x_0(n) = \frac{1}{2}(x(n) - x(-n))$
at $n = -3$; $x(n) + x(-n) = 0 + 2 = 2$	$n = -3$; $x(n) - x(-n) = 0 - 2 = -2$
$n = -2$; $0 + 2 = 2$	$n = -2$; $= 0 - 2 = -2$
$n = -1$; $0 + 4 = 4$	$n = -1$; $= 0 - 4 = -4$
$n = 0$; $6 + 6 = 12$	$n = 0$; $= 6 - 6 = 0$
$n = 1$; $4 + 0 = 4$	$n = 1$; $= 4 - 0 = 4$
$n = 2$; $2 + 0 = 2$	$n = 2$; $= 2 - 0 = 2$
$n = 3$; $2 + 0 = 2$	$n = 3$; $= 2 - 0 = 2$
$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$	$x_0(n) = \frac{1}{2}[x(n) - x(-n)]$
$x_e(n) = \{1, \ 1, \ 2, \ \underset{\uparrow}{6}, \ 2, \ 1, \ 1\}$	$x_0(n) = \{-1, \ -1, \ -2, \ \underset{\uparrow}{0}, \ 2, \ 1, \ 1\}$

E2.3. a) ***Consider the analog signal x(t) = 2sin80pt. If the sampling frequency is 60 Hz, find the sampled version of discrete time signal x(n). Also find an alias frequency corresponding to F_s = 60 Hz.***

Solution

$$x(n) = x(t)\Big|_{t=nT=\frac{n}{F_s}}$$

$$\therefore \ x(n) = 2\sin 80\pi t\Big|_{t=\frac{n}{60}} = 2\sin 80\pi \times \frac{n}{60} = 2\sin\left(\frac{4\pi n}{3}\right)$$

Now, $2\sin\left(\frac{4\pi}{3}n + 2\pi n\right) = 2\sin\left(\frac{10\pi n}{3}\right)$

The frequency of $2\sin\left(\frac{10\pi n}{3}\right)$ is, $f = \frac{5}{3}$

Also, $f = \frac{F}{F_s} \quad \Rightarrow \quad F = fF_s = \frac{5}{3} \times 60 = 100$

Hence for F_s = 60 Hz, F = 100 Hz is an alias frequency.

b) ***Consider the analog signals $x_1(t)$ = 4cos2p(30t), $x_2(t)$ = 4cos2p(5t). Find a sampling frequency so that 30 Hz signal is an alias of 5 Hz signal.***

Solution

Let the sampling frequency be, $F_s = 30 - 5 = 25$ Hz

$$\therefore \ x_1(n) = x_1(t)\Big|_{t=nT=\frac{n}{F_s}} = 4\cos 2\pi\left(30 \times \frac{n}{25}\right)$$

$$\therefore \ x_1(n) = 4\cos\frac{12\pi}{5}n = 4\cos\left(2\pi n + \frac{2\pi n}{5}\right) = 4\cos\frac{2\pi}{5}n$$

$$x_2(n) = x_2(t)\Big|_{t=nT=\frac{n}{F_s}} = 4\cos 2\pi\left(5 \times \frac{n}{25}\right) = 4\cos\frac{2\pi}{5}n$$

c) ***Consider the analog signal, x(t) = 3sin40pt – sin100pt + 2cos50pt***

Determine the minimum sampling frequency and the sampled version of analog signal at this frequency. Sketch the waveform and show the sampling points.Comment on the result.

Solution

$$x(t) = 3\sin 40\pi t - \sin 100\pi t + 2\cos 50\pi t \ \equiv \ x(t) = 3\sin 2\pi F_1 t - \sin 2\pi F_2 t + 2\cos 2\pi F_3 t$$

$$\therefore \ F_1 = \frac{40}{2} = 20 \text{ Hz} \ ; \quad F_2 = \frac{100}{2} = 50 \text{ Hz} \ ; \quad F_3 = \frac{50}{2} = 25 \text{ Hz}$$

The maximum analog frequency in the signal is 50 Hz.

The minimum sampling frequency should be twice that of this maximum analog frequency.

$$F_s \ \geq\ 2\,F_{max} \ \Rightarrow\ F_s \ \geq\ 2\times 50$$

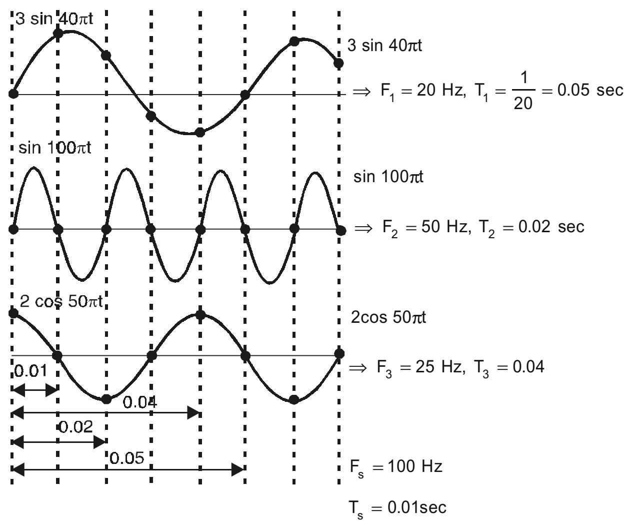
Let, $F_s = 100\,\text{Hz}$

$$\therefore\ x(nT) = x(t)\Big|_{t=nT=\frac{n}{F_s}}$$

$$x(nT) = 3\sin 40\pi\times\frac{n}{100} - \sin 100\pi\frac{n}{100} + 2\cos 50\pi\times\frac{n}{100} = 3\sin\frac{2\pi n}{5} - \sin\pi n + 2\cos\frac{\pi}{2}n$$

$\sin \pi n = 0$, for integer values of n.

$$\therefore\ x(nT) = 3\sin\frac{2\pi n}{5} + 2\cos\frac{\pi n}{2}$$



In the analog signal x(nT), the component $\sin 100\pi t$ will give always zero samples when sampled at 100Hz for any value of n. This is the drawback in sampling at nyquist rate, which is $F_s = 2\,F_{max}$.

E2.4. *Determine whether the following signals are energy or power signals.*

a) $x(n) = \left(\frac{5}{9}\right)^n u(n)$

Solution

$$x(n) = \left(\frac{5}{9}\right)^n u(n) \quad \text{for all n.}$$

$$\therefore\ x(n) = (0.55)^n \ ;\ n \geq 0$$

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty}|x(n)|^2 = \sum_{n=0}^{\infty}\left|(0.55)^n\right|^2 = \sum_{n=0}^{\infty}\left((0.55)^2\right)^n = \sum_{n=0}^{\infty}(0.302)^n$$

$$\therefore\quad E = \sum_{n=0}^{\infty}(0.302)^n = \frac{1}{1-0.302} = 1.43 \text{ Joules}$$

$$\text{Power, } P = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=0}^{N}\left((0.55)^2\right)^n = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=0}^{N}(0.302)^n$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\ \frac{(0.302)^{N+1}-1}{0.302-1} = \frac{1}{\infty}\times\frac{0-1}{-0.698} = 0$$

P is zero and E is finite.

So x(n) is energy signal.

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

b) $x(n) = \left|\cos\frac{3\pi}{4}n\right|$

Solution

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=-\infty}^{+\infty} \left|\cos\frac{3\pi}{4}n\right|^2 = \sum_{n=-\infty}^{+\infty}\left(\frac{1+\cos 2\times\frac{3\pi}{4}n}{2}\right)$$

$$= \left(\frac{1}{2}\right)\sum_{n=-\infty}^{+\infty}\left(1+\cos\frac{3\pi}{2}n\right) = \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty}1^n + \sum_{n=-\infty}^{+\infty}\cos\frac{3\pi}{2}n\right) = \frac{1}{2}(\infty+0) = \infty$$

$$\text{Power, } P = \underset{N\to\infty}{Lt}\frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2 = \underset{N\to\infty}{Lt}\frac{1}{2N+1}\sum_{n=-N}^{N}\cos^2\left(\frac{3\pi}{4}n\right)$$

$$= \underset{N\to\infty}{Lt}\frac{1}{2N+1}\sum_{n=-N}^{N}\frac{\left(1+\cos 2\times\frac{3\pi}{4}n\right)}{2} = \underset{N\to\infty}{Lt}\frac{1}{2N+1}\frac{1}{2}\left[\sum_{n=-N}^{+N}1^n + \sum_{n=-N}^{+N}\cos\frac{3\pi}{2}n\right]$$

$$= \underset{N\to\infty}{Lt}\frac{1}{2N+1}\frac{1}{2}\left[\underbrace{1+1+1\,.....\,1}_{N\text{ terms}} + 1 + \underbrace{1+1+1\,.....\,+1}_{N\text{ terms}} + 0\right]$$

$$= \underset{N\to\infty}{Lt}\frac{1}{2N+1}\times\frac{1}{2}[2N+1] = \frac{1}{2} = 0.5$$

Since P is finite and E is infinite, x(n) is power signal.

It can be shown that $\cos\frac{3\pi}{2}n$ is periodic and sum of samples of one period of periodic cosine signal is zero.

$$\cos\frac{3\pi}{2}(n+N) = \cos\left(\frac{3\pi n}{2}+\frac{3\pi N}{2}\right)$$

Let, $\frac{3\pi N}{2} = 2\pi M$

$\therefore\ N = \frac{4M}{3}$

Let, $M = 3$, Now, $N = 4$

$\therefore\ \cos\frac{3\pi}{2}n$ is periodic with period 4 samples.

$n=0;\ \cos\frac{3\pi}{2}n = 1$	$n=4;\ \cos\frac{3\pi}{2}n = 1$
$n=1;\ \cos\frac{3\pi}{2}n = 0$	$n=5;\ \cos\frac{3\pi}{2}n = 0$
$n=2;\ \cos\frac{3\pi}{2}n = -1$	$n=6;\ \cos\frac{3\pi}{2}n = -1$
$n=3;\ \cos\frac{3\pi}{2}n = 0$	$n=7;\ \cos\frac{3\pi}{2}n = 0$

c) $x(n) = u(2n)$

Solution

$$E = \sum_{n=-\infty}^{+\infty}|x(n)|^2 = \sum_{n=0}^{\infty}u(2n)^2 = \sum_{n=\text{even}}u(n) = 1+1+1+1\ .....\ \infty = \infty$$

$$P = \underset{N\to\infty}{Lt}\frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2 = \underset{N\to\infty}{Lt}\frac{1}{2N+1}\sum_{n=0}^{N}u(2n)^2 = \underset{N\to\infty}{Lt}\frac{1}{2N+1} = [\underbrace{1+1+......+1}_{1+\frac{N}{2}\text{ terms}}]$$

$$= \underset{N\to\infty}{Lt}\frac{1}{(2N+1)}\left(1+\frac{N}{2}\right) = \underset{N\to\infty}{Lt}\frac{N\left(\frac{1}{N}+\frac{1}{2}\right)}{N\left(2+\frac{1}{N}\right)} = \frac{\frac{1}{2}+\frac{1}{\infty}}{2+\frac{1}{\infty}} = \frac{\frac{1}{2}}{2} = \frac{1}{4}.$$

Since P is finite, E is infinite, x(n) is power signal.

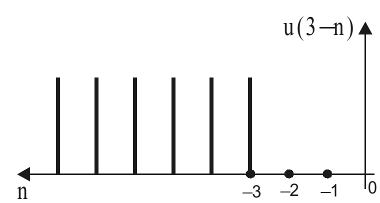
d) $x(n) = 2\,u(3-n)$

Solution

$$E = \sum_{n=-\infty}^{+\infty}|x(n)|^2 = \sum_{n=-\infty}^{-3}(2\,u(3-n))^2 = \sum_{n=-\infty}^{-3}4\,[\underbrace{.......1+1+1}_{\text{infinite terms}}] = \infty$$

$$P = \underset{N\to\infty}{Lt}\frac{1}{(2N+1)}\sum_{n=-N}^{+N}|x(n)|^2 = \underset{N\to\infty}{Lt}\frac{1}{(2N+1)}\sum_{n=N}^{-3}4\,u(3-n)$$

$$= \underset{N\to\infty}{Lt}\frac{1}{(2N+1)}4\,[\underbrace{1+1+1......+1}_{N-2\text{ terms}}] = \underset{N\to\infty}{Lt}\frac{4}{(2N+1)}[N-2]$$



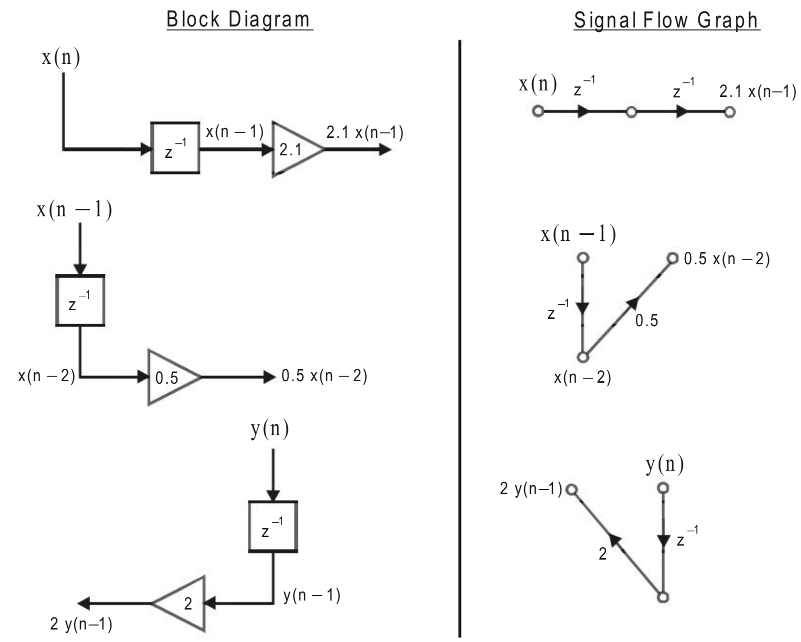
$$\therefore \quad P = \underset{N\to\infty}{Lt} \frac{N \times 4\left(1-\frac{2}{N}\right)}{N\left(2+\frac{1}{N}\right)} = \frac{4\left(1-\frac{2}{\infty}\right)}{2+\frac{1}{\infty}} = \frac{4}{2} = 2$$

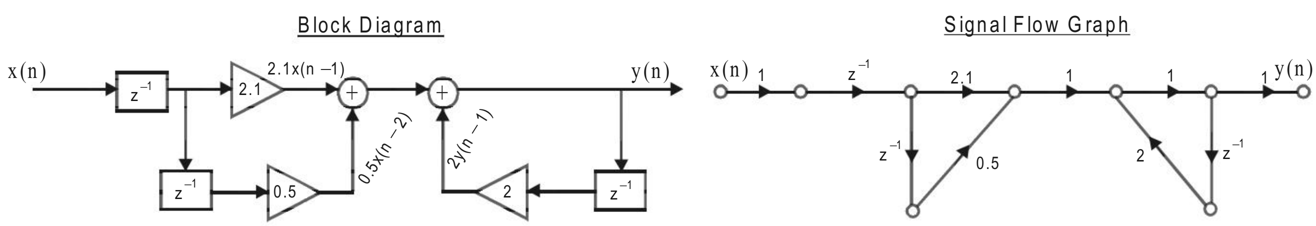
Since P is finite, E is infinite x(n) is power signal.

E2.5. ***Construct the block diagram and signal flow graph of the discrete time systems whose input-output relations are described by the following difference equations.***

a) $y(n) = 2y(n-1) + 2.1x(n-1) + 0.5x(n-2).$

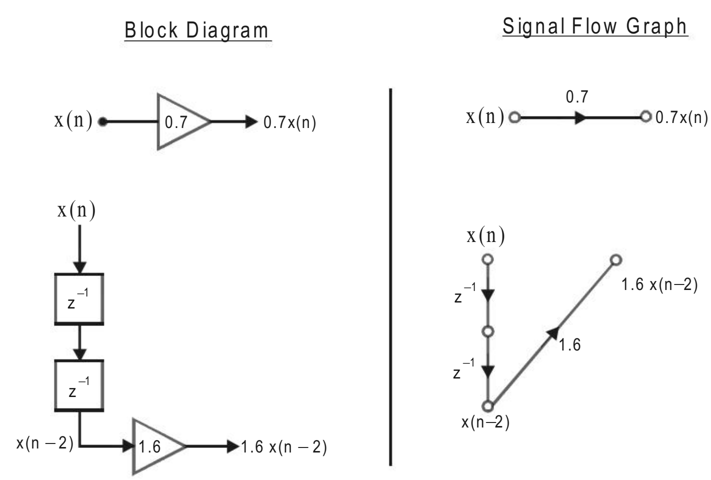
Solution



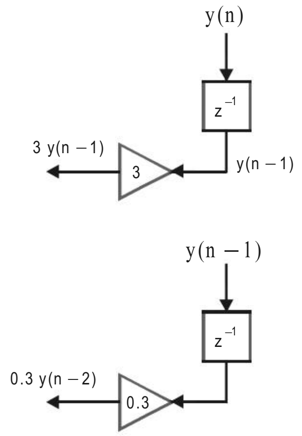


b) $y(n) = 1.6x(n-2) + 0.7x(n) + 3y(n-1) + 0.3y(n-2).$

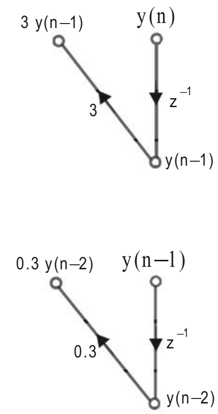
Solution



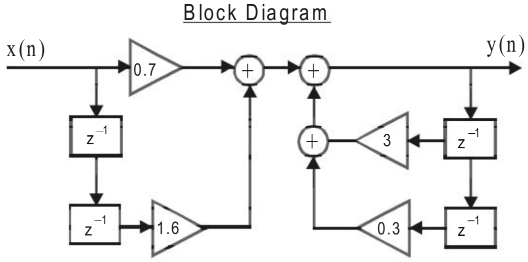
Block Diagram



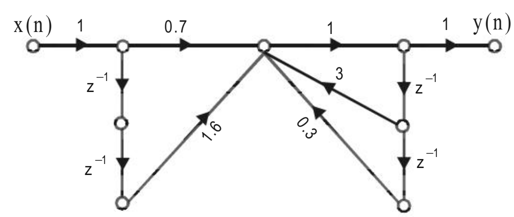
Signal Flow Graph



Block Diagram



Signal Flow Graph



E2.6. Determine the response of the discrete time systems governed by the following difference equations.

a) $y(n) = 0.1\,y(n-1) + x(n-1) + 0.7\,x(n)$;

$x(n) = 2^{-n}u(n)$; $y(-1) = -1$

Solution

$$y(n) = 0.1\,y(n-1) + x(n-1) + 0.7\,x(n)$$

$$\therefore\ y(n) - 0.1\,y(n-1) = 0.7\,x(n) + x(n-1) \qquad .....(1)$$

Homogeneous solution

When the input is zero the equation (1) can be written as,

$$y(n) - 0.1\,y(n-1) = 0 \qquad .....(2)$$

On substituting $y(n) = \lambda^n$ in equation (2) we get,

$$\lambda^n - 0.1\,\lambda^{(n-1)} = 0$$

$$\therefore\ \lambda^{(n-1)}(\lambda - 0.1) = 0 \ \Rightarrow\ \lambda = 0.1$$

The homogeneous solution $y_h(n)$ is given by,

$$y_h(n) = C\lambda^n = C(0.1)^n \text{ for } n \geq 0 = C(0.1)^n\,u(n) \qquad .....(3)$$

Particular solution

Given that, $x(n) = \left(\frac{1}{2}\right)^n u(n)$; $\therefore\ y(n) = K\left(\frac{1}{2}\right)^n u(n)$

Using the above values for x(n) and y(n) in equation(1) we get,

$$K\left(\frac{1}{2}\right)^n u(n) - 0.1K\left(\frac{1}{2}\right)^{(n-1)} u(n-1) = 0.7 \times \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{2}\right)^{n-1} u(n-1) \qquad .....(4)$$

To determine the value of 'K' evaluate equation(4) for n = 1.

$$K\left(\frac{1}{2}\right)^1 u(1) - 0.1K\left(\frac{1}{2}\right)^0 u(0) = 0.7\left(\frac{1}{2}\right)^1 u(1) + \left(\frac{1}{2}\right)^0 u(0)$$

$$0.5K - 0.1K = 0.35 + 1 \ \Rightarrow\ 0.4K = 1.35 \ \Rightarrow\ K = \frac{1.35}{0.4} = 3.375$$

$\therefore$ The particular solution $y_p(n)$ is given by,

$$y_p(n) = K\left(\frac{1}{2}\right)^n u(n) = 3.375 \times \left(\frac{1}{2}\right)^n u(n) \qquad .....(5)$$

Total response

$\therefore$ Response, $y(n) = y_h(n) + y_p(n)$

$$y(n) = \left[C(0.1)^n + 3.375 \times \left(\frac{1}{2}\right)^n\right]u(n) \quad \text{(or)} \quad y(n) = C(0.1)^n + 3.375\left(\frac{1}{2}\right)^n \;;\; \text{for } n \geq 0 \qquad .....(6)$$

At $n = 0$, from equation (1) we get,

$$y(0) - 0.1y(-1) = 0.7\,x(0) + x(-1) \qquad .....(7)$$

Given that : $y(-1) = -1$ and $x(n) = 2^{-n}u(n)$

$\therefore$ $x(0) = 1$ and $x(-1) = 0$

On substituting the above values in equation (7) we get,

$$y(0) + 0.1 = 0.7$$

$$y(0) = 0.7 - 0.1$$

$$\therefore\; y(0) = 0.6$$

Put $n = 0$ and $y(0) = 0.6$ in equation (6).

$$y(0) = C(0.1)^0 + 3.375 \times \left(\frac{1}{2}\right)^0 = C + 3.375$$

$$0.6 = C + 3.375$$

$$C = 0.6 - 3.375$$

$$C = -2.775$$

$\therefore$ The total response is given by,

$$y(n) = \left[-2.775\,(0.1)^n + 3.375\left(\frac{1}{2}\right)^n\right]u(n)$$

---

b) $y(n) + 2.1\,y(n-1) + 0.2\,y(n-2) = x(n) + 0.56\,x(n-1)$; $x(n) = u(n)$; $y(-2) = 1$; $y(-1) = -3$.

Solution

$$y(n) + 2.1\,y(n-1) + 0.2\,y(n-2) = x(n) + 0.56\,x(n-1) \qquad .....(1)$$

Homogeneous Solution

When the input is zero, the equation(1) can be written as,

$$y(n) + 2.1\,y(n-1) + 0.2\,y(n-2) = 0 \qquad .....(2)$$

substitute $y(n) = \lambda^n$ in equation (2)

$$\therefore\; \lambda^n + 2.1\,\lambda^{n-1} + 0.2\,\lambda^{n-2} = 0$$

$$\lambda^{(n-2)}\,[\lambda^2 + 2.1\,\lambda + 0.2] = 0$$

The characteristic equation is,

$$\lambda^2 + 2.1\,\lambda + 0.2 = 0 \quad \Rightarrow \quad (\lambda + 0.1)\,(\lambda + 2) = 0$$

$\therefore$ The roots are, $\lambda_1 = -0.1$, $\lambda_2 = -2$.

The homogenous solution $y_h(n)$ is given by,

$$y_h(n) = C_1\,\lambda_1^n + C_2\,\lambda_2^n$$

$$y_h(n) = C_1(-0.1)^n + C_2(-2)^n \quad \text{for } n \geq 0$$

$$= \left[C_1(-0.1)^n + C_2(-2)^n\right]u(n) \qquad .....(3)$$

Particular Solution

Given that , $x(n) = u(n)$; $\therefore$ $y(n) = K\,u(n)$. .....(4)

Using the above values for x(n) and y(n) in equation(1) we get,

$$K\,u(n) + 2.1\,K\,u(n-1) + 0.2\,K\,u(n-2) = u(n) + 0.56\,u(n-1) \qquad .....(5)$$

To find 'K' evaluate equation (5), for n = 2.

$$\therefore\ K\,u(2) + 2.1\,K\,u(1) + 0.2\,K\,u(0) = u(2) + 0.56\,u(1)$$

$$K + 2.1\,K + 0.2\,K = 1 + 0.56$$

$$3.3K = 1.56 \quad \Rightarrow \quad K = \frac{1.56}{3.3} = 0.47$$

$$\therefore\ y_p(n) = 0.47\,u(n)$$

$\therefore$ Total response,

$$y(n) = y_h(n) + y_p(n)$$

$$y(n) = [C_1(-0.1)^n + C_2(-2)^n + 0.47]\,u(n)$$

$$y(n) = C_1(-0.1)^n + C_2(-2)^n + 0.47 \quad \text{for } n \geq 0. \qquad .....(6)$$

At n = 0 from equation (1) we get,

$$y(0) + 2.1\,y(-1) + 0.2\,y(-2) = x(0) + 0.56\,x(-1) \qquad .....(7)$$

Given, $y(-1) = -3$, Also, $x(n) = u(n)$

$y(-2) = 1$ $\therefore$ $x(0) = 1$ and $x(-1) = 0$.

On substituting the above values in equation (7),

$$y(0) + 2.1\,(-3) + 0.2\,(1) = 1 + 0.$$

$$y(0) - 6.3 + 0.2 = 1 \ \Rightarrow\ y(0) - 6.1 = 1$$

$$\therefore\ y(0) = 1 + 6.1 = 7.1$$

At n = 1 from equation (1) we get,

$$y(1) + 2.1\,y(0) + 0.2\,y(-1) = x(1) + 0.56\,x(0)$$

We know that, $y(0) = 7.1$, $x(0) = 1$

$y(-1) = -3$, $x(1) = 1$

$$\therefore\ y(1) + 2.1\,(7.1) + 0.2\,(-3) = 1 + 0.56 \ \Rightarrow\ y(1) + 14.31 = 1.56$$

$$\therefore\ y(1) = 1.56 - 14.31 = -12.75$$

Put n = 0 and y(0) = 7.1 in eqaution(6).

$$y(0) = C_1(-0.1)^0 + C_2(-2)^0 + 0.47$$

$$7.1 = C_1 + C_2 + 0.47$$

$$\therefore\ C_1 + C_2 = 7.1 - 0.47$$

$$\therefore\ C_1 + C_2 = 6.63 \qquad .....(8)$$

Put n= 1 and y(1) = –12.75 in equation(6).

$$y(1) = C_1(-0.1)^1 + C_2(-2)^1 + 0.47$$

$$-12.75 = -0.1C_1 - 2\,C_2 + 0.47$$

$$0.1C_1 + 2C_2 = 12.75 + 0.47$$

$$\therefore\ 0.1C_1 + 2\,C_2 = 13.22 \qquad .....(9)$$

$$\begin{array}{lcrcrcl}
\text{Equation (8)} \times 2 & \Rightarrow & 2\,C_1 & + & 2\,C_2 & = & 13.26 \\
\text{Equation (9)} & \Rightarrow & 0.1\,C_1 & + & 2\,C_2 & = & 13.22 \\
 & & (-) & & (-) & & (-) \\
\hline
 & & 1.9\,C_1 & & & = & -0.04 \\
\hline
\end{array}$$

$$C_1 = \frac{-0.04}{1.9} = -0.02$$

$$\therefore\ C_2 = 6.63 - C_1 = 6.63 + 0.02 = 6.65$$

$$\therefore\ y(n) = -0.02\,(-0.1)^n + 6.65\,(-2)^n + 0.47, \quad \text{for } n \geq 0.$$

$$= [0.47 - 0.02\,(-0.1)^n + 6.65\,(-2)^n]\,u(n).$$

E2.7. Test the following systems for time invariance.

a) y(n) = x(n + 1) + x(n + 2)

Solution

Given that, $y(n) = \mathcal{H}\{x(n)\} = x(n + 1) + x(n + 2)$

Response for Delayed Input

$$y(n-m) = \mathcal{H}\{x(n-m)\} = x(n-m+1) + x(n-m+2)$$

Response for Unshifted Input

$$y(n) = \mathcal{H}\{x(n)\} = x(n + 1) + x(n + 2)$$

Delayed Response

$$y_d(n) = z^{-m}\,\mathcal{H}\{x(n)\} = z^{-m}\left[x(n+1) + x(n+2)\right] = z^{-m}\,x(n+1) + z^{-m}\,x(n+2)$$

$$= x(n-m+1) + x(n-m+2)$$

Here, $y(n-m) = y_d(n)$

Hence system is time invariant.

b) y(n) = n a^x(n)

Solution

Given that, $y(n) = \mathcal{H}\{x(n)\} = y(n) = n\,a^{x(n)}$

Response for Delayed Input

$$y(n-m) = \mathcal{H}\{x(n-m)\} = (n-m)\,a^{x(n-m)}$$

Delayed Response

$$y_d(n) = z^{-m}\,\mathcal{H}\{x(n)\} = z^{-m}\left[n\,a^{x(n)}\right] = n\,a^{x(n-m)}$$

Here, $y(n-m) \neq y_d(n)$

Hence the system is time variant.

c) y(n) = x²(n + 2) + C

Solution

Given that, $y(n) = \mathcal{H}\{x(n)\} = x^2(n + 2) + C$

Response for Delayed Input

$$y(n-m) = x^2(n-m+2) + C$$

Delayed Response

$$y_d(n) = z^{-m}\,\mathcal{H}\{x(n)\} = z^{-m}\left[x^2(n+2) + C\right] = z^{-m}\,x^2(n+2) + C = x^2(n-m+2) + C$$

Here, $y(n-m) = y_d(n)$

Hence the systems is time invariant.

d) y(n) = (n – 1) x²(n) + C

Solution

Given that, $y(n) = \mathcal{H}\{x(n)\} = (n - 1)x^2(n) + C$

Response for Delayed Input

$$y(n-m) = (n-m-1)x^2(n-m) + C$$

Delayed Response

$$y_d(n) = z^{-m}\,\mathcal{H}\{x(n)\} = z^{-m}\left[(n-1)\,x^2(n) + C\right] = (n-1)\,x^2(n-m) + C.$$

Here, $y(n-m) \neq y_d(n)$

Hence the systen is time variant.

E2.8. Test the following systems for linearity.

a) y(n) = x²(n) + x³(n – 1)

Solution

Let, '$\mathcal{H}$' be the system,

$$\therefore\; y(n) = \mathcal{H}\{x(n)\} = x^2(n) + x^3(n-1)$$

Consider two signals, $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be responses of system 'H' for inputs $x_1(n)$ and $x_2(n)$.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = x_1^2(n) + x_1^3(n-1)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = x_2^2(n) + x_2^3(n-1)$$

$$\therefore\ a_1\, y_1(n) + a_2\, y_2(n) = a_1\, [x_1^2(n) + x_1^3(n-1)] + a_2\, [x_2^2(n) + x_2^3(n-1)] \qquad .....(1)$$

Consider a linear combination of inputs.

$$a_1\, x_1(n) + a_2\, x_2(n) = x_3(n)$$

$$\therefore\ y_3(n) = \mathcal{H}\{x_3(n)\} = x_3^2(n) + x^3(n-1)$$

$$= [a_1\, x_1(n) + a_2\, x_2(n)]^2 + [a_1\, x_1(n-1) + a_2\, x_2(n-1)]^3 \qquad .....(2)$$

From equations (1) and (2) we can say that,

$$y_3(n) \neq a_1\, y_1(n) + a_2\, y_2(n)$$

Hence the system is non-linear.

---

b) $y(n) = bx(n+2) + n\, e^{x(n)}$

Solution

Let '$\mathcal{H}$' be the system.

$$\therefore\ y(n) = \mathcal{H}\{x(n)\} = b\, x(n+2) + n\, e^{x(n)}$$

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be their respective outputs.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = b\, x_1(n+2) + ne^{x_1(n)}$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = b\, x_2(n+2) + ne^{x_2(n)}$$

$$\therefore\ a_1\, y_1(n) + a_2\, y_2(n) = a_1\left(b\, x_1(n+2) + ne^{x_1(n)}\right) + a_2\left(b\, x_2(n+2) + ne^{x_2(n)}\right) \qquad .....(1)$$

Consider a linear combination of inputs.

$$\therefore\ a_1 x_1(n) + a_2\, x_2(n) = x_3(n)$$

$$\therefore\ y_3(n) = \mathcal{H}\{x_3(n)\} = b\, x_3(n+2) + ne^{x_3(n)}$$

$$= b\left[a_1 x_1(n+2) + a_2\, x_2(n+2)\right] + ne^{[a_1 x_1(n) + a_2 x_2(n)]}$$

$$= a_1 b\, x_1(n+2) + a_2\, b\, x_2(n+2) + ne^{a_1 x_1(n)}\, e^{a_2 x_2(n)} \qquad .....(2)$$

From equations (1) and (2) we get,

$$y_3(n) \neq a_1 y_1(n) + a_2 y_2(n)$$

Hence the system is non-linear system.

---

c) $y(n) = a\sqrt{x(n)} + b\, x(n)$

Solution

Let '$\mathcal{H}$' be the system.

$$\therefore\ y(n) = \mathcal{H}\{x(n)\} = a\sqrt{x(n)} + b\, x(n)$$

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be their respective outputs.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = a\sqrt{x_1(n)} + b\, x_1(n)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = a\sqrt{x_2(n)} + b\, x_2(n)$$

$$\therefore\ a_1\, y_1(n) + a_2\, y_2(n) = a_1\left[a\sqrt{x_1(n)} + b\, x_1(n)\right] + a_2\left[a\sqrt{x_2(n)} + b\, x_2(n)\right] \qquad .....(1)$$

Consider a linear combination of inputs.

$$\therefore\ a_1 x_1(n) + a_2\, x_2(n) = x_3(n)$$

$$\therefore\ y_3(n) = \mathcal{H}\{x_3(n)\} = a\sqrt{x_3(n)} + b\, x_3(n)$$

$$= a\sqrt{a_1 x_1(n) + a_2\, x_2(n)} + b\left[a_1 x_1(n) + a_2\, x_2(n)\right] \qquad .....(2)$$

From equations (1) and (2) we can say that,

$$y_3(n) \neq a_1\, y_1(n) + a_2\, y_2(n)$$

Hence the system is non-linear system.

d) $y(n) = \sqrt{x(n)} + \dfrac{1}{\sqrt{x(n)}}$

Solution

Let '$\mathcal{H}$' be the system.

$$\therefore \; y(n) = \mathcal{H}\{x(n)\} = \sqrt{x(n)} + \frac{1}{\sqrt{x(n)}}$$

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be their respective outputs.

$$\therefore \; y_1(n) = \mathcal{H}\{x_1(n)\} = \sqrt{x_1(n)} + \frac{1}{\sqrt{x_1(n)}} \quad ; \quad y_2(n) = \mathcal{H}\{x_2(n)\} = \sqrt{x_2(n)} + \frac{1}{\sqrt{x_2(n)}}$$

$$\therefore \; a_1\,y_1(n) + a_2\,y_2(n) = a_1\sqrt{x_1(n)} + \frac{a_1}{\sqrt{x_1(n)}} + a_2\sqrt{x_2(n)} + \frac{a_2}{\sqrt{x_2(n)}} \qquad .....(1)$$

Consider linear combination of inputs.

$$\therefore \; a_1x_1(n) + a_2x_2(n) = x_3(n)$$

$$\therefore \; y_3(n) = \mathcal{H}\{x_3(n)\} = \sqrt{x_3(n)} + \frac{1}{\sqrt{x_3(n)}} = \sqrt{a_1x_1(n) + a_2x_2(n)} + \frac{1}{\sqrt{a_1x_1(n) + a_2x_2(n)}} \qquad .....(2)$$

From equations (1) and (2) we get,

$$y_3(n) \neq a_1\,y_1(n) + a_2\,y_2(n)$$

Hence the system is non-linear system.

---

e) $y(n) = \displaystyle\sum_{m=-1}^{N} b_m\,x(n+m) + \sum_{m=0}^{M} c_m\,y(n+m)$

Solution

Let, $\mathcal{H}$ be the system represented by the given equation.

$$\therefore \; y(n) = \mathcal{H}\{x(n)\} = \sum_{m=-1}^{N} b_m\,x(n+m) + \sum_{m=0}^{M} c_m\,y(n+m)$$

Consider two signals, $x_1(n)$ and $x_2(n)$. Let $y_1(n)$ and $y_2(n)$ be the respective outputs.

$$y_1(n) = \mathcal{H}\{x_1(n)\} = \sum_{m=-1}^{N} b_m x_1(n+m) + \sum_{m=0}^{M} c_m y_1(n+m)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = \sum_{m=-1}^{N} b_m x_2(n+m) + \sum_{m=0}^{M} c_m y_2(n+m)$$

$$a_1\,y_1(n) + a_2\,y_2(n) = a_1\left[\sum_{m=-1}^{N} b_m x_1(n+m) + \sum_{m=0}^{M} c_m y_1(n+m)\right] + a_2\left[\sum_{m=-1}^{N} b_m x_2(n+m) + \sum_{m=0}^{M} c_m y_2(n+m)\right] \qquad .....(1)$$

Now consider linear combination of inputs

$$a_1x_1(n) + a_2x_2(n) = x_3(n)$$

$$\therefore \; y_3(n) = \mathcal{H}\{x_3(n)\} = \sum_{m=-1}^{N} b_m\,x_3(n+m) + \sum_{m=0}^{M} c_m\,y_3(n+m)$$

$$= \sum_{m=-1}^{N} b_m\left[a_1x_1(n+m) + a_2x_2(n+m)\right] + \sum_{m=0}^{M} c_m\,y_3(n+m)$$

$$= a_1\sum_{m=-1}^{N} b_m\,x_1(n+m) + a_2\sum_{m=-1}^{M} b_m\,x_2(n-m) + \sum_{m=0}^{M} c_m\,y_3(n+m) \qquad .....(2)$$

By time invariant property,

If $y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2x_2(n)\}$; then, $y_3(n+m) = \mathcal{H}\{a_1\,x_1(n+m) + a_2x_2(n+m)\}$

If $y_2(n) = \mathcal{H}\{x_2(n)\}$, then $y_2(n+m) = \mathcal{H}\{x_2(n+m)\}$

If $y_1(n) = \mathcal{H}\{x_1(n)\}$, then $y_1(n + m) = \mathcal{H}\{x_1(n + m)\}$

$$\therefore \quad y_3(n + m) = \mathcal{H}\{a_1 x_1(n + m) + a_2 x_2(n + m)\} = a_1 \mathcal{H}\{x_1(n + m)\} + a_2 \mathcal{H}\{x_2(n + m)\}$$

$$= a_1 y_1(n + m) + a_2 y_2(n + m) \qquad .....(3)$$

Using equation (3), the equation(2) can be written as,

$$y_3(n) = a_1 \sum_{m=-1}^{N} b_m x_1(n+m) + a_2 \sum_{m=-1}^{N} b_m x_2(n+m) + \sum_{m=0}^{M} c_m\left[a_1 y_1(n+m) + a_2 y_2(n+m)\right]$$

$$= a_1 \sum_{m=-1}^{N} b_m x_1(n+m) + a_2 \sum_{m=-1}^{N} b_m x_2(n+m) + a_1 \sum_{m=0}^{M} c_m y_1(n+m) + a_2 \sum_{m=0}^{M} c_m y_2(n+m)$$

$$= a_1 \left(\sum_{m=-1}^{N} b_m x_1(n+m) + \sum_{m=0}^{M} c_m y_1(n+m)\right)$$

$$+ a_2 \left(\sum_{m=-1}^{N} b_m x_2(n-m) + \sum_{m=0}^{M} c_m y_2(n+m)\right) \qquad .....(4)$$

From equations (1) and (4) we can say that,

$$y_3(n) = a_1 y_1(n) + a_2 y_2(n)$$

Hence the system is linear.

E2.9. Test the causality of the following systems.

a) $y(n) = x(n) - x(-n - 2) + x(n - 1)$

Solution

When, $n = -2$, $y(-2) = x(-2) - x(0) + x(-3)$

$n = -1$, $y(-1) = x(-1) - x(-1) + x(-2)$

$n = 0$, $y(0) = x(0) - x(-2) + x(-1)$

$n = 1$, $y(1) = x(1) - x(-3) + x(0)$

$n = 2$, $y(2) = x(2) - x(-4) + x(1)$

For $n \le -2$, the system response depends on future input.

Hence the system is noncausal.

---

b) $y(n) = a\, x(2n) + x(n^2)$

Solution

When, $n = -1$, $y(-1) = a\, x(-2) + x(1)$; Response depends on future input

$n = 0$, $y(0) = a\, x(0) + x(0)$

$n = 1$, $y(0) = a\, x(2) + x(1)$; Response depends on future input.

Except $n = 0$ for all other values of n, the response depends on future input.

Hence the system is noncausal.

---

c) $y(n) = \sum_{m=-1}^{n} x(m) + \sum_{m=-\infty}^{n} x(2m)$

Solution

$$n = 0, \quad y(0) = \sum_{m=-1}^{0} x(m) + \sum_{m=-\infty}^{0} x(2m) \Rightarrow y(0) = x(-1) + x(0) + ... + x(-4) + x(-2) + x(0).....$$

$\therefore$ y(0) depends on present and past inputs.

$$n = 1, \quad y(1) = \sum_{m=-1}^{1} x(m) + \sum_{m=-\infty}^{1} x(2m) \Rightarrow y(1) = x(-1) + x(0) + x(1) + ... + x(-4) + x(-2) + x(0) + x(2)$$

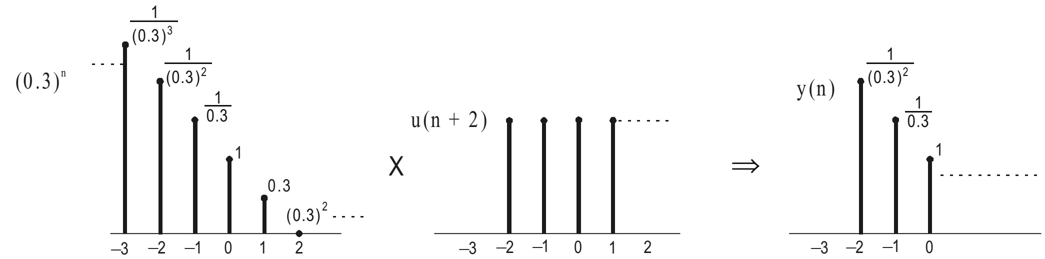
y(1) depends on future input x(2).

The system response depends on future input for $n > 0$.

Hence it is noncausal system.

d) $y(n) = (0.3)^n\, u(n+2)$ $\boxed{\textit{Note : For causality } y(n) = 0 \;;\; \text{for } n < 0.}$

Solution



Here $y(n) \neq 0$ for $n < 0$. Therefore the system is noncausal.

e) $y(n) = \sum\limits_{k=-4}^{4} x(n-k)$

Solution

$$y(n) = \sum_{k=-4}^{4} x(n-k) = x(n+4) + x(n+3) + x(n+2) + x(n+1) + x(n) + x(n-1) + x(n-2) + x(n-3) + x(n-4)$$

For any value of n the system response depends on future inputs.

Hence, the system is noncausal.

E2.10. Test the stability of the following discrete time systems.

a) $y(n) = x^2(n) + x(n+1)$

Solution

The given system involves squaring operation and so it is nonlinear.

The operations performed by system is squaring and shifting.

A bounded input signal will remain bounded even after squaring and shifting.

Hence the system is BIBO stable.

b) $y(n) = n\, x(n-1)$

Solution

The given system involves multiplication by n and so it is time variant system.

If x(n) doesnot tend to "0" as n tends to infinity then the system is unstable.

c) $h(n) = (0.4)^n\, u(n+3)$

Solution

Here, $h(n) = 0.4^n\, u(n+3) = 0.4^n$; For $n = -3$ to $+\infty$

$$\sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-3}^{\infty} (0.4)^n = \sum_{n=-3}^{-1} (0.4)^n + \sum_{n=0}^{\infty} (0.4)^n = (0.4)^{-3} + (0.4)^{-2} + (0.4)^{-1} + \frac{1}{1-0.4} = 26.04 = \text{Constant}$$

Hence it is stable system.

d) $h(n) = (8)^n\, u(4-n)$

Solution

Here, $h(n) = 8^n u(4-n) = 8^n$; for $n = -\infty$ to $+4$

For stability, $\sum\limits_{n=-\infty}^{\infty} |h(n)| < \infty$

$$\therefore \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{4} (8)^n = \sum_{n=-\infty}^{0} (8)^n + \sum_{n=1}^{4} (8)^n$$

$$= \sum_{n=0}^{\infty} (8)^{-n} + 8^1 + 8^2 + 8^3 + 8^4$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^n + 4680 = \sum_{n=0}^{\infty} (0.125)^n + 4680 = \frac{1}{1-0.125} + 4680$$

$$= 4681.14 = \text{Constant}$$

Hence it is stable system.

e) $y(n) = x(n-3)$

If $x(n) = \delta(n)$, then $y(n) = h(n)$

$\therefore \; h(n) = \delta(n-3)$

$$\therefore \sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} |\delta(n-3)| = 1$$

$\delta(n-3) = 1$, only when $n = 3$, and zero for all other values of n

Hence the system is stable.

E2.11. *Determine the range of values of 'a' and 'b' for the stability of LTI system with impulse response,*

$$h(n) = \begin{cases} (-4a)^n & ; \; n \geq 0 \\ (2b)^{-n} & ; \; n < 0 \end{cases}$$

Solution

The condition to be satisfied for the stability of the system is,

$$\sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{-1} \left|(2b)^{-n}\right| + \sum_{n=0}^{\infty} \left|(-4a)^n\right|$$

$$= \sum_{n=-\infty}^{-1} |2b|^{-n} + \sum_{n=0}^{\infty} |4a|^n$$

$$= \sum_{n=1}^{\infty} |2b|^n + \sum_{n=0}^{\infty} |4a|^n$$

$$= \sum_{n=0}^{\infty} |2b|^n - |2b|^0 + \sum_{n=0}^{\infty} |4a|^n$$

If $0 < |2b| < 1$, then $\displaystyle\sum_{n=0}^{\infty} |2b|^n = \frac{1}{1-|2b|}$

If $0 < |4a| < 1$, then $\displaystyle\sum_{n=0}^{\infty} |4a|^n = \frac{1}{1-|4a|}$

$$\therefore \sum_{n=-\infty}^{+\infty} |h(n)| = \frac{1}{1-|2b|} - 1 + \frac{1}{1-|4a|} = \text{Constant}$$

$\therefore$ Condition for stability is,

$$0 < |2b| < 1 \quad \Rightarrow \quad 0 < |b| < \frac{1}{2}$$

$$0 < |4a| < 1 \quad \Rightarrow \quad 0 < |a| < \frac{1}{4}$$

E2.12. *a) Determine the impulse response for the cascade of two LTI systems having impulse responses,*

$$h_1(n) = \left(\frac{1}{7}\right)^n u(n) \text{ and } h_2(n) = \delta(n-3)$$

Solution

The impulse response of the cascade system is given by,

$$h(n) = h_1(n) * h_2(n) = h_2(n) * h_1(n)$$

$$= \sum_{m=-\infty}^{\infty} h_2(m)\, h_1(n-m) \; ; \quad \text{'m' is dummy variable}$$

$$\therefore \; h(n) = \sum_{m=0}^{\infty} h_2(m)\, h_1(n-m) = \sum_{m=0}^{\infty} \delta(m-3) \left(\frac{1}{7}\right)^{n-m} = \sum_{m=0}^{\infty} \delta(m-3) \left(\frac{1}{7}\right)^{n} \left(\frac{1}{7}\right)^{-m}$$

$$= \left(\frac{1}{7}\right)^{n} \sum_{m=0}^{\infty} \delta(m-3) \left(\frac{1}{7}\right)^{-m}$$

The product of $\delta(m-3)$ and $\left(\frac{1}{7}\right)^{-m}$ will be nonzero, only when $m = 3$.

$$\therefore \; h(n) = \left(\frac{1}{7}\right)^{n} \left(\frac{1}{7}\right)^{-3} \quad \text{for } n \geq 3$$

$$h(n) = \left(\frac{1}{7}\right)^{n-3} u(n-3) \quad \text{for all n.}$$

b) Determine the overall impulse response of the interconnected discrete time system shown in fig E2.12.

Take, $h_1(n) = \left(\frac{1}{3}\right)^n u(n)\ ;\ h_2(n) = \left(\frac{1}{6}\right)^n u(n)\ ;\ h_3(n) = \left(\frac{1}{9}\right)^n u(n)$

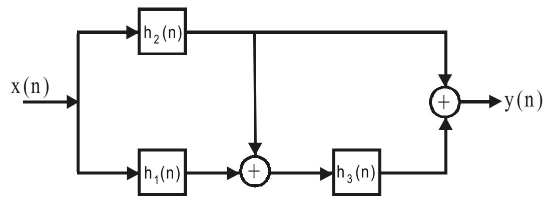
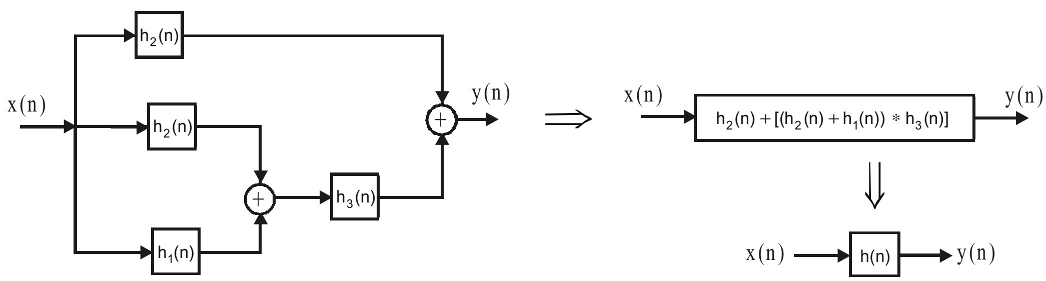


Fig E2.12

Solution

The given system can be redrawn as,



$$h(n) = h_2(n) + \left[(h_1(n)+h_2(n)) * h_3(n)\right] = h_2(n) + (h_1(n) * h_3(n)) + (h_2(n) * h_3(n))$$

Evaluation of $h_1(n) * h_3(n)$

$$h_1(n) * h_3(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_3(n-m)$$

$$= \sum_{m=0}^{n} h_1(m)\, h_3(n-m) = \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{9}\right)^{n-m} = \left(\frac{1}{9}\right)^n \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m 9^m = \left(\frac{1}{9}\right)^n \sum_{m=0}^{n} \left(\frac{9}{3}\right)^m$$

$$= \left(\frac{1}{9}\right)^n \frac{\left(\frac{9}{3}\right)^{n+1} - 1}{\frac{9}{3} - 1}$$

$$= \left(\frac{1}{9}\right)^n \frac{\left(\frac{9}{3}\right)^n \frac{9}{3} - 1}{\frac{9}{3} - 1} = \left(\frac{1}{9}\right)^n \frac{\left(\frac{9}{3}\right)^n \frac{9}{3} - 1}{2} = \left(\frac{1}{9}\right)^n \left[\frac{1}{2}\left(\frac{9}{3}\right)^n \frac{9}{3} - \frac{1}{2}\right]$$

$$= \frac{3}{2}\left(\frac{1}{9}\right)^n \left(\frac{9}{3}\right)^n - \frac{1}{2}\left(\frac{1}{9}\right)^n = \frac{3}{2}\left(\frac{1}{3}\right)^n - \frac{1}{2}\left(\frac{1}{9}\right)^n \quad \text{for } n \geq 0.$$

$$= \frac{3}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{1}{2}\left(\frac{1}{9}\right)^n u(n) \quad \text{for all 'n'}$$

Evaluation of $h_2(n) * h_3(n)$

$$h_2(n) * h_3(n) = \sum_{m=-\infty}^{+\infty} h_2(m)\, h_3(n-m) = \sum_{m=0}^{n} h_2(m)\, h_3(n-m) = \sum_{m=0}^{n} \left(\frac{1}{6}\right)^m \left(\frac{1}{9}\right)^{n-m} = \left(\frac{1}{9}\right)^n \sum_{m=0}^{n} \left(\frac{1}{6}\right)^m \left(\frac{1}{9}\right)^{-m}$$

$$= \left(\frac{1}{9}\right)^n \sum_{m=0}^{n} \left(\frac{1}{6}\right)^m 9^m = \left(\frac{1}{9}\right)^n \sum_{m=0}^{n} \left(\frac{3}{2}\right)^m = \left(\frac{1}{9}\right)^n \frac{\left(\frac{3}{2}\right)^{n+1} - 1}{\frac{3}{2} - 1} = \left(\frac{1}{9}\right)^n \left[\frac{\left(\frac{3}{2}\right)^{n+1} - 1}{\frac{1}{2}}\right]$$

$$= \left(\frac{1}{9}\right)^n \left[2\left(\frac{3}{2}\right)^n \frac{3}{2} - 2\right] = \left(\frac{1}{9}\right)^n \left[\left(\frac{3}{2}\right)^n 3 - 2\right] = \left(\frac{1}{9}\right)^n \left(\frac{3}{2}\right)^n 3 - 2\left(\frac{1}{9}\right)^n$$

$$= \left(\frac{3}{18}\right)^n 3 - 2\left(\frac{1}{9}\right)^n = 3\left(\frac{1}{6}\right)^n - 2\left(\frac{1}{9}\right)^n = 3\left(\frac{1}{6}\right)^n u(n) - \left(\frac{1}{9}\right)^n u(n) \quad \text{for all n.}$$

Overall Impulse Response

Now the overall impulse response h(n) is given by,

$$h(n) = h_2(n) + (h_1(n) * h_3(n)) + (h_2(n) * h_3(n))$$

$$h(n) = \left(\frac{1}{6}\right)^n u(n) + \left[\left(\frac{3}{2}\right)\left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)\left(\frac{1}{9}\right)^n u(n)\right] + \left[3\left(\frac{1}{6}\right)^n u(n) - \left(\frac{1}{9}\right)^n u(n)\right]$$

$$\Rightarrow \quad h(n) = 4\left(\frac{1}{6}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{9}\right)^n u(n) + \frac{3}{2}\left(\frac{1}{3}\right)^n u(n) = \left[4\left(\frac{1}{6}\right)^n - \frac{3}{2}\left(\frac{1}{9}\right)^n + \frac{3}{2}\left(\frac{1}{3}\right)^n\right] u(n)$$

E2.13. Determine the response of an LTI system whose and impulse response h(n) and input x(n) are given by,

a) $h(n) = \{1,\ 4,\ \underset{\uparrow}{1},\ -2,\ 1\}$

$x(n) = \{1,\ \underset{\uparrow}{3},\ 5,\ -1,\ -2\}$

Solution

The response y(n) of the system is given by convolution of x(n) and h(n).

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m)$$

Input sequence starts at $n = -1$

Impulse response starts at $n = -2$

Therefore the output sequence start at, $n = -1 + (-2) = -3$

The output consists of $5 + 5 - 1 = 9$ samples.

The 9 samples of output sequence are computed by table method as shown below.

m	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
x(m)					1	3	5	–1	–2				
h(m)				1	4	1	–2	1					
h(–m)				1	–2	1	4	1					
h(–3–m)	1	–2	1	4	1								
h(–2–m)		1	–2	1	4	1							
h(–1–m)			1	–2	1	4	1						
h(0 – m)				1	–2	1	4	1					
h(1 – m)					1	–2	1	4	1				
h(2 – m)						1	–2	1	4	1			
h(3 – m)							1	–2	1	4	1		
h(4 – m)								1	–2	1	4	1	
h(5 – m)									1	–2	1	4	1

When $n = -3$; $y(-3) = \sum_{m=-5}^{3} x(m)\, h(-3-m) = 0+0+0+0+1+0+0+0+0 = 1$

When $n = -2$; $y(-2) = \sum_{m=-4}^{3} x(m)\, h(-2-m) = 0+0+0+4+3+0+0+0 = 7$

When $n = -1$; $y(-1) = \sum_{m=-3}^{3} x(m)\, h(-1-m) = 0+0+1+12+5+0+0 = 18$

When $n = 0$; $y(0) = \sum_{m=-2}^{3} x(m)\, h(0-m) = 0-2+3+20-1+0 = 20$

When $n = 1$; $y(1) = \sum_{m=-1}^{3} x(m)\, h(1-m) = 1-6+5-4-2 = -6$

When n = 2 ; $y(2) = \sum_{m=-1}^{4} x(m)\, h(2-m) = 0+3-10-1-8+0 = -16$

When n = 3 ; $y(3) = \sum_{m=-1}^{5} x(m)\, h(3-m) = 0+0+5+2-2+0+0 = 5$

When n = 4 ; $y(4) = \sum_{m=-1}^{6} x(m)\, h(4-m) = 0+0+0-1+4+0+0+0 = 3$

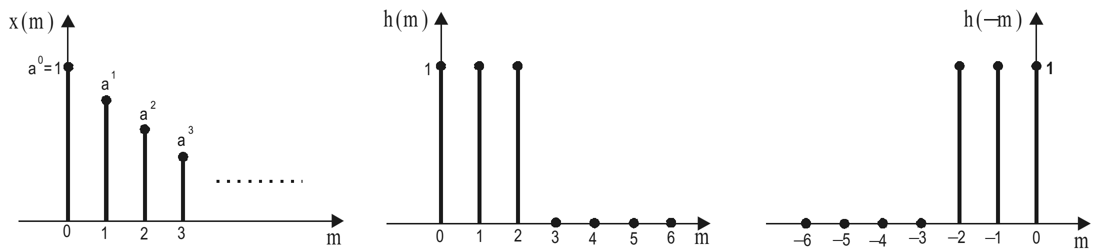
When n = 5 ; $y(5) = \sum_{m=-1}^{7} x(m)\, h(5-m) = 0+0+0+0-2+0+0+0+0 = -2$

$$\therefore\ y(n) = \{1,\ 7,\ 18,\ \underset{\uparrow}{20},\ -6,\ -16,\ 5,\ 3,\ -2\}$$

E2.13. b) $h(n) = \begin{cases} 1\ ; & 0 \le n \le 2 \\ 0\ ; & n \ge 3 \end{cases}$

$x(n) = a^n u(n)\ ;\ |a| < 1$

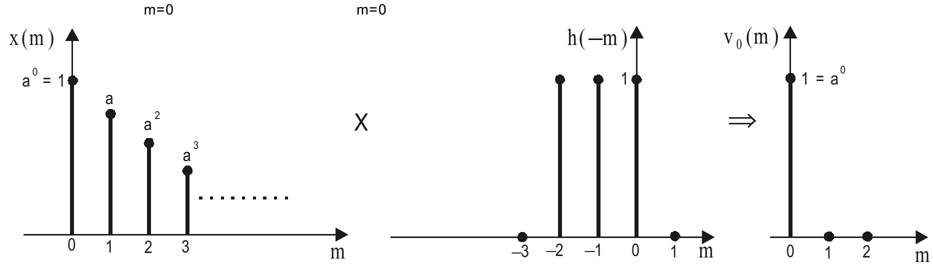
Solution



By convolution formula,

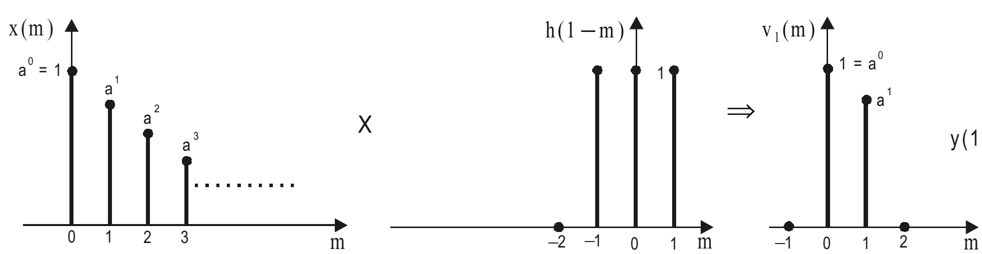
$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\, h(n-m)$$

When n = 0 ; $y(0) = \sum_{m=0}^{\infty} x(m)\, h(0-m) = \sum_{m=0}^{\infty} v_0(m)$



$\therefore\ y(0) = a^0$

When n = 1 ; $y(1) = \sum_{m=0}^{\infty} x(m)\, h(1-m) = \sum_{m=0}^{\infty} v_1(m)$



$y(1) = a^0 + a^1$

Similarly,

When, n = 2 ; $y(2) = a^0 + a^1 + a^2$

When, n = 3 ; $y(3) = a^1 + a^2 + a^3$

When, n = 4 ; $y(4) = a^2 + a^3 + a^4$

When, n = 5 ; $y(5) = a^3 + a^4 + a^5$

$$\therefore\ y(n) = \sum_{k=0}^{n} a^k\ ; \quad \text{for } n = 0,\ 1,\ 2$$

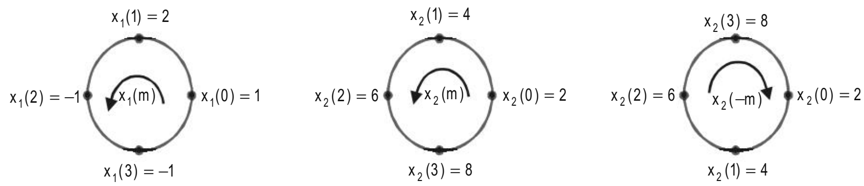
$$= \sum_{k=n-2}^{n} a^k\ ; \quad \text{for } n > 2$$

E2.14. Perform circular convolution of the two sequences,

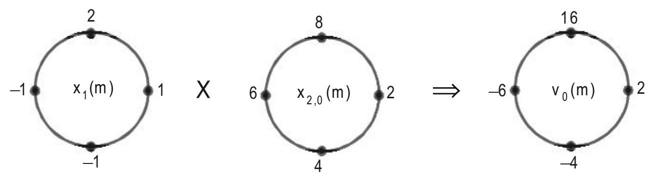
a) $x_1(n) = \{\underset{\uparrow}{1}, \ 2, \ -1 \ \ -1\}$ ***and*** $x_2(n) = \{\underset{\uparrow}{2}, \ 4, \ 6, \ 8\}$

Solution

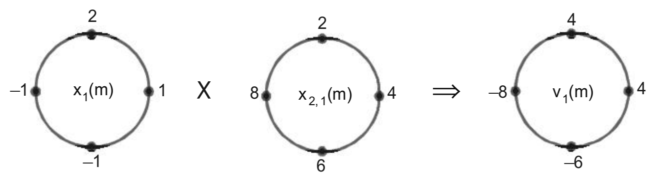
$$\text{Let} \quad x_3(n) = x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)\ x_{2,n}(m) \ ; \ x_{2,n}(m) = x_2((n-m))_N \ ; \ N = 4$$



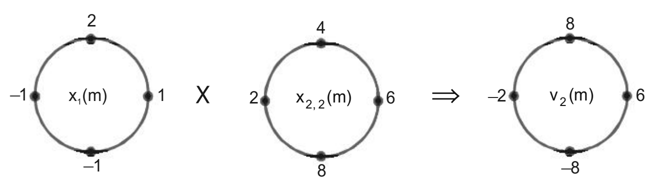
$$\text{When } n = 0 \ ; \quad x_3(0) = \sum_{m=0}^{3} x_1(m)\ x_{2,0}(m) = \sum_{m=0}^{3} v_0(m) = 2 + 16 - 6 - 4 = 8$$



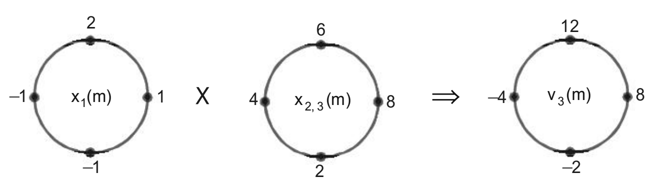
$$\text{When } n = 1 \ ; \quad x_3(1) = \sum_{m=0}^{3} x_1(m)\ x_{2,1}(m) = \sum_{m=0}^{3} v_1(m) = 4 + 4 - 8 - 6 = -6$$



$$\text{When } n = 2 ; \quad x_3(2) = \sum_{m=0}^{3} x_1(m)\ x_{2,2}(m) = \sum_{m=0}^{3} v_2(m) = 6 + 8 - 2 - 8 = 4$$



$$\text{When } n = 3 \ ; \quad x_3(3) = \sum_{m=0}^{3} x_1(m)\ x_{2,3}(m) = \sum_{m=0}^{3} v_3(m) = 8 + 12 - 4 - 2 = 14$$



$$x_3(n) = \{\underset{\uparrow}{8}, \ -6, \ 4, \ 14\}$$

b) Perform the circular convolution of the two sequences,

$x_1(n) = \{\underset{\uparrow}{0}, \ 0.6, \ -1, \ 1.5, \ 2\}$; $x_2(n) = \{\underset{\uparrow}{-2}, \ 3, \ 0.2, \ 0.7, \ 0.8\}$

Solution

The response $x_3(n)$ of the system is given by convolution of $x_1(n)$ and $x_2(n)$.

$$x_3(n) = x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N = \sum_{m=0}^{4} x_1(m)\, x_2((n-m))_5$$
$$= \sum_{m=0}^{4} x_1(m)\, x_{2,n}(m)$$

m	–4	–3	–2	–1	0	1	2	3	4
$x_1(m)$					0	0.6	–1	1.5	2
$x_2(m)$					–2	3	0.2	0.7	0.8
$x_2((-m))_5 = x_{2,0}(m)$	0.8	0.7	0.2	3	–2	**0.8**	**0.7**	**0.2**	**3**
$x_2((1-m))_5 = x_{2,1}(m)$		0.8	0.7	0.2	3	–2	**0.8**	**0.7**	**0.2**
$x_2((2-m))_5 = x_{2,2}(m)$			0.8	0.7	0.2	3	–2	**0.8**	**0.7**
$x_2((3-m))_5 = x_{2,3}(m)$				0.8	0.7	0.2	3	–2	**0.8**
$x_2((4-m))_5 = x_{2,4}(m)$					0.8	0.7	0.2	3	–2

When $n = 0$;

$$x_3(0) = \sum_{m=0}^{4} x_1(m)\, x_{2,0}(m) = x_1(0)\, x_{2,0}(0) + x_1(1)\, x_{2,0}(1) + x_1(2)\, x_{2,0}(2) + x_1(3)\, x_{2,0}(3) + x_1(4)\, x_{2,0}(4)$$
$$= (0 \times -2) + (0.6 \times 0.8) + (-1 \times 0.7) + (1.5 \times 0.2) + (2 \times 3) = 6.08$$

Similarly

When $n = 1$; $x_3(1) = (0 \times 3) + (0.6 \times -2) + (-1 \times 0.8) + (1.5 \times 0.7) + (2 \times 0.2) = -0.55$

When $n = 2$; $x_3(2) = (0 \times 0.2) + (0.6 \times 3) + (-1 \times -2) + (1.5 \times 0.8) + (2 \times 0.7) = 6.4$

When $n = 3$; $x_3(3) = (0 \times 0.7) + (0.6 \times 0.2) + (-1 \times 3) + (1.5 \times -2) + (2 \times 0.8) = -4.28$

When $n = 4$; $x_3(4) = (0 \times 0.8) + (0.6 \times 0.7) + (-1 \times 0.2) + (1.5 \times 3) + (2 \times -2) = 0.72$

$$\therefore\ x_3(n) = \{\underset{\uparrow}{6.08},\ -0.55,\ 6.4,\ -4.28,\ 0.72\}$$

E2.15. The input x(n) and impulse response h(n) of an LTI system are given by,

$$x(n) = \{-1,\ \underset{\uparrow}{1},\ -1,\ 1,\ -1,\ 1\} \quad \text{and} \quad h(n) = \{\underset{\uparrow}{-0.5},\ 0.5,\ -1,\ 0.5,\ -1,\ -2\}$$

Find the response of the system using,

a) Linear Convolution

b) Circular Convolution

Solution

a) Response of LTI System Using Linear Convolution

Let, $y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m)$

$$= \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m) \ ; \text{ where } h_n(m) = h(n-m)$$

$x(n)$ starts at $n = -1$ and $h(n)$ starts at $n = 0$.

$\therefore$ $y(n)$ will start at $n = 0 + (-1) = -1$

Length of $x(n)$ is 6 and $h(n)$ is 6.

Hence length of $y(n)$ is $6 + 6 - 1 = 11$. Also $y(n)$ ends at $n = 0 + (-1) + (6 + 6 - 2) = 9$

The 11 samples of y(n) computed by table method as shown below.

m	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8	9
x(m)						–1	1	–1	1	–1	1					
h(m)							–0.5	0.5	–1	0.5	–1	–2				
h(–m)		–2	–1	0.5	–1	0.5	–0.5									
$h_{-1}(-m)$	–2	–1	0.5	–1	0.5	–0.5										
$h_0(m)$		–2	–1	0.5	–1	0.5	–0.5									
$h_1(m)$			–2	–1	0.5	–1	0.5	–0.5								
$h_2(m)$				–2	–1	0.5	–1	0.5	–0.5							
$h_3(m)$					–2	–1	0.5	–1	0.5	–0.5						
$h_4(m)$						–2	–1	0.5	–1	0.5	–0.5					
$h_5(m)$							–2	–1	0.5	–1	0.5	–0.5				
$h_6(m)$								–2	–1	0.5	–1	0.5	–0.5			
$h_7(m)$									–2	–1	0.5	–1	0.5	–0.5		
$h_8(m)$										–2	–1	0.5	–1	0.5	–0.5	
$h_9(m)$											–2	–1	0.5	–1	0.5	–0.5

When n = –1, $y(-1) = \sum_{m=-6}^{4} x(m)\, h_{-1}(m)$

$$= x(-6)\, h_{-1}(-6) + x(-5)\, h_{-1}(-5) + x(-4) h_{-1}(-4) + x(-3)\, h_{-1}(-3) + x(-2)\, h_{-1}(-2) + x(-1) h_{-1}(-1)$$

$$+ x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2) h_{-1}(2) + x(3)\, h_{-1}(3) + x(4)\, h_{-1}(4)$$

$$= 0 + 0 + 0 + 0 + 0 + (-1 \times -0.5) + 0 + 0 + 0 + 0 + 0 = 0.5$$

When n = 0 ; $y(0) = \sum_{m=-5}^{4} x(m)\, h_0(m) = 0+0+0+0+(-1\times 0.5)+(1\times -0.5)+0+0+0+0 = -1$

When n = 1 ; $y(1) = \sum_{m=-4}^{4} x(m)\, h_1(m) = 0+0+0+(-1\times -1)+(1\times 0.5)+(-1\times -0.5)+0+0+0 = 2$

When n = 2 ; $y(2) = \sum_{m=-3}^{4} x(m)\, h_2(m) = 0+0+(-1\times 0.5)+(1\times -1)+(-1\times 0.5)+(1\times -0.5)+0+0 = -2.5$

When n = 3 ; $y(3) = \sum_{m=-2}^{4} x(m)\, h_3(m) = 0+(-1\times -1)+(1\times 0.5)+(-1\times -1)+(1\times 0.5)+(-1\times -0.5)+0 = 3.5$

When n = 4 ; $y(4) = \sum_{m=-1}^{4} x(m)\, h_4(m) = (-1\times -2)+(1\times -1)+(-1\times 0.5)+(1\times -1)+(-1\times 0.5)+(1\times -0.5) = -1.5$

When n = 5 ; $y(5) = \sum_{m=-1}^{5} x(m)\, h_5(m) = 0+(1\times -2)+(-1\times -1)+(1\times 0.5)+(-1\times -1)+(1\times 0.5)+0 = 1$

When n = 6 ; $y(6) = \sum_{m=-1}^{6} x(m)\, h_6(m) = 0+0+(-1\times -2)+(1\times -1)+(-1\times 0.5)+(1\times -1)+0+0 = -0.5$

When n = 7 ; $y(7) = \sum_{m=-1}^{7} x(m)\, h_7(m) = 0+0+0+(1\times -2)+(-1\times -1)+(1\times 0.5)+0+0+0 = -0.5$

When n = 8 ; $y(8) = \sum_{m=-1}^{8} x(m)\, h_8(m) = 0+0+0+0+(-1\times -2)+(1\times -1)+0+0+0+0 = 1$

When n = 9 ; $y(9) = \sum_{m=-1}^{9} x(m)\, h_9(m) = 0+0+0+0+0+(1\times -2)+0+0+0+0+0 = -2$

The response of LTI system y(n) is,

$$y(n) = \{0.5,\ \underset{\uparrow}{-1},\ 2,\ -2.5,\ 3.5,\ -1.5,\ 1,\ -0.5,\ -0.5,\ 1,\ -2\}$$

b) Response of LTI system using circular convolution

The response y(n) is 11-point sequence. The y(n) start at n = –1 and end of n = 9. Hence both x(n) and h(n) should be converted to 11-point sequence such that they start at n = –1 and end at n = 9 by appending zeros for missing samples.

$$\therefore \quad x(n) = \{-1,\ \underset{\uparrow}{1},\ -1,\ 1,\ -1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0\}$$

$$h(n) = \{0,\ \underset{\uparrow}{-0.5},\ 0.5,\ -1,\ 0.5,\ -1,\ -2,\ 0,\ 0,\ 0,\ 0\}$$

Now, $y(n) = x(n) \circledast h(n) = \sum\limits_{m=-1}^{9} x(m)\, h((n-m))_{11} = \sum\limits_{m=-1}^{9} x(m)\, h_n(m)$; where $h_n(m) = h((n-m))_{11}$

The 11 samples of y(n) are computed by table method as shown below.

m	–10	–9	–8	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8	9
x(m)										–1	1	–1	1	–1	1	0	0	0	0	0
h(m)										0	–0.5	0.5	–1	0.5	–1	–2	0	0	0	0
h(–m)		0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0								
$h_{-1}(m)$	0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0	**0**	**0**	**0**	**0**	**–2**	**–1**	**0.5**	**–1**	**0.5**
$h_0(m)$		0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0	**0**	**0**	**0**	**0**	**–2**	**–1**	**0.5**	**–1**
$h_1(m)$			0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0	**0**	**0**	**0**	**0**	**–2**	**–1**	**0.5**
$h_2(m)$				0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0	**0**	**0**	**0**	**0**	**–2**	**–1**
$h_3(m)$					0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0	**0**	**0**	**0**	**0**	**–2**
$h_4(m)$						0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0	**0**	**0**	**0**	**0**
$h_5(m)$							0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0	**0**	**0**	**0**
$h_6(m)$								0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0	**0**	**0**
$h_7(m)$									0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0	**0**
$h_8(m)$										0	0	0	0	–2	–1	0.5	–1	0.5	–0.5	0
$h_9(m)$	**0**	**0**	**0**	**–2**	**–1**	**0.5**	**–1**	**0.5**	**–0.5**	**0**	0	0	0	0	–2	–1	0.5	–1	0.5	–0.5

When n = –1 ; $y(-1) = \sum\limits_{m=-1}^{9} x(m)\, h_{-1}(m) = (-1 \times -0.5) + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0.5$

When n = 0 ; $y(0) = \sum\limits_{m=-1}^{9} x(m)\, h_0(m) = (-1 \times 0.5) + (1 \times -0.5) + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = -1$

When n = 1 ; $y(1) = \sum\limits_{m=-1}^{9} x(m)\, h_1(m) = (-1 \times -1) + (1 \times 0.5) + (-1 \times -0.5) + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 2$

When n = 2 ; $y(2) = \sum\limits_{m=-1}^{9} x(m)\, h_2(m) = (-1 \times 0.5) + (1 \times -1) + (-1 \times 0.5) + (1 \times -0.5) + 0 + 0 + 0 + 0 + 0 + 0 + 0 = -2.5$

When n = 3 ; $y(3) = \sum\limits_{m=-1}^{9} x(m)\, h_3(m) = (-1 \times -1) + (1 \times 0.5) + (-1 \times -1) + (1 \times 0.5) + (-1 \times -0.5) + 0 + 0 + 0 + 0 + 0 + 0 = 3.5$

When n = 4 ; $y(4) = \sum\limits_{m=-1}^{9} x(m)\, h_4(m) = (-1 \times -2) + (1 \times -1) + (-1 \times 0.5) + (1 \times -1) + (-1 \times 0.5) + (1 \times -0.5) + 0 + 0 + 0 + 0 + 0 = -1.5$

When n = 5 ; $y(5) = \sum\limits_{m=-1}^{9} x(m)\, h_5(m) = 0 + (1 \times -2) + (-1 \times -1) + (1 \times 0.5) + (-1 \times -1) + (1 \times 0.5) + 0 + 0 + 0 + 0 + 0 = 1$

When n = 6 ; $y(6) = \sum\limits_{m=-1}^{9} x(m)\, h_6(m) = 0 + 0 + (-1 \times -2) + (1 \times -1) + (-1 \times 0.5) + (1 \times -1) + 0 + 0 + 0 + 0 + 0 = -0.5$

When n = 7 ; $y(7) = \sum\limits_{m=-1}^{9} x(m)\, h_7(m) = 0 + 0 + 0 + (1 \times -2) + (-1 \times -1) + (1 \times 0.5) + 0 + 0 + 0 + 0 + 0 = -0.5$

When n = 8 ; $y(8) = \sum_{m=-1}^{9} x(m)\,h_8(m) = 0+0+0+0+0+(-1\times-2)+(1\times-1)+0+0+0+0+0=1$

When n = 9 ; $y(9) = \sum_{m=-1}^{9} x(m)\,h_9(m) = 0+0+0+0+0+(1\times-2)+0+0+0+0+0=-2$

The response of LTI system y(n) is,

$$y(n) = \{0.5,\ \underset{\uparrow}{-1},\ 2,\ -2.5,\ 3.5,\ -1.5,\ 1,\ -0.5,\ -0.5,\ 1,\ -2\}$$

E2.16. Perform linear convolution of the following sequences by,

i) Overlap add method

ii) Overlap save method

$$x(n) = \{1,\ -1,\ 2\ \ 1,\ -1,\ 2,\ 1,\ -1,\ 2\}$$
$$h(n) = \{2,\ 3,\ -1\}$$

Solution

Overlap Add Method

$$x(n) = \{1,\ -1,\ 2,\ 1,\ -1,\ 2,\ 1,\ -1,\ 2\}$$

$x_1(n)$	$x_2(n)$	$x_3(n)$
$x_1(n) = 1,\ n=0$	$x_2(n) = 1,\ n=3$	$x_3(n) = 1,\ n=6$
$= -1,\ n=1$	$= -1,\ n=4$	$= -1,\ n=7$
$= 2,\ n=2$	$= 2,\ n=5$	$= 2,\ n=8$

$$h(n) = \{2,\ 3,\ -1\}$$

Let, $y_1(n)$, $y_2(n)$, $y_3(n)$ be output of linear convolution of $x_1(n)$, $x_2(n)$, $x_3(n)$ with h(n) respectively.

Here, h(n) starts at $n_h = 0$.

$x_1(n)$ starts at, $n = n_1 = 0$ $\therefore$ $y_1(n)$ starts at, $n = 0 + 0 = 0$

$x_2(n)$ starts at, $n = n_2 = 3$ $\therefore$ $y_2(n)$ starts at, $n = 3 + 0 = 3$

$x_3(n)$ starts at, $n = n_3 = 6$ $\therefore$ $y_3(n)$ starts at, $n = 6 + 0 = 6$

Here, $N_1 = 9$, $N_2 = 3$, $N_3 = 3$

$N_2 - 1 = 2$

$N_2 + N_3 - 1 = 5$

Convolution output of each section will consists of 3 + 3 – 1 = 5 samples.

Convolution of Section - 1

m	–2	–1	0	1	2	3	4
x(m)			1	–1	2		
h(m)			2	3	–1		
$h(-m)=h_0(m)$	–1	3	2				
$h_1(m)$		–1	3	2			
$h_2(m)$			–1	3	2		
$h_3(m)$				–1	3	2	
$h_4(m)$					–1	3	2

$$y_1(n) = x_1(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\,h(n-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\,h_n(m)\ ;\quad n = 0,\ 1,\ 2,\ 3,\ 4$$

where $h_n(m) = h(n-m)$

When n = 0 ; $y_1(0) = \sum x_1(m)\,h_0(m) = 0+0+2+0+0 = 2$

When n = 1 ; $y_1(1) = \sum x_1(m)\,h_1(m) = 0+3-2+0 = 1$

When n = 2 ; $y_1(2) = \sum x_1(m)\,h_2(m) = -1-3+4 = 0$

When n = 3 ; $y_1(3) = \sum x_1(m)\,h_3(m) = 0+1+6+0 = 7$

When n = 4 ; $y_1(4) = \sum x_1(m)\,h_4(m) = 0+0-2+0+0 = -2$

$$\therefore\ y_1(n) = \{\underset{\substack{\uparrow \\ n=0}}{2},\ 1,\ 0,\ 7,\ -2\}$$

Convolution of sections 2 and 3

The convolution of section –2 and 3 are identical to that of section -1 except the starting value of n.

$$\therefore\ y_2(n) = \{\underset{\substack{\uparrow \\ n=3}}{2},\ 1,\ 0,\ 7,\ -2\}$$

$$\therefore\ y_3(n) = \{\underset{\substack{\uparrow \\ n=6}}{2},\ 1,\ 0,\ 7,\ -2\}$$

Overall Output

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	2	1	0	7	–2						
$y_2(n)$				2	1	0	7	–2			
$y_3(n)$							2	1	0	7	–2
$y(n)$	2	1	0	9	–1	0	9	–1	0	7	–2

$$y(n) = \{2,\ 1,\ 0,\ 9,\ -1,\ 0,\ 9,\ -1,\ 0,\ 7,\ -2\}$$

Overlap save Method

$$x(n) = \{1,\ -1,\ 2,\ 1,\ -1,\ 2,\ 1,\ -1,\ 2\}$$

$$h(n) = \{2,\ 3,\ -1\}$$

$$N_1 = 9, \quad N_2 = 3, \quad \text{Let } N_3 = 3$$

$x_1(n)$	$x_2(n)$	$x_3(n)$
$x_1(n) = 1,\ n=0$	$x_2(n) = 1,\ n=3$	$x_3(n) = 1,\ n=6$
$= -1,\ n=1$	$= -1,\ n=4$	$= -1,\ n=7$
$= 2,\ n=2$	$= 2,\ n=5$	$= 2,\ n=8$

Let, $y_1(n)$, $y_2(n)$ and $y_3(n)$ be output of linear convolution of $x_1(n)$, $x_2(n)$ and $x_3(n)$ with h(n) respectively.

Now each output will consists of 3 + 3 – 1 = 5 samples. Hence convert $x_1(n)$, $x_2(n)$, $x_3(n)$ and h(n) to 5 sample sequence as shown below.

$x_1(n)$	$x_2(n)$	$x_3(n)$
$x_1(n) = 1,\ n=0$	$x_2(n) = 1,\ n=3$	$x_3(n) = 1,\ n=6$
$= -1,\ n=1$	$= -1,\ n=4$	$= -1,\ n=7$
$= 2,\ n=2$	$= 2,\ n=5$	$= 2,\ n=8$
$= 1,\ n=3$	$= 1,\ n=6$	$= 0,\ n=9$
$= -1,\ n=4$	$= -1,\ n=7$	$= 0,\ n=10$

h(n) = {2, 3, –1, 0, 0}

Now perform circular convolution of each section with h(n).

$x_1(n)$ starts at, n = 0 ; \ $y_1(n)$ starts at, n = 0.

$x_2(n)$ starts at, n = 3 ; \ $y_2(n)$ starts at, n = 3.

$x_3(n)$ starts at, n = 6 ; \ $y_3(n)$ starts at, n = 6.

Convolution of Section 1

m	–4	–3	–2	–1	0	1	2	3	4
$x_1(m)$					1	–1	2	1	–1
h(m)					2	3	–1	0	0
$h(-m)= h_0(m)$	0	0	–1	3	2	**0**	**0**	**–1**	**3**
$h_1(m)$		0	0	–1	3	2	**0**	**0**	**–1**
$h_2(m)$			0	0	–1	3	2	**0**	**0**
$h_3(m)$				0	0	–1	3	2	**0**
$h_4(m)$					0	0	–1	3	2

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m=0}^{N-1} x_1(m)\ h((n-m))_n = \sum_{m=0}^{4} x_1(m)\ h_n(m)\ ; \quad \text{where } h_n(m) = h((n-m))_5$$

When $n = 0$; $y_1(0) = \sum_{m=0}^{4} x_1(m)\, h_0(m) = 2 + 0 + 0 - 1 - 3 = -2$

When $n = 1$; $y_1(1) = \sum_{m=0}^{4} x_1(m)\, h_1(m) = 3 - 2 + 0 + 0 + 1 = 2$

When $n = 2$; $y_1(2) = \sum_{m=0}^{4} x_1(m)\, h_2(m) = -1 - 3 + 4 + 0 + 0 = 0$

When $n = 3$; $y_1(3) = \sum_{m=0}^{4} x_1(m)\, h_3(m) = 0 + 1 + 6 + 2 + 0 = 9$

When $n = 4$; $y_1(4) = \sum_{m=0}^{4} x_1(m)\, h_4(m) = 0 + 0 - 2 + 3 - 2 = -1$

$$\therefore\ y_1(n) = \{ \underset{\substack{\uparrow \\ n=0}}{-2},\ 2,\ 0,\ 9,\ -1 \}$$

Convolution of Section 2

The output of convolution of section -2 will be identical to that of section-1 except the starting value of n.

$$\therefore\ y_2(n) = \{ \underset{\substack{\uparrow \\ n=3}}{-2},\ 2,\ 0,\ 9,\ -1 \}$$

Convolution of Section 3

m	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8	9	10
$x_3(m)$											1	–1	2	0	0
$h(m)$					2	3	–1	0	0						
$h_0(m)$	0	0	–1	3	2										
$h_6(m)$							0	0	–1	3	2	**0**	**0**	**–1**	**3**
$h_7(m)$								0	0	–1	3	2	**0**	**0**	**–1**
$h_8(m)$									0	0	–1	3	2	**0**	**0**
$h_9(m)$										0	0	–1	3	2	**0**
$h_{10}(m)$											0	0	–1	3	2

$$y_3(n) = x_3(n) \circledast h(n) = \sum_{m=6}^{10} x_3(m)\ h((n-m))_5 = \sum_{m=6}^{10} x_3(m)\ h_n(m)\ ; \quad \text{where } h_n(m) = h((n-m))_5$$

When $n = 6$; $y_3(6) = \sum_{m=6}^{10} x_3(m)\, h_6(m) = 2 + 0 + 0 + 0 + 0 = 2$

When $n = 7$; $y_3(7) = \sum_{m=6}^{10} x_3(m)\, h_7(m) = 3 - 2 + 0 + 0 + 0 = 1$

When $n = 8$; $y_3(8) = \sum_{m=6}^{10} x_3(m)\, h_8(m) = -1 - 3 + 4 + 0 + 0 = 0$

When $n = 9$; $y_3(9) = \sum_{m=6}^{10} x_3(m)\, h_9(m) = 0 + 1 + 6 + 0 + 0 = 7$

When $n = 10$; $y_3(10) = \sum_{m=6}^{10} x_3(m)\, h_{10}(m) = 0 + 0 - 2 + 0 + 0 = -2$

$$\therefore\ y_3(n) = \{ \underset{\substack{\uparrow \\ n=6}}{2},\ 1,\ 0,\ 7,\ -2 \}$$

Overall Output

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	~~–2~~	~~2~~	0	9	–1						
$y_2(n)$				~~–2~~	~~2~~	0	9	–1			
$y_3(n)$							~~2~~	~~1~~	0	7	–2
$y(n)$	*	*	0	9	–1	0	9	–1	0	7	–2

$y(n) = \{*,\ *,\ 0,\ 9,\ -1,\ 0,\ 9,\ -1,\ 0, 7,\ -2\}$

Hence both the results are same except the first $(N_2 - 1)$ samples.

E2.17. Perform crosscorrelation of the sequences,

$$x(n) = \{\underset{\uparrow}{-1},\ 2,\ 3\ \ -4\} \quad \textit{and} \quad y(n) = \{2,\ \underset{\uparrow}{-1},\ \ -3\}$$

Solution

The crosscorrelation sequence $r_{xy}(m)$ is given by,

$$r_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-m)$$

The x(n) starts at n = 0, and has 4 samples.

\ $n_1 = 0,\ \ N_1 = 4$

The y(n) start at, n = –1 and has 3 samples.

\ $n_2 = -1\ ,\ \ N_2 = 3$

Now $r_{xy}(m)$ will have, $N_1 + N_2 - 1 = 4 + 3 - 1 = 6$ samples.

The initial value of $m = m_i = n_1 - (n_2 + N_2 - 1) = 0\ - (-1 + 3 - 1) = -1$

The final value of $m = m_f = m_i + (N_1 + N_2 - 2) = -1 + (4 + 3 - 2) = 4$

The 6 samples of crosscorrelation sequence are computed using table method as shown below.

n	–2	–1	0	1	2	3	4	5
$x(n)$			–1	2	3	–4		
$y(n)$		2	–1	–3				
$y(n-(-1)) = y_{-1}(n)$	2	–1	–3					
$y(n-0) = y_0(n)$		2	–1	–3				
$y(n-1) = y_1(n)$			2	–1	–3			
$y(n-2) = y_2(n)$				2	–1	–3		
$y(n-3) = y_3(n)$					2	–1	–3	
$y(n-4) = y_4(n)$						2	–1	–3

Each sample of $r_{xy}(m)$ is given by,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\ y(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\ y_m(n)\ ; \quad \text{where } y_m(n) = y(n-m)$$

$$\text{When } m = -1\ ;\ r_{xy}(-1) = \sum_{n=-2}^{3} x(n)\ y_{-1}(n)\ =\ 0+0+3+0+0+0 = 3$$

$$\text{When } m = 0\ \ ;\ r_{xy}(0)\ =\ \sum_{n=-1}^{3} x(n)\ y_0(n)\ =\ 0+1-6+0+0 = -5$$

$$\text{When } m = 1\ \ ;\ r_{xy}(1)\ =\ \sum_{n=0}^{3} x(n)\ y_1(n)\ =\ -2-2-9+0 = -13$$

$$\text{When } m = 2\ \ ;\ r_{xy}(2)\ =\ \sum_{n=0}^{3} x(n)\ y_2(n)\ =\ 0+4-3+12 = 13$$

When m = 3 ; $r_{xy}(3) = \sum_{n=0}^{4} x(n)\, y_3(n) \;=\; 0+0+6+4+0 = 10$

When m = 4 ; $r_{xy}(4) = \sum_{n=0}^{5} x(n)\, y_4(n) \;=\; 0+0+0-8+0+0 = -8$

$$\therefore\; r_{xy}(m) = \left\{\; 3,\;\; \underset{\uparrow}{-5},\;\; -13,\;\; 13,\;\; 10,\;\; -8 \right\}$$

E2.18. Determine the autocorrelation sequence for x(n) = {1, $\underline{4}$, 3, –5, 2}

Solution

The autocorrelation sequence $r_{xx}(m)$ is given by,

$$r_{xx}(m) \;=\; \sum_{n=-\infty}^{+\infty} x(n)\, x(n-m)$$

The x(n) starts at n = –1, and has 5 samples.

\ $n_x = -1$, and N = 5

The $r_{xx}(m)$ will have, 2N –1 = 10 – 1 = 9 samples

The initial value of m = m_i = – (N – 1) = – (5 – 1) = –4

The final value of m = m_f = m_i + (2 N – 2) = –4 + (10 – 2) = 4

n	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
x(n)					1	4	3	–5	2				
$x_{-4}(n)$	1	4	3	–5	2								
$x_{-3}(n)$		1	4	3	–5	2							
$x_{-2}(n)$			1	4	3	–5	2						
$x_{-1}(n)$				1	4	3	–5	2					
$x_0(n)$					1	4	3	–5	2				
$x_1(n)$						1	4	3	–5	2			
$x_2(n)$							1	4	3	–5	2		
$x_3(n)$								1	4	3	–5	2	
$x_4(n)$									1	4	3	–5	2

Each sample of autocorrelation sequence $r_{xx}(m)$ is given by,

$$r_{xx}(m) \;=\; \sum_{n=-\infty}^{+\infty} x(n)\, x(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\, x_m(n)\,; \quad \text{where } x_m(n) = x(n-m)$$

When m = –4 ; $r_{xx}(-4) = \sum_{n=-5}^{3} x(n)\, x_{-4}(n) \;=\; 0+0+0+0+2+0+0+0+0 = 2$

When m = –3 ; $r_{xx}(-3) = \sum_{n=-4}^{3} x(n)\, x_{-3}(n) \;=\; 0+0+0-5+8+0+0+0 = 3$

When m = –2 ; $r_{xx}(-2) = \sum_{n=-3}^{3} x(n)\, x_{-2}(n) \;=\; 0+0+3-20+6+0+0 = -11$

When m = –1 ; $r_{xx}(-1) = \sum_{n=-2}^{3} x(n)\, x_{-1}(n) \;=\; 0+4+12-15-10+0 = -9$

When m = 0 ; $r_{xx}(0) \;=\; \sum_{n=-1}^{3} x(n)\, x_0(n) \;=\; 1+16+9+25+4 = 55$

When m = 1 ; $r_{xx}(1) \;=\; \sum_{n=-1}^{4} x(n)\, x_1(n) \;=\; 0+4+12-15-10+0 = -9$

When m = 2 ; $r_{xx}(2) \;=\; \sum_{n=-1}^{5} x(n)\, x_2(n) \;=\; 0+0+3-20+6+0+0 = -11$

When m = 3 ; $r_{xx}(3) = \sum_{n=-1}^{6} x(n)\, x_3(n) = 0+0+0-5+8+0+0+0 = 3$

When m = 4 ; $r_{xx}(3) = \sum_{n=-1}^{7} x(n)\, x_4(n) = 0+0+0+0+2+0+0+0+0 = 2$

$$\therefore\; r_{xx}(m) = \{\, 2,\; 3,\; -11,\; -9,\; \underset{\uparrow}{55},\; -9,\; -11,\; 3,\; 2 \}$$

E2.19. ***Find the inverse system for the following discrete time system***

$$y(n) = \sum_{p=0}^{n} c^p\, x(p-2) \quad ; \quad for \quad n \ge 0.$$

Solution

Given that, $y(n) = \sum_{p=0}^{n} c^p x(p-2) \quad ; \quad \text{for } n \ge 0$

When n = 0 ; $y(0) = \sum_{p=0}^{0} c^p x(p-2) = c^0 x(-2) = x(-2) \quad \Rightarrow \quad x(-2) = y(0)$

When n = 1 ; $y(1) = \sum_{p=0}^{1} c^p x(p-2) = c^0 x(-2) + c^1 x(-1)$

$= x(-2) + c\,x(-1)$

$= y(0) + c\,x(-1) \qquad \Rightarrow \quad x(-1) = \frac{1}{c}[y(1) - y(0)]$

When n = 2 ; $y(2) = \sum_{p=0}^{2} c^p x(p-2) = c^0 x(-2) + c^1 x(-1) + c^2 x(0)$

$= x(-2) + c x(-1) + c^2 x(0)$

$= y(0) + y(1) - y(0) + c^2 x(0)$

$= y(1) + c^2 x(0) \qquad \Rightarrow \quad x(0) = \frac{1}{c^2}[y(2) - y(1)]$

Therefore, in general, $x(n) = \frac{1}{c^{n+2}}\,[y(n+2) - y(n+1)]$; for $n \ge -1$ with initial condition $x(-2) = y(0)$.

E2.20. ***A discrete time system is excited by an input x(n), and the response is,*** $y(n) = \{\underset{\uparrow}{4},\; 3,\; 6,\; 7.5,\; 3,\; 30,\; -8\}$. ***If the impulse response of the system is*** $h(n) = \{\underset{\uparrow}{2},\; 4,\; -2\}$, ***then what will be the input to the system?***

Solution

Let, N_1 = Number of samples in x(n)

N_2 = Number of samples in h(n)

N_3 = Number of samples in y(n)

Now, $N_3 = N_1 + N_2 - 1 \;\Rightarrow\; N_1 = N_3 - N_2 + 1 = 7 - 3 + 1 = 5$ samples

Each sample of x(n) is given by,

$$x(n) = \frac{1}{h(0)}\left[y(n) - \sum_{m=0}^{n-1} x(m)\, h(n-m)\right]$$

When n = 0 ; $x(0) = \frac{y(0)}{h(0)} = \frac{4}{2} = 2$

When n = 1 ; $x(1) = \frac{1}{h(0)}\left[y(1) - \sum_{m=0} x(m)\, h(1-m)\right] = \frac{1}{h(0)}\,[y(1) - x(0)\, h(1)] = \frac{1}{2}\,[3 - (2\times 4)] = -2.5$

When n = 2 ; $x(2) = \frac{1}{h(0)}\left[y(2) - \sum_{m=0}^{1} x(m)\, h(2-m)\right] = \frac{1}{h(0)}\,[y(2) - x(0)\, h(2) - x(1)\, h(1)] = \frac{1}{2}\,[6 - (2\times -2) - (-2.5\times 4)] = 10$

When n = 3 ; $x(3) = \frac{1}{h(0)}\left[y(3) - \sum_{m=0}^{2} x(m)\, h(3-m)\right] = \frac{1}{h(0)}\left[y(3) - x(0)\,h(3) - x(1)\,h(2) - x(2)\,h(1)\right]$

$= \frac{1}{2}\left[7.5 - (2\times 0) - (-2.5\times -2) - (10\times 4)\right] = -18.75$

When n = 4 ; $x(4) = \frac{1}{h(0)}\left[y(4) - \sum_{m=0}^{2} x(m)\, h(4-m)\right] = \frac{1}{h(0)}\left[y(4) - x(0)\,h(4) - x(1)\,h(3) - x(2)\,h(2) - x(3)\,h(1)\right]$

$= \frac{1}{2}\left[3 - (2\times 0) - (-2.5\times 0) - (10\times -2) - (-18.75\times 4)\right] = 49$

$\therefore\ x(n) = \{x(0),\ x(1),\ x(2),\ x(3),\ x(4)\} = \{\underset{\uparrow}{2},\ -2.5,\ 10,\ -18.75,\ 49\}$

E2.21. Perform circular correlation of the sequences, $x(n) = \{-1,\ 1,\ 2,\ 6\}$ and $y(n) = \{4,\ -2,\ -1,\ 2\}$

Solution

Let $\bar{r}_{xy}(m)$ be the sequence obtained by circular correlation of x(n) and y(n).

The circular correlation is given by,

$$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\, y((n-m))_N = \sum_{n=0}^{N-1} x(n)\, y_m(n), \quad \text{where } y_m(m) = y((n-m))_N$$

Here, N = 4, The circular correlation is performed by table method as shown below.

q	0	1	2	3	4	5	6	7
x(n)	–1	1	2	6				
y(n)	4	–2	–1	2				
$y_0(n)$	4	–2	–1	2	**4**	**–2**	**–1**	**2**
$y_1(n)$	**2**	4	–2	–1	2	**4**	**–2**	**–1**
$y_2(n)$	**–1**	**2**	4	–2	–1	2	**4**	**–2**
$y_3(n)$	**–2**	**–1**	**2**	4	–2	–1	2	**4**

When m = 0 ; $\bar{r}_{xy}(0) = \sum_{n=0}^{3} x(n)\, y_0(n) = -4 - 2 - 2 + 12 = 4$

When m = 1 ; $\bar{r}_{xy}(1) = \sum_{n=0}^{3} x(n)\, y_1(n) = -2 + 4 - 4 - 6 = -8$

When m = 2 ; $\bar{r}_{xy}(2) = \sum_{n=0}^{3} x(n)\, y_2(n) = 1 + 2 + 8 - 12 = -1$

When m = 3 ; $\bar{r}_{xy}(3) = \sum_{n=0}^{3} x(n)\, y_3(n) = 2 - 1 + 4 + 24 = 29$

$\therefore\ \bar{r}_{xy}(m) = \{4,\ -8,\ -1,\ 29\}$

Chapter 3

Z-Transform

3.1 Introduction

Transform techniques are an important tool in the analysis of signals and systems. The Laplace transforms are popularly used for analysis of continuous time signals and systems. Similarly Z-transform plays an important role in analysis and representation of discrete time signals and systems. The Z-transform provides a method for the analysis of discrete time signals and systems in the frequency domain which is generally more efficient than its time domain analysis.

The ***Z-transform*** of x(n) will convert the time domain signal x(n) to z-domain signal X(z), where the signal becomes a function of complex variable z.

The complex variable z is defined as,

$$z = u + jv = r\,e^{j\omega}$$

where, u = Real part of z ; v = Imaginary part of z

$$r = \sqrt{u^2 + v^2} = \text{Magnitude of } z$$

$$\omega = \tan^{-1}\frac{v}{u} = \text{Phase or Argument of } z$$

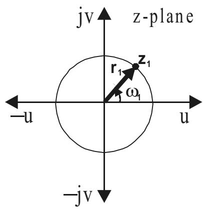


Fig 3.1 : *z-plane.*

The u and v takes value from $-\infty$ to $+\infty$. A two dimensional complex plane with values of u on horizontal axis and values of v on vertical axis as shown in fig 3.1 is called z-plane. A circle with radius r_1 in z-plane represents all values of z_1 having same magnitude r_1 with variable phase ω_1, where $\omega_1 = 0$ to 2π.

History of Z-Transform

A transform of a sampled signal or sequence was defined in 1947 by W. Hurewicz as,

$$z[f(kT)] = \sum_{k=0}^{\infty} f(kT)\,z^{-k}$$

which was later denoted in 1952 as Z-transform by a sampled-data control group at Columbia University led by professor John R. Raggazini and including L.A. Zadeh, E.I. Jury, R.E. Kalman, J.E. Bertram, B. Friedland and G.F. Franklin, (Source : www.ling.upenn.edu).

Definition of Z-Transform

Let, x(n) = Discrete time signal

X(z) = Z-transform of x(n)

The ***Z-transform*** of a discrete time signal, x(n) is defined as,

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}} \quad ; \quad \text{where, z is a complex variable.} \qquad .....(3.1)$$

The Z-transform of x(n) is symbolically denoted as,

$Z\{x(n)\}$; where, Z is the operator that represents Z-transform.

$$\boxed{\therefore\ X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}}$$

Since the time index n is defined for both positive and negative values, the discrete time signal x(n) in equation (3.1) is considered to be two-sided and the transform is called ***two-sided Z-transform***. If the signal x(n) is one-sided signal, [i.e., x(n) is defined only for positive value of n] then the Z-transform is called ***one-sided Z-transform.***

The one-sided Z-transform of x(n) is defined as,

$$\boxed{X(z) = Z\{x(n)\} = \sum_{n=0}^{+\infty} x(n)\, z^{-n}} \qquad .....(3.2)$$

The computation of X(z) involves summation of infinite terms which are functions of z. Hence it is possible that the infinite series may not converge to finite value for certain values of z. Therefore for every X(z) there will be a set of values of z for which X(z) can be computed. Such a set of values will lie in a particular region of z-plane and this region is called Region Of Convergence (ROC) of X(z).

Inverse Z-Transform

Let, X(z) be Z-transform of x(n). Now the signal x(n) can be uniquely determined from X(z) and its region of convergence (ROC).

The ***inverse Z-transform*** of X(z) is defined as,

$$\boxed{x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz} \qquad .....(3.3)$$

The inverse Z-transform of X(z) is symbolically denoted as,

$Z^{-1}\{X(z)\}$; where, Z^{-1} is the operator that represents the inverse Z-transform

$$\boxed{\therefore\ x(n) = Z^{-1}\{X(z)\} = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz}$$

We also refer x(n) and X(z) as a Z-transform pair and this relation is expressed as,

$$\boxed{x(n) \underset{Z^{-1}}{\overset{Z}{\rightleftharpoons}} X(z)}$$

Proof :

Consider the definition of Z-transform of x(n),

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{k=-\infty}^{+\infty} x(k)\, z^{-k} \qquad \boxed{\text{Let } n \rightarrow k}$$

$$X(z)\, z^{n-1} = \sum_{k=-\infty}^{+\infty} x(k)\, z^{-k}\, z^{n-1} \qquad \boxed{\text{Multiply both sides by } z^{n-1}}$$

Let us integrate the above equation on both sides over a closed contour "C" within the ROC of X(z) which encloses the origin.

$$\therefore \oint_c X(z)\, z^{n-1}\, dz = \oint_c \sum_{k=-\infty}^{+\infty} x(k)\, z^{n-1-k} dz$$

$$= \sum_{k=-\infty}^{+\infty} x(k) \oint_c z^{n-1-k} dz \qquad \boxed{\text{Interchanging the order of summation and integration.}}$$

$$= 2\pi j \sum_{k=-\infty}^{+\infty} x(k) \frac{1}{2\pi j} \oint_c z^{n-1-k} dz \qquad \boxed{\text{Multiply and divide by } 2\pi j.} \quad .....(3.4)$$

By Cauchy integral theorem,

$$\frac{1}{2\pi j} \oint_c z^{n-1-k}\, dz = 1 \quad ; \quad k = n$$

$$= 0 \quad ; \quad k \neq n$$

On applying Cauchy integral theorem the equation (3.4) reduces to,

$$\boxed{\sum_{k=-\infty}^{+\infty} x(k)\Big|_{n=k} = x(n)}$$

$$\oint_c X(z)\, z^{n-1}\, dz = 2\pi j\, x(n)$$

$$\therefore \quad x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

Geometric Series

The Z-transform of a discrete time signal involves convergence of geometric series. Hence the following two geometric series sum formula will be useful in evaluating Z-transform.

1. Infinite geometric series sum formula.

   If C is a complex constant and $0 < |C| < 1$, then,

$$\boxed{\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}} \qquad .....(3.5)$$

2. Finite geometric series sum formula.

   If C is a complex constant and,

When $C \neq 1$, $\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C} = \frac{C^N-1}{C-1}$ or $\sum_{n=0}^{N} C^n = \frac{C^{N+1}-1}{C-1}$	.....(3.6)
When $C = 1$, $\sum_{n=0}^{N-1} C^n = N$ or $\sum_{n=0}^{N} C^n = N+1$	.....(3.7)

Note : The infinite geometric series sum formula requires that the magnitude of C be strictly less than unity, but the finite geometric series sum formula is valid for any value of C.

3.2 Region of Convergence

Since the Z-transform is an infinite power series, it exists only for those values of z for which the series converges. The ***region of convergence, (ROC)*** of X(z) is the set of all values of z, for which X(z) attains a finite value. The ROC for the following six types of signals are discussed here.

Case i : Finite duration, right-sided (causal) signal

Case ii : Finite duration, left-sided (anticausal) signal

Case iii : Finite duration, two-sided (noncausal) signal

Case iv : Infinite duration, right-sided (causal) signal

Case v : Infinite duration, left-sided (anticausal) signal

Case vi : Infinite duration, two-sided (noncausal) signal

***Case i :* Finite duration, right-sided (causal) signal**

Let, x(n) be a finite duration signal with N-samples, defined in the range $0 \le n \le (N-1)$.

$$\therefore \; x(n) = \{x(0), x(1), x(2), ...x(N-1)\}$$

Now, the Z-transform of x(n) is,

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + ... + x(N-1)\, z^{-(N-1)}$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + ... + \frac{x(N-1)}{z^{N-1}}$$

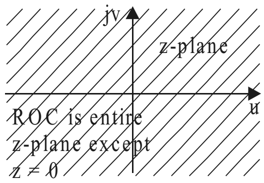


Fig 3.2 : *ROC of finite duration causal signal.*

In the above summation, when z = 0, all the terms except the first term become infinite. Hence the X(z) exists for all values of z, except z = 0. *Therefore, the ROC of finite duration right-sided (or causal signal) is entire z-plane except z = 0.*

***Case ii :* Finite duration, left-sided (anticausal) signal**

Let, x(n) be a finite duration signal with N-samples, defined in the range $-(N-1) \le n \le 0$.

$$\therefore \; x(n) = \{x(-(N-1)), ..., x(-2), x(-1), x(0)\}$$

Now, the Z-transform of x(n) is,

$$X(z) = \sum_{n=-(N-1)}^{0} x(n)\, z^{-n}$$

$$= x(-(N-1))\, z^{(N-1)} + ... + x(-2)z^2 + x(-1)z + x(0)$$

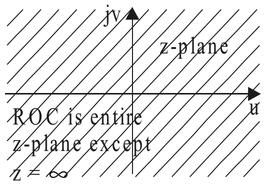


Fig 3.3 : *ROC of finite duration anticausal signal.*

In the above summation, when $z = \infty$, all the terms except the last term become infinite. Hence the X(z) exists for all values of z, except, $z = \infty$. *Therefore, the ROC of X(z) is entire z-plane, except $z = \infty$.*

***Case iii :* Finite duration, two-sided (noncausal) signal**

Let, x(n) be a finite duration signal with N-samples, defined in the range $-M \le n \le +M$, where, $M = \frac{N-1}{2}$

$$\therefore \; x(n) = \{x(-M), ..., x(-2), x(-1), x(0), x(1), x(2), ...x(M)\}$$

Now, the $\mathcal{Z}$-transform of x(n) is,

$$X(z) = \sum_{n=-M}^{+M} x(n)\, z^{-n}$$

$$= x(-M)\, z^{M} + \ldots\ldots + x(-2)z^{2} + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\ldots + x(M)\, z^{-M}$$

$$= x(-M)\, z^{M} + \ldots\ldots + x(-2)z^{2} + x(-1)z + x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^{2}} + \ldots\ldots + \frac{x(M)}{z^{M}}$$

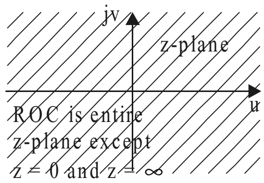


***Fig 3.4 :** ROC of finite duration two-sided signal.*

In the above summation, when z = 0, the terms with negative power of z attain infinity and when $z = \infty$, the terms with positive power of z attain infinity. Hence X(z) converges for all values of z, except z = 0 and $z = \infty$. *Therefore, the ROC is entire z-plane, except z = 0 and $z = \infty$.*

***Case - iv :** Infinite duration, right-sided (causal) signal*

Let, $x(n) = r_1^{\,n}$; $n \geq 0$

Now, the $\mathcal{Z}$-transform of x(n) is,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} r_1^{n}\, z^{-n} = \sum_{n=0}^{\infty} \left(r_1\, z^{-1}\right)^{n}$$

> Using infinite geometric series sum formula
> $$\sum_{n=0}^{\infty} C^{n} = \frac{1}{1-C}$$
> if, $0 < |C| < 1$

If, $0 < |r_1\, z^{-1}| < 1$, then $\displaystyle\sum_{n=0}^{\infty} (r_1\, z^{-1})^{n} = \frac{1}{1 - r_1\, z^{-1}}$

$$\therefore\ X(z) = \frac{1}{1 - r_1\, z^{-1}} = \frac{1}{1 - \dfrac{r_1}{z}} = \frac{1}{\dfrac{z - r_1}{z}} = \frac{z}{z - r_1}$$

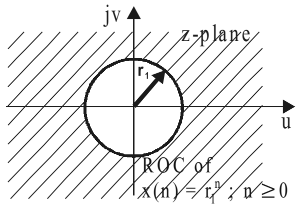


***Fig 3.5 :** ROC of infinite duration right-sided signal.*

Here the condition to be satisfied for the convergence of X(z) is,

$$0 < |r_1\, z^{-1}| < 1$$

$$\therefore\ |r_1\, z^{-1}| < 1 \ \Rightarrow\ \frac{|r_1|}{|z|} < 1 \ \Rightarrow\ |z| > |r_1|$$

The term $|r_1|$ represents a circle of radius r_1 in z-plane as shown in fig 3.5. From the above analysis we can say that, X(z) converges for all points external to the circle of radius r_1 in z-plane. *Therefore, the ROC of X(z) is exterior of the circle of radius r_1 in z-plane as shown in fig 3.5.*

***Case v :** Infinite duration, left-sided (anticausal) signal*

Let, $x(n) = r_2^{\,n}$; $n \leq 0$

Now, the $\mathcal{Z}$-transform of x(n) is,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{0} r_2^{n}\, z^{-n} = \sum_{n=0}^{+\infty} r_2^{-n}\, z^{n} = \sum_{n=0}^{+\infty} (r_2^{-1}\, z)^{n}$$

If, $0 < |r_2^{-1}\, z| < 1$, then $\displaystyle\sum_{n=0}^{\infty} (r_2^{-1}\, z)^{n} = \frac{1}{1 - r_2^{-1}\, z}$

$$\therefore\ X(z) = \frac{1}{1 - r_2^{-1}\, z} = \frac{1}{1 - \dfrac{z}{r_2}} = \frac{1}{\dfrac{r_2 - z}{r_2}} = \frac{r_2}{r_2 - z} = -\frac{r_2}{z - r_2}$$

> Using infinite geometric series sum formula
> $$\sum_{n=0}^{\infty} C^{n} = \frac{1}{1-C}$$
> if, $0 < |C| < 1$

Here the condition to be satisfied for the convergence of X(z) is,

$$0 < |r_2^{-1}\, z| < 1$$

$$\therefore\ |r_2^{-1}\, z| < 1 \ \Rightarrow\ \frac{|z|}{|r_2|} < 1 \ \Rightarrow\ |z| < |r_2|$$

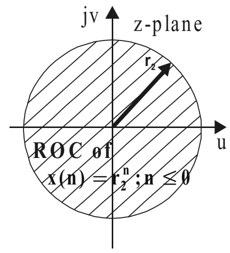


***Fig 3.6 :** ROC of infinite duration left-sided signal.*

The term $|r_2|$ represents a circle of radius r_2 in z-plane as shown in fig 3.6. From the above analysis we can say that X(z) converges for all points internal to the circle of radius r_2 in z-plane. *Therefore, the ROC of X(z) is interior of the circle of radius r_2 as shown in fig 3.6.*

***Case vi:** Infinite duration, two-sided (noncausal) signal*

Let, $x(n) = r_1^n\, u(n) + r_2^n\, u(-n)$

Now, the Z-transform of x(n) is,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{0} r_2^n\, z^{-n} + \sum_{n=0}^{+\infty} r_1^n\, z^{-n} = \sum_{n=0}^{+\infty} r_2^{-n}\, z^{n} + \sum_{n=0}^{+\infty} r_1^n\, z^{-n}$$

$$= \sum_{n=0}^{+\infty} (r_2^{-1}\, z)^n + \sum_{n=0}^{+\infty} (r_1\, z^{-1})^n$$

$$= \frac{1}{1 - r_2^{-1}\, z} + \frac{1}{1 - r_1\, z^{-1}}$$

Infinite geometric series sum formula
$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C} \quad \text{if,}\ \ 0 < |C| < 1$$

Using infinite geometric series sum formula
if, $0 < |r_2^{-1}\, z| < 1$, and, $0 < |r_1\, z^{-1}| < 1$

The term $\sum_{n=0}^{\infty} (r_2^{-1}\, z)^n$ converges if,

$$0 < |r_2^{-1}\, z| < 1$$

$$\therefore\ |r_2^{-1}\, z| < 1 \ \Rightarrow\ \frac{|z|}{|r_2|} < 1 \ \Rightarrow\ |z| < |r_2|$$

The term $\sum_{n=0}^{\infty} (r_1\, z^{-1})^n$ converges if,

$$0 < |r_1\, z^{-1}| < 1$$

$$\therefore\ |r_1\, z^{-1}| < 1 \ \Rightarrow\ \frac{|r_1|}{|z|} < 1 \ \Rightarrow\ |z| > |r_1|$$

The term $|r_2|$ represents a circle of radius r_2 and $|r_1|$ represents a circle of radius r_1 in z-plane. If $|r_2| > |r_1|$ then there will be a region between two circles as shown in fig 3.7. Now the X(z) will converge for all points in the region between two circles (because the first term of X(z) converges for $|z| < |r_2|$ and the second term of X(z) converges for $|z| > |r_1|$). Hence the *ROC is the region between two circles of radius r_1 and r_2* as shown in fig 3.7.

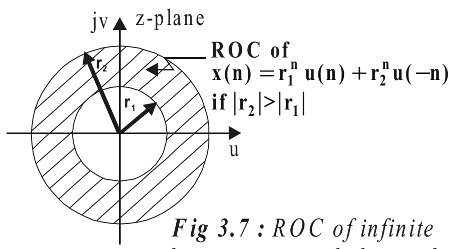
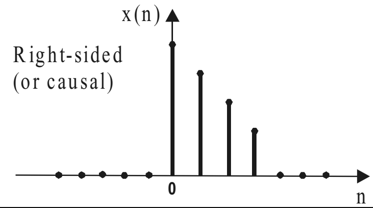
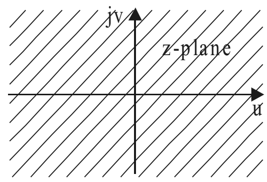
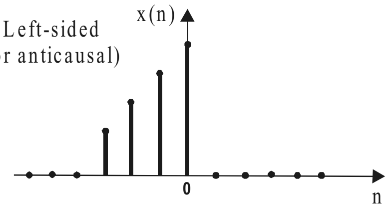
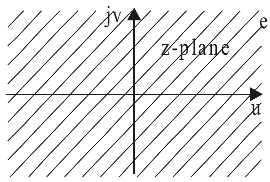
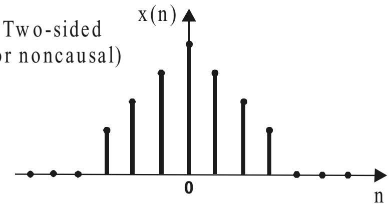
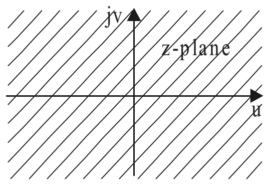
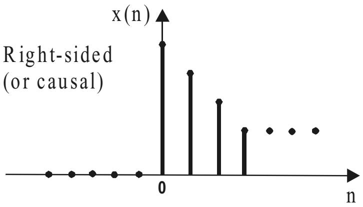
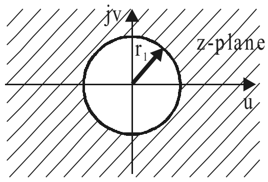
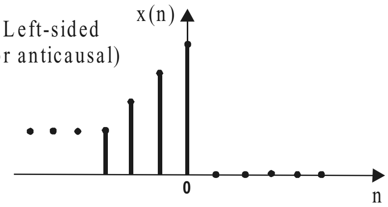
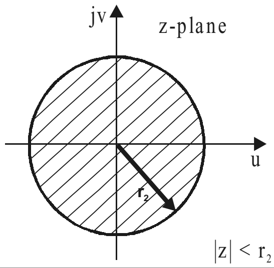
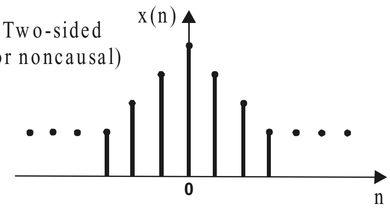
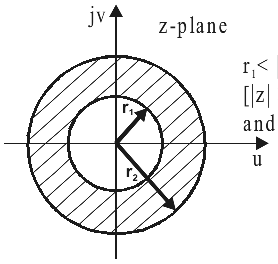


***Fig 3.7 :** ROC of infinite duration two-sided signal.*

Table 3.1 : Summary of ROC of Discrete Time Signals

Sequence	ROC
Finite, right-sided (causal)	Entire z-plane except $z = 0$
Finite, left-sided (anticausal)	Entire z-plane except $z = \infty$
Finite, two-sided (noncausal)	Entire z-plane except $z = 0$ and $z = \infty$
Infinite, right-sided (causal)	Exterior of circle of radius r_1, where $\lvert z\rvert > r_1$
Infinite, left-sided (anticausal)	Interior of circle of radius r_2, where $\lvert z\rvert < r_2$
Infinite, two-sided (noncausal)	The area between two circles of radius r_2 and r_1 where, $r_2 > r_1$, and $r_1 < \lvert z\rvert < r_2$, (i.e., $\lvert z\rvert > r_1$, and, $\lvert z\rvert < r_2$)

Table 3.2 : Characteristic Families of Signals and Corresponding ROC

Signal	ROC in z-plane
Finite Duration Signals	
Right-sided (or causal) 	 Entire z-plane except z = 0
Left-sided (or anticausal) 	 Entire z-plane except z = ∞
Two-sided (or noncausal) 	 Entire z-plane except z = 0 and z = ∞
Infinite Duration Signals	
Right-sided (or causal) 	 $\|z\|>r_1$
Left-sided (or anticausal) 	 $\|z\| < r_2$
Two-sided (or noncausal) 	 $r_1 < \|z\| < r_2$ [$\|z\| > r_1$ and $\|z\| < r_2$]

Example 3.1

Determine the Z-transform and their ROC of the following discrete time signals.

a) $x(n) = \{\underset{\uparrow}{3}, 4, 2, 7\}$ **b)** $x(n) = \{6, 8, 9, \underset{\uparrow}{3}\}$ **c)** $x(n) = \{2, 4, \underset{\uparrow}{6}, 8, 10\}$

Solution

a) Given that, $x(n) = \{\underset{\uparrow}{3}, 4, 2, 7\}$

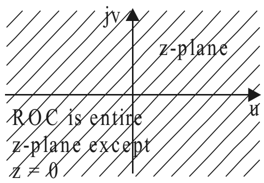
i.e., $x(0) = 3$; $x(1) = 4$; $x(2) = 2$; $x(3) = 7$; and $x(n) = 0$ for $n < 0$ and for $n > 3$.

By the definition of Z-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The given sequence is a finite duration sequence defined in the range n = 0 to 3, hence the limits of summation is changed to n = 0 to n = 3.

$$\therefore X(z) = \sum_{n=0}^{3} x(n)\, z^{-n}$$



$$= x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3}$$

$$= 3 + 4z^{-1} + 2z^{-2} + 7z^{-3}$$

$$= 3 + \frac{4}{z} + \frac{2}{z^2} + \frac{7}{z^3}$$

In X(z), when z = 0, except the first terms all other terms will become infinite. Hence X(z) will be finite for all values of z, except z = 0. Therefore, the ROC is entire z-plane except z = 0.

b) Given that, $x(n) = \{6, 8, 9, \underset{\uparrow}{3}\}$

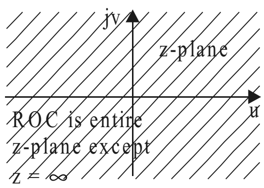
i.e, $x(-3) = 6$; $x(-2) = 8$; $x(-1) = 9$; $x(0) = 3$; and $x(n) = 0$ for $n < -3$ and for $n > 0$.

By the definition of Z-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The given sequence is a finite duration sequence defined in the range n = –3 to 0, hence the limits of summation is changed to n = –3 to 0.

$$\therefore X(z) = \sum_{n=-3}^{0} x(n)\, z^{-n}$$



$$= x(-3)\, z^{3} + x(-2)\, z^{2} + x(-1)\, z + x(0)$$

$$= 6z^{3} + 8z^{2} + 9z + 3$$

In X(z), when z = ¥, except the last term all other terms become infinite. Hence X(z) will be finite for all values of z, except z = ¥. Therefore, the ROC is entire z-plane except z = ¥.

c) Given that, $x(n) = \{2, 4, \underset{\uparrow}{6}, 8, 10\}$

i.e, $x(-2) = 2$; $x(-1) = 4$; $x(0) = 6$; $x(1) = 8$; $x(2) = 10$ and $x(n) = 0$ for $n < -2$ and for $n > 2$.

By the definition of Z-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

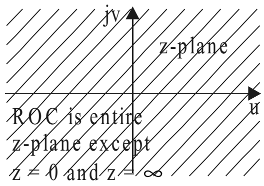
The given sequence is a finite duration sequence defined in the range n = –2 to +2, hence the limits of summation is changed to n = –2 to n = 2.

$$\therefore\ X(z) = \sum_{n=-2}^{2} x(n)\, z^{-n}$$

$$= x(-2)\, z^{2} + x(-1)\, z^{1} + x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2}$$

$$= 2z^{2} + 4z + 6 + 8z^{-1} + 10z^{-2}$$

$$= 2z^{2} + 4z + 6 + \frac{8}{z} + \frac{10}{z^{2}}$$



In X(z), when z = 0, the terms with negative power of z will become infinite and when z = ¥, the terms with positive power of z will become infinite. Hence X(z) will be finite for all values of z except when z = 0 and z = ¥. Therefore, the ROC is entire z-plane except z = 0 and z = ¥.

Example 3.2

Determine the Z-transform and their ROC of the following discrete time signals.

a) $x(n) = u(n)$ **b)** $x(n) = 0.3^n\, u(n)$ **c)** $x(n) = 0.8^n u(-n-1)$ **d)** $x(n) = 0.3^n\, u(n) + 0.8^n\, u(-n-1)$

Solution

a) Given that, x(n) = u(n)

The u(n) is a discrete unit step signal, which is defined as,

u(n) = 1 ; for n ³ 0
= 0 ; for n < 0

Infinite geometric series sum formula
$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$; if, $0 < \lvert C \rvert < 1$

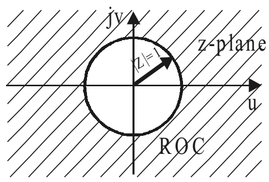
By the definition of Z-transform,

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1-z^{-1}}$$

Using infinite geometric series sum formula.

$$= \frac{1}{1-1/z} = \frac{z}{z-1}$$



Here the condition for convergence is, $0 < |z^{-1}| < 1$.

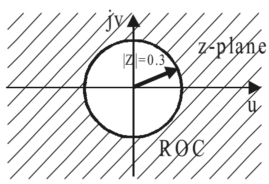
$$\therefore\ |z^{-1}| < 1 \quad \Rightarrow \quad \frac{1}{|z|} < 1 \quad \Rightarrow \quad |z| > 1$$

The term |z| = 1 represents a circle of unit radius in z-plane. Therefore, the ROC is exterior of unit circle in z-plane.

b) Given that, x(n) = 0.3ⁿ u(n)

The u(n) is a discrete unit step signal, which is defined as,

u(n) = 1 ; for n ³ 0
= 0 ; for n < 0
\ x(n) = 0.3^n ; for n ³ 0
= 0 ; for n < 0



By the definition of Z-transform,

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} 0.3^n\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(0.3z^{-1}\right)^n = \frac{1}{1 - 0.3z^{-1}}$$

Using infinite geometric series sum formula.

$$\therefore\ X(z) = \frac{1}{1 - 0.3\,\frac{1}{z}} = \frac{z}{z - 0.3}$$

Here the condition for convergence is, $0 < |0.3\ z^{-1}| < 1$.

$$\therefore\ |0.3\ z^{-1}| < 1 \ \Rightarrow\ \frac{0.3}{|z|} < 1 \ \Rightarrow\ |z| > 0.3$$

The term $|z| = 0.3$ represents a circle of radius 0.3 in z-plane. Therefore, the ROC is exterior of circle with radius 0.3 in z-plane.

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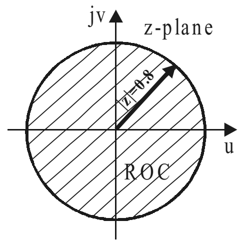
c) Given that, $x(n) = 0.8^n\, u(-n-1)$



The $u(-n-1)$ is a discrete unit step signal, which is defined as,

$$u(-n-1) = 0 \quad ; \text{ for } n \geq 0$$
$$= 1 \quad ; \text{ for } n \leq -1$$
$$\therefore\ x(n) = 0 \quad ; \text{ for } n \geq 0$$
$$= 0.8^n \ ; \text{ for } n \leq -1$$

By the definition of Ƶ-transform,

$$Ƶ\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{-1} 0.8^n\, z^{-n}$$

$$= \sum_{n=1}^{\infty} 0.8^{-n}\, z^n = \sum_{n=1}^{\infty} (0.8^{-1}\, z)^n = \sum_{n=0}^{\infty} (0.8^{-1}\, z)^n - 1 \qquad \boxed{(0.8^{-1}\, z)^0 = 1}$$

$$= \frac{1}{1 - (0.8^{-1}\, z)} - 1 \qquad \boxed{\text{Using infinite geometric series sum formula.}}$$

$$= \frac{1}{1 - \frac{z}{0.8}} - 1 = \frac{0.8}{0.8 - z} - 1 = \frac{0.8 - 0.8 + z}{0.8 - z} = \frac{z}{0.8 - z} = -\frac{z}{z - 0.8}$$

Here the condition for convergence is, $0 < |0.8^{-1}\, z| < 1$.

$$\therefore\ |0.8^{-1}\, z| < 1 \ \Rightarrow\ \frac{|z|}{0.8} < 1 \ \Rightarrow\ |z| < 0.8$$

The term $|z| = 0.8$, represents a circle of radius 0.8 in z-plane. Therefore, the ROC is interior of the circle of radius 0.8 in z-plane.

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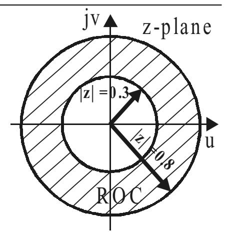
d) Given that, $x(n) = 0.3^n\, u(n) + 0.8^n\, u(-n-1)$



$$X(z) = Ƶ\{x(n)\} = Ƶ\{0.3^n\, u(n) + 0.8^n\, u(-n-1)\}$$

$$= Ƶ\{0.3^n\, u(n)\} + Ƶ\{0.8^n\, u(-n-1)\} \qquad \boxed{\text{Using linearity property.}}$$

$$= \frac{z}{z - 0.3} - \frac{z}{z - 0.8} \qquad \boxed{\text{Using the results of (b) and (c).}}$$

$$= \frac{z(z - 0.8) - z(z - 0.3)}{(z - 0.3)(z - 0.8)} = \frac{z^2 - 0.8z - z^2 + 0.3z}{z^2 - 0.8z - 0.3z + 0.24} = \frac{-0.5z}{z^2 - 1.1z + 0.24}$$

Here the condition for convergence of $0.3^n\, u(n)$ is,

$$0 < |0.3\ z^{-1}| < 1 \ \Rightarrow\ |z| > 0.3$$

and the condition for convergence of $0.8^n\, u(-n-1)$ is,

$$0 < |0.8^{-1}\, z| < 1 \ \Rightarrow\ |z| < 0.8$$

The term $|z| = 0.8$, represents a circle of radius 0.8 in z-plane and the term $|z| = 0.3$ represents a circle of radius 0.3 in z-plane. Hence the common region of convergence for both the terms of x(n) is the region in between the circles of radius $|z| = 0.8$ and $|z| = 0.3$ in the z-plane.

3.3 Properties of Ƶ-Transform

1. Linearity property

The linearity property of Ƶ-transform states that the Ƶ-transform of linear weighted combination of discrete time signals is equal to similar linear weighted combination of Ƶ-transform of individual discrete time signals.

Let, $Ƶ\{x_1(n)\} = X_1(z)$ and $Ƶ\{x_2(n)\} = X_2(z)$ then by linearity property,

$Ƶ\{a_1x_1(n) + a_2x_2(n)\} = a_1X_1(z) + a_2X_2(z)$; where, a_1 and a_2 are constants.

Proof :

By definition of Ƶ-transform,

$$X_1(z) = Ƶ\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n} \qquad .....(3.8)$$

$$X_2(z) = Ƶ\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad .....(3.9)$$

$$\therefore\ Ƶ\{a_1x_1(n)+a_2x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[a_1x_1(n)+a_2x_2(n)\right] z^{-n} = \sum_{n=-\infty}^{+\infty} \left[a_1x_1(n)\, z^{-n} + a_2x_2(n)\, z^{-n}\right]$$

$$= \sum_{n=-\infty}^{+\infty} a_1x_1(n)\, z^{-n} + \sum_{n=-\infty}^{+\infty} a_2x_2(n) z^{-n} = a_1 \sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n} + a_2 \sum_{n=-\infty}^{+\infty} x_2(n) z^{-n}$$

$$= a_1\, X_1(z) + a_2X_2(z)$$

Using equations (3.8) and (3.9).

2. Shifting property

Case i: ***Two-sided Ƶ-transform***

The shifting property of Ƶ-transform states that, Ƶ-transform of a shifted signal shifted by m-units of time is obtained by multiplying z^m to Ƶ-transform of unshifted signal.

Let, $Ƶ\{x(n)\} = X(z)$

Now, by shifting property,

$Ƶ\{x(n-m)\} = z^{-m}\, X(z)$

$Ƶ\{x(n+m)\} = z^{m}\, X(z)$

Proof :

By definition of Ƶ-transform,

$$X(z) = Ƶ\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(3.10)$$

$$\therefore\ Ƶ\{x(n-m)\} = \sum_{n=-\infty}^{+\infty} x(n-m)\, z^{-n}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{-(m+p)}$$

Let, n – m = p, \ n = p + m
when n ® -¥, p ® -¥
when n ® +¥, p ® +¥

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{-m}\ z^{-p}$$

$$= z^{-m} \sum_{p=-\infty}^{+\infty} x(p)\, z^{-p} = z^{-m} \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

Let, p → n

$$= z^{-m}\, X(z)$$

Using equation (3.10).

By definition of Ƶ-transform,

$$\mathcal{Z}\{x(n+m)\} = \sum_{n=-\infty}^{+\infty} x(n+m)\, z^{-n}$$

Let, $n+m=p$, $\therefore\ n=p-m$
when $n \rightarrow -\infty$, $p \rightarrow -\infty$
when $n \rightarrow +\infty$, $p \rightarrow +\infty$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{-(p-m)}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{-p} z^{m}$$

$$= z^{m} \sum_{p=-\infty}^{+\infty} x(p)\, z^{-p} = z^{m} \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

Let, $p \rightarrow n$

$$= z^{m}\, X(z)$$

Using equation (3.10).

***Case ii:* One-sided Ƶ-transform**

Let x(n) be a discrete time signal defined in the range $0 < n < \infty$.

Let, $\mathcal{Z}\{x(n)\} = X(z)$

Now by shifting property,

$$\mathcal{Z}\left\{x(n-m)\right\} = z^{-m}\, X(z) + \sum_{i=1}^{m} x(-i)\, z^{-(m-i)}$$

$$\mathcal{Z}\left\{x(n+m)\right\} = z^{m}\, X(z) - \sum_{i=0}^{m-1} x(i)\, z^{(m-i)}$$

Proof :

By definition of one-sided Ƶ-transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{+\infty} x(n)\, z^{-n} \qquad .....(3.11)$$

$$\therefore\ \mathcal{Z}\{x(n-m)\} = \sum_{n=0}^{+\infty} x(n-m)\, z^{-n}$$

Multiply by z^{m} and z^{-m}

$$= \sum_{n=0}^{+\infty} x(n-m)\, z^{-n} z^{m} z^{-m}$$

$$= z^{-m} \sum_{n=0}^{+\infty} x(n-m)\, z^{-(n-m)}$$

Let, $n-m=p$,
when $n \rightarrow 0$, $p \rightarrow -m$
when $n \rightarrow +\infty$, $p \rightarrow +\infty$

$$= z^{-m} \sum_{p=-m}^{+\infty} x(p)\, z^{-p}$$

$$= z^{-m} \sum_{p=0}^{+\infty} x(p)\, z^{-p} + z^{-m} \sum_{p=-m}^{-1} x(p)\, z^{-p}$$

$$= z^{-m} \sum_{p=0}^{+\infty} x(p)\, z^{-p} + z^{-m} \sum_{p=1}^{m} x(-p)\, z^{p}$$

Let $p = n$, in first summation.
Let $p = i$, in second summation.

$$= z^{-m} \sum_{n=0}^{+\infty} x(n)\, z^{-n} + z^{-m} \sum_{i=1}^{m} x(-i)\, z^{i}$$

Using equation (3.11).

$$= z^{-m}\, X(z) + \sum_{i=1}^{m} x(-i)\, z^{-(m-i)} \qquad .....(3.12)$$

Note : In equation (3.12) if x(−i) for i = 1 to m are zero then the shifting property of one-sided Ƶ-transform for delayed signal will be same as that for two-sided Ƶ-transform.

By definition of one-sided $\mathcal{Z}$-transform,

$$\mathcal{Z}\{x(n+m)\} = \sum_{n=0}^{+\infty} x(n+m)\, z^{-n}$$

$$= \sum_{n=0}^{+\infty} x(n+m)\, z^{-n}\, z^{m}\, z^{-m} \qquad \boxed{\text{Multiply by } z^m \text{ and } z^{-m}}$$

$$= z^m \sum_{n=0}^{+\infty} x(n+m)\, z^{-(n+m)}$$

$$= z^m \sum_{p=m}^{+\infty} x(p)\, z^{-p} \qquad \boxed{\begin{array}{l}\text{Let, } n+m=p,\\ \text{when } n \rightarrow 0 \ ,\ p \rightarrow m\\ \text{when } n \rightarrow +\infty,\ p \rightarrow +\infty\end{array}}$$

$$= z^m \sum_{p=0}^{+\infty} x(p)\, z^{-p} - z^m \sum_{p=0}^{m-1} x(p)\, z^{-p}$$

$$= z^m \sum_{n=0}^{+\infty} x(n)\, z^{-n} - z^m \sum_{i=0}^{m-1} x(i)\, z^{-i} \qquad \boxed{\begin{array}{l}\text{Let } p=n, \text{ in first summation.}\\ \text{Let } p=i, \text{ in second summation.}\end{array}}$$

$\boxed{\text{Using equation (3.11).}}$

$$= z^m\, X(z) - \sum_{i=0}^{m-1} x(i)\, z^{m-i} \qquad .....(3.13)$$

Note : In equation (3.13) if x(i) for i = 0 to m – 1 are zero then the shifting property of one-sided $\mathcal{Z}$-transform for advanced signal will be same as that for two-sided $\mathcal{Z}$-transform.

3. *Multiplication by n (or Differentiation in z-domain)*

If $\mathcal{Z}\{x(n)\} = X(z)$

then $\mathcal{Z}\left\{nx(n)\right\} = -z\,\dfrac{d}{dz}\,X(z)$

In general,

$$\mathcal{Z}\left\{n^m\, x(n)\right\} = \left(-z\,\frac{d}{dz}\right)^m X(z)$$

$$= \underbrace{-z\,\frac{d}{dz}\left(-z\frac{d}{dz}\left(.....\left(-z\,\frac{d}{dz}\left(-z\,\frac{d}{dz}\right.\right.\right.\right.}_{m\,-\,\text{times}}\left.\left.\left.\left. X(z)\right)\right)\,.....\ \right)\right)$$

Proof :

By definition of $\mathcal{Z}$-transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(3.14)$$

$$\therefore\ \mathcal{Z}\{n\, x(n)\} = \sum_{n=-\infty}^{+\infty} n\, x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} n\, x(n)\, z^{-n}\, z\, z^{-1} \qquad \boxed{\text{Multiply by z and } z^{-1}}$$

$$= -z \sum_{n=-\infty}^{+\infty} x(n)\left[-n\, z^{-n-1}\right]$$

$$= -z \sum_{n=-\infty}^{+\infty} x(n)\left[\frac{d}{dz}\, z^{-n}\right] \qquad \boxed{\frac{d}{dz}\, z^{-n} = -n\, z^{-n-1}}$$

$$= -z\,\frac{d}{dz}\sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad \boxed{\begin{array}{l}\text{Interchanging summation}\\ \text{and differentiation.}\end{array}}$$

$$= -z\,\frac{d}{dz}\, X(z) \qquad \boxed{\text{Using equation (3.14).}}$$

4. *Multiplication by an exponential sequence, a^n (or Scaling in z-domain)*

If $\mathcal{Z}\{x(n)\} = X(z)$

then $\mathcal{Z}\left\{a^n\, x(n)\right\} = X(a^{-1}z)$

Proof :

By definition of Z-transform,

$$\mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(3.15)$$

$$\therefore\ \mathcal{Z}\{a^n\, x(n)\} = \sum_{n=-\infty}^{+\infty} a^n\, x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, (a^{-1}z)^{-n} \qquad .....(3.16)$$

$$= X(a^{-1}z)$$

The equation (3.16) is similar to the form of equation (3.15).

5. *Time reversal*

If $\mathcal{Z}\{x(n)\} = X(z)$

then $\mathcal{Z}\{x(-n)\} = X(z^{-1})$

Proof :

By definition of Z-transform,

$$\mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(3.17)$$

$$\therefore\ \mathcal{Z}\{x(-n)\} = \sum_{n=-\infty}^{+\infty} x(-n)\, z^{-n}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{p}$$

Let, $p = -n$
when $n \to -\infty$, $p \to +\infty$
when $n \to +\infty$, $p \to -\infty$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, (z^{-1})^{-p} \qquad .....(3.18)$$

$$= X(z^{-1})$$

The equation (3.18) is similar to the form of equation (3.17).

6. *Conjugation*

If $\mathcal{Z}\{x(n)\} = X(z)$

then $\mathcal{Z}\{x^*(n)\} = X^*(z^*)$

Proof :

By definition of Z-transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(3.19)$$

$$\therefore\ \mathcal{Z}\{x^*(n)\} = \sum_{n=-\infty}^{+\infty} x^*(n)\, z^{-n}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x(n)\, (z^*)^{-n}\right]^* \qquad .....(3.20)$$

$$= \left[X(z^*)\right]^*$$

$$= X^*(z^*)$$

The equation (3.20) is similar to the form of equation (3.19).

7. Convolution theorem

If $\mathcal{Z}\{x_1(n)\} = X_1(z)$

and $\mathcal{Z}\{x_2(n)\} = X_2(z)$

then $\mathcal{Z}\{x_1(n) * x_2(n)\} = X_1(z)\, X_2(z)$

$$\text{where, } x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) \qquad .....(3.21)$$

Proof :

By definition of $\mathcal{Z}$-transform,

$$X_1(z) = \mathcal{Z}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n} \qquad .....(3.22)$$

$$X_2(z) = \mathcal{Z}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad .....(3.23)$$

$$\therefore\ \mathcal{Z}\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{+\infty} \big[x_1(n) * x_2(n)\big]\, z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \left[\sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\right] z^{-n}$$

Using equation (3.21).

$$= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\, z^{-n}\, z^{-m}\, z^{m}$$

Multiply by z^m and z^{-m}

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, z^{-m} \sum_{n=-\infty}^{+\infty} x_2(n-m)\, z^{-(n-m)}$$

Let, $n - m = p$
when n® -¥, p® -¥
when n® +¥, p® +¥

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, z^{-m} \sum_{p=-\infty}^{+\infty} x_2(p)\, z^{-p}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n}\right]\left[\sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n}\right]$$

Let $m = n$, in first summation.
Let $p = n$, in second summation.

$$= X_1(z)\, X_2(z)$$

Using equations (3.22) and (3.23).

8. Correlation property

If $\mathcal{Z}\{x(n)\} = X(z)$ and $\mathcal{Z}\{y(n)\} = Y(z)$

then $\mathcal{Z}\{r_{xy}(m)\} = X(z)\, Y(z^{-1})$

$$\text{where, } r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) \qquad .....(3.24)$$

Proof :

By definition of $\mathcal{Z}$-transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(3.25)$$

$$Y(z) = \mathcal{Z}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\, z^{-n} \qquad .....(3.26)$$

$$
\begin{aligned}
\therefore\ \mathcal{Z}\{r_{xy}(m)\} &= \sum_{m=-\infty}^{+\infty} r_{xy}(m)\, z^{-m} \\
&= \sum_{m=-\infty}^{+\infty} \left[\sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)\right] z^{-m} && \boxed{\text{Using equation (3.24).}} \\
&= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)\, z^{-m} z^{-n} z^{n} && \boxed{\text{Multiply by } z^{n} \text{ and } z^{-n}} \\
&= \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \sum_{m=-\infty}^{+\infty} y(n-m)\, z^{(n-m)} \\
&= \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \sum_{p=-\infty}^{+\infty} y(p)\, z^{p} && \boxed{\begin{array}{l}\text{Let, } n-m=p \quad \therefore\ m=n-p \\ \text{when } m \to -\infty,\ \ p \to +\infty, \\ \text{when } m \to +\infty,\ \ p \to -\infty.\end{array}} \\
&= \left[\sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}\right]\left[\sum_{p=-\infty}^{+\infty} y(p)\, (z^{-1})^{-p}\right] \\
&= X(z)\ Y(z^{-1}) && \boxed{\begin{array}{c}\text{Using equations (3.25)} \\ \text{and (3.26).}\end{array}}
\end{aligned}
$$

9. *Initial value theorem*

Let x(n) be an one-sided signal defined in the range $0 \le n \le \infty$.

Now, if $\mathcal{Z}\{x(n)\} = X(z)$,

then the initial value of x(n) [i.e., x(0)] is given by,

$$x(0) = \underset{z \to \infty}{\mathrm{Lt}}\ X(z)$$

Proof :

By definition of one-sided Z - transfrom,

$$X(z) = \sum_{n=0}^{+\infty} x(n)\, z^{-n}$$

On expanding the above summation we get,

$$
\begin{aligned}
X(z) &= x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + \ldots\ldots \\
\therefore\ X(z) &= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \ldots\ldots
\end{aligned}
$$

On taking limit $z \to \infty$ in the above equation we get,

$$
\begin{aligned}
\underset{z \to \infty}{\mathrm{Lt}}\ X(z) &= \underset{z \to \infty}{\mathrm{Lt}} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \ldots\ldots\right] \\
&= x(0) + 0 + 0 + 0 + \ldots\ldots \\
\therefore\ x(0) &= \underset{z \to \infty}{\mathrm{Lt}}\ X(z)
\end{aligned}
$$

10. Final value theorem

Let x(n) be a one-sided signal defined in the range 0 £ n £ ¥.

Now, if $\mathcal{Z}\{x(n)\} = X(z)$,

then the final value of x(n) [i.e., x(¥)] is given by,

$$x(\infty) = \underset{z \to 1}{\text{Lt}} \; (1 - z^{-1})\, X(z) \quad \text{or} \quad x(\infty) = \underset{z \to 1}{\text{Lt}} \left(\frac{z-1}{z}\right) X(z)$$

Proof :

By definition of one-sided $\mathcal{Z}$-transfrom,

$$\mathcal{Z}\{x(n)\} = \sum_{n=0}^{+\infty} x(n)\, z^{-n} \qquad .....(3.27)$$

$$\therefore \; \underset{\text{(LHS)}}{\mathcal{Z}\{x(n-1) - x(n)\}} = \sum_{n=0}^{+\infty} \underset{\text{(RHS)}}{\big[x(n-1) - x(n)\big]\, z^{-n}}$$

$$\begin{aligned}
\text{LHS} &= \mathcal{Z}\{x(n-1) - x(n)\} \\
&= \mathcal{Z}\{x(n-1)\} - \mathcal{Z}\{x(n)\} && \boxed{\text{Using linearity property.}} \\
&= z^{-1}\, X(z) + x(-1) - X(z) && \boxed{\text{Using shifting property and equation (3.27).}} \\
&= x(-1) - (1 - z^{-1})\, X(z) \\
&= \underset{z \to 1}{\text{Lt}} \Big[x(-1) - (1 - z^{-1})\, X(z)\Big] && \boxed{\text{Taking limit } z \to 1} \\
&= x(-1) - \underset{z \to 1}{\text{Lt}} \; (1 - z^{-1})\, X(z) && .....(3.28)
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \sum_{n=0}^{+\infty} \big[x(n-1) - x(n)\big] z^{-n} \\
&= \underset{z \to 1}{\text{Lt}} \sum_{n=0}^{+\infty} \big[x(n-1) - x(n)\big]\, z^{-n} && \boxed{\text{Taking limit z ® 1}} \\
&= \sum_{n=0}^{+\infty} \big[x(n-1) - x(n)\big] && \boxed{\text{On applying limit z ® 1, the term } z^{-n} \text{ becomes unity.}} \\
&= \underset{p \to \infty}{\text{Lt}} \sum_{n=0}^{p} \big[x(n-1) - x(n)\big] && \boxed{\text{Changing the summation index from 0 to p and then taking limit p ® ¥.}} \\
&= \underset{p \to \infty}{\text{Lt}} \left[\begin{array}{l} \big[x(-1) - x(0)\big] + \big[x(0) - x(1)\big] + \big[x(1) - x(2)\big] + \,..... \\ ..... + \big[x(p-2) - x(p-1)\big] + \big[x(p-1) - x(p)\big] \end{array}\right] \\
&= \underset{p \to \infty}{\text{Lt}} \big[x(-1) - x(p)\big] \\
&= x(-1) - x(\infty) && .....(3.29)
\end{aligned}$$

On equating equation (3.29) with (3.28) we get,

$$x(-1) - x(\infty) = x(-1) - \underset{z \to 1}{\text{Lt}} \; (1 - z^{-1})\, X(z)$$

$$\therefore \; x(\infty) = \underset{z \to 1}{\text{Lt}} \; (1 - z^{-1})\, X(z)$$

11. Complex convolution theorem (or Multiplication in time domain)

Let, $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$.

Now, the complex convolution theorem states that,

$$\mathcal{Z}\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi j} \oint_C X_1(v)\; X_2\left(\frac{z}{v}\right) v^{-1}\, dv$$

where, v is a dummy variable used for contour integration

Proof :

Let, $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$.

Now, by definition of inverse Ƶ-transform,

$$x_1(n) = \frac{1}{2\pi j} \oint_C X_1(z)\, z^{n-1}\, dz = \frac{1}{2\pi j} \oint_C X_1(v)\, v^{n-1}\, dv \qquad \boxed{\text{let, } z = v} \qquad .....(3.30)$$

Now, by definition of Ƶ-transform,

$$X_2(z) = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad .....(3.31)$$

Using the definition of Ƶ-transform, the $\mathcal{Z}\{x_1(n)\, x_2(n)\}$ can be written as,

$$\begin{aligned}
\mathcal{Z}\{x_1(n)\, x_2(n)\} &= \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2(n)\, z^{-n} \\
&= \sum_{n=-\infty}^{+\infty} \left[\frac{1}{2\pi j} \oint_C X_1(v)\, v^{n-1}\, dv\right] x_2(n)\, z^{-n} && \boxed{\text{Using equation (3.30).}} \\
&= \frac{1}{2\pi j} \oint_C X_1(v) \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n}\, v^{n}\, v^{-1}\, dv && \boxed{\begin{array}{l}\text{Interchanging the order of}\\ \text{summation and integration.}\end{array}} \\
&= \frac{1}{2\pi j} \oint_C X_1(v) \left[\sum_{n=-\infty}^{+\infty} x_2(n) \left(\frac{z}{v}\right)^{-n}\right] v^{-1}\, dv \\
&= \frac{1}{2\pi j} \oint_C X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1}\, dv && \boxed{\text{Using equation (3.31).}}
\end{aligned}$$

12. Parseval's relation

If $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$.

Then the Parseval's relation states that,

$$\sum_{n=-\infty}^{+\infty} x_1(n)\; x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(z)\, X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz$$

Proof :

Let, $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$.

Now, by definition of inverse $\mathcal{Z}$-transform,

$$x_1(n) = \frac{1}{2\pi j}\oint_C X_1(z)\, z^{n-1}\, dz = \frac{1}{2\pi j}\oint_C X_1(v)\, v^{n-1}\, dv \qquad \boxed{\text{let, } z = v} \qquad .....(3.32)$$

Now, by definition of $\mathcal{Z}$-transform,

$$\mathcal{Z}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad .....(3.33)$$

Using the definition of $\mathcal{Z}$ - transform, the $\mathcal{Z}\left\{x_1(n)\, x_2^*(n)\right\}$ can be written as,

$$\mathcal{Z}\left\{x_1(n)\, x_2^*(n)\right\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)\, z^{-n} \qquad .....(3.34)$$

On substituting for $x_1(n)$ from equation (3.32) in equation (3.34) we can write,

$$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)\, z^{-n} = \sum_{n=-\infty}^{+\infty}\left[\frac{1}{2\pi j}\oint_C X_1(v)\, v^{n-1}\, dv\right] x_2^*(n)\, z^{-n}$$

$$= \frac{1}{2\pi j}\oint_C X_1(v)\left[\sum_{n=-\infty}^{+\infty} x_2^*(n)\, z^{-n}\, v^{n}\right] v^{-1}\, dv \qquad \boxed{\text{Interchanging the order of summation and integration.}}$$

$$= \frac{1}{2\pi j}\oint_C X_1(v)\left[\sum_{n=-\infty}^{+\infty} x_2^*(n)\left(\frac{z}{v}\right)^{-n}\right] v^{-1}\, dv$$

$$= \frac{1}{2\pi j}\oint_C X_1(v)\left[\sum_{n=-\infty}^{+\infty} x_2(n)\left(\frac{z^*}{v^*}\right)^{-n}\right]^* v^{-1}\, dv$$

$$= \frac{1}{2\pi j}\oint_C X_1(v)\, X_2^*\left(\frac{z^*}{v^*}\right) v^{-1}\, dv \qquad \boxed{\text{using equation (3.33).}}$$

Let us take limit z ® 1 in the above equation,

$$\therefore \lim_{z\to 1}\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)\, z^{-n} = \lim_{z\to 1}\frac{1}{2\pi j}\oint_C X_1(v)\, X_2^*\left(\frac{z^*}{v^*}\right) v^{-1}\, dv$$

$$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv$$

$$\therefore \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(z)\, X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz \qquad \boxed{\text{let } v = z}$$

Table 3.3 : Summary of Properties of Ƶ-Transform

Note : $X(z) = Ƶ\{x(n)\}$; $X_1(z) = Ƶ\{x_1(n)\}$; $X_2(z) = Ƶ\{x_2(n)\}$; $Y(z) = Ƶ\{y(n)\}$

Property		**Discrete time signal**	**Ƶ-transform**
Linearity		$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$
Shifting (m ³ 0)	x(n); for n ³ 0	$x(n-m)$	$z^{-m} X(z) + \sum_{i=1}^{m} x(-i)\, z^{-(m-i)}$
		$x(n+m)$	$z^{m} X(z) - \sum_{i=0}^{m-1} x(i)\, z^{m-i}$
	x(n); for all n	$x(n-m)$	$z^{-m} X(z)$
		$x(n+m)$	$z^{m} X(z)$
Multiplication by n^m (or differentiation in z-domain)		$n^m x(n)$	$\left(-z \frac{d}{dz}\right)^m X(z)$
Scaling in z-domain (or multiplication by a^n)		$a^n x(n)$	$X(a^{-1} z)$
Time reversal		$x(-n)$	$X(z^{-1})$
Conjugation		$x^*(n)$	$X^*(z^*)$
Convolution		$x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)$	$X_1(z) X_2(z)$
Corrrelation		$r_{xy}(m) = \sum_{m=-\infty}^{+\infty} x(n)\, y(n-m)$	$X(z) Y(z^{-1})$
Initial value		$x(0) = \underset{z\to\infty}{\text{Lt}}\, X(z)$	
Final value		$x(\infty) = \underset{z\to 1}{\text{Lt}}\, (1 - z^{-1})\, X(z)$ $= \underset{z\to 1}{\text{Lt}}\, \frac{(z-1)}{z} X(z)$ if $X(z)$ is analytic for $\lvert z\rvert > 1$	
Complex convolution theorem		$x_1(n)\, x_2(n)$	$\frac{1}{2\pi j} \oint_C X_1(v)\, X_2\!\left(\frac{z}{v}\right) v^{-1}\, dv$
Parseval's relation		$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(z)\, X_2^*\!\left(\frac{1}{z^*}\right) z^{-1}\, dz$	

Example 3.3

Find the one-sided $\mathcal{Z}$-transform of the following discrete time signals.

a) $x(n) = n\,a^{(n-1)}$ **b)** $x(n) = n^3$

Solution

a) Given that, $x(n) = n\,a^{(n-1)}$

Let, $x_1(n) = a^n$

By definition of one-sided $\mathcal{Z}$-transform,

$$X_1(z) = \sum_{n=0}^{\infty} x_1(n)\, z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(a\, z^{-1}\right)^n = \frac{1}{1 - a\,z^{-1}} = \frac{z}{z-a}$$

Using infinite geometric series sum formula.

Let, $x_1(n-1) = a^{n-1}$

By shifting property,

$$\mathcal{Z}\{x_1(n-1)\} = z^{-1} X_1(z) = z^{-1} \frac{z}{z-a} = \frac{1}{z-a}$$

Given that, $x(n) = n\,a^{n-1}$

If $\mathcal{Z}\{x(n)\} = X(z)$ then $\mathcal{Z}\{n\,x(n)\} = -z\dfrac{d}{dz}X(z)$

$$\mathcal{Z}\{x(n)\} = \mathcal{Z}\{n\,a^{n-1}\} = \mathcal{Z}\{n\,x_1(n-1)\} = -z\frac{d}{dz}X_1(z)$$

$$= -z\frac{d}{dz}\frac{1}{z-a} = -z \times \frac{-1}{(z-a)^2} = \frac{z}{(z-a)^2}$$

b) Given that, $x(n) = n^3$

Let us multiply the given discrete time signal by a discrete unit step signal,

\ $x(n) = n^3\,u(n)$

e c n e u q N e s d e d i s -a ev n o s t a i t o r g e m u il y l p i t l u M

By the property of $\mathcal{Z}$-transform, we get,

$$\mathcal{Z}\{n^m u(n)\} = \left(-z\frac{d}{dz}\right)^m U(z)$$

$$\text{where, } U(z) = \mathcal{Z}\{u(n)\} = \frac{z}{z-1}$$

$$\therefore\ -z\frac{d}{dz}U(z) = -z\left[\frac{d}{dz}\left(\frac{z}{z-1}\right)\right] = -z\left[\frac{z-1-z}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

$$d\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

$$\therefore\ \left(-z\frac{d}{dz}\right)^2 U(z) = -z\frac{d}{dz}\left[-z\frac{d}{dz}U(z)\right] = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = -z\left(\frac{(z-1)^2 - z\times 2(z-1)}{(z-1)^4}\right)$$

$$= -z\left(\frac{(z-1)\,(z-1-2z)}{(z-1)^4}\right) = -z\left(\frac{-(z+1)}{(z-1)^3}\right) = \frac{z(z+1)}{(z-1)^3}$$

$$d\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

$$\therefore\left(-z\frac{d}{dz}\right)^3 U(z) = -z\frac{d}{dz}\left[-z\frac{d}{dz}\right]^2 U(z) = -z\frac{d}{dz}\left[\frac{z(z+1)}{(z-1)^3}\right] = -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$= -z\left(\frac{(z-1)^3\,(2z+1) - (z^2+z)3\,(z-1)^2}{(z-1)^6}\right) = -z\left(\frac{(z-1)^2\left[(z-1)\,(2z+1) - (z^2+z)3\right]}{(z-1)^6}\right)$$

$$= -z\frac{\left(2z^2 + z - 2z - 1 - 3z^2 - 3z\right)}{(z-1)^4} = -z\frac{\left(-z^2 - 4z - 1\right)}{(z-1)^4} = \frac{z\left(z^2 + 4z + 1\right)}{(z-1)^4}$$

$$\therefore\ \mathcal{Z}\left\{n^3 u(n)\right\} = \left(-z\frac{d}{dz}\right)^3 U(z) = \frac{z\left(z^2 + 4z + 1\right)}{(z-1)^4}$$

Example 3.4

Find the one-sided Z-transform of the discrete time signals generated by mathematically sampling the following continuous time signals.

a) t^2 **b)** $\sin\Omega_o t$ **c)** $\cos\Omega_o t$

Solution

a) Given that, $x(t) = t^2$

The discrete time signal is generated by replacing t by nT, where T is the sampling time period.

$$\therefore \quad x(n) = (nT)^2 = n^2 T^2 = n^2\, g(n)$$

where, $g(n) = T^2$

If $\mathcal{Z}\{g(n)\} = G(z)$ then $\mathcal{Z}\{n^m g(n)\} = \left(-z\dfrac{d}{dz}\right)^m G(z)$

By the definition of one-sided Z-transform we get,

$$G(z) = \mathcal{Z}\{g(n)\} = \mathcal{Z}\{T^2\} = \sum_{n=0}^{\infty} T^2 z^{-n} = T^2 \sum_{n=0}^{\infty} (z^{-1})^n = T^2\left(\frac{1}{1 - z^{-1}}\right) = \frac{T^2 z}{z - 1}$$

By the multiplication by n^m property of Z-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{n^2\, g(n)\} = \left(-z\frac{d}{dz}\right)^2 G(z) = -z\frac{d}{dz}\left(-z\frac{d}{dz}\, G(z)\right)$$

$d\dfrac{u}{v} = \dfrac{v\,du - u\,dv}{v^2}$

$$= -z\frac{d}{dz}\left(-z\frac{d}{dz}\,\frac{T^2 z}{z-1}\right) = -z\frac{d}{dz}\left(-z \times \frac{(z-1)\,T^2 - T^2 z}{(z-1)^2}\right)$$

$$= -z\frac{d}{dz}\left(\frac{z\,T^2}{(z-1)^2}\right) = -z \times \frac{(z-1)^2\, T^2 - zT^2 \times 2(z-1)}{(z-1)^4}$$

$$= -z \times \frac{(z-1)\,(zT^2 - T^2 - 2zT^2)}{(z-1)^4} = -z \times \frac{-zT^2 - T^2}{(z-1)^3} = \frac{zT^2(z+1)}{(z-1)^3}$$

---

b) Given that, $x(t) = \sin\Omega_0 t$

The discrete time signal is generated by replacing t by nT, where T is the sampling time period.

$$\therefore \quad x(n) = \sin(\Omega_0 nT) = \sin\omega n\,; \quad \text{where } \omega = \Omega_0 T$$

By the definition of one-sided Z-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} \sin\omega n \times z^{-n}$$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$$= \sum_{n=0}^{\infty} \frac{e^{j\omega n} - e^{-j\omega n}}{2j}\, z^{-n} = \frac{1}{2j}\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} - \frac{1}{2j}\sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{j\omega} z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-j\omega} z^{-1}\right)^n$$

$$= \frac{1}{2j}\,\frac{1}{1 - e^{j\omega} z^{-1}} - \frac{1}{2j}\,\frac{1}{1 - e^{-j\omega} z^{-1}}$$

Using infinite geometric series sum formula.

$$= \frac{1}{2j}\,\frac{z}{z - e^{j\omega}} - \frac{1}{2j}\,\frac{z}{z - e^{-j\omega}}$$

$$= \frac{z\,(z - e^{-j\omega}) - z\,(z - e^{j\omega})}{2j\,(z - e^{j\omega})\,(z - e^{-j\omega})} = \frac{z^2 - z\,e^{-j\omega} - z^2 + z\,e^{j\omega}}{2j\,(z^2 - z\,e^{-j\omega} - z\,e^{j\omega} + e^{j\omega}\,e^{-j\omega})}$$

$$= \frac{z\,(e^{j\omega} - e^{-j\omega})/2j}{z^2 - z\,(e^{j\omega} + e^{-j\omega}) + 1}$$

$$= \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1} \quad ; \quad \text{where } \omega = \Omega_0 T$$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$
$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

c) Given that, $x(t) = \cos\Omega_0 t$

The discrete time signal is generated by replacing t by nT, where T is the sampling time period.7

$\therefore$ $x(n) = \cos(\Omega_0 nT) = \cos\omega n$; where $\omega = \Omega_0 T$

By the definition of one-sided Z-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} \cos\omega n \times z^{-n}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{e^{j\omega n} + e^{-j\omega n}}{2}\, z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n$$

$$= \frac{1}{2}\frac{1}{1 - e^{j\omega} z^{-1}} + \frac{1}{2}\frac{1}{1 - e^{-j\omega} z^{-1}}$$

Using infinite geometric series sum formula.

$$= \frac{1}{2}\frac{z}{z - e^{j\omega}} + \frac{1}{2}\frac{z}{z - e^{-j\omega}}$$

$$= \frac{z(z - e^{-j\omega}) + z(z - e^{j\omega})}{2(z - e^{j\omega})(z - e^{-j\omega})} = \frac{z^2 - z e^{-j\omega} + z^2 - z e^{j\omega}}{2(z^2 - z e^{-j\omega} - z e^{j\omega} + e^{j\omega} e^{-j\omega})}$$

$$= \frac{2z^2 - z(e^{j\omega} + e^{-j\omega})}{2[z^2 - z(e^{j\omega} + e^{-j\omega}) + 1]} = \frac{z^2 - z(e^{j\omega} + e^{-j\omega})/2}{z^2 - z(e^{j\omega} + e^{-j\omega}) + 1}$$

$$= \frac{z(z - \cos\omega)}{z^2 - 2z\cos\omega + 1} \quad ; \quad \text{where } \omega = \Omega_0 T$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Example 3.5

Find the one-sided Z-transform of the discrete time signals generated by mathematically sampling the following continuous time signals.

a) $e^{-at}\cos\Omega_0 t$ **b)** $e^{-at}\sin\Omega_0 t$

Solution

a) Given that, $x(t) = e^{-at}\cos\Omega_0 t$

The discrete time signal x(n) is generated by replacing t by nT, where T is the sampling time period.

$\therefore$ $x(n) = e^{-anT}\cos\Omega_0 nT = e^{-anT}\cos\omega n$; where $\omega = \Omega_0 T$

By the definition of one-sided Z-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{\infty} e^{-anT}\cos\omega n\, z^{-n} = \sum_{n=0}^{\infty} e^{-anT}\left(\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right) z^{-n}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-aT} e^{j\omega} z^{-1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-aT} e^{-j\omega} z^{-1}\right)^n$$

$$= \frac{1}{2}\frac{1}{1 - e^{-aT} e^{j\omega} z^{-1}} + \frac{1}{2}\frac{1}{1 - e^{-aT} e^{-j\omega} z^{-1}}$$

Using infinite geometric series sum formula,

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$$

$$= \frac{1}{2}\frac{1}{1 - e^{j\omega}/z e^{aT}} + \frac{1}{2}\frac{1}{1 - e^{-j\omega}/z e^{aT}}$$

$$= \frac{1}{2}\left[\frac{z e^{aT}}{z e^{aT} - e^{j\omega}} + \frac{z e^{aT}}{z e^{aT} - e^{-j\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{z\,e^{aT}\left(z\,e^{aT} - e^{-j\omega}\right) + z\,e^{aT}\left(z\,e^{aT} - e^{j\omega}\right)}{\left(z\,e^{aT} - e^{j\omega}\right)\left(z\,e^{aT} - e^{-j\omega}\right)}\right]$$

$$= \frac{z\,e^{aT}}{2}\left[\frac{z\,e^{aT} - e^{-j\omega} + z\,e^{aT} - e^{j\omega}}{\left(z\,e^{aT}\right)^2 - z\,e^{aT}e^{-j\omega} - z\,e^{aT}e^{j\omega} + e^{j\omega}\,e^{-j\omega}}\right]$$

$$= \frac{z\,e^{aT}}{2}\left[\frac{2z\,e^{aT} - \left(e^{j\omega} + e^{-j\omega}\right)}{z^2\,e^{2aT} - z\,e^{aT}\left(e^{j\omega} + e^{-j\omega}\right) + 1}\right]$$

$$= \left[\frac{z\,e^{aT}\left(z\,e^{aT} - \cos\omega\right)}{z^2\,e^{2aT} - 2z\,e^{aT}\cos\omega + 1}\right] \quad ; \quad \text{where } \omega = \Omega_0 T$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

b) Given that, $x(t) = e^{-at}\sin\Omega_0 t$

The discrete time signal x(n) is generated by replacing t by nT, where T is the sampling time period.

\ $x(n) = e^{-anT}\sin W_0\, n\,T = e^{-anT}\sin wn$; where $w = W_0 T$

By the definition of one-sided Z-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{\infty} e^{-anT}\sin\omega n\; z^{-n} = \sum_{n=0}^{\infty} e^{-anT}\left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) z^{-n}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-aT}e^{j\omega}z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-aT}e^{-j\omega}z^{-1}\right)^n$$

Infinite geometric series sum formula, $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$

$$= \frac{1}{2j}\,\frac{1}{1 - e^{-aT}e^{j\omega}z^{-1}} - \frac{1}{2j}\,\frac{1}{1 - e^{-aT}e^{-j\omega}z^{-1}}$$

$$= \frac{1}{2j}\,\frac{1}{1 - e^{j\omega}/z\,e^{aT}} - \frac{1}{2j}\,\frac{1}{1 - e^{-j\omega}/z\,e^{aT}}$$

$$= \frac{1}{2j}\,\frac{z\,e^{aT}}{z\,e^{aT} - e^{j\omega}} - \frac{1}{2j}\,\frac{z\,e^{aT}}{z\,e^{aT} - e^{-j\omega}}$$

$$= \frac{1}{2j}\left[\frac{z\,e^{aT}\left(z\,e^{aT} - e^{-j\omega}\right) - z\,e^{aT}\left(z\,e^{aT} - e^{j\omega}\right)}{\left(z\,e^{aT} - e^{j\omega}\right)\left(z\,e^{aT} - e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2j}\left[\frac{\left(z\,e^{aT}\right)\left[z\,e^{aT} - e^{-j\omega} - z\,e^{aT} + e^{j\omega}\right]}{\left(z\,e^{aT}\right)^2 - z\,e^{aT}e^{-j\omega} - z\,e^{aT}e^{j\omega} + e^{j\omega}\,e^{-j\omega}}\right]$$

$$= \left[\frac{z\,e^{aT}\left(e^{j\omega} - e^{-j\omega}\right)/\,2j}{z^2\,e^{2aT} - z\,e^{aT}\left(e^{j\omega} + e^{-j\omega}\right) + 1}\right]$$

$$= \frac{z\,e^{aT}\sin\omega}{z^2\,e^{2aT} - 2z e^{aT}\cos\omega + 1} \quad ; \quad \text{where } \omega = \Omega_0 T$$

$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$	
$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$	

Example 3.6

Find the initial value, x(0) and final value, x(¥) of the following z-domain signals.

a) $X(z) = \frac{1}{1 - z^{-2}}$ **b)** $\frac{2 - 4z^{-1}}{1 + 2z^{-1} - 3z^{-2}}$ **c)** $X(z) = \frac{1 - 3z^{-1}}{1 - 3.6z^{-1} + 1.8z^{-2}}$

Solution

a) Given that, $X(z) = \dfrac{1}{1 - z^{-2}}$

By initial value theorem of Z-transform we get,

$$x(0) = \underset{z \to \infty}{Lt}\ X(z) = \underset{z \to \infty}{Lt}\ \frac{1}{1 - z^{-2}} = \underset{z \to \infty}{Lt}\ \frac{1}{1 - \frac{1}{z^2}} = \frac{1}{1 - \frac{1}{\infty}} = \frac{1}{1 - 0} = 1$$

By final value theorem of Z-transform we get,

$$x(\infty) = \underset{z \to 1}{Lt}\ (1 - z^{-1})\ X(z) = \underset{z \to 1}{Lt}\ (1 - z^{-1})\ \frac{1}{1 - z^{-2}}$$

$\boxed{a^2 - b^2 = (a + b)\ (a - b)}$

$$= \underset{z \to 1}{Lt}\ \cancel{(1 - z^{-1})}\ \frac{1}{\cancel{(1 - z^{-1})}\ (1 + z^{-1})} = \underset{z \to 1}{Lt}\ \frac{1}{(1 + z^{-1})} = \frac{1}{1 + 1^{-1}} = \frac{1}{2}$$

---

b) Given that, $X(z) = \dfrac{2 - 4z^{-1}}{1 + 2z^{-1} - 3z^{-2}}$

By initial value theorem of Z-transform we get,

$$x(0) = \underset{z \to \infty}{Lt}\ X(z) = \underset{z \to \infty}{Lt}\ \frac{2 - 4z^{-1}}{1 + 2z^{-1} - 3z^{-2}} = \underset{z \to \infty}{Lt}\ \frac{2 - \frac{4}{z}}{1 + \frac{2}{z} - \frac{3}{z^2}}$$

$$= \frac{2 - \frac{4}{\infty}}{1 + \frac{2}{\infty} - \frac{3}{\infty}} = \frac{2 - 0}{1 + 0 + 0} = 2$$

$\boxed{\text{The roots of quadratic } z^2 + 2z - 3 = 0 \text{ are, } z = \frac{-2 \pm \sqrt{2^2 - 4 \times (-3)}}{2} = \frac{-2 \pm 4}{2} = 1, -3}$

By final value theorem of Z-transform we get,

$$x(\infty) = \underset{z \to 1}{Lt}\ (1 - z^{-1})\ X(z) = \underset{z \to 1}{Lt}\ (1 - z^{-1})\ \frac{2 - 4z^{-1}}{1 + 2z^{-1} - 3z^{-2}}$$

$$= \underset{z \to 1}{Lt}\ \frac{2z^{-2}(z - 1)(z - 2)}{z^{-2}(z^2 + 2z - 3)} = \underset{z \to 1}{Lt}\ \frac{2\cancel{(z - 1)}(z - 2)}{\cancel{(z - 1)}\ (z + 3)} = \underset{z \to 1}{Lt}\ \frac{2(z - 2)}{z + 3} = \frac{2(1 - 2)}{1 + 3} = \frac{-2}{4} = -0.5$$

---

c) Given that, $X(z) = \dfrac{1 - 3\,z^{-1}}{1 - 3.6z^{-1} + 1.8z^{-2}}$

By initial value theorem of Z-transform we get,

$$x(0) = \underset{z \to \infty}{Lt}\ X(z) = \underset{z \to \infty}{Lt}\ \frac{1 - 3z^{-1}}{1 - 3.6z^{-1} + 1.8z^{-2}} = \underset{z \to \infty}{Lt}\ \frac{1 - \frac{3}{z}}{1 - \frac{3.6}{z} + \frac{1.8}{z^2}}$$

$$= \frac{1 - \frac{3}{\infty}}{1 - \frac{3.6}{\infty} + \frac{1.8}{\infty}} = \frac{1 - 0}{1 - 0 + 0} = 1$$

$\boxed{\text{The roots of quadratic } z^2 - 3.6z + 1.8 = 0 \text{ are, } z = \frac{3.6 \pm \sqrt{3.6^2 - 4 \times 1.8}}{2} = \frac{3.6 \pm 2.4}{2} = 3,\ 0.6}$

By final value theorem of Z-transform we get,

$$x(\infty) = \underset{z \to 1}{Lt}\ (1 - z^{-1})\ X(z)$$

$$= \underset{z \to 1}{Lt}\ (1 - z^{-1})\ \frac{1 - 3z^{-1}}{1 - 3.6z^{-1} + 1.8z^{-2}} = \underset{z \to 1}{Lt}\ \frac{z^{-2}(z - 1)(z - 3)}{z^{-2}(z^2 - 3.6z + 1.8)}$$

$$= \underset{z \to 1}{Lt}\ \frac{(z - 1)\ \cancel{(z - 3)}}{\cancel{(z - 3)}\ (z - 0.6)} = \underset{z \to 1}{Lt}\ \frac{z - 1}{z - 0.6} = \frac{1 - 1}{1 - 0.6} = \frac{0}{0.4} = 0$$

Table 3.4 : Some Common Z-transform Pairs

x(t)	x(n)	X(z) With positive power of z	X(z) With negative power of z	ROC
	$\delta(n)$	1	1	Entire z-plane
	u(n) or 1	$\frac{z}{z-1}$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
	$a^n\, u(n)$	$\frac{z}{z-a}$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
	$n\, a^n\, u(n)$	$\frac{az}{(z-a)^2}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
	$n^2\, a^n\, u(n)$	$\frac{az\,(z+a)}{(z-a)^3}$	$\frac{az^{-1}\,(1 + az^{-1})}{(1 - az^{-1})^3}$	$\lvert z\rvert > \lvert a\rvert$
	$-\,a^n u(-n-1)$	$\frac{z}{z-a}$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
	$-na^n\, u(-n-1)$	$\frac{az}{(z-a)^2}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$t\,u(t)$	$nT\,u(nT)$	$\frac{Tz}{(z-1)^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
$t^2\,u(t)$	$(nT)^2 u(nT)$	$\frac{T^2 z\,(z+1)}{(z-1)^3}$	$\frac{T^2 z^{-1}\,(1+z^{-1})}{(1-z^{-1})^3}$	$\lvert z\rvert > 1$
$e^{-at}\,u(t)$	$e^{-anT}\,u(nT)$	$\frac{z}{z-e^{-aT}}$	$\frac{1}{1-e^{-aT}\,z^{-1}}$	$\lvert z\rvert > \lvert e^{-aT}\rvert$
$te^{-at}\,u(t)$	$nTe^{-anT}\,u(nT)$	$\frac{z\,T\,e^{-aT}}{(z-e^{-aT})^2}$	$\frac{z^{-1}\,T\,e^{-aT}}{(1-e^{-aT}\,z^{-1})^2}$	$\lvert z\rvert > \lvert e^{-aT}\rvert$
$\sin\Omega_0 t\; u(t)$	$\sin\Omega_0 nT\; u(nT)$ $= \sin wn\, u(nT)$ where, $w = W_0 T$	$\frac{z\sin\omega}{z^2-2z\cos\omega+1}$	$\frac{z^{-1}\sin\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\lvert z\rvert > 1$
$\cos\Omega_0 t\; u(t)$	$\cos\Omega_0 nT\; u(nT)$ $= \cos wn\, u(nT)$ where, $w = W_0 T$	$\frac{z\,(z-\cos\omega)}{z^2-2z\cos\omega+1}$	$\frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\lvert z\rvert > 1$

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3.4 Poles and Zeros of Rational Function of z

Let, X(z) be $\mathcal{Z}$-transform of x(n). When X(z) is expressed as a ratio of two polynomials in z or z^{-1}, then X(z) is called a ***rational function*** of z.

Let X(z) be expressed as a ratio of two polynomials in z, as shown below.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + ... + a_N z^{-N}} \qquad .....(3.35)$$

where, N(z) = Numerator polynomial of X(z)

D(z) = Denominator polynomial of X(z)

In equation (3.35) let us scale the coefficients of numerator polynomial by b_0 and that of denominator polynomial by a_0, and then convert the polynomials to positive power of z as shown below.

$$X(z) = \frac{b_0\left(1 + \frac{b_1}{b_0} z^{-1} + \frac{b_2}{b_0} z^{-2} + \frac{b_3}{b_0} z^{-3} + ... + \frac{b_M}{b_0} z^{-M}\right)}{a_0\left(1 + \frac{a_1}{a_0} z^{-1} + \frac{a_2}{a_0} z^{-2} + \frac{a_3}{a_0} z^{-3} + ... + \frac{a_N}{a_0} z^{-N}\right)}$$

$$= G\,\frac{z^{-M}\left(z^M + \frac{b_1}{b_0} z^{M-1} + \frac{b_2}{b_0} z^{M-2} + \frac{b_3}{b_0} z^{M-3} + ... \frac{b_M}{b_0}\right)}{z^{-N}\left(z^N + \frac{a_1}{a_0} z^{N-1} + \frac{a_2}{a_0} z^{N-2} + \frac{a_3}{a_0} z^{N-3} + ... \frac{a_N}{a_0}\right)} \qquad \boxed{\text{Let, } M = N}$$

$$= G\,\frac{(z - z_1)(z - z_2)(z - z_3)\, ...\, (z - z_N)}{(z - p_1)(z - p_2)(z - p_3)\, ...\, (z - p_N)} \qquad .....(3.36)$$

where, z_1, z_2, z_3, ...z_N are roots of numerator polynomial

p_1,p_2, p_3, ...p_N are roots of denominator polynomial

G is a scaling factor.

In equation (3.36) if the value of z is equal to one of the roots of the numerator polynomial, then the function X(z) will become zero.

Therefore the roots of numerator polynomial z_1, z_2, z_3, ...z_N are called zeros of X(z). Hence the ***zeros*** are defined as values z at which the function X(z) become zero.

In equation (3.36) if the value of z is equal to one of the roots of the denominator polynomial then the funcion X(z) will become infinite. Therefore the roots of denominator polynomial p_1, p_2, p_3, ...p_N are called poles of X(z). Hence the ***poles*** are defined as values of z at which the function X(z) become infinite.

Since the function X(z) attains infinite values at poles, the ROC of X(z) does not include poles.

In a realizable system, the number of zeros will be less than or equal to number of poles. Also for every zero, we can associate one pole (the missing zeros are assumed to exist at infinity).

Let z_i be the zero associated with the pole p_i. If we evaluate |X(z)| for various values of z, then |X(z)| will be zero for $z = z_i$ and infnite for $z = p_i$. Hence the plot of |X(z)| in a three-dimensional plane will look like a pole (or pillar-like structure) and so the point $z = p_i$ is called a pole.

3.4.1 Representation of Poles and Zeros in z-Plane

The complex variable, z is defined as,

$z = u + jv$

where, u = Real part of z

v = Imaginary part of z

Hence the z-plane is a complex plane, with u on real axis and v on imaginary axis (Refer fig 3.1 in section 3.1). In the z-plane, the zeros are marked by small circle "o" and the poles are marked by letter "x".

For example consider a rational function of z shown below.

$$X(z) = \frac{0.5 - 0.4\,z^{-1} + 0.06\,z^{-2}}{2 + 1.6\,z^{-1} + 0.64\,z^{-2}}$$

$$= \frac{0.5\left(1 - \frac{0.4}{0.5}z^{-1} + \frac{0.06}{0.5}z^{-2}\right)}{2\left(1 + \frac{1.6}{2}z^{-1} + \frac{0.64}{2}z^{-2}\right)} = \frac{0.25\,(1 - 0.8z^{-1} + 0.12\,z^{-2})}{1 + 0.8z^{-1} + 0.32z^{-2}}$$

$$= \frac{0.25\,z^{-2}(z^2 - 0.8z + 0.12)}{z^{-2}(z^2 + 0.8z + 0.32)} = \frac{0.25\,(z - 0.2)\,(z - 0.6)}{(z + 0.4 - j0.4)\,(z + 0.4 + j0.4)} \quad .....(3.37)$$

The roots of quadratic, $z^2 - 0.8z + 0.12 = 0$ are,

$$z = \frac{0.8 \pm \sqrt{0.8^2 - 4 \times 0.12}}{2} = \frac{0.8 \pm 0.4}{2} = 0.6,\ 0.2$$

$$\therefore \quad z^2 - 0.8z + 0.12 = (z - 0.6)\,(z - 0.2)$$

The roots of quadratic, $z^2 + 0.8z + 0.32 = 0$ are,

$$z = \frac{-0.8 \pm \sqrt{0.8^2 - 4 \times 0.32}}{2} = \frac{-0.8 \pm \sqrt{-0.64}}{2} = \frac{-0.8 \pm j0.8}{2} = -0.4 \pm j0.4$$

$$\therefore \quad z^2 + 0.8z + 0.32 = (z + 0.4 - j0.4)\,(z + 0.4 + j0.4)$$

The zeros of X(z) are roots of numerator polynomial, which has two roots.

Therefore, the zeros of X(z) are,

$z_1 = 0.6,\ z_2 = 0.2$

The poles of X(z) are roots of denominator polynomial, which has two roots.

Therefore, the poles of X(z) are,

$p_1 = -0.4 + j0.4,\ p_2 = -0.4 - j0.4$

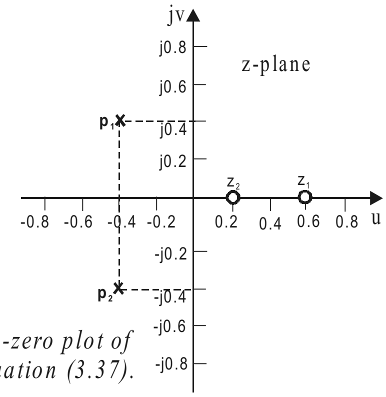


Fig 3.8 : Pole-zero plot of X(z) of equation (3.37).

The pole-zero plot of X(z) is shown in fig 3.8.

3.4.2 ROC of Rational Function of z

Case i: Right-sided (causal) signal

Let x(n) be a right-sided (causal) signal defined as,

$$x(n) = r_1^n\, u(n) + r_2^n\, u(n) + r_3^n\, u(n) \quad ; \quad \text{where } r_1 < r_2 < r_3$$

$$\mathcal{Z}\{a^n\, u(n)\} = \frac{z}{z-a} \quad \text{with ROC } |z| > |a|$$

Now, the $\mathcal{Z}$-transform of x(n) is,

$$X(z) = \frac{z}{z-r_1} + \frac{z}{z-r_2} + \frac{z}{z-r_3}$$

$$= \frac{N(z)}{(z-r_1)\,(z-r_2)\,(z-r_3)}$$

where, $N(z) = z(z-r_2)\,(z-r_3) + z(z-r_1)\,(z-r_3) + z(z-r_1)\,(z-r_2)$

The poles of X(z) are,

$$p_1 = r_1,\; p_2 = r_2,\; p_3 = r_3$$

The convergence criteria for X(z) are,

$$|z| > |r_1| \;\; ; \;\; |z| > |r_2| \;\; ; \;\; |z| > |r_3|$$

Since $r_1 < r_2 < r_3$, the ROC is exterior of the circle of radius r_3 in z-plane as shown in fig.3.9. In terms of poles of X(z) we can say that the *ROC is exterior of a circle, whose radius is equal to the magnitude of outer most pole (i.e., pole with largest magnitude) of X(z).*

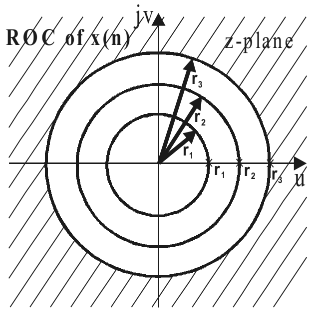


Fig 3.9 : *ROC of $x(n) = r_1 u(n) + r_2 u(n) + r_3 u(n)$ where $r_1 < r_2 < r_3$.*

Case ii: Left-sided (anticausal) signal

Let x(n) be a left-sided (anticausal) signal defined as,

$$x(n) = -r_1^n\, u(-n-1) - r_2^n\, u(-n-1) - r_3^n\, u(-n-1) \quad ; \quad \text{where } r_1 < r_2 < r_3$$

Now, the $\mathcal{Z}$-transform of x(n) is,

$$\mathcal{Z}\{-a^n\, u(-n-1)\} = \frac{z}{z-a} \quad \text{with ROC } |z| < |a|$$

$$X(z) = \frac{z}{z-r_1} + \frac{z}{z-r_2} + \frac{z}{z-r_3}$$

$$= \frac{N(z)}{(z-r_1)\,(z-r_2)\,(z-r_3)}$$

where, $N(z) = z(z-r_2)\,(z-r_3) + z(z-r_1)\,(z-r_3) + z(z-r_1)\,(z-r_2)$

The poles of X(z) are,

$$p_1 = r_1,\; p_2 = r_2,\; p_3 = r_3$$

The convergence criteria for X(z) are,

$$|z| < |r_1| \;\; ; \;\; |z| < |r_2| \;\; ; \;\; |z| < |r_3|$$

Since $r_1 < r_2 < r_3$, the ROC is interior of the circle of radius r_1 in z-plane as shown in fig.3.10. In terms of poles of X(z) we can say that the *ROC is interior of a circle, whose radius is equal to the magnitude of inner most pole (i.e., pole with smallest magnitude) of X(z).*

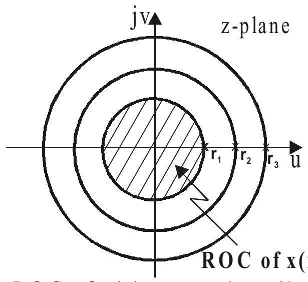


Fig 3.10 : *ROC of $x(n) = -r_1 u(-n-1) - r_2 u(-n-1) - r_3 u(-n-1)$ where $r_1 < r_2 < r_3$.*

***Case iii:* Two-sided (noncausal) signal**

Let x(n) be two-sided signal defined as,

$$x(n) = r_1^n\, u(n) + r_2^n\, u(n) - r_3^n\, u(-n-1) - r_4^n\, u(-n-1) \quad ; \quad \text{where } r_1 < r_2 < r_3 < r_4$$

Now, the Z-transform of x(n) is,

$$X(z) = \frac{z}{z-r_1} + \frac{z}{z-r_2} + \frac{z}{z-r_3} + \frac{z}{z-r_4} = \frac{N(z)}{(z-r_1)\,(z-r_2)\,(z-r_3)\,(z-r_4)}$$

$$\text{where, } N(z) = z(z-r_2)\,(z-r_3)\,(z-r_4) + z(z-r_1)\,(z-r_3)\,(z-r_4) + z(z-r_1)\,(z-r_2)\,(z-r_4) + z(z-r_1)\,(z-r_2)\,(z-r_3)$$

The poles of X(z) are,

$$p_1 = r_1\ ;\ p_2 = r_2\ ;\ p_3 = r_3\ ;\ p_4 = r_4$$

The convergence criteria for X(z) are,

$$|z| > |r_1|\ ;\ |z| > |r_2|\ ;\ |z| < |r_3|\ ;\ |z| < |r_4|$$

Since $r_1 < r_2 < r_3 < r_4$, the ROC is the region inbetween the circles of radius r_2 and r_3 as shown in fig 3.11. Let r_x be the magnitude of largest pole of causal signal and let r_y be the magnitude of smallest pole of anticausal signal and let $r_x < r_y$. Now in terms of poles of X(z) we can say that the *ROC is the region in between two circles of radius r_x and r_y, where $r_x < r_y$.*

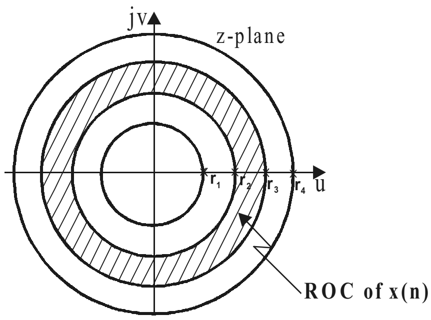


Fig 3.11 : *ROC of $x(n) = r_1 u(n) + r_2 u(n) - r_3 u(-n-1) - r_4 (-n-1)$.*

3.4.3 Properties of ROC

The various concepts of ROC that has been discussed in sections 3.2 and 3.4.2 are summarized as properties of ROC and given below.

Property - 1 : The ROC of X(z) is a ring or disk in z-plane, with centre at origin.

Property - 2 : If x(n) is finite duration right-sided (causal) signal, then the ROC is entire z- plane except z = 0.

Property - 3 : If x(n) is finite duration left-sided (anticausal) signal, then the ROC is entire z-plane except $z = \infty$.

Property - 4 : If x(n) is finite duration two-sided (noncausal) signal, then the ROC is entire z-plane except z = 0 and $z = \infty$.

Property - 5 : If x(n) is infinite duration right-sided (causal) signal, then the ROC is exterior of a circle of radius r_1.

Property - 6 : If x(n) is infinite duration left-sided (anticausal) signal, then the ROC is interior of a circle of radius r_2.

Property - 7 : If x(n) is infinite duration two-sided (noncausal) signal, then the ROC is the region in between two circles of radius r_1 and r_2.

Property - 8 : If X(z) is rational, [where X(z) is Z-transform of x(n)], then the ROC does not include any poles of X(z).

Property - 9 : If X(z) is rational, [where X(z) is Z-transform of x(n)], and if x(n) is right-sided, then the ROC is exterior of a circle whose radius corresponds to the pole with largest magnitude.

Property - 10 : If X(z) is rational, [where X(z) is Z-transform of x(n)], and if x(n) is left-sided, then the ROC is interior of a circle whose radius corresponds to the pole with smallest magnitude.

Property - 11 : If X(z) is rational, [where X(z) is Z-transform of x(n)], and if x(n) is two-sided, then the ROC is region in between two circles whose radius corresponds to the pole of causal part with largest magnitude and the pole of anticausal part with smallest magnitude.

3.5 Inverse Z-Transform

Let X(z) be Z-transform of the discrete time signal x(n).The inverse Z-transform is the process of recovering the discrete time signal x(n) from its Z-transform X(z). The signal x(n) can be uniquely determined from X(z) and its ROC.

The inverse Z-transform can be determined by the following three methods.

1. Direct evaluation by contour integration (or residue method).
2. Partial fraction expansion method.
3. Power series expansion method.

3.5.1 Inverse Z-Transform by Contour Integration or Residue Method

Let, X(z) be Z-transform of x(n).

Now by definition of inverse Z-transform,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1} dz \qquad .....(3.38)$$

Using partial fraction expansion technique the function $X(z)\, z^{n-1}$ can be expressed as shown below.

$$X(z)\, z^{n-1} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + \,.....\, + \frac{A_N}{z-p_N} \qquad .....(3.39)$$

where, $p_1, p_2, p_3, \,.....\, p_N$ are poles of $X(z)\, z^{n-1}$ and $A_1, A_2, A_3, \,.....\, A_N$ are residues.

The residue A_1 is obtained by multiplying the equation (3.39) by $(z-p_1)$ and letting $z = p_1$.

Similarly other residues are evaluated.

$$\therefore\; A_1 = (z-p_1)\, X(z)\, z^{n-1}\Big|_{z=p_1} \qquad .....(3.40.1)$$

$$A_2 = (z-p_2)\, X(z)\, z^{n-1}\Big|_{z=p_2} \qquad .....(3.40.2)$$

$$A_3 = (z-p_3)\, X(z)\, z^{n-1}\Big|_{z=p_3} \qquad .....(3.40.3)$$

$$\vdots$$

$$A_N = (z-p_N)\, X(z)\, z^{n-1}\Big|_{z=p_N} \qquad .....(3.40.N)$$

Using equation (3.39) the equation (3.38) can be written as,

$$x(n) = \frac{1}{2\pi j} \oint_C \left[\frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + \,.....\, + \frac{A_N}{z-p_N} \right] dz$$

$$= \frac{1}{2\pi j} \left[A_1 \oint_C \frac{dz}{z-p_1} + A_2 \oint_C \frac{dz}{z-p_2} + A_3 \oint_C \frac{dz}{z-p_3} + \,.....\, + A_N \oint_C \frac{dz}{z-p_N} \right]$$

$$.....(3.41)$$

If, $G(z) = \dfrac{1}{z - p_0}$, then by ***Cauchy's integral theorem***,

$$\oint_C G(z)\,dz = \oint_C \frac{1}{z - p_0}\,dz \begin{array}{ll} = 2\pi j & ;\ \text{if } p_0 \text{ is a point inside the contour C in z - plane.} \\ = 0 & ;\ \text{if } p_0 \text{ is a point outside the contour C in z - plane.} \end{array}$$

Using Cauchy's integral theorem, the equation (3.41) can be written as shown below.

$$\begin{aligned} x(n) &= \frac{1}{2\pi j}\left[A_1\, 2\pi j + A_2\, 2\pi j + A_3\, 2\pi j + \ldots.. + A_N\, 2\pi j\right] \\ &= A_1 + A_2 + A_3 + \ldots.. + A_N \qquad .....(3.42) \\ &= \text{Sum of residues of } X(z)\, z^{n-1} \end{aligned}$$

On substituting for residues from equation (3.40.1) to (3.40.N) in equation(3.42), we get,

$$\begin{aligned} x(n) &= (z - p_1)\, X(z)\, z^{n-1}\Big|_{z = p_1} + (z - p_2)\, X(z)\, z^{n-1}\Big|_{z = p_2} \\ &\quad + (z - p_3)\, X(z)\, z^{n-1}\Big|_{z = p_3} + \ldots.. + (z - p_N)\, X(z)\, z^{n-1}\Big|_{z = p_N} \end{aligned}$$

$$\boxed{\therefore\ x(n) = \sum_{i=1}^{N}\left[(z - p_i)\, X(z)\, z^{n-1}\Big|_{z = p_i}\right]} \qquad .....(3.43)$$

where, N = Number or poles of $X(z)\,z^{n-1}$ lying inside the contour C.

Using equation (3.43), by considering only the poles lying inside the contour C, the inverse Z-transform can be evaluated. For a stable system the contour C is the unit circle in z-plane.

3.5.2 Inverse Z-Transform by Partial Fraction Expansion Method

Let X(z) be Z-transform of x(n), and X(z) be a rational function of z. Now the function X(z) can be expressed as a ratio of two polynomials in z as shown below. (Refer equation 3.35).

$$X(z) = \frac{N(z)}{D(z)} \qquad .....(3.44)$$

where, N(z) = Numerator polynomial of X(z)

D(z) = Denominator polynomial of X(z)

Let us divide both sides of equation (3.44) by z and express equation (3.44) as shown below.

$$\frac{X(z)}{z} = \frac{N(z)}{z\,D(z)}$$

$$\therefore\ \frac{X(z)}{z} = \frac{Q(z)}{D(z)} \qquad .....(3.45)$$

$$\text{where, } Q(z) = \frac{N(z)}{z}$$

Note : It is convenient, if we consider $\frac{X(z)}{z}$ rather than X(z) for inverse Z-transform by partial fraction expansion method.

On factorizing the denominator polynomial of equation (3.45) we get,

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)\,(z-p_2)\,(z-p_3)\;.....\;(z-p_N)} \quad .....(3.46)$$

where, $p_1, p_2, p_3, ...p_N$ are roots of denominator polynomial [as well as poles of X(z)].

The equation (3.46) can be expressed as a series of sum terms by partial fraction expansion technique as shown below.

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + \;.....\; + \frac{A_N}{z-p_N}$$

where, $A_1, A_2, A_3, ...\ A_N$ are residues.

$$\therefore\ X(z) = A_1 \frac{z}{z-p_1} + A_2 \frac{z}{z-p_2} + A_3 \frac{z}{z-p_3} + \;.....\; + A_N \frac{z}{z-p_N} \quad .....(3.47)$$

Now, the inverse Z-transform of equation (3.47) is obtained by comparing each term with standard Z-transform pair. The two popular Z-transform pairs useful for inverse Z-transform of equation (3.47) are given below.

If a^n is a causal (or right-sided) signal then,

$$\mathcal{Z}\left\{a^n\, u(n)\right\} = \frac{z}{z-a}\ ;\ \text{with ROC } |z| > |a|$$

If a^n is an anticausal (or left-sided) signal then,

$$\mathcal{Z}\left\{-a^n\, u(-n-1)\right\} = \frac{z}{z-a}\ ;\ \text{with ROC } |z| < |a|$$

Let r_1 be the magnitude of the largest pole and let the ROC be $|z| > r_1$ (where r_1 is radius of a circle in z-plane), then each term of equation (3.47) gives rise to a causal sequence, and so the inverse Z-transform of equation (3.47) will be as shown in equation (3.48).

$$x(n) = A_1\, p_1^{\,n}\, u(n) + A_2\, p_2^{\,n}\, u(n) + A_3\, p_3^{\,n}\, u(n) + \;.....\; + A_N\, p_N^{\,n}\, u(n) \quad .....(3.48)$$

Let r_2 be the magnitude of the smallest pole and let ROC be $|z| < r_2$ (where r_2 is radius of a circle in z-plane), then each term of equation (3.47) give rise to an anticausal sequence, and so the inverse Z-transform of equation (3.47) will be as shown in equation (3.49).

$$x(n) = -A_1\, p_1^{\,n}\, u(-n-1) - A_2\, p_2^{\,n}\, u(-n-1) - A_3\, p_3^{\,n}\, u(-n-1) - \;.....\; - A_N\, p_N^{\,n}\, u(-n-1) \quad .....(3.49)$$

Sometimes the specified ROC will be in between two circles of radius r_x and r_y, where $r_x < r_y$. [i.e., ROC is $r_x < |z| < r_y$]. Now in this case, the terms with magnitude of pole less than r_x will give rise to causal signal and the terms with magnitude of pole greater than r_y will give rise to anticausal signal so that the inverse Z-transform of X(z) will give a two-sided signal. [Refer section 3.4.2, case iii].

Evaluation of Residues

The coefficients of the denominator polynomial D(z) are assumed real and so the roots of the denominator polynomial are real and/or complex conjugate pairs (i.e., complex roots will occur only in conjugate pairs). Hence on factorizing the denominator polynomial we get the following cases. [The roots of the denominator polynomial are poles of X(z)].

Case i : *When roots (or poles) are real and distinct.*

Case ii : *When roots (or poles) have multiplicity.*

Case iii : *When roots (or poles) are complex conjugate.*

Case i : When roots (or poles) are real and distinct

In this case $\frac{X(z)}{z}$ can be expressed as,

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)\,(z-p_2)\,\ldots\ldots\,(z-p_N)}$$

$$= \frac{A_1}{(z-p_1)} + \frac{A_2}{(z-p_2)} + \ldots.. + \frac{A_N}{(z-p_N)}$$

where, A_1, A_2 ...... A_N are residues and p_1, p_2, ... p_N are poles.

The residue A_1 is evaluated by multiplying both sides of $\frac{X(z)}{z}$ by $(z-p_1)$ and letting $z = p_1$. Similarly other residues are evaluated.

$$\therefore\ A_1 = (z-p_1)\left.\frac{X(z)}{z}\right|_{z=p_1}$$

$$A_2 = (z-p_2)\left.\frac{X(z)}{z}\right|_{z=p_2}$$

$$\vdots$$

$$A_N = (z-p_N)\left.\frac{X(z)}{z}\right|_{z=p_N}$$

Case ii : When roots (or poles) have multiplicity

Let one pole have a multiplicity of q (i.e., repeats q times). In this case $\frac{X(z)}{z}$ is expressed as,

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)\,(z-p_2)\ldots\ldots(z-p_x)^q\ldots\ldots(z-p_N)}$$

$$= \frac{A_1}{(z-p_1)} + \frac{A_2}{(z-p_2)} + \ldots..$$

$$+ \frac{A_{x0}}{(z-p_x)^q} + \frac{A_{x1}}{(z-p_x)^{q-1}} + \ldots\ldots + \frac{A_{x(q-1)}}{(z-p_x)} + \ldots.. + \frac{A_N}{(z-p_N)}$$

where, A_{x0}, A_{x1}, ... $A_{x(q-1)}$ are residues of repeated root (or pole), $z = p_x$

The residues of distinct real roots are evaluated as explained in case i.

The residue A_{xr} of repeated root is obtained as shown below.

$$A_{xr} = \frac{1}{r!}\,\frac{d^r}{dz^r}\left.\left[(z-p_x)^q\,\frac{X(z)}{z}\right]\right|_{z=p_x} \quad ; \quad \text{where,}\ \ r = 0,\ 1,\ 2,\ldots.(q-1)$$

Case iii : When roots (or poles) are complex conjugate

Let $\frac{X(z)}{z}$ has one pair of complex conjugate pole. In this case $\frac{X(z)}{z}$ can be expressed as,

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)\,(z-p_2)......(z^2+az+b)......(z-p_N)}$$

$$= \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + ... + \frac{A_x}{z-(x+jy)} + \frac{A_x^*}{z-(x-jy)} + ... + \frac{A_N}{z-p_N}$$

The residues of real and nonrepeated roots are evaluated as explained in case i.

The residue A_x is evaluated as that of case i and the residue A_x^* is the conjugate of A_x.

3.5.3 Inverse Z-Transform by Power Series Expansion Method

Let X(z) be Z-transform of x(n), and X(z) be a rational function of z as shown below.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + ... + a_N z^{-N}}$$

On dividing the numerator polynomial N(z) by denominator polynomial D(z) we can express X(z) as a power series of z. It is possible to express X(z) as positive power of z or as negative power of z or with both positive and negative power of z as shown below.

Case i : $$X(z) = \frac{N(z)}{D(z)} = c_0 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + ... \qquad .....(3.50.1)$$

Case ii : $$X(z) = \frac{N(z)}{D(z)} = d_0 + d_1 z^{1} + d_2 z^{2} + d_3 z^{3} + ... \qquad .....(3.50.2)$$

Case iii : $$X(z) = \frac{N(z)}{D(z)} = ... + e_{-3} z^{3} + e_{-2} z^{2} + e_{-1} z + e_0 + e_1 z^{-1} + e_2 z^{-2} + e_3 z^{-3} + ... \qquad .....(3.50.3)$$

The case-i power series of z is obtained when the ROC is exterior of a circle of radius r in z-plane (i.e., ROC is $|z| > r$).

The case-ii power series of z is obtained when the ROC is interior of a circle of radius r in z-plane (i.e., ROC is $|z| < r$).

The case-iii power series of z is obtained when the ROC is in between two circles of radius r_1 and r_2 in z-plane (i.e., ROC is $r_1 < |z| < r_2$).

By the definition of Z-transform, we get,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

On expanding the summation we get,

$$X(z) = ... x(-3)\, z^3 + x(-2)\, z^2 + x(-1)\, z^1 + x(0)\, z^0 + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + ... \qquad .....(3.51)$$

On comparing the coefficients of z of equations (3.50) and (3.51), the samples of x(n) are determined. [i.e., the coefficient of z^i is the i^{th} sample, x(i) of the signal x(n)].

Note : The different methods of evaluation of inverse Z-transform of a function X(z) will result in different type of mathematical expressions. But the inverse Z-transform is unique for a specified ROC and so on evaluating the expressions for each value of n, we may get a same signal.

Example 3.7

Determine the inverse Z-transform of the function, $X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$ by the following three methods and prove that the inverse Z-transform is unique.

1. Residue Method
2. Partial Fraction Expansion Method
3. Power Series Expansion Method

The roots of quadratic $z^2 - 3z + 2 = 0$ are,

$$z = \frac{3 \pm \sqrt{3^2 - 4 \times 2}}{2} = \frac{3 \pm 1}{2} = 2,\ 1$$

Solution

Method-1 : Residue Method

Given that, $X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = \frac{z^{-2}(3z^2 + 2z + 1)}{z^{-2}(z^2 - 3z + 2)} = \frac{3z^2 + 2z + 1}{z^2 - 3z + 2}$.

Let us divide the numerator polynomial by denominator polynomial and express X(z) as shown below.

$$X(z) = \frac{3z^2 + 2z + 1}{z^2 - 3z + 2} = 3 + \frac{11z - 5}{z^2 - 3z + 2}$$

$$= 3 + \frac{11z - 5}{(z - 1)\,(z - 2)}$$

$$\begin{array}{r|l} & \quad 3 \\ z^2 - 3z + 2 & 3z^2 + 2z + 1 \\ & 3z^2 - 9z + 6 \\ & (-) \quad (+) \quad (-) \\ \hline & \qquad 11z - 5 \end{array}$$

Let, $X_1(z) = 3$ and $X_2(z) = \frac{11z - 5}{z^2 - 3z + 2}$; $\therefore\ X(z) = X_1(z) + X_2(z)$

$$x(n) = Z^{-1}\{X(z)\} = Z^{-1}\{X_1(z)\} + Z^{-1}\{X_2(z)\}$$

$$= Z^{-1}\{3\} + Z^{-1}\{X_2(z)\}$$

$$= 3\,\delta(n) + \sum_{i=1}^{N} \left[(z - p_i)\, X_2(z)\, z^{n-1} \Big|_{z = p_i} \right]$$

Using residue theorem.

$$= 3\,\delta(n) + \cancel{(z-1)} \frac{11z - 5}{\cancel{(z-1)}\,(z - 2)}\, z^{n-1} \Big|_{z=1} + \cancel{(z-2)} \frac{11z - 5}{(z - 1)\,\cancel{(z-2)}}\, z^{n-1} \Big|_{z=2}$$

$$= 3\,\delta(n) + \frac{11 - 5}{1 - 2}\,(1)^{n-1} + \frac{11 \times 2 - 5}{2 - 1}\, 2^{n-1}$$

$$\therefore\ x(n) = 3\,\delta(n) - 6\,u(n-1) + 17(2)^{n-1}\,u(n-1) = 3\,\delta(n) + \left[-6 + 17(2)^{n-1}\right] u(n-1)$$

When $n = 0$, $x(0) = 3 - 0 + 0 = 3$

When $n = 1$, $x(1) = 0 - 6 + 17 \times 2^0 = 11$

When $n = 2$, $x(2) = 0 - 6 + 17 \times 2^1 = 28$

When $n = 3$, $x(3) = 0 - 6 + 17 \times 2^2 = 62$

When $n = 4$, $x(4) = 0 - 6 + 17 \times 2^3 = 130$

$\therefore\ x(n) = \{\underset{\uparrow}{3}, 11, 28, 62, 130, ...\}$

Method-2 : Partial Fraction Expansion Method

Given that, $X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = \frac{z^{-2}(3z^2 + 2z + 1)}{z^{-2}(z^2 - 3z + 2)} = \frac{3z^2 + 2z + 1}{(z - 1)\,(z - 2)}$.

$$\therefore\ \frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z - 1)\,(z - 2)}$$

$$\text{Let,}\quad \frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z-1)(z-2)} = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-2}$$

$$\text{Now,}\quad A_1 = z\frac{X(z)}{z}\bigg|_{z=0} = \cancel{z}\,\frac{3z^2 + 2z + 1}{\cancel{z}(z-1)(z-2)}\bigg|_{z=0} = \frac{0+0+1}{(0-1)(0-2)} = 0.5$$

$$A_2 = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \cancel{(z-1)}\,\frac{3z^2 + 2z + 1}{z\cancel{(z-1)}(z-2)}\bigg|_{z=1} = \frac{3+2+1}{1\times(1-2)} = -6$$

$$A_3 = (z-2)\frac{X(z)}{z}\bigg|_{z=2} = \cancel{(z-2)}\,\frac{3z^2 + 2z + 1}{z(z-1)\cancel{(z-2)}}\bigg|_{z=2} = \frac{3\times 2^2 + 2\times 2 + 1}{2\times(2-1)} = 8.5$$

$$\frac{X(z)}{z} = \frac{0.5}{z} - \frac{6}{z-1} + \frac{8.5}{z-2}$$

$$\therefore\ X(z) = 0.5 - 6\,\frac{z}{z-1} + 8.5\,\frac{z}{z-2}$$

On taking inverse Z-transform of X(z) we get,

$Z\{\delta(n)\} = 1$
$Z\{u(n)\} = \dfrac{z}{z-1}$
$Z\{a^n u(n)\} = \dfrac{z}{z-a}$

$$x(n) = 0.5\,d(n) - 6\,u(n) + 8.5\,(2)^n u(n) = 0.5\,d(n) + [-6 + 8.5(2)^n]\,u(n)$$

When $n = 0$, $x(0) = 0.5 - 6 + 8.5$ ´ $2^0 = 3$

When $n = 1$, $x(1) = 0 - 6 + 8.5$ ´ $2^1 = 11$

When $n = 2$, $x(2) = 0 - 6 + 8.5$ ´ $2^2 = 28$

When $n = 3$, $x(3) = 0 - 6 + 8.5$ ´ $2^3 = 62$

When $n = 4$, $x(4) = 0 - 6 + 8.5$ ´ $2^4 = 130$

\ $x(n) = \{3, 11, 28, 62, 130, ...\}$

Method-3 : Power Series Expansion Method

$$\text{Given that,}\quad X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

Let us divide the numerator polynomial by denominator polynomial as shown below.

$$\begin{array}{r|l}
 & 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \ldots\ldots \\
\hline
1 - 3z^{-1} + 2z^{-2} & 3 + 2z^{-1} + z^{-2} \\
 & 3 - 9z^{-1} + 6z^{-2} \\
 & (-)\ \ (+)\qquad (-) \\
\hline
 & \quad 11z^{-1} - 5z^{-2} \\
 & \quad 11z^{-1} - 33z^{-2} + 22z^{-3} \\
 & \quad (-)\qquad (+)\qquad (-) \\
\hline
 & \qquad 28z^{-2} - 22z^{-3} \\
 & \qquad 28z^{-2} - 84z^{-3} + 56z^{-4} \\
 & \qquad (-)\qquad (+)\qquad (-) \\
\hline
 & \qquad\quad 62z^{-3} - 56z^{-4} \\
 & \qquad\quad 62z^{-3} - 186z^{-4} + 124z^{-5} \\
 & \qquad\quad (-)\qquad (+)\qquad (-) \\
\hline
 & \qquad\qquad 130z^{-4} - 124z^{-5} \\
 & \qquad\qquad \vdots
\end{array}$$

$$\therefore\ X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \ \ldots.. \qquad .....(1)$$

Let, x(n) be inverse Ƶ-transform of X(z).

Now, by definition of Ƶ-transform,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

$$= \,.....+ x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + x(4)\, z^{-4} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$x(0) = 3$

$x(1) = 11$

$x(2) = 28$

$x(3) = 62$

$x(4) = 130$ and so on.

\ $x(n) = \{\underline{3}, 11, 28, 62, 130, ...\}$

<u>Conclusion</u> : It is observed that the results of all the three methods are same.

Example 3.8

Determine the inverse Ƶ-transform of the following z-domain functions.

a) $X(z) = \dfrac{3z^2 + 2z + 1}{z^2 + 4z + 3}$ **b)** $X(z) = \dfrac{z - 0.6}{z^2 + z + 2}$ **c)** $X(z) = \dfrac{2z - 4}{(z - 1)\,(z + 2)^2}$

Solution

a) Given that, $\mathbf{X(z) = \dfrac{3z^2 + 2z + 1}{z^2 + 4z + 3}}$

On dividing the numerator by denominator, the X(z) can be expressed as shown below.

$$X(z) = \frac{3z^2 + 2z + 1}{z^2 + 4z + 3} = 3 + \frac{-10z - 8}{z^2 + 4z + 3} = 3 + \frac{-10z - 8}{(z + 1)\,(z + 3)}$$

$$\begin{array}{r|l} & \quad 3 \\ \hline z^2 + 4z + 3 & 3z^2 + 2z + 1 \\ & 3z^2 + 12z + 9 \\ & \underline{(-) \quad (-) \quad (-)} \\ & \quad -10z - 8 \end{array}$$

By partial fraction expansion we get, $X(z) = 3 + \dfrac{A_1}{z + 1} + \dfrac{A_2}{z + 3}$

$$A_1 = \cancel{(z+1)}\, \frac{-10z - 8}{\cancel{(z + 1)}\,(z + 3)}\Bigg|_{z = -1} = \frac{-10z - 8}{z + 3}\Bigg|_{z = -1} = \frac{10 - 8}{-1 + 3} = \frac{2}{2} = 1$$

$$A_2 = \cancel{(z+3)}\, \frac{-10z - 8}{(z + 1)\,\cancel{(z + 3)}}\Bigg|_{z = -3} = \frac{-10z - 8}{z + 1}\Bigg|_{z = -3} = \frac{-10 \times (-3) - 8}{-3 + 1} = -11$$

$$\therefore\ X(z) = 3 + \frac{1}{z + 1} - \frac{11}{z + 3} = 3 + \frac{1}{z}\,\frac{z}{z - (-1)} - 11\,\frac{1}{z}\,\frac{z}{z - (-3)}$$

Multiply and divide by z.

$$= 3 + z^{-1}\,\frac{z}{z - (-1)} - 11z^{-1}\,\frac{z}{z - (-3)}$$

On taking inverse Ƶ-transform of X(z) we get,

$$x(n) = 3\,\delta(n) + (-1)^{n-1}\, u(n - 1) - 11(-3)^{n-1}\, u(n - 1)$$

$$= 3\,\delta(n) + \left[(-1)^{n-1} - 11(-3)^{n-1}\right] u(n - 1)$$

$Ƶ\{\delta(n)\} = 1$

$Ƶ\{a^n\, u(n)\} = \dfrac{z}{z - a}$

If $Ƶ\{a^n\, u(n)\} = \dfrac{z}{z - a}$

then by time shifting property,

$Ƶ\{a^{(n-1)}\, u(n - 1)\} = z^{-1}\, \dfrac{z}{z - a}$

When $n = 0$, $x(0) = 3 + 0 + 0 = 3$

When $n = 1$, $x(1) = 0 + 1 - 11 = -10$

When $n = 2$, $x(2) = 0 - 1 + 33 = 32$

When $n = 3$, $x(3) = 0 + 1 - 99 = -98$

When $n = 4$, $x(4) = 0 - 1 + 297 = 296$

$\therefore\ x(n) = \{3, -10, 32, -98, 296, ...\}$

Alternate Method

$$X(z) = \frac{3z^2 + 2z + 1}{z^2 + 4z + 3}$$

$$\therefore \frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z^2 + 4z + 3)} = \frac{3z^2 + 2z + 1}{z(z + 1)(z + 3)}$$

By partial fraction expansion technique $\frac{X(z)}{z}$ can be expressed as,

$$\frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z + 1)(z + 3)} = \frac{A_1}{z} + \frac{A_2}{z + 1} + \frac{A_3}{z + 3}$$

$$A_1 = z\frac{X(z)}{z}\bigg|_{z=0} = \cancel{z}\,\frac{3z^2 + 2z + 1}{\cancel{z}(z + 1)(z + 3)}\bigg|_{z=0} = \frac{0 + 0 + 1}{(0 + 1)(0 + 3)} = \frac{1}{3}$$

$$A_2 = (z + 1)\frac{X(z)}{z}\bigg|_{z=-1} = \cancel{(z + 1)}\,\frac{3z^2 + 2z + 1}{z\cancel{(z + 1)}(z + 3)}\bigg|_{z=-1} = \frac{3(-1)^2 + 2(-1) + 1}{-1 \times (-1 + 3)} = -1$$

$$A_3 = (z + 3)\frac{X(z)}{z}\bigg|_{z=-3} = \cancel{(z + 3)}\,\frac{3z^2 + 2z + 1}{z(z + 1)\cancel{(z + 3)}}\bigg|_{z=-3} = \frac{3(-3)^2 + 2(-3) + 1}{-3 \times (-3 + 1)} = \frac{22}{6} = \frac{11}{3}$$

$$\therefore \frac{X(z)}{z} = \frac{1}{3}\,\frac{1}{z} - \frac{1}{z + 1} + \frac{11}{3}\,\frac{1}{z + 3}$$

$$\therefore X(z) = \frac{1}{3} - \frac{z}{z + 1} + \frac{11}{3}\,\frac{z}{z + 3}$$

$$= \frac{1}{3} - \frac{z}{z - (-1)} + \frac{11}{3}\,\frac{z}{z - (-3)}$$

$\mathcal{Z}\{\delta(n)\} = 1$

$\mathcal{Z}\{a^n u(n)\} = \frac{z}{z - a}$

On taking inverse $\mathcal{Z}$-transform of X(z) we get,

$$x(n) = \frac{1}{3}\,\delta(n) - (-1)^n u(n) + \frac{11}{3}(-3)^n u(n) = \frac{1}{3}\,\delta(n) + [-(-1)^n + \frac{11}{3}(-3)^n]\,u(n)$$

When $n = 0$, $x(0) = \frac{1}{3} - 1 + \frac{11}{3} = 3$

When $n = 1$, $x(1) = 0 + 1 + \frac{11}{3} \times -3 = -10$

When $n = 2$, $x(2) = 0 - 1 + \frac{11}{3} \times (-3)^2 = 32$

When $n = 3$, $x(3) = 0 + 1 + \frac{11}{3} \times (-3)^3 = -98$

When $n = 4$, $x(4) = 0 - 1 + \frac{11}{3} \times (-3)^4 = 296$

$\therefore\ x(n) = \{3, -10, 32, -98, 296, ...\}$

Note: The closed form expression of x(n) in the two methods look different, but on evaluating x(n) for various values of n we get same signal x(n).

b) Given that, $X(z) = \dfrac{z - 0.6}{z^2 + z + 2}$

The roots of the quadratic $z^2 + z + 2 = 0$ are,

$$z = \frac{-1 \pm \sqrt{1 - 4 \times 2}}{2} = \frac{-1 \pm j\sqrt{7}}{2} = -0.5 \pm j1.323$$

$$X(z) = \frac{z - 0.6}{z^2 + z + 2} = \frac{z - 0.6}{(z + 0.5 - j1.323\,(z + 0.5 + j1.323)}$$

By partial fraction expansion we get,

$$X(z) = \frac{A}{z + 0.5 - j1.323} + \frac{A^*}{z + 0.5 + j1.323}$$

$$A = \cancel{(z + 0.5 - j1.323)}\,\frac{z - 0.6}{\cancel{(z + 0.5 - j1.323)}\,(z + 0.5 + j1.323)}\Bigg|_{z = -0.5 + j1.323}$$

$$= \frac{z - 0.6}{(z + 0.5 + j1.323)}\Bigg|_{z = -0.5 + j1.323} = \frac{-0.5 + j1.323 - 0.6}{-0.5 + j1.323 + 0.5 + j1.323}$$

$$= \frac{-1.1 + j1.323}{j2.646} = \frac{-1.1}{j2.646} + \frac{j1.323}{j2.646} = 0.5 + j0.416$$

$$\therefore \quad A^* = (0.5 + j0.416)^* = (0.5 - j0.416)$$

Multiply and divide by z

$$\therefore\ X(z) = \frac{0.5 + j0.416}{z + 0.5 - j1.323} + \frac{0.5 - j0.416}{z + 0.5 + j1.323}$$

$$= (0.5 + j0.416)\,\frac{1}{z}\,\frac{z}{z + 0.5 - j1.323} + (0.5 - j0.416)\,\frac{1}{z}\,\frac{z}{z + 0.5 + j1.323}$$

$$= (0.5 + j0.416)z^{-1}\,\frac{z}{z - (-0.5 + j1.323)} + (0.5 - j0.416)\,z^{-1}\,\frac{z}{z - (-0.5 - j1.323)}$$

On taking inverse Z-transform of X(z) we get,

If $Z\{a^n\,u(n)\} = \dfrac{z}{z - a}$ then by time shifting property, $Z\{a^{(n-1)}\,u(n-1)\} = z^{-1}\,\dfrac{z}{z-a}$

$$x(n) = (0.5 + j0.416)\,(-0.5 + j1.323)^{(n-1)}u(n-1) + (0.5 - j0.416)\,(-0.5 - j1.323)^{(n-1)}u(n-1)$$

Alternatively the above result can be expressed as shown below.

$180^\circ = \pi\,\text{rad}$ $\quad\therefore\ 1^\circ = \dfrac{\pi}{180}\,\text{rad}$

$$0.5 + j0.416 = 0.5 + j0.416 = 0.65\ \angle 39.7^\circ = 0.65\,\angle 0.22\pi$$

$$0.5 - j0.416 = 0.5 - j0.416 = 0.65\ \angle{-39.7^\circ} = 0.65\,\angle{-0.22\pi}$$

$$-0.5 + j1.323 = 1.414\,\angle 110.7^\circ = 1.414\ \angle 0.61\pi$$

$$-0.5 - j1.323 = 1.414\,\angle{-110.7^\circ} = 1.414\ \angle{-0.61\pi}$$

$$\therefore\ x(n) = [0.65\,\angle 0.22\pi]\ [1.414\,\angle 0.61\pi]^{(n-1)}\ u(n-1) + [0.65\,\angle{-0.22\pi}]\ [1.414\,\angle{-0.61\pi}]^{(n-1)}u(n-1)$$

$$= [0.65\,\angle 0.22\pi]\ [1.414^{(n-1)}\angle 0.61\pi\,(n-1)]\ u(n-1) + [0.65\,\angle{-0.22\pi}]\ [1.414^{(n-1)}\ \angle{-0.61\pi}\,(n-1)]\,u(n-1)$$

$\therefore\ 39.7^\circ = \dfrac{39.7}{180}\pi = 0.22\pi\,\text{rad}$

$110.7^\circ = \dfrac{110.7}{180}\pi = 0.61\pi\,\text{rad}$

$$= 0.65\,(1.414)^{(n-1)}\ \angle(0.22\pi + 0.61\pi n - 0.61\pi)\,u(n-1) + 0.65\ (1.414)^{(n-1)}\ \angle(-0.22\pi - 0.61\pi n + 0.61\pi)\,u(n-1)$$

$$= 0.65\,(1.414)^{(n-1)}\ [1\,\angle(0.61n - 0.4)\pi + 1\,\angle{-(0.61n - 0.4)\pi}]\ u(n-1)$$

$$= 0.65\,(1.414)^{(n-1)}\ [\cos((0.61n - 0.4)\pi) + j\sin((0.61n - 0.4)\pi) + \cos((0.61\,n - 0.4)\pi) - j\sin((0.61n - 0.4)\pi)\,]\ u(n-1)$$

$$= 0.65\,(1.414)^{(n-1)}\ 2\cos((0.61n - 0.4)\pi)\ u(n-1)$$

$$= 1.3\,(1.414)^{(n-1)}\cos((0.61n - 0.4)\pi)\ u(n-1)$$

c) Given that, $X(z) = \dfrac{2z - 4}{(z-1)(z+2)^2}$

By partial fraction expansion we get,

$$X(z) = \frac{2z-4}{(z-1)(z+2)^2} = \frac{A_1}{z-1} + \frac{A_2}{(z+2)^2} + \frac{A_3}{(z+2)}$$

$$A_1 = \cancel{(z-1)}\frac{2z-4}{\cancel{(z-1)}(z+2)^2}\Bigg|_{z=1} = \frac{2z-4}{(z+2)^2}\Bigg|_{z=1} = \frac{2-4}{(1+2)^2} = \frac{-2}{9} = -0.22$$

$$A_2 = \cancel{(z+2)^2}\frac{2z-4}{(z-1)\cancel{(z+2)^2}}\Bigg|_{z=-2} = \frac{2z-4}{z-1}\Bigg|_{z=-2} = \frac{2\times -2-4}{-2-1} = \frac{-8}{-3} = 2.67$$

$$A_3 = \frac{d}{dz}\left[\cancel{(z+2)^2}\frac{2z-4}{(z-1)\cancel{(z+2)^2}}\right]\Bigg|_{z=-2} = \frac{d}{dz}\left[\frac{2z-4}{z-1}\right]\Bigg|_{z=-2}$$

$\boxed{d\dfrac{u}{v} = \dfrac{v\,du - u\,dv}{v^2}}$

$$= \frac{2(z-1) - (2z-4)}{(z-1)^2}\Bigg|_{z=-2} = \frac{2(-2-1) - (2\times -2-4)}{(-2-1)^2} = \frac{2}{9} = 0.22$$

$\boxed{\text{Multiply and divide by } z}$

$$\therefore X(z) = \frac{-0.22}{z-1} + \frac{2.67}{(z+2)^2} + \frac{0.22}{z+2} = -0.22\frac{1}{z}\frac{z}{z-1} + \frac{1}{z}\frac{2.67z}{(z+2)^2} + 0.22\frac{1}{z}\frac{z}{z+2}$$

$$= -0.22z^{-1}\frac{z}{z-1} + \frac{2.67}{-2}z^{-1}\frac{-2z}{(z-(-2))^2} + 0.22z^{-1}\frac{z}{z-(-2)}$$

$\boxed{\text{Multiply and divide by } -2}$

> $\mathcal{Z}\{u(n)\} = \dfrac{z}{z-1}$; $\mathcal{Z}\{a^n u(n)\} = \dfrac{z}{z-a}$; $\mathcal{Z}\{na^n u(n)\} = \dfrac{az}{(z-a)^2}$
>
> If $\mathcal{Z}\{x(n)\} = X(z)$ then by time shifting property $\mathcal{Z}\{x(n-1)\} = z^{-1}X(z)$
>
> $\therefore \mathcal{Z}\{u(n-1)\} = z^{-1}\dfrac{z}{z-1}$; $\mathcal{Z}\{a^{(n-1)}u(n-1)\} = z^{-1}\dfrac{z}{z-a}$
>
> and $\mathcal{Z}\{(n-1)\,a^{(n-1)}u(n-1)\} = z^{-1}\dfrac{az}{(z-a)^2}$

On taking inverse $\mathcal{Z}$-transform of X(z) using standard transform and shifting property we get,

$$x(n) = -0.22\,u(n-1) - 1.335(n-1)(-2)^{n-1}u(n-1) + 0.22(-2)^{n-1}u(n-1)$$

$$= \left[-0.22 + [-1.335(n-1) + 0.22]\,(-2)^{n-1}\right]u(n-1)$$

Example 3.9

Determine the inverse $\mathcal{Z}$-transform of the following function.

a) $X(z) = \dfrac{1}{1 + 4.5z^{-1} + 3.5z^{-2}}$

b) $X(z) = \dfrac{z^2}{z^2 - z + 0.5}$

c) $X(z) = \dfrac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$

d) $X(z) = \dfrac{1}{(1+z^{-1})(1-z^{-1})^2}$

Solution

a) Given that, $X(z) = \dfrac{1}{1 + 4.5z^{-1} + 3.5z^{-2}}$

> The roots of quadratic $z^2 + 4.5z + 3.5 = 0$ are,
>
> $z = \dfrac{-4.5 \pm \sqrt{4.5^2 - 4\times 3.5}}{2} = \dfrac{-4.5 \pm 2.5}{2} = -1,\ -3.5$

$$X(z) = \frac{1}{1 + 4.5z^{-1} + 3.5z^{-2}} = \frac{1}{1 + \dfrac{4.5}{z} + \dfrac{3.5}{z^2}} = \frac{z^2}{z^2 + 4.5z + 3.5} = \frac{z^2}{(z+1)(z+3.5)}$$

$$\therefore \frac{X(z)}{z} = \frac{z}{(z+1)(z+3.5)}$$

By partial fraction expansion, X(z)/z can be expressed as,

$$\frac{X(z)}{z} = \frac{A_1}{z+1} + \frac{A_2}{z+3.5}$$

$$A_1 = (z+1)\left.\frac{X(z)}{z}\right|_{z=-1} = \cancel{(z+1)}\left.\frac{z}{\cancel{(z+1)}(z+3.5)}\right|_{z=-1} = \frac{-1}{-1+3.5} = -0.4$$

$$A_2 = (z+3.5)\left.\frac{X(z)}{z}\right|_{z=-3.5} = \cancel{(z+3.5)}\left.\frac{z}{(z+1)\cancel{(z+3.5)}}\right|_{z=-3.5} = \frac{-3.5}{-3.5+1} = 1.4$$

$$\therefore \frac{X(z)}{z} = \frac{-0.4}{z+1} + \frac{1.4}{z+3.5}$$

$$\therefore X(z) = \frac{-0.4z}{z+1} + \frac{1.4z}{z+3.5} = \frac{-0.4z}{z-(-1)} + \frac{1.4z}{z-(-3.5)}$$

$$\boxed{Ƶ\{a^n\,u(n)\} = \frac{z}{z-a}\ ;\ \text{ROC } |z|>|a|}$$

On taking inverse Ƶ-transform of X(z), we get,

$$x(n) = -0.4(-1)^n u(n) + 1.4(-3.5)^n u(n) = [-0.4(-1)^n + 1.4(-3.5)^n]\,u(n)$$

---

b) Given that, $X(z) = \dfrac{z^2}{z^2 - z + 0.5}$

$$X(z) = \frac{z^2}{z^2 - z + 0.5} = \frac{z^2}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

$$\therefore \frac{X(z)}{z} = \frac{z}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

The roots of quadratic $z^2 - z + 0.5 = 0$ are,
$$z = \frac{1 \pm \sqrt{1 - 4\times 0.5}}{2} = 0.5 \pm j\,0.5$$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{A}{z - 0.5 - j0.5} + \frac{A^*}{z - 0.5 + j0.5}$$

$$A = (z - 0.5 - j0.5)\left.\frac{X(z)}{z}\right|_{z=0.5+j0.5}$$

$$= \cancel{(z - 0.5 - j0.5)}\left.\frac{z}{\cancel{(z - 0.5 - j0.5)}(z - 0.5 + j0.5)}\right|_{z=0.5+j0.5}$$

$$= \frac{0.5 + j0.5}{0.5 + j0.5 - 0.5 + j0.5} = \frac{0.5 + j0.5}{j1.0} = -j(j0.5 + 0.5) = 0.5 - j0.5$$

$$\therefore A^* = (0.5 - j0.5)^* = 0.5 + j0.5$$

$$\therefore \frac{X(z)}{z} = \frac{0.5 - j0.5}{z - 0.5 - j0.5} + \frac{0.5 + j0.5}{z - 0.5 + j0.5}$$

$$X(z) = \frac{(0.5 - j0.5)\,z}{z - (0.5 + j0.5)} + \frac{(0.5 + j0.5)\,z}{z - (0.5 - j0.5)}$$

$$\boxed{z\{a^n\,u(n)\} = \frac{z}{z-a}\ ;\ \text{ROC } |z|>|a|}$$

On taking inverse Ƶ-transform of X(z) we get,

$$x(n) = (0.5 - j0.5)(0.5 + j0.5)^n u(n) + (0.5 + j0.5)(0.5 - j0.5)^n u(n)$$

Alternatively the above result can be expressed as shown below.

$$180^\circ = \pi\,\text{rad}\ ;\quad \therefore 1^\circ = \frac{\pi}{180}\text{rad}$$
$$\therefore 45^\circ = \frac{45}{180}\pi = 0.25\pi\,\text{rad}$$

Here, 0.5 + j0.5 = 0.707Ð45° = 0.707Ð0.25p

0.5 – j0.5 = 0.707Ð–45° = 0.707Ð – 0.25p

\ x(n) = [0.707Ð – 0.25p] [0.707 Ð0.25p]^n u(n) + [0.707Ð0.25p] [0.707Ð – 0.25p]^n u(n)

= [0.707Ð–0.25p] [0.707^n Ð0.25pn] u(n) + [0.707 Ð0.25p] [0.707^n Ð–0.25pn] u(n)

= 0.707$^{(n+1)}$ Ð(0.25p (n – 1)) u(n) + 0.707$^{(n+1)}$ Ð(–0.25p(n – 1)) u(n)

= 0.707$^{(n+1)}$ [1Ð0.25p (n – 1) + 1Ð –0.25p(n – 1)]u(n)

= 0.707$^{(n+1)}$ [cos (0.25p (n – 1)) + j sin (0.25p(n – 1)) + cos (0.25p(n – 1)) – j sin (0.25p(n – 1))] u(n)

= 0.707$^{(n+1)}$ 2 cos (0.25p (n – 1)) u(n)

c) Given that, $\mathbf{X(z) = \dfrac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}}$

The roots of the quadratic $z^2 - z + 0.5 = 0$ are,

$$z = \frac{1 \pm \sqrt{1 - 4 \times 0.5}}{2}$$
$$= 0.5 \pm j0.5$$

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5\,z^{-2}} = \frac{z^{-1}(z + 1)}{z^{-2}(z^2 - z + 0.5)}$$
$$= \frac{z\,(z + 1)}{(z^2 - z + 0.5)} = \frac{z\,(z + 1)}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)}$$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{(z + 1)}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)} = \frac{A}{z - 0.5 - j0.5} + \frac{A^*}{z - 0.5 + j0.5}$$

$$A = (z - 0.5 - j0.5)\left.\frac{X(z)}{z}\right|_{z = 0.5 + j0.5}$$

$$= \cancel{(z - 0.5 - j0.5)}\left.\frac{(z + 1)}{\cancel{(z - 0.5 - j0.5)}\,(z - 0.5 + j0.5)}\right|_{z = 0.5 + j0.5}$$

$$= \frac{0.5 + j0.5 + 1}{0.5 + j0.5 - 0.5 + j0.5} = \frac{1.5 + j0.5}{j1} = -j(j0.5 + 1.5) = 0.5 - j1.5$$

$$A^* = (0.5 - j1.5)^* = 0.5 + j1.5$$

$$\therefore \quad \frac{X(z)}{z} = \frac{0.5 - j1.5}{z - 0.5 - j0.5} + \frac{0.5 + j1.5}{z - 0.5 + j0.5}$$

$$\mathcal{Z}\{a^n\,u(n)\} = \frac{z}{z - a}$$

$$X(z) = (0.5 - j1.5)\,\frac{z}{z - (0.5 + j0.5)} + (0.5 + j1.5)\,\frac{z}{z - (0.5 - j0.5)}$$

On taking inverse $\mathcal{Z}$-transform of X(z) we get,

$$x(n) = (0.5 - j1.5)\,(0.5 + j0.5)^n\,u(n) + (0.5 + j1.5)\,(0.5 - j0.5)^n\,u(n)$$

Alternatively the above result can be expressed as shown below.

$$180^\circ = \pi\,\text{rad}\;;\quad \therefore 1^\circ = \frac{\pi}{180}\,\text{rad}$$
$$\therefore 71.6^\circ = \frac{71.6}{180}\pi = 0.4\pi\,\text{rad}$$
$$\therefore 45^\circ = \frac{45}{180}\pi = 0.25\pi\,\text{rad}$$

Here, $0.5 - j1.5 = 1.581\angle{-71.6^\circ} = 1.581\angle{-0.4\pi}$

$0.5 + j1.5 = 1.581\angle 71.6^\circ = 1.581\angle 0.4\pi$

$0.5 + j0.5 = 0.707\angle 45^\circ = 0.707\angle 0.25\pi$

$0.5 - j0.5 = 0.707\angle{-45^\circ} = 0.707\angle{-0.25\pi}$

$$\therefore\; x(n) = [1.581\angle{-0.4\pi}]\,[0.707\angle 0.25\pi]^n\,u(n) + [1.581\angle 0.4\pi]\,[0.707\angle{-0.25\pi}]^n\,u(n)$$
$$= [1.581\angle{-0.4\pi}]\,[0.707^n\angle 0.25\pi n]\,u(n) + [1.581\angle 0.4\pi]\,[0.707^n\angle{-0.25\pi n}]\,u(n)$$
$$= 1.581\,(0.707)^n\,[1\angle \pi(0.25n - 0.4) + 1\angle{-\pi(0.25n - 0.4)}]\,u(n)$$
$$= 1.581\,(0.707)^n\,[\cos(\pi(0.25n - 0.4)) + j\sin(\pi(0.25n - 0.4)) + \cos(\pi(0.25n - 0.4))$$
$$- j\sin(\pi(0.25n - 0.4))]\,u(n)$$
$$= 1.581\,(0.707)^n\,2\cos(\pi(0.25\,n - 0.4))\,u(n)$$
$$= 3.162\,(0.707)^n\cos(\pi(0.25n - 0.4))\,u(n)$$

---

d) Given that, $\mathbf{X(z) = \dfrac{2}{(1 + z^{-1})\,(1 - z^{-1})^2}}$

$$X(z) = \frac{2}{(1 + z^{-1})\,(1 - z^{-1})^2} = \frac{2}{z^{-1}(z + 1)\,z^{-2}(z - 1)^2} = \frac{2z^3}{(z + 1)\,(z - 1)^2}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{2z^2}{(z + 1)\,(z - 1)^2}$$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z}=\frac{A_1}{z+1}+\frac{A_2}{(z-1)^2}+\frac{A_3}{z-1}$$

$$A_1=(z+1)\left.\frac{X(z)}{z}\right|_{z=-1}=\cancel{(z+1)}\left.\frac{2z^2}{\cancel{(z+1)}\,(z-1)^2}\right|_{z=-1}=\left.\frac{2z^2}{(z-1)^2}\right|_{z=-1}=\frac{2(-1)^2}{(-1-1)^2}=\frac{2}{4}=0.5$$

$$A_2=(z-1)^2\left.\frac{X(z)}{z}\right|_{z=1}=\cancel{(z-1)^2}\left.\frac{2z^2}{(z+1)\,\cancel{(z-1)^2}}\right|_{z=1}=\left.\frac{2z^2}{z+1}\right|_{z=1}=\frac{2}{1+1}=1$$

$$A_3=\frac{d}{dz}\left[(z-1)^2\frac{X(z)}{z}\right]\Bigg|_{z=1}=\frac{d}{dz}\left[\cancel{(z-1)^2}\frac{2z^2}{(z+1)\,\cancel{(z-1)^2}}\right]\Bigg|_{z=1}\qquad \boxed{d\frac{u}{v}=\frac{v\,du-u\,dv}{v^2}}$$

$$=\frac{d}{dz}\left[\frac{2z^2}{z+1}\right]\Bigg|_{z=1}=\left.\frac{(z+1)\,4z-2z^2}{(z+1)^2}\right|_{z=1}=\frac{(1+1)\times4-2}{(1+1)^2}=\frac{6}{4}=1.5$$

$$\therefore\ \frac{X(z)}{z}=\frac{0.5}{z+1}+\frac{1}{(z-1)^2}+\frac{1.5}{z-1}$$

$$\therefore X(z)=0.5\frac{z}{z-(-1)}+\frac{z}{(z-1)^2}+1.5\frac{z}{z-1}$$

On taking inverse Ƶ-transform of X(z) we get,

$$x(n)=0.5(-1)^n u(n)+n\,u(n)+1.5\,u(n)$$

$$=[0.5(-1)^n+n+1.5]\,u(n)$$

$$\boxed{\begin{aligned}&Ƶ\{a^n u(n)\}=\frac{z}{z-a}\\&Ƶ\{n\,u(n)\}=\frac{z}{(z-1)^2}\\&Ƶ\{u(n)\}=\frac{z}{z-1}\end{aligned}}$$

Example 3.10

Determine the inverse Ƶ-transform of $X(z)=\dfrac{1}{1-4.5\,z^{-1}+3.5\,z^{-2}}$

(a) if ROC : $|z|>3.5$ **(b)** if ROC : $|z|<1.0$.

Solution

The roots of quadratic $z^2-4.5z+3.5=0$ are,

$$z=\frac{4.5\pm\sqrt{4.5^2-4\times3.5}}{2}=\frac{4.5\pm2.5}{2}=3.5,\ 1$$

Given that, $X(z)=\dfrac{1}{1-4.5z^{-1}+3.5z^{-2}}=\dfrac{1}{z^{-2}(z^2-4.5z+3.5)}=\dfrac{z^2}{(z-3.5)\,(z-1)}$

The poles of X(z) are, z = 3.5 and z = 1.0.

a) <u>When ROC is |z| > 3.5</u>

In this case, the ROC is exterior of circle whose radius corresponds to largest pole. Hence x(n) will be a causal signal. (Refer section 3.4.2).

Let us express X(z) as a power series expansion in negative powers of z, by dividing the numerator of X(z) by its denominator as shown below.

$$\begin{array}{r|l}
 & 1+4.5\,z^{-1}+16.75\,z^{-2}+59.625\,z^{-3}+209.6875\,z^{-4}+\dots\dots\dots \\
\hline
1-4.5\,z^{-1}+3.5\,z^{-2} & 1 \\
 & 1-4.5\,z^{-1}+3.5\,z^{-2} \\
 & (-)\quad(+)\qquad\quad(-) \\
\hline
 & \quad 4.5\,z^{-1}-3.5\,z^{-2} \\
 & \quad 4.5\,z^{-1}-20.25\,z^{-2}+15.75\,z^{-3} \\
 & \quad (-)\qquad\quad(+)\qquad\qquad(-) \\
\hline
 & \qquad\qquad 16.75\,z^{-2}-15.75\,z^{-3} \\
 & \qquad\qquad 16.75\,z^{-2}-75.375\,z^{-3}+58.625\,z^{-4} \\
 & \qquad\qquad (-)\qquad\quad(+)\qquad\qquad(-) \\
\hline
 & \qquad\qquad\qquad\qquad 59.625\,z^{-3}-58.625\,z^{-4} \\
 & \qquad\qquad\qquad\qquad 59.625\,z^{-3}-268.3125\,z^{-4}+208.6875\,z^{-5} \\
 & \qquad\qquad\qquad\qquad (-)\qquad\quad(+)\qquad\qquad(-) \\
\hline
 & \qquad\qquad\qquad\qquad\qquad\qquad 209.6875\,z^{-4}-208.6875\,z^{-5} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad \vdots
\end{array}$$

$$\therefore \; X(z) = \frac{1}{1 - 4.5\,z^{-1} + 3.5\,z^{-2}}$$

$$= 1 + 4.5\,z^{-1} + 16.75\,z^{-2} + 59.625\,z^{-3} + 209.6875\,z^{-4} + \ldots.. \qquad \ldots..(1)$$

If X(z) is Z-transform of x(n) then, by the definition of Z-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$$

For a causal signal,

$$X(z) = \sum_{n=0}^{\infty} x(n)\,z^{-n}$$

On expanding the summation we get,

$$X(z) = x(0)\,z^{0} + x(1)\,z^{-1} + x(2)\,z^{-2} + x(3)\,z^{-3} + x(4)\,z^{-4} + \ldots\ldots.. \qquad \ldots..(2)$$

On comparing the two power series of X(z) [equations (1) and (2)], we get,

$$x(0) = 1\,;\; x(1) = 4.5\,;\;\; x(2) = 16.75;\;\; x(3) = 59.625;\; x(4) = 209.6875\,;\; \ldots\ldots.$$

$$x(n) = \{\underset{\uparrow}{1},\; 4.5,\; 16.75,\; 59.625,\; 209.6875, \ldots.\}$$

b) When ROC is $|z| < 1.0$

In this case, the ROC is interior of circle whose radius corresponds to smallest pole. Hence x(n) will be an anticausal signal. (Refer section 3.4.2).

Let us express X(z) as a power series expansion in positive powers of z. Therefore, rewrite the denominator polynomial of X(z) in the reverse order and then the numerator, is divided by the denominator as shown below.

$$
\begin{array}{r|l}
 & 0.286z^{2} + 0.368z^{3} + 0.391z^{4} + 0.398z^{5} + 1.4z^{6} + \ldots.. \\
\hline
3.5\,z^{-2} - 4.5\,z^{-1} + 1 & 1 \\
 & \underset{(-)}{1} \;\underset{(+)}{-}\; 1.287z \;\underset{(-)}{+}\; 0.286z^{2} \\
\hline
 & \quad 1.287z - 0.286z^{2} \\
 & \quad \underset{(-)}{1.287z} \underset{(+)}{-} 1.656z^{2} \underset{(-)}{+} 0.368z^{3} \\
\hline
 & \qquad 1.37z^{2} - 0.368z^{3} \\
 & \qquad \underset{(-)}{1.37z^{2}} \underset{(+)}{-} 1.76z^{3} \underset{(-)}{+} 0.391z^{4} \\
\hline
 & \qquad\quad 1.392z^{3} - 0.391z^{4} \\
 & \qquad\quad \underset{(-)}{1.392z^{3}} \underset{(+)}{-} 1.791z^{4} \underset{(-)}{+} 0.398z^{5} \\
\hline
 & \qquad\qquad 1.4z^{4} - 0.398z^{5} \\
 & \qquad\qquad \vdots
\end{array}
$$

$$\therefore X(z) = \frac{1}{1 - 4.5\,z^{-1} + 3.5\,z^{-2}} = \frac{1}{3.5\,z^{-2} - 4.5\,z^{-1} + 1}$$

$$= 0.286z^{2} + 0.368z^{3} + 0.391z^{4} + 0.398z^{5} + 1.4z^{6} + \ldots.. \qquad \ldots..(3)$$

If X(z) is Z-transform of x(n) then, by the definition of Z-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$$

For an anticausal signal, $X(z) = \sum_{n=-\infty}^{0} x(n)\,z^{-n}$

On expanding the summation we get,

$$X(z) = \ldots..\; x(-6)\,z^{6} + x(-5)\,z^{5} + x(-4)\,z^{4} + x(-3)\,z^{3} + x(-2)\,z^{2} + x(-1)\,z + x(0) \qquad \ldots..(4)$$

On comparing the two power series of X(z) [equations (3) and (4)], we get,

$$x(0) = 0 \; ; \; x(-1) = 0 \; ; \; x(-2) = 0.286 \; ; \; x(-3) = 0.368 \; ; \; x(-4) = 0.391 \; ;$$
$$x(-5) = 0.398 \; ; \; x(-6) = 1.4 \; ; \; .....$$

$$\therefore \; x(n) = \{........, \; 1.4, \; 0.398, \; 0.391, \; 0.368, \; 0.286, \; 0, \; \underset{\uparrow}{0} \}$$

Example 3.11

Determine the inverse Z-transform of $X(z) = \dfrac{1}{1 - 0.8\,z^{-1} + 0.12\,z^{-2}}$

a) if ROC is, $|z| > 0.6$ **b)** if ROC is, $|z| < 0.2$ **c)** if ROC is, $0.2 < |z| < 0.6$

Solution

Given that, $X(z) = \dfrac{1}{1 - 0.8\,z^{-1} + 0.12\,z^{-2}} = \dfrac{1}{z^{-2}(z^2 - 0.8z + 0.12)} = \dfrac{z^2}{(z - 0.6)\,(z - 0.2)}$

$$\therefore \; \frac{X(z)}{z} = \frac{z}{(z - 0.6)\,(z - 0.2)}$$

The roots of quadratic $z^2 - 0.8z + 0.12 = 0$ are,

$$z = \frac{0.8 \pm \sqrt{0.8^2 - 4 \times 0.12}}{2} = \frac{0.8 \pm 0.4}{2} = 0.6, \; 0.2$$

By partial fraction expansion technique we get,

$$\frac{X(z)}{z} = \frac{z}{(z - 0.6)\,(z - 0.2)} = \frac{A_1}{z - 0.6} + \frac{A_2}{z - 0.2}$$

$$A_1 = (z - 0.6)\left.\frac{X(z)}{z}\right|_{z\,=\,0.6} = \cancel{(z - 0.6)}\left.\frac{z}{\cancel{(z - 0.6)}\,(z - 0.2)}\right|_{z\,=\,0.6} = \frac{0.6}{0.6 - 0.2} = 1.5$$

$$A_2 = (z - 0.2)\left.\frac{X(z)}{z}\right|_{z\,=\,0.2} = \cancel{(z - 0.2)}\left.\frac{z}{(z - 0.6)\,\cancel{(z - 0.2)}}\right|_{z\,=\,0.2} = \frac{0.2}{0.2 - 0.6} = -0.5$$

$$\therefore \; \frac{X(z)}{z} = \frac{1.5}{z - 0.6} - \frac{0.5}{z - 0.2}$$

$$\therefore \; X(z) = 1.5\,\frac{z}{z - 0.6} - 0.5\,\frac{z}{z - 0.2}$$

Now, the poles of X(z) are $p_1 = 0.6, \; p_2 = 0.2$

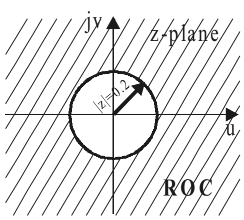
a) ROC is |z| > 0.6

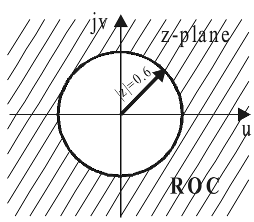
The specified ROC is exterior of the circle whose radius corresponds to the largest pole, hence x(n) will be a causal (or right- sided) signal. (Refer section 3.4.2).

$$\therefore \; x(n) = 1.5(0.6)^n\,u(n) - 0.5\,(0.2)^n\,u(n)$$

$$\mathcal{Z}\,\{a^n\,u(n)\} = \frac{z}{z - a} \; ; \; \text{ROC } |z|>|a|$$

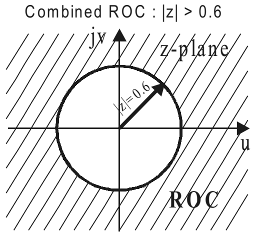
$$\mathcal{Z}\left\{0.2^n\;u(n)\right\} = \frac{z}{z - 0.2}$$
with ROC : $|z| > 0.2$





$$\mathcal{Z}\left\{0.6^n\;u(n)\right\} = \frac{z}{z - 0.6}$$
with ROC : $|z| > 0.6$

Combined ROC : $|z| > 0.6$



b) ROC is |z| < 0.2

The specified ROC is interior of the circle whose radius corresponds to the smallest pole, hence x(n) will be an anticausal (or left-sided) signal. (Refer section 3.4.2).

\ $x(n) = 1.5(-(0.6)^n\, u(-n-1)) \; - 0.5\, [-(0.2)^n\; u(-n-1)]$

$\quad = -1.5\,(0.6)^n\; u(-n-1) \; + 0.5\,(0.2)^n\; u(-n-1)$

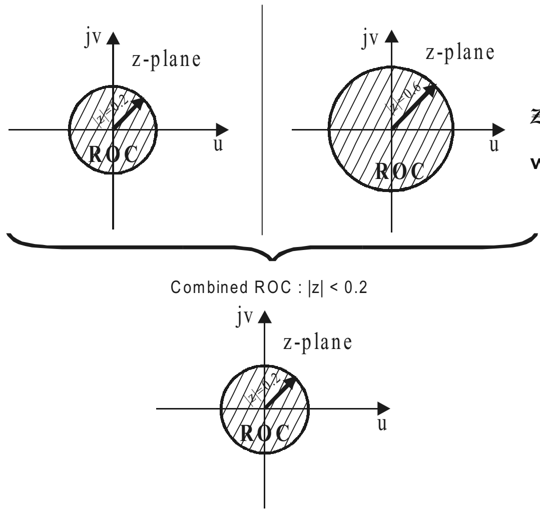
$$\boxed{\mathcal{Z}\,\{-a^n\, u(-n-1)\} = \frac{z}{z-a}\;\; ; \;\text{ROC } |z|<|a|}$$

$\mathcal{Z}\left\{-0.2^n\; u(-n-1)\right\} = \dfrac{z}{z-0.2}$

with ROC : |z| < 0.2

$\mathcal{Z}\left\{-0.6^n\; u(-n-1)\right\} = \dfrac{z}{z-0.6}$

with ROC : |z| < 0.6



c) ROC is 0.2 < |z| < 0.6

The specified ROC is the region in between two circles of radius 0.2 and 0.6. Hence the term corresponds to the pole, $p_1 = 0.6$ will be anticausal signal (because |z| < 0.6) and the term corresponds to the pole, $p_2 = 0.2$, will be a causal signal (because |z| > 0.2). (Refer section 3.4.2).

\ $x(n) = 1.5(-(0.6)^n\, u(-n-1)) \; - 0.5\,(0.2)^n\; u(n)$

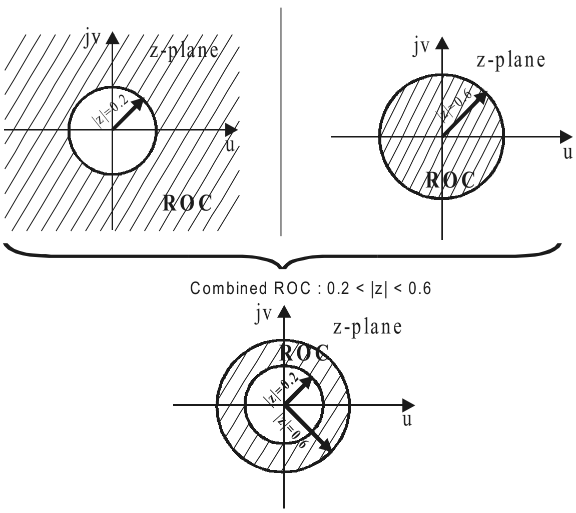
$\quad = -1.5(0.6)^n\, u(-n-1)) \; - 0.5\,(0.2)^n\; u(n)$

$\mathcal{Z}\left\{0.2^n\; u(n)\right\} = \dfrac{z}{z-0.2}$

with ROC : |z| > 0.2

$\mathcal{Z}\left\{-0.6^n\; u(-n-1)\right\} = \dfrac{z}{z-0.6}$

with ROC : |z| < 0.6



3.6 Analysis of LTI Discrete Time System Using Z-Transform

3.6.1 Transfer Function of LTI Discrete Time System

Let x(n) be the input and y(n) be the output of an LTI discrete time system. The mathematical equation governing the input-output relation of an LTI discrete time system is given by, (refer Chapter 2, equation (2.17)).

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m) \qquad .....(3.52)$$

The equation (3.52) is a constant coefficient difference equation and N is the order of the system.

Let us take Z-transform of equation (3.52) with zero initial conditions (i.e., y(n) = 0 for n < 0 and x(n) = 0 for n < 0).

$$\therefore\ \mathcal{Z}\{y(n)\} = \mathcal{Z}\left\{-\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)\right\}$$

$$= \mathcal{Z}\left\{-\sum_{m=1}^{N} a_m\, y(n-m)\right\} + \mathcal{Z}\left\{\sum_{m=0}^{M} b_m\, x(n-m)\right\}$$

$$= -\sum_{m=1}^{N} a_m\, \mathcal{Z}\{y(n-m)\} + \sum_{m=0}^{M} b_m\, \mathcal{Z}\{x(n-m)\} \qquad .....(3.53)$$

Let y(n) = 0 for n < 0, now if $\mathcal{Z}\{y(n)\} = Y(z)$, then $\mathcal{Z}\{y(n-m)\} = z^{-m}\, Y(z)$ (Using shifting property).
Let x(n) = 0 for n < 0, now if $\mathcal{Z}\{x(n)\} = X(z)$, then $\mathcal{Z}\{x(n-m)\} = z^{-m}\, X(z)$ (Using shifting property).

Using shifting property of Z-transform, the equation (3.53) is written as shown below.

$$Y(z) = -\sum_{m=1}^{N} a_m\, z^{-m}\, Y(z) + \sum_{m=0}^{M} b_m\, z^{-m}\, X(z)$$

$$Y(z) + \sum_{m=1}^{N} a_m\, z^{-m}\, Y(z) = \sum_{m=0}^{M} b_m\, z^{-m}\, X(z)$$

$$Y(z)\left[1 + \sum_{m=1}^{N} a_m\, z^{-m}\right] = \sum_{m=0}^{M} b_m\, z^{-m}\, X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m\, z^{-m}}{1 + \sum_{m=1}^{N} a_m\, z^{-m}}$$

On expanding the summations in the above equation we get,

$$\boxed{\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \,.....\, + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \,.....\, + a_N z^{-N}}} \qquad .....(3.54)$$

The ***transfer function*** of a discrete time system is defined as the ratio of Z-transform of output and Z-transform of input. Hence the equation (3.54) is the transfer function of an LTI discrete time system.

The equation (3.54) is a rational function of z^{-1} (i.e., ratio of two polynomials in z^{-1}). The numerator and denominator polynomials of equation (3.54) are converted to positive power of z and then expressed in the factorized form as shown in equation (3.55). [Refer equation (3.36)].

$$\frac{Y(z)}{X(z)} = G\,\frac{(z - z_1)(z - z_2)(z - z_3)\,.....\,(z - z_N)}{(z - p_1)(z - p_2)(z - p_3)\,.....\,(z - p_N)} \qquad \boxed{\text{Let } M = N} \qquad .....(3.55)$$

where, $z_1, z_2, z_3, \,.....\, z_N$ are roots of numerator polynomial (or zeros of discrete time system)

$p_1, p_2, p_3, \,.....\, p_N$ are roots of denominator polynomial (or poles of discrete time system).

3.6.2 Impulse Response and Transfer Function

Let, x(n) = Input of an LTI discrete time system

y(n) = Output or Response of the LTI discrete time system for the input x(n)

h(n) = Impulse response (i.e., response for impulse input)

Now, the response y(n) of the discrete time system is given by convolution of input and impulse response. [Refer Chapter 2, equation (2.33)].

$$\therefore\ y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m)$$

On taking $\mathcal{Z}$-transform of the above equation we get,

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n) * h(n)\}$$

Using convolution property of $\mathcal{Z}$-transform, the above equation can be written as,

> If $\mathcal{Z}\{x(n)\} = X(z)$
> and $\mathcal{Z}\{h(n)\} = H(z)$
> then by convolution property,
> $\mathcal{Z}\{x(n) * h(n)\} = X(z)\,H(z)$

$$Y(z) = X(z)\,H(z)$$

$$\therefore\ H(z) = \frac{Y(z)}{X(z)} \qquad .....(3.56)$$

$$\therefore\ H(z) = \frac{Y(z)}{X(z)} = G\,\frac{(z-z_1)(z-z_2)(z-z_3)\,.....\,(z-z_N)}{(z-p_1)(z-p_2)(z-p_3)\,.....\,(z-p_N)}$$

Using equation (3.55).

From equation (3.56) we can conclude that the ***transfer function*** of an LTI discrete time system is also given by $\mathcal{Z}$-transform of the impulse response.

Alternatively, we can say that the inverse $\mathcal{Z}$-transform of transfer function is the impulse response of the system.

$$\boxed{\therefore\ \text{Impulse reponse, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{Y(z)}{X(z)}\right\}}$$

Using equation (3.56).

3.6.3 Response of LTI Discrete Time System Using $\mathcal{Z}$-Transform

In general, the input-output relation of an LTI (Linear Time Invariant) discrete time system is represented by the constant coefficient difference equation shown below, [equation (3.52)].

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\text{(or)} \quad \sum_{m=0}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m) \ \text{ with } a_o = 1 \qquad .....(3.57)$$

The solution of the above difference equation (equation (3.57)) is the (total) response y(n) of LTI discrete time system, which consists of two parts. In signals and systems the two parts of the solution y(n) are called zero-input response $y_{zi}(n)$ and zero-state response $y_{zs}(n)$.

$$\boxed{\therefore\ \text{Response, } y(n) = y_{zi}(n) + y_{zs}(n)} \qquad .....(3.58)$$

Zero-input Response (or Free Response or Natural Response) Using $\mathcal{Z}$-Transform

The ***zero-input response*** $y_{zi}(n)$ is mainly due to initial output (or initial stored energy) in the system. The zero-input response is obtained from system equation [equation (3.57)] when input x(n) = 0.

On substituting $x(n) = 0$ and $y(n) = y_{zi}(n)$ in equation (3.57) we get,

$$\sum_{m=0}^{N} a_m \, y_{zi}(n-m) = 0 \; ; \quad \text{with } a_o = 1$$

On taking $\mathcal{Z}$-transform of the above equation with non-zero initial conditions for output we can form an equation for $Y_{zi}(z)$. The zero-input response $y_{zi}(n)$ of a discrete time system is given by inverse $\mathcal{Z}$-transform of $Y_{zi}(z)$.

Zero-State Response (or Forced Response) Using $\mathcal{Z}$-Transform

The ***zero-state response*** $y_{zs}(n)$ is the response of the system due to input signal and with zero initial output. The zero-state response is obtained from the difference equation governing the system [equation(3.57)] for specific input signal $x(n)$ for $n \geq 0$ and with zero initial output.

On substituting $y(n) = y_{zs}(n)$ in equation (3.57) we get,

$$\sum_{m=0}^{N} a_m \, y_{zs}(n-m) = \sum_{m=0}^{M} b_m \, x(n-m) \; ; \quad \text{with } a_o = 1$$

On taking $\mathcal{Z}$-transform of the above equation with zero initial conditions for output [i.e., $y_{zs}(n)$] and nonzero initial values for input [i.e., $x(n)$] we can form an equation for $Y_{zs}(z)$. The zero-state response $y_{zs}(n)$ of a discrete time system is given by inverse $\mathcal{Z}$-transform of $Y_{zs}(z)$.

Total Response

The ***total response*** $y(n)$ is the response of the system due to input signal and initial output (or intial stored energy). The total response is obtained from the difference equation governing the system [equation(3.57)] for specific input signal $x(n)$ for $n \geq 0$ and with nonzero initial conditions.

On taking $\mathcal{Z}$-transform of equation (3.57) with nonzero initial conditions for both input and output, and then substituting for $X(z)$ we can form an equation for $Y(z)$. The total response $y(n)$ is given by inverse $\mathcal{Z}$-transform of $Y(z)$. Alternatively, the total response $y(n)$ is given by sum of zero-input response $y_{zi}(n)$ and zero-state response $y_{zs}(n)$.

$$\boxed{\therefore \text{ Total response, } \; y(n) = y_{zi}(n) + y_{zs}(n)}$$

3.6.4 Convolution and Deconvolution Using $\mathcal{Z}$-Transform

Convolution

The ***convolution*** operation is performed to find the response $y(n)$ of an LTI discrete time system from the input $x(n)$ and impulse response $h(n)$.

$\therefore$ Response, $y(n) = x(n) * h(n)$

On taking $\mathcal{Z}$-transform of the above equation we get,

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n) * h(n)\} \qquad \boxed{\text{Using convolution property.}}$$

$$\therefore Y(z) = X(z)\,H(z) \qquad \text{.....(3.59)}$$

$\therefore$ Response, $y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\{X(z)\,H(z)\}$

Procedure :
1. Take $\mathcal{Z}$-transform of $x(n)$ to get $X(z)$.
2. Take $\mathcal{Z}$-transform of $h(n)$ to get $H(z)$.
3. Get the product $X(z)\,H(z)$.
4. Take inverse $\mathcal{Z}$-transform of the product $X(z)\,H(z)$.

Deconvolution

The ***deconvolution*** operation is performed to extract the input x(n) of an LTI system from the response y(n) of the system.

From equation (3.59) get,

$$X(z) = \frac{Y(z)}{H(z)}$$

On taking inverse $\mathcal{Z}$-transform of the above equation we get,

$$\text{Input, } x(n) = \mathcal{Z}^{-1}\{X(z)\} = \mathcal{Z}^{-1}\left\{\frac{Y(z)}{H(z)}\right\}$$

Procedure :
1. Take $\mathcal{Z}$-transform of y(n) to get Y(z).
2. Take $\mathcal{Z}$-transform of h(n) to get H(z).
3. Divide Y(z) by H(z) to get X(z), [i.e., X(z) = Y(z) / H(z)].
4. Take inverse $\mathcal{Z}$-transform of X(z) to get x(n).

3.6.5 Stability in z-Domain

Location of Poles for Stability

Let, h(n) be the impulse response of an LTI discrete time system. Now, if h(n) satisfies the condition,

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty \qquad .....(3.60)$$

then the LTI discrete time system is stable. [Refer Chapter 2, equation (2.24)].

The stability condition of equation (3.60) can be transformed as a condition on location of poles of transfer function of the LTI discrete time system in z-plane.

Let, $h(n) = a^n\, u(n)$

$$\text{Now, } \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} \left|a^n\, u(n)\right| = \sum_{n=0}^{\infty} a^n$$

If |a| is such that, $0 < |a| < 1$, then $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ = constant, and so the system is stable.

If $|a| > 1$, then $\sum_{n=0}^{\infty} a^n = \infty$ and so the system is unstable.

$$\text{Now, } H(z) = \mathcal{Z}\{h(n)\} = \mathcal{Z}\left\{a^n\, u(n)\right\} = \frac{z}{z-a}$$

Here H(z) has pole at z = a.

If |a| < 1, then the pole will lie inside the unit circle and if |a| > 1, then the pole will lie outside the unit circle. Therefore we can say that, ***for a stable discrete time system the poles should lie inside the unit circle.*** The various types of impulse response of LTI discrete time system and their transfer functions and the locations of poles are summarized in table 3.5.

Table 3.5 : Impulse Response and Location of Poles

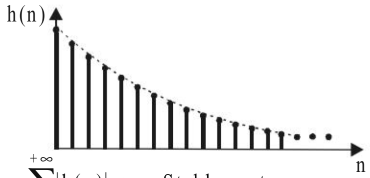
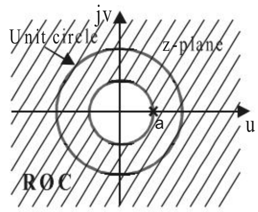
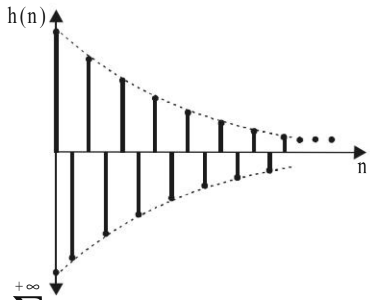
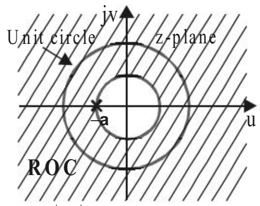
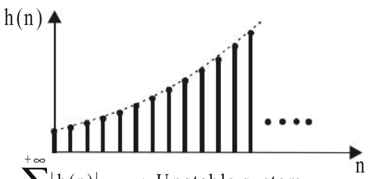
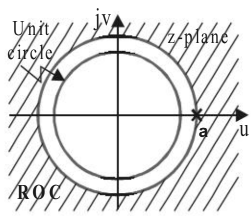
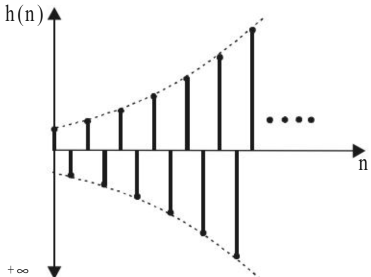
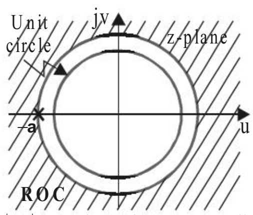
Impulse response h(n)	Transfer function H(z)	Location of poles in z-plane and ROC
$h(n) = a^n u(n)\ ;\ 0 < a < 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert < \infty$; Stable system	$H(z) = \dfrac{z}{z-a}$ ROC is $\lvert z \rvert > a$ pole at $z = a$	 Since $0 < a < 1$, the pole $z = a$, lies inside the unit circle. The ROC contains the unit circle.
$h(n) = (-a)^n u(n)\ ;\ 0 < \lvert -a \rvert < 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert < \infty$; Stable system	$H(z) = \dfrac{z}{z+a}$ ROC is $\lvert z \rvert > \lvert -a \rvert$ pole at $z = -a$	 Since $0 < \lvert -a \rvert < 1$, the pole at $z = -a$, lies inside the unit circle. The ROC contains the unit circle.
$h(n) = a^n u(n)\ ;\ a > 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; Unstable system	$H(z) = \dfrac{z}{z-a}$ ROC is $\lvert z \rvert > a$ pole at $z = a$	 Since $a > 1$, the pole at $z = a$, lies outside the unit circle. The ROC does not contain the unit circle.
$h(n) = (-a)^n u(n)\ ;\ \lvert -a \rvert > 1$  $\sum_{n=0}^{+\infty} \lvert h(n) \rvert = \infty$; Unstable system	$H(z) = \dfrac{z}{z+a}$ ROC is $\lvert z \rvert > \lvert -a \rvert$ pole at $z = -a$	 Since $\lvert -a \rvert > 1$, the pole at $z = -a$, lies outside the unit circle. The ROC does not contain the unit circle.

Table 3.5 : Continued....

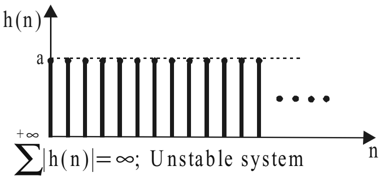
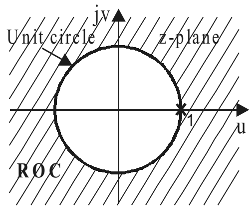
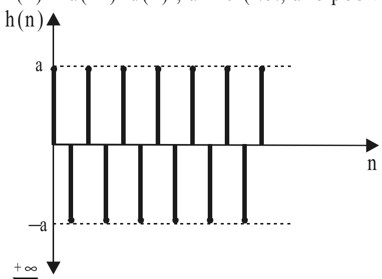
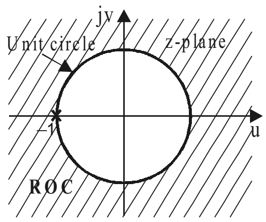
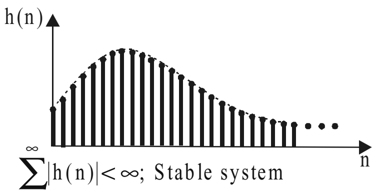
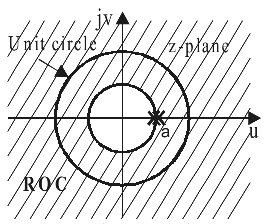
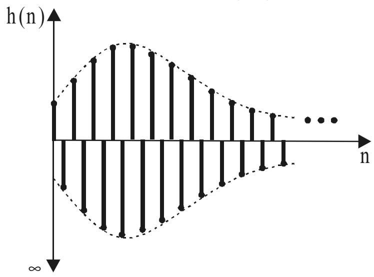
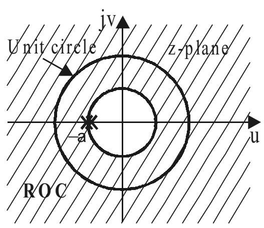
Impulse response h(n)	Transfer function H(z)	Location of poles in z-plane and ROC
$h(n) = a\,u(n)$; $a > 0$ (i.e., a is positive)  $\sum_{n=0}^{+\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{az}{z-1}$ ROC is $\lvert z\rvert > 1$ pole at $z = 1$	 The pole z = 1 lies on the unit circle. The ROC does not contain the unit circle.
$h(n) = a(-1)^n\,u(n)$; $a > 0$ (i.e., a is positive)  $\sum_{n=0}^{+\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{az}{z+1}$ ROC is $\lvert z\rvert > 1$ pole at $z = -1$	 The pole at z = –1 lies on the unit circle. The ROC does not contain the unit circle.
$h(n) = n\,a^n\,u(n)$; $0 < a < 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert < \infty$; Stable system	$H(z) = \dfrac{az}{(z-a)^2}$ ROC is $\lvert z\rvert > a$ Two poles at $z = a$	 Since $0 < a < 1$, the two poles at $z = a$ lie inside the unit circle. The ROC contains the unit circle.
$h(n) = n\,(-a)^n\,u(n)$; $0 < \lvert -a\rvert < 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert < \infty$; Stable system	$H(z) = \dfrac{az}{(z+a)^2}$ ROC is $\lvert z\rvert > a$ Two poles at $z = -a$	 Since $0 < \lvert -a\rvert < 1$, the two poles at $z = -a$ lie inside the unit circle. The ROC contains the unit circle.

Table 3.5 : Continued....

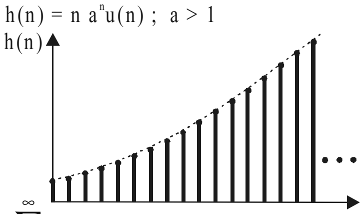
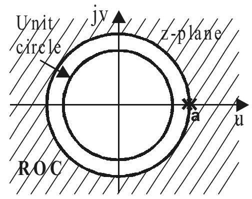
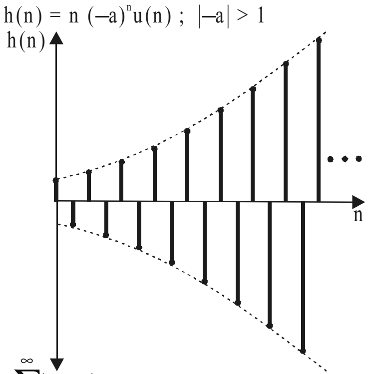
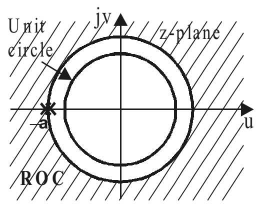
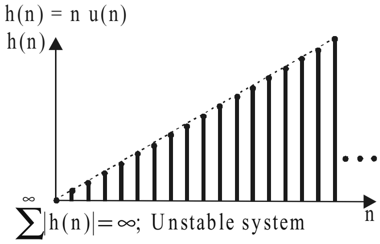
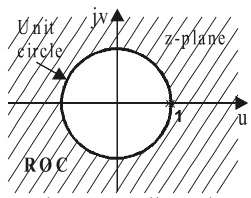
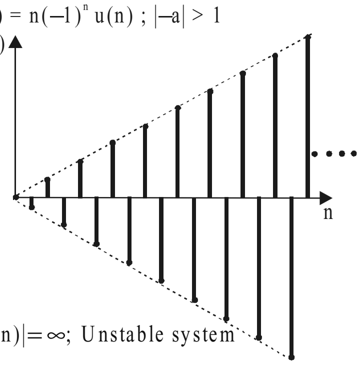
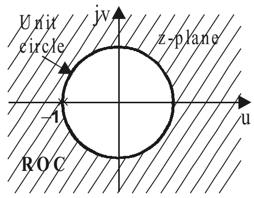
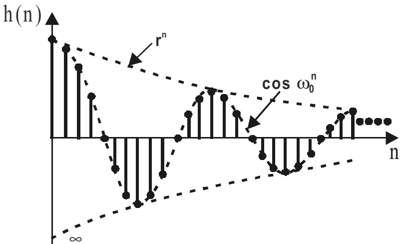
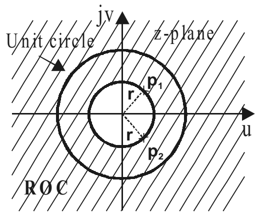
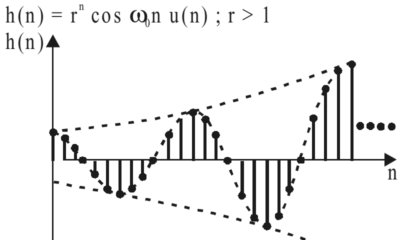
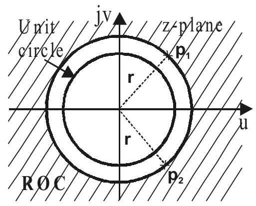
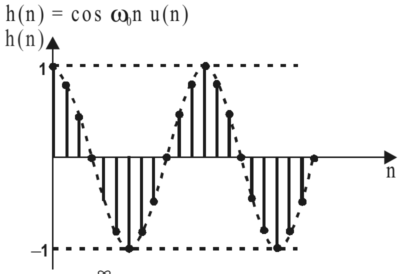
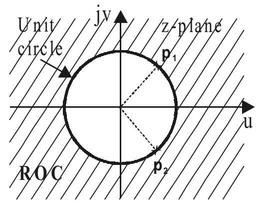
Impulse response h(n)	Transfer function H(z)	Location of poles in z-plane and ROC
$h(n) = n\,a^n u(n)\ ;\ a > 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{az}{(z - a)^2}$ ROC is $\lvert z\rvert > a$ Two poles at $z = a$	 Since a> 1, the two poles at z = a lie outside the unit circle. The ROC does not contain the unit circle.
$h(n) = n\,(-a)^n u(n)\ ;\ \lvert -a\rvert > 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{az}{(z + a)^2}$ ROC is $\lvert z\rvert > \lvert -a\rvert$ Two poles at $z = -a$	 Since $\lvert -a\rvert > 1$, the two poles at z = –a lie outside the unit circle. The ROC does not contain the unit circle.
$h(n) = n\,u(n)$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{z}{(z - 1)^2}$ ROC is $\lvert z\rvert > 1$ Two poles at $z = 1$	 The two poles at z = 1, lie on the unit circle. The ROC does not contain the unit circle.
$h(n) = n(-1)^n u(n)\ ;\ \lvert -a\rvert > 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{z}{(z + 1)^2}$ ROC is $\lvert z\rvert > 1$ Two poles at $z = -1$	 The two poles at z = –1, lie on the unit circle. The ROC does not contain the unit circle.

Table 3.5 : Continued....

Impulse response h(n)	Transfer function H(z)	Location of poles in z-plane and ROC
$h(n) = r^n \cos \omega_0 n\, u(n)\ ;\ 0 < r < 1$  $\sum_{n=0}^{\infty}\lvert h(n)\rvert < \infty$; Stable system	$H(z) = \dfrac{z(z - r\cos\omega_0)}{(z - r\cos\omega_0 - jr\sin\omega_0)(z - r\cos\omega_0 + jr\sin\omega_0)}$ ROC is $\lvert z\rvert > r$. A pair of conjugate poles at $z = p_1 = r\cos w_0 + jr\sin w_0$ $z = p_2 = r\cos w_0 - jr\sin w_0$	 Since $0 < r < 1$, the conjugate pole pairs lie inside the unit circle. The ROC contains the unit circle.
$h(n) = r^n \cos \omega_0 n\, u(n)\ ;\ r > 1$  $\sum_{n=0}^{\infty}\lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{z(z - r\cos\omega_0)}{(z - r\cos\omega_0 - jr\sin\omega_0)(z - r\cos\omega_0 + jr\sin\omega_0)}$ ROC is $\lvert z\rvert > r$. A pair of conjugate poles at $z = p_1 = r\cos w_0 + jr\sin w_0$ $z = p_2 = r\cos w_0 - jr\sin w_0$	 Since $r > 1$, the conjugate pole pairs lie outside the unit circle. The ROC does not contains the unit circle.
$h(n) = \cos \omega_0 n\, u(n)$  $\sum_{n=0}^{\infty}\lvert h(n)\rvert = 0$; Stable system	$H(z) = \dfrac{z(z - \cos\omega_0)}{(z - \cos\omega_0 - j\sin\omega_0)(z - \cos\omega_0 + j\sin\omega_0)}$ ROC is $\lvert z\rvert > 1$. A pair of conjugate poles on unit circle at, $z = p_1 = \cos w_0 + j\sin w_0$ $z = p_2 = \cos w_0 - j\sin w_0$	 Since conjugate pole pairs lie on the circle. The ROC does not contains the unit circle.

ROC of a Stable System

Let, H(z) be Z-transform of h(n). Now, by definition of Z-transform we get,

$$H(z) = \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n}$$

Let us evaluate H(z) for z = 1.

$$\therefore\ H(z) = \sum_{n=-\infty}^{+\infty} h(n)$$

On taking absolute value on both sides we get,

$$\left|H(z)\right| = \left|\sum_{n=-\infty}^{+\infty} h(n)\right| \quad \Rightarrow \quad \left|H(z)\right| = \sum_{n=-\infty}^{+\infty} \left|h(n)\right|$$

For a stable LTI discrete time system,

$$\sum_{n=-\infty}^{+\infty} \left|h(n)\right| < \infty \quad \Rightarrow \quad \left|H(z)\right| < \infty$$

Therefore, we can conclude that z = 1 will be a point in the ROC of a stable system. Hence *for a stable discrete time system the ROC of impulse response should include the unit circle.*

General Condition for Stability in z-plane

On combining the condition for location of poles and the ROC we can say that *for a stable LTI discrete time system the poles should lie inside the unit circle and the unit circle should be included in ROC of impulse response of the system.*

3.7 Relation Between Laplace Transform and Z-Transform

3.7.1 Impulse Train Sampling of Continuous Time Signal

Consider a periodic impulse train p(t) shown in fig 3.12a, with period T. The pulse train can be mathematically expressed as shown in equation (3.61).

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad \text{..... (3.61)}$$

When a continuous time signal x(t) is multiplied by the impulse train p(t), the product signal will have impulses. A continuous time signal x(t) and the product of x(t) and p(t) are shown in fig 3.12b and fig 3.12c respectively. In fig 3.12c, the magnitudes of the impulses are equal to magnitude of x(t), and so the product signal is impulse sampled version of x(t), with sampling period T. Let us denote the product signal as $x_p(t)$ and it is mathematically expressed as shown in equation (3.62).

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT) \qquad \text{..... (3.62)}$$

where, x(nT) are samples of x(t) at t = nT

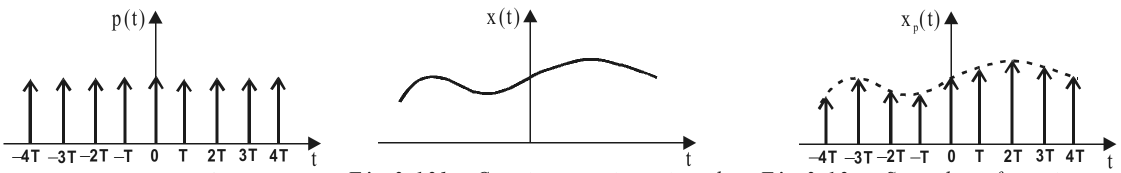


Fig 3.12a : Impulse train. *Fig 3.12b : Continuous time signal.* *Fig 3.12c : Samples of continuous time signal.*

Fig 3.12 : Impulse sampling of continuous time signal.

3.7.2 Transformation From Laplace Transform to Z-Transform

Let x(t) be a continuous time signal, and $x_p(t)$ be its impulse sampled version of discrete time signal. From equation (3.62) we get,

$\mathcal{L}\{\delta(t)\} = 1$
If $\mathcal{L}\{x(t)\} = X(s)$ then by time shifting property $\mathcal{L}\{x(t-a)\} = e^{-as}X(s)$

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{x_P(t)\} = X_P(s) = \mathcal{L}\left\{\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)\right\} = \sum_{n=-\infty}^{\infty} x(nT)\,\mathcal{L}\{\delta(t-nT)\}$$

$$\therefore X_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\,e^{-nsT} = \sum_{n=-\infty}^{\infty} x(nT)\left(e^{sT}\right)^{-n} \qquad .....(3.63)$$

where $X_p(s)$ is Laplace transform of $x_p(t)$.

Let us take a transformation, $e^{sT} = z$.

On substituting, $e^{sT} = z$, in equation (3.63) we get,

$$X_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\,z^{-n} \qquad ....(3.64)$$

The Z-transform of x(nT), using the definition of Z-transform is given by,

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT)\,z^{-n} \qquad ....(3.65)$$

On comparing equations (3.64) and (3.65) we can say that, if a discrete time signal x(nT) is a sampled version of x(t), then *Z-transform of the discrete time signal can be obtained from Laplace transform of sampled version of x(t), by choosing the transformation, $e^{sT} = z$.* This transformation is also called ***impulse invariant transformation***.

3.7.3 Relation Between s-Plane and z-Plane

Consider a point s_1 in s-plane as shown in fig 3.13. Now the transformation,

$$e^{s_1T} = z_1 \qquad .....(3.66)$$

will transform the point s_1 to a corresponding point z_1 in z-plane.

Let the coordinates of s_1 be s$_1$ and W$_1$ as shown in fig 3.13.

$$\therefore\ s_1 = \sigma_1 + j\Omega_1 \qquad .....(3.67)$$

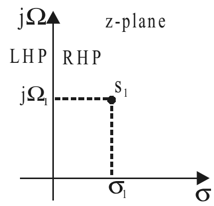


Fig 3.13 : *s-plane.*

Using equation (3.67) the equation (3.66) can be written as,

$$z_1 = e^{(\sigma_1 + j\Omega_1)T} = e^{\sigma_1T}\,e^{j\Omega_1T} \qquad .....(3.68)$$

On separating the magnitude and phase of equation (3.68) we get,

$$|z_1| = e^{\sigma_1T}\ ; \quad \angle z_1 = j\Omega_1T \qquad .....(3.69)$$

From equation (3.69) the following observations can be made.

1. If s$_1$ < 0 (i.e., s$_1$ is negative), then the point-s_1 lies on Left Half (LHP) of s-plane. In this case, $|z_1| < 1$, hence the corresponding point-z_1 will lie inside the unit circle in z-plane.
2. If s$_1$ = 0 (i.e., real part is zero), then the point-s_1 lies on imaginary axis of s-plane. In this case, $|z_1| = 1$, hence the corresponding point-z_1 will lie on the unit circle in z-plane.

3. If $\sigma_1 > 0$ (i.e., σ_1 is positive), then the point-s_1 lies on the Right Half (RHP) of s-plane. In this case $|z_1| > 1$, hence the corresponding point-z_1 will lie outside the unit circle in z-plane.

The above discussions are applicable for mapping of any point on s-plane to z-plane.

In general all points of s-plane, described by the equation,

$$s_1 = \sigma_1 + j\Omega_1 + j\frac{2\pi k}{T}, \quad \text{for } k = 0, \ \pm 1, \ \pm 2 \, ... \qquad ..... (3.70)$$

map as a single point in the z-plane described by equation,

$e^{\pm j2\pi k} = 1$; for integer k

$$z_1 = e^{\left(\sigma_1 + j\Omega_1 + \frac{j2\pi k}{T}\right)T} = e^{\sigma_1 T} e^{j\Omega_1 T} e^{j2\pi k} = e^{\sigma_1 T} e^{j\Omega_1 T} \qquad ..... (3.71)$$

The equation (3.70) represents a strip of width $2\pi/T$ in the s-plane for values of imaginary part of s in the range $-\pi/T \le \Omega \le +\pi/T$ is mapped into the entire z-plane. Similarly the strip of width $2\pi/T$ in the s-plane for values of imaginary part of s in the range $\pi/T \le \Omega \le 3\pi/T$ is also mapped into the entire z-plane. Likewise the strip of width $2\pi/T$ in the s-plane for values of imaginary part of s in the range $-3\pi/T \le \Omega \le -\pi/T$ is also mapped into the entire z-plane.

In general any strip of width $2\pi/T$ in the s-plane for values of imaginary part of s in the range $(2k-1)\pi/T \le \Omega \le (2k+1)\pi/T$, where k is an integer, is mapped into the entire z-plane. Therefore we can say that the transformation, $e^{sT} = z$, leads to many-to-one mapping, (and does not provide one-to-one mapping).

In this mapping, *the left half portion of each strip in s-plane maps into the interior of the unit circle in z-plane, right half portion of each strip in s-plane maps into the exterior of the unit circle in z-plane and the imaginary axis of each strip in s-plane maps into the unit circle in z-plane* as shown in fig 3.14.

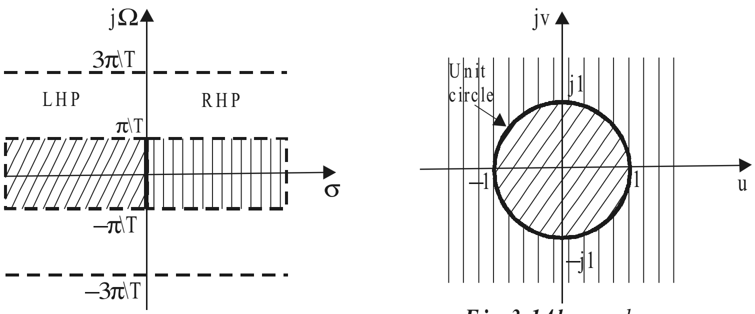


Fig 3.14a : *s-plane.* ***Fig 3.14b :*** *z-plane.*

Fig 3.14 : *Mapping of s-plane into z-plane.*

Relation Between Frequency of Continuous Time and Discrete Time Signal

Let, Ω = Frequency of continuous time signal in rad/sec.

ω = Frequency of discrete time signal in rad/sample.

Let, $z = re^{j\omega}$ be a point on z-plane, and $s = \sigma + j\Omega$, be a corresponding point in s-plane.

Consider the transformation,

$$z = e^{sT} \qquad ..... (3.72)$$

Put, $z = r\,e^{j\omega}$ and $s = \sigma + j\Omega$ in equation (3.72)

$$\therefore\ r\,e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$r\,e^{j\omega} = e^{\sigma T}\,e^{j\Omega T} \qquad .....(3.73)$$

On equating the imaginary part on either side of equation (3.73) we get,

$$\boxed{\omega = \Omega T} \quad \text{or} \quad \boxed{\Omega = \frac{\omega}{T}} \qquad .....(3.74)$$

When the transformation $e^{sT} = z$ is employed, the equation (3.74) can be used to compute the frequency of discrete time signal for a given frequency of continuous time signal and viceversa.The frequency of discrete time signal ω is unique over the range $(-\pi, +\pi)$, and so the mapping $\omega = \Omega T$ implies that the frequency of continuous time signal in the interval $-\pi/T \le \Omega \le +\pi/T$ maps into the corresponding values of frequency of discrete time signal in the interval $-\pi \le \omega \le +\pi$.

The mapping of s-plane to z-plane, using the transformation, $e^{sT} = z$ is not one-to-one. Therefore in general, the interval $(2k-1)\pi/T \le \Omega \le (2k+1)\pi/T$, where k is an integer, maps into the corresponding values of $-\pi \le \omega \le +\pi$. Thus the mapping of the frequency of continuous time signal Ω to the frequency of discrete time signal ω is many-to-one. This reflects the effects of aliasing due to sampling.

Example 3.12

Determine the impulse response h(n) for the system described by the second-order difference equation,

$y(n) + 4y(n-1) + 3y(n-2) = x(n-1)$.

Solution

The difference equation governing the system is,

$$y(n) + 4y(n-1) + 3y(n-2) = x(n-1)$$

Let us take $\mathcal{Z}$-transform of the difference equation governing the system with zero initial conditions.

$$\therefore\ \mathcal{Z}\{y(n) + 4y(n-1) + 3y(n-2)\} = \mathcal{Z}\{x(n-1)\}$$

$$\mathcal{Z}\{y(n)\} + 4\,\mathcal{Z}\{y(n-1)\} + 3\,\mathcal{Z}\{y(n-2)\} = \mathcal{Z}\{x(n-1)\}$$

$$Y(z) + 4z^{-1}\,Y(z) + 3z^{-2}\,Y(z) = z^{-1}\,X(z)$$

$$(1 + 4z^{-1} + 3z^{-2})\,Y(z) = z^{-1}\,X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 + 4z^{-1} + 3z^{-2}}$$

If $\mathcal{Z}\{x(n)\} = X(z)$ then by shifting property $\mathcal{Z}\{x(n-m)\} = z^{-m}\,X(z)$
If $\mathcal{Z}\{y(n)\} = Y(z)$ then by shifting property $\mathcal{Z}\{y(n-m)\} = z^{-m}\,Y(z)$

We know that, $\dfrac{Y(z)}{X(z)} = H(z)$

$$\therefore\ H(z) = \frac{z^{-1}}{1 + 4z^{-1} + 3z^{-2}} = \frac{z^{-1}}{z^{-2}(z^2 + 4z + 3)} = \frac{z}{(z+1)\,(z+3)}$$

Using partial fraction expansion technique we can write,

$$\frac{H(z)}{z} = \frac{1}{(z+1)\,(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$A = \cancel{(z+1)}\,\frac{1}{\cancel{(z+1)}\,(z+3)}\Bigg|_{z=-1} = \frac{1}{-1+3} = \frac{1}{2} = 0.5$$

$$B = \cancel{(z+3)}\,\frac{1}{(z+1)\,\cancel{(z+3)}}\Bigg|_{z=-3} = \frac{1}{-3+1} = -\frac{1}{2} = -0.5$$

The roots of quadratic $z^2 + 4z + 3$ are, $z = \dfrac{-4 \pm \sqrt{4^2 - 4\times 3}}{2} = \dfrac{-4 \pm 2}{2} = -1,\ -3$

$$\therefore\ \frac{H(z)}{z} = \frac{0.5}{z+1} - \frac{0.5}{z+3} \quad \Rightarrow \quad H(z) = \frac{0.5z}{z+1} - \frac{0.5z}{z+3}$$

The impulse response h(n) is given by inverse Z-transform of H(z).

$$\text{Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{0.5z}{z+1} - \frac{0.5z}{z+3}\right\} = 0.5\,\mathcal{Z}^{-1}\left\{\frac{z}{z-(-1)}\right\} - 0.5\,\mathcal{Z}^{-1}\left\{\frac{z}{z-(-3)}\right\}$$

$$= 0.5(-1)^n\,u(n) - 0.5\,(-3)^n\,u(n) = 0.5[(-1)^n - (-3)^n]\,u(n)$$

Example 3.13

Find the transfer function and unit sample response of the second-order difference equation with zero initial condition,

$y(n) = x(n) - 0.25y(n-2)$.

Solution

The difference equation governing the system is,

$$y(n) = x(n) - 0.25\,y(n-2)$$

Let us take Z-transform of the difference equation governing the system with zero initial condition.

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n) - 0.25\,y(n-2)\}$$

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n)\} - 0.25\,\mathcal{Z}\{y(n-2)\}$$

$$Y(z) = X(z) - 0.25\,z^{-2}\,Y(z)$$

$$Y(z) + 0.25z^{-2}\,Y(z) = X(z)$$

$$[1 + 0.25z^{-2}]\,Y(z) = X(z)$$

$\mathcal{Z}\{x(n)\} = X(z)$
$\mathcal{Z}\{y(n)\} = Y(z)$ $\mathcal{Z}\{y(n-2)\} = z^{-2}\,Y(z)$ (Using shifting property)

$$\therefore \text{ Transfer function, } \frac{Y(z)}{X(z)} = \frac{1}{1+0.25z^{-2}}$$

We know that, $\dfrac{Y(z)}{X(z)} = H(z)$

$(a+b)\,(a-b) = a^2 - b^2$	$j^2 = -1$

$$\therefore\; H(z) = \frac{1}{1 + 0.25\,z^{-2}} = \frac{1}{z^{-2}(z^2 + 0.25)} = \frac{z^2}{(z + j\,0.5)\,(z - j\,0.5)}$$

Using partial fraction expansion technique we can write,

$$\frac{H(z)}{z} = \frac{z}{(z + j0.5)\,(z - j0.5)} = \frac{A}{z + j0.5} + \frac{A^*}{z - j0.5} \quad ; \quad \text{where } A^* \text{ is conjugate of A.}$$

$$A = (z + j0.5)\,\frac{H(z)}{z}\bigg|_{z=-j0.5} = \cancel{(z + j0.5)}\,\frac{z}{\cancel{(z + j0.5)}\,(z - j0.5)}\bigg|_{z=-j0.5}$$

$$= \frac{z}{z - j0.5}\bigg|_{z=-j0.5} = \frac{-j0.5}{-j0.5 - j0.5} = \frac{-j0.5}{2(-j0.5)} = \frac{1}{2} = 0.5$$

$$\therefore\; A^* = 0.5$$

$$\frac{H(z)}{z} = \frac{A}{z + j0.5} + \frac{A^*}{z - j0.5} = \frac{0.5}{z + j0.5} + \frac{0.5}{z - j0.5}$$

$$\therefore\; H(z) = \frac{0.5z}{z + j0.5} + \frac{0.5z}{z - j0.5} = \frac{0.5z}{z - (-j0.5)} + \frac{0.5z}{z - j0.5}$$

The impulse response is obtained by taking inverse Z-transform of H(z).

$$\therefore \text{ Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{0.5z}{z-(-j0.5)} + \frac{0.5z}{z-j0.5}\right\}$$

$$= 0.5\left[\mathcal{Z}^{-1}\left\{\frac{z}{z-(-j0.5)}\right\} + \mathcal{Z}^{-1}\left\{\frac{z}{z-j0.5}\right\}\right]$$

$$= 0.5\left[(-j0.5)^n\ u(n) + (j0.5)^n\ u(n)\right]$$

$$\boxed{\mathcal{Z}\{a^n\,u(n)\} = \frac{z}{z-a}}$$

Alternatively the impulse response can be expressed as shown below.

$$-j0.5 = 0.5\angle-90° = 0.5\angle-\pi/2 = 0.5\ \angle-0.5\pi$$

$$+j0.5 = 0.5\angle 90° = 0.5\angle\pi/2 = 0.5\ \angle 0.5\pi$$

$$\therefore\ h(n) = 0.5\left[(0.5\angle-0.5\pi)^n + (0.5\angle 0.5\pi)^n\right]u(n)$$

$$= 0.5\left[0.5^n\angle-0.5n\pi + 0.5^n\angle 0.5n\pi\right]u(n)$$

$$= 0.5\,(0.5)^n\left[\cos 0.5n\pi - j\sin 0.5n\pi + \cos 0.5n\pi + j\sin 0.5n\pi\right]u(n)$$

$$= 0.5\,(0.5)^n\,[2\cos 0.5n\pi]\,u(n)$$

$$= 0.5^n\cos(0.5n\pi)\ u(n)$$

$$180° = \pi \text{ rad}$$

$$\therefore\ 1° = \frac{\pi}{180}\text{ rad}$$

$$\therefore\ 90° = 90\times\frac{\pi}{180} = 0.5\pi\text{ rad}$$

Example 3.14

Determine the impulse response sequence of the discrete time LTI system defined by,

$y(n) - 4y(n-1) + 4y(n-2) = x(n) - 5x(n-3)$.

Solution

The difference equation governing the LTI system is,

$$y(n) - 4y(n-1) + 4y(n-2) = x(n) - 5x(n-3)$$

Let us assume that the initial conditions are zero.

On taking $\mathcal{Z}$-transform of the difference equation governing the system we get,

$\mathcal{Z}\{x(n)\} = X(z)$, \	$\mathcal{Z}\{ax(n-m)\} = az^{-m}X(z)$
$\mathcal{Z}\{y(n)\} = Y(z)$, \	$\mathcal{Z}\{ay(n-m)\} = az^{-m}Y(z)$

$$\mathcal{Z}\{y(n) - 4y(n-1) + 4y(n-2)\} = \mathcal{Z}\{x(n) - 5x(n-3)\}$$

$$\mathcal{Z}\{y(n)\} - 4\,\mathcal{Z}\{y(n-1)\} + 4\mathcal{Z}\{y(n-2)\} = \mathcal{Z}\{x(n)\} - 5\,\mathcal{Z}\{x(n-3)\}$$

$$Y(z) - 4z^{-1}\,Y(z) + 4z^{-2}\,Y(z) = X(z) - 5z^{-3}\,X(z)$$

$$[1 - 4z^{-1} + 4z^{-2}]\,Y(z) = [1 - 5z^{-3}]\,X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{1-5z^{-3}}{1-4z^{-1}+4z^{-2}}$$

We know that, $\dfrac{Y(z)}{X(z)} = H(z)$

$$\therefore\ H(z) = \frac{1-5z^{-3}}{1-4z^{-1}+4z^{-2}} = \frac{1-5z^{-3}}{z^{-2}(z^2-4z+4)} = \frac{z^2-5z^{-1}}{(z-2)^2}$$

$$= \frac{z^2}{(z-2)^2} - \frac{5z^{-1}}{(z-2)^2} = \frac{1}{2}z\,\frac{2z}{(z-2)^2} - \frac{5}{2}z^{-2}\,\frac{2z}{(z-2)^2}$$

$\mathcal{Z}\{a^n u(n)\} = \dfrac{z}{z-a}$

$\mathcal{Z}\{na^n u(n)\} = \dfrac{az}{(z-a)^2}$

If $\mathcal{Z}\{x(n)\} = X(z)$ then by shifting property $\mathcal{Z}\{x(n\pm m)\} = z^{\pm m}X(z)$

$$\boxed{(a-b)^2 = a^2 - 2ab + b^2}$$

The impulse response is obtained by taking inverse $\mathcal{Z}$-transform of H(z).

$$\therefore\ \text{Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{2}z\,\frac{2z}{(z-2)^2} - \frac{5}{2}z^{-2}\,\frac{2z}{(z-2)^2}\right\}$$

$$= \frac{1}{2}\mathcal{Z}^{-1}\left\{z\,\frac{2z}{(z-2)^2}\right\} - \frac{5}{2}\mathcal{Z}^{-1}\left\{z^{-2}\,\frac{2z}{(z-2)^2}\right\}$$

$$= \frac{1}{2}(n+1)\,(2)^{n+1}u(n+1) - \frac{5}{2}(n-2)\,(2)^{n-2}u(n-2)$$

Example 3.15

Find the impulse response of the system described by the difference equation,
$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$.

Solution

$\mathcal{Z}\{y(n)\} = Y(z)$; \	$\mathcal{Z}\{y(n-m)\} = z^{-m}\,Y(z)$
$\mathcal{Z}\{x(n)\} = X(z)$; \	$\mathcal{Z}\{x(n-m)\} = z^{-m}\,X(z)$

The difference equation governing the LTI system is,

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

On taking $\mathcal{Z}$-transform we get,

$$Y(z) - 3z^{-1}Y(z) - 4\,z^{-2}Y(z) = X(z) + 2z^{-1}\,X(z)$$

$$[1 - 3z^{-1} - 4z^{-2}]\;Y(z) = [1 + 2z^{-1}]\;X(z)$$

$$\therefore\;\frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}}$$

The roots of the quadratic,
$z^2 - 3z - 4 = 0$ are,
$$z = \frac{3 \pm \sqrt{3^2 + 4 \times 4}}{2} = 4 \text{ or } -1$$

We know that $\frac{Y(z)}{X(z)} = H(z)$

$$\therefore\; H(z) = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}} = \frac{z^{-2}(z^2 + 2z)}{z^{-2}(z^2 - 3z - 4)} = \frac{z^2 + 2z}{(z - 4)\,(z + 1)}$$

By partial fraction expansion technique,

$$\frac{H(z)}{z} = \frac{z + 2}{(z-4)(z+1)} = \frac{A}{z-4} + \frac{B}{z+1}$$

$$A = (z-4)\left.\frac{H(z)}{z}\right|_{z=4} = \cancel{(z-4)}\left.\frac{z+2}{\cancel{(z-4)}\,(z+1)}\right|_{z=4} = \left.\frac{z+2}{z+1}\right|_{z=4} = \frac{4+2}{4+1} = \frac{6}{5} = 1.2$$

$$B = (z+1)\left.\frac{H(z)}{z}\right|_{z=-1} = \cancel{(z+1)}\left.\frac{z+2}{(z-4)\,\cancel{(z+1)}}\right|_{z=-1} = \left.\frac{z+2}{z-4}\right|_{z=-1} = \frac{-1+2}{-1-4} = \frac{1}{-5} = -0.2$$

$$\therefore\;\frac{H(z)}{z} = \frac{A}{z-4} + \frac{B}{z+1} = \frac{1.2}{z-4} - \frac{0.2}{z+1}$$

$$\therefore\; H(z) = 1.2\,\frac{z}{z-4} - 0.2\frac{z}{z+1} = 1.2\left(\frac{z}{z-4}\right) - 0.2\left(\frac{z}{z-(-1)}\right)$$

$$\mathcal{Z}\left\{\frac{z}{z - a}\right\} = a^n$$

The impulse response is obtained by taking inverse $\mathcal{Z}$-transform of H(z).

\ Impulse response, $h(n) = 1.2(4)^n\,u(n) - 0.2(-1)^n\,u(n)$

Example 3.16

Determine the steady state response for the system with impulse function, $h(n) = (j0.8)^n\,u(n)$ for an input, $x(n) = \cos(pn)\;u(n)$.

Solution

Let y(n) be the steady state response of the system, which is given by convolution of x(n) and h(n).

\ Steady state response, $y(n) = x(n) * h(n)$

On taking $\mathcal{Z}$-transform of the above equation we get,

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n) * h(n)\}$$

Using convolution property.

\ $Y(z) = X(z)\,H(z)$

\ $y(n) = \mathcal{Z}^{-1}\{X(z)\,H(z)\}$

Given that, $h(n) = (j0.8)^n\ u(n)$

$$\mathcal{Z}\{a^n u(n)\} = \frac{z}{z-a}$$

$$\mathcal{Z}\{\cos(\omega n)\ u(n)\} = \frac{z(z-\cos\omega)}{z^2 - 2z\cos\omega + 1}$$

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \frac{z}{z - j0.8}$$

Given that, $x(n) = \cos(\pi n)\ u(n)$

$\cos \pi = -1$

$$\therefore\ X(z) = \mathcal{Z}\{x(n)\} = \frac{z(z - \cos\pi)}{z^2 - 2z\cos\pi + 1} = \frac{z(z+1)}{z^2 + 2z + 1} = \frac{z(z+1)}{(z+1)^2} = \frac{z}{z+1}$$

$$\therefore\ Y(z) = X(z)\,H(z) = \frac{z}{z+1} \times \frac{z}{z-j0.8} = \frac{z^2}{(z+1)(z-j0.8)}$$

By partial fraction expansion technique we can write,

$$\frac{Y(z)}{z} = \frac{z}{(z+1)(z-j0.8)} = \frac{A}{z+1} + \frac{B}{z-j0.8}$$

$$A = (z+1)\left.\frac{Y(z)}{z}\right|_{z=-1} = \cancel{(z+1)}\left.\frac{z}{\cancel{(z+1)}(z-j0.8)}\right|_{z=-1} = \left.\frac{z}{z-j0.8}\right|_{z=-1} = \frac{-1}{-1-j0.8}$$

$$= \frac{-1}{-1-j0.8} \times \frac{-1+j0.8}{-1+j0.8} = \frac{1-j0.8}{1^2+0.8^2} = \frac{1-j0.8}{1.64} = 0.61 - j0.49$$

$$B = (z-j0.8)\left.\frac{Y(z)}{z}\right|_{z=j0.8} = \cancel{(z-j0.8)}\left.\frac{z}{(z+1)\cancel{(z-j0.8)}}\right|_{z=j0.8} = \left.\frac{z}{z+1}\right|_{z=j0.8} = \frac{j0.8}{j0.8+1}$$

$$= \frac{j0.8}{1+j0.8} \times \frac{1-j0.8}{1-j0.8} = \frac{j0.8-(j0.8)^2}{1^2+0.8^2} = \frac{0.64+j0.8}{1.64} = 0.39 + j0.49$$

$$\therefore\ \frac{Y(z)}{z} = \frac{A}{z+1} + \frac{B}{z-j0.8} = \frac{0.61 - j0.49}{z+1} + \frac{0.39 + j0.49}{z - j0.8}$$

$$\therefore\ Y(z) = (0.61 - j0.49)\frac{z}{z+1} + (0.39 + j0.49)\frac{z}{z-j0.8}$$

$$= (0.61 - j0.49)\frac{z}{z-(-1)} + (0.39 + j0.49)\frac{z}{z-j0.8}$$

$$\mathcal{Z}\{a^n\ u(n)\} = \frac{z}{z-a}$$

The steady state response is obtained by taking inverse $\mathcal{Z}$-transform of Y(z).

\ Steady state response, $y(n) = (0.61 - j0.49)\,(-1)^n\,u(n) + (0.39 + j0.49)\,(j0.8)^n\,u(n)$

Alternatively the steady state response can be expressed as shown below.

Here, $0.61 - j0.49 = 0.78\,\angle -38.2^\circ = 0.78\,\angle -0.21\pi$

$0.39 + j0.49 = 0.63\,\angle 51.5^\circ = 0.63\,\angle 0.29\pi$

$-1 = 1\,\angle 180^\circ = 1\,\angle \pi$

$j0.8 = 0.8\,\angle 90^\circ = 0.8\,\angle 0.5\pi$

$$\therefore\ y(n) = 0.78\,\angle -0.21\pi\,[1\angle\pi]^n\,u(n) + 0.63\angle 0.29\pi\,[0.8\angle 0.5\pi]^n\,u(n)$$

$$= 0.78\,\angle -0.21\pi\,1^n\angle n\pi\,u(n) + 0.63\angle 0.29\pi\ 0.8^n\angle 0.5n\pi\,u(n)$$

$$= 0.78\,\angle(n-0.21)\pi\,u(n) + 0.63\,(0.8)^n\angle(0.5n+0.29)\pi\,u(n)$$

$180^\circ = \pi$ rad

$\therefore\ 1^\circ = \dfrac{\pi}{180}$ rad

$38.2^\circ = \dfrac{38.2}{180}\pi = 0.21\pi$ rad

$51.5^\circ = \dfrac{51.5}{180}\pi = 0.29\pi$ rad

$90^\circ = \dfrac{90}{180}\pi = 0.5\pi$ rad

Example 3.17

Obtain and sketch the impulse response of shift invariant system described by,
$y(n) = 0.4\ x(n) + x(n-1) + 0.2\ x(n-2) + x(n-3) + 0.6\ x(n-4)$.

Solution

The difference equation governing the system is,

$y(n) = 0.4\ x(n) + x(n-1) + 0.2\ x(n-2) + x(n-3) + 0.6\ x(n-4)$

On taking Z-transform we get,

$$Y(z) = 0.4X(z) + z^{-1}X(z) + 0.2z^{-2}X(z) + z^{-3}X(z) + 0.6z^{-4}X(z)$$

$$Y(z) = [0.4 + z^{-1} + 0.2z^{-2} + z^{-3} + 0.6z^{-4}]\ X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \left[0.4 + z^{-1} + 0.2z^{-2} + z^{-3} + 0.6z^{-4}\right]$$

If $\mathcal{Z}\{x(n)\} = X(z)$ then by shifting property
$\mathcal{Z}\ \{x(n-k)\} = z^{-k}\ X(z)$

We know that, $\dfrac{Y(z)}{X(z)} = H(z)$

$$\therefore\ H(z) = 0.4 + z^{-1} + 0.2z^{-2} + z^{-3} + 0.6z^{-4} \qquad .....(1)$$

By the definition of one sided Z-transform we get,

$$H(z) = \sum_{n=0}^{+\infty} h(n)z^{-n}$$

$$= h(0)\ z^{0} + h(1)\ z^{-1} + h(2)\ z^{-2} + h(3)\ z^{-3} + h(4)\ z^{-4} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$h(0) = 0.4$	$h(3) = 1$
$h(1) = 1$	$h(4) = 0.6$
$h(2) = 0.2$	$h(n) = 0$; for $n < 0$ and $n > 4$

\ Impulse response, $h(n) = \{0.4,\ 1.0,\ 0.2,\ 1.0,\ 0.6\}$

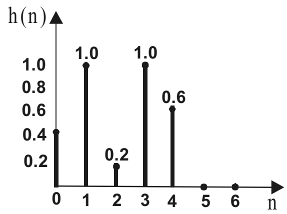


Fig 1: Graphical representation of impulse response h(n).

Example 3.18

Determine the response of discrete time LTI system governed by the difference equation, $y(n) = -0.8\ y(n-1) + x(n)$, when the input is unit step and initial condition, a) $y(-1) = 0$ and b) $y(-1) = 2/9$.

Solution

Given that, $x(n) = u(n)$; $\therefore\ X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{u(n)\} = \dfrac{z}{z-1}$ .....(1)

Given that, $y(n) = -0.8\ y(n-1) + x(n)$

\ $y(n) + 0.8\ y(n-1) = x(n)$

On taking Z-transform of above equation we get,

If $\mathcal{Z}\{y(n)\} = Y(z)$
then $\mathcal{Z}\{y(n-1)\} = z^{-1}\ Y(z) - y(-1)$

$$Y(z) + 0.8\left[z^{-1}\ Y(z) + y(-1)\right] = X(z)$$

Using equation (1).

$$Y(z)\ \left[1 + 0.8\ z^{-1}\right] + 0.8\ y(-1) = \frac{z}{z-1}$$

$$Y(z)\left(1 + \frac{0.8}{z}\right) = \frac{z}{z-1} - 0.8\ y(-1)$$

$$Y(z)\left(\frac{z+0.8}{z}\right) = \frac{z}{z-1} - 0.8\ y(-1)$$

$$\therefore Y(z) = \frac{z^2}{(z-1)\ (z+0.8)} - 0.8\ \frac{z\ y(-1)}{z+0.8}$$

Let, $P(z) = \dfrac{z^2}{(z-1)(z+0.8)} \quad \Rightarrow \quad \dfrac{P(z)}{z} = \dfrac{z}{(z-1)(z+0.8)}$

Let, $\dfrac{z}{(z-1)(z+0.8)} = \dfrac{A}{z-1} + \dfrac{B}{z+0.8}$

$$A = \frac{z}{\cancel{(z-1)}(z+0.8)} \times \cancel{(z-1)}\Big|_{z=1} = \frac{1}{1+0.8} = \frac{1}{1.8} = \frac{10}{18} = \frac{5}{9}$$

$$B = \frac{z}{(z-1)\cancel{(z+0.8)}} \times \cancel{(z+0.8)}\Big|_{z=-0.8} = \frac{-0.8}{-0.8-1} = \frac{-0.8}{-1.8} = \frac{8}{18} = \frac{4}{9}$$

$$\therefore \frac{P(z)}{z} = \frac{5}{9}\,\frac{1}{z-1} + \frac{4}{9}\,\frac{1}{z+0.8} \quad \Rightarrow \quad P(z) = \frac{5}{9}\,\frac{z}{z-1} + \frac{4}{9}\,\frac{z}{z+0.8}$$

$$\therefore Y(z) = \frac{5}{9}\,\frac{z}{z-1} + \frac{4}{9}\,\frac{z}{z+0.8} - 0.8\,\frac{z\,y(-1)}{z+0.8} \qquad .....(2)$$

a) When y(–1) = 0

From equation (2), when y(–1) = 0, we get,

$$Y(z) = \frac{5}{9}\,\frac{z}{z-1} + \frac{4}{9}\,\frac{z}{z+0.8}$$

$$\therefore \text{Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{\frac{5}{9}\,\frac{z}{z-1} + \frac{4}{9}\,\frac{z}{z+0.8}\right\}$$

$$= \frac{5}{9}\,u(n) + \frac{4}{9}\,(-0.8)^n\,u(n)$$

b) When y(–1) = 2/9

From equation (2), when y(–1) = 2/9, we get,

$$Y(z) = \frac{5}{9}\,\frac{z}{z-1} + \frac{4}{9}\,\frac{z}{z+0.8} - 0.8 \times \frac{2}{9}\,\frac{z}{z+0.8} = \frac{5}{9}\,\frac{z}{z-1} + \frac{2.4}{9}\,\frac{z}{z+0.8}$$

$$= \frac{5}{9}\,\frac{z}{z-1} + \frac{24}{90}\,\frac{z}{z+0.8} = \frac{5}{9}\,\frac{z}{z-1} + \frac{12}{45}\,\frac{z}{z+0.8}$$

$$\therefore \text{Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{\frac{5}{9}\,\frac{z}{z-1} + \frac{12}{45}\,\frac{z}{z+0.8}\right\}$$

$$= \left[\frac{5}{9} + \frac{12}{45}(-0.8)^n\right] u(n)$$

Note : Compare the result with example 2.8 of Chapter 2.

Example 3.19

Determine the response of LTI discrete time system governed by the difference equation, y(n) – 0.2 y(n – 1) – 0.03 y(n – 2) = x(n) + 0.4 x(n – 1) for the input, x(n) = 0.2^n u(n) and with initial condition, y(–2)= 0, y(–1) = 0.5.

Solution

Given that, x(n) = 0.2^n u(n) ; $\quad \therefore X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{0.2^n\,u(n)\} = \dfrac{z}{z-0.2}$ .....(1)

Given that, y(n) – 0.2 y(n – 1) – 0.03 y(n – 2) = x(n) +0. 4 x(n – 1)

On taking $\mathcal{Z}$-transform of above equation we get,

$$Y(z) - 0.2\left[z^{-1}Y(z) + y(-1)\right] - 0.03\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right] = X(z) + 0.4\left[z^{-1}X(z) + x(-1)\right] \qquad .....(2)$$

If $\mathcal{Z}\{y(n)\} = Y(z)$, then $\mathcal{Z}\{y(n-1)\} = z^{-1}Y(z) + y(-1)$

and $\mathcal{Z}\{y(n-2)\} = z^{-2}Y(z) + z^{-1}y(-1) + y(-2)$

Given that, $y(-2) = 0$, $y(-1) = 0.5$

$$\left.\begin{aligned} x(n) &= 0.2^n\, u(n) = 0.2^n \ ; \ \text{for} \quad n \ge 0 \\ &= 0 \qquad ; \ \text{for} \quad n < 0 \end{aligned}\right. \quad \Rightarrow \quad x(-1) = 0$$

On substituting the above initial conditions in equation (2) we get,

$$Y(z) - 0.2\,z^{-1}\,Y(z) - 0.2 \times 0.5 - 0.03\,z^{-2}\,Y(z) - 0.03\,z^{-1} \times 0.5 + 0 = X(z) + 0.4\,z^{-1}\,X(z) + 0$$

$$Y(z) - \frac{0.2}{z}Y(z) - 0.1 - \frac{0.03}{z^2}Y(z) - \frac{0.015}{z} = X(z) + \frac{0.4}{z}X(z)$$

$$\therefore \ Y(z)\left(1 - \frac{0.2}{z} - \frac{0.03}{z^2}\right) - \left(\frac{0.015}{z} + 0.1\right) = X(z)\left(1 + \frac{0.4}{z}\right)$$

$$Y(z)\left(\frac{z^2 - 0.2\,z - 0.03}{z^2}\right) - \left(\frac{0.015 + 0.1z}{z}\right) = \left(\frac{\cancel{z}}{z - 0.2}\right)\left(\frac{z + 0.4}{\cancel{z}}\right)$$

Using equation (1).

$$Y(z)\frac{(z-0.3)\,(z+0.1)}{z^2} = \frac{z+0.4}{z-0.2} + \frac{0.015+0.1z}{z}$$

$$Y(z)\frac{(z-0.3)\,(z+0.1)}{z^2} = \frac{z(z+0.4) \ + (0.015+0.1z)\,(z-0.2)}{z(z-0.2)}$$

$$Y(z)\frac{(z-0.3)\,(z+0.1)}{z^2} = \frac{z^2 + 0.4\,z + 0.015\,z - 0.003 + 0.1z^2 - 0.02\,z}{z(z-0.2)}$$

The roots of quadratic $z^2 - 0.2z - 0.03 = 0$ are,

$$z = \frac{0.2 \pm \sqrt{0.2^2 - 4 \times (-0.03)}}{2} = \frac{0.2 \pm 0.4}{2} = 0.3, \ -0.1$$

$$Y(z)\frac{(z-0.3)\,(z+0.1)}{z^2} = \frac{1.1z^2 + 0.395z - 0.003}{z(z-0.2)}$$

$$Y(z) = \frac{1.1z^2 + 0.395z - 0.003}{z(z-0.2)} \times \frac{z^2}{(z-0.3)\,(z+0.1)}$$

$$= \frac{z(1.1z^2 + 0.395z - 0.003)}{(z-0.2)\,(z-0.3)\,(z+0.1)}$$

Let, $\dfrac{Y(z)}{z} = \dfrac{1.1z^2 + 0.395z - 0.003}{(z-0.2)\,(z-0.3)\,(z+0.1)} = \dfrac{A}{z-0.2} + \dfrac{B}{z-0.3} + \dfrac{C}{z+0.1}$

$$A = \left.\frac{1.1z^2 + 0.395z - 0.003}{\cancel{(z-0.2)}\,(z-0.3)\,(z+0.1)} \times \cancel{(z-0.2)}\right|_{z=0.2} = \left.\frac{1.1z^2 + 0.395z - 0.003}{(z-0.3)\,(z+0.1)}\right|_{z=0.2}$$

$$= \frac{1.1 \times 0.2^2 + 0.395 \times 0.2 - 0.003}{(0.2-0.3)\,(0.2+0.1)} = -4$$

$$B = \left.\frac{1.1z^2 + 0.395z - 0.003}{(z-0.2)\,\cancel{(z-0.3)}\,(z+0.1)} \times \cancel{(z-0.3)}\right|_{z=0.3} = \left.\frac{1.1z^2 + 0.395z - 0.003}{(z-0.2)\,(z+0.1)}\right|_{z=0.3}$$

$$= \frac{1.1 \times 0.3^2 + 0.395 \times 0.3 - 0.003}{(0.3-0.2)\,(0.3+0.1)} = 5.3625$$

$$C = \left.\frac{1.1z^2 + 0.395z - 0.003}{(z-0.2)\,(z-0.3)\,\cancel{(z+0.1)}} \times \cancel{(z+0.1)}\right|_{z=-0.1} = \left.\frac{1.1z^2 + 0.395z - 0.003}{(z-0.2)\,(z-0.3)}\right|_{z=-0.1}$$

$$= \frac{1.1 \times (-0.1)^2 + 0.395 \times (-0.1) - 0.003}{(-0.1-0.2)\,(-0.1-0.3)} = -0.2625$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{-4}{z-0.2} + \frac{5.3625}{z-0.3} - \frac{0.2625}{z+0.1} \quad \Rightarrow \quad Y(z) = -4\frac{z}{z-0.2} + 5.3625\frac{z}{z-0.3} - 0.2625\frac{z}{z-(-0.1)}$$

$$\therefore \text{ Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{-4\frac{z}{z-0.2} + 5.3625\frac{z}{z-0.3} - 0.2625\frac{z}{z-(-0.1)}\right\}$$

$$= -4(0.2)^n u(n) + 5.3625(0.3)^n u(n) - 0.2625(-0.1)^n u(n)$$

$$= \left[-4(0.2)^n + 5.3625(0.3)^n - 0.2625(-0.1)^n\right]u(n)$$

Note : Compare the result with example 2.9 of Chapter 2.

Example 3.20

Find the response of the time invariant system with impulse response, h(n) = {1, 2, –1, –2} to an input signal, x(n) = {1, 2, 3, 4}.

Solution

Let, y(n) = Response or Output of an LTI system.

The response of an LTI system is given by the convolution of input signal and impulse response.

$\therefore \; y(n) = x(n) * h(n)$

By convolution property
$\mathcal{Z}\{x(n) * h(n)\} = X(z)\,H(z)$

On taking $\mathcal{Z}$-transform we get,

$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n) * h(n)\}$

$\therefore \; Y(z) = X(z)\,H(z)$

Given that, x(n) = {1, 2, 3, 4}

By definition of one-sided $\mathcal{Z}$-transform,

$$X(z) = \sum_{n=0}^{\infty} x(n)\,z^{-n} = \sum_{n=0}^{3} x(n)\,z^{-n} = x(0)\,z^0 + x(1)\,z^{-1} + x(2)\,z^{-2} + x(3)\,z^{-3}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

Given that, h(n) = { 1, 2, –1, –2}

By definition of one-sided $\mathcal{Z}$-transform,

$$H(z) = \sum_{n=0}^{\infty} h(n)\,z^{-n} = \sum_{n=0}^{3} h(n)\,z^{-n} = h(0)\,z^0 + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3}$$

$$= 1 + 2z^{-1} - z^{-2} - 2z^{-3}$$

$$\begin{aligned}
\therefore \; Y(z) &= X(z)\,H(z) \\
&= [1 + 2z^{-1} + 3z^{-2} + 4z^{-3}]\;[1 + 2z^{-1} - z^{-2} - 2z^{-3}] \\
&= 1 + 2z^{-1} - z^{-2} - 2z^{-3} \\
&\quad + 2z^{-1} + 4z^{-2} - 2z^{-3} - 4z^{-4} \\
&\quad + 3z^{-2} + 6z^{-3} - 3z^{-4} - 6z^{-5} \\
&\quad + 4z^{-3} + 8z^{-4} - 4z^{-5} - 8z^{-6} \\
&= 1 + 4z^{-1} + 6z^{-2} + 6z^{-3} + z^{-4} - 10z^{-5} - 8z^{-6} \qquad .....(1)
\end{aligned}$$

By definition of one-sided $\mathcal{Z}$-transform we get,

$$Y(z) = \sum_{n=0}^{\infty} y(n)\,z^{-n}$$

$$= y(0)\,z^0 + y(1)\,z^{-1} + y(2)\,z^{-2} + y(3)\,z^{-3} + y(4)\,z^{-4} + y(5)\,z^{-5} + y(6)\,z^{-6} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

y(0) = 1	y(2) = 6	y(4) = 1	y(6) = –8
y(1) = 4	y(3) = 6	y(5) = –10	

$\therefore$ The response of the system, y(n) = {1, 4, 6, 6, 1, –10, –8}

Example 3.21

Using Z-transform, perform deconvolution of the response, $y(n) = \{1,\ \ 4,\ \ 6,\ \ 6,\ \ 1,\ \ -10,\ \ -8\}$ and impulse response $h(n) = \{1,\ \ 2,\ \ -1,\ \ -2\}$ to extract the input x(n).

Solution

Given that, $y(n) = \{1,\ \ 4,\ \ 6,\ \ 6,\ \ 1,\ \ -10,\ \ -8\}$

$$\therefore\ Y(z) = Z\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\ z^{-n} = \sum_{n=0}^{6} y(n)\ z^{-n}$$

$$= y(0) + y(1)\ z^{-1} + y(2)\ z^{-2} + y(3)\ z^{-3} + y(4)\ z^{-4} + y(5)\ z^{-5} + y(6)\ z^{-6}$$

$$= 1 + 4z^{-1} + 6z^{-2} + 6z^{-3} + z^{-4} - 10z^{-5} - 8z^{-6}$$

Given that, $h(n) = \{1,\ \ 2,\ \ -1,\ \ -2\}$

$$\therefore H(z) = Z\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\ z^{-n} = \sum_{n=0}^{3} h(n)\ z^{-n} = h(0) + h(1)\ z^{-1} + h(2)\ z^{-2} + h(3)\ z^{-3}$$

$$= 1 + 2z^{-1} - z^{-2} - 2z^{-3}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore X(z) = \frac{Y(z)}{H(z)} = \frac{1 + 4z^{-1} + 6z^{-2} + 6z^{-3} + z^{-4} - 10z^{-5} - 8z^{-6}}{1 + 2z^{-1} - z^{-2} - 2z^{-3}}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} \qquad .....(1)$$

$$\begin{array}{r|l}
 & 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} \\
\hline
1 + 2z^{-1} - z^{-2} - 2z^{-3} & 1 + 4z^{-1} + 6z^{-2} + 6z^{-3} + z^{-4} - 10z^{-5} - 8z^{-6} \\
 & \underset{(-)}{1} \underset{(-)}{+} 2z^{-1} \underset{(+)}{-} z^{-2} \underset{(+)}{-} 2z^{-3} \\
\cline{2-2}
 & 2z^{-1} + 7z^{-2} + 8z^{-3} + z^{-4} \\
 & \underset{(-)}{2z^{-1}} \underset{(-)}{+} 4z^{-2} \underset{(+)}{-} 2z^{-3} \underset{(+)}{-} 4z^{-4} \\
\cline{2-2}
 & 3z^{-2} + 10z^{-3} + 5z^{-4} - 10z^{-5} \\
 & \underset{(-)}{3z^{-2}} \underset{(-)}{+} 6z^{-3} \underset{(+)}{-} 3z^{-4} \underset{(+)}{-} 6z^{-5} \\
\cline{2-2}
 & 4z^{-3} + 8z^{-4} - 4z^{-5} - 8z^{-6} \\
 & \underset{(-)}{4z^{-3}} \underset{(-)}{+} 8z^{-4} \underset{(+)}{-} 4z^{-5} \underset{(+)}{-} 8z^{-6} \\
\cline{2-2}
 & 0 \\
\cline{2-2}
\end{array}$$

By the definition of Z-transform,

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\ z^{-n}$$

On expanding the above summation we get,

$$X(z) = \ ....\ + x(0) + x(1)\ z^{-1} + x(2)\ z^{-2} + x(3)\ z^{-3} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$x(0) = 1 \quad ; \quad x(1) = 2 \quad ; \quad x(2) = 3 \quad ; \quad x(3) = 4$

$\therefore$ Input, $x(n) = \{\underset{\uparrow}{1},\ \ 2,\ \ 3,\ \ 4\}$

Example 3.22

An LTI system is described by the equation, $y(n) = x(n) + 0.8\ x(n-1) + 0.8\ x(n-2) - 0.49\ y(n-2)$. Determine the transfer function of the system. Sketch the poles and zeros on the z-plane.

Solution

Given that, $y(n) = x(n) + 0.8\ x(n-1) + 0.8\ x(n-2) - 0.49\ y(n-2)$

On taking $\mathcal{Z}$-transform we get,

$\mathcal{Z}\{y(n)\} = Y(z);\ \backslash\ \mathcal{Z}\{y(n-m)\} = z^{-m}Y(z)$
$\mathcal{Z}\{x(n)\} = X(z);\ \backslash\ \mathcal{Z}\{x(n-m)\} = z^{-m}X(z)$

$$Y(z) = X(z) + 0.8z^{-1}X(z) + 0.8z^{-2}X(z) - 0.49z^{-2}Y(z)$$

$$Y(z) + 0.49z^{-2}Y(z) = X(z) + 0.8z^{-1}X(z) + 0.8z^{-2}X(z)$$

$$[1 + 0.49z^{-2}]\ Y(z) = [1 + 0.8z^{-1} + 0.8z^{-2}]\ X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{1 + 0.8z^{-1} + 0.8z^{-2}}{1 + 0.49z^{-2}} \quad .....(1)$$

The equation(1) is the transfer function of the LTI system.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.8z^{-1} + 0.8z^{-2}}{1 + 0.49z^{-2}} = \frac{z^{-2}(z^2 + 0.8z + 0.8)}{z^{-2}(z^2 + 0.49)}$$

$$= \frac{z^2 + 0.8z + 0.8}{z^2 + 0.49}$$

The poles are the roots of the denominator polynomial,

$$z^2 + 0.49 = 0$$

$$\backslash\ z^2 = -0.49$$

$$z = \pm\sqrt{-0.49} = \pm j0.7$$

\ The poles are, $p_1 = j0.7$, $p_2 = -j0.7$

The zeros are the roots of the numerator polynomial,

$$z^2 + 0.8z + 0.8 = 0$$

$$z = \frac{-0.8 \pm \sqrt{0.8^2 - 4 \times 0.8}}{2} = \frac{-0.8 \pm \sqrt{-2.56}}{2}$$

$$= \frac{-0.8 \pm j0.16}{2} = -0.4 \pm j0.8$$

\ The zeros are, $z_1 = -0.4 + j0.8$ and $z_2 = -0.4 - j0.8$

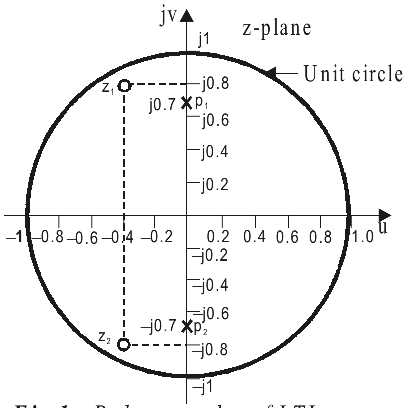


Fig 1 : Pole-zero plot of LTI system.

$$\therefore\ H(z) = \frac{z^2 + 0.8z + 0.8}{z^2 + 0.49} = \frac{(z + 0.4 - j0.8)\,(z + 0.4 + j0.8)}{(z - j0.7)\,(z + j0.7)}$$

The fig1 Shows the location of poles and zeros on the z-plane. The poles are marked as "X" and Zeros as "O".

Example 3.23

Determine the step response of an LTI system whose impulse response h(n) is given by,

$h(n) = a^{-n}\,u(-n);\ 0 < a < 1.$

Solution

$\because u(-n) = 1$	$;\ n \le 0$
$= 0$	$;\ n > 0$

On taking $\mathcal{Z}$-transform of impulse response h(n) we get,

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=-\infty}^{\infty} h(n)\,z^{-n} = \sum_{n=-\infty}^{0} a^{-n}\,z^{-n} = \sum_{n=0}^{\infty} a^n z^n = \sum_{n=0}^{\infty} (az)^n \quad ....(1)$$

If |az| < 1, then using infinite geometric series sum formula,

$$H(z) = \sum_{n=0}^{\infty} (az)^n = \frac{1}{1 - a\,z} = \frac{-\frac{1}{a}}{z - \frac{1}{a}}\ ;\ \text{ROC } |z| < \left|\frac{1}{a}\right|$$

Here, $|a\,z| < 1 \quad \Rightarrow \quad |z| < \left|\frac{1}{a}\right|$

$\therefore$ ROC is $|z| < \left|\frac{1}{a}\right|$. Since $|a| < 1$, $\left|\frac{1}{a}\right| > 1$, and so ROC includes unit circle.

The step input, u(n) = 1 ; n ³ 0

= 0 ; n < 0

On taking Z-transform of unit step signal we get,

Refer table 3.4

$$U(z) = Z\{u(n)\} = \frac{z}{z-1} \;;\; \text{ROC } |z| > 1 \qquad .....(2)$$

Let y(n) be step response. Now the step response is given by convolution of step input, u(n) and impulse response, h(n).

\ y(n) = u(n) * h(n)

By convolution property, Z{u(n) * h(n)} = U(z) H(z)

On taking Z-transform we get,

Z{y(n)} = Z{u(n) * h(n)}

\ Y(z) = U(z) H(z)

On substituting for U(z) and H(z) from equations (1) and (2) respectively we get,

$$Y(z) = U(z)\ H(z) = \left(\frac{z}{z-1}\right)\left(\frac{-1/a}{z-1/a}\right)$$

By partial fraction expansion we can write,

$$\frac{Y(z)}{z} = \frac{-1/a}{(z-1)(z-1/a)} = \frac{A}{z-1} + \frac{B}{z-1/a}$$

$$A = (z-1)\left.\frac{Y(z)}{z}\right|_{z=1} = \left.\frac{-1/a}{z-1/a}\right|_{z=1} = \frac{-1/a}{1-1/a} = \frac{-1/a}{\frac{a-1}{a}} = \frac{-1}{a-1} = \frac{1}{1-a}$$

$$B = (z-1/a)\left.\frac{Y(z)}{z}\right|_{z=1/a} = \left.\frac{-1/a}{z-1}\right|_{z=1/a} = \frac{-1/a}{1/a-1} = \frac{-1/a}{\frac{1-a}{a}} = \frac{-1}{1-a}$$

$$\therefore \frac{Y(z)}{z} = \frac{1}{(1-a)}\frac{1}{(z-1)} - \frac{1}{(1-a)}\frac{1}{(z-1/a)}$$

$$\therefore Y(z) = \frac{1}{(1-a)}\frac{z}{(z-1)} - \frac{1}{(1-a)}\frac{z}{(z-1/a)}$$

$$Z\{-u(-n-1)\} = \frac{z}{z-1}$$

$$Z\{-b^n\, u(-n-1)\} = \frac{z}{z-b}$$

Note : Since impulse response is anticausal, the step response is also anticausal.

On taking inverse Z-transform of Y(z) we get step response.

$$\therefore \text{Step response, } y(n) = -\frac{1}{1-a}\,u(-n-1) + \frac{1}{1-a}\left(\frac{1}{a}\right)^n u(-n-1) = \left[\left(\frac{1}{a}\right)^n - 1\right]\left(\frac{1}{1-a}\right)u(-n-1)$$

Example 3.24

Test the stability of the first-order system governed by the equation, y(n) = x(n) + b y(n–1), where |b| < 1.

Solution

Z{y(n)} = Y(z) ; Z{y(n – 1)} = z^{-1}Y(z) ; Z{x(n)} = X(z)

Given that, y(n) = x(n) + b y(n – 1)

On taking Z-transform we get,

$$Y(z) = X(z) + b\,z^{-1}\,Y(z) \quad Þ \quad Y(z) - b\,z^{-1}\,Y(z) = X(z) \quad Þ \quad (1 - b\,z^{-1})\,Y(z) = X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1}{1 - b\,z^{-1}}$$

We know that, Y(z)/X(z) is equal to H(z).

$$\therefore\ H(z) = \frac{1}{1 - b\ z^{-1}} = \frac{1}{z^{-1}(z - b)} = \frac{z}{z - b}$$

On taking inverse Z-transform of H(z) we get the impulse response h(n).

$$\boxed{\mathcal{Z}\left\{a^n u(n)\right\} = \frac{z}{z-a}}$$

\ Impulse response, $h(n) = b^n\ u(n)$

The condition to be satisfied for the stability of the system is, $\sum\limits_{n=-\infty}^{\infty} |h(n)| < \infty$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |b^n| = \sum_{n=0}^{\infty} |b|^n$$

Since |b| < 1, using the infinite geometric series sum formula we can write,

$$\sum_{n=0}^{\infty} |b|^n = \frac{1}{1-|b|}$$

$$\therefore\ \sum_{n=-\infty}^{\infty} |h(n)| = \frac{1}{1-|b|} = \text{constant}$$

Infinite geometric series sum formula
$\sum\limits_{n=0}^{\infty} C^n = \frac{1}{1-C}\ ;\ \text{if, } 0 < \lvert C \rvert < 1$

The term 1/(1–|b|) is less than infinity and so the system is stable.

Example 3.25

Using Z-transform, find the autocorrelation of the causal sequence, $x(n) = a^n\ u(n)$, $-1 < a < 1$.

Solution

Given that, $x(n) = a^n\ u(n)$

$$\therefore\ X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\left\{a^n u(n)\right\} = \frac{1}{1 - a\,z^{-1}} = \frac{z}{z-a}$$

$$\therefore\ X(z^{-1}) = X(z)\Big|_{z=z^{-1}} = \frac{1}{1 - a\,z} = -\frac{1}{a}\ \frac{1}{z - 1/a}$$

Let, $r_{xx}(m)$ be autocorrelation sequence.

By correlation property of Z - transform,

$$\mathcal{Z}\{r_{xx}(m)\} = X(z)\ X(z^{-1}) = \frac{z}{z-a} \times \frac{1}{-a}\ \frac{1}{z - 1/a} = -\frac{1}{a}\ \frac{z}{(z-a)\ (z - 1/a)}$$

Let, $\dfrac{z}{(z-a)\ (z-1/a)} = \dfrac{A}{z-a} + \dfrac{B}{z-1/a}$

$$A = \frac{z}{\cancel{(z-a)}\ (z-1/a)} \times \cancel{(z-a)}\Big|_{z=a} = \frac{z}{z-1/a}\Bigg|_{z=a} = \frac{a}{a-1/a} = \frac{a}{\frac{a^2-1}{a}} = \frac{a^2}{a^2-1}$$

$$B = \frac{z}{(z-a)\ \cancel{(z-1/a)}} \times \cancel{(z-1/a)}\Big|_{z=1/a} = \frac{z}{z-a}\Bigg|_{z=1/a} = \frac{1/a}{1/a - a} = \frac{1/a}{\frac{1-a^2}{a}} = \frac{1}{1-a^2} = \frac{-1}{a^2-1}$$

$$\therefore\ \mathcal{Z}\{r_{xx}(m)\} = -\frac{1}{a}\left(\frac{a^2}{a^2-1}\ \frac{1}{z-a} - \frac{1}{a^2-1}\ \frac{1}{z-1/a}\right) = -\frac{a}{a^2-1}\ \frac{1}{z-a} + \frac{1}{a(a^2-1)}\ \frac{1}{z-1/a}$$

$$\therefore \; r_{xx}(m) = \mathcal{Z}^{-1}\left\{-\frac{a}{a^2-1}\;\frac{1}{z-a} + \frac{1}{a(a^2-1)}\;\frac{1}{z-1/a}\right\}$$

$$= -\frac{a}{a^2-1}\;\mathcal{Z}^{-1}\left\{z^{-1}\frac{z}{z-a}\right\} + \frac{1}{a(a^2-1)}\;\mathcal{Z}^{-1}\left\{z^{-1}\frac{z}{z-1/a}\right\}$$

$$= -\frac{a}{a^2-1}\;a^{(n-1)}\;u(n-1) + \frac{1}{a(a^2-1)}\left(\frac{1}{a}\right)^{n-1} u(n-1)$$

$$= \frac{1}{a^2-1}\left[\frac{1}{a}\left(\frac{1}{a}\right)^{n-1} - a\,(a)^{n-1}\right] u(n-1) = \frac{1}{a^2-1}\left[\left(\frac{1}{a}\right)^{n} - a^{n}\right] u(n-1)$$

$z^{-1}\,z = z^0 = 1$

If $\mathcal{Z}\{x(n)\} = X(z)$ then by shifting property $\mathcal{Z}\{x(n-m)\} = z^{-m}\,X(z)$

$\mathcal{Z}\left\{a^n u(n)\right\} = \dfrac{z}{z-a}$

$\mathcal{Z}\left\{a^{n-1}u(n-1)\right\} = z^{-1}\dfrac{z}{z-a}$

3.8 Structures for Realization of LTI Discrete Time Systems in z-Domain

A discrete time system is a system that accepts a discrete time signal as input and processes it, and delivers the processed discrete time signal as output. Mathematically, a discrete time system is represented by a difference equation. Physically, a discrete time system is realized or implemented either as a digital hardware (like special purpose Microprocessor / Microcontroller) or as a software running on a digital hardware (like PC-Personal Computer).

The processing of the discrete time signal by the digital hardware involves mathematical operations like addition, multiplication and delay. Also the calculations are performed either by using fixed point arithmetic or floating point arithmetic. The time taken to process the discrete time signal and the computational complexity, depends on number of calculations involved and the type of arithmetic used for computation. These issues are addressed in structures for realization of discrete time systems.

From the implementation point of view, the discrete time systems are basically classified as IIR and FIR systems. The various structures proposed for IIR and FIR systems, attempt to reduce the computational complexity, errors in computation, memory requirement and finite word length effects in computations.

Discrete Time IIR System

Let, H(z) = Transfer function of discrete time IIR system.

The general form of transfer function of IIR system is,

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \,.....\, + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \,.....\, + a_N z^{-N}}$$

Let, X(z) = Input of the discrete time system in z-domain.

Y(z) = Output of the discrete time system in z-domain.

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \,.....\, + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \,.....\, + a_N z^{-N}} \qquad .....(3.75)$$

On cross multiplying the equation (3.75) we get,

$$\left[1 + a_1 z^{-1} + a_2 z^{-2} + \,.....\, + a_N z^{-N}\right] Y(z) = \left[b_0 + b_1 z^{-1} + b_2 z^{-2} + \,.....\, + b_M z^{-M}\right] X(z)$$

$$Y(z) + a_1 z^{-1}\,Y(z) + a_2 z^{-2}\,Y(z) + \,.....\, + a_N z^{-N} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \,.....\, + b_M z^{-M} X(z)$$

On taking inverse $\mathcal{Z}$-transform of the above equation we get,

If $\mathcal{Z}\{x(n)\} = X(z)$ then, $\mathcal{Z}\{x(n-k)\} = z^{-k}X(z)$

$$y(n) + a_1\, y(n-1) + a_2\, y(n-2) + \,.....\, + a_N\, y(n-N)$$
$$= b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \,.....\, + b_M\, x(n-M)$$
$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - \,.....\, - a_N\, y(n-N)$$
$$+ b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \,.....\, + b_M\, x(n-M)$$
$$\therefore y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m) \qquad .....(3.76)$$

The equation (3.75) is the transfer function of discrete time IIR system and the equation (3.76) is the time domain difference equation governing discrete time IIR system. From equation (3.76), it is observed that the output at any time n depends on past outputs and so the IIR systems are recursive systems.

Discrete Time FIR system

Let, H(z) = Transfer function of discrete time FIR system.

The general form of transfer function of FIR system is,

$$H(z) = b_0 + b_1\, z^{-1} + b_2\, z^{-2} + \,.....\, + b_{N-1}\, z^{-(N-1)}$$

Let, X(z) = Input of the discrete time system in z-domain.

Y(z) = Output of the discrete time system in z-domain.

$$\therefore H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1\, z^{-1} + b_2\, z^{-2} + \,.....\, + b_{N-1}\, z^{-(N-1)} \qquad .....(3.77)$$

On cross multiplying the equation (3.77) we get,

$$Y(z) = \left[b_0 + b_1\, z^{-1} + b_2\, z^{-2} + \,.....\, + b_{N-1}\, z^{-(N-1)}\right] X(z)$$
$$= b_0 X(z) + b_1\, z^{-1}\, X(z) + b_2\, z^{-2}\, X(z) + \,.....\, + b_{N-1}\, z^{-(N-1)}\, X(z)$$

On taking inverse $\mathcal{Z}$-transform of the above equation we get,

$$y(n) = b_0 x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \,.....\, + b_{N-1}\, x(n-(N-1))$$
$$\therefore y(n) = \sum_{m=0}^{N-1} b_m x(n-m) \qquad .....(3.78)$$

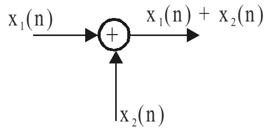
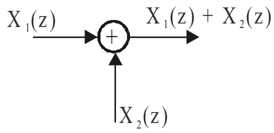
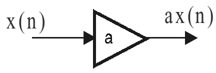
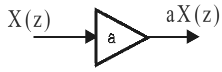
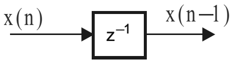
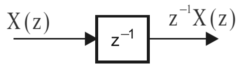
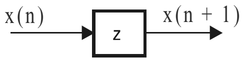
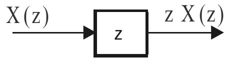
The equation (3.77) is the transfer function of discrete time FIR system and the equation (3.78) is the time domain difference equation governing discrete time FIR system. From equation (3.78), it is observed that the output at any time n does not depend on past outputs and so the FIR systems are nonrecursive systems.

Basic Elements of Block Diagram

The difference equations of IIR and FIR systems can be viewed as a computational procedure (or algorithm) to determine the output signal y(n) from the input signal x(n). The computations in the above difference equation of a system can be arranged into various equivalent sets of difference equations.

For each set of equations, we can construct a block diagram consisting of adder, constant multiplier, unit delay element and Unit advance element. Such block diagrams are referred to as realization of system or equivalently as structure for realizing system.The basic elements used to construct block diagrams are listed in table 3.6.

Table 3.6 : Basic elements of block diagram in time domain and z-domain

Elements of block diagram	Time domain representation	z-domain representation
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

3.9 Structures for Realization of IIR Systems

In general, the time domain representation of an N^th^ order IIR system is,

$$y(n) = -\sum_{m=1}^{N} a_m y(n-m) + \sum_{m=0}^{M} b_m x(n-m)$$

and the z-domain representation of an N^th^ order IIR system is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}$$

The above two representations of IIR system can be viewed as a computational procedure (or algorithm) to determine the output sequence y(n) from the input sequence x(n). Also, in the above representations the value of M gives the number of zeros and the value of N gives the number of poles of the IIR system.

The computations in the above equation can be arranged into various equivalent sets of difference equations, which leads to different types of structures for realizing IIR systems.

Some of the structures of the system gives a direct relation between the time domain equation and the z-domain equation.

The different types of structures for realizing the IIR systems are,

1. Direct form-I structure
2. Direct form-II structure
3. Cascade form structure
4. Parallel form structure

3.9.1 Direct Form-I Structure of IIR System

Consider the difference equation governing an IIR system.

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\begin{aligned} y(n) = & -a_1\, y(n-1) - a_2\, y(n-2) - \ldots\ldots - a_N\, y(n-N) \\ & + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \ldots\ldots + b_M\, x(n-M) \end{aligned}$$

On taking $\mathcal{Z}$-transform of the above equation we get,

$$\begin{aligned} Y(z) = & -a_1 z^{-1}\, Y(z) - a_2 z^{-2}\, Y(z) - \ldots\ldots - a_N z^{-N}\, Y(z) \\ & + b_0\, X(z) + b_1 z^{-1}\, X(z) + b_2 z^{-2}\, X(z) + \ldots\ldots + b_M z^{-M}\, X(z) \end{aligned} \qquad \ldots\ldots(3.79)$$

The equation of Y(z) [equation (3.79)] can be directly represented by a block diagram as shown in fig 3.15 and this structure is called direct form-I structure. The direct form-I structure provides a direct relation between time domain and z-domain equations. The direct form-I structure requires separate delays (z^{-1}) for input and output samples. Hence for realizing direct form-I structure more memory is required.

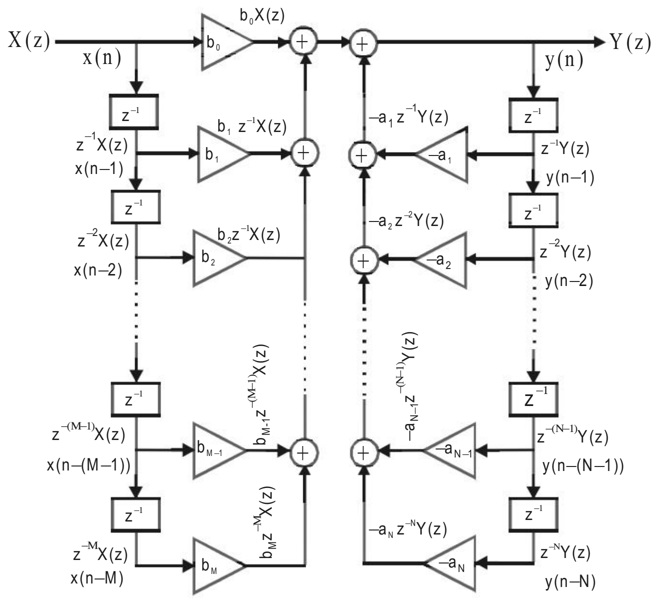


Fig 3.15 : *Direct form-I structure of IIR system.*

From the direct form-I structure it is observed that the realization of an N^{th} order discrete time system with M number of zeros and N number of poles, involves M+N+1 number of multiplications and M+N number of additions. Also this structure involves M+N delays and so M+N memory locations are required to store the delayed signals.

When the number of delays in a structure is equal to the order of the system, the structure is called ***canonic structure***. In direct form-I structure the number of delays is not equal to order of the system and so direct form-I structure is ***noncanonic structure***.

3.9.2 Direct Form-II Structure of IIR System

An alternative structure called direct form-II structure can be realized which uses less number of delay elements than the direct form-I structure.

Consider the general difference equation governing an IIR system.

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - .....- a_N\, y(n-N)$$
$$+ b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + ... + b_M\, x(n-M)$$

On taking Z-transform of the above equation we get,

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - ..... - a_N z^{-N} Y(z)$$
$$+ b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... + b_M z^{-M} X(z)$$

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + ... + a_N z^{-N} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... + b_M z^{-M} X(z)$$

$$Y(z)\left[1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}\right]$$
$$= X(z)\left[b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}$$

Let, $$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)}$$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}} \qquad .....(3.80)$$

$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M} \qquad .....(3.81)$$

On cross multiplying equation (3.80) we get,

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + ... + a_N z^{-N} W(z) = X(z)$$

$$\therefore\; W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - ..... - a_N z^{-N} W(z) \qquad .....(3.82)$$

On cross multiplying equation (3.81) we get,

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + ... + b_M z^{-M} W(z) \qquad .....(3.83)$$

The equations (3.82) and (3.83) represent the IIR system in z-domain and can be realized by a direct structure called direct form-II structure as shown in fig 3.16. In direct form-II structure the number of delays is equal to order of the system and so the direct form-II structure is canonic structure.

From the direct form-II structure it is observed that the realization of an N^{th} order discrete time system with M number of zeros and N number of poles, involves M+N+1 number of multiplications and M+N number of additions. In a realizable system, $N \geq M$, and so the number of delays in direct form-II structure will be equal to N. Hence, when a system is realized using direct form-II structure, N memory locations are required to store the delayed signals.

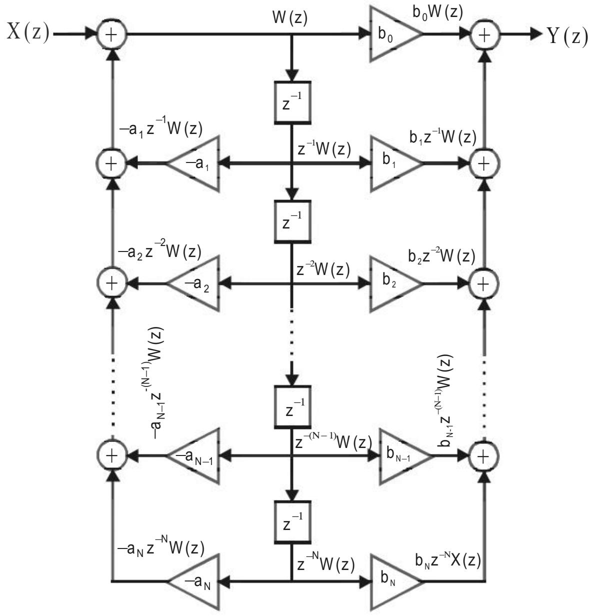


Fig 3.16 : *Direct form-II structure of IIR system for N = M.*

Conversion of Direct Form-I Structure to Direct Form-II Structure

The direct form-I structure can be converted to direct form-II structure by considering the direct form-I structure as cascade of two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in fig 3.17. By linearity property the order of cascading can be interchanged as shown in fig 3.18 and fig 3.19.

In fig 3.19 we can observe that the input to the delay elements in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same and so the output of delay elements in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same. Therefore instead of having separate delays for $\mathcal{H}_1$ and $\mathcal{H}_2$, a single set of delays can be used. Hence the delays can be merged to combine the cascaded systems to a single system and the resultant structure will be direct form-II structure as that of fig 3.16.

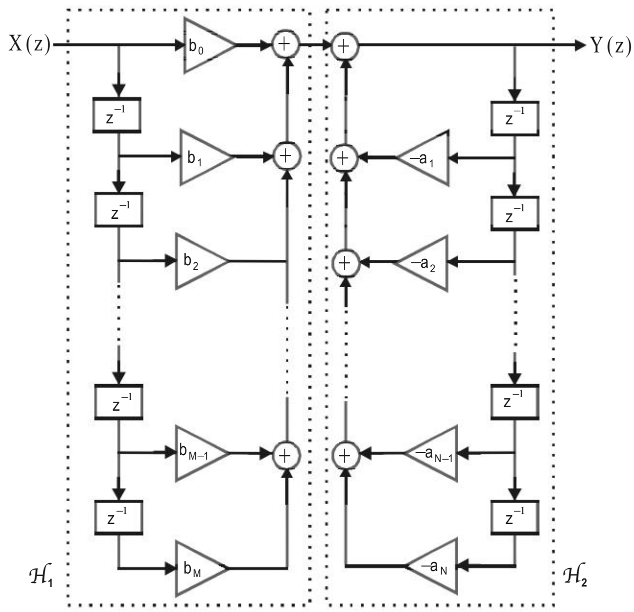


Fig 3.17 : *Direct form-I structure as cascade of two systems.*

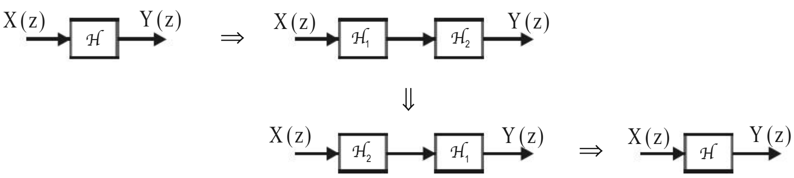


Fig 3.18 : *Conversion of Direct form-I structure to Direct form-II structure.*

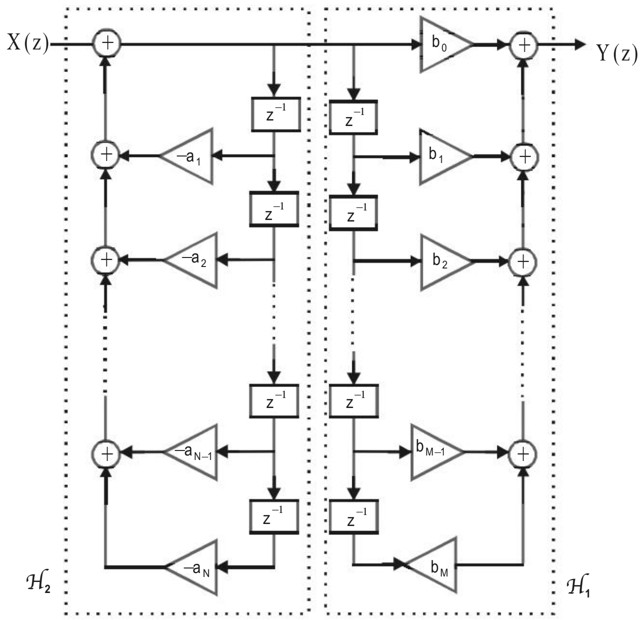


Fig 3.19 : Direct form-I structure after interchanging the order of cascading.

3.9.3 Cascade Form Realization of IIR System

The transfer function H(z) can be expressed as a product of a number of second-order or first-order sections, as shown in equation (3.84).

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \times H_2(z) \times H_3(z) \,.....\, H_m(z) = \prod_{i=1}^{m} H_i(z) \qquad .....(3.84)$$

where, $H_i(z) = \dfrac{c_{0i} + c_{1i}\,z^{-1} + c_{2i}\,z^{-2}}{d_{0i} + d_{1i}\,z^{-1} + d_{2i}\,z^{-2}}$ Second-order section

or, $H_i(z) = \dfrac{c_{0i} + c_{1i}\,z^{-1}}{d_{0i} + d_{1i}\,z^{-1}}$ First-order section

The individual second-order or first-order sections can be realized either in direct form-I or direct form-II structures. The overall system is obtained by cascading the individual sections as shown in fig 3.20. The number of calculations and the memory requirement depends on the realization of individual sections.

Fig 3.20 : Cascade form realization of IIR system.

The difficulty in cascade structure are,

1. Decision of pairing poles and zeros.
2. Deciding the order of cascading the first and second-order sections.
3. Scaling multipliers should be provided between individual sections to prevent the system variables from becoming too large or too small.

3.9.4 Parallel Form Realization of IIR System

The transfer function H(z) of a discrete time system can be expressed as a sum of first and second-order sections, using partial fraction expansion technique as shown in equation (3.85).

$$H(z) = \frac{Y(z)}{X(z)} = C + H_1(z) + H_2(z) + \ldots + H_m(z) \qquad \ldots(3.85)$$

$$= C + \sum_{i=1}^{m} H_i(z)$$

$$\text{where, } H_i(z) = \frac{c_{0i} + c_{1i}\, z^{-1}}{d_{0i} + d_{1i}\, z^{-1} + d_{2i}\, z^{-2}} \quad \boxed{\text{Second-order section}}$$

$$\text{or} \quad H_i(z) = \frac{c_{0i}}{d_{0i} + d_{1i}\, z^{-1}} \quad \boxed{\text{First-order section}}$$

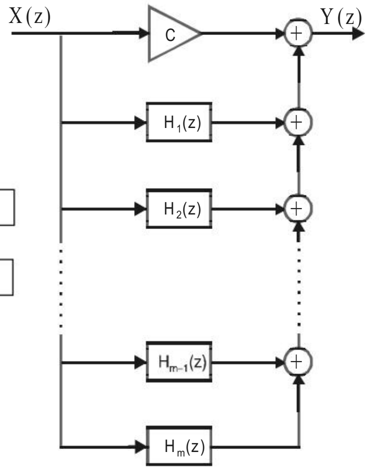


Fig 3.21 : Parallel form realization of IIR system.

The individual first and second-order sections can be realized either in direct form-I or direct form-II structures. The overall system is obtained by connecting the individual sections in parallel as shown in fig 3.21.The number of calculations and the memory requirement depends on the realization of individual sections.

Example 3.26

Obtain the direct form-I, direct form-II, cascade and parallel form realizations of the LTI system governed by the equation,

$$y(n) = -\frac{3}{8}\, y(n-1) + \frac{3}{32}\, y(n-2) + \frac{1}{64}\, y(n-3) + x(n) + 3\, x(n-1) + 2\, x(n-2)$$

Solution

Direct Form-I

Given that,

$$y(n) = -\frac{3}{8}\, y(n-1) + \frac{3}{32}\, y(n-2) + \frac{1}{64}\, y(n-3) + x(n) + 3\, x(n-1) + 2\, x(n-2) \qquad \ldots(1)$$

On taking Z-transform of equation(1) we get,

$$Y(z) = -\frac{3}{8}\, z^{-1}\, Y(z) + \frac{3}{32}\, z^{-2}\, Y(z) + \frac{1}{64}\, z^{-3}\, Y(z) + X(z) + 3z^{-1}\, X(z) + 2z^{-2}\, X(z) \qquad \ldots(2)$$

The direct form-I structure can be obtained from equation (2), as shown in fig 1.

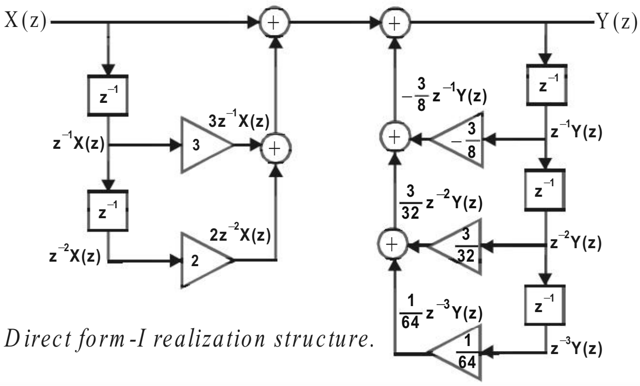


Fig 1 : Direct form-I realization structure.

Direct Form-II

Consider equation (2).

$$Y(z) = -\frac{3}{8}z^{-1}\,Y(z) + \frac{3}{32}z^{-2}\,Y(z) + \frac{1}{64}z^{-3}\,Y(z) + X(z) + 3z^{-1}\,X(z) + 2z^{-2}\,X(z)$$

$$Y(z) + \frac{3}{8}z^{-1}\,Y(z) - \frac{3}{32}z^{-2}\,Y(z) - \frac{1}{64}z^{-3}\,Y(z) = X(z) + 3z^{-1}\,X(z) + 2z^{-2}\,X(z)$$

$$Y(z)\left[1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}\right] = X(z)\left[1 + 3z^{-1} + 2z^{-2}\right]$$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \qquad .....(3)$$

Let, $\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)}\;\frac{Y(z)}{W(z)}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \qquad .....(4)$$

$$\frac{Y(z)}{W(z)} = 1 + 3z^{-1} + 2z^{-2} \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$W(z) + \frac{3}{8}z^{-1}\,W(z) - \frac{3}{32}z^{-2}\,W(z) - \frac{1}{64}z^{-3}\,W(z) = X(z)$$

or $$W(z) = X(z) - \frac{3}{8}z^{-1}\,W(z) + \frac{3}{32}z^{-2}\,W(z) + \frac{1}{64}z^{-3}\,W(z) \qquad .....(6)$$

On cross multiplying equation (5) we get,

$$Y(z) = W(z) + 3z^{-1}\,W(z) + 2z^{-2}\,W(z) \qquad .....(7)$$

The equations (6) and (7) can be realized by a direct form-II structure as shown in fig 2.

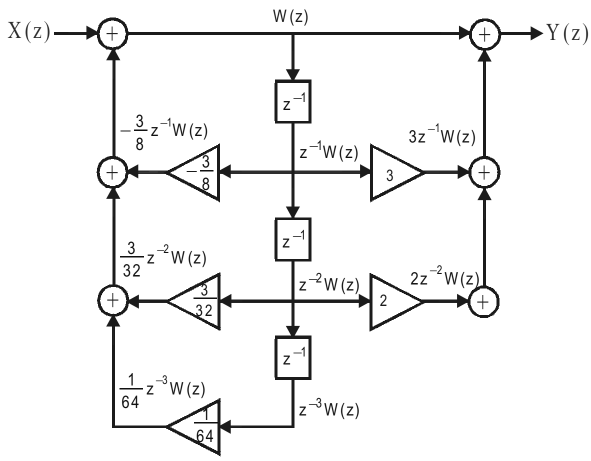


Fig 2 : Direct form-II realization structure.

Cascade Form

Consider equation (3).

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \qquad .....(8)$$

The numerator and denominator polynomials of equation (8) should be expressed in the factored form.

Consider the numerator polynomial of equation (8).

$$\begin{aligned} 1 + 3z^{-1} + 2z^{-2} &= z^{-2}(z^2 + 3z + 2) \\ &= z^{-2}(z + 1)(z + 2) = z^{-1}(z + 1)\, z^{-1}(z + 2) \\ &= \left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right) \end{aligned} \qquad .....(9)$$

Consider the denominator polynomial of equation (8)

$$1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3} = z^{-3}\left(z^3 + \frac{3}{8}z^2 - \frac{3}{32}z - \frac{1}{64}\right) \qquad .....(10)$$

$$= z^{-3}\left(z + \frac{1}{8}\right)\left(z^2 + \frac{2}{8}z - \frac{8}{64}\right)$$

–1/8	1	3/8	–3/32	–1/64
	–	–1/8	–2/64	+1/64
	1	2/8	–8/64	0

z = –1/8 is one of the root of equation (10).

The roots of quadratic,

$$z^2 + \frac{2}{8}z - \frac{8}{64} = 0 \text{ are,}$$

$$z = \frac{-\frac{2}{8} \pm \sqrt{\left(\frac{2}{8}\right)^2 - 4\left(-\frac{8}{64}\right)}}{2}$$

$$= \frac{-\frac{2}{8} \pm \sqrt{\frac{4}{64} + \frac{32}{64}}}{2} = \frac{-\frac{2}{8} \pm \frac{6}{8}}{2} = \frac{-1}{8} \pm \frac{3}{8} = \frac{2}{8}, \frac{-4}{8} = \frac{1}{4}, \frac{-1}{2}$$

$$\therefore 1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3} = z^{-3}\left(z + \frac{1}{8}\right)\left(z + \frac{1}{2}\right)\left(z - \frac{1}{4}\right)$$

$$= z^{-1}\left(z + \frac{1}{8}\right) z^{-1}\left(z + \frac{1}{2}\right) z^{-1}\left(z - \frac{1}{4}\right)$$

$$= \left(1 + \frac{1}{8}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right) \qquad .....(11)$$

From equations(8), (9) and (11) we can write,

$$H(z) = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} = \frac{(1 + z^{-1})(1 + 2z^{-1})}{\left(1 + \frac{1}{8}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} \qquad .....(12)$$

Since there are three first-order factors in the denominator of equation (12), H(z) can be expressed as a product of 3 sections as shown in equation (13).

$$\text{Let,} \quad H(z) = \frac{1 + z^{-1}}{1 + \frac{1}{8}z^{-1}} \times \frac{1 + 2z^{-1}}{1 + \frac{1}{2}z^{-1}} \times \frac{1}{1 - \frac{1}{4}z^{-1}} = H_1(z) \times H_2(z) \times H_3(z) \qquad .....(13)$$

$$\text{where,} \quad H_1(z) = \frac{1 + z^{-1}}{1 + \frac{1}{8}z^{-1}} \;;\; H_2(z) = \frac{1 + 2z^{-1}}{1 + \frac{1}{2}z^{-1}} \text{ and } H_3(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

The transfer function $H_1(z)$ can be realized in direct form-II structure using equations (14) and (15), as shown in fig 3.

$$\text{Let, } H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{W_1(z)}{X(z)}\,\frac{Y_1(z)}{W_1(z)} = \frac{1 + z^{-1}}{1 + \frac{1}{8}z^{-1}}$$

$$\text{where, } \frac{W_1(z)}{X(z)} = \frac{1}{1+\frac{1}{8}z^{-1}} \quad \text{and} \quad \frac{Y_1(z)}{W_1(z)} = 1 + z^{-1}$$

$$\therefore\ W_1(z) = X(z) - \frac{1}{8}z^{-1}W_1(z) \qquad .....(14)$$

$$Y_1(z) = W_1(z) + z^{-1}W_1(z) \qquad .....(15)$$

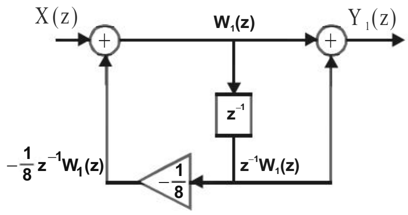


Fig 3 : Direct form-II structure of $H_1(z)$.

The transfer function $H_2(z)$ can be realized in direct form-II structure using equations (16) and (17), as shown in fig 4.

$$\text{Let, } H_2(z) = \frac{Y_2(z)}{Y_1(z)} = \frac{W_2(z)}{Y_1(z)}\,\frac{Y_2(z)}{W_2(z)} = \frac{1+2z^{-1}}{1+\frac{1}{2}z^{-1}}$$

$$\text{where, } \frac{W_2(z)}{Y_1(z)} = \frac{1}{1+\frac{1}{2}z^{-1}} \quad \text{and} \quad \frac{Y_2(z)}{W_2(z)} = 1 + 2z^{-1}$$

$$\therefore\ W_2(z) = Y_1(z) - \frac{1}{2}z^{-1}W_2(z) \qquad .....(16)$$

$$Y_2(z) = W_2(z) + 2z^{-1}W_2(z) \qquad .....(17)$$

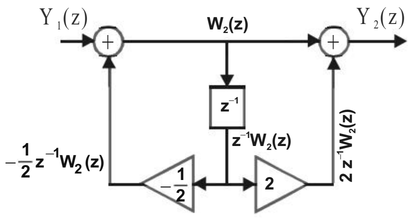


Fig 4 : Direct form-II structure of $H_2(z)$.

The transfer function $H_3(z)$ can be realized in direct form-II structure using equation (18), as shown in fig 5.

$$\text{Let, } H_3(z) = \frac{Y(z)}{Y_2(z)} = \frac{1}{1-\frac{1}{4}z^{-1}}$$

$$\therefore\ Y(z) - \frac{1}{4}z^{-1}Y(z) = Y_2(z)$$

$$Y(z) = Y_2(z) + \frac{1}{4}z^{-1}Y(z) \qquad .....(18)$$

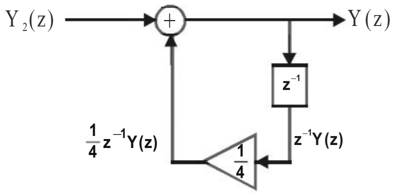


Fig 5 : Direct form-II structure of $H_3(z)$.

The cascade structure of the given LTI system is obtained by connecting the individual sections shown in fig 3, fig 4 and fig 5 in cascade as shown in fig 6.

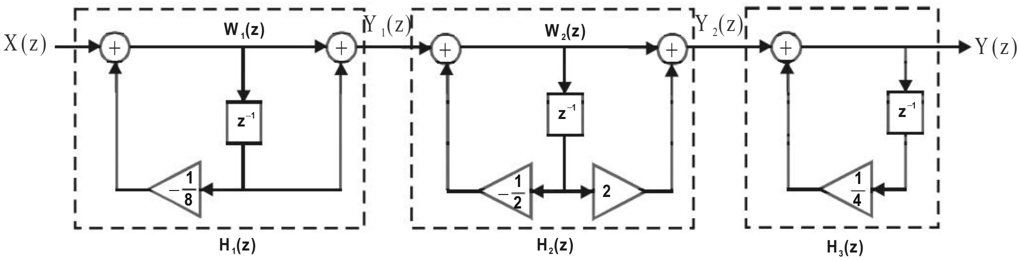


Fig 6 : Cascade realization of the system.

Parallel Form

Consider the equation (12).

$$H(z) = \frac{\left(1+z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

By partial fraction expansion,

$$H(z) = \frac{\left(1+z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} = \frac{A}{1+\frac{1}{8}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}} + \frac{C}{1-\frac{1}{4}z^{-1}}$$

$$A = \left.\frac{\left(1+z^{-1}\right)\left(1+2z^{-1}\right)}{\cancel{\left(1+\frac{1}{8}z^{-1}\right)}\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} \times \cancel{\left(1+\frac{1}{8}z^{-1}\right)}\right|_{z^{-1}=-8} = \frac{(1-8)\,(1-16)}{(1-4)\,(1+2)} = -\frac{105}{9} = -\frac{35}{3}$$

$$B = \left.\frac{\left(1+z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1+\frac{1}{8}z^{-1}\right)\cancel{\left(1+\frac{1}{2}z^{-1}\right)}\left(1-\frac{1}{4}z^{-1}\right)} \times \cancel{\left(1+\frac{1}{2}z^{-1}\right)}\right|_{z^{-1}=-2} = \frac{(1-2)\,(1-4)}{\left(1-\frac{1}{4}\right)\left(1+\frac{1}{2}\right)} = \frac{(-1)\times(-3)}{\frac{3}{4}\times\frac{3}{2}} = \frac{8}{3}$$

$$C = \left.\frac{\left(1+z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\cancel{\left(1-\frac{1}{4}z^{-1}\right)}} \times \cancel{\left(1-\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=4} = \frac{(1+4)\,(1+8)}{\left(1+\frac{1}{2}\right)(1+2)} = \frac{5\times 9}{\frac{3}{2}\times 3} = 10$$

$$\therefore\ H(z) = \frac{-\frac{35}{3}}{1+\frac{1}{8}z^{-1}} + \frac{\frac{8}{3}}{1+\frac{1}{2}z^{-1}} + \frac{10}{1-\frac{1}{4}z^{-1}} = H_1(z) + H_2(z) + H_3(z)$$

$$\text{where,}\quad H_1(z) = \frac{-\frac{35}{3}}{1+\frac{1}{8}z^{-1}}\ ;\qquad H_2(z) = \frac{\frac{8}{3}}{1+\frac{1}{2}z^{-1}}\ ;\qquad H_3(z) = \frac{10}{1-\frac{1}{4}z^{-1}}$$

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)}\ ;\quad H_1(z) = \frac{Y_1(z)}{X(z)};\quad H_2(z) = \frac{Y_2(z)}{X(z)};\quad H_3(z) = \frac{Y_3(z)}{X(z)}$$

$$\therefore\ H(z) = H_1(z) + H_2(z) + H_3(z) \quad\Rightarrow\quad \frac{Y(z)}{X(z)} = \frac{Y_1(z)}{X(z)} + \frac{Y_2(z)}{X(z)} + \frac{Y_3(z)}{X(z)}$$

$$\therefore\ Y(z) = Y_1(z) + Y_2(z) + Y_3(z)$$

The transfer function $H_1(z)$ can be realized in direct form-I structure using equation (19) as shown in fig 7.

$$\text{Let,}\ H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{-\frac{35}{3}}{1+\frac{1}{8}z^{-1}}$$

On cross multiplying and rearranging we get,

$$Y_1(z) = -\frac{1}{8}z^{-1}\,Y_1(z) - \frac{35}{3}\,X(z) \qquad .....(19)$$

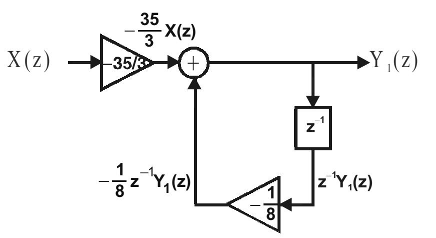


Fig 7 : Direct form-I structure of $H_1(z)$.

The transfer function $H_2(z)$ can be realized in direct form-I structure using equation (20) as shown in fig 8.

$$\text{Let,}\quad H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{\frac{8}{3}}{1 + \frac{1}{2}z^{-1}}$$

On cross multiplying and rearranging we get,

$$Y_2(z) = -\frac{1}{2}z^{-1}\,Y_2(z) + \frac{8}{3}\,X(z) \qquad .....(20)$$

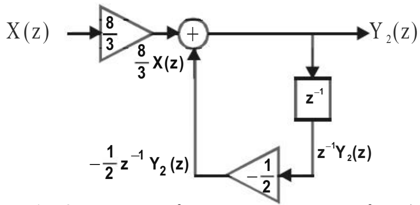


Fig 8 : Direct form-I structure of $H_2(z)$.

The transfer function $H_3(z)$ can be realized in direct form-I structure using equation (21) as shown in fig 9.

$$\text{Let,}\quad H_3(z) = \frac{Y_3(z)}{X(z)} = \frac{10}{1 - \frac{1}{4}z^{-1}}$$

On cross multiplying and rearranging we get,

$$Y_3(z) = \frac{1}{4}z^{-1}\,Y_3(z) + 10\,X(z) \qquad .....(21)$$

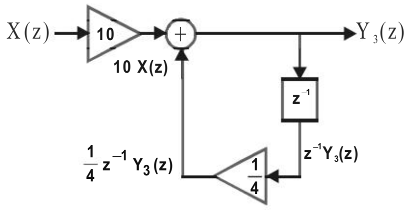


Fig 9 : Direct form-I structure of $H_3(z)$.

The overall structure is obtained by connecting the individual sections shown in fig 7, fig 8 and fig 9 in parallel as shown in fig 10.

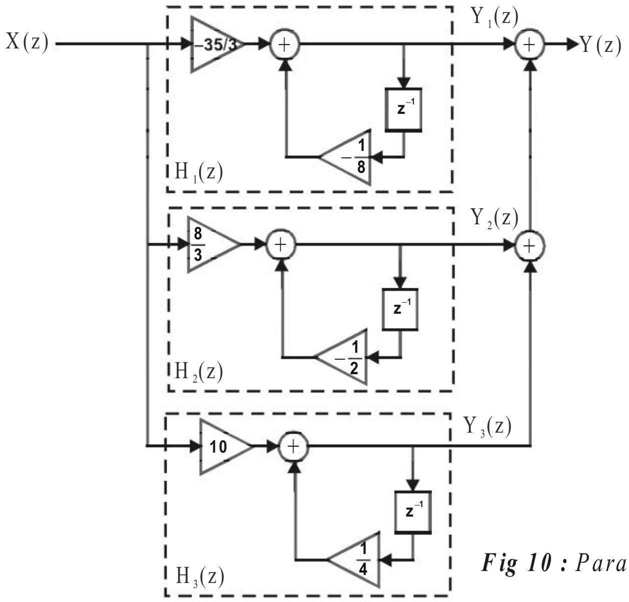


Fig 10 : Parallel form realization.

Example 3.27

Find the direct form-I and direct form-II realizations of a discrete time system represented by transfer function,

$$H(z) = \frac{2z^3 - 4z^2 + 11z - 8}{(z-8)\left(z^2 - z + 3\right)}$$

Solution

Direct Form-I

Let, $H(z) = \dfrac{Y(z)}{X(z)}$; where, $Y(z)$ = Output and $X(z)$ = Input.

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{2z^3 - 4z^2 + 11z - 8}{(z-8)\left(z^2 - z + 3\right)} = \frac{2z^3 - 4z^2 + 11z - 8}{z^3 - z^2 + 3z - 8z^2 + 8z - 24}$$

$$= \frac{2z^3 - 4z^2 + 11z - 8}{z^3 - 9z^2 + 11z - 24} = \frac{z^3\left(2 - 4z^{-1} + 11z^{-2} - 8z^{-3}\right)}{z^3\left(1 - 9z^{-1} + 11z^{-2} - 24z^{-3}\right)}$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{2 - 4z^{-1} + 11z^{-2} - 8z^{-3}}{1 - 9z^{-1} + 11z^{-2} - 24z^{-3}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y(z) - 9z^{-1}\,Y(z) + 11z^{-2}\,Y(z) - 24z^{-3}\,Y(z) = 2X(z) - 4z^{-1}\,X(z) + 11z^{-2}\,X(z) - 8z^{-3}\,X(z)$$

$$\therefore\ Y(z) = 2X(z) - 4z^{-1}\,X(z) + 11z^{-2}\,X(z) - 8z^{-3}\,X(z)$$
$$+ 9z^{-1}\,Y(z) - 11z^{-2}\,Y(z) + 24z^{-3}\,Y(z) \qquad .....(2)$$

The direct form-I structure can be obtained from equation (2) as shown in fig 1.

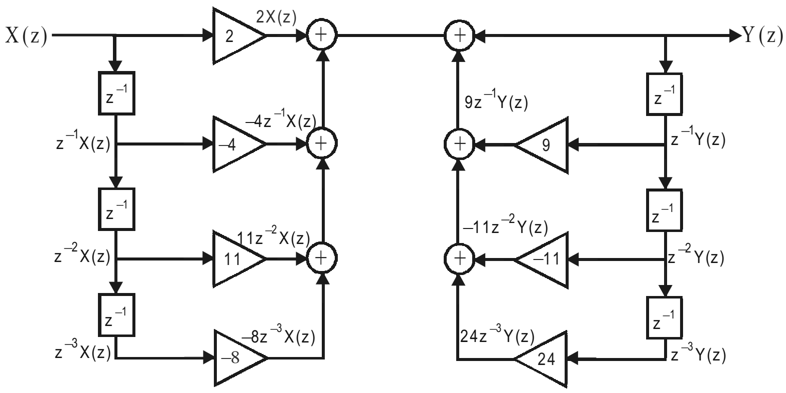


Fig 1 : Direct form-I realization.

Direct Form-II

From equation (1) we get,

$$\frac{Y(z)}{X(z)} = \frac{2 - 4z^{-1} + 11z^{-2} - 8z^{-3}}{1 - 9z^{-1} + 11z^{-2} - 24z^{-3}}$$

Let, $\dfrac{Y(z)}{X(z)} = \dfrac{W(z)}{X(z)}\,\dfrac{Y(z)}{W(z)}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 - 9z^{-1} + 11z^{-2} - 24z^{-3}} \qquad .....(3)$$

$$\frac{Y(z)}{W(z)} = 2 - 4z^{-1} + 11z^{-2} - 8z^{-3} \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$W(z) - 9z^{-1}\,W(z) + 11z^{-2}\,W(z) - 24z^{-3}\,W(z) = X(z)$$

$$\therefore\ W(z) = X(z) + 9z^{-1}\,W(z) - 11z^{-2}\,W(z) + 24z^{-3}\,W(z) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = 2W(z) - 4z^{-1}\,W(z) + 11z^{-2}\,W(z) - 8z^{-3}\,W(z) \qquad .....(6)$$

The equations (5) and (6) can be realized by a direct form-II Structure as shown in fig 2.

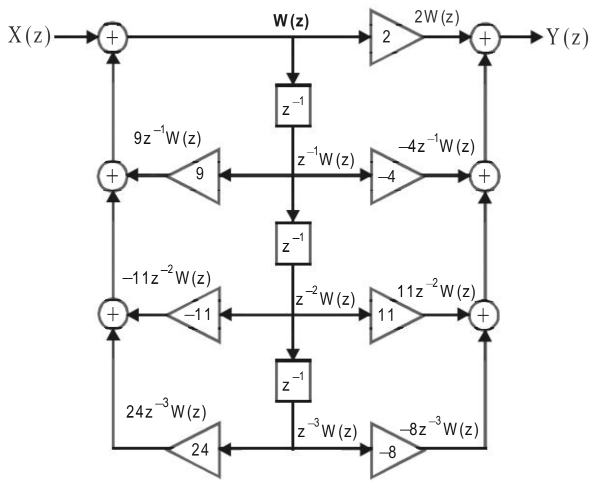


Fig 2 : *Direct form-II realization.*

Example 3.28

Find the digital network in direct form-I and II for the system described by the difference equation,

$$y(n) = x(n) + 0.3\, x(n-1) - 0.4\, x(n-2) - 0.8\, y(n-1) + 0.7\, y(n-2).$$

Solution

Given that, $y(n) = x(n) + 0.3\, x(n-1) - 0.4\, x(n-2) - 0.8\, y(n-1) + 0.7\, y(n-2)$

On taking Z-transform we get,

$$Y(z) = X(z) + 0.3z^{-1}X(z) - 0.4z^{-2}X(z) - 0.8z^{-1}Y(z) + 0.7z^{-2}\ Y(z) \qquad .....(1)$$

The direct form-I digital network can be realized using equation (1) as shown in fig 1.

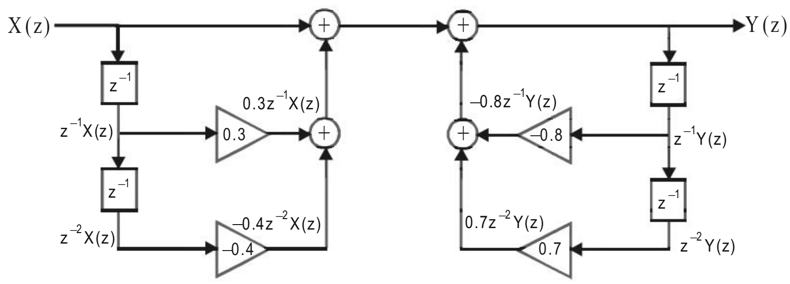


Fig 1 : *Direct form-I digital network.*

On rearranging equation (1) we get,

$$Y(z) + 0.8z^{-1}Y(z) - 0.7z^{-2}Y(z) = X(z) + 0.3z^{-1}X(z) - 0.4z^{-2}X(z)$$

$$[1+0.8z^{-1} - 0.7z^{-2}]\ Y(z) = [1 + 0.3z^{-1} - 0.4z^{-2}]\ X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1+0.3z^{-1} - 0.4z^{-2}}{1+0.8z^{-1} - 0.7z^{-2}} \qquad .....(2)$$

The equation (2) is the transfer function of the system.

Let, $\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)}\frac{Y(z)}{W(z)}$

where, $\frac{W(z)}{X(z)} = \frac{1}{1+0.8z^{-1}-0.7z^{-2}}$ ..... (3)

$\frac{Y(z)}{W(z)} = 1+0.3z^{-1}-0.4z^{-2}$ ..... (4)

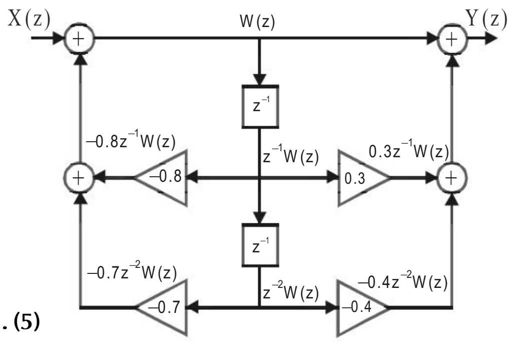


Fig 2 : Direct form-II digital network.

On cross multiplying equation (3) we get,

$W(z) + 0.8z^{-1}W(z) + 0.7z^{-2}W(z) = X(z)$

$\therefore\ W(z) = X(z) - 0.8z^{-1}W(z) - 0.7z^{-2}W(z)$ ..... (5)

On cross multiplying equation (4) we get,

$Y(z) = W(z) + 0.3z^{-1}W(z) - 0.4z^{-2}W(z)$ ..... (6)

The direct form-II digital network is realized using equations (5) and (6) as shown in fig 2.

Example 3.29

Realize the digital network described by H(z) in two ways. $H(z) = \frac{1 - a\ \cos\omega_0\ z^{-1}}{1 - 2a\ \cos\omega_0\ z^{-1} + a^2\ z^{-2}}$

Solution

Let, $H(z) = \frac{Y(z)}{X(z)} = \frac{1 - a\ \cos\omega_0\ z^{-1}}{1 - 2a\ \cos\omega_0\ z^{-1} + a^2\ z^{-2}}$

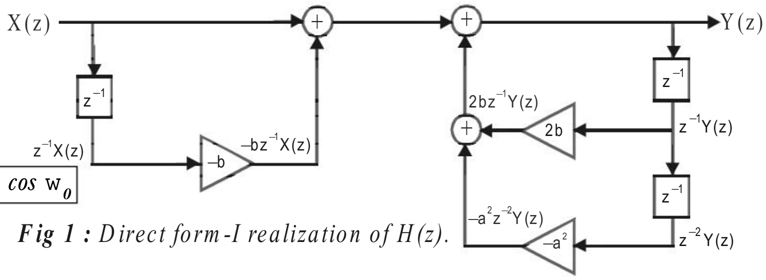
On cross multiplying we get,

$Y(z) - 2a\cos w_0 z^{-1}\,Y(z) + a^2 z^{-2}\,Y(z) = X(z) - a\cos w_0 z^{-1}\,X(z)$

$\therefore\ Y(z) = X(z) - a\cos w_0 z^{-1}\,X(z) + 2a\cos w_0 z^{-1}\,Y(z) - a^2 z^{-2}\,Y(z)$

Let, $a\cos w_0 = b$. $\therefore\ Y(z) = X(z) - bz^{-1}\,X(z) + 2bz^{-1}\,Y(z) - a^2 z^{-2}\,Y(z)$ ..... (1)

The equation (1) can be used to construct direct form-I structure of H(z) as shown in fig 1.



Note : $b = a\cos w_0$

Fig 1 : Direct form-I realization of H(z).

Consider the direct form-I structure as cascade of two systems $H_1(z)$ and $H_2(z)$ as shown in fig 2.

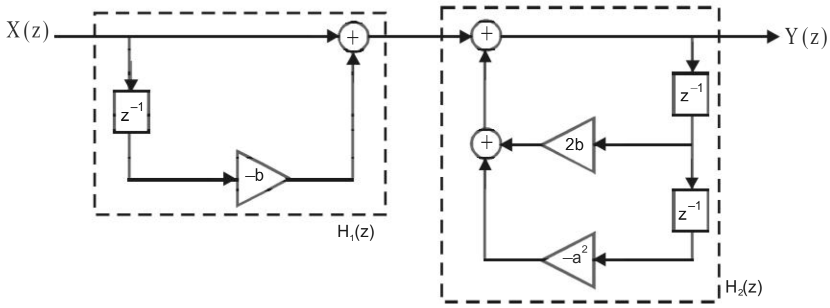


Fig 2 : Direct form-I structure as cascade of two systems.

In an LT1 system, by linearity property, the order of cascading can be changed. Hence the systems $H_1(z)$ and $H_2(z)$ are interchanged and the fig 2 is redrawn as shown in fig 3.

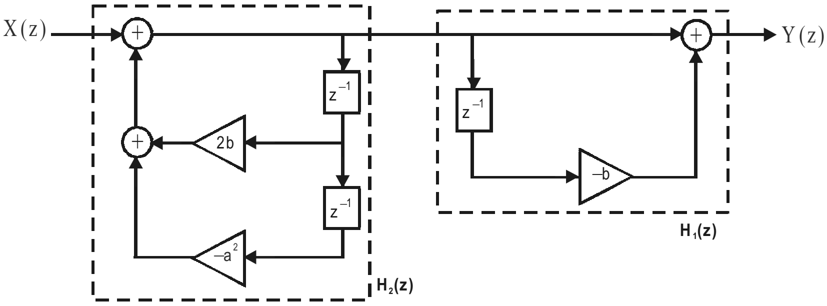


Fig 3 : Direct form-I structure with $H_1(z)$ and $H_2(z)$ interchanged.

Since the input to delay elements in both the systems $H_1(z)$ and $H_2(z)$ of fig 3 are same, the outputs will also be same. Hence the delays can be combined and the resultant structure is direct form-II structure, which is shown in fig 4.

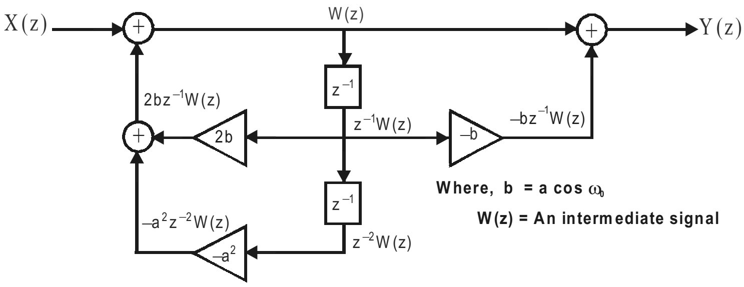


Fig 4 : Direct form-II structure of H(z).

Example 3.30

Realize the given system in cascade and parallel forms.

$$H(z) = \frac{1+0.25z^{-1}}{\left(1-2z^{-1}+0.25z^{-2}\right)\left(1-3z^{-1}+0.2z^{-2}\right)}$$

Solution

Cascade Form

Let us realize the system as cascade of two second-order systems.

$$H(z) = \frac{1+0.25z^{-1}}{\left(1-2z^{-1}+0.25z^{-2}\right)\left(1-3z^{-1}+0.2z^{-2}\right)} = \frac{1}{1-2z^{-1}+0.25z^{-2}} \times \frac{1+0.25z^{-1}}{1-3z^{-1}+0.2z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

where, $H_1(z) = \dfrac{1}{1-2z^{-1}+0.25z^{-2}}$; $H_2(z) = \dfrac{1+0.25z^{-1}}{1-3z^{-1}+0.2z^{-2}}$

Let, $$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-2z^{-1}+0.25z^{-2}} \quad .....(1)$$

On cross multiplying equation (1) we get,

$$Y_1(z) - 2z^{-1}\,Y_1(z) + 0.25z^{-2}\,Y_1(z) = X(z)$$

$$\therefore\ Y_1(z) = X(z) + 2z^{-1}\,Y_1(z) - 0.25z^{-2}\,Y_1(z) \qquad .....(2)$$

Using equation (2) the direct form-II structure of $H_1(z)$ is realized as shown in fig 1.

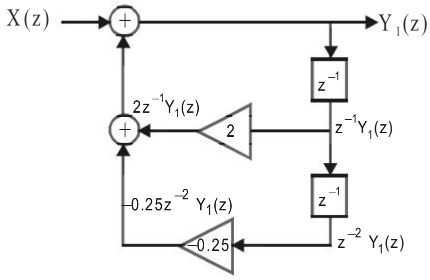


Fig 1 : Direct form-II structure of $H_1(z)$.

$$\text{Let,}\quad H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1 + 0.25z^{-1}}{1 - 3z^{-1} + 0.2z^{-2}}$$

$$\text{Let,}\quad \frac{Y(z)}{Y_1(z)} = \frac{W_2(z)}{Y_1(z)}\ \frac{Y(z)}{W_2(z)}$$

$$\text{where,}\quad \frac{W_2(z)}{Y_1(z)} = \frac{1}{1 - 3z^{-1} + 0.2z^{-2}} \qquad .....(3)$$

$$\frac{Y(z)}{W_2(z)} = 1 + 0.25z^{-1} \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$W_2(z) - 3z^{-1}\,W_2(z) + 0.2z^{-2}\,W_2(z) = Y_1(z)$$

$$\therefore\ W_2(z) = Y_1(z) + 3z^{-1}\,W_2(z) - 0.2z^{-2}\,W_2(z) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = W_2(z) + 0.25z^{-1}\,W_2(z) \qquad .....(6)$$

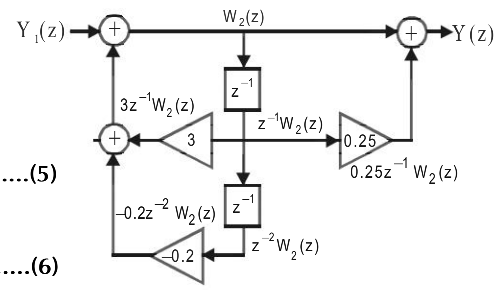


Fig 2 : Direct form-II structure of $H_2(z)$.

Using equations (5) and (6) the direct form-II structure of $H_2(z)$ is realized as shown in fig 2.

The cascade structure of H(z) is obtained by connecting the structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

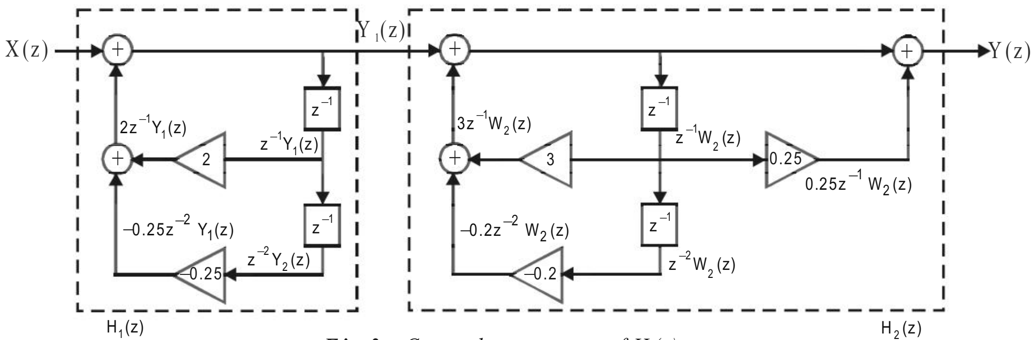


Fig 3 : Cascade structure of H(z).

Parallel Realization

$$\text{Given that,}\ H(z) = \frac{1 + 0.25z^{-1}}{\left(1 - 2z^{-1} + 0.25z^{-2}\right)\left(1 - 3z^{-1} + 0.2z^{-2}\right)}$$

By partial fraction expansion we can write,

$$H(z) = \frac{1 + 0.25z^{-1}}{\left(1 - 2z^{-1} + 0.25z^{-2}\right)\left(1 - 3z^{-1} + 0.2z^{-2}\right)} = \frac{A + Bz^{-1}}{1 - 2z^{-1} + 0.25z^{-2}} + \frac{C + Dz^{-1}}{1 - 3z^{-1} + 0.2z^{-2}} \qquad .....(7)$$

On cross multiplying equation (7) we get,

$$1+0.25z^{-1} = (A+Bz^{-1})\left(1-3z^{-1}+0.2\,z^{-2}\right)+(C+Dz^{-1})\left(1-2z^{-1}+0.25\,z^{-2}\right)$$

$$1+0.25z^{-1} = A-3Az^{-1}+0.2Az^{-2}+Bz^{-1}-3Bz^{-2}+0.2Bz^{-3}+C-2Cz^{-1}+0.25Cz^{-2}$$
$$+Dz^{-1}-2Dz^{-2}+0.25Dz^{-3}$$

$$1+0.25z^{-1} = (A+C)+(-3A+B-2C+D)\,z^{-1}+\left(0.2A-3B+0.25C-2D\right)z^{-2}$$
$$+\left(0.2B+0.25D\right)z^{-3} \qquad .....(8)$$

On equating the constants in equation (8) we get,

$$A + C = 1 \quad \Rightarrow \quad C = 1 - A$$

On equating the coefficients of z^{-3} in equation (8) we get,

$$0.2B+0.25D = 0 \qquad \therefore\ 0.25D = -0.2B \quad \Rightarrow \quad D = -\frac{0.2}{0.25}B = -0.8B$$

On equating the coefficients of z^{-1} in equation (8) we get,

$$-3A+B-2C+D = 0.25$$

On substituting $C = 1 - A$ and $D = -0.8B$ in the above equation we get,

$$-3A+B-2(1-A)+\left(-0.8B\right) = 0.25 \quad \Rightarrow \quad -A+0.2B = 2.25 \quad \Rightarrow \quad A = 0.2B-2.25$$

On equating the coefficients of z^{-2} in equation (8) we get,

$$0.2A-3B+0.25C-2D = 0$$

On substituting $C = 1 - A$, and $D = -0.8B$ in the above equation we get,

$$0.2A-3B+0.25(1-A)-2(-0.8B) = 0 \quad \Rightarrow \quad -0.05A-1.4B = -0.25$$

On substituting $A = 0.2B - 2.25$ in the above equation we get,

$$-0.05\,(0.2B-2.25)-1.4B = -0.25 \quad \Rightarrow \quad -1.41B = -0.3625 \quad \Rightarrow \quad B = \frac{0.3625}{1.41} = 0.26$$

$$\therefore\ A = 0.2B-2.25 = 0.2 \times 0.26-2.25 = -2.2$$

$$\therefore\ C = 1-A = 1+2.2 = 3.2$$

$$\therefore\ D = -0.8B = -0.8 \times 0.26 = -0.21$$

$$\therefore\ H(z) = \frac{A+Bz^{-1}}{1-2z^{-1}+0.25z^{-2}}+\frac{C+Dz^{-1}}{1-3z^{-1}+0.2z^{-2}} = \frac{-2.2+0.26z^{-1}}{1-2z^{-1}+0.25z^{-2}}+\frac{3.2-0.21z^{-1}}{1-3z^{-1}+0.2z^{-2}}$$

Let, $H(z) = \dfrac{-2.2+0.26z^{-1}}{1-2z^{-1}+0.25z^{-2}}+\dfrac{3.2-0.21z^{-1}}{1-3z^{-1}+0.2z^{-2}} = H_1(z) + H_2(z)$

where, $H_1(z) = \dfrac{-2.2 + 0.26z^{-1}}{1 - 2z^{-1} + 0.25z^{-2}}$

$H_2(z) = \dfrac{3.2 - 0.21z^{-1}}{1-3z^{-1} + 0.2z^{-2}}$

Let, $H(z) = \dfrac{Y(z)}{X(z)}$; $H_1(z) = \dfrac{Y_1(z)}{X(z)}$; $H_2(z) = \dfrac{Y_2(z)}{X(z)}$

$$\therefore\ H(z) = H_1(z)+H_2(z) \quad \Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{Y_1(z)}{X(z)}+\frac{Y_2(z)}{X(z)} \quad \Rightarrow \quad Y(z) = Y_1(z)+Y_2(z)$$

Realization of $H_1(z)$

$$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{-2.2 + 0.26z^{-1}}{1 - 2z^{-1} + 0.25z^{-2}}$$

Let, $\frac{Y_1(z)}{X(z)} = \frac{W_1(z)}{X(z)} \frac{Y_1(z)}{W_1(z)}$

where, $$\frac{W_1(z)}{X(z)} = \frac{1}{1 - 2z^{-1} + 0.25z^{-2}} \qquad .....(9)$$

$$\frac{Y_1(z)}{W_1(z)} = -2.2 + 0.26z^{-1} \qquad .....(10)$$

On cross multiplying equation (9) we get,

$$W_1(z) - 2z^{-1} W_1(z) + 0.25z^{-2} W_1(z) = X(z)$$

$$\therefore \; W_1(z) = X(z) + 2z^{-1} W_1(z) - 0.25z^{-2} W_1(z) \qquad .....(11)$$

On cross multiplying equation (10) we get,

$$Y_1(z) = -2.2W_1(z) + 0.26z^{-1} W_1(z) \qquad .....(12)$$

The direct form-II structure of system $H_1(z)$ can be realized using equations (11) and (12) as shown in fig 4.

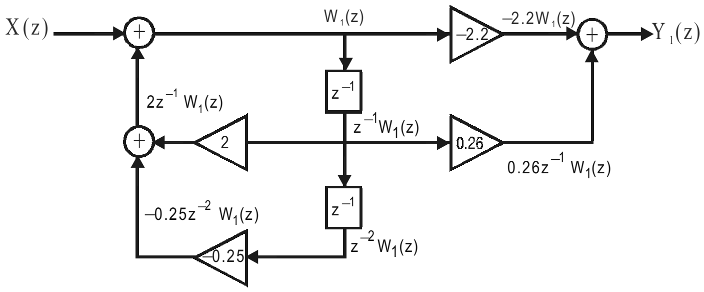


Fig 4 : Direct form-II structure of $H_1(z)$.

Realization of $H_2(z)$

$$H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{3.2 - 0.21z^{-1}}{1 - 3z^{-1} + 0.2z^{-2}}$$

Let, $\frac{Y_2(z)}{X(z)} = \frac{W_2(z)}{X(z)} \frac{Y_2(z)}{W_2(z)}$

where, $$\frac{W_2(z)}{X(z)} = \frac{1}{1 - 3z^{-1} + 0.2z^{-2}} \qquad .....(13)$$

$$\frac{Y_2(z)}{W_2(z)} = 3.2 - 0.21z^{-1} \qquad .....(14)$$

On cross multiplying equation (13) we get,

$$W_2(z) - 3z^{-1} W_2(z) + 0.2z^{-2} W_2(z) = X(z)$$

$$\therefore \; W_2(z) = X(z) + 3z^{-1} W_2(z) - 0.2z^{-2} W_2(z) \qquad .....(15)$$

On cross multiplying equation (14) we get,

$$Y_2(z) = 3.2\, W_2(z) - 0.21z^{-1} W_2(z) \qquad ..... (16)$$

The direct form-II structure of system $H_2(z)$ can be realized using equations (15) and (16) as shown in fig 5.

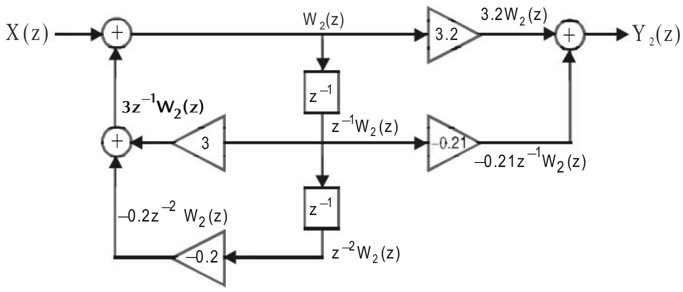


Fig 5 : Direct form-II structure of $H_2(z)$.

The parallel form structure of H(z) is obtained by connecting the direct form-II structure of $H_1(z)$ and $H_2(z)$ in parallel as shown in fig 6.

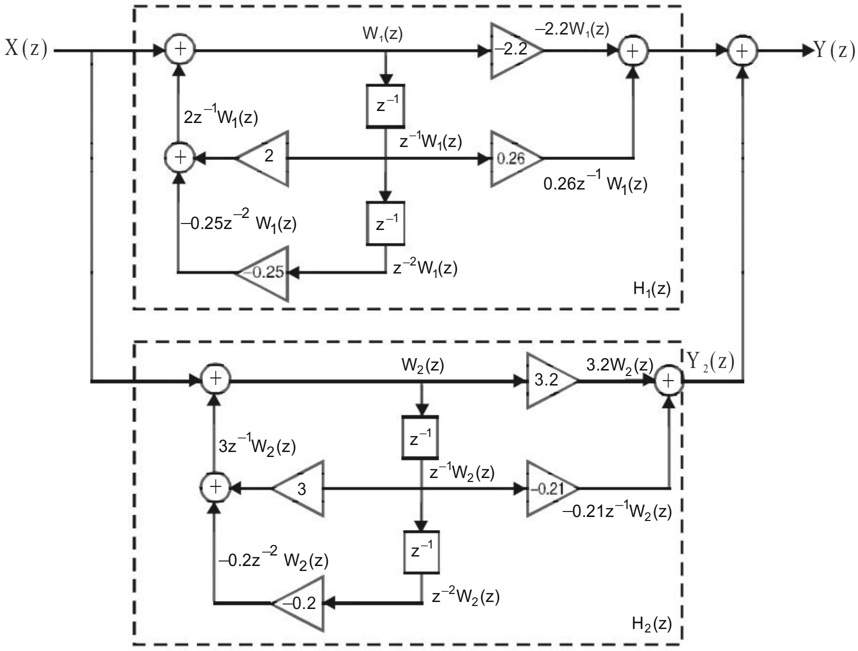


Fig 6 : Parallel form realization of system H(z).

Example 3.31

Obtain the cascade realization of the system, $H(z) = \dfrac{2 + 3z^{-1} + 4z^{-2}}{\left(1+\frac{1}{7}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{1}{9}z^{-1}\right)}$

Solution

Given that, $H(z) = \dfrac{2 + 3z^{-1} + 4z^{-2}}{\left(1+\frac{1}{7}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{1}{9}z^{-1}\right)}$

On examining the roots of numerator polynomial it is found that the roots are complex conjugate. Hence H(z) can be realized as cascade of one first-order and one second-order system.

$2 + 3z^{-1} + 4z^{-2} = 2z^{-2}(z^2 + 1.5z + 2)$

The roots of quadratic,

$z^2 + 1.5z + 2 = 0$ are,

$$z = \frac{-1.5 \pm \sqrt{1.5^2 - 4 \times 2}}{2} = \frac{-1.5 \pm j2.4}{2}$$

$$\therefore\ H(z) = \frac{1}{1-\frac{1}{4}z^{-1}} \times \frac{2+3z^{-1}+4z^{-2}}{\left(1+\frac{1}{7}z^{-1}\right)\left(1+\frac{1}{9}z^{-1}\right)} = \frac{1}{1-\frac{1}{4}z^{-1}} \times \frac{2+3z^{-1}+4z^{-2}}{1+\frac{16}{63}z^{-1}+\frac{1}{63}z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

$$\text{where,}\quad H_1(z) = \frac{1}{1-\frac{1}{4}z^{-1}} \quad \text{and} \quad H_2(z) = \frac{2+3z^{-1}+4z^{-2}}{1+\frac{16}{63}z^{-1}+\frac{1}{63}z^{-2}}$$

$$\text{Let,}\quad H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-\frac{1}{4}z^{-1}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y_1(z) - \frac{1}{4}z^{-1}Y_1(z) = X(z) \quad \Rightarrow \quad Y_1(z) = X(z) + \frac{1}{4}z^{-1}Y_1(z) \qquad .....(2)$$

The direct form-II structure of $H_1(z)$ can be obtained from equation (2) as shown in fig 1.

$$\text{Let,}\quad H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{2+3z^{-1}+4z^{-2}}{1+\frac{16}{63}z^{-1}+\frac{1}{63}z^{-2}}$$

$$\text{Let,}\quad \frac{Y(z)}{Y_1(z)} = \frac{W_2(z)}{Y_1(z)}\ \frac{Y(z)}{W_2(z)}$$

$$\text{where,}\quad \frac{W_2(z)}{Y_1(z)} = \frac{1}{1+\frac{16}{63}z^{-1}+\frac{1}{63}z^{-2}} \qquad .....(3)$$

$$\frac{Y(z)}{W_2(z)} = 2+3z^{-1}+4z^{-2} \qquad .....(4)$$

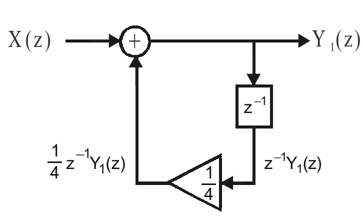


Fig 1 : Direct form-II structure of $H_1(z)$.

On cross multiplying equation (3) we get,

$$W_2(z) + \frac{16}{63}z^{-1}W_2(z) + \frac{1}{63}z^{-2}W_2(z) = Y_1(z)$$

$$\therefore\ W_2(z) = Y_1(z) - \frac{16}{63}z^{-1}W_2(z) - \frac{1}{63}z^{-2}W_2(z) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = 2W_2(z) + 3z^{-1}W_2(z) + 4z^{-2}W_2(z) \qquad .....(6)$$

The direct form-II structure of $H_2(z)$ can be obtained using equations (5) and (6) as shown in fig 2.

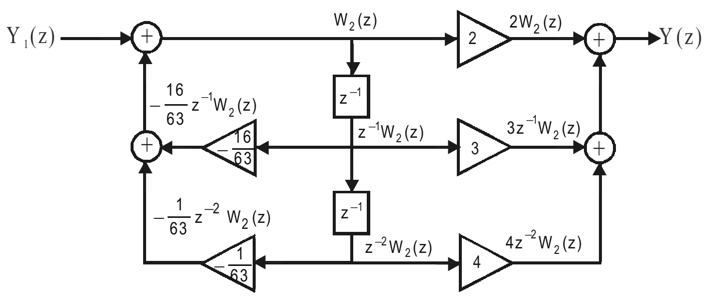


Fig 2 : Direct form-II structure of $H_2(z)$.

The cascade realization of H(z) is obtained by connecting the direct form-II structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

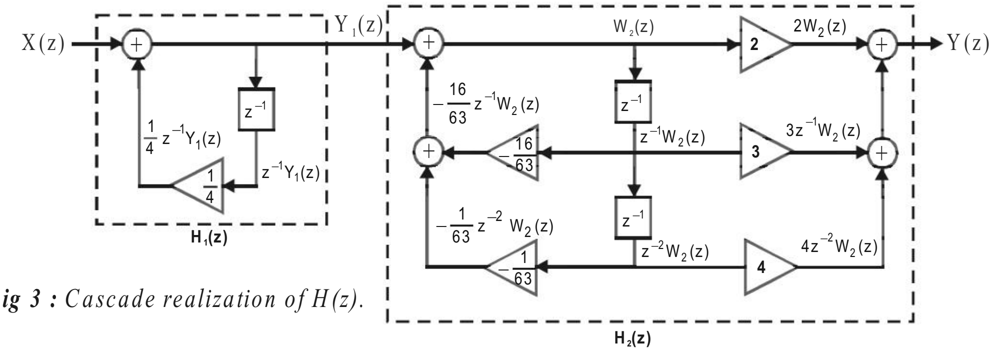


Fig 3 : Cascade realization of H(z).

Example 3.32

The transfer function of a system is given by, $H(z) = \dfrac{(2-z^{-1})(1-z^{-1})^2}{(1-2z^{-1})(5-3z^{-1}+2z^{-2})}$

Realize the system in cascade and parallel structures.

> $5 - 3z^{-1} + 2z^{-2} = z^{-2}(5z^2 - 3z + 2)$
>
> The roots of quadratic,
>
> $5z^2 - 3z + 2 = 0$ are,
>
> $z = \dfrac{3 \pm \sqrt{3^2 - 4\times5\times2}}{2} = -\dfrac{3 \pm j5.6}{2}$

Solution

Cascade Realization

Given that, $H(z) = \dfrac{(2-z^{-1})(1-z^{-1})^2}{(1-2z^{-1})(5-3z^{-1}+2z^{-2})}$

On examining the roots of the quadratic factor in the denominator it is observed that the roots are complex conjugate. Hence the system has to be realized as cascade of one first-order section and one second-order section.

$$\therefore\ H(z) = \frac{2-z^{-1}}{1-2z^{-1}} \times \frac{(1-z^{-1})^2}{5-3z^{-1}+2z^{-2}} = \frac{2-z^{-1}}{1-2z^{-1}} \times \frac{1-2z^{-1}+z^{-2}}{5-3z^{-1}+2z^{-2}}$$

Let, $H(z) = H_1(z)$ ´ $H_2(z)$

where, $H_1(z) = \dfrac{2-z^{-1}}{1-2z^{-1}}$ and $H_2(z) = \dfrac{1-2z^{-1}+z^{-2}}{5-3z^{-1}+2z^{-2}}$

Let, $H_1(z) = \dfrac{Y_1(z)}{X(z)} = \dfrac{2-z^{-1}}{1-2z^{-1}}$ ..... (1)

On cross multiplying equation (1) we get,

$Y_1(z) - 2z^{-1}Y_1(z) = 2X(z) - z^{-1}X(z)$

$\therefore\ Y_1(z) = 2X(z) - z^{-1}X(z) + 2z^{-1}Y_1(z)$ ..... (2)

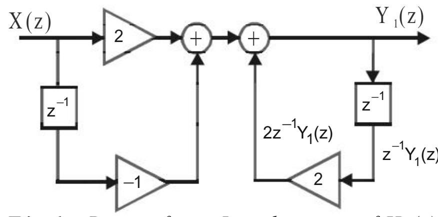


Fig 1 : Direct form-I realization of $H_1(z)$.

The direct form-I structure of $H_1(z)$ can be drawn using equation (2) as shown in fig 1.

Let, $H_2(z) = \dfrac{Y(z)}{Y_1(z)} = \dfrac{1-2z^{-1}+z^{-2}}{5-3z^{-1}+2z^{-2}}$ .....(3)

On cross multiplying equation (3) we get

$$5Y(z) - 3z^{-1}Y(z) + 2z^{-2}Y(z) = Y_1(z) - 2z^{-1}Y_1(z) + z^{-2}Y_1(z)$$

$$\therefore\ Y(z) = \frac{1}{5}Y_1(z) - \frac{2}{5}z^{-1}Y_1(z) + \frac{1}{5}z^{-2}Y_1(z) + \frac{3}{5}z^{-1}Y(z) - \frac{2}{5}z^{-2}Y(z) \qquad .....(4)$$

The direct form-I structure of $H_2(z)$ can be drawn using equation (4) as shown in fig 2.

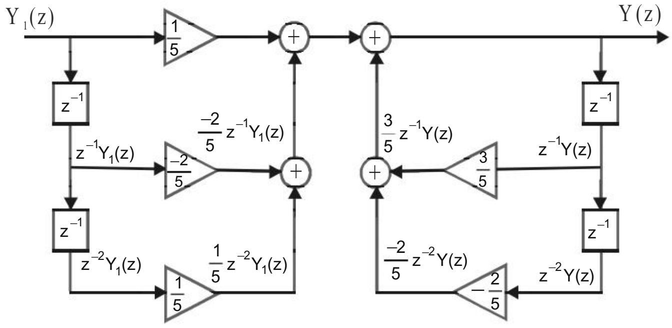


Fig 2 : The direct form-I structure of $H_2(z)$.

The cascade realization of H(z) is obtained by connecting the direct form-I structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

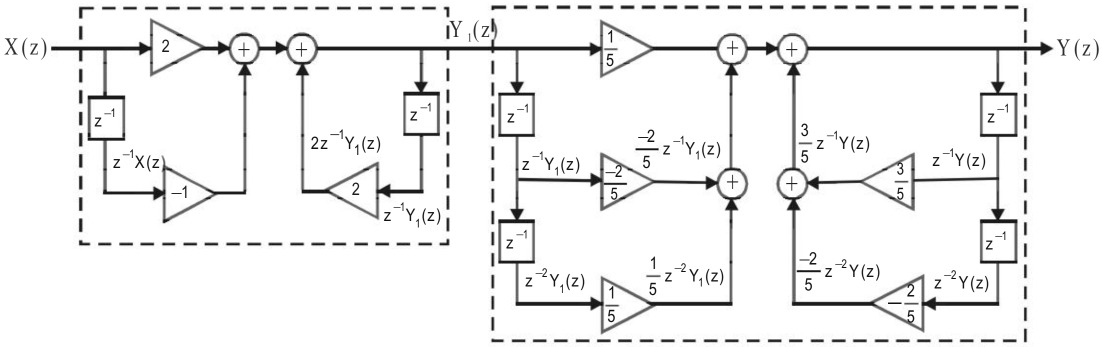


Fig 3 : Cascade realization of H(z).

Parallel Realization

$$\text{Given that, } H(z) = \frac{(2 - z^{-1})(1 - z^{-1})^2}{(1-2z^{-1})(5-3z^{-1}+2z^{-2})} \qquad .....(5)$$

$$= \frac{(2 - z^{-1})(1 - 2z^{-1} + z^{-2})}{(1-2z^{-1})(5-3z^{-1}+2z^{-2})} = \frac{2-4z^{-1}+2z^{-2}-z^{-1}+2z^{-2}-z^{-3}}{5-3z^{-1}+2z^{-2}-10z^{-1}+6z^{-2}-4z^{-3}}$$

$$\therefore H(z) = \frac{2-5z^{-1}+4z^{-2}-z^{-3}}{5-13z^{-1}+8z^{-2}-4z^{-3}}$$

$$= 0.4 + \frac{0.2z^{-1}+0.8z^{-2}+0.6z^{-3}}{5-13z^{-1}+8z^{-2}-4z^{-3}}$$

$$\begin{array}{r|l} & 0.4 \\ \hline 5-13z^{-1}+8z^{-2}-4z^{-3} & 2-5z^{-1}+4z^{-2}-z^{-3} \\ & 2-5.2z^{-1}+3.2z^{-2}-1.6z^{-3} \\ & (-)\quad(+)\quad(-)\quad(+) \\ \hline & 0.2z^{-1}+0.8z^{-2}+0.6z^{-3} \end{array}$$

$$= 0.4 + z^{-1}\left[\frac{0.2+0.8z^{-1}+0.6z^{-2}}{(1-2z^{-1})(5-3z^{-1}+2z^{-2})}\right] \qquad .....(6)$$

Using equation (5).

By partial fraction expansion we can write,

$$\frac{0.2 + 0.8z^{-1} - 0.6z^{-2}}{\left(1 - 2z^{-1}\right)\left(5 - 3z^{-1} + 2z^{-2}\right)} = \frac{A}{1 - 2z^{-1}} + \frac{B \; + \; C\,z^{-1}}{5 - 3z^{-1} + 2z^{-2}} \qquad .....(7)$$

On cross multiplying equation (7) we get,

$$0.2 + 0.8z^{-1} + 0.6z^{-2} = A\left(5 - 3z^{-1} + 2z^{-2}\right) + (B \; + \; C\,z^{-1})\left(1 - 2z^{-1}\right)$$

$$0.2 + 0.8z^{-1} + 0.6z^{-2} = 5A - 3Az^{-1} + 2Az^{-2} + B - 2B\,z^{-1} + Cz^{-1} - 2Cz^{-2} \qquad .....(8)$$

$$0.2 + 0.8z^{-1} + 0.6z^{-2} = (5A + B) + (-3A - 2B + C)z^{-1} + (2A - 2C)z^{-2}$$

On equating constants of equation (8),

$5A + B = 0.2$

$\backslash \quad B = 0.2 - 5A$

On equating coefficients of z^{-1} of equation (8),

$-3A - 2B + C = 0.8$

Put, $B = 0.2 - 5A$

$\backslash \quad -3A - 2(0.2 - 5A) + C = 0.8$

$-3A - 0.4 + 10A + C = 0.8$

$\backslash \quad C = 1.2 - 7A$

On equating coefficients of z^{-2} of equation (8),

$2A - 2C = 0.6$

$2A - 2(1.2 - 7A) = 0.6$

$2A - 2.4 + 14A = 0.6$

$\therefore \quad 16A = 3 \Rightarrow A = \dfrac{3}{16}$

Here, $A = \dfrac{3}{16}$

$$\therefore \; B = 0.2 - 5A = 0.2 - 5 \times \frac{3}{16} = \frac{2}{10} - \frac{15}{16} = \frac{32 - 150}{160} = -\frac{118}{160} = -\frac{59}{80}$$

$$\therefore \; C = 1.2 - 7A = 1.2 - 7 \times \frac{3}{16} = \frac{12}{10} - \frac{21}{16} = \frac{192 - 210}{160} = -\frac{18}{160} = -\frac{9}{80}$$

From equations (6) and (7) we can write,

$$H(z) = 0.4 + z^{-1}\left[\frac{A}{1 - 2z^{-1}} + \frac{B + C\,z^{-1}}{5 - 3z^{-1} + 2z^{-2}}\right] = 0.4 + \frac{\frac{3}{16}z^{-1}}{1 - 2z^{-1}} + \frac{-\frac{59}{80}z^{-1} - \frac{9}{80}z^{-2}}{5 - 3z^{-1} + 2z^{-2}}$$

Let, $$H(z) = 0.4 + \frac{\frac{3}{16}z^{-1}}{1 - 2z^{-1}} + \frac{-\frac{59}{80}z^{-1} - \frac{9}{80}z^{-2}}{5 - 3z^{-1} + 2z^{-2}} = H_1(z) + H_2(z) + H_3(z)$$

where, $H_1(z) = 0.4$; $\quad H_2(z) = \dfrac{\frac{3}{16}z^{-1}}{1 - 2z^{-1}}$; $\quad H_3(z) = \dfrac{-\frac{59}{80}z^{-1} - \frac{9}{80}z^{-1}}{5 - 3z^{-1} + 2z^{-2}}$

Let, $H(z) = \dfrac{Y(z)}{X(z)}$; $\quad H_1(z) = \dfrac{Y_1(z)}{X(z)}$; $\quad H_2(z) = \dfrac{Y_2(z)}{X(z)}$; $\quad H_3(z) = \dfrac{Y_3(z)}{X(z)}$

$$\therefore \; H(z) = H_1(z) + H_2(z) + H_3(z) \;\Rightarrow\; \frac{Y(z)}{X(z)} = \frac{Y_1(z)}{X(z)} + \frac{Y_2(z)}{X(z)} + \frac{Y_3(z)}{X(z)}$$

$$\therefore Y(z) = Y_1(z) + Y_2(z) + Y_3(z)$$

Realization of $H_1(z)$

$$H_1(z) = \frac{Y_1(z)}{X(z)} = 0.4 \quad \Rightarrow \quad Y_1(z) = 0.4\;X(z)$$

Using the above equation, the direct form-I structure of $H_1(z)$ is drawn as shown in fig 4.

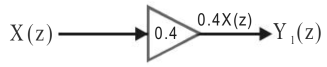


Fig 4 : Direct form-I structure of $H_1(z)$.

Realization of $H_2(z)$

$$H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{\frac{3}{16} z^{-1}}{1 - 2z^{-1}}$$

On cross multiplying the above equation we get,

$$Y_2(z) - 2z^{-1} Y_2(z) = \frac{3}{16} z^{-1} X(z)$$

$$\therefore \quad Y_2(z) = \frac{3}{16} z^{-1} X(z) + 2z^{-1} Y_2(z) \qquad \text{..... (9)}$$

Using equation (9) the direct form-I structure of $H_2(z)$ is drawn as shown in fig 5.

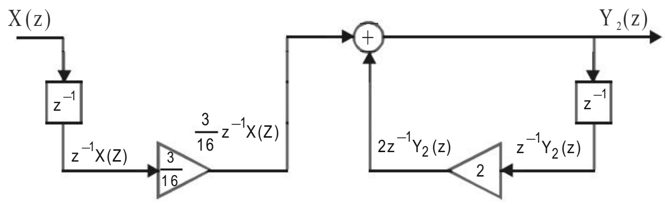


Fig 5 : Direct form-I structure of $H_2(z)$.

Realization of $H_3(z)$

$$H_3(z) = \frac{Y_3(z)}{X(z)} = \frac{-\frac{59}{80} z^{-1} - \frac{9}{80} z^{-2}}{5 - 3z^{-1} + 2z^{-2}}$$

On cross multiplying the above equation we get,

$$5Y_3(z) - 3z^{-1} Y_3(z) + 2z^{-2} Y_3(z) = -\frac{59}{80} z^{-1} X(z) - \frac{9}{80} z^{-2} X(z)$$

$$\therefore \; Y_3(z) = -\frac{59}{400} z^{-1} X(z) - \frac{9}{400} z^{-2} X(z) + \frac{3}{5} z^{-1} Y_3(z) - \frac{2}{5} z^{-2} Y_3(z) \qquad \text{.....(10)}$$

Using equation (10) the direct form-I structure of $H_2(z)$ is drawn as shown in fig 6.

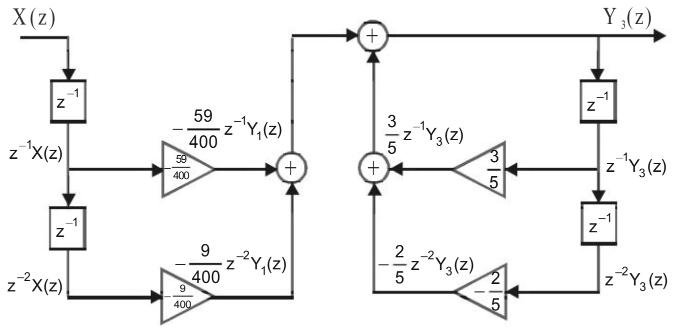


Fig 6 : Direct form-I structure of $H_3(z)$.

Parallel Structure

The parallel structure of H(z) is obtained by connecting the direct form-I structure of $H_1(z)$, $H_2(z)$ and $H_3(z)$ as shown in fig 7.

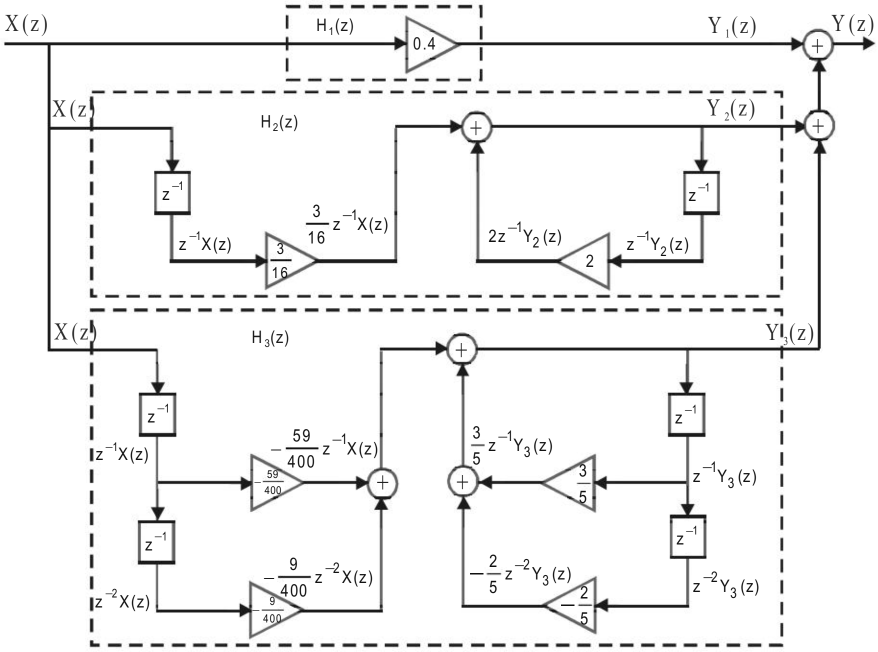


Fig 7 : The parallel structure of H(z).

Example 3.33

An LTI System is described by the equation, $y(n) - \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) = x(n)$.

Determine the cascade realization structure of the system.

Solution

Given that, $y(n) - \frac{1}{2}y(n-1) - \frac{1}{4}y(n-2) = x(n)$

On taking Z-transform we get,

$$Y(z) - \frac{1}{2} z^{-1} Y(z) - \frac{1}{4} z^{-2} Y(z) = X(z)$$

$$\left(1 - \frac{1}{2} z^{-1} - \frac{1}{4} z^{-2}\right) Y(z) = X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1} - \frac{1}{4} z^{-2}}$$

The roots of quadratic

$z^2 - 0.5z - 0.25 = 0$ are,

$$z = \frac{0.5 \pm \sqrt{0.5^2 + 4 \times 0.25}}{2}$$

$$= \frac{0.5 \pm 1.118}{2} = 0.809, \ -0.309$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1} - \frac{1}{4} z^{-2}} = \frac{1}{z^{-2}\left(z^2 - 0.5z - 0.25\right)}$$

$$= \frac{1}{z^{-2}(z - 0.809)(z + 0.309)} = \frac{1}{(1 - 0.809z^{-1})(1 + 0.309z^{-1})}$$

Let, $H(z) = H_1(z)\,H_2(z)$

where, $H_1(z) = \frac{1}{1 - 0.809z^{-1}}$; $\quad H_2(z) = \frac{1}{1 + 0.309z^{-1}}$

Let, $H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-0.809z^{-1}}$ .....(1)

On cross multiplying equation (1) we get,

$Y_1(z) - 0.809z^{-1}\,Y_1(z) = X(z)$

$\therefore\ Y_1(z) = X(z) + 0.809z^{-1}\,Y_1(z)$ ..... (2)

The direct form-I structure of $H_1(z)$ is obtained using equation (2) as shown in fig 1.

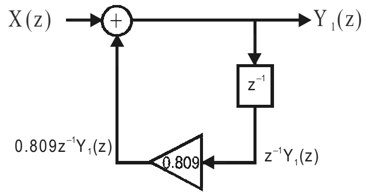


Fig 1 : Direct form-I structure of $H_1(z)$.

Let, $H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{1 + 0.309z^{-1}}$ ..... (3)

On cross multiplying equation (3) we get,

$Y(z) + 0.309\,z^{-1}\,Y(z) = Y_1(z)$

$Y(z) = Y_1(z) - 0.309\,z^{-1}\,Y(z)$ ..... (4)

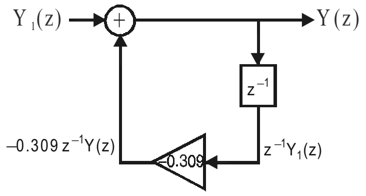


Fig 2 : Direct form-I structure of $H_2(z)$.

The direct form-I structure of $H_2(z)$ is obtained using equation (4) as shown in fig 2.The cascade structure is obtained by connecting the direct form structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

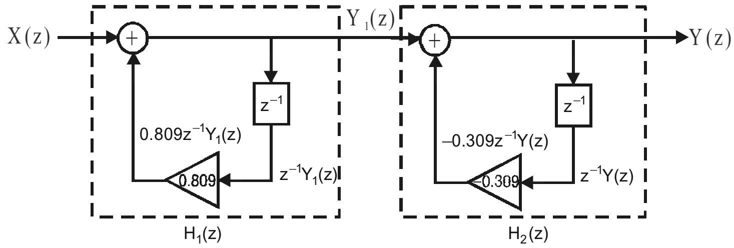


Fig 3 : Cascade structure.

3.10 Structures for Realization of FIR Systems

In general, the time domain representation of an N^{th} order FIR system is,

$$y(n) = \sum_{m=0}^{N-1} b_m\,x(n-m) = b_0\,x(n) + b_1\,x(n-1) + b_2\,x(n-2) + ... + b_{N-1}\,x(n-(N-1))$$

and the z-domain representation of a FIR system is,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)}$$

The above two representations of FIR system can be viewed as a computational procedure (or algorithm) to determine the output sequence y(n) from the input sequence x(n). Also in the above representations the value of N gives the number of zeros of the FIR system. The computations in the above equation can be arranged into various equivalent sets of difference equations, which leads to different types of structures for realizing FIR systems. Some of the structures of the system gives a direct relation between time domain equation and z-domain equation.

The different types of structures for realizing FIR systems are,

1. Direct form realization
2. Cascade realization
3. Linear phase realization

3.10.1 Direct Form Realization of FIR System

Consider the difference equation governing a FIR system,

If $\mathcal{Z}\{x(n)\} = X(z)$ then, $\mathcal{Z}\{x(n-k)\} = z^{-k}X(z)$

$$y(n) = \sum_{m=0}^{N-1} b_m\, x(n-m)$$

$$= b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \,.....\, + b_{N-1}\, x(n-(N-1))$$

On taking $\mathcal{Z}$-transform of the above equation we get,

$$Y(z) = b_0\, X(z) + b_1\, z^{-1}\, X(z) + b_2\, z^{-2}\, X(z) + b_3\, z^{-3}\, X(z) + \,.....\, + b_{N-2}\, z^{-(N-2)}\, X(z) + b_{N-1}\, z^{-(N-1)}\, X(z) \qquad .....(3.86)$$

The equation of Y(z) [equation (3.86)] can be directly represented by a block diagram as shown in fig 3.22 and this structure is called direct form structure. The direct form structure provides a direct relation between time domain and z-domain equations.

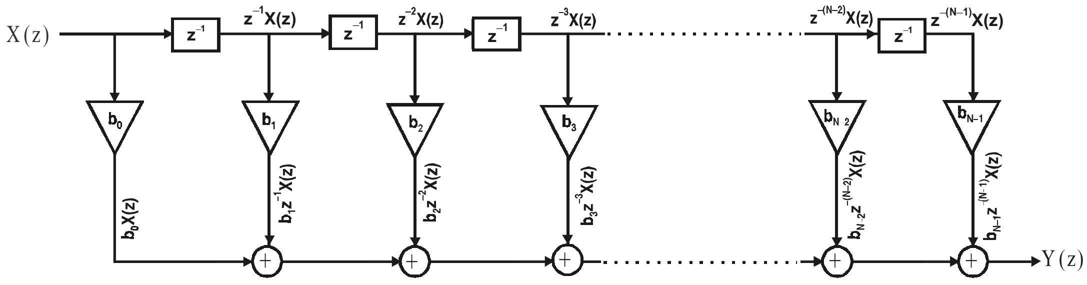


Fig 3.22 : Direct form structure of FIR system.

From the direct form structure it is observed that the realization of an N^{th} order FIR discrete time system involves N number of multiplications and N-1 number of additions. Also the structure involves N-1 delays and so N-1 memory locations are required to store the delayed signals.

3.10.2 Cascade Form Realization of FIR System

Consider the transfer function of a FIR system,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1\, z^{-1} + b_2\, z^{-2} + \,.....\, + b_{N-1}\, z^{-(N-1)}$$

The transfer function of FIR system is $(N-1)^{th}$ order polynomial in z. This polynomial can be factorized into first and second-order factors and the transfer function can be expressed as a product of first and second-order factors or sections as shown in equation (3.87).

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \times H_2(z) \times H_3(z) \,.....\, H_m(z) = \prod_{i=1}^{m} H_i(z) \qquad ...(3.87)$$

where, $H_i(z) = c_{0i} + c_{1i}\, z^{-1} + c_{2i}\, z^{-2}$ Second-order section

or, $H_i(z) = c_{0i} + c_{1i}\, z^{-1}$ First-order section

The individual second-order or first-order sections can be realized either in direct form structure or linear phase structure. The overall system is obtained by cascading the individual sections as shown in fig 3.23. The number of calculations and the memory requirement depends on the realization of individual sections.

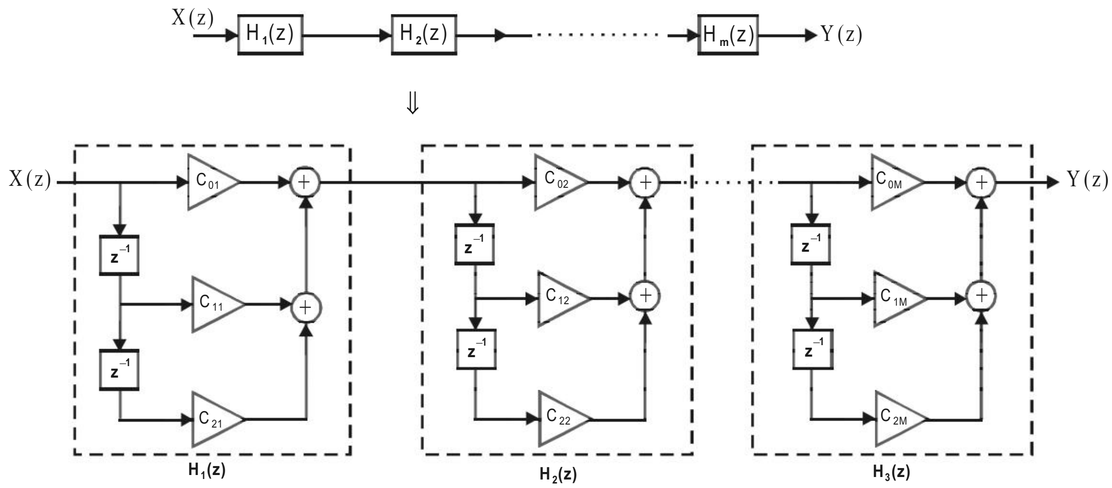


Fig 3.23 : Cascade structure of FIR system.

3.10.3 Linear Phase Realization of FIR System

Consider the impulse response h(n) of FIR system,

$$h(n) = \{b_0,\ b_1,\ b_2,\ \ldots\ldots\ldots\ldots\ldots\ b_{N-1}\}$$

In FIR system, for linear phase response the impulse response should be symmetrical.

The condition for symmetry is,

$$h(n) = h(N-1-n)$$

Proof :	
Let, N = 7, \ h(n) = h(6–n)	Let, N = 8, \ h(n) = h(7–n)
n = 0, 1, 2, 3, 4, 5, 6	n = 0, 1, 2, 3, 4, 5, 6, 7
When n = 0; h(0) = h(6)	When n = 0; h(0) = h(7)
When n = 1; h(1) = h(5)	When n = 1; h(1) = h(6)
When n = 2; h(2) = h(4)	When n = 2; h(2) = h(5)
When n = 3; h(3) = h(3)	When n = 3; h(3) = h(4)

When the impulse response is symmetric, the samples of impulse response will satisfy the condition,

$$b_n = b_{N-1-n}$$

By using the above symmetry condition it is possible to reduce the number of multipliers required for the realization of FIR system. Hence, the linear phase realization is also called ***realization with minimum number of multipliers.***

Consider the transfer function of FIR system,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots.. + b_{N-1} z^{-(N-1)}$$

The linear phase realization of the FIR system using the above equation for even and odd values of N are discussed below.

Case i : When N is even

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)}$$

$$= \sum_{m=0}^{N-1} b_m z^{-m} = \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=\frac{N}{2}}^{N-1} b_m z^{-m}$$

Dividing the summation of N terms into two summations with N/2 terms.

Let, p = N–1–m, ∴ m = N–1–p

When, $m = \frac{N}{2}$; $p = N-1-\frac{N}{2} = \frac{N}{2}-1$

When, m = N–1; p = N–1–(N–1) = 0

Let, m = p in the second summation

$$\therefore \frac{Y(z)}{X(z)} = \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{p=0}^{\frac{N}{2}-1} b_{N-1-p} z^{-(N-1-p)}$$

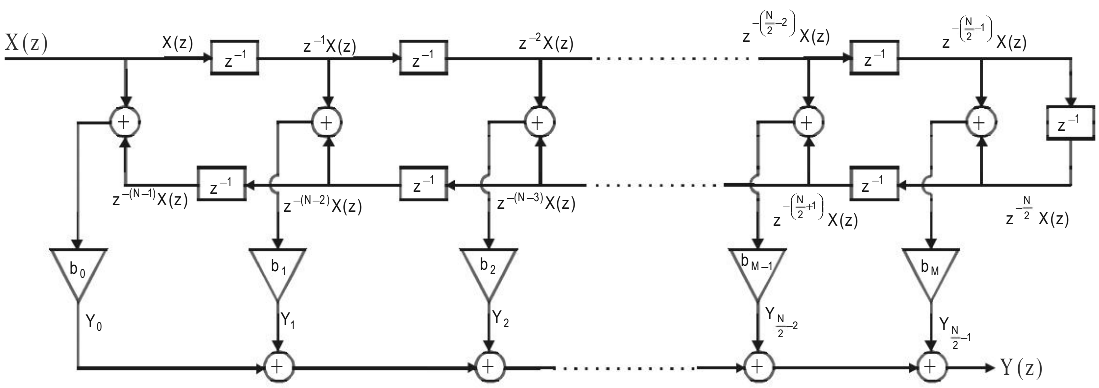
Let, p = m

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=0}^{\frac{N}{2}-1} b_{N-1-m} z^{-(N-1-m)}$$

When impulse response is symmetric, $b_m = b_{N-1-m}$

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-(N-1-m)}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m \left[z^{-m} + z^{-(N-1-m)}\right]$$



where, $M = \frac{N}{2}-1$; $Y_0 = b_0\left[X(z) + z^{-(N-1)} X(z)\right]$; $Y_{\frac{N}{2}-2} = b_{\frac{N}{2}-2}\left[z^{-\left(\frac{N}{2}-2\right)} X(z) + z^{-\left(\frac{N}{2}+1\right)} X(z)\right]$

$Y_1 = b_1\left[z^{-1} X(z) + z^{-(N-2)} X(z)\right]$; $Y_{\frac{N}{2}-1} = b_{\frac{N}{2}-1}\left[z^{-\left(\frac{N}{2}-1\right)} X(z) + z^{-\frac{N}{2}} X(z)\right]$

Fig 10.24 : *Direct form realization of a linear phase FIR system when N is even.*

$$\therefore\ Y(z) = b_0\left[X(z) + z^{-(N-1)}X(z)\right] + b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right] + ...$$

$$+\ b_{\frac{N}{2}-2}\left[z^{-\left(\frac{N}{2}-2\right)}X(z) + z^{-\left(\frac{N}{2}+1\right)}X(z)\right] + b_{\frac{N}{2}-1}\left[z^{-\left(\frac{N}{2}-1\right)}X(z) + z^{-\frac{N}{2}}X(z)\right]$$

When N is even, the above equation can be used to construct the direct form structure of linear phase FIR system with minimum number of multipliers, as shown in fig 3.24. From the direct form linear phase structure it is observed that the realization of an N^{th} order FIR discrete time system for even values of N involves N/2 number of multiplications and N-1 number of additions. Also the structure involves N-1 delays and so N-1 memory locations are required to store the delayed signals.

Case ii : When N is odd

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)} = \sum_{m=0}^{N-1} b_m z^{-m}$$

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=\frac{N+1}{2}}^{N-1} b_m z^{-m}$$

Dividing the summation of N terms into two summations with $\frac{N-1}{2}$ terms.

Let, p = N–1–m, $\therefore$ m = N–1–p

When, $m = \frac{N+1}{2}$; $p = N-1-\frac{N+1}{2} = \frac{N-3}{2}$

When, m = N–1 ; p = N–1 – (N–1) = 0

$$\therefore\ \frac{Y(z)}{X(z)} = \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{p=0}^{\frac{N-3}{2}} b_{N-1-p}\, z^{-(N-1-p)}$$

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_{N-1-m}\, z^{-(N-1-m)}$$

Let, p = m

When impulse response is symmetric, $b_m = b_{N-1-m}$

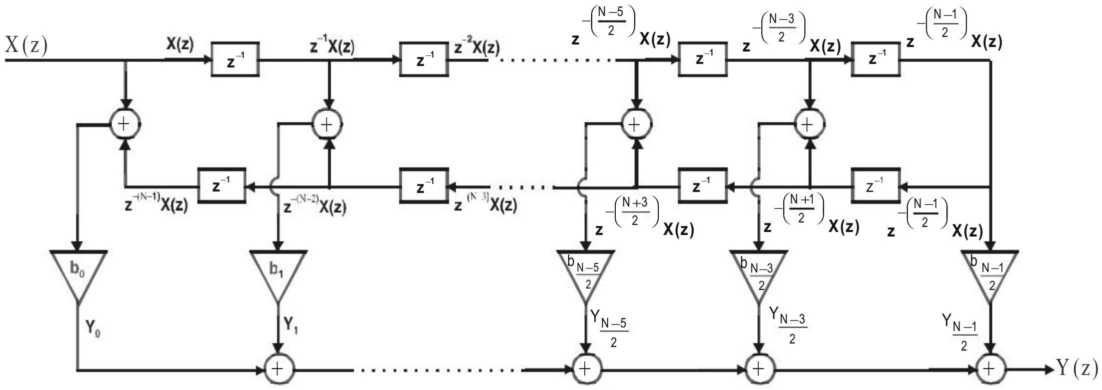
$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_m\, z^{-(N-1-m)}$$

$$= b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_m\left[z^{-m} + z^{-(N-1-m)}\right]$$

$$\therefore\ Y(z) = b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} X(z) + b_0\left[X(z) + z^{-(N-1))}X(z)\right] + b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right] +$$

$$.....\ +\ b_{\frac{N-5}{2}}\left[z^{-\left(\frac{N-5}{2}\right)}X(z) + z^{-\left(\frac{N+3}{2}\right)}X(z)\right] + b_{\frac{N-3}{2}}\left[z^{-\left(\frac{N-3}{2}\right)}X(z) + z^{-\left(\frac{N+1}{2}\right)}X(z)\right]$$

When N is odd, the above equation can be used to construct the direct form structure of linear phase FIR system with minimum number of multipliers, as shown in fig 3.25.



$$Y_0 = b_0\left[X(z) + z^{-(N-1)}X(z)\right]; \quad Y_{\frac{N-5}{2}} = b_{\frac{N-5}{2}}\left[z^{-\left(\frac{N-5}{2}\right)}X(z) + z^{-\left(\frac{N-3}{2}\right)}X(z)\right]; \quad Y_{\frac{N-1}{2}} = b_{\frac{N-1}{2}}\, z^{-\left(\frac{N-1}{2}\right)}X(z)$$

$$Y_1 = b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right]; \quad Y_{\frac{N-3}{2}} = b_{\frac{N-3}{2}}\left[z^{-\left(\frac{N-3}{2}\right)}X(z) + z^{-\left(\frac{N-1}{2}\right)}X(z)\right]$$

Fig 3.25 : *Direct form realization of a linear phase FIR system when N is odd.*

From the direct form linear phase structure it is observed that the realization of an N^{th} order FIR discrete time system for odd values of N involves (N+1)/2 number of multiplications and N-1 number of additions. Also the structure involves N-1 delays and so N-1 memory locations are required to store the delayed signals.

Example 3.34

Draw the direct form structure of the FIR system described by the transfer function,

$$H(z) = 1 + \frac{1}{2}z^{-1} + \frac{3}{8}z^{-2} + \frac{5}{4}z^{-3} + \frac{1}{2}z^{-4} + \frac{7}{8}z^{-5}$$

Solution

$$\text{Let,} \quad H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{2}z^{-1} + \frac{3}{8}z^{-2} + \frac{5}{4}z^{-3} + \frac{1}{2}z^{-4} + \frac{7}{8}z^{-5}$$

$$\therefore \quad Y(z) = X(z) + \frac{1}{2}z^{-1}X(z) + \frac{3}{8}z^{-2}X(z) + \frac{5}{4}z^{-3}X(z) + \frac{1}{2}z^{-4}X(z) + \frac{7}{8}z^{-5}X(z) \quad .....(1)$$

The direct form structure of FIR system can be obtained directly from equation (1).

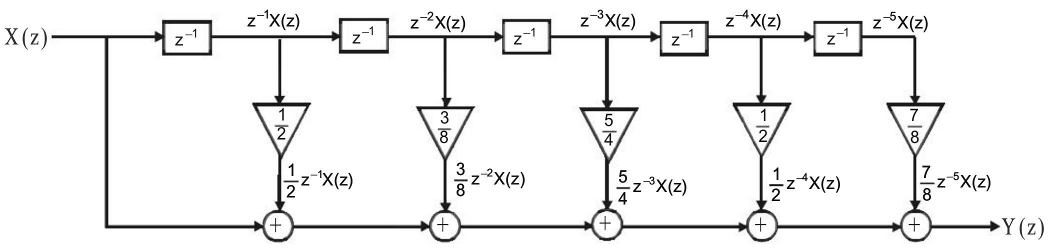


Fig 1 : *Direct form structure of H(z).*

Example 3.35

Realize the following system with minimum number of multipliers.

a) $H(z) = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$

b) $H(z) = \frac{1}{3} + \frac{1}{4}z^{-1} + \frac{3}{2}z^{-2} + \frac{3}{2}z^{-3} + \frac{1}{4}z^{-4} + \frac{1}{3}z^{-5}$

c) $H(z) = \left(\frac{1}{5} + \frac{1}{2}z^{-1} + \frac{1}{5}z^{-2}\right)\left(\frac{1}{7} + \frac{1}{4}z^{-1} + \frac{1}{7}z^{-2}\right)$

Solution

a) Given that, $\mathbf{H(z) = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}}$ ..... (1)

By the definition of Z-transform we get,

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + \ldots.. \qquad ..... (2)$$

On comparing equations (1) and (2) we get,

Impulse response, $h(n) = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right\}$

Here h(n) satisfies the condition h(n) = h(N – 1 – n) and so impulse response is symmetrical. Hence the system has linear phase and can be realized with minimum number of multipliers.

Let, $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$

$$\therefore\ Y(z) = \frac{1}{4}X(z) + \frac{1}{2}z^{-1}X(z) + \frac{3}{4}z^{-2}X(z) + \frac{1}{2}z^{-3}X(z) + \frac{1}{4}z^{-4}X(z)$$

$$= \frac{1}{4}\left[X(z) + z^{-4}X(z)\right] + \frac{1}{2}\left[z^{-1}X(z) + z^{-3}X(z)\right] + \frac{3}{4}z^{-2}X(z) \qquad .....(3)$$

The direct form structure of linear phase FIR system is constructed using equation (3) as shown in fig 1.

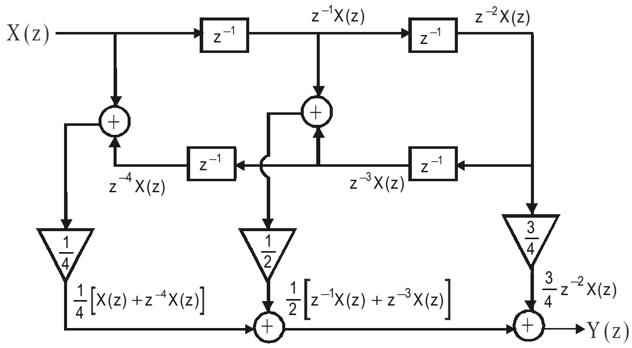


Fig 1 : Linear phase realization of H(z).

b) Given that, $\mathbf{H(z) = \frac{1}{3} + \frac{1}{4}z^{-1} + \frac{3}{2}z^{-2} + \frac{3}{2}z^{-3} + \frac{1}{4}z^{-4} + \frac{1}{3}z^{-5}}$

Let, $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{3} + \frac{1}{4}z^{-1} + \frac{3}{2}z^{-2} + \frac{3}{2}z^{-3} + \frac{1}{4}z^{-4} + \frac{1}{3}z^{-5}$

$$\therefore\ Y(z) = \frac{1}{3}X(z) + \frac{1}{4}z^{-1}X(z) + \frac{3}{2}z^{-2}X(z) + \frac{3}{2}z^{-3}X(z) + \frac{1}{4}z^{-4}X(z) + \frac{1}{3}z^{-5}X(z)$$

$$= \frac{1}{3}\left[X(z) + z^{-5}X(z)\right] + \frac{1}{4}\left[z^{-1}X(z) + z^{-4}X(z)\right] + \frac{3}{2}\left[z^{-2}X(z) + z^{-3}X(z)\right] \qquad .....(4)$$

The direct form realization of H(z) with minimum number of multipliers (i.e., linear phase realization) is obtained using equation (4) as shown in fig 2.

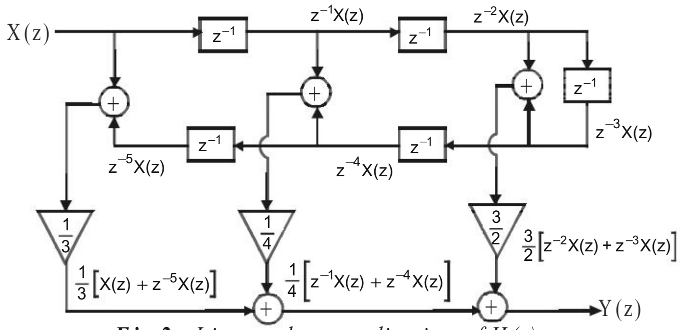


Fig 2 : Linear phase realization of H(z).

c) Given that, $H(z) = \left(\frac{1}{5}+\frac{1}{2}z^{-1}+\frac{1}{5}z^{-2}\right)\left(\frac{1}{7}+\frac{1}{4}z^{-1}+\frac{1}{7}z^{-2}\right)$

The given system can be realized as cascade of two second-order systems. Each system can be realized with minimum number of multipliers.

Let, $H(z) = H_1(z)\,H_2(z)$

where, $H_1(z) = \frac{1}{5}+\frac{1}{2}z^{-1}+\frac{1}{5}z^{-2}$; $H_2(z) = \frac{1}{7}+\frac{1}{4}z^{-1}+\frac{1}{7}z^{-2}$

Let, $H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{5}+\frac{1}{2}z^{-1}+\frac{1}{5}z^{-2}$

$$\therefore\ Y_1(z) = \frac{1}{5}X(z)+\frac{1}{2}z^{-1}X(z)+\frac{1}{5}z^{-2}X(z)$$

$$= \frac{1}{5}\left[X(z)+z^{-2}X(z)\right]+\frac{1}{2}z^{-1}X(z) \quad .....(5)$$

The linear phase realization structure of $H_1(z)$ is obtained using equation (5) as shown in fig 3.

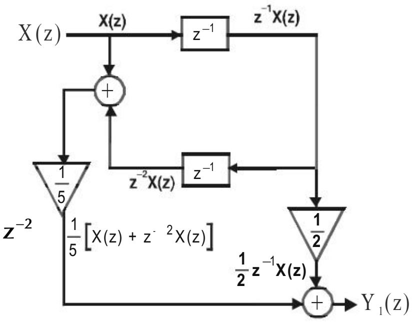


Fig 3 : Linear phase realization of $H_1(z)$.

Let, $H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{7}+\frac{1}{4}z^{-1}+\frac{1}{7}z^{-2}$

$$\therefore\ Y(z) = \frac{1}{7}Y_1(z)+\frac{1}{4}z^{-1}Y_1(z)+\frac{1}{7}z^{-2}Y_1(z)$$

$$= \frac{1}{7}\left[Y_1(z)+z^{-2}Y_1(z)\right]+\frac{1}{4}z^{-1}Y_1(z) \quad .....(6)$$

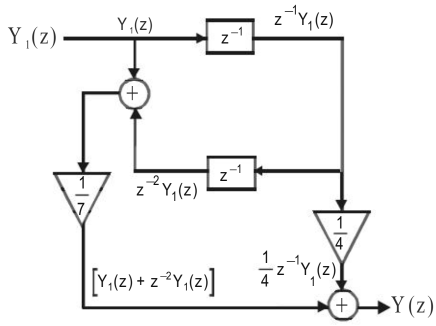


Fig 4 : Linear phase realization of $H_2(z)$.

The linear phase realization structure of $H_2(z)$ is obtained using equation (6) as shown in fig 4.

The linear phase structure of H(z) is obtained by connecting the linear phase realization structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 5.

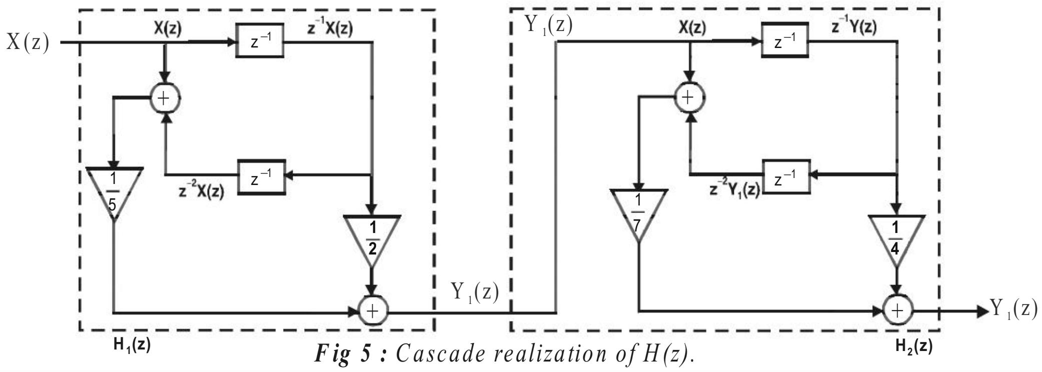


Fig 5 : Cascade realization of H(z).

3.11 Summary of Important Concepts

1. The $\mathcal{Z}$-transform provides a method for analysis of discrete time signals and systems in frequency domain.
2. The ROC of X(z) is a set of all values of z, for which X(z) attains a finite value.
3. Since ROC is a set of values of z, it will be a ring or disk in z-plane, with centre at origin.
4. The zeros are defined as values of z at which the function X(z) becomes zero.
5. The poles are defined as values of z at which the function X(z) becomes infinite.
6. In a realizable system, the number of zeros will be less than or equal to number of poles.
7. If x(n) is finite duration right-sided (causal) signal, then the ROC is entire z-plane except z = 0.
8. If x(n) is finite duration left-sided (anticausal) signal, then the ROC is entire z-plane except $z = \infty$.
9. If x(n) is finite duration two-sided (noncausal) signal, then the ROC is entire z-plane except z = 0 and $z = \infty$.
10. If x(n) is infinite duration right-sided (causal) signal, then the ROC is exterior of a circle of radius r_1.
11. If x(n) is infinite duration left-sided (anticausal) signal, then the ROC is interior of a circle of radius r_2.
12. If x(n) is infinite duration two-sided (noncausal) signal, then the ROC is the region in between two circles of radius r_1 and r_2.
13. If X(z) is rational, [where X(z) is $\mathcal{Z}$-transform of x(n)], then the ROC does not include any poles of X(z).
14. If X(z) is rational, [where X(z) is $\mathcal{Z}$-transform of x(n)], and if x(n) is right-sided, then ROC is exterior of a circle whose radius corresponds to pole with largest magnitude.
15. If X(z) is rational, [where X(z) is $\mathcal{Z}$-transform of x(n)], and if x(n) is left-sided, then ROC is interior of a circle whose radius corresponds to pole with smallest magnitude.
16. If X(z) is rational, [where X(z) is $\mathcal{Z}$-transform of x(n)], and if x(n) is two-sided, then ROC is region in between two circles whose radii corresponds to pole of causal part with largest magnitude and pole of anticausal part with smallest magnitude.
17. The inverse $\mathcal{Z}$-transform is the process of recovering the discrete time signal x(n) from its $\mathcal{Z}$-transform X(z).
18. The transfer function of an LTI discrete time system is defined as the ratio of $\mathcal{Z}$-transform of output and $\mathcal{Z}$-transform of input.
19. The transfer function of an LTI discrete time system is also given by $\mathcal{Z}$-transform of the impulse response.
20. The inverse $\mathcal{Z}$-transform of transfer function is the impulse response of the system.
21. The zero-input response $y_{zi}(n)$ is mainly due to initial output (or initial stored energy) in the system.
22. The zero-state response $y_{zs}(n)$ is the response of the system due to input signal and with zero initial output.
23. The total response y(n) is the response of the system due to input signal and initial output (or initial stored energy).
24. The convolution operation is performed to find the response y(n) of an LTI discrete time system from the input x(n) and impulse response h(n).
25. The deconvolution operation is performed to extract the input x(n) of an LTI system from the response y(n) and impulse response h(n) of the system.
26. A point-s_1 on left half of s-plane (LHP), will map as a point-z_1 inside the unit circle in z-plane.
27. A point-s_1 on imaginary axis of s-plane, will map as a point-z_1 on the unit circle in z-plane.
28. A point-s_1 on the right half of s-plane (RHP), will map as a point-z_1 outside the unit circle in z-plane.
29. The mapping of s-plane to z-plane, using the transformation, $e^{sT} = z$ is not one-to-one.
30. The mapping of frequency of continuous time signal Ω to the frequency of discrete time signal ω is many-to-one.
31. Mathematically, a discrete time system is represented by a difference equation.

32. Physically, a discrete time system is realized or implemented either as a digital hardware or as a software running on a digital hardware.
33. The processing of the discrete time signal by the digital hardware involves mathematical operations like addition, multiplication, and delay.
34. The time taken to process the discrete time signal and the computational complexity, depends on number of calculations involved and the type of arithmetic used for computation.
35. The various structures proposed for IIR and FIR systems, attempt to reduce the computational complexity, errors in computation and the memory requirement of the system.
36. When a discrete time system is designed by considering all the infinite samples of the impulse response, then the system is called IIR (Infinite Impulse Response) system.
37. When a discrete time system is designed by choosing only finite samples (usually N-samples) of the impulse response, then the system is called FIR (Finite Impulse Response) system.
38. The IIR systems are recursive systems, whereas the FIR systems are nonrecursive systems.
39. The direct form-I structure of IIR system offers a direct relation between time domain and z-domain equations.
40. Since separate delays are employed for input and output samples, realizing IIR system using direct form-I structure require more memory.
41. The direct form-I and II structure realization of an N^{th} order IIR discrete time system involves M+N+1 number of multiplications and M+N number of additions.
42. The direct form-I structure realization of an N^{th} order IIR discrete time system involves M+N delays and so M+N memory locations are required to store the delayed signals.
43. In a realizable N^{th} order IIR discrete time system, the direct form-II structure realization involves N delays and so N memory locations are required to store the delayed signals.
44. In canonic structure, the number of delays will be equal to the order of the system.
45. The direct form-II structure of IIR system is canonic whereas the direct form-I structure is noncanonic.
46. In cascade realization of IIR system, the N^{th} order transfer function is divided into first and second-order sections and they are realized in direct form-I or II structure and then connected in cascade.
47. In parallel realization of IIR system, the N^{th} order transfer function is divided into first and second-order sections and they are realized in direct form-I or II structure and then connected in parallel.
48. In cascade and parallel realization of IIR systems, the number of calculations and the memory requirement depends on the realization of individual sections.
49. Direct form structure of FIR system provides a direct relation between time domain and z-domain equations.
50. The realization of an N^{th} order FIR discrete time system using direct form structure and linear phase structure involves N number of multiplications and N-1 number of additions.
51. The realization of an N^{th} order FIR discrete time system using direct form structure involves N-1 delays and so N-1 memory locations are required to store the delayed signals.
52. The condition for symmetry of impulse response of FIR system is, $h(n) = h(N-1-n)$.
53. The linear phase realization is also called realization with minimum number of multipliers.
54. In cascade realization of FIR system, the N^{th} order transfer function is divided into first and second-order sections and they are realized in direct form or linear phase structure and then connected in cascade.
55. The direct form linear phase realization structure of an N^{th} order FIR discrete time system for even values of N involves N/2 number of multiplications, and N-1 number of additions.
56. The direct form linear phase realization structure of an N^{th} order FIR discrete time system for odd values of N involves (N+1)/2 number of multiplications, and N-1 number of additions.

3.12 Short Questions and Answers

Infinite geometric series sum formula,

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C} \quad ; \text{ if, } 0 < |C| < 1$$

Q3.1 ***Find the Ƶ-transform of $a^n u(n)$.***

By the definition of Ƶ-transform,

$$Ƶ\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a\,z^{-1})^n = \frac{1}{1-az^{-1}} = \frac{1}{1-a/z} = \frac{z}{z-a}$$

Q3.2 ***Find the Ƶ-transform of $e^{-anT} u(n)$.***

By the definition of Ƶ-transform,

$$Ƶ\{e^{-anT} u(n)\} = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = \sum_{n=0}^{\infty} (e^{-aT} z^{-1})^n = \frac{1}{1-e^{-aT}\,z^{-1}} = \frac{1}{1-e^{-aT}/z} = \frac{z}{z-e^{-aT}}$$

Q3.3 ***Find the Ƶ-transform of x(n) defined as,***

$x(n) = b^n \quad ; \quad 0 \le n \le N-1$

$\quad = 0 \quad ; \quad$ otherwise

Solution

Finite geometric series sum formula,

$$\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}$$

By the definition of Ƶ-transform,

$$Ƶ\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=0}^{N-1} b^n\, z^{-n}$$

$$= \sum_{n=0}^{N-1} (b\,z^{-1})^n = \frac{1-(b\,z^{-1})^N}{1-b\,z^{-1}} = \frac{1-b^N z^{-N}}{1-b\,z^{-1}} = \frac{z^{-N}\left(z^N - b^N\right)}{z^{-1}(z-b)} = \frac{z^{-N+1}\left(z^N - b^N\right)}{z-b}$$

Q3.4 ***Find the Ƶ-transform of $x(n) = a^{n+1} u(n+1)$.***

Solution

Given that, $x(n) = a^{n+1} u(n+1) = a^{n+1} \quad ; \quad n \ge -1$

By the definition of Ƶ-transform,

$$Ƶ\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=-1}^{+\infty} a^{n+1}\, z^{-n} = a^{n+1}\, z^{-n}\Big|_{n=-1} + \sum_{n=0}^{+\infty} a^{n+1}\, z^{-n} = a^0\, z + \sum_{n=0}^{+\infty} a^n\, a\, z^{-n}$$

$$= z + a \sum_{n=0}^{+\infty} \left(a\, z^{-1}\right)^n = z + a\,\frac{1}{1-a\,z^{-1}} = z + \frac{az}{z-a} = \frac{z(z-a)+az}{z-a} = \frac{z^2}{z-a}$$

Q3.5 ***Determine the inverse Ƶ-transform of $X(z) = \log(1 + az^{-1}) \quad ; \quad |z| > |a|$***

Solution

Given that, $X(z) = \log(1 + az^{-1}) \quad ; \quad |z| > |a|$

Let, $x(n) = Ƶ^{-1}\{X(z)\}$

Since ROC is exterior of a circle of radius "a", the x(n) should be a causal signal.

By differentiation property of Ƶ-transform we get,

$$Ƶ\{n\,x(n)\} = -z\,\frac{d}{dz}\,X(z)$$

$$= -z\,\frac{d}{dz}\left[\log\,(1+a\,z^{-1})\right] = -z\,\frac{1}{1+a\,z^{-1}}(-a\,z^{-2}) = \frac{a\,z^{-1}}{1+a\,z^{-1}}$$

$$= \frac{a\,z^{-1}}{z^{-1}(z+a)} = \frac{a}{z+a} = a\,z^{-1}\,\frac{z}{z-(-a)}$$

If $Ƶ\{x(n)\} = X(z)$ then by shifting property $Ƶ\{x(n-m)\} = z^{-m} X(z)$

$$\therefore \quad n\,x(n) = Ƶ^{-1}\left\{a\,z^{-1}\,\frac{z}{z-(-a)}\right\} = a(-a)^{n-1}\,u(n-1)$$

$$\therefore \quad x(n) = \frac{a}{n}\,(-a)^{n-1}\,u(n-1)$$

Q3.6 ***Determine x(0) if the Ƶ-transform of x(n) is*** $X(z) = \dfrac{2z^2}{(z+3)\,(z-4)}$.

Solution

By initial value theorem of Ƶ-transform,

$$x(0) = \underset{z\to\infty}{\text{Lt}}\; X(z) = \underset{z\to\infty}{\text{Lt}}\; \frac{2z^2}{(z+3)\,(z-4)}$$

$$= \underset{z\to\infty}{\text{Lt}}\; \frac{2z^2}{z^2\left(1+\dfrac{3}{z}\right)\left(1-\dfrac{4}{z}\right)} = \underset{z\to\infty}{\text{Lt}}\; \frac{2}{\left(1+\dfrac{3}{z}\right)\left(1-\dfrac{4}{z}\right)} = \frac{2}{\left(1+\dfrac{3}{\infty}\right)\left(1-\dfrac{4}{\infty}\right)} = \frac{2}{(1+0)\,(1-0)} = 2$$

Q3.7 ***Determine the Ƶ-transform of x(n) = (n − 3) u(n).***

Solution

$$\boxed{Ƶ\{u(n)\} = \frac{z}{z-1}}$$

$$\boxed{Ƶ\{n\,x(n)\} = -z\frac{d}{dz}\,X(z)}$$

$$\boxed{d\frac{u}{v} = v\,du - u\,dv}$$

$$Ƶ\{x(n)\} = Ƶ\{(n-3)\,u(n)\} = Ƶ\{n\,u(n) - 3\,u(n)\}$$

$$= Ƶ\{n\,u(n)\} - 3\,Ƶ\{u(n)\}$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) - 3\,\frac{z}{z-1} = -z\,\frac{z-1-z}{(z-1)^2} - \frac{3z}{z-1}$$

$$= \frac{z}{(z-1)^2} - \frac{3z}{z-1} = \frac{z-3z(z-1)}{(z-1)^2} = \frac{z-3z^2+3z}{(z-1)^2} = \frac{-3z^2+4z}{(z-1)^2} = \frac{z(4-3z)}{(z-1)^2}$$

Q3.8 ***Determine the transfer function of the LTI system defined by the equation,***
y(n) − 0.5 y(n − 1) = x(n) + 0.4 x(n − 1)

Solution

Given that, $y(n) - 0.5\,y(n-1) = x(n) + 0.4\,x(n-1)$

On taking Ƶ-transform we get,

$$Y(z) - 0.5\,z^{-1}Y(z) = X(z) + 0.4\,z^{-1}X(z) \quad\Rightarrow\quad Y(z)\left[1-0.5\,z^{-1}\right] = X(z)\left[1+0.4\,z^{-1}\right]$$

$\therefore$ Transfer function, $\dfrac{Y(z)}{X(z)} = \dfrac{1+0.4\,z^{-1}}{1-0.5\,z^{-1}}$

Q3.9 ***The transfer function of a system is given by, $H(z) = 1 - z^{-1}$. Find the response of the system for any input, x(n).***

Solution

Given that, $H(z) = 1 - z^{-1}$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$\therefore$ Response in z - domain, $\quad Y(z) = H(z)\,X(z) = (1-z^{-1})\,X(z) = X(z) - z^{-1}\,X(z)$

$\therefore$ Response in time domain, $\;y(n) = Ƶ^{-1}\{Y(z)\} = Ƶ^{-1}\{X(z) - z^{-1}\,X(z)\} = x(n) - x(n-1)$

Q3.10 ***An LTI system is governed by equation, y(n) = −2 y(n − 2) − 0.5 y(n − 1) + 3 x(n − 1) + 5 x(n). Determine the transfer function of the system.***

Solution

Given that, $y(n) = -2\,y(n-2) - 0.5\,y(n-1) + 3\,x(n-1) + 5\,x(n)$

On taking Ƶ-transform of above equation we get,

$$Y(z) = -2\,z^{-2}\,Y(z) - 0.5\,z^{-1}\,Y(z) + 3\,z^{-1}\,X(z) + 5\,X(z)$$

$$Y(z) + 2\,z^{-2}\,Y(z) + 0.5\,z^{-1}\,Y(z) = 3\,z^{-1}\,X(z) + 5\,X(z)$$

$$Y(z)\left[1 + 2\,z^{-2} + 0.5\,z^{-1}\right] = \left[3\,z^{-1} + 5\right]X(z)$$

$$\therefore\ \text{Transfer function, } H(z) = \frac{Y(z)}{X(z)} = \frac{3z^{-1} + 5}{1 + 0.5\,z^{-1} + 2\,z^{-2}} = \frac{5\,z^2 + 3\,z}{z^2 + 0.5\,z + 2}$$

Q3.11 ***The transfer function of an LTI system is*** $H(z) = \dfrac{z-1}{(z-2)\,(z+3)}$***. Determine the impulse response.***

Solution

$$H(z) = \frac{z-1}{(z-2)\,(z+3)} = \frac{A}{z-2} + \frac{B}{z+3}$$

$$A = \frac{z-1}{\cancel{(z-2)}\,(z+3)} \times \cancel{(z-2)}\Big|_{z=2} = \frac{z-1}{z+3}\Bigg|_{z=2} = \frac{2-1}{2+3} = \frac{1}{5}$$

$$B = \frac{z-1}{(z-2)\,\cancel{(z+3)}} \times \cancel{(z+3)}\Big|_{z=-3} = \frac{z-1}{z-2}\Bigg|_{z=-3} = \frac{-3-1}{-3-2} = \frac{-4}{-5} = \frac{4}{5}$$

$$\therefore\ H(z) = \frac{1}{5}\,\frac{1}{z-2} + \frac{4}{5}\,\frac{1}{z+3}$$

$$\boxed{\mathcal{Z}\left\{a^n\,u(n)\right\} = \frac{z}{z-a}}$$

$$\boxed{\mathcal{Z}\left\{a^{(n-1)}\,u(n-1)\right\} = z^{-1}\,\frac{z}{z-a}}$$

$$\text{Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{5}\,\frac{1}{z-2} + \frac{4}{5}\,\frac{1}{z+3}\right\}$$

$$= \mathcal{Z}^{-1}\left\{\frac{1}{5}\,z^{-1}\,\frac{z}{z-2} + \frac{4}{5}\,z^{-1}\,\frac{z}{z-(-3)}\right\}$$

$$= \frac{1}{5}\,2^{(n-1)}\,u(n-1) + \frac{4}{5}\,(-3)^{(n-1)}\,u(n-1) = \frac{1}{5}\left[2^{(n-1)} + 4\,(-3)^{(n-1)}\right]u(n-1)$$

Q3.12 ***Determine the response of LTI system governed by the equation, y(n) – 0.5 y(n – 1) = x(n), for input x(n) = 5ⁿ u(n), and initial condition y(–1) = 2.***

Solution

$$\text{Given that, } x(n) = 5^n u(n) \quad ; \qquad \therefore X(z) = \mathcal{Z}\{u(n)\} = \frac{z}{z-5}$$

Given that, $y(n) - 0.5\,y(n-1) = x(n)$,

On taking $\mathcal{Z}$-transform of above equation we get,

$$Y(z) - 0.5\left[z^{-1}Y(z) + y(-1)\right] = X(z)$$

$$Y(z) - 0.5\left[z^{-1}Y(z) + 2\right] = \frac{z}{z-5}$$

$$Y(z) - 0.5\,z^{-1}Y(z) - 1 = \frac{z}{z-5} \quad\Rightarrow\quad Y(z)\left[1 - \frac{0.5}{z}\right] = \frac{z}{z-5} + 1 \quad\Rightarrow\quad Y(z)\left[\frac{z-0.5}{z}\right] = \frac{z+z-5}{z-5}$$

$$\therefore\quad Y(z) = \frac{z(2\,z-5)}{(z-0.5)\,(z-5)} \quad\Rightarrow\quad \frac{Y(z)}{z} = \frac{2\,z-5}{(z-0.5)\,(z-5)}$$

$$\text{Let, } \frac{Y(z)}{z} = \frac{2\,z-5}{(z-0.5)\,(z-5)} = \frac{A}{z-0.5} + \frac{B}{z-5}$$

$$A = \frac{2z-5}{\cancel{(z-0.5)}\,(z-5)} \times \cancel{(z-0.5)}\Big|_{z=0.5} = \frac{2\times 0.5-5}{0.5-5} = \frac{-4}{-4.5} = \frac{40}{45} = \frac{8}{9}$$

$$B = \frac{2z-5}{(z-0.5)\,\cancel{(z-5)}} \times \cancel{(z-5)}\Big|_{z=5} = \frac{2\times 5-5}{5-0.5} = \frac{5}{4.5} = \frac{50}{45} = \frac{10}{9}$$

$$\therefore \frac{Y(z)}{z} = \frac{8}{9}\;\frac{1}{z-0.5} + \frac{10}{9}\;\frac{1}{z-5} \qquad \Rightarrow \qquad Y(z) = \frac{8}{9}\;\frac{z}{z-0.5} + \frac{10}{9}\;\frac{z}{z-5}$$

$$\therefore \text{ Response, } y(n) = Ƶ^{-1}\{Y(z)\} = Ƶ^{-1}\left\{\frac{8}{9}\;\frac{z}{z-0.5} + \frac{10}{9}\;\frac{z}{z-5}\right\}$$

$$\boxed{Ƶ\left\{a^n\,u(n)\right\} = \frac{z}{z-a}}$$

$$= \frac{8}{9}\;0.5^n\;u(n) + \frac{10}{9}\;5^n\;u(n) \;= \left[\frac{8}{9}\;0.5^n + \frac{10}{9}\;5^n\right] u(n)$$

Q3.13 ***A signal x(t) = a^t is sampled at a frequency of 1/T Hz in the range −¥ < t < 0. Find the Ƶ-transform of the sampled version of the signal.***

Solution

Given that, $x(t) = a^t$; −¥ < t < 0

The sampled version of the signal x(nT) is given by, $x(nT) = a^{nT}$; −¥ < nT < 0

Now the Ƶ - transform of x(nT) is,

$$Ƶ\{x(nT)\} = \sum_{n=-\infty}^{+\infty} x(nT)\,z^{-n} = \sum_{n=-\infty}^{0} a^{nT}\;z^{-n} = \sum_{n=0}^{\infty} a^{-nT}\;z^{n}$$

$$= \sum_{n=0}^{\infty}\left(a^{-T}z\right)^n = \frac{1}{1-a^{-T}z} = \frac{1}{1-z/a^T} = \frac{a^T}{a^T-z}$$

Q3.14 ***The transfer function of a system is given by,*** $H(z) = \dfrac{1}{1-0.5\,z^{-1}} + \dfrac{1}{1-2\,z^{-1}}$ ***. Determine the stability and causality of the system for a) ROC : |z| > 2 ; b) ROC : |z| < 0.5.***

Solution

a) ROC is |z| > 2

When ROC is |z| > 2, the impulse response h(n) should be right-sided signal.

$$\therefore \text{ Impulse response, } h(n) = Ƶ^{-1}\{H(z)\} = Ƶ^{-1}\left\{\frac{1}{1-0.5z^{-1}} + \frac{1}{1-2z^{-1}}\right\} = \left(0.5^n + 2^n\right)\,u(n)$$

1. The ROC does not include unit circle. Hence the system is unstable.
2. The impulse response is right-sided signal. Hence the system is causal.

b) ROC is |z| < 0.5

When ROC is |z| < 0.5, the impulse response h(n) should be left-sided signal.

$$\therefore \text{ Impulse response, } h(n) = Ƶ^{-1}\{H(z)\} = Ƶ^{-1}\left\{\frac{1}{1-0.5z^{-1}} + \frac{1}{1-2z^{-1}}\right\} = \left(-0.5^n - 2^n\right)\,u(-n-1)$$

1. The ROC does not include unit circle. Hence the system is unstable.
2. The impulse response is left-sided sequence. Hence the system is anticausal.

Q3.15 ***Determine the stability and causality of the system described by the transfer function,***

$$H(z) = \frac{1}{1-0.25\,z^{-1}} + \frac{1}{1-2\,z^{-1}} \quad \text{for ROC} : 0.25 < |z| < 2.$$

Solution

Given that, ROC is $0.25 < |z| < 2$

When ROC is $0.25 < |z| < 2$, the impulse response h(n) is two-sided signal. Since $|z| > 0.25$, the term with pole $z = 0.25$ corresponds to right-sided signal. Since $|z| < 2$, the term with pole $z = 2$ corresponds to left-sided signal.

$$\therefore \text{ Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{1-0.25\,z^{-1}} + \frac{1}{1-2\,z^{-1}}\right\} = 0.25^n\,u(n) - 2^n\,u(-n-1)$$

1. The ROC includes the unit circle. Hence the system is stable.
2. The impulse response is two-sided noncausal signal. Hence the system is noncausal.

Q3.16 ***Using $\mathcal{Z}$-transform, determine the response of the LTI system with impulse response, $h(n) = \{1, \ -1, \ 1\}$, for an input $x(n) = \{-2, \ 3, \ 1\}$.***

Solution

Given that, $x(n) = \{-2, \ 3, \ 1\}$

$$\therefore X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\,z^{-n} = \sum_{n=0}^{2} x(n)\,z^{-n} = x(0) + x(1)\,z^{-1} + x(2)\,z^{-2} = -2 + 3\,z^{-1} + z^{-2}$$

Given that, $h(n) = \{1, \ -1, \ 1\}$

$$\therefore H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\,z^{-n} = \sum_{n=0}^{2} h(n)\,z^{-n} = h(0) + h(1)\,z^{-1} + h(2)\,z^{-2} = 1 - z^{-1} + z^{-2}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore Y(z) = X(z)\,H(z) = \left(-2 + 3\,z^{-1} + z^{-2}\right) \times \left(1 - z^{-1} + z^{-2}\right)$$

$$= -2 + 2\,z^{-1} - 2\,z^{-2} + 3\,z^{-1} - 3\,z^{-2} + 3\,z^{-3} + z^{-2} - z^{-3} + z^{-4}$$

$$= -2 + 5\,z^{-1} - 4\,z^{-2} + 2\,z^{-3} + z^{-4} \qquad .....(1)$$

By definition of $\mathcal{Z}$-transform,

$$Y(z) = \mathcal{Z}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\,z^{-n}$$

On expanding the above summation we get,

$$Y(z) = ...+ y(0) + y(1)\,z^{-1} + y(2)\,z^{-2} + y(3)\,z^{-3} + y(4)\,z^{-4} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$$y(0) = -2 \quad ; \quad y(1) = 5 \quad ; \quad y(2) = -4 \quad ; \quad y(3) = 2 \quad ; \quad y(4) = 1$$

$\therefore$ Response, $y(n) = \{-2, \ 5, \ -4, \ 2, \ 1\}$

Q3.17 ***Using $\mathcal{Z}$-transform, perform deconvolution of response $y(n) = \{-2, \ 5, \ -4, \ 2, \ 1\}$ and impulse response $h(n) = \{1, \ -1, \ 1\}$, to extract the input x(n).***

Solution

Given that, $y(n) = \{-2, \ 5, \ -4, \ 2, \ 1\}$

$$Y(z) = \mathcal{Z}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\,z^{-n} = \sum_{n=0}^{4} y(n)\,z^{-n}$$

$$= y(0) + y(1)\,z^{-1} + y(2)\,z^{-2} + y(3)\,z^{-3} + y(4)\,z^{-4} = -2 + 5\,z^{-1} - 4z^{-2} + 2\,z^{-3} + z^{-4}$$

Given that, $h(n) = \{1, \ \ -1, \ \ 1\}$

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n} = \sum_{n=0}^{2} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} = 1 - z^{-1} + z^{-2}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore\ X(z) = \frac{Y(z)}{H(z)} = \frac{-2 + 5z^{-1} - 4z^{-2} + 2z^{-3} + z^{-4}}{1 - z^{-1} + z^{-2}}$$

$$= -2 + 3z^{-1} + z^{-2} \qquad .....(1)$$

$$\begin{array}{r|l} & -2 + 3z^{-1} + z^{-2} \\ \hline 1 - z^{-1} + z^{-2} & -2 + 5z^{-1} - 4z^{-2} + 2z^{-3} + z^{-4} \\ & \underset{(+)}{-}\,2 \underset{(-)}{+} 2z^{-1} \underset{(+)}{-} 2z^{-2} \\ \hline & 3z^{-1} - 2z^{-2} + 2z^{-3} \\ & \underset{(-)}{}3z^{-1} \underset{(+)}{-} 3z^{-2} \underset{(-)}{+} 3z^{-3} \\ \hline & z^{-2} - z^{-3} + z^{-4} \\ & \underset{(-)}{}z^{-2} \underset{(+)}{-} z^{-3} \underset{(-)}{+} z^{-4} \\ \hline & 0 \end{array}$$

By definition of $\mathcal{Z}$ - transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

On expanding the above summation we get,

$$X(z) = ...+ x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$$x(0) = -2 \quad ; \quad x(1) = 3 \quad ; \quad x(2) = 1$$

$\therefore$ Input, $x(n) = \{-2, \ \ 3, \ \ 1\}$

Q3.18 ***In an LTI system the impulse response $h(n) = C^n$ for n £ 0. Determine the range of values of C, for which the system is stable.***

Solution

Given that, $h(n) = C^n$ for n £ 0.

$$\therefore\ \sum_{n=-\infty}^{+\infty} h(n) = \sum_{n=-\infty}^{0} C^n = \sum_{n=0}^{\infty} C^{-n} = \sum_{n=0}^{+\infty} (C^{-1})^n$$

$$\text{If,}\quad 0 < \left|C^{-1}\right| < 1, \quad \text{then} \quad \sum_{n=0}^{+\infty} (C^{-1})^n = \frac{1}{1 - C^{-1}}$$

$$\text{If,}\quad \left|C^{-1}\right| > 1, \quad \text{then} \quad \sum_{n=0}^{+\infty} (C^{-1})^n = \infty$$

$$\therefore\ \text{For stability,}\ \left|C^{-1}\right| < 1 \quad \Rightarrow \quad \frac{1}{C} < 1 \quad \Rightarrow \quad C > 1$$

Q3.19 ***Using Z-transform, determine the response of the LTI system with impulse response $h(n) = 0.4^n\, u(n)$, for an input $x(n) = 0.2^n\, u(n)$.***

Solution

Given that, $x(n) = 0.2^n\, u(n)$.

$$\therefore\ X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{0.2^n\ u(n)\} = \frac{z}{z - 0.2}$$

Given that, $h(n) = 0.4^n\ u(n)$

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \mathcal{Z}\{0.4^n\ u(n)\} = \frac{z}{z - 0.4}$$

We know that, $H(z) = \frac{Y(z)}{X(z)}$

$$\therefore Y(z) = X(z)\,H(z) = \frac{z}{z-0.2} \times \frac{z}{z-0.4} = \frac{z^2}{(z-0.2)\,(z-0.4)}$$

Let, $\frac{Y(z)}{z} = \frac{z}{(z-0.2)\,(z-0.4)} = \frac{A}{z-0.2} + \frac{B}{z-0.4}$

$$A = \frac{z}{\cancel{(z-0.2)}\,(z-0.4)} \times \cancel{(z-0.2)}\Big|_{z=0.2} = \frac{0.2}{0.2-0.4} = \frac{0.2}{-0.2} = -1$$

$$B = \frac{z}{(z-0.2)\,\cancel{(z-0.4)}} \times \cancel{(z-0.4)}\Big|_{z=0.4} = \frac{0.4}{0.4-0.2} = \frac{0.4}{0.2} = 2$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{-1}{z-0.2} + \frac{2}{z-0.4} \qquad \Rightarrow \qquad Y(z) = -\frac{z}{z-0.2} + 2\frac{z}{z-0.4}$$

Response, $y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{-\frac{z}{z-0.2} + 2\frac{z}{z-0.4}\right\}$

$$= -(0.2)^n\,u(n) + 2\,(0.4)^n\,u(n) = \left[2\,(0.4)^n - (0.2)^n\right] u(n)$$

Q3.20 ***Using $\mathcal{Z}$-transform perform deconvolution of response, $y(n) = 2\,(0.4)^n\,u(n) - (0.2)^n\,u(n)$ and impulse response, $h(n) = 0.4^n\,u(n)$, to extract the input $x(n)$.***

<u>Solution</u>

Given that, $y(n) = 2\,(0.4)^n\,u(n) - (0.2)^n\,u(n)$

$$\therefore \quad Y(z) = \mathcal{Z}\{y(n)\} = \mathcal{Z}\left\{2\,(0.4)^n u(n) - (0.2)^n\,u(n)\right\}$$

$$= \frac{2z}{z-0.4} - \frac{z}{z-0.2} = \frac{2z(z-0.2) - z(z-0.4)}{(z-0.4)\,(z-0.2)} = \frac{2z^2 - 0.4z - z^2 + 0.4z}{(z-0.4)\,(z-0.2)} = \frac{z^2}{(z-0.4)\,(z-0.2)}$$

Given that, $h(n) = 0.4^n\,u(n)$

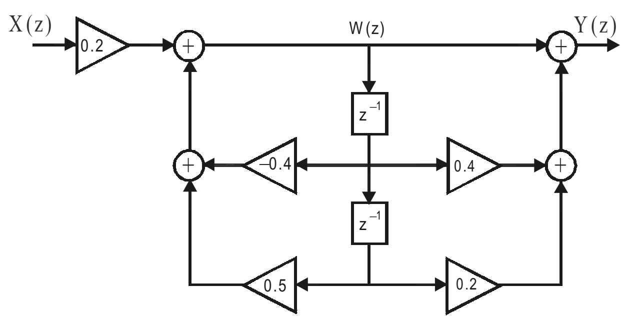
$$\therefore \quad H(z) = \mathcal{Z}\{h(n)\} = \mathcal{Z}\left\{0.4^n u(n)\right\} = \frac{z}{z-0.4}$$

We know that, $H(z) = \frac{Y(z)}{X(z)}$

$$\therefore \quad X(z) = \frac{Y(z)}{H(z)} = Y(z) \times \frac{1}{H(z)} = \frac{z^2}{\cancel{(z-0.4)}\,(z-0.2)} \times \frac{\cancel{z-0.4}}{z} = \frac{z}{z-0.2}$$

$$\therefore \quad \text{Input, } x(n) = \mathcal{Z}^{-1}\{X(z)\} = \mathcal{Z}^{-1}\left\{\frac{z}{z-0.2}\right\} = 0.2^n\,u(n)$$

Q3.21 ***Obtain the transfer function for the following structure.***



Solution

The following z-domain equations can be obtained from the given direct form-II structure.

$$W(z) = -0.4z^{-1}W(z) + 0.5z^{-2}W(z) + 0.2\,X(z)$$

$$\therefore W(z) + 0.4z^{-1}W(z) - 0.5z^{-2}W(z) = 0.2\,X(z) \quad \Rightarrow \quad \frac{W(z)}{X(z)} = \frac{0.2}{1 + 0.4\,z^{-1} - 0.5\,z^{-2}}$$

$$Y(z) = W(z) + 0.4\,z^{-1}W(z) + 0.2\,z^{-2}W(z) \quad \Rightarrow \quad \frac{Y(z)}{W(z)} = 1 + 0.4z^{-1} + 0.2z^{-2}$$

The given direct form-II digital network can be realized by the transfer function,

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.2\,(1 + 0.4z^{-1} + 0.2z^{-2})}{1 + 0.4z^{-1} - 0.5z^{-2}}$$

Q3.22 ***Realize the following FIR system with minimum number of multipliers.***

$h(n) = \{-0.5,\ 0.8,\ -0.5\}$

Solution

Given that, $h(n) = \{-0.5,\ 0.8,\ -0.5\}$

On taking Z- transform,

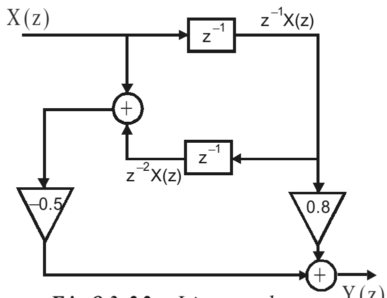
$$H(z) = \sum_{n=0}^{\infty} h(n)\,z^{-n} = \sum_{n=0}^{2} h(n)\,z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} = -0.5 + 0.8\,z^{-1} - 0.5\,z^{-2}$$

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = -0.5 + 0.8\,z^{-1} - 0.5\,z^{-2}$$

$$\therefore Y(z) = -0.5\,X(z) + 0.8\,z^{-1}\,X(z) - 0.5\,z^{-2}\,X(z)$$

$$= -0.5\,[\,X(z) + z^{-2}\,X(z)\,] + 0.8\,z^{-1}\,X(z)$$



FigQ3.22 : *Linear phase realization.*

The linear phase structure is drawn using the above equation as shown in fig Q3.22.

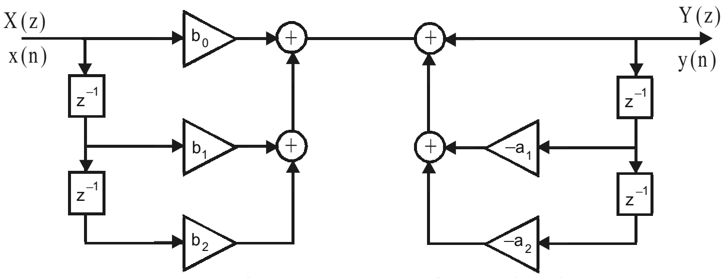
Q3.23 ***The transfer function of an IIR system has 'Z' number of zeros and 'P' number of poles. How many number of additions, multiplications and memory locations are required to realize the system in direct form-I and direct form-II.***

The realization of IIR system with Z zeros and P poles in direct form-I and II structure, involves Z+P number of additions and Z+P+1 number of multiplications. The direct form-I structure requires Z+P memory locations whereas the direct form-II structure requires only P number of memory locations.

Q3.24 ***What are the factors that influence the choice of structure for realization of an LTI system?***

The factors that influence the choice of realization structure are computational complexity, memory requirements, finite word length effects, parallel processing and pipelining of computations.

Q3.25 ***Draw the direct form-I structure of second-order IIR system with equal number of poles and zeros.***



FigQ3.25 : *Direct form-I structure of second-order IIR system.*

Q3.26 ***An LTI system is described by the difference equation, $y(n) = a_1 y(n-1) + x(n) + b_1 x(n-1)$. Realize it in direct form-I structure and convert to direct form-II structure.***

<u>Solution</u>

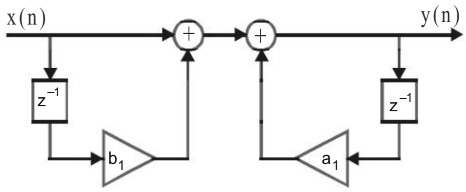
Given that, $y(n) = a_1 y(n-1) + x(n) + b_1 x(n-1)$.

Using the given equation the direct form-I structure is drawn as shown in fig Q3.26a.

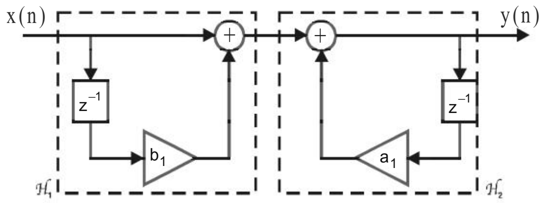
Direct form-I structure can be considered as cascade of two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in fig Q3.26b.

By linearity property, order of cascading can be changed as shown in fig Q3.26c.

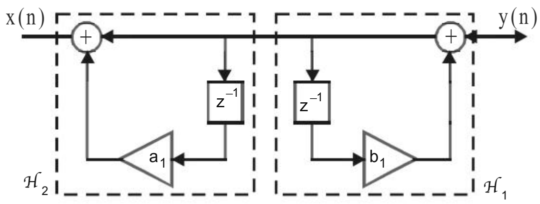
In fig Q3.26c, we can observe that the input to the delay in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same and so the output of delays will be same. Hence the delays can be combined to get direct form-II structure as shown in fig Q3.26d.



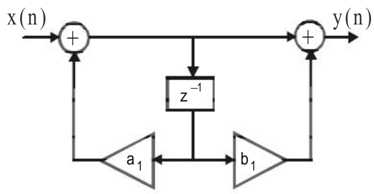
FigQ3.26a : *Direct form-I structure.*



FigQ3.26b : *Direct form-I structure as cascade of two systems .*



FigQ3.26c : *Direct form-I structure after interchanging the order of cascading.*



FigQ3.26d : *Direct form-II structure.*

Q3.27 ***What is the advantage in cascade and parallel realization of IIR systems ?***

In digital implementation of LTI system the coefficients of the difference equation governing the system are quantized. While quantizing the coefficients the value of poles may change. This will end up in a frequency response different to that of desired frequency response.

These effects can be avoided or minimized, if the LTI system is realized in cascade or parallel structure. [i.e, The sensitivity of frequency response characteristics to quantization of the coefficients is minimized]

Q3.28 ***Compare the direct form-I and II structures of an IIR systems, with M zeros and N poles.***

Direct form-I	***Direct form-II***
1. Separate delay for input and output.	1. Same delay for input and output.
2. M + N + 1 multiplications are involved.	2. M + N + 1 multiplications are involved.
3. M + N additions are involved.	3. M + N additions are involved.
4. M + N delays are involved.	4. N delays are involved.
5. M + N memory locations are required.	5. N memory locations are required.
6. Noncanonical structure.	6. Canonical structure.

Q3.29 Compare the direct form and linear phase structures of an N^{th} order FIR system.

Direct form	*Linear phase*
1. Impulse response need not be symmetric.	1. Impulse response should be symmetric.
2. N multiplications are involved.	2. N/2 or (N+1)/2 multiplications are involved.
3. N-1 additions and delays are involved.	3. N-1 additions and delays are involved.
4. N-1 memory locations are required.	4. N-1 memory locations are required.

Q3.30 What is the advantage in linear phase realization of FIR systems ?

The advantage in the linear phase realization structure is that it involves minimum number of multiplications. In linear phase realization of N^{th} order FIR system, the number of multiplications for even values of N will be N/2 and for odd values of N will be (N+1)/2, whereas the direct form realization involves N multiplications.

3.13 MATLAB Programs

Program 3.1

Write a MATLAB program to find one-sided Z-transform of the following standard causal signals.

a) n b) a^n c) na^n d) e^{-anT}

```
%Program to find the Z-transform of some standard signals

  clear all
  syms n T a real;     %Let n, T, a be real variable
  syms z complex;      %Let z be complex variable

 %(a)
  x = n;
  disp('(a) z-transform of "n" is');
  ztrans(x)

 %(b)
  x = a^n;
  disp('(b) z-transform of  "a^n" is');
  ztrans(x)

 %(c)
  x=n*(a^n);
  disp('(c) z-transform of  "n(a^n)" is');
  ztrans(x)

 %(d)
  x=exp(-a*n*T);
  disp('(d) z-transform of "exp(-a*n*T)" is');
  ztrans(x)
```

OUTPUT

```
(a) z-transform of "n" is
      ans =
            z/(z-1)^2
(b) z-transform of  "a^n" is
      ans =
            z/a/(z/a-1)
```

```
    (c) z-transform of  "n(a^n)" is
            ans =
                    z*a/(z-a)^2
    (d) z-transform of "exp(-a*n*T)" is
            ans =
                    z/exp(-a*T)/(z/exp(-a*T)-1)
```

Program 3.2

Write a MATLAB program to find z-transform of the following causal signals.
a) 0.5^n b) $1+n(0.4)^{(n-1)}$

```
%********** program to determine z-transform of given signals
clear all
syms n real; %Let n be real variable

%(a)
x1=0.5^n;
disp('(a) z-transform of "0.5^n" is');
X1=ztrans(x1)

%(b)
x2=1+n*(0.4^(n-1));
disp('(b) z-transform of "1+n*(0.4^(n-1))" is');
X2=ztrans(x2)
```

OUTPUT

```
    (a) z-transform of "0.5^n" is
            X1 =
                  2*z/(2*z-1)
    (b) z-transform of "1+n*(0.4^(n-1))" is
            X2 =
                  z/(z-1)+25*z/(5*z-2)^2
```

Program 3.3

Write a MATLAB program to find inverse z-transform of the following z-domain signals.

a) $1/(1-1.5z^{-1}+0.5z^{-2})$ b) $1/((1+z^{-1})(1-z^{-1})^2)$

```
%***********Program to determine the inverse z-transform
syms n z

X=1/(1-1.5*(z^(-1))+0.5*(z^(-2)));
disp('Inverse z-transform of 1/(1-1.5z^-1+0.5z^-2)is');
x=iztrans(X,z,n);
simplify(x)

X=1/((1+(z^(-1)))*((1-(z^(-1))^2)));
disp('Inverse z-transform of 1/((1+z^-1)*(1-z^-1)^2))is');
x=iztrans(X,z,n);
simplify(x)
```

OUTPUT

```
Inverse z-transform of 1/(1-1.5z^-1+0.5z^-2) is
        ans =
                2-2^(-n)
Inverse z-transform of 1/((1+z^-1)*(1-z^-1)^2)) is
        ans =
                3/4*(-1)^n+1/2*(-1)^n*n+1/4
```

Program 3.4

Write a MATLAB program to perform convolution of signals, $x_1(n) = (0.4)^n u(n)$ and $x_2(n) = (0.5)^n u(n)$, using Z-transform, and then to perform deconvolution using the result of convolution to extract $x_1(n)$ and $x_2(n)$.

```
%*** Program to perform convolution and deconvolution using z-transform

clear all;
syms n z
x1n=0.4^n;
x2n=0.5^n;

X1z=ztrans(x1n);
X2z=ztrans(x2n);
X3z=X1z*X2z;                    %product of z-transform of inputs
con12=iztrans(X3z);
disp('Convolution of x1(n) and x2(n) is');
simplify(con12)                 % convolution output

decon_X1z=X3z/X1z;
decon_x1n=iztrans(decon_X1z);
disp('The signal x1(n) obtained by deconvolution is');
simplify(decon_x1n)

decon_X2z=X3z/X2z;
decon_x2n=iztrans(decon_X2z);
disp('The signal x2(n) obtained by deconvolution is');
simplify(decon_x2n)
```

OUTPUT

```
Convolution of x1(n) and x2(n) is
      ans =
            5*2^(-n)-4*2^n*5^(-n)
The signal x1(n) obtained by deconvolution is
      ans =
           2^(-n)
The signal x2(n) obtained by deconvolution is
      ans =
           2^n*5^(-n)
```

Program 3.5

Write a MATLAB program to find residues and poles of z-domain signal, $(3z^2+2z+1)/(z^2-3z+2)$

```
%*** Program to find partial fraction expansion of rational
% function of z

clear all
H=tf('z');
Ts=0.1;

b=[3 2 1 ];         %Numerator coefficients
a=[1 -3 2];         %Denominator coefficients

disp('The given transfer function is,');
H=tf([b], [a],Ts)

disp('The residues, poles and direct terms of given TF are,');
disp('(r - residue ; p - poles ; k - direct terms)');
[r,p,k]=residue(b,a)

disp('The num. and den. coefficients extracted from r,p,k,');
[b,a]=residue(r,p,k)
```

OUTPUT

```
The given transfer function is,
      Transfer function:
                       3 z^2 + 2 z + 1
                        ---------------
                        z^2 - 3 z + 2
Sampling time: 0.1

The residues, poles and direct terms of given TF are,
(r - residue ; p - poles ; k - direct terms)
      r =
          17
          -6
      p =
           2
           1
      k =
           3
The num. and den. coefficients extracted from r,p,k are,
      b =
            3     2     1
      a =
            1    -3     2
```

Program 3.6

Write a MATLAB program to find poles and zeros of z-domain signal, $(z^2+0.8z+0.8)/(z^2+0.49)$, and sketch the pole zero plot.

```
% Program to determine poles and zeros of rational function of Z and
% to plot the poles and zeros in z-plane

clear all
syms z

num_coeff=[1 0.8 0.8];            %find the factors of z^2+0.8z+0.8
disp('Roots of numerator polynomial z^2+0.8z+0.8 are zeros.');
zeros=roots(num_coeff)

den_coeff=[1 0 0.49];             %find the factors of z^2+0.49
disp('Roots of denominator polynomial z^2+0.49 are poles.');
poles=roots(den_coeff)

H=tf('z');
Ts=0.1;

H=tf([num_coeff],[den_coeff],Ts);
zgrid on;
pzmap(H);                          %Pole-zero plot
```

OUTPUT

```
Roots of numerator polynomial z^2+0.8z+0.8 are zeros.
      zeros =
            -0.4000 + 0.8000i
            -0.4000 - 0.8000i

Roots of denominator polynomial z^2+0.49 are poles.
      poles =
            0 + 0.7000i
            0 - 0.7000i

The pole-zero plot is shown in fig P3.6.
```

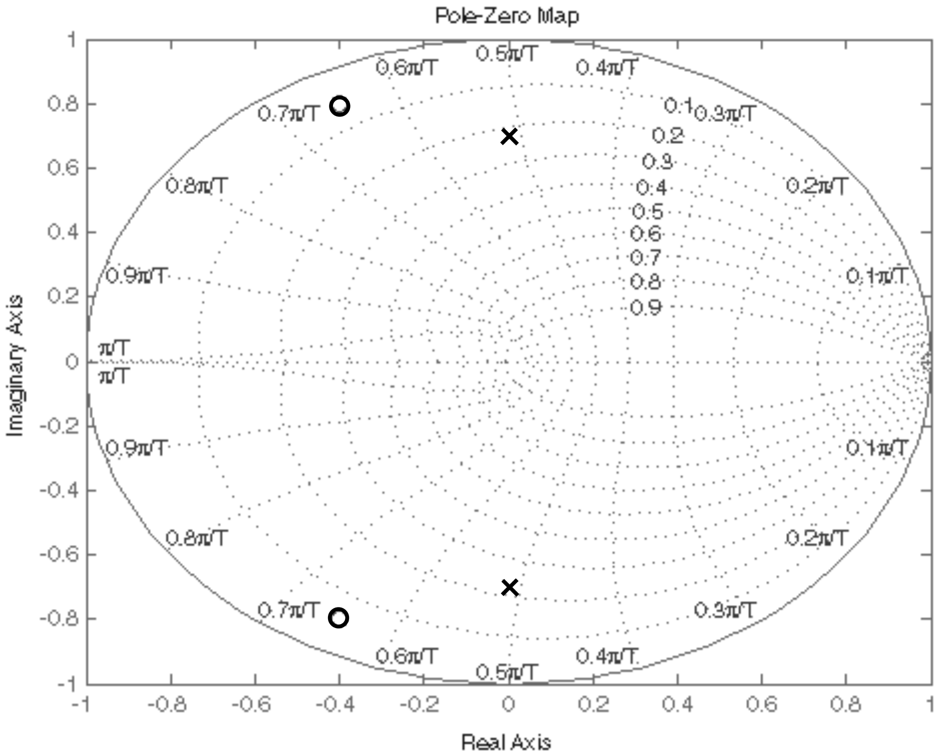


Fig P3.6 *: Pole-Zero plot of program 3.6.*

Program 3.7

Write a MATLAB program to compute and sketch the impulse response of discrete time system governed by transfer function, $H(z)=1/(1-0.8z^{-1}+0.16z^{2})$.

```
%******* Program to find impulse response of a discrete time system
clear all
syms z n

H=1/(1-0.8*(z^(-1))+0.16*(z^(-2)));
disp('Impulse response h(n) is');
h=iztrans(H);           %compute impulse response
simplify(h)

N=15;
b=[0 0 1];              %numerator coefficients
a=[1 -0.8 0.16];        %denominator coefficients
[H,n]=impz(b,a,N);      %compute N samples of impulse response

stem(n,H);              %sketch impulse response
xlabel('n');
ylabel('h(n)');
```

OUTPUT

```
 Impulse response h(n) is
        ans =
              2^n*5^(-n)+2^n*5^(-n)*n
```

The sketch of impulse response is shown in fig P3.3.

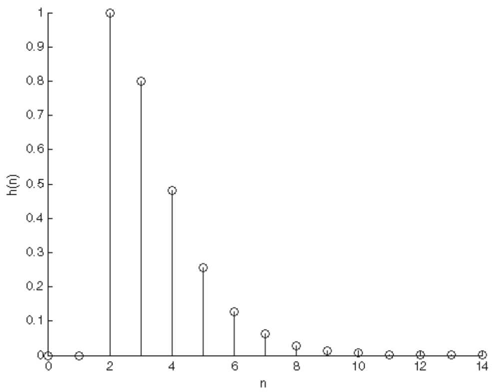


***Fig P3.7** : Impulse response of program 3.7.*

3.14 Exercises

I. Fill in the blanks with appropriate words

1. The ———— of X(z) is the set of all values of z, for which X(z) attains a finite value.
2. The transformation ——— maps the s-plane into z-plane.
3. The ——— of s-plane can be mapped into the ——— of the unit circle in z-plane.
4. The ratio of Z-transform of output to Z-transform of input is called _______ of the system.
5. In the mapping $z = e^{sT}$, the _______ poles of s-plane are mapped into _______ of unit circle in z-plane.
6. In impulse invariant mapping the _______ poles of s-plane are mapped into _______ of unit circle in z-plane.
7. In impulse invariant mapping the poles on the imaginary axis in s-plane are mapped on the _______ in z-plane.
8. In _______ transformation any strip of width 2p/T in s-plane is mapped into the entire z-plane.
9. The phenomena of high frequency components acquiring the identity of low frequency components is called _______.
10. For a causal LTI discrete time system the ROC should be _______ the circle of radius whose value corresponds to pole with _______ magnitude.
11. If X(z) is rational, then the ROC does not include _______ of X(z).
12. The sequences multiplied by u(–n) are _______ and defined for _______.
13. The inverse Z-transform of transfer function is _______ of the system.
14. If Z-transform of x(n) is X(z), then Z-transform of x*(n) is _______.
15. The Z-transform of a shifted signal, shifted by 'q' units of time is obtained by _______ to Z-transform of unshifted signal.
16. In IIR systems, the ___________ structure will give direct relation between time domain and z-domain.
17. When number of delays is equal to order of the system, the structure is called ___________.
18. The direct form realization of IIR system with M zeros and N poles involves ___________ multiplications.
19. The direct form-II realization of N^{th} order IIR system requires ___________ delays and memory locations.
20. The direct form realization of N^{th} order FIR system involves ___________ additions.
21. ___________ realization is called realization with minimum number of multipliers

Answers

1. region of convergence	8. impulse invariant	15. multiplying z^q
2. s = (1/T) *l*n z	9. aliasing	16. direct form-I
3. left half, interior	10. outside, largest	17. canonic structure
4. transfer function	11. poles	18. M + N + 1
5. left half, interior	12. anticausal sequences, $n \le 0$	19. N
6. right half, exterior	13. impulse response	20. N – 1
7. unit circle	14. X*(z*)	21. linear phase

II. State whether the following statements are True/False

1. The Z-transform exists only for those values of z for which X(z) is finite.
2. When the input is an impulse sampled signal, the z-domain transfer function can be directly obtained from s-domain transfer function.
3. The $j\Omega$ axis in s-plane maps into the unit circle of z-plane in the clockwise direction.
4. The left half of s-plane maps into the interior of the unit circle in z-plane.
5. The system is unstable if all the poles of transfer function lies inside the unit circle in z-plane.
6. The Z-transform of impulse response gives the transfer function of LTI system.
7. If X(z) and H(z) are Z-transform of input and impulse response respectively, then the response of LTI system is given by inverse Z-transform of the product X(z) H(z).
8. For a stable LTI continuous time system the poles should lie on the right half of s-plane.
9. For a stable LTI discrete time system the poles should lie on the unit circle.
10. If $\mathcal{Z}\{x(n)\} = X(z)$, then $\mathcal{Z}\left\{n^m x(n)\right\} = -z\left(\frac{d}{dz}\right)^m X(z)$.
11. The direct form-I structure of IIR system employs same delay for input and output samples.
12. In direct form-II realization of IIR system, N memory locations are required to store delayed signals.
13. In parallel or cascade realization, the memory requirement depends on realization of individual sections.
14. Scaling multipliers has to be provided between individual sections of cascade structure.
15. The linear phase realization of N^{th} order FIR system for odd values of N involves N/2 multiplications.
16. For linear phase realization of FIR system, the impulse response should be symmetric.

Answers

1. True	5. False	9. False	13. True
2. True	6. True	10. False	14. True
3. False	7. True	11. False	15. False
4. True	8. False	12. True	16. True

III. Choose the right answer for the following questions

1. ***The impulse response,*** $h(n) = 1 \quad ; \quad n = 0$

$$= -(1-b)\, b^{n-1} \quad ; \quad n \geq 1, \text{ can be represented as,}$$

a) d(n) **b)** $u(n) - (1-b)\, b^{n-1}\, u(n-1)$

c) d(n) $- (1-b)\, b^{n-1}\, u(n-1)$ **d)** $u(n) - (1-b)\, b^{n-1}\, u(n)$

---

2. ***The Z-transform of*** $a^{-n}\, u(-n-1)$ ***is,***

a) $\dfrac{-z}{z - 1/a}$ **b)** $\dfrac{z}{z - 1/a}$ **c)** $\dfrac{z}{z-a}$ **d)** $\dfrac{-z}{z-a}$

---

3. ***The ROC of the sequence*** $x(n) = u(-n)$ ***is,***

a) $|z| > 1$ **b)** $|z| < 1$ **c)** no ROC **d)** $-1 < |z| < 1$

---

4. ***The inverse Z-transform of*** $\dfrac{3}{z-4}$, $|z| > 4$ ***is,***

a) $3(4)^n\, u(n-1)$ **b)** $3(4)^{n-1}\, u(n)$ **c)** $3(4)^{n-1}\, u(n+1)$ **d)** $3(4)^{n-1}\, u(n-1)$

---

5. ***ROC of x(n) contains,***

a) poles **b)** zeros **c)** no poles **d)** no zeros

---

6. ***The inverse Z-transform of*** $X(z) = e^{a/z}$, $|z| > 0$ ***is,***

a) $x(n) = \dfrac{-a^n}{n!}\, u(n)$ **b)** $x(n) = \dfrac{a^n}{n!}\, u(n)$ **c)** $x(n) = \dfrac{a^{n-1}}{n!}\, u(n-1)$ **d)** none of the above

---

7. ***The Z-transform of*** $x(n) = [u(n) - u(n-3)]$, ***for ROC*** $|z| > 1$ ***is,***

a) $X(z) = \dfrac{z - z^{-3}}{z-1}$ **b)** $X(z) = \dfrac{z^{-2}}{(z-1)^2}$ **c)** $X(z) = \dfrac{z - 4z^{-2} + 3z^{-3}}{(z-1)^2}$ **d)** $X(z) = \dfrac{z - z^{-2}}{z-1}$

---

8. ***The system function*** $H(z) = \dfrac{z^3 - 2z^2 + z}{z^2 + 0.25z + 0.125}$ ***is,***

a) causal **b)** noncausal **c)** unstable but causal **d)** cannot be defined

---

9. ***If all the poles of the system function H(z) have magnitude smaller than 1, then the system will be,***

a) stable **b)** unstable **c)** BIBO stable **d)** a and c

---

10. ***If*** $x(n) = \{0.5, \ -0.25, \ 1\}$, ***then Z-transform of the signal is,***

a) $\dfrac{z^2}{0.5z^2 - 0.25z + 1}$ **b)** $\dfrac{z^2}{z^2 - 0.5z + 0.25}$ **c)** $\dfrac{0.5z^2 - 0.25z + 1}{z^2}$ **d)** $\dfrac{2z^2 + 4z + 1}{z^2}$

---

11. ***The ROC of the signal*** $x(n) = a^n$ ***for*** $-5 < n < 5$ ***is,***

a) entire z-plane **b)** entire z-plane except z = 0 and z = ¥

c) entire z-plane except z = 0 **d)** entire z-plane except z = ¥

---

12. ***If Z-transform of x(n) is X(z) then Z-transform of x(–n) is,***

a) $-X(z)$ **b)** $X(-z)$ **c)** $-X(z^{-1})$ **d)** $X(z^{-1})$

---

13. The inverse Z-transform of X(z) can be defined as,

a) $x(n) = \frac{1}{2\pi}\oint_c X(z)\, z^{n-1}\, dz$ **b)** $x(n) = \frac{1}{2j}\oint_c X(z)\, z^{n-1}\, dz$

c) $x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz$ **d)** $x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{-n}\, dz$

14. The Z-transform is a,

a) finite series **b)** infinite power series **c)** geometric series **d)** both a and c

15. If the Z-transform of x(n) is X(z), then Z-transform of $(0.5)^n$ x(n) is,

a) X(0.5 z) **b)** $X(0.5^{-1} z)$ **c)** $X(2^{-1} z)$ **d)** X(2z)

16. The Z-transform of correlation of the sequences x(n) and y(n) is,

a) $X^*(z)\, Y^*(z^{-1})$ **b)** $X(z)\, Y(z^{-1})$ **c)** $X(z) * Y(z)$ **d)** $X(z^{-1})\, Y(z^{-1})$

17. The parseval's relation states that if Z{x_1 (n)} = $X_1(z)$ and Z{x_2 (n)} = $X_2(z)$ then $\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)$ is,

a) $\frac{1}{2\pi}\oint_c X_1(z)\; X_2^*\left(\frac{1}{z}\right) z^{-1}\, dz$ **b)** $\frac{1}{2\pi}\oint_c X_1(z)\; X_2\left(\frac{1}{z^*}\right) z^{-1}\, dz$

c) $\frac{1}{2\pi j}\oint_c X_1(z)\; X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz$ **d)** $\frac{1}{2\pi}\oint_c X_1(z)\; X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz$

18. For a stable LTI discrete time system poles should lie ——— and unit circle should be ———— .

a) outside unit circle, included in ROC **b)** inside unit circle, outside of ROC
c) inside unit circle, included in ROC **d)** outside unit circle, outside of ROC

19. An LTI system with impulse response, h(n) = $(-a)^n$ u(n) and –a < –1 will be,

a) stable system **b)** unstable system
c) anticausal system **d)** neither stable nor causal

20. If X(z) has a single pole on the unit circle, on negative real axis then, x(n) is,

a) signed constant sequence **b)** signed decaying sequence
c) signed growing sequence **d)** constant sequence

21. The Z-transform of x(n) = $-na^n$ u(–n – 1) is,

a) $X(z) = \frac{a\,z}{(z-a)^2}$ **b)** $\frac{a\,z(z+a)}{(z-a)^3}$ **c)** $X(z) = \frac{a\,z^{-1}}{(1-a\,z^{-1})^2}$ **d)** both a and c

22. The ROC for $x(n) \overset{Z}{\underset{Z^{-1}}{\longleftrightarrow}} X(z)$ is R_1, then ROC of $a^n x(n) \overset{Z}{\underset{Z^{-1}}{\longleftrightarrow}} X\left(\frac{z}{a}\right)$ is,

a) $\frac{R_1}{a}$ **b)** aR_1 **c)** R_1 **d)** $\frac{1}{R_1}$

23. The Z-transform of a ramp function x(n) = n u(n) is,

a) $X(z) = \frac{z}{(z-1)^2}$; ROC is $|z| > 1$ **b)** $X(z) = \frac{-z}{(z-1)^2}$; ROC is $|z| > 1$

c) $X(z) = \frac{z}{(z-1)^2}$; ROC is $|z| < 1$ **d)** $X(z) = \frac{-z}{(z-1)^2}$; ROC is $|z| < 1$

24. By impulse invariant transformation, if x(nT) is sampled version of x(t), then Z{x(nT)} is,

a) $\mathcal{L}\{x(nT)\}\big|_{z=e^{sT}}$ **b)** $\mathcal{L}^{-1}\{x(nT)\}\big|_{z=e^{-sT}}$ **c)** $\mathcal{L}\{x(nT)\}\big|_{z=e^{-sT}}$ **d)** $\mathcal{L}^{-1}\{x(nT)\}\big|_{z=e^{sT}}$

25. The Z-transform of $x(n)=\left[\sin\frac{\pi}{2}n\right]u(n)$ ***is,***

a) $\frac{z}{z+1}$ **b)** $\frac{z^2}{z^2+1}$ **c)** $\frac{1}{z+1}$ **d)** $\frac{z}{z^2+1}$

26. The factor that influence the choice of realization of structure is,

a) memory requirements **b)** computational complexity
c) parallel processing and pipelining **d)** all the above

27. The structure that uses separate delays for input and output samples is,

a) direct form-II **b)** direct form-I
c) cascade form **d)** parallel form

28. The linear phase realization structure is used to represent,

a) FIR systems **b)** IIR systems
c) both FIR and IIR systems **d)** all discrete time systems

29. The effect of quantization of coefficients on the frequency response is minimized in,

a) cascade realization **b)** parallel realization
c) direct form structure **d)** both a and b

30. The direct form-I and II structures of IIR system will be identical in,

a) all pole system **b)** all zero system
c) both a and b **d)** first-order and second-order systems

31. The condition for symmetry of impulse response of FIR system is,

a) h(n) = h(N–1) **b)** h(n) = h(N+1)
c) h(n) = h(N–n) **d)** h(n) = h(N–1–n)

32. The linear phase realization is used in FIR systems in order to minimize,

a) multipliers **b)** memory **c)** delays **d)** adders

33. Which one of the following FIR system has linear phase response?

a) y(n) = 0.4 x(n) + 0.1 x(n–1) + 0.5 x(n–2) **b)** y(n) = 0.3 x(n) + x(n–1) + 3.0 x(n–2)
c) y(n) = 0.5 x(n) + 0.7 x(n–1) **d)** y(n) = 0.6x(n) + 0.6 x(n–1)

34. The quantization error increases, when the order of the system 'N' increases in case of,

a) direct form realization **b)** cascade or parallel form realization
c) all IIR systems **d)** all FIR systems

35. The number of memory locations required to realize the system, $H(z)=\frac{1+z^{-2}+2z^{-3}}{1+z^{-2}+z^{-4}}$ ***is,***

a) 8 **b)** 7 **c)** 2 **d)** 10

36. Number of multipliers and adders required for direct form realization of N^{th} order FIR system are,

a) N, N+1 **b)** N, N–1 **c)** N+1, N **d)** N–1, N+1

37. The realization of linear phase FIR system for odd values of 'N' needs,

a) $\frac{N}{2}$ multipliers **b)** $\frac{N+1}{2}$ multipliers **c)** N–1 multipliers **d)** N multipliers

Answers

1. c	7. d	13. c	19. a	25. d	31. d	37. b
2. a	8. b	14. b	20. a	26. d	32. a	
3. b	9. a	15. b	21. d	27. b	33. d	
4. d	10. c	16. b	22. a	28. a	34. a	
5. c	11. b	17. c	23. a	29. d	35. b	
6. b	12. d	18. c	24. a	30. c	36. b	

IV. Answer the following questions

1. Define one-sided and two-sided Z-transform.
2. What is region of convergence (ROC)?
3. State the final value theorem with regard to Z-transform.
4. State the initial value theorem with regard to Z-transform.
5. Define Z-transform of unit step signal.
6. What are the different methods available for inverse Z-transform?
7. When the z-domain transfer function of the system can be directly obtained from s-domain transfer function?
8. Define the transfer function of an LTI system.
9. Write the transfer function of N^{th} order LTI system.
10. What is impulse invariant transformation?
11. How is a point in s-plane mapped to z-plane in impulse invariant transformation?
12. Why is an impulse invariant transformation not considered to be one-to-one?
13. Give the importance of convolution and deconvolution operations using Z-transform.
14. Give the conditions for stability of an LTI discrete time system in z-plane.
15. Explain when an LTI dicrete time system will be causal.
16. Define ROC for various finite and infinite discrete time signals.
17. Explain the shifting property of a discrete time signal defined in the range 0 < n < ¥ with an example.
18. What are all the properties of ROC of a rational function of z?
19. State and prove the convolution property of Z-transform.
20. State and prove the linearity property of Z-transform.
21. What are the various issues that are addressed by realization structures?
22. What are the basic elements used to construct the realization structures of discrete time system?
23. List the different types of structures for realization of IIR systems.

24. Draw the direct form-I structure of an N^{th} order IIR system with equal number of poles and zeros.
25. Draw the direct form-II structure of an N^{th} order IIR system with equal number of poles and zeros.
26. Explain the conversion of direct form-I structure to direct form-II structure with an example.
27. What are the difficulties in cascade realization?
28. Explain the realization of cascade structure of an IIR system.
29. Explain the realization of parallel structure of an IIR system.
30. What are the different types of structure for realization of FIR systems?
31. Draw the direct form structure of an N^{th} order FIR system.
32. What is the necessary condition for Linear phase realization of FIR system?
33. Draw the linear phase realization structure of an Nth order FIR system when 'N' is even.
34. Draw the linear phase realization structure of an Nth order FIR system when 'N' is odd.
35. Explain the realization of cascade structure of a FIR system.

V. Solve the following problems

E3.1 ***Determine the Z-transform and their ROC of the following discrete time signals.***

a) $x(n) = \{\underset{\uparrow}{4}, \ 2, \ 8, \ 5\}$ **b)** $x(n) = \{3, \ 0, \ 0, \ 4, \ 45, \ \underset{\uparrow}{1}\}$

c) $x(n) = \{2, \ 1, \ 1, \ \underset{\uparrow}{2}, \ 5, \ 8, \ 2\}$ **d)** $x(n) = -0.2^n \ u(n-1)$

e) $x(n) = (0.6)^n \ u(n) + (0.7)^n \ u(-n-1)$ **f)** $x(n) = (0.9)^{|n|}$

E3.2 ***Find the one-sided Z-transform of the following discrete time signals.***

a) $x(n) = n^2 \ 5^n \ u(n)$ **b)** $x(n) = n(0.5)^{n+4}$ **c)** $x(n) = (0.5)^{n-2} \left[u(n) - u(n-2)\right]$

E3.3 ***Find the one-sided Z-transform of the discrete signals generated by mathematically sampling the following continuous time signals.***

a) $x(t) = 4\,t\,e^{-0.6t} \ u(t)$ **b)** $x(t) = 2\,t^3 \ u(t)$

E3.4 ***Find the time domain initial value x(0) and final value x(¥) of the following z-domain functions.***

a) $X(z) = \dfrac{0.5}{\left(1 - z^{-1}\right)^2 \left(1 + z^{-1}\right)}$ **b)** $X(z) = \dfrac{z^3}{(z-1)\left(z^2 - 0.2\right)}$

E3.5 ***Determine the inverse Z-transform of the following functions using contour integral method.***

a) $X(z) = \dfrac{(2z-1)z}{(z-1)^2}$ **b)** $X(z) = \dfrac{z^2 + z}{(z-2)^2}$ **c)** $X(z) = \dfrac{(1 - e^{-a})\,z}{(z-1)\,(z - e^{-a})}$

E3.6 ***Determine the inverse Z-transform of the following functions by partial fraction method.***

a) $X(z) = \dfrac{z^2}{(z+1)\,(z+2)^2}$ **b)** $X(z) = \dfrac{2z^2 - z}{z^3 - 5z^2 + 8z - 4}$ **c)** $X(z) = \dfrac{z(z^2+3)}{(z^2+1)^2}$

E3.7 ***Determine the inverse Z-transform of the function,*** $X(z) = \dfrac{2 - z^{-1}}{\left[1 - \frac{1}{4} z^{-1}\right]\left[1 - \frac{1}{3} z^{-1}\right]}$

a) ROC : $|z| > \dfrac{1}{3}$, **b)** ROC : $|z| < \dfrac{1}{4}$, **c)** ROC : $\dfrac{1}{4} < |z| < \dfrac{1}{3}$.

E3.8 ***Determine the inverse Z-transform of the following function using power series method.***

$$X(z)=\frac{z}{2z^2-3z+1}$$

a) ROC : $|z|<0.5$, **b)** ROC : $|z|>1$

E3.9 ***Determine the inverse Z-transform for the following functions using power series method.***

a) $X(z)=\dfrac{z^2+z}{z^2-2z+1}$; ROC : $|z|>1$ **b)** $X(z)=\dfrac{1-\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}$; ROC : $|z|>\dfrac{1}{3}$

E3.10 ***Determine the transfer function and impulse response for the systems described by the following equations.***

a) $y(n)+2y(n-1)-3y(n-2)=x(n-1)$ **b)** $y(n)-\dfrac{7}{4}y(n-1)+\dfrac{5}{8}y(n-2)=2x(n)$

c) $y(n)=0.2x(n)-5x(n-1)+0.6y(n-1)-0.08y(n-2)$

d) $y(n)\ -\dfrac{3}{2}y(n-1)=x(n)+\dfrac{2}{3}x(n-1)$

E3.11 ***A discrete time LTI system is characterized by the transfer function,*** $H(z)=\dfrac{z\,(6\,z-8)}{\left(z-\frac{1}{2}\right)(z-3)}$.

Specify the ROC of H(z) and determine h(n) for the system to be, (i) stable, (ii) causal.

E3.12 ***Determine the unit step response of the discrete time LTI system, whose input and output relation is described by the difference equation, y(n) + 7 y(n–1) = x(n), where the initial condition is, y(–1) = 1.***

E3.13 ***Determine the response of discrete time LTI system governed by the following difference equation, 4 y(n) + 5 y(n –1) + y(n – 2) = x(n) ; with initial conditions, y(–2) = –2 and y(–1) = 1, for the input*** $x(n)=(0.5)^n\,u(n)$.

E3.14 ***An LTI system has the impulse response h(n) defined by*** $h(n)=x_1(n-1)*x_2(n)$ ***. The Z-transform of the two signals*** $x_1(n)$ ***and*** $x_2(n)$ ***are*** $X_1(z)=2-4z^{-1}$ ***and*** $X_2(z)=1+5z^{-2}$ ***respectively. Determine the output of the system for the input*** $\delta(n-1)$.

E3.15 ***Obtain the direct form-I, direct form-II, cascade and parallel form realizations of the LTI system governed by the equation,***

$$y(n)=-\frac{3}{4}\,y(n-1)+\frac{1}{2}\,y(n-2)+\frac{1}{4}\,y(n-3)+\ x(n)+4x(n-1)+3x(n-2)$$

E3.16 ***Realize the direct form-I, II structures of the IIR system represented by the transfer function,***

$$H(z)=\frac{(z+5)}{(z+0.4)\,(z+0.5)\,(z+0.6)}$$

E3.17 ***Determine the direct form-I, II, cascade and parallel realization of the following LTI system.***

$$H(z)=\frac{(z^3-8z^2+13z-5)}{(z-0.75)\,(z^2+z-0.25)}$$

E3.18 ***Realize the cascade and parallel structures of the system governed by the difference equation,***

$$y(n) - \frac{3}{10}y(n-1) - \frac{1}{10}y(n-2) = x(n) + \frac{1}{9}x(n-1)$$

E3.19 ***Draw the direct form structure of the FIR systems described by the following equations,***

a) $y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2) + \frac{1}{6}x(n-3) + \frac{1}{8}x(n-4)$

b) $y(n) = 0.2\,x(n) + 0.25\,x(n-1) + 0.3x(n-2) - 0.35x(n-3)$
$\quad - 0.4\,x(n-4) - 0.45x(n-5) - 0.5x(n-6)$

E3.20 ***Realize the following FIR systems with minimum number of multipliers.***

a) $H(z) = 0.2 + 0.4z^{-1} + 0.6z^{-2} + 0.4z^{-3} + 0.2z^{-4}$

b) $H(z) = \left(0.3 + \frac{1}{9}z^{-1} + 0.3\,z^{-2}\right)\left(0.5 - \frac{1}{7}z^{-1} + 0.5z^{-2}\right)$

c) $y(n) = -\frac{1}{8}x(n) + \frac{3}{4}x(n-1) + \frac{3}{2}x(n-2) + \frac{3}{4}x(n-3) - \frac{1}{8}x(n-4)$

Answers

E3.1

a) $X(z) = 4 + \frac{2}{z} + \frac{8}{z^2} + \frac{5}{z^3}$
ROC is entire z - plane except at $z = 0$.

b) $X(z) = 3z^5 + 4z^2 + 45z + 1$
ROC is entire z - plane except at $z = \infty$.

c) $X(z) = 2z^3 + 1z^2 + z + 2 + 5z^{-1} + 8z^{-2} + 2z^{-3}$
ROC is entire z - plane except at $z = 0$ and $z = \infty$.

d) $X(z) = \frac{-0.2}{z-0.2}$; ROC is exterior of the circle of radius 0.2 in z - plane.

e) $X(z) = \frac{-0.1z}{(z-0.6)\,(z-0.7)}$; ROC is $0.6 < |z| < 0.7$

f) $X(z) = \frac{-0.21z}{(z-0.9)\,(z-1.11)}$; ROC is $0.9 < |z| < 1.11$

E3.2

a) $X(z) = \frac{5z(z+5)}{(z-5)^3}$ **b)** $X(z) = \frac{0.5^5\,z}{(z-0.5)^2}$ **c)** $X(z) = \frac{4z^2-1}{z(z-0.5)}$

E3.3

a) $X(z) = \frac{4zT\,e^{-0.6T}}{(z-e^{-0.6T})^2}$ **b)** $X(z) = \frac{2T^3 z\left(z^2+4z+1\right)}{(z-1)^4}$

E3.4

a) Initial value, $x(0) = 0.5$
Final value, $x(\infty) = \infty$

b) Initial value, $x(0) = 1$
Final value, $x(\infty) = 1.25$

E3.5

a) $x(n) = [n+2]\,u(n)$ **b)** $x(n) = (n+1)\,2^n\,u(n) + n\,2^{(n-1)}\,u(n-1)$

c) $x(n) = \left(1 - e^{-an}\right)u(n)$

E3.6

a) $x(n) = \left[(-2)^n - (-1)^n - n(-2)^n\right]u(n)$ **b)** $x(n) = \left[1 + (1.5n-1)2^n\right]u(n)$

c) $x(n) = \left[\left[j(-j)^n - j^n\right] + \frac{n}{2j}\left[(-j)^n - j^n\right]\right]u(n)$

E3.7

a) $x(n) = \left[6\left(\frac{1}{4}\right)^n - 4\left(\frac{1}{3}\right)^n\right] u(n)$

b) $x(n) = \left[-6\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{3}\right)^n\right] u(-n-1)$

c) $x(n) = 6\left(\frac{1}{4}\right)^n u(n) + 4\left(\frac{1}{3}\right)^n u(-n-1)$

E3.8

a) $x(n)\{.....31,\ 15,\ 7,\ 3,\ \underset{\uparrow}{1}\}$

b) $x(n) = \left\{\underset{\uparrow}{0},\ \frac{1}{2},\ \frac{3}{4},\ \frac{7}{8},\ \frac{15}{16},\ \frac{31}{32}, ...\right\}$

E3.9

a) $\{\underset{\uparrow}{1},\ 3,\ 5,\ 7, ...\}$

b) $x(n) = \left\{\underset{\uparrow}{1},\ -\frac{2}{3},\ \frac{2}{9},\ \frac{-2}{27},\ \frac{2}{81},\ ......\right\}$

E3.10

a) $H(z) = \dfrac{z}{z^2 + 2z - 3}$; $h(n) = \frac{1}{4}\left[1 - (-3)^n\right] u(n)$

b) $H(z) = \dfrac{2z^2}{z^2 - \frac{7}{4}z + \frac{5}{8}}$; $h(n) = \frac{1}{3}\left[-4\left(\frac{1}{2}\right)^n + 10\left(\frac{5}{4}\right)^n\right] u(n)$

c) $H(z) = \dfrac{0.2z^2 - 5z}{z^2 - 0.6z + 0.08}$; $h(n) = \left[24.8\ (0.2)^n - 24.6\ (0.4)^n\right] u(n)$

d) $H(z) = \dfrac{1 + \frac{2}{3}z^{-1}}{1 - \frac{3}{2}z^{-1}}$; $h(n) = \left(\frac{3}{2}\right)^n u(n) + \frac{2}{3}\left(\frac{3}{2}\right)^{n-1} u(n-1)$

E3.11

i) Stable system

ROC : $0.5 < |z| < 3$; $h(n) = 2\left(\frac{1}{2}\right)^n u(n) - 4(3)^n u(-n-1)$

ii) Causal system

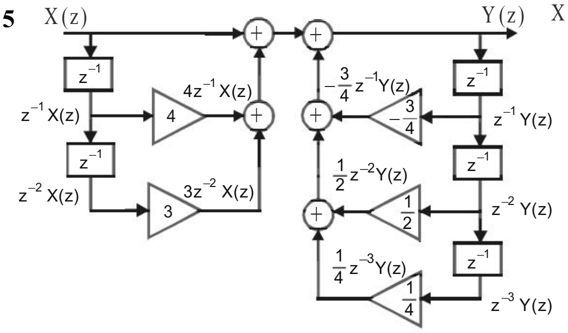
ROC : $|z| > 3$; $h(n) = 2\left(\frac{1}{2}\right)^n u(n) + 4(3)^n u(n)$

E3.12 $y(n) = \frac{1}{8}\left[1 - 49(-7)^n\right] u(n)$

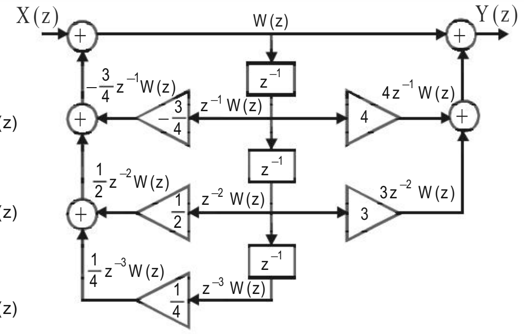
E3.13 $y(n) = \left[0.056(0.5)^n - 0.444(-1)^n - 0.111(-0.25)^n\right] u(n)$

E3.14 $y_1(n) = \{\underset{\uparrow}{0},\ 0,\ 2,\ -4,\ 10,\ -20\}$

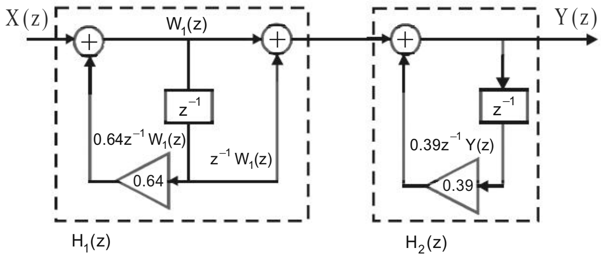
E3.15



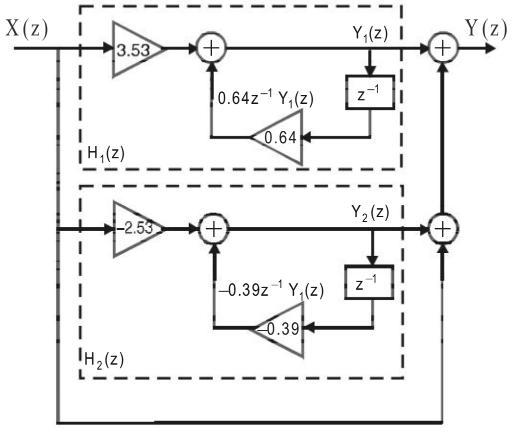
FigE3.15.1 : Direct form-I structure.



FigE3.15.2 : Direct form-II structure.

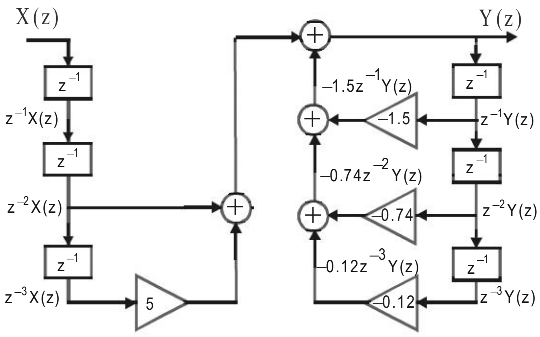


FigE3.15.3 : Cascade structure.

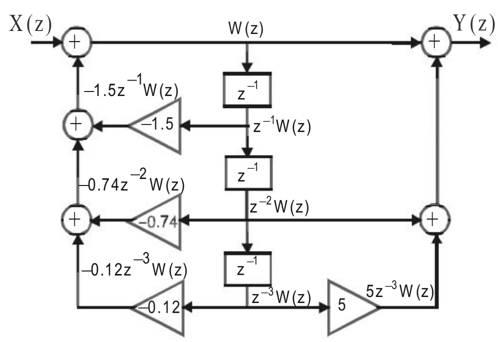


FigE3.15.4 : Parallel structure.

E3.16

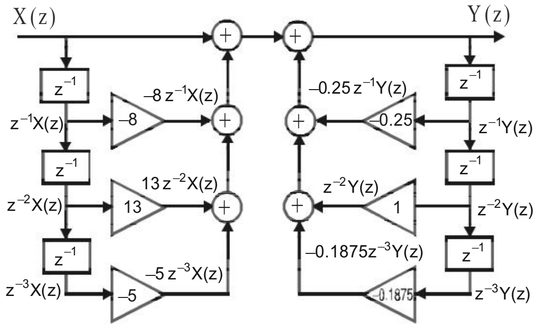


FigE3.16.1 : Direct form-I structure.

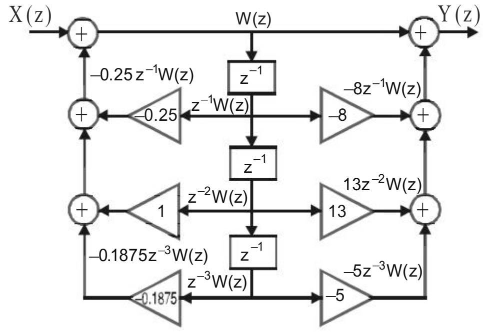


FigE3.16.2 : Direct form-II structure.

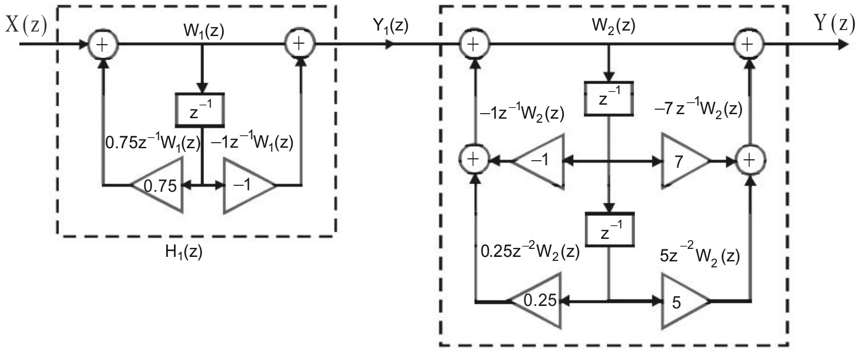
E3.17



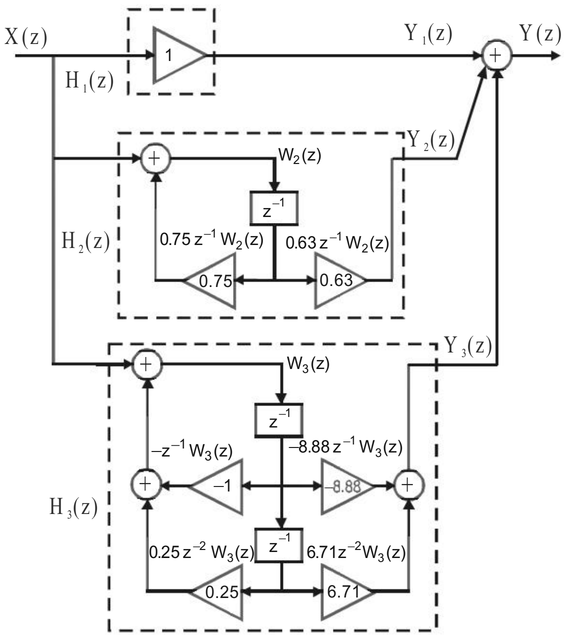
***FigE3.17.1 :** Direct form-I structure.*



***FigE3.17.2 :** Direct form-II structure.*

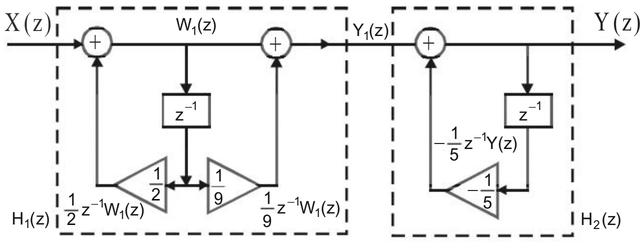


***FigE.3.17.3 :** Cascade structure.*

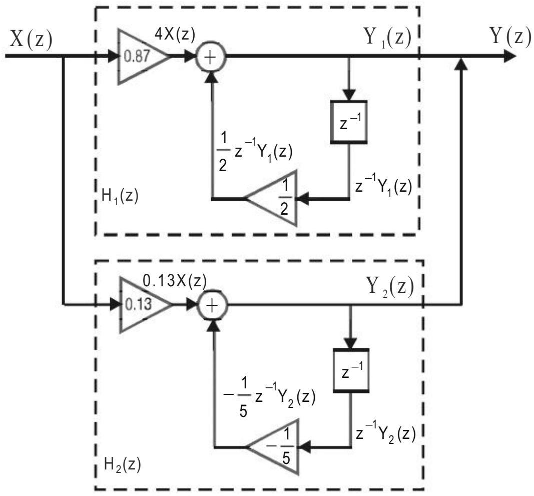


***FigE3.17.4 :** Parallel structure.*

E3.18

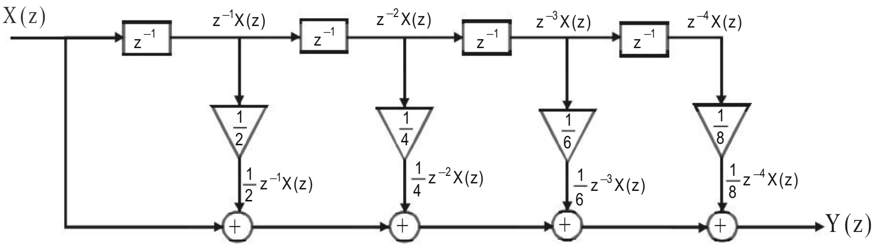


FigE3.18.1 : Cascade structure.



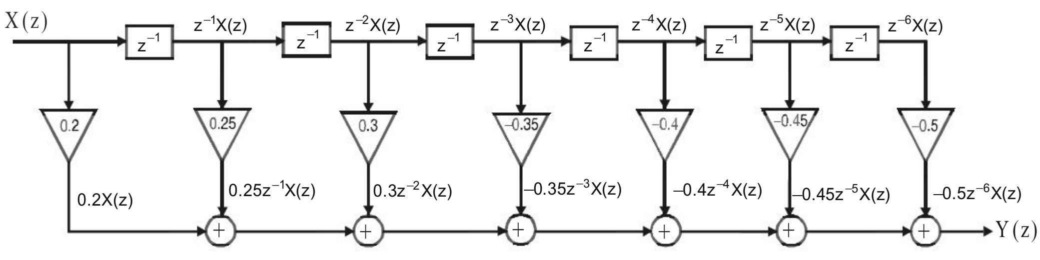
FigE3.18.2 : Parallel structure.

E3.19 a)



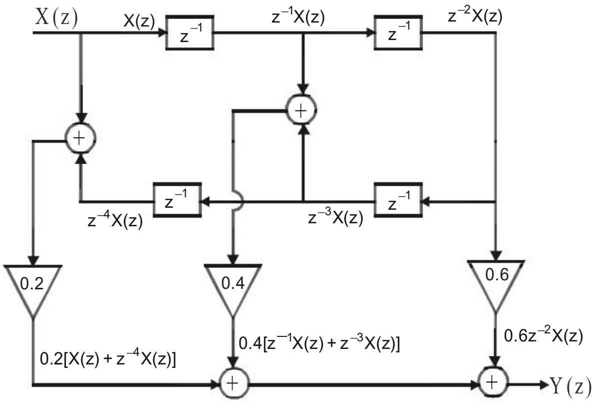
FigE3.19 a : Direct form structure.

E3.19 b)

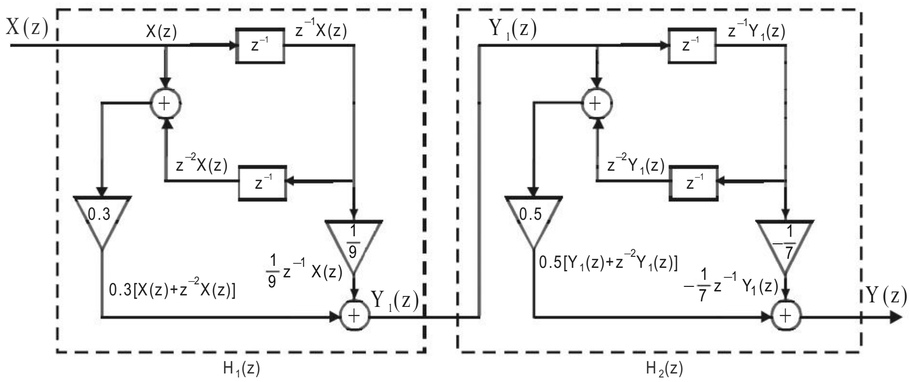


FigE3.19 b : Direct form structure.

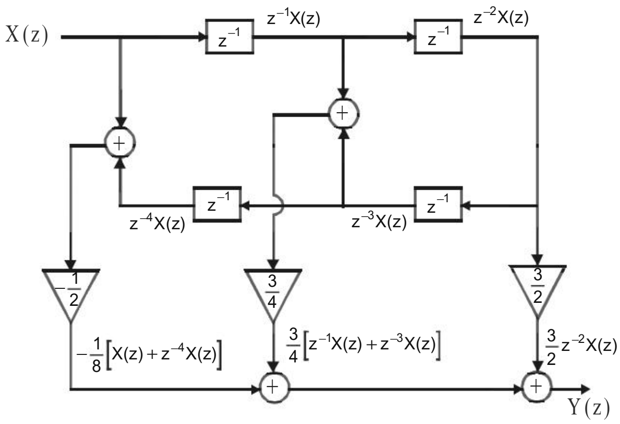
E3.20



***FigE3.20a :** Linear phase structure.*



***FigE3.20b:** Cascade of linear phase structure.*



***FigE3.20c :** Linear phase structure.*

Solution for Exercise Problems

E3.1 ***Determine the Ƶ-transform and their ROC of the following discrete signals.***

a) $x(n) = \{\underset{\uparrow}{4}, 2, 8, 5\}$

Solution

x(0) = 4; x(1) = 2; x(2) = 8; x(3) = 5,

and x(n) = 0 for n < 0 and for n>3.

By definition,

$$Ƶ\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The given sequence is finite duration sequence and so the limits of summation is changed to, n = 0 and n = 3.

$$\therefore\ X(z) = \sum_{n=0}^{3} x(n) z^{-n} = x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= 4 + 2z^{-1} + 8z^{-2} + 5z^{-3} = 4 + \frac{2}{z} + \frac{8}{z^2} + \frac{5}{z^3}$$

Here, X(z) will be finite for all values of z, except z = 0.

$\therefore$ ROC is entire z-plane except at z = 0.

---

b) $x(n) = \{3, 0, 0, 4, 45, \underset{\uparrow}{1}\}$

Solution

x(-5) = 3; x(-4) = 0; x(-3) = 0; x(-2) = 4; x(-1) = 45; x(0) = 1.

$$X(z) = \sum_{n=-5}^{0} x(n) z^{-n} = x(-5)z^{5} + x(-4)z^{4} + x(-3)z^{3} + x(-2)z^{2} + x(-1)z^{1} + x(0)z^{0}$$

$$= 3z^{5} + 0 + 0 + 4z^{2} + 45z + 1 = 3z^{5} + 4z^{2} + 45z + 1$$

Here, X(z) will be finite for all values of z, except z = ∞.

$\therefore$ ROC is entire z-plane except at z = ∞.

---

c) $x(n) = \{2, 1, 1, \underset{\uparrow}{2}, 5, 8, 2\}$

Solution

$$X(z) = \sum_{n=-3}^{3} x(n) z^{-n} = 2z^{3} + 1z^{2} + z + 2 + 5z^{-1} + 8z^{-2} + 2z^{-3}$$

$\therefore$ ROC is entire z-plane except at z=0 and z = ∞.

---

d) $x(n) = -0.2^{n}\, u(n-1)$

Solution

$x(n) = -0.2^{n}u(n-1) = -0.2^{n}\ ;\ n \geq 1$

$$Ƶ\{x(n)\} = X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=1}^{\infty} -(0.2)^{n} z^{-n} = -\left[\sum_{n=1}^{\infty} (0.2)^{n} z^{-n} + 0.2^{0} z^{0} - 0.2^{0} z^{0}\right]$$

$$= -\left[\sum_{n=0}^{\infty} \left(0.2z^{-1}\right)^{n} - 1\right]$$

If $|0.2z^{-1}| < 1$, then using infinite geometric series sum formula we can write,

$$X(z) = -\left[\frac{1}{1-\left(0.2z^{-1}\right)} - 1\right]$$

$$\therefore\ X(z) = 1 - \frac{1}{1-\left(0.2z^{-1}\right)} = 1 - \frac{z}{z-0.2} = \frac{z-0.2-z}{z-0.2} = \frac{-0.2}{z-0.2}$$

Here, $\left|0.2z^{-1}\right| < 1 \Rightarrow \frac{0.2}{|z|} < 1$

$\therefore\ |z| > 0.2$

\ ROC is exterior of the circle of radius 0.2 in z-plane.

---

e) $x(n) = (0.6)^n\ u(n) + (0.7)^n\ u(-n-1)$

Solution

$$X(z) = \sum_{n=-\infty}^{\infty}\left[(0.6)^n u(n) + (0.7)^n u(-n-1)\right]z^{-n} = \sum_{n=0}^{\infty}(0.6)^n z^{-n} + \sum_{n=-\infty}^{-1}(0.7)^n z^{-n}$$

$$= \sum_{n=0}^{\infty}(0.6)^n z^{-n} + \sum_{n=1}^{\infty}(0.7)^{-n} z^{n} = \sum_{n=0}^{\infty}\left(0.6\,z^{-1}\right)^n + \sum_{n=1}^{\infty}\left(0.7^{-1}\,z\right)^n$$

$$= \sum_{n=0}^{\infty}\left(0.6\,z^{-1}\right)^n + \left(0.7^{-1}z\right)^0 + \sum_{n=1}^{\infty}\left(0.7^{-1}\,z\right)^n - \left(0.7^{-1}z\right)^0$$

$$= \sum_{n=0}^{\infty}\left(0.6z^{-1}\right)^n + \sum_{n=0}^{\infty}\left(0.7^{-1}z\right)^n - 1$$

If $\left|0.6z^{-1}\right| < 1$ and $\left|0.7^{-1}z\right| < 1$, then using infinite geometric series sum formula we can write,

$$X(z) = \frac{1}{1-0.6z^{-1}} + \frac{1}{1-0.7^{-1}z} - 1 = \frac{z}{z-0.6} + \frac{0.7}{0.7-z} - 1$$

$$= \frac{z}{z-0.6} + \frac{0.7-(0.7-z)}{0.7-z} = \frac{z}{z-0.6} + \frac{z}{0.7-z}$$

$$= \frac{z}{z-0.6} - \frac{z}{z-0.7}$$

$$= \frac{z(z-0.7)-z(z-0.6)}{(z-0.7)(z-0.6)} = \frac{-0.1z}{(z-0.7)(z-0.6)}$$

Here, $\left|0.6z^{-1}\right| < 1 \Rightarrow \frac{0.6}{|z|} < 1$

Also, $\left|0.7^{-1}z\right| < 1 \Rightarrow \frac{|z|}{0.7} < 1$

\ $|z| > 0.6$ and $|z| < 0.7$

\ ROC is region in between circles of radius 0.6 and 0.7 in z-plane.

---

f) $x(n) = (0.9)^{|n|}$

Solution

$x(n) = (0.9)^{|n|}$ is a two-sided sequence.

\ $x(n) = 0.9^{|n|} = 0.9^{-n}\,u(-n-1) + 0.9^{n}\,u(n)$

$$\therefore X(z) = \sum_{n=-\infty}^{+\infty} x(n)z^{-n} = \sum_{n=-\infty}^{+\infty}\left[0.9^{-n}u(-n-1) + 0.9^n u(n)\right]z^{-n}$$

$$= \sum_{n=-\infty}^{-1}(0.9)^{-n}z^{-n} + \sum_{n=0}^{\infty}0.9^n z^{-n} = \sum_{n=1}^{\infty}(0.9z)^n + \sum_{n=0}^{\infty}\left(0.9z^{-1}\right)^n$$

$$= (0.9\,z)^0 + \sum_{n=1}^{\infty}(0.9\,z)^n - (0.9\,z)^0 + \sum_{n=0}^{\infty}\left(0.9\,z^{-1}\right)^n = \sum_{n=0}^{\infty}(0.9\,z)^n - 1 + \sum_{n=0}^{\infty}\left(0.9\,z^{-1}\right)^n$$

If $|0.9z| < 1$ and $\left|0.9z^{-1}\right| < 1$, then using infinite geometric series sum formula we can write,

$$X(z) = \frac{1}{1-0.9z} - 1 + \frac{1}{1-0.9z^{-1}} = \frac{1}{-0.9(z-1/0.9)} - 1 + \frac{z}{z-0.9} = \frac{-1.11}{z-1.11} - 1 + \frac{z}{z-0.9}$$

$$= \frac{-1.11-(z-1.11)}{z-1.11} + \frac{z}{z-0.9} = \frac{-z}{z-1.11} + \frac{z}{z-0.9} = \frac{-z(z-0.9)+z(z-1.11)}{(z-1.11)\,(z-0.9)} = \frac{-0.21z}{(z-1.11)\,(z-0.9)}$$

Here, $|0.9z| < 1 \quad \Rightarrow \quad \frac{|z|}{1/0.9} < 1 \quad \Rightarrow \quad \frac{|z|}{1.11} < 1$

Also, $\left|0.9z^{-1}\right| < 1 \quad \Rightarrow \quad \frac{0.9}{|z|} < 1$

$\backslash$ $|z| > 0.9$ and $|z| < 1.11$

$\backslash$ ROC is region in between circles of radius 0.9 and 1.11 in z-plane.

E3.2 Find the one-sided $\mathcal{Z}$-transform of the following discrete time signals.

a) $x(n) = n^2\, 5^n\, u(n)$

Solution

Let, $x_1(n) = 5^n\, u(n)$

By definition, $X_1(z) = \sum_{n=0}^{\infty} 5^n z^{-n} = \sum_{n=0}^{\infty} (5z^{-1})^n = \frac{1}{1-5z^{-1}} = \frac{z}{z-5}$

$$\therefore\ X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{n^2\, 5^n\} = \mathcal{Z}\{n^2\, x_1(n)\} = -z\frac{d}{dz}\left[-z\frac{d}{dz}X_1(z)\right] = -z\frac{d}{dz}\left[-z\frac{d}{dz}\left(\frac{z}{z-5}\right)\right]$$

$$= -z\frac{d}{dz}\left[-z\left(\frac{(z-5)-z}{(z-5)^2}\right)\right] = -z\frac{d}{dz}\left[\frac{5z}{(z-5)^2}\right] = -z\left[\frac{(z-5)^2\times 5-(5z\times 2\,(z-5))}{(z-5)^4}\right]$$

$$= -z\left[\frac{(z-5)\times 5-10z}{(z-5)^3}\right] = -z\left[\frac{5z-25-10z}{(z-5)^3}\right] = -z\left[\frac{-5z-25}{(z-5)^3}\right] = \frac{5z(z+5)}{(z-5)^3}$$

---

b) $x(n) = n\,(0.5)^{n+4}\,u(n)$

Solution

Let, $x_1(n) = (0.5)^{n+4}\,u(n) = 0.5^4\, 0.5^n\, u(n)$

$$\therefore\ X_1(z) = \mathcal{Z}\{x_1(n)\} = \mathcal{Z}\left[0.5^4\, 0.5^n u(n)\right] = 0.5^4\, \mathcal{Z}\{0.5^n u(n)\} = 0.5^4\, \frac{z}{z-0.5}$$

By property of $\mathcal{Z}$ - transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{n(0.5)^{n+4}u(n)\} = \mathcal{Z}\{nx_1(n)\} = -z\frac{d}{dz}X_1(z)$$

$$= -z\frac{d}{dz}\left[0.5^4\,\frac{z}{z-0.5}\right] = -0.5^4 z\frac{d}{dz}\left[\frac{z}{z-0.5}\right] = -0.5^4\, z\left[\frac{z-0.5-z}{(z-0.5)^2}\right] = \frac{0.5^5 z}{(z-0.5)^2}$$

---

c) $x(n) = (0.5)^{n-2}\,[u(n)-u(n-2)]$

Solution

$$X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\left\{(0.5)^{n-2}\ u(n)\right\} - \mathcal{Z}\left\{(0.5)^{n-2}\ u(n-2)\right\}$$

$$= \mathcal{Z}\left\{(0.5)^n u(n)\times(0.5)^{-2}\right\} - \mathcal{Z}\left\{(0.5)^{(n-2)} u(n-2)\right\}$$

$$= \frac{1}{0.5^2}\left[\frac{z}{z-0.5}\right] - z^{-2}\left[\frac{z}{z-0.5}\right]$$

$$= \frac{4z}{z-0.5} - \frac{1}{z(z-0.5)} = \frac{4z^2-1}{z(z-0.5)}$$

E3.3 Find the one-sided $\mathcal{Z}$-transform of the discrete signals generated by mathematically sampling the following continuous time signals.

a) $x(t) = 4t\, e^{-0.6t}\, u(t)$

Solution

Let, $t = nT$

$$x(nT) = 4nT\, e^{-0.6nT}\, u(nT) = ng(n)$$

where, $g(n) = 4\,T\,e^{-0.6nT}\,u(nT) = 4Te^{-0.6nT}$; n ³ 0

$$\therefore\ G(z) = \mathcal{Z}\{g(n)\} = \sum_{n=0}^{\infty} 4\,T\,e^{-0.6nT} z^{-n} = 4T\sum_{n=0}^{\infty} e^{-0.6nT} z^{-n} = 4T\sum_{n=0}^{\infty}\left(e^{-0.6T}z^{-1}\right)^n$$

$$= 4T\frac{1}{1-e^{-0.6T}z^{-1}} = 4T\frac{1}{z^{-1}\left(z-e^{-0.6T}\right)} = \frac{4Tz}{z-e^{-0.6T}}$$

Now, $X(z) = Z\{ng(n)\} = -z\frac{d}{dz}G(z) = -z\frac{d}{dz}\left[\frac{4Tz}{z-e^{-0.6T}}\right] = -4zT\frac{\left(z-e^{-0.6T}\right)-z(1)}{\left(z-e^{-0.6T}\right)^2}$

$$= \frac{-4zT\left[z-e^{-0.6T}-z\right]}{\left(z-e^{-0.6T}\right)^2} = \frac{4zTe^{-0.6T}}{\left(z-e^{-0.6T}\right)^2}$$

b) $x(t)=2t^3u(t)$

Solution

Let, t = nT.

\ x(nT) = 2 (nT)^{3}u(nT) = 2 n^{3}T^{3}u(nT) = n^{3}g(n)

where, g(n) = 2T^3 u(nT) = 2T^3 ; n ³ 0

$$\therefore\ G(z) = Z\{g(n)\} = \sum_{n=0}^{\infty} 2T^3 z^{-n} = 2T^3 \sum_{n=0}^{\infty} z^{-n} = 2T^3\frac{1}{1-z^{-1}} = 2T^3\frac{z}{z-1}$$

$$\text{Now, } X(z) = Z\{x(nT)\} = Z\{n^3 g(n)\} = \left(-z\frac{d}{dz}\right)^3 G(z) = -z\frac{d}{dz}\left[-z\frac{d}{dz}\left(-z\frac{d}{dz}G(z)\right)\right]$$

$$\text{Here, } -z\frac{d}{dz}G(z) = -z\frac{d}{dz}\left[2T^3\frac{z}{z-1}\right] = -2zT^3\frac{d}{dz}\left[\frac{z}{z-1}\right] = -2zT^3\left[\frac{z-1-z}{(z-1)^2}\right] = \frac{2zT^3}{(z-1)^2}$$

$$\therefore\ -z\frac{d}{dz}\left(-z\frac{d}{dz}G(z)\right) = -z\frac{d}{dz}\left[\frac{2zT^3}{(z-1)^2}\right] = -2zT^3\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right] = -2zT^3\left[\frac{(z-1)^2-z2(z-1)}{(z-1)^4}\right]$$

$$= -2zT^3\left[\frac{z-1-2z}{(z-1)^3}\right] = \frac{-2zT^3(-z-1)}{(z-1)^3} = \frac{2zT^3(z+1)}{(z-1)^3}$$

$$\therefore\ X(z) = -z\frac{d}{dz}\left[-z\frac{d}{dz}\left(-z\frac{d}{dz}G(z)\right)\right] = -z\frac{d}{dz}\left[\frac{2zT^3(z+1)}{(z-1)^3}\right]$$

$$= -2zT^3\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right] = -2zT^3\left[\frac{(z-1)^3(2z+1)-(z^2+z)\,3(z-1)^2}{(z-1)^6}\right]$$

$$= -2zT^3\left[\frac{(z-1)(2z+1)-3(z^2+z)}{(z-1)^4}\right] = -2zT^3\left[\frac{2z^2+z-2z-1-3z^2-3z}{(z-1)^4}\right]$$

$$= -2zT^3\left[\frac{-z^2-4z-1}{(z-1)^4}\right] = \frac{2zT^3(z^2+4z+1)}{(z-1)^4}$$

E3.4 Find the time domain initial value, x(0) and final value, x(¥) of the following z-domain functions.

a) $X(z) = \dfrac{0.5}{\left(1-z^{-1}\right)^2\left(1+z^{-1}\right)}$

Solution

By initial value theorem,

$$x(0) = \underset{z\to\infty}{Lt}\ X(z) = \underset{z\to\infty}{Lt}\frac{0.5}{\left(1-z^{-1}\right)^2\left(1+z^{-1}\right)}$$

$$= \underset{z\to\infty}{Lt}\frac{0.5}{\left(1-\frac{1}{z}\right)^2\left(1+\frac{1}{z}\right)} = \frac{0.5}{\left(1-\frac{1}{\infty}\right)^2\left(1+\frac{1}{\infty}\right)}$$

$$= \frac{0.5}{(1-0)^2(1+0)} = 0.5$$

By final value theorem,

$$x(\infty) = \underset{z\to1}{Lt}\left(1-z^{-1}\right)X(z) = \underset{z\to1}{Lt}\left(1-z^{-1}\right)\times\frac{0.5}{\left(1-z^{-1}\right)^2\left(1+z^{-1}\right)}$$

$$= \underset{z\to1}{Lt}\frac{0.5}{\left(1-z^{-1}\right)\left(1+z^{-1}\right)} = \frac{0.5}{\left(1-1^{-1}\right)\left(1+1^{-1}\right)} = \frac{0.5}{0\times2} = \infty$$

b) $X(z) = \dfrac{z^3}{(z-1)\left(z^2-0.2\right)}$

Solution

By initial value theorem,

$$x(0) = \underset{z\to\infty}{Lt}\ X(z) = \underset{z\to\infty}{Lt}\left[\frac{z^3}{(z-1)(z^2-0.2)}\right] = \underset{z\to\infty}{Lt}\ \frac{z^3}{z\left(1-\frac{1}{z}\right)z^2\left(1-\frac{0.2}{z^2}\right)} = \frac{1}{\left(1-\frac{1}{\infty}\right)\left(1-\frac{0.2}{\infty}\right)} = \frac{1}{(1-0)\,(1-0)} = \frac{1}{1\times 1} = 1$$

By final value theorem,

$$x(\infty) = \underset{z\to1}{Lt}\ \left(1-z^{-1}\right)X(z) = \underset{z\to1}{Lt}\ \left(1-z^{-1}\right)\left[\frac{z^3}{(z-1)(z^2-0.2)}\right]$$

$$= \underset{z\to1}{Lt}\ z^{-1}\cancel{(z-1)}\left[\frac{z^3}{\cancel{(z-1)}(z^2-0.2)}\right] = \frac{1}{(1-0.2)} = \frac{1}{0.8} = 1.25$$

E3.5 Determine the inverse Z-transform of the following functions using contour integral method.

a) $X(z) = \dfrac{(2z-1)z}{(z-1)^2}$

Solution

Given, $X(z) = \dfrac{(2z-1)z}{(z-1)^2}$

$$x(n) = \text{sum of residues of } X(z)z^{n-1}$$

$$= \text{sum of residues of } \frac{(2z-1)z}{(z-1)^2}z^{n-1}$$

$$= \text{sum of residues of } \frac{(2z-1)z^n}{(z-1)^2}$$

$$= \frac{1}{1!}\frac{d}{dz}\left[\cancel{(z-1)^2}\frac{(2z-1)z^n}{\cancel{(z-1)^2}}\right]\Bigg|_{z=1} = \frac{d}{dz}\left[\left(2z^{n+1}-z^n\right)\right]\Big|_{z=1} = \frac{d}{dz}\left[2z^{n+1}-z^n\right]\Big|_{z=1}$$

$$= \left[2(n+1)z^n - nz^{n-1}\right]\Big|_{z=1}$$

$$= 2(n+1)(1)^n - n(1)^{n-1} = 2n+2-n$$

$$\therefore\ \ x(n) = (n+2)\,u(n)$$

---

b) $X(z) = \dfrac{z^2+z}{(z-2)^2}$

Solution

$$x(n) = \text{sum of residues of } X(z)z^{n-1}$$

$$= \text{sum of residues of } \frac{z^2+z}{(z-2)^2}z^{n-1}$$

$$= \text{sum of residues of } \frac{z^{n+1}+z^n}{(z-2)^2}$$

$$\therefore\ \ x(n) = \frac{1}{1!}\frac{d}{dz}\left[\cancel{(z-2)^2}\times\frac{\left(z^{n+1}+z^n\right)}{\cancel{(z-2)^2}}\right]\Bigg|_{z=2}$$

$$= \frac{d}{dz}\left[z^{n+1}+z^n\right]\Big|_{z=2} = \left[(n+1)(z^n)+nz^{(n-1)}\right]\Big|_{z=2} = (n+1)(2)^n + n2^{(n-1)}$$

$$\therefore\ \ x(n) = (n+1)(2)^n u(n) + n2^{(n-1)}u(n-1)$$

c) $X(z) = \dfrac{\left(1-e^{-a}\right)z}{(z-1)\left(z-e^{-a}\right)}$

Solution

$$x(n) = \text{sum of residues of } X(z)z^{n-1}$$

$$= \text{sum of residues of } \frac{(1-e^{-a})z}{(z-1)\left(z-e^{-a}\right)}z^{n-1}$$

$$= \text{sum of residues of } \frac{\left(1-e^{-a}\right)z^{n}}{(z-1)\left(z-e^{-a}\right)}$$

$$= \cancel{(z-1)}\frac{\left(1-e^{-a}\right)z^{n}}{\cancel{(z-1)}\left(z-e^{-a}\right)}\Bigg|_{z=1} + \cancel{\left(z-e^{-a}\right)}\frac{\left(1-e^{-a}\right)z^{n}}{(z-1)\cancel{\left(z-e^{-a}\right)}}\Bigg|_{z=e^{-a}}$$

$$= \frac{\left(1-e^{-a}\right)(1)^{n}}{\left(1-e^{-a}\right)} + \frac{\left(1-e^{-a}\right)(e)^{-an}}{\left(e^{-a}-1\right)} = \frac{\cancel{\left(1-e^{-a}\right)}(1)^{n}}{\cancel{\left(1-e^{-a}\right)}} + \frac{\cancel{\left(1-e^{-a}\right)}(e)^{-an}}{-\cancel{\left(1-e^{-a}\right)}} = 1-e^{-an}$$

$$\therefore\ x(n) = \left(1-e^{-an}\right)u(n)\ ;\quad \text{for } n \geq 0$$

E3.6 ***Determine the inverse Z-transform of the following functions using partial fraction method.***

a) $X(z) = \dfrac{z^{2}}{(z+1)(z+2)^{2}}$

Solution

$$\frac{X(z)}{z} = \frac{z}{(z+1)(z+2)^{2}}$$

By partial fraction expansion,

$$\frac{X(z)}{z} = \frac{A_1}{(z+1)} + \frac{A_2}{(z+2)} + \frac{A_3}{(z+2)^{2}}$$

$$A_1 = \frac{z}{\cancel{(z+1)}(z+2)^{2}} \times \cancel{(z+1)}\Bigg|_{z=-1} = \frac{-1}{(-1+2)^{2}} = -1$$

$$A_2 = \frac{d}{dz}\left[\frac{z}{(z+1)\cancel{(z+2)^{2}}} \times \cancel{(z+2)^{2}}\right]_{z=-2} = \frac{d}{dz}\left[\frac{z}{z+1}\right]\Bigg|_{z=-2} = \frac{(z+1)-z(1)}{(z+1)^{2}}\Bigg|_{z=-2}$$

$$= \frac{1}{(z+1)^{2}}\Bigg|_{z=-2} = \frac{1}{(-2+1)^{2}} = 1$$

$$A_3 = \frac{z}{(z+1)\cancel{(z+2)^{2}}} \times \cancel{(z+2)^{2}}\Bigg|_{z=-2} = \frac{-2}{-2+1} = \frac{-2}{-1} = 2$$

$$\therefore\ \frac{X(z)}{z} = \frac{-1}{z+1} + \frac{1}{z+2} + \frac{2}{(z+2)^{2}}$$

$a^{n}u(n) \xrightarrow{Z} \dfrac{z}{z-a}$
$na^{n}u(n) \xrightarrow{Z} \dfrac{az}{(z-a)^{2}}$

$$\therefore\ X(z) = -1 \times \frac{z}{z+1} + \frac{z}{z+2} + 2 \times \frac{z}{(z+2)^{2}}$$

$$= -1 \times \frac{z}{z-(-1)} + \frac{z}{z-(-2)} + \frac{2z}{(z-(-2))^{2}}$$

$$\therefore\ x(n) = Z^{-1}\{X(z)\} = Z^{-1}\left\{-1 \times \frac{z}{z-(-1)} + \frac{z}{z-(-2)} + \frac{2z}{(z-(-2))^{2}}\right\}$$

$$= -1 \times Z^{-1}\left\{\frac{z}{z-(-1)}\right\} + Z^{-1}\left\{\frac{z}{z-(-2)}\right\} + Z^{-1}\left\{\frac{2z}{(z-(-2))^{2}}\right\}$$

$$= -(-1)^{n}u(n) + (-2)^{n}u(n) - n(-2)^{n}u(n)$$

$$= \left[-(-1)^{n} + (-2)^{n} - n(-2)^{n}\right]u(n)$$

b) $X(z) = \dfrac{2z^2 - z}{z^3 - 5z^2 + 8z - 4}$

Solution

$$\frac{X(z)}{z} = \frac{2z-1}{z^3-5z^2+8z-4}$$

Here, z = 1 is one of the root of the polynomial $z^3 - 5z^2 + 8z - 4 = 0$

Hence, $z^3 - 5z^2 + 8z - 4 = (z-1)(z^2 - 4z + 4)$

$$\begin{array}{r|l} & z^2 - 4z + 4 \\ \hline z-1 & z^3 - 5z^2 + 8z - 4 \\ & \underset{(-)}{z^3}\ \underset{(+)}{-z^2} \\ \hline & -4z^2 + 8z - 4 \\ & \underset{(+)}{-4z^2}\ \underset{(-)}{+4z} \\ \hline & 4z - 4 \\ & \underset{(-)}{4z}\ \underset{(+)}{-4} \\ \hline & 0 \end{array}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{2z-1}{(z-1)\left(z^2-4z+4\right)} = \frac{2z-1}{(z-1)(z-2)^2}$$

By partial fraction expansion,

$\boxed{(a-b)^2 = a^2 - 2ab + b^2}$

$$\frac{X(z)}{z} = \frac{A_1}{z-1} + \frac{A_2}{z-2} + \frac{A_3}{(z-2)^2}$$

$$A_1 = \frac{2z-1}{\cancel{(z-1)}(z-2)^2} \times \cancel{(z-1)}\Bigg|_{z=1} = \frac{2\times1-1}{(1-2)^2} = 1$$

$$A_2 = \frac{d}{dz}\left[\frac{2z-1}{(z-1)\cancel{(z-2)^2}} \times \cancel{(z-2)^2}\right]\Bigg|_{z=2} = \frac{d}{dz}\left[\frac{2z-1}{z-1}\right]\Bigg|_{z=2}$$

$$= \left[\frac{2(z-1)-(2z-1)(1)}{(z-1)^2}\right]\Bigg|_{z=2} = \frac{2(2-1)-(2\times2-1)}{(2-1)^2} = \frac{2-3}{1} = -1$$

$$A_3 = \frac{2z-1}{(z-1)\cancel{(z-2)^2}} \times \cancel{(z-2)^2}\Bigg|_{z=2} = \frac{2\times2-1}{2-1} = \frac{3}{1} = 3$$

$$\therefore \quad \frac{X(z)}{z} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{3}{(z-2)^2}$$

$$X(z) = 1\times\frac{z}{z-1} - 1\times\frac{z}{z-2} + 3\times\frac{z}{(z-2)^2} = \frac{z}{z-1} - \frac{z}{z-2} + \frac{3}{2}\times\frac{2z}{(z-2)^2}$$

$$\therefore \quad x(n) = \mathcal{Z}^{-1}\{X(z)\} = \mathcal{Z}^{-1}\left\{\frac{z}{z-1} - \frac{z}{z-2} + \frac{3}{2}\,\frac{2z}{(z-2)^2}\right\}$$

$$= \mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} - \mathcal{Z}^{-1}\left\{\frac{z}{z-2}\right\} + \frac{3}{2}\,\mathcal{Z}^{-1}\left\{\frac{2z}{(z-2)^2}\right\}$$

$$= u(n) - (2)^n u(n) + 1.5n\,(2)^n u(n)$$

$$= u(n) - (2)^n u(n) + 1.5n\,(2)^n u(n)$$

$$= \left[1 + (1.5n - 1)2^n\right] u(n)$$

---

c) $X(z) = \dfrac{z(z^2+3)}{\left(z^2+1\right)^2}$

Solution

$$\frac{X(z)}{z} = \frac{z^2+3}{\left(z^2+1\right)^2} = \frac{z^2+3}{(z+j)^2(z-j)^2}$$

$$z^2 + 1 = 0$$
$$\therefore \quad z^2 = -1$$
$$\therefore \quad z = \pm\sqrt{-1} = \pm j$$
$$\therefore \quad z^2 + 1 = (z+j)\,(z-j)$$

By partial fraction expansion,

$$\frac{X(z)}{z} = \frac{A_1}{z+j} + \frac{A_2}{(z+j)^2} + \frac{A_1^*}{z-j} + \frac{A_2^*}{(z-j)^2}$$

$$A_1 = \frac{d}{dz}\left[\frac{z^2+3}{\cancel{(z+j)^2}(z-j)^2} \times \cancel{(z+j)^2}\right]\Bigg|_{z=-j} = \left[\frac{(z-j)^2(2z) - \left(z^2+3\right)2(z-j)}{(z-j)^4}\right]\Bigg|_{z=-j}$$

$$= \frac{(z-j)(2z) - \left(z^2+3\right)2}{(z-j)^3}\Bigg|_{z=-j} = \frac{(-2j)2(-j) - (-1+3)2}{(-j-j)^3} = \frac{-4-(4)}{(-2j)^3} = \frac{-8}{8j} = \frac{-1}{j} = j$$

$$A_2 = \frac{z^2+3}{\cancel{(z+j)^2}(z-j)^2} \times \cancel{(z+j)^2}\bigg|_{z=-j} = \frac{z^2+3}{(z-j)^2}\bigg|_{z=-j} = \frac{-1+3}{(-j-j)^2} = \frac{2}{-4} = -\frac{1}{2}$$

$$A_1^* = (j)^* = -j$$

$$A_2^* = \left(-\frac{1}{2}\right)^* = -\frac{1}{2}$$

$$\therefore\ \frac{X(z)}{z} = \frac{j}{z+j} + \frac{\frac{-1}{2}}{(z+j)^2} + \frac{-j}{z-j} + \frac{\frac{-1}{2}}{(z-j)^2}$$

$$\therefore\ X(z) = j\,\frac{z}{z+j} - \frac{1}{2}\,\frac{z}{(z+j)^2} - j\frac{z}{z-j} - \frac{1}{2}\,\frac{z}{(z-j)^2}$$

$$= j\,\frac{z}{z-(-j)} - \frac{1}{2(-j)}\,\frac{-jz}{(z-(-j))^2} - j\,\frac{z}{z-j} - \frac{1}{2j}\,\frac{jz}{(z-j)^2}$$

$$\therefore\ x(n) = \mathcal{Z}^{-1}\{X(z)\}$$

$$= \mathcal{Z}^{-1}\left\{j\,\frac{z}{z-(-j)} + \frac{1}{2j}\,\frac{-jz}{(z-(-j))^2} - j\,\frac{z}{z-j} - \frac{1}{2j}\,\frac{jz}{(z-j)^2}\right\}$$

$$= j\mathcal{Z}^{-1}\left\{\frac{z}{z-(-j)}\right\} + \frac{1}{2j}\mathcal{Z}^{-1}\left\{\frac{-jz}{(z-(-j))^2}\right\} - j\mathcal{Z}^{-1}\left\{\frac{z}{z-j}\right\} - \frac{1}{2j}\mathcal{Z}^{-1}\left\{\frac{jz}{(z-j)^2}\right\}$$

$$= j(-j)^n u(n) + \frac{1}{2j}\,n(-j)^n\,u(n) - j(j)^n\,u(n) - \frac{1}{2j}\,n\,j^n\,u(n)$$

$$= \left[j\left[(-j)^n - j^n\right] + \frac{n}{2j}\left[(-j)^n - j^n\right]\right]u(n)$$

E3.7. Determine the inverse Z-transform of the function,

$$X(z) = \frac{2-z^{-1}}{\left[1-(1/4)z^{-1}\right]\left[1-(1/3)z^{-1}\right]}$$

(a) ROC : $|z| > \frac{1}{3}$***;*** ***(b) ROC :*** $|z| < \frac{1}{4}$***;*** ***(c) ROC :*** $\frac{1}{4} < |z| < \frac{1}{3}$.

Solution

Given, $X(z) = \dfrac{2-z^{-1}}{\left[1-(1/4)z^{-1}\right]\left[1-(1/3)z^{-1}\right]} = \dfrac{z^{-1}(2z-1)}{z^{-2}\left[z-(1/4)\right]\left[z-(1/3)\right]} = \dfrac{z(2z-1)}{\left[z-(1/4)\right]\left[z-(1/3)\right]}$

Now, $\dfrac{X(z)}{z} = \dfrac{(2z-1)}{(z-1/4)(z-1/3)} = \dfrac{A_1}{z-1/4} + \dfrac{A_2}{z-1/3}$

$$\therefore\ A_1 = \cancel{(z-1/4)} \times \frac{(2z-1)}{\cancel{(z-1/4)}(z-1/3)}\bigg|_{z=1/4} = \frac{2\times(1/4)-1}{(1/4)-(1/3)} = \frac{-1/2}{\frac{3-4}{12}} = \frac{-1/2}{-1/12} = -\frac{1}{2}\times\left(-\frac{12}{1}\right) = 6$$

$$A_2 = \cancel{(z-1/3)} \times \frac{(2z-1)}{(z-1/4)\cancel{(z-1/3)}}\bigg|_{z=1/3} = \frac{2\times(1/3)-1}{(1/3)-(1/4)} = \frac{-1/3}{\frac{4-3}{12}} = \frac{-1/3}{1/12} = -\frac{1}{3}\times\frac{12}{1} = -4$$

$$\therefore\ \frac{X(z)}{z} = \frac{6}{z-1/4} + \frac{-4}{z-1/3}$$

$$\therefore\ X(z) = \frac{6z}{z-1/4} - \frac{4z}{z-1/3}$$

$$\therefore\ x(n) = \mathcal{Z}^{-1}\{X(z)\} = \mathcal{Z}^{-1}\left\{\frac{6z}{z-1/4} - \frac{4z}{z-1/3}\right\} = 6\mathcal{Z}^{-1}\left\{\frac{z}{z-1/4}\right\} - 4\mathcal{Z}^{-1}\left\{\frac{z}{z-1/3}\right\}$$

a) ROC : |z| >1/3 - ROC is exterior of the circle whose radius is given by largest pole. Therefore x(n) is causal or right-sided signal.

$$\therefore x(n) = 6(1/4)^n - 4(1/3)^n\ ; \quad \text{for } n \ge 0$$

$$= \left[6(1/4)^n - 4(1/3)^n\right]u(n)$$

(b) ROC : | z | < 1/4 - ROC is interior of the circle whose radius is given by smallest pole. Therefore, x(n) is anticausal or left-sided signal.

$$\therefore \quad x(n) = \left[-6(1/4)^n u(-n-1) - 4\left(-(1/3)^n u(-n-1)\right)\right] = \left[-6(1/4)^n + 4(1/3)^n\right] u(-n-1)$$

(c) ROC : 1/4 <| z | < 1/3 - ROC is the region in between two circles of radius 1/4 and 1/3. Therefore, the term with pole = 1/3 will be anticausal and the term with pole = 1/4 will be causal.

$$\therefore x(n) = 6(1/4)^n u(n) + 4(1/3)^n u(-n-1)$$

E3.8. ***Determine the inverse Z-transform of the following function using power series method.***

$$X(z) = \frac{z}{2z^2 - 3z + 1}$$

Solution

Given that, $X(z) = \dfrac{z}{2z^2 - 3z + 1} = \dfrac{z}{2(z^2 - 1.5z + 0.5)} = \dfrac{z}{2(z-1)(z-0.5)}$

If ROC is |z| < 0.5, then x(n) will be anticausal signal.

If ROC is |z| > 1, then x(n) will be causal signal.

(a) ROC : | z | < 0.5

$$X(z) = \frac{z}{2z^2 - 3z + 1} = \frac{z}{1 - 3z + 2z^2}$$

$$= 1 + 3z + 7z^2 + 15z^3 + 31z^4 + \ldots\ldots$$

$$= \ldots\ldots + 31z^4 + 15z^3 + 7z^2 + 3z + 1 \qquad \ldots..(1)$$

$$\begin{array}{r|l}
 & 1 + 3z + 7z^2 + 15z^3 + 31z^4 + \ldots\ldots \\
\hline
1 - 3z + 2z^2 & 1 \\
 & \underset{(-)}{}\ \underset{(+)}{1} - \underset{(-)}{3z} + 2z^2 \\
 & 3z - 2z^2 \\
 & \underset{(-)}{3z} - \underset{(+)}{9z^2} + \underset{(-)}{6z^3} \\
 & 7z^2 - 6z^3 \\
 & \underset{(-)}{7z^2} - \underset{(+)}{21z^3} + \underset{(-)}{14z^4} \\
 & 15z^3 - 14z^4 \\
 & \underset{(-)}{15z^3} - \underset{(+)}{45z^4} + \underset{(-)}{30z^5} \\
 & 31z^4 - 30z^5 \ldots.
\end{array}$$

By definition of Z - transform,

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} = \ldots.. + x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + \ldots\ldots \qquad \ldots..(2)$$

On comparing (1) and (2),

$$x(-4) = 31, \quad x(-3) = 15, \quad x(-2) = 7, \quad x(-1) = 3, \quad x(0) = 1$$

$$\therefore \quad x(n) = \{\ldots..31,\ 15,\ 7,\ 3,\ \underset{\uparrow}{1}\}$$

(b) ROC : | z | > 1

$$X(z) = \frac{z}{2z^2 - 3z + 1}$$

$$= \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{15}{16}z^{-4} + \frac{31}{32}z^{-5} + \ldots\ldots \qquad \ldots..(3)$$

$$\begin{array}{r|l}
 & \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{15}{16}z^{-4} + \frac{31}{32}z^{-5} + \ldots\ldots \\
\hline
2z^2 - 3z + 1 & z \\
 & \underset{(-)}{z} - \underset{(+)}{\frac{3}{2}} + \underset{(-)}{\frac{1}{2}z^{-1}} \\
 & \frac{3}{2} - \frac{1}{2}z^{-1} \\
 & \underset{(-)}{\frac{3}{2}} - \underset{(+)}{\frac{9}{4}z^{-1}} + \underset{(-)}{\frac{3}{4}z^{-2}} \\
 & \frac{7}{4}z^{-1} - \frac{3}{4}z^{-2} \\
 & \underset{(-)}{\frac{7}{4}z^{-1}} - \underset{(+)}{\frac{21}{8}z^{-2}} + \underset{(-)}{\frac{7}{8}z^{-3}} \\
 & \frac{15}{8}z^{-2} - \frac{7}{8}z^{-3} \\
 & \underset{(-)}{\frac{15}{8}z^{-2}} - \underset{(+)}{\frac{45}{16}z^{-3}} + \underset{(-)}{\frac{15}{16}z^{-4}} \\
 & \frac{31}{16}z^{-3} - \frac{15}{16}z^{-4} \ldots.
\end{array}$$

By definition of Z - transform,

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

$$= \ldots.. + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5} + \ldots\ldots \qquad \ldots..(4)$$

On comparing (3) and (4),

$$x(n) = \left\{\underset{\uparrow}{0},\ \frac{1}{2},\ \frac{3}{4},\ \frac{7}{8},\ \frac{15}{16},\ \frac{31}{32},\ \ldots..\right\}$$

E3.9. Determine the inverse Z-transform of the following functions using power series method.

a) $X(z) = \dfrac{z^2+z}{z^2-2z+1}$; ***ROC :*** $|z| > 1$

Solution

$$X(z) = \frac{z^2+z}{z^2-2z+1} = \frac{z^2+z}{(z-1)^2}$$

Since ROC is $|z| > 1$, x(n) is causal or right-sided.

$$\therefore\ X(z) = \frac{z^2+z}{z^2-2z+1} = 1+3z^{-1}+5z^{-2}+7z^{-3}+..... \qquad .....(1)$$

$1+3z^{-1}+5z^{-2}+7z^{-3}......$

$z^2-2z+1 \,\big)\, z^2+z$

z^2-2z+1

$(-)\ (+)\ (-)$

$3z-1$

$3z-6+3z^{-1}$

$(-)\ (+)\ (-)$

$5-3z^{-1}$

$5-10z^{-1}+5z^{-2}$

$(-)\ (+)\ (-)$

$7z^{-1}-5z^{-2}....$

By definition of Z - transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)z^{-n} = ...+x(0)+x(1)z^{-1}+x(2)z^{-2}+x(3)z^{-3}+...... \qquad .....(2)$$

On comparing (1) and (2) we get,

$x(0) = 1,\ \ x(1) = 3,\ \ x(2) = 5,\ \ x(3) = 7$ and so on.

$$\therefore\ x(n) = \{\underset{\uparrow}{1},\ 3,\ 5,\ 7,\ .....\}$$

b) $X(z) = \dfrac{1-\left(\frac{1}{3}\right)z^{-1}}{1+\left(\frac{1}{3}\right)z^{-1}}$; ***ROC :*** $|z| > \dfrac{1}{3}$

Solution

Given that , ROC : | z | > 1/3 , \ x(n) is causal.

$$X(z) = \frac{1-\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}} = 1-\frac{2}{3}z^{-1}+\frac{2}{9}z^{-2}-\frac{2}{27}z^{-3}+\frac{2}{81}z^{-4}+...... \qquad .....(1)$$

$1-\frac{2}{3}z^{-1}+\frac{2}{9}z^{-2}-\frac{2}{27}z^{-3}+\frac{2}{81}z^{-4}+......$

$1+\frac{1}{3}z^{-1} \,\big)\, 1-\frac{1}{3}z^{-1}$

$1+\frac{1}{3}z^{-1}$

$(-)\ (-)$

$-\frac{2}{3}z^{-1}$

$-\frac{2}{3}z^{-1}-\frac{2}{9}z^{-2}$

$(+)\ (+)$

$\frac{2}{9}z^{-2}$

$\frac{2}{9}z^{-2}+\frac{2}{27}z^{-3}$

$(-)\ (-)$

$-\frac{2}{27}z^{-3}$

$-\frac{2}{27}z^{-3}-\frac{2}{81}z^{-4}$

$(+)\ (+)$

$\frac{2}{81}z^{-4}$

By definition of Z - transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)z^{-n} = ...+x(0)+x(1)z^{-1}+x(2)z^{-2}$$

$$+x(3)z^{-3}+x(4)z^{-4}+...... \qquad .....(2)$$

On comparing (1) and (2) we get,

$$x(0)=1,\ \ x(1)=-\frac{2}{3},\ \ x(2)=\frac{2}{9},\ \ x(3)=-\frac{2}{27},\ \ x(4)=\frac{2}{81},\ ......$$

$$\therefore\ x(n) = \left\{\underset{\uparrow}{1},\ -\frac{2}{3},\ \frac{2}{9},\ -\frac{2}{27},\ \frac{2}{81},\ .....\right\}$$

E3.10. Determine the transfer function and impulse response for the systems described by the following equations.

a) $y(n)+2y(n-1)-3y(n-2)=x(n-1).$

Solution

Given that, $y(n)+2y(n-1)-3y(n-2)=x(n-1)$

On taking Z - transform of above equation we get,

$$Y(z)+2z^{-1}Y(z)-3z^{-2}Y(z) = z^{-1}X(z)$$

$$\left(1+2z^{-1}-3z^{-2}\right)Y(z) = z^{-1}X(z)$$

$$\text{Transfer Function, } H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1+2z^{-1}-3z^{-2}}$$

$$\text{Let, } H(z) = \frac{z^{-1}}{1+2z^{-1}-3z^{-2}} = \frac{z^{-1}}{z^{-2}\left[z^2+2z-3\right]} = \frac{z}{z^2+2z-3}$$

Impulse response, $h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\dfrac{z}{z^2+2z-3}\right\} = \mathcal{Z}^{-1}\left\{\dfrac{z}{(z-1)(z+3)}\right\}$

Let, $\dfrac{H(z)}{z} = \dfrac{1}{(z-1)(z-3)} = \dfrac{A}{z-1} + \dfrac{B}{z+3}$

$$A = \cancel{(z-1)}\frac{1}{\cancel{(z-1)}(z+3)}\Bigg|_{z=1} = \frac{1}{1+3} = \frac{1}{4}$$

$$B = \cancel{(z+3)}\frac{1}{(z-1)\cancel{(z+3)}}\Bigg|_{z=-3} = \frac{1}{-3-1} = -\frac{1}{4}$$

$$\therefore\ H(z) = \frac{1}{4}\frac{z}{(z-1)} - \frac{1}{4}\frac{z}{(z+3)}$$

$$\therefore\ h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{4}\frac{z}{(z-1)} - \frac{1}{4}\frac{z}{(z+3)}\right\} = \frac{1}{4}\mathcal{Z}^{-1}\left\{\frac{z}{z-1} - \frac{z}{z-(-3)}\right\}$$

$$= \frac{1}{4}\left[u(n) - (-3)^n u(n)\right] = \frac{1}{4}\left[1-(-3)^n\right]u(n)$$

---

b) $y(n) - \frac{7}{4}y(n-1) + \frac{5}{8}y(n-2) = 2x(n)$

Solution

Given that, $y(n)-\frac{7}{4}y(n-1)+\frac{5}{8}y(n-2) = 2x(n)$

On taking $\mathcal{Z}$ - transform of above equation we get,

$$Y(z) - \frac{7}{4}z^{-1}Y(z) + \frac{5}{8}z^{-2}Y(z) = 2X(z)$$

$$\left(1-\frac{7}{4}z^{-1}+\frac{5}{8}z^{-2}\right)Y(z) = 2X(z)$$

Transfer Function, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{2}{1-\frac{7}{4}z^{-1}+\frac{5}{8}z^{-2}}$

Let, $H(z) = \dfrac{2}{1-\frac{7}{4}z^{-1}+\frac{5}{8}z^{-2}} = \dfrac{2}{z^{-2}\left[z^2-\frac{7}{4}z+\frac{5}{8}\right]} = \dfrac{2z^2}{z^2-\frac{7}{4}z+\frac{5}{8}}$

$$\frac{H(z)}{z} = \frac{2z}{z^2-\frac{7}{4}z+\frac{5}{8}} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{5}{4}}$$

The roots of quadratic,

$$z = \frac{\frac{7}{4} \pm \sqrt{\frac{7^2}{4^2} - 4\times\frac{5}{8}}}{2} = \frac{\frac{7}{4} \pm \sqrt{\frac{49-40}{16}}}{2} = \frac{7}{8} \pm \frac{1}{2}\times\frac{3}{4} = \frac{7}{8} \pm \frac{3}{8} = \frac{10}{8},\ \frac{4}{8} = \frac{5}{4},\ \frac{1}{2}$$

$$A = \cancel{\left(z-\frac{1}{2}\right)}\times\frac{2z}{\cancel{\left(z-\frac{1}{2}\right)}\left(z-\frac{5}{4}\right)}\Bigg|_{z=\frac{1}{2}} = \frac{2\times\frac{1}{2}}{\frac{1}{2}-\frac{5}{4}} = \frac{1}{\frac{2-5}{4}} = \frac{1}{-3/4} = -\frac{4}{3}$$

$$B = \cancel{\left(z-\frac{5}{4}\right)}\times\frac{2z}{\left(z-\frac{1}{2}\right)\cancel{\left(z-\frac{5}{4}\right)}}\Bigg|_{z=\frac{5}{4}} = \frac{2\times\frac{5}{4}}{\frac{5}{4}-\frac{1}{2}} = \frac{5/2}{\frac{5-2}{4}} = \frac{5/2}{3/4} = \frac{5}{2}\times\frac{4}{3} = \frac{10}{3}$$

$$\therefore\ \frac{H(z)}{z} = -\frac{4}{3}\frac{1}{\left(z-\frac{1}{2}\right)} + \frac{10}{3}\frac{1}{\left(z-\frac{5}{4}\right)} \quad\Rightarrow\quad H(z) = -\frac{4}{3}\frac{z}{\left(z-\frac{1}{2}\right)} + \frac{10}{3}\frac{z}{\left(z-\frac{5}{4}\right)}$$

$\therefore$ Impulse response, $h(n) = \mathcal{Z}^{-1}\{H(z)\} = -\dfrac{4}{3}\left(\dfrac{1}{2}\right)^n u(n) + \dfrac{10}{3}\left(\dfrac{5}{4}\right)^n u(n)$

$$= \frac{1}{3}\left[-4\left(\frac{1}{2}\right)^n + 10\left(\frac{5}{4}\right)^n\right]u(n)$$

---

c) $y(n) = 0.2x(n) - 5x(n-1) + 0.6\,y(n-1) - 0.08y(n-2)$

Solution

Given that, $y(n) = 0.2x(n) - 5x(n-1) + 0.6y(n-1) - 0.08y(n-2)$

$\therefore\ y(n) - 0.6y(n-1) + 0.08y(n-2) = 0.2x(n) - 5x(n-1)$

On taking Z - transform of above equation we get,

$$Y(z) - 0.6z^{-1}Y(z) + 0.08z^{-2}Y(z) = 0.2X(z) - 5z^{-1}X(z)$$

$$\left[1 - 0.6z^{-1} + 0.08z^{-2}\right]Y(z) = \left[0.2 - 5z^{-1}\right]X(z)$$

Transfer Function, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{0.2 - 5z^{-1}}{1 - 0.6z^{-1} + 0.08z^{-2}}$

The roots of quadratic, $z^2 - 0.6z + 0.08$ are,

$$z = 0.6 \pm \frac{\sqrt{(-0.6)^2 - 4(0.08)}}{2}$$

$$z = \frac{0.6 \pm 0.2}{2}$$

$$= 0.4,\ 0.2$$

Let, $H(z) = \dfrac{0.2 - 5z^{-1}}{1 - 0.6z^{-1} + 0.08z^{-2}} = \dfrac{z^{-1}[0.2z - 5]}{z^{-2}\left[z^2 - 0.6z + 0.08\right]} = \dfrac{0.2z^2 - 5z}{z^2 - 0.6z + 0.08}$

$$\therefore\ \frac{H(z)}{z} = \frac{0.2z - 5}{z^2 - 0.6z + 0.08} = \frac{0.2z - 5}{(z-0.4)(z-0.2)} = \frac{A}{z-0.4} + \frac{B}{z-0.2}$$

$$A = \cancel{(z-0.4)} \times \frac{0.2z - 5}{\cancel{(z-0.4)}\,(z-0.2)}\Bigg|_{z=0.4} = \frac{0.2 \times 0.4 - 5}{0.4 - 0.2} = -24.6$$

$$B = \cancel{(z-0.2)} \times \frac{0.2z - 5}{(z-0.4)\,\cancel{(z-0.2)}}\Bigg|_{z=0.2} = \frac{0.2 \times 0.2 - 5}{0.2 - 0.4} = 24.8$$

$$\frac{H(z)}{Z} = \frac{-24.6}{z-0.4} + \frac{24.8}{z-0.2}$$

$$\therefore\ H(z) = -24.6\frac{z}{(z-0.4)} + 24.8\frac{z}{(z-0.2)}$$

Impulse response, $h(n) = Z^{-1}\{H(z)\} = -24.6(0.4)^n u(n) + 24.8(0.2)^n u(n)$

$$= \left[24.8(0.2)^n - 24.6(0.4)^n\right]u(n)$$

---

d) $y(n) - \dfrac{3}{2}y(n-1) = x(n) + \dfrac{2}{3}x(n-1)$

Solution

Given that, $y(n) - \dfrac{3}{2}y(n-1) = x(n) + \dfrac{2}{3}x(n-1)$

On taking Z - transform of above equation we get,

$$Y(z) - \frac{3}{2}z^{-1}Y(z) = X(z) + \frac{2}{3}z^{-1}X(z)$$

$$\left(1 - \frac{3}{2}z^{-1}\right)Y(z) = X(z)\left(1 + \frac{2}{3}z^{-1}\right)$$

Transfer function, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1 + \frac{2}{3}z^{-1}}{1 - \frac{3}{2}z^{-1}}$

Let, $H(z) = \dfrac{1 + \frac{2}{3}z^{-1}}{1 - \frac{3}{2}z^{-1}} = \dfrac{z + \frac{2}{3}}{z - \frac{3}{2}} = \dfrac{z}{z - \frac{3}{2}} + \dfrac{\frac{2}{3}}{z - \frac{3}{2}} = \dfrac{z}{z - \frac{3}{2}} + \dfrac{2}{3}z^{-1}\dfrac{z}{z - \frac{3}{2}}$

Impulse response, $h(n) = Z^{-1}\{H(z)\} = \left(\dfrac{3}{2}\right)^n u(n) + \dfrac{2}{3}\left(\dfrac{3}{2}\right)^{n-1} u(n-1)$

E3.11. A discrete time LTI system is characterised by the transfer function,

$$H(z) = \frac{z(6z - 8)}{\left(z - \frac{1}{2}\right)(z - 3)}$$

Specify the ROC of H(z) and determine h(n) for the system to be, (i) Stable, (ii) Causal.

Solution

Given that $H(z) = \dfrac{z(6z-8)}{\left(z-\frac{1}{2}\right)(z-3)}$

By partial fraction,

$$\frac{H(z)}{z} = \frac{6z-8}{\left(z-\frac{1}{2}\right)(z-3)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-3}$$

$$A = \cancel{\left(z-\frac{1}{2}\right)} \times \left.\frac{6z-8}{\cancel{\left(z-\frac{1}{2}\right)}(z-3)}\right|_{z=\frac{1}{2}} = \frac{\frac{6}{2}-8}{\frac{1}{2}-3} = \frac{-\frac{10}{2}}{-\frac{5}{2}} = \frac{10}{5} = 2$$

$$B = \cancel{(z-3)} \times \left.\frac{6z-8}{\left(z-\frac{1}{2}\right)\cancel{(z-3)}}\right|_{z=3} = \frac{18-8}{3-\frac{1}{2}} = \frac{10}{\frac{5}{2}} = \frac{20}{5} = 4$$

$$\therefore \quad \frac{H(z)}{z} = \frac{2}{z-\frac{1}{2}} + \frac{4}{z-3} \quad \Rightarrow \quad H(z) = \frac{2z}{z-\frac{1}{2}} + 4\,\frac{z}{z-3}$$

Now the poles are z = 0.5, z = 3.

(i) Stable System :-

For stable system the ROC should include unit circle. Therefore, the ROC will be the region in between circles of radius 0.5 and 3.

\ ROC : $0.5 < |z| < 3$

The term with pole = 0.5 will be causal and the term with pole = 3 will be anticausal.

$$\therefore \quad h(n) = Z^{-1}\{H(z)\} = 2\left(\frac{1}{2}\right)^n u(n) - 4(3)^n\, u(-n-1)$$

(ii) Causal :-

For causal system the ROC should be exterior of the circle corresponding to largest pole.

\ ROC : $|z| > 3$.

$$\therefore h(n) = Z^{-1}\{H(z)\} = 2\left(\frac{1}{2}\right)^n u(n) + 4(3)^n\, u(n)$$

E3.12. Determine the unit step response of the discrete time LTI system whose input and output relation is described by the difference equation,

$$y(n) + 7\,y(n-1) = x(n),$$

where the initial condition is, y(–1) = 1.

Solution

Given that, $y(n) + 7y(n-1) = x(n)$; $y(-1) = 1$.

Input is unit step, i.e., $x(n) = u(n)$
$\therefore \; X(z) = Z\{x(n)\} = Z\{u(n)\} = \dfrac{z}{z-1}$

On taking Z-transform of above equation we get,

$$Y(z) + 7\left[Y(z)\,z^{-1} + y(-1)\right] = \frac{z}{z-1}$$

$$Y(z)\left[1+7z^{-1}\right] + 7y(-1) = \frac{z}{z-1}$$

$$Y(z)\left[\frac{z+7}{z}\right] + 7 = \frac{z}{z-1}$$

$$Y(z) = \left(\frac{z}{z-1} - 7\right) \times \frac{z}{z+7} = \frac{z(7-6z)}{(z-1)\,(z+7)}$$

By partial fraction,

$$\frac{Y(z)}{z} = \frac{7-6z}{(z-1)(z+7)} = \frac{A}{z-1} + \frac{B}{z+7}$$

$$A = \cancel{(z-1)} \times \frac{7-6z}{\cancel{(z-1)}(z+7)}\Bigg|_{z=1} = \frac{7-6}{1+7} = \frac{1}{8}$$

$$B = \cancel{(z+7)} \times \frac{7-6z}{(z-1)\cancel{(z+7)}}\Bigg|_{z=-7} = \frac{7-6\times(-7)}{-7-1} = -\frac{49}{8}$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{1}{8}\frac{1}{z-1} - \frac{49}{8}\frac{1}{z+7} \quad \Rightarrow \quad Y(z) = \frac{1}{8}\frac{z}{z-1} - \frac{49}{8}\frac{z}{z+7}$$

$$\therefore \quad y(n) = \mathcal{Z}^{-1}\{y(z)\} = \frac{1}{8}\,u(n) - \frac{49}{8}\,(-7)^n u(n) = \frac{1}{8}\left[1-49(-7)^n\right] u(n)$$

E3.13. Determine the response of discrete time LTI system governed by the following difference equation,

$4y(n) + 5\,y(n-1) + y(n-2) = x(n)$; with initial conditions, $y(-2) = -2$; $y(-1) = 1$, for the input $x(n) = (0.5)^n u(n)$.

<u>Solution</u>

Given that, $4y(n) + 5y(n-1) + y(n-2) = x(n)$

On taking $\mathcal{Z}$-transform of above equation we get,

$$4Y(z) + 5\left[Y(z)\,z^{-1} + y(-1)\right] + \left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right] = X(z)$$

$$4Y(z) + 5z^{-1}Y(z) + z^{-2}Y(z) + 5y(-1) + z^{-1}y(-1) + y(-2) = X(z)$$

$$Y(z)\left[4 + 5z^{-1} + z^{-2}\right] + 5(1) + z^{-1}(1) - 2 = X(z)$$

$$Y(z)\left[4 + 5z^{-1} + z^{-2}\right] + z^{-1} + 3 = \frac{z}{z-0.5}$$

$$x(n) = 0.5^n u(n)$$
$$X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{0.5^n u(n)\} = \frac{z}{z-0.5}$$

$$Y(z)\left[4 + \frac{5}{z} + \frac{1}{z^2}\right] = \frac{z}{z-0.5} - z^{-1} - 3$$

$$Y(z)\left[\frac{4z^2+5z+1}{z^2}\right] = \frac{z}{z-0.5} - \frac{1}{z} - 3 \quad \Rightarrow \quad Y(z)\left[\frac{4z^2+5z+1}{z^2}\right] = \frac{z^2-(z-0.5)-3z(z-0.5)}{z(z-0.5)}$$

$$\therefore \quad 4Y(z)\left[\frac{z^2+\frac{5}{4}z+\frac{1}{4}}{z^2}\right] = \frac{z^2 - z + 0.5 - 3z^2 + 1.5z}{z(z-0.5)}$$

$$4Y(z)\left[\frac{z^2+1.25z+0.25}{z^2}\right] = \frac{-2z^2+0.5z+0.5}{z(z-0.5)}$$

The roots of quadratic,
$z^2 + 1.25z + 0.25 = 0$ are,

$$z = \frac{-1.25 \pm \sqrt{1.25^2 - 4\times 0.25}}{2} = \frac{-1.25 \pm 0.75}{2} = -0.25,\ -1$$

$$\therefore \quad Y(z) = \frac{-2z^2+0.5z+0.5}{z(z-0.5)} \times \frac{z^2}{4\left(z^2+1.25z+0.25\right)}$$

$$= \frac{0.25z\left(-2z^2+0.5z+0.5\right)}{(z-0.5)(z^2+1.25z+0.25)} = \frac{z\left(-0.5z^2+0.125z+0.125\right)}{(z-0.5)(z+1)(z+0.25)}$$

$$\frac{Y(z)}{z} = \frac{-0.5z^2+0.125z+0.125}{(z-0.5)(z+1)(z+0.25)} = \frac{A}{z-0.5} + \frac{B}{z+1} + \frac{C}{z+0.25}$$

$$A = \cancel{(z-0.5)} \times \frac{-0.5z^2+0.125z+0.125}{\cancel{(z-0.5)}(z+1)(z+0.25)}\Bigg|_{z=0.5} = \frac{-0.5\times0.5^2+0.125\times0.5+0.125}{(0.5+1)(0.5+0.25)} = \frac{0.0625}{1.125} = 0.056$$

$$B = \cancel{(z+1)} \times \frac{-0.5z^2+0.125z+0.125}{(z-0.5)\cancel{(z+1)}(z+0.25)}\Bigg|_{z=-1} = \frac{-0.5(-1)^2+0.125(-1)+0.125}{(-1-0.5)(-1+0.25)} = \frac{-0.5}{1.125} = -0.444$$

$$C = \cancel{(z+0.25)} \times \frac{-0.5z^2+0.125z+0.125}{(z-0.5)(z+1)\cancel{(z+0.25)}}\Bigg|_{z=-0.25} = \frac{-0.5(-0.25)^2+0.125(-0.25)+0.125}{(-0.25-0.5)(-0.25+1)} = \frac{0.0625}{-0.5625} = -0.111$$

$$\therefore \quad \frac{Y(z)}{z} = \frac{0.056}{z-0.5} - \frac{0.444}{z+1} - \frac{0.111}{z+0.25}$$

$$\therefore\ Y(z) = 0.056\frac{z}{z-0.5} + -0.444\frac{z}{z+1} - 0.111\frac{z}{z+0.25}$$

On taking inverse Z-transform,

$$y(n) = \left[0.056\,(0.5)^n - 0.444(-1)^n - 0.111(-0.25)^n\right]u(n)$$

E3.14. An LTI system has the impulse response h(n) defined by $h(n) = x_1(n-1) * x_2(n)$. The Z-transform of the two signals $x_1(n)$ and $x_2(n)$ are, $X_1(z) = 2 - 4z^{-1}$ and $X_2(z) = 1 + 5z^{-2}$ respectively. Determine the output of the system for input $\delta(n-1)$.

Solution

Given that, $X_1(z) = 2 - 4z^{-1}$; $X_2(z) = 1 + 5z^{-2}$

$$h(n) = x_1(n-1) * x_2(n)$$

By convolution property,

$$\mathcal{Z}\{h(n)\} = \mathcal{Z}\{x_1(n-1)\} \times \mathcal{Z}\{x_2(n)\}$$

$$H(z) = z^{-1}X_1(z)\,X_2(z)$$

$$= z^{-1}\left(2-4z^{-1}\right)\left(1+5z^{-2}\right) = z^{-1}\left[2+10z^{-2} - 4z^{-1} - 20z^{-3}\right]$$

$$= 2z^{-1} - 4z^{-2} + 10z^{-3} - 20z^{-4}$$

Let, $y_1(n)$ be the response for input $\delta(n-1) = x(n)$.

Now, $y_1(n) = x(n) * h(n)$

By convolution property,

$$\mathcal{Z}\{y_1(n)\} = \mathcal{Z}\{x(n)\} \times \mathcal{Z}\{h(n)\}$$

$$Y_1(z) = z^{-1}\,H(z)$$

$$= z^{-1}\left[2z^{-1} - 4z^{-2} + 10z^{-3} - 20z^{-4}\right]$$

$$= 2z^{-2} - 4z^{-3} + 10z^{-4} - 20z^{-5} \qquad .....(1)$$

$$\boxed{\begin{aligned}\mathcal{Z}\{x(n)\} &= \mathcal{Z}\{\delta(n-1)\}\\ &= z^{-1}\end{aligned}}$$

By definition of Z-transform,

$$Y_1(z) = \mathcal{Z}\{y_1(n)\} = \sum_{n=-\infty}^{+\infty} y_1(n)z^{-n} = \;.....\; + y_1(0) + y_1(1)z^{-1} + y_1(2)z^{-2} + y_1(3)z^{-3}$$

$$+ y_1(4)z^{-4} + y_1(5)z^{-5} + \;...... \qquad .....(2)$$

On comparing (1) and (2) we get,

$$y_1(0) = 0,\ \ y_1(1) = 0,\ \ y_1(2) = 2,\ \ y_1(3) = -4,\ \ y_1(4) = 10,\ \ y_1(5) = -20$$

$$\therefore\ y_1(n) = \{\underset{\uparrow}{0},\ 0,\ 2,\ -4,\ 10,\ -20\}$$

E3.15. Obtain the direct form-I, direct form-II, cascade and parallel form realizations of the LTI system governed by the equation,

$$\boldsymbol{y(n) = -\frac{3}{4}y(n-1) + \frac{1}{2}y(n-2) + \frac{1}{4}y(n-3) + x(n) + 4x(n-1) + 3x(n-2)}$$

Solution

Direct Form – I

Given that,

$$y(n) = -\frac{3}{4}y(n-1) + \frac{1}{2}y(n-2) + \frac{1}{4}y(n-3) + x(n) + 4x(n-1) + 3x(n-2)$$

Taking Z-transform,

$$Y(z) = -\frac{3}{4}z^{-1}Y(z) + \frac{1}{2}z^{-2}Y(z) + \frac{1}{4}z^{-3}Y(z) + X(z) + 4z^{-1}X(z) + 3z^{-2}X(z) \qquad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

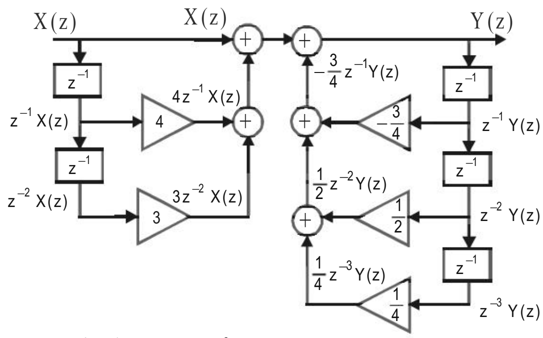


Fig 1 : Direct form-I structure.

Direct Form – II

Consider equation (1),

$$Y(z) = -\frac{3}{4}z^{-1}Y(z) + \frac{1}{2}z^{-2}Y(z) + \frac{1}{4}z^{-3}Y(z) + X(z) + 4z^{-1}X(z) + 3z^{-2}X(z)$$

$$Y(z) + \frac{3}{4}z^{-1}Y(z) - \frac{1}{2}z^{-2}Y(z) - \frac{1}{4}z^{-3}Y(z) = X(z) + 4z^{-1}X(z) + 3z^{-2}X(z)$$

$$Y(z)\left[1 + \frac{3}{4}z^{-1} - \frac{1}{2}z^{-2} - \frac{1}{4}z^{-3}\right] = X(z)\left[1 + 4z^{-1} + 3z^{-2}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 4z^{-1} + 3z^{-2}}{1 + \frac{3}{4}z^{-1} - \frac{1}{2}z^{-2} - \frac{1}{4}z^{-3}} \qquad .....(2)$$

Let, $\dfrac{Y(z)}{X(z)} = \dfrac{Y(z)}{W(z)}\,\dfrac{W(z)}{X(z)}$

Let, $\dfrac{W(z)}{X(z)} = \dfrac{1}{1 + \frac{3}{4}z^{-1} - \frac{1}{2}z^{-2} - \frac{1}{4}z^{-3}}$

$$\therefore\ W(z) + \frac{3}{4}z^{-1}W(z) - \frac{1}{2}z^{-2}W(z) - \frac{1}{4}z^{-3}W(z) = X(z)$$

$$W(z) = X(z) - \frac{3}{4}z^{-1}W(z) + \frac{1}{2}z^{-2}W(z) + \frac{1}{4}z^{-3}W(z) \qquad .....(3)$$

Let, $\dfrac{Y(z)}{W(z)} = 1 + 4z^{-1} + 3z^{-2} \quad \Rightarrow \quad Y(z) = W(z) + 4z^{-1}W(z) + 3z^{-2}W(z)$ .....(4)

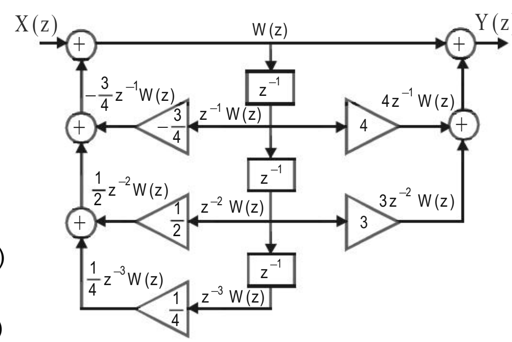


Fig 2 : Direct form-II structure.

Using equations (3) and (4), the direct form-II structure is drawn as shown in fig 2.

Cascade Form

Consider equation (2),

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1 + 4z^{-1} + 3z^{-2}}{1 + \frac{3}{4}z^{-1} - \frac{1}{2}z^{-2} - \frac{1}{4}z^{-3}} = \frac{z^{-2}\left[z^2 + 4z + 3\right]}{z^{-3}\left[z^3 + 0.75z^2 - 0.5z - 0.25\right]} = \frac{z\cancel{(z+1)}(z+3)}{\cancel{(z+1)}(z^2 - 0.25z - 0.25)}$$

$$= \frac{z\,(z+3)}{(z-0.64)\,(z+0.39)} = \frac{1 + 3z^{-1}}{\left(1 - 0.64z^{-1}\right)\left(1 + 0.39z^{-1}\right)} \qquad .....(5)$$

Let, $H(z) = H_1(z)\,H_2(z)$

where, $H_1(z) = \dfrac{1 + 3z^{-1}}{1 - 0.64z^{-1}}$; $H_2(z) = \dfrac{1}{1 + 0.39z^{-1}}$

Let, $H_1(z) = \dfrac{Y_1(z)}{X(z)} = \dfrac{W_1(z)}{X(z)}\,\dfrac{Y_1(z)}{W_1(z)}$

Let, $\dfrac{W_1(z)}{X(z)} = \dfrac{1}{1 - 0.64z^{-1}} \quad \Rightarrow \quad W_1(z) - 0.64z^{-1}W_1(z) = X(z)$

$$\therefore\ W_1(z) = X(z) + 0.64z^{-1}W_1(z) \qquad .....(6)$$

$z = -1$ is one of the root of equation

$z^3 + 0.75z^2 - 0.5z - 0.25 = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 0.75 & -0.5 & -0.25 \\ & \downarrow & -1.00 & +0.25 & +0.25 \\ \hline & 1 & -0.25 & -0.25 & 0 \end{array}$$

$\therefore\ z^3 + 0.75z^2 - 0.5z - 0.25$

$= (z+1)\,(z^2 - 0.25z - 0.25)$

The roots of quadratic $z^2 - 0.25z - 0.25 = 0$ are,

$$z = \frac{0.25 \pm \sqrt{0.25^2 - 4(-0.25)}}{2}$$

$$= \frac{0.25 \pm 1.03}{2} = 0.64,\ -0.39$$

Let, $\dfrac{Y_1(z)}{W_1(z)} = 1 + 3z^{-1}$

$\therefore\ Y_1(z) = W_1(z) + 3z^{-1}W_1(z)$ .....(7)

Using equations (6) and (7), the direct form-II structure of $H_1(z)$ is drawn as shown in fig 3.

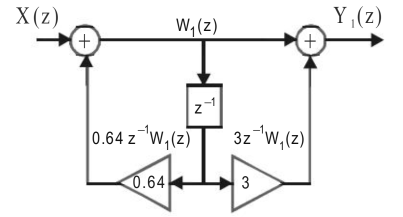


Fig 3 : Direct form-II structure of $H_1(z)$.

Let, $H_2(z) = \dfrac{Y(z)}{Y_1(z)} = \dfrac{1}{1 + 0.39z^{-1}} \Rightarrow Y(z) + 0.39z^{-1}Y(z) = Y_1(z)$

$\therefore\ Y(z) = Y_1(z) - 0.39z^{-1}Y(z)$ .....(8)

Using equation (8), the direct form-II structure of $H_2(z)$ is drawn as shown in fig 4.

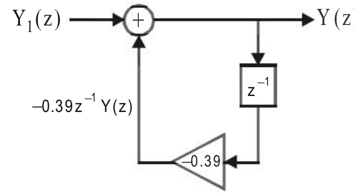


Fig 4: Direct form-II structure of $H_2(z)$.

The cascade structure is obtained by connecting $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 5.

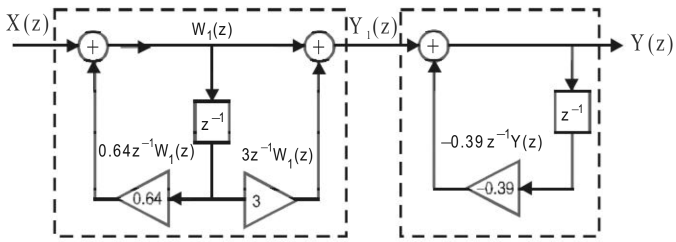


Fig 5 : Cascade structure.

Parallel Form

Consider the transfer function of the system [equation (5)].

$$H(z) = \frac{1 + 3z^{-1}}{\left(1 - 0.64z^{-1}\right)\left(1 + 0.39z^{-1}\right)} = \frac{A}{1 - 0.64z^{-1}} + \frac{B}{1 + 0.39z^{-1}}$$

$$A = \frac{1 + 3z^{-1}}{\cancel{\left(1 - 0.64z^{-1}\right)}\left(1 + 0.39z^{-1}\right)} \times \cancel{\left(1 - 0.64z^{-1}\right)}\Bigg|_{z^{-1} = \frac{1}{0.64}} = \frac{1 + 3 \times \dfrac{1}{0.64}}{1 + 0.39 \times \dfrac{1}{0.64}} = 3.53$$

$$B = \frac{1 + 3z^{-1}}{\left(1 - 0.64z^{-1}\right)\cancel{\left(1 + 0.39z^{-1}\right)}} \times \cancel{\left(1 + 0.39z^{-1}\right)}\Bigg|_{z^{-1} = \frac{-1}{0.39}} = \frac{1 + 3 \times \dfrac{-1}{0.39}}{1 - 0.64 \times \dfrac{-1}{0.39}} = -2.53$$

$$H(z) = \frac{3.53}{1 - 0.64z^{-1}} - \frac{2.53}{1 + 0.39z^{-1}}$$

Let, $H_1(z) = \dfrac{3.53}{1 - 0.64z^{-1}}$; $H_2(z) = \dfrac{-2.53}{1 + 0.39z^{-1}}$

Let, $H_1(z) = \dfrac{Y_1(z)}{X(z)} = \dfrac{3.53}{1 - 0.64z^{-1}}$

$\therefore\ Y_1(z) - 0.64z^{-1}Y_1(z) = 3.53X(z)$

$\therefore\ Y_1(z) = 3.53X(z) + 0.64z^{-1}Y_1(z)$ .....(9)

Let, $H_2(z) = \dfrac{Y_2(z)}{X(z)} = \dfrac{-2.53}{1 + 0.39z^{-1}}$

$\therefore\ Y_2(z) + 0.39z^{-1}\,Y_2(z) = -2.53X(z)$

$\therefore\ Y_2(z) = -2.53X(z) - 0.39z^{-1}Y_2(z)$ .....(10)

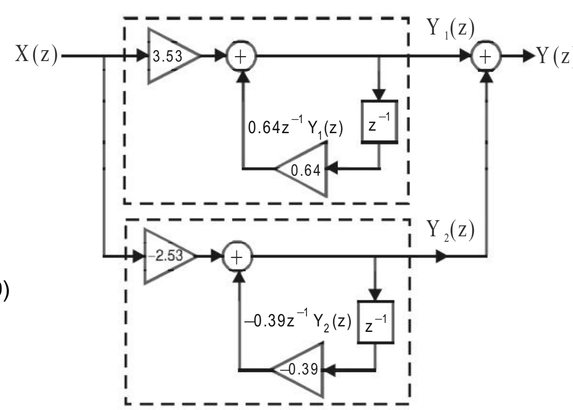


Fig 6 : Parallel structure.

Using equations (9) and (10), the parallel structure is drawn as shown in fig 6.

E3.16. Realize direct form-I, II structures of the IIR system represented by transfer function,

$$H(z) = \frac{z+5}{(z+0.4)(z+0.5)(z+0.6)}$$

Solution

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)} = \frac{z+5}{(z+0.4)(z+0.5)(z+0.6)} = \frac{z+5}{(z^2+0.9z+0.2)\ (z+0.6)}$$

$$= \frac{z+5}{z^3+0.9z^2+0.2z+0.6z^2+0.54z+0.12}$$

$$= \frac{z+5}{z^3+1.5z^2+0.74z+0.12}$$

$$= \frac{z\left(1+5z^{-1}\right)}{z^3\left[1+1.5z^{-1}+0.74z^{-2}+0.12z^{-3}\right]}$$

$$= \frac{z^{-2}\left(1+5z^{-1}\right)}{1+1.5z^{-1}+0.74z^{-2}+0.12z^{-3}}$$

$$\therefore\quad \frac{Y(z)}{X(z)} = \frac{z^{-2}+5z^{-3}}{1+1.5z^{-1}+0.74z^{-2}+0.12z^{-3}} \qquad .....(1)$$

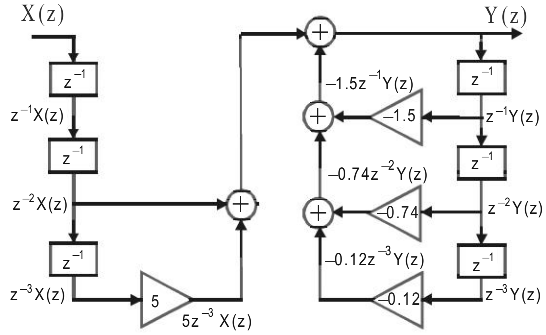


Fig 1 : *Direct form-I structure.*

On cross multiplying equation (1) we get,

$$Y(z)+1.5z^{-1}Y(z)+0.74z^{-2}Y(z)+0.12z^{-3}Y(z) = z^{-2}X(z)+5z^{-3}X(z)$$

\ $Y(z) = -1.5z^{-1}Y(z) - 0.74z^{-2}Y(z) - 0.12z^{-3}Y(z) + z^{-2}X(z) + 5z^{-3}X(z)$ .....(2)

Using equation (2), the direct form-I structure is drawn as shown in fig 1.

Let us express the transfer function of equation (1) as,

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)}\,\frac{Y(z)}{W(z)}$$

$$\text{Let,}\quad \frac{W(z)}{X(z)} = \frac{1}{1+1.5z^{-1}+0.74z^{-2}+0.12z^{-3}}$$

$$\therefore\quad W(z) = X(z) - 1.5z^{-1}W(z) - 0.74z^{-2}W(z) - 0.12z^{-3}W(z) \qquad .....(3)$$

$$\text{Let,}\quad \frac{Y(z)}{W(z)} = z^{-2}+5z^{-3} \quad\Rightarrow\quad Y(z) = z^{-2}W(z)+5z^{-3}W(z) \qquad .....(4)$$

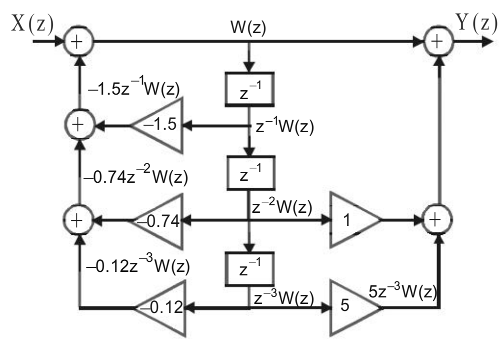


Fig 2 : *Direct form-II structure.*

Using equations (3) and (4), the direct form-II sturcuture is drawn as shown in fig 2.

E3.17. Determine the direct form-I, II, cascade and parallel realization of the following LTI system.

$$H(z) = \frac{z^3-8z^2+13z-5}{(z-0.75)\left(z^2+z-0.25\right)}$$

Solution

Direct Form – I

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)} = \frac{z^3-8z^2+13z-5}{(z-0.75)(z^2+z-0.25)} = \frac{z^3\left[1-8z^{-1}+13z^{-2}-5z^{-3}\right]}{z^3\left[1-0.75z^{-1}\right]\left[1+z^{-1}-0.25z^{-2}\right]} = \frac{1-8z^{-1}+13z^{-2}-5z^{-3}}{1+z^{-1}-0.25z^{-2}-0.75z^{-1}-0.75z^{-2}+0.1875z^{-3}}$$

$$\therefore\quad \frac{Y(z)}{X(z)} = \frac{1-8z^{-1}+13z^{-2}-5z^{-3}}{1+0.25z^{-1}-z^{-2}+0.1875z^{-3}} \qquad .....(1)$$

On cross multiplying equation (1) and rearranging we get,

$$Y(z) = -0.25z^{-1}Y(z)+z^{-2}Y(z)-0.1875z^{-3}Y(z) + X(z)-8z^{-1}X(z)+13z^{-2}X(z)-5z^{-3}X(z) \qquad .....(2)$$

Using equation (2), the direct form-I structure is drawn as shown in fig 1.

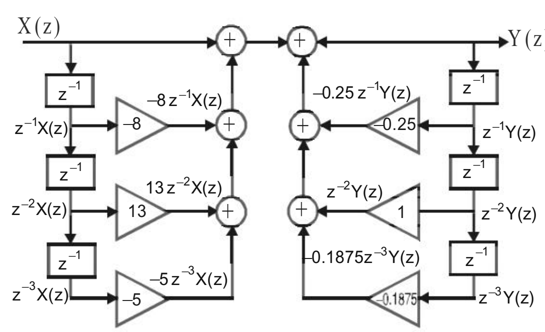


Fig 1 : *Direct form-I structure.*

Direct Form - II

Let us express the transfer function of equation (1) as shown below.

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)}\frac{Y(z)}{W(z)} = \frac{1-8z^{-1}+13z^{-2}-5z^{-3}}{1+0.25z^{-1}-z^{-2}+0.1875z^{-3}}$$

Let, $\dfrac{W(z)}{X(z)} = \dfrac{1}{1+0.25z^{-1}-z^{-2}+0.1875z^{-3}}$

On cross multiplying the above equation and rearranging we get,

$$W(z) = X(z) - 0.25z^{-1}W(z) + z^{-2}W(z) - 0.1875z^{-3}W(z) \quad .....(3)$$

Let, $\dfrac{Y(z)}{W(z)} = 1-8z^{-1}+13z^{-2}-5z^{-3}$

$$\therefore\ Y(z) = W(z) - 8z^{-1}W(z) + 13z^{-2}W(z) - 5z^{-3}W(z) \quad .....(4)$$

Using equations (3) and (4), the direct form-II structure is realized as shown in fig 2.

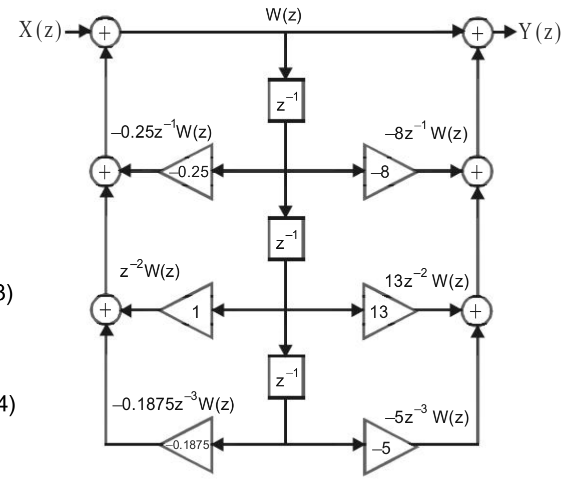


Fig 2 : Direct form-II structure.

Cascade Form

Given that, $H(z) = \dfrac{z^3-8z^2+13z-5}{(z-0.75)\ (z^2+z-0.25)} = \dfrac{(z-1)\left(z^2-7z+5\right)}{(z-0.75)\left(z^2+z-0.25\right)}$

$$= \frac{z^3\left(1-z^{-1}\right)\left(1-7z^{-1}+5z^{-2}\right)}{z^3\left(1-0.75z^{-1}\right)\left(1+z^{-1}-0.25z^{-2}\right)}$$

$$\therefore\ H(z) = \frac{\left(1-z^{-1}\right)\left(1-7z^{-1}+5z^{-2}\right)}{\left(1-0.75z^{-1}\right)\left(1+z^{-1}-0.25z^{-2}\right)}$$

$z = 1$, is one of the root of the polynomial,

$z^3 - 8z^2 + 13z - 5 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -8 & 13 & -5 \\ & \downarrow & 1 & -7 & 5 \\ \hline & 1 & -7 & 5 & 0 \end{array}$$

Let, $H(z) = \dfrac{(1-z^{-1})(1-7z^{-1}+5z^{-2})}{\left(1-0.75z^{-1}\right)\left(1+z^{-1}-0.25z^{-2}\right)} = H_1(z)\ H_2(z)$

where, $H_1(z) = \dfrac{1-z^{-1}}{1-0.75z^{-1}}$; $H_2(z) = \dfrac{1-7z^{-1}+5z^{-2}}{1+z^{-1}-0.25z^{-2}}$

Let, $H_1(z) = \dfrac{Y_1(z)}{X(z)} = \dfrac{W_1(z)}{X(z)}\dfrac{Y_1(z)}{W_1(z)} = \dfrac{1-z^{-1}}{1-0.75z^{-1}}$

Let, $\dfrac{W_1(z)}{X(z)} = \dfrac{1}{1-0.75z^{-1}} \Rightarrow W_1(z) = X(z) + 0.75z^{-1}W_1(z)$ .....(5)

Let, $\dfrac{Y_1(z)}{W_1(z)} = 1 - z^{-1} \quad \Rightarrow Y_1(z) = W_1(z) - z^{-1}W_1(z)$ .....(6)

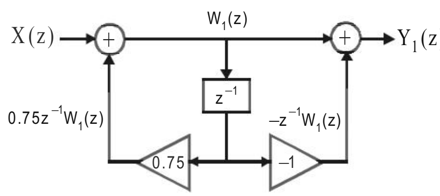


Fig 3.

Using equations (5) and (6), the direct form-II structure of $H_1(z)$ is realized as shown in fig 3.

Let, $H_2(z) = \dfrac{Y(z)}{Y_1(z)} = \dfrac{W_2(z)}{Y_1(z)}\dfrac{Y(z)}{W_2(z)} = \dfrac{1-7z^{-1}+5z^{-2}}{1+z^{-1}-0.25z^{-2}}$

Let, $\dfrac{W_2(z)}{Y_1(z)} = \dfrac{1}{1+z^{-1}-0.25z^{-2}} \Rightarrow W_2(z) = Y_1(z) - z^{-1}W_2(z) + 0.25z^{-2}W_2(z)$ .....(7)

Let $\dfrac{Y(z)}{W_2(z)} = 1-7z^{-1}+5z^{-2} \quad \Rightarrow Y(z) = W_2(z) - 7z^{-1}W_2(z) + 5z^{-2}W_2(z)$ .....(8)

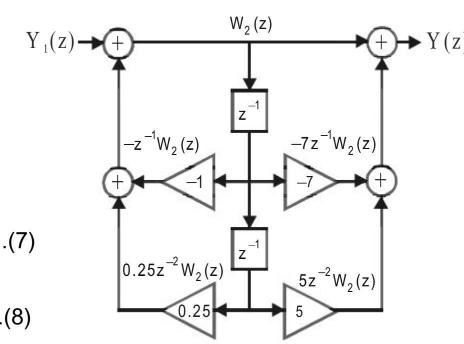


Fig 4.

Using equations (7) and (8), the direct form-II structure of $H_2(z)$ is drawn as shown in fig 4.

The cascade structure is obtained by connecting the structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 5.

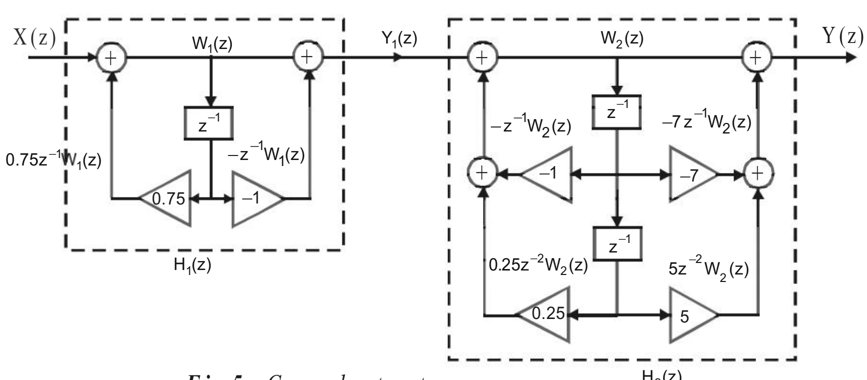


Fig 5 : Cascade structure.

Parallel Form

Given that, $$H(z)=\frac{z^3-8z^2+13z-5}{(z-0.75)\left(z^2+z-0.25\right)}=\frac{z^3-8z^2+13z-5}{z^3+z^2-0.25z-0.75z^2-0.75z+0.1875}$$

$$=\frac{z^3-8z^2+13z-5}{z^3+0.25z^2-z+0.1875}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline z^3+0.25z^2-z+0.1875 & z^3-\ \ 8z^2+\ 13z-\ \ 5 \\ & z^3+0.25z^2-\ \ z+0.1875 \\ (-) & (-)\quad (+)\quad (-) \\ \hline & -8.25z^2+14z-5.1875 \end{array}$$

$$\therefore\ H_1(z)=1+\frac{-8.25z^2+14z-5.1875}{z^3+0.25z^2-z+0.1875}$$

$$=1+\frac{-8.25z^2+14z-5.1875}{(z-0.75)\left(z^2+z-0.25\right)}$$

Let, $$\frac{-8.25z^2+14z-5.1875}{(z-0.75)\left(z^2+z-0.25\right)}=\frac{A}{z-0.75}+\frac{Bz+C}{z^2+z-0.25}$$

On cross multiplying we get,

$-8.25z^2 + 14z - 5.1875 = A(z^2 + z-0.25) + (Bz + C)(z - 0.75)$

$-8.25z^2 + 14z - 5.1875 = Az^2 + Az - 0.25A + Bz^2 - 0.75Bz + Cz - 0.75C$

On equating coefficients of z^2 we get,

$-8.25 = A + B$

\ $B = -8.25 - A$

On equating coefficients of z we get,

$14 = A - 0.75B + C$

Put, $B = -8.25 - A$

\ $14 = A - 0.75(-8.25 - A) + C$

$14 = A + 6.1875 + 0.75A + C$

\ $C = 14 - 6.1875 - 1.75A$

$= 7.8125 - 1.75A$

On equating constants we get,

$-5.1875 = -0.25A - 0.75C$

Put, $C = 7.8125 - 1.75A$

\ $-5.1875 = -0.25A - 0.75(7.8125 - 1.75A)$

$-5.1875 = -0.25A - 5.8593 + 1.3125A$

$1.3125A - 0.25A = -5.1875 + 5.8593$

$1.0625A = 0.6718$

$\therefore\ A=\frac{0.6718}{1.0625}=0.63$

Here, $A = 0.63$, \ $B = -8.25 - A$

$= -8.88$

\ $C = 7.8125 - 1.75A$

$= 7.8125 - 1.75$ ´ $0.63 = 6.71$

$$\therefore\ H(z)=1+\frac{0.63}{z-0.75}+\frac{-8.88z+6.71}{z^2+z-0.25}$$

$$=1+\frac{0.63}{z\left(1-0.75z^{-1}\right)}+\frac{-8.88z+6.71}{z^2\left(1+z^{-1}-0.25z^{-2}\right)}=1+\frac{0.63z^{-1}}{1-0.75z^{-1}}+\frac{-8.88z^{-1}+6.71z^{-2}}{1+z^{-1}-0.25z^{-2}}$$

Let, $H(z) = H_1(z) + H_2(z) + H_3(z)$

where, $H_1(z)=1$; $H_2(z)=\dfrac{0.63z^{-1}}{1-0.75z^{-1}}$; $H_3(z)=\dfrac{-8.88z^{-1}+6.71z^{-2}}{1+z^{-1}-0.25z^{-2}}$

Let, $H_1(z)=\dfrac{Y_1(z)}{X(z)}=1$

$\therefore\ Y_1(z)=X(z)$ .....(9)

Using equation (9), the $H_1(z)$ is realized as shown in fig 6.

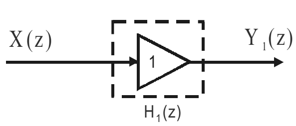


Fig 6.

$$\text{Let, } H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{0.63z^{-1}}{1-0.75z^{-1}}$$

$$\therefore \quad Y_2(z) = 0.63z^{-1}X(z) + 0.75z^{-1}Y_2(z) \qquad .....(10)$$

Using equation (10), the $H_2(z)$ is realized as shown in fig 7.

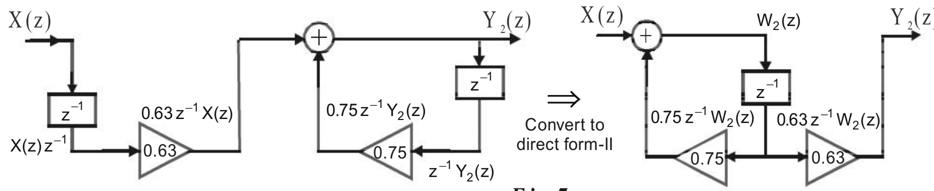


Fig 7.

$$\text{Let, } H_3(z) = \frac{Y_3(z)}{X(z)} = \frac{-8.88z^{-1}+6.71z^{-2}}{1+z^{-1}-0.25z^{-2}}$$

$$\therefore \quad Y_3(z) = -8.88z^{-1}X(z) + 6.71z^{-2}X(z) - z^{-1}Y_3(z) + 0.25z^{-2}Y_3(z) \qquad .....(11)$$

Using equation (11), the $H_3(z)$ is realized as shown in fig 8.

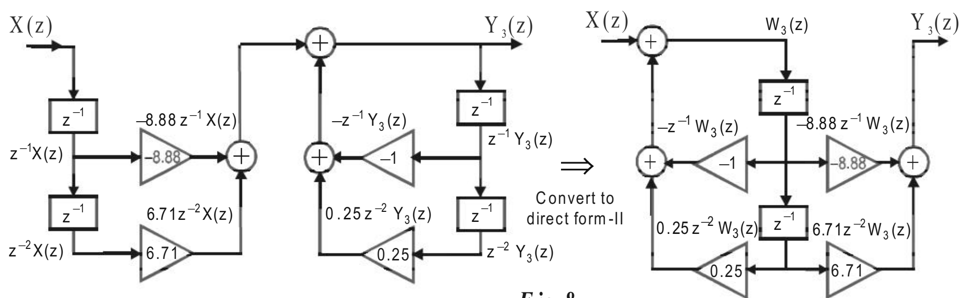


Fig 8.

The parallel structure is obtained by connecting the structures of $H_1(z)$, $H_2(z)$ and $H_3(z)$ in parallel as shown in fig 9.

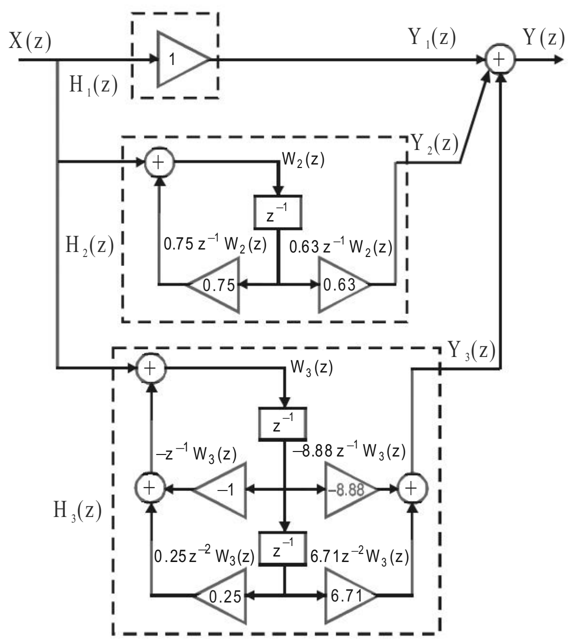


Fig 9 : Parallel structure.

E3.18. Realize the cascade and parallel structures of the system governed by the difference equation,

$$y(n) - \frac{3}{10}y(n-1) - \frac{1}{10}y(n-2) = x(n) + \frac{1}{9}x(n-1)$$

Solution

On taking $\mathcal{Z}$-transform of given equation we get,

$$Y(z) - \frac{3}{10}z^{-1}Y(z) - \frac{1}{10}z^{-2}Y(z) = X(z) + \frac{1}{9}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{3}{10}z^{-1} - \frac{1}{10}z^{-2}\right] = X(z)\left[1 + \frac{1}{9}z^{-1}\right]$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{9}z^{-1}}{1-\frac{3}{10}z^{-1}-\frac{1}{10}z^{-2}}=\frac{1+\frac{1}{9}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{5}z^{-1}\right)} \quad .....(1)$$

$$\left(1-\frac{3}{10}z^{-1}-\frac{1}{10}z^{-2}\right)=z^{-2}\left(z^2-\frac{3}{10}z-\frac{1}{10}\right)$$

The roots of quadratic,

$z^2-\frac{3}{10}z-\frac{1}{10}=0$ are,

$$z=\frac{\frac{3}{10}\pm\sqrt{\left(\frac{3}{10}\right)^2-4\left(\frac{-1}{10}\right)}}{2}$$

$$=\frac{1}{2}\left(\frac{3}{10}\pm\frac{7}{10}\right)=\frac{1}{2},\ -\frac{1}{5}$$

$$\therefore\ z^{-2}\left(z^2-\frac{3}{10}z-\frac{1}{10}\right)=z^{-2}\left(z-\frac{1}{2}\right)\left(z+\frac{1}{5}\right)$$

$$=\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{5}z^{-1}\right)$$

Cascade Form

Let, $H(z) = H_1(z)\,H_2(z)$

$$\text{where, } H_1(z)=\frac{1+\frac{1}{9}z^{-1}}{1-\frac{1}{2}z^{-1}}\ ;\ H_2(z)=\frac{1}{1+\frac{1}{5}z^{-1}}$$

$$\text{Let, } H_1(z)=\frac{Y_1(z)}{X(z)}=\frac{W_1(z)}{X(z)}\,\frac{Y_1(z)}{W_1(z)}=\frac{1+\frac{1}{9}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$\text{Let, } \frac{W_1(z)}{X(z)}=\frac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow W_1(z)=X(z)+\frac{1}{2}z^{-1}W_1(z) \quad .....(2)$$

$$\text{Let, } \frac{Y_1(z)}{W_1(z)}=1+\frac{1}{9}z^{-1} \Rightarrow Y_1(z)=W_1(z)+\frac{1}{9}z^{-1}W_1(z) \quad .....(3)$$

Using equations (2) and (3), the direct form-II structure of $H_1(z)$ is drawn as shown in fig 1.

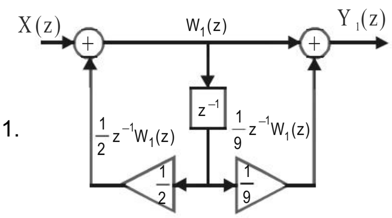


Fig 1.

$$\text{Let, } H_2(z)=\frac{Y(z)}{Y_1(z)}=\frac{1}{1+\frac{1}{5}z^{-1}} \Rightarrow Y(z)=Y_1(z)-\frac{1}{5}z^{-1}Y(z) \quad .....(4)$$

Using equation (4), the $H_2(z)$ is realized as shown in fig 2.

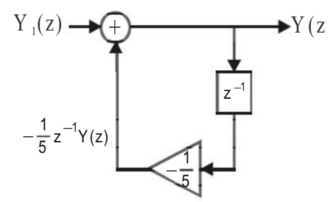


Fig 2.

The cascade structure is drawn by connecting $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3

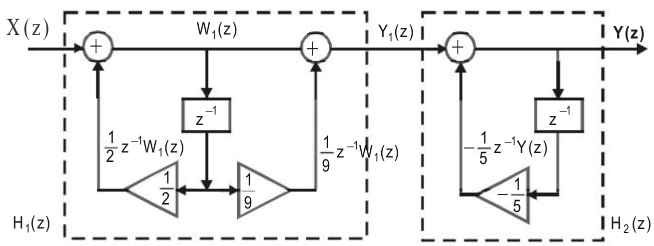


Fig 3 : Cascade structure.

Parallel Form

Consider equation (1),

$$H(z)=\frac{1+\frac{1}{9}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{5}z^{-1}\right)}=\frac{A}{1-\frac{1}{2}z^{-1}}+\frac{B}{1-\frac{1}{5}z^{-1}}$$

$$A=\frac{1+\frac{1}{9}z^{-1}}{\cancel{\left(1-\frac{1}{2}z^{-1}\right)}\left(1+\frac{1}{5}z^{-1}\right)}\times\cancel{\left(1-\frac{1}{2}z^{-1}\right)}\Bigg|_{z^{-1}=2}=\frac{1+\frac{1}{9}\times 2}{1+\frac{1}{5}\times 2}=\frac{1.2222}{1.4}=0.87$$

$$B=\frac{1+\frac{1}{9}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\cancel{\left(1+\frac{1}{5}z^{-1}\right)}}\times\cancel{\left(1+\frac{1}{5}z^{-1}\right)}\Bigg|_{z^{-1}=-5}=\frac{1+\frac{1}{9}(-5)}{1-\frac{1}{2}(-5)}=\frac{0.4444}{3.5}=0.13$$

$$\therefore\ H(z)=\frac{0.87}{1-\frac{1}{2}z^{-1}}+\frac{0.13}{1+\frac{1}{5}z^{-1}}$$

$$\text{Let, } H_1(z)=\frac{Y_1(z)}{X(z)}=\frac{0.87}{1-\frac{1}{2}z^{-1}} \Rightarrow Y_1(z)=0.87\,X(z)+\frac{1}{2}z^{-1}Y_1(z) \quad .....(5)$$

Using equation (5), the $H_1(z)$ is realized as shown in fig 4.

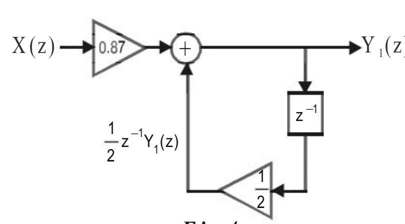


Fig 4.

Let, $H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{0.13}{1+\frac{1}{5}z^{-1}} = \frac{0.13}{\left(1+\frac{1}{5}z^{-1}\right)} \Rightarrow Y_2(z) = 0.13X(z) - \frac{1}{5}z^{-1}Y_2(z)$ .....(6)

Using equation (6), the $H_2(z)$ is realized as shown in fig 5.

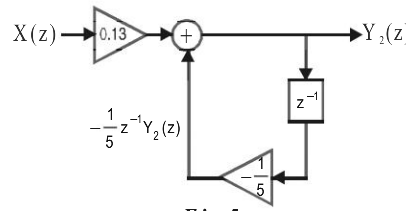


Fig 5.

The parallel structure is drawn by connecting $H_1(z)$ and $H_2(z)$ is parallel as shown in fig 6.

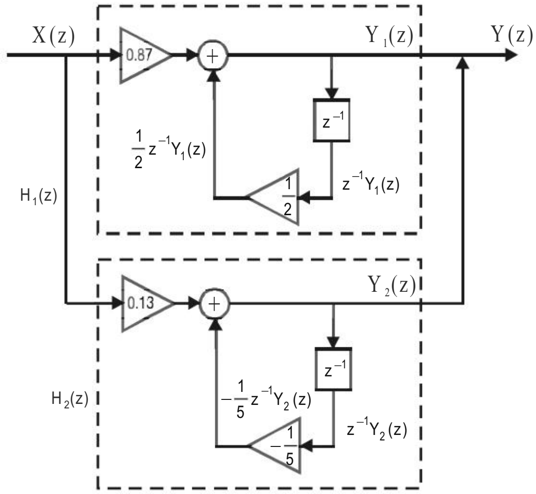


Fig 6 : *Parallel structure.*

E3.19. Draw the direct form structure of the FIR systems described by the following equations.

a) $y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2) + \frac{1}{6}x(n-3) + \frac{1}{8}x(n-4)$

Solution

On taking Z-transform of given equation we get,

$$Y(z) = X(z) + \frac{1}{2}z^{-1}X(z) + \frac{1}{4}z^{-2}X(z) + \frac{1}{6}z^{-3}X(z) + \frac{1}{8}z^{-4}X(z) \quad .....(1)$$

Using equation (1), the direct form structure of FIR system is drawn as shown in fig 1.

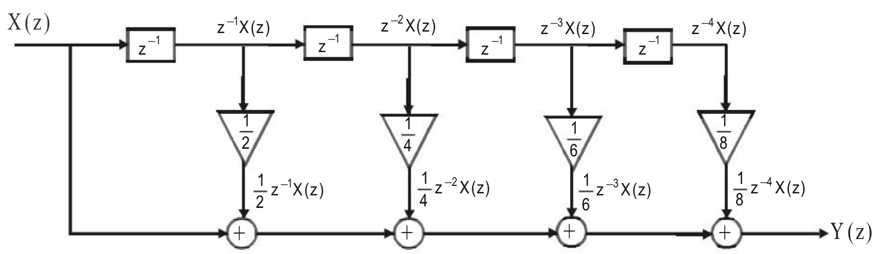


Fig 1 : *Direct form structure of FIR system.*

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b) $y(n) = 0.2\,x(n) + 0.25\,x(n-1) + 0.3\,x(n-2) - 0.35\,x(n-3) - 0.4\,x(n-4) - 0.45\,x(n-5) - 0.5\,x(n-6)$

Solution

On taking Z-transform of given equation we get,

$$Y(z) = 0.2X(z) + 0.25z^{-1}X(z) + 0.3z^{-2}X(z) - 0.35z^{-3}X(z) - 0.4\,z^{-4}X(z) - 0.45z^{-5}X(z) - 0.5z^{-6}X(z) \quad .....(2)$$

Using equation (2), the direct form structure of FIR system is drawn as shown in fig 2.

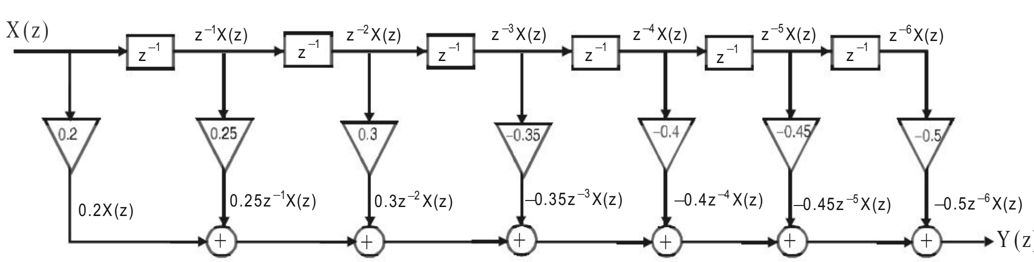


Fig 2 : *Direct form structure of FIR system.*

E3.20. Realize the following FIR systems with minimum number of multipliers.

a) $H(z) = 0.2 + 0.4z^{-1} + 0.6z^{-2} + 0.4z^{-3} + 0.2z^{-4}$

Solution

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)} = 0.2 + 0.4z^{-1} + 0.6z^{-2} + 0.4z^{-3} + 0.2z^{-4}$$

$$\therefore\; Y(z) = 0.2X(z) + 0.4z^{-1}X(z) + 0.6z^{-2}X(z) + 0.4z^{-3}X(z) + 0.2z^{-4}X(z)$$

$$= 0.2\left[X(z) + z^{-4}X(z)\right] + 0.4\left[z^{-1}X(z) + z^{-3}X(z)\right] + 0.6z^{-2}X(z) \qquad .....(1)$$

Using equation (1), the linear phase realization structure of FIR system is drawn as shown in fig 1. (The linear phase structure requires minimum number of multipliers).

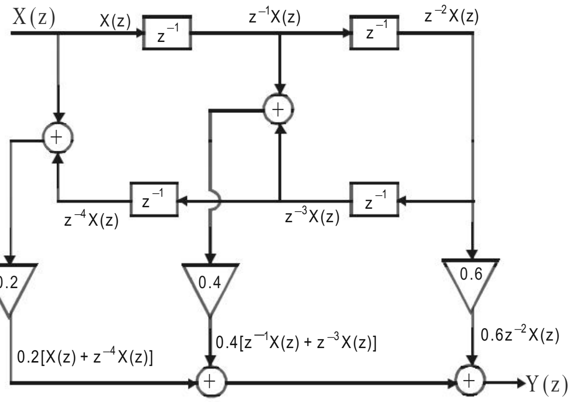


Fig 1 : Linear phase realization of equation (1).

b) $H(z) = \left(0.3 + \frac{1}{9}z^{-1} + 0.3z^{-2}\right)\left(0.5 - \frac{1}{7}z^{-1} + 0.5z^{-2}\right)$

Solution

Let, $\quad H(z) = H_1(z)\, H_2(z)$

$$\text{Let,}\quad H_1(z) = \frac{Y_1(z)}{X(z)} = 0.3 + \frac{1}{9}z^{-1} + 0.3z^{-2}$$

$$\therefore\; Y_1(z) = 0.3X(z) + \frac{1}{9}z^{-1}X(z) + 0.3z^{-2}X(z)$$

$$= 0.3\left[X(z) + z^{-2}X(z)\right] + \frac{1}{9}z^{-1}X(z) \qquad .....(2)$$

Using equation (2), the linear phase realization of $H_1(z)$ is obtained as shown in fig 2.

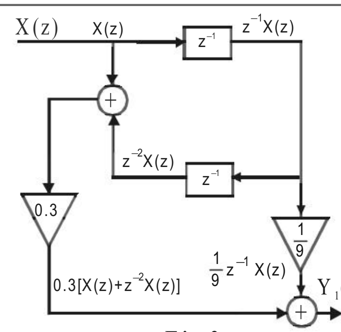


Fig 2.

$$\text{Let,}\quad H_2(z) = \frac{Y(z)}{Y_1(z)} = 0.5 - \frac{1}{7}z^{-1} + 0.5z^{-2}$$

$$\therefore\; Y(z) = 0.5Y_1(z) - \frac{1}{7}z^{-1}Y_1(z) + 0.5z^{-2}Y_1(z)$$

$$= 0.5\left[Y_1(z) + z^{-2}Y_1(z)\right] - \frac{1}{7}z^{-1}Y_1(z) \qquad .....(3)$$

Using equation (3), the linear phase realization of $H_2(z)$ is obtained as shown in fig 3.

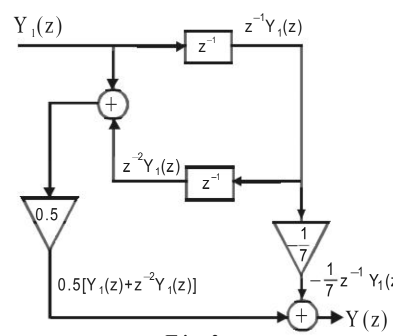


Fig 3.

Cascade Realization of H(z)

The cascade realization of H(z) with minimum number of multipliers is obtained by connecting the structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 4.

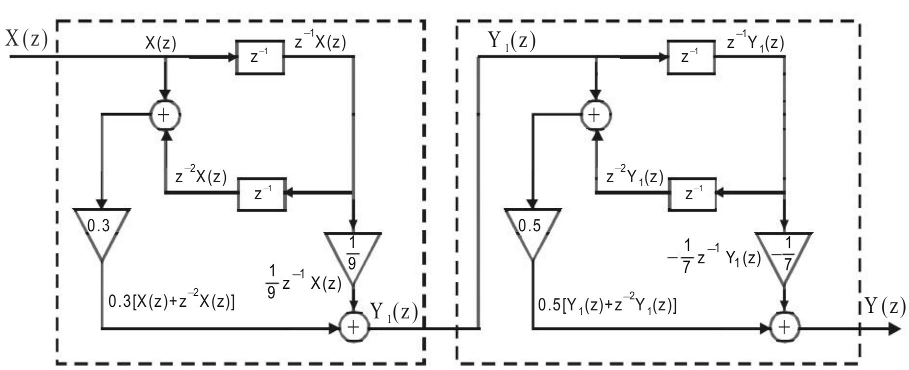


Fig 4.

c) $y(n) = \frac{-1}{8}x(n) + \frac{3}{4}x(n-1) + \frac{3}{2}x(n-2) + \frac{3}{4}x(n-3) - \frac{1}{8}x(n-4)$

Solution

On taking Ƶ-transform of given equation we get,

$$Y(z) = -\frac{1}{8}X(z) + \frac{3}{4}z^{-1}X(z) + \frac{3}{2}z^{-2}X(z) + \frac{3}{4}z^{-3}X(z) - \frac{1}{8}z^{-4}X(z)$$

$$\therefore \; Y(z) = -\frac{1}{8}\left[X(z) + z^{-4}X(z)\right] + \frac{3}{4}\left[z^{-1}X(z) + z^{-3}X(z)\right] + \frac{3}{2}z^{-2}X(z) \qquad .....(4)$$

Using equation (4), the linear phase structure of FIR system is drawn as shown in fig 5.

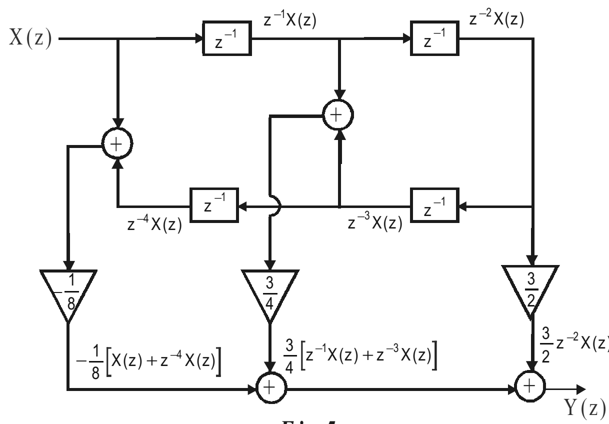


Fig 5.

Chapter 4

Fourier Series and Fourier Transform of Discrete Time Signals

4.1 Introduction

A periodic discrete time signal with fundamental period N can be decomposed into N harmonically related frequency components. The summation of the frequency components gives the Fourier series representation of periodic discrete time signal, in which the discrete time signal is represented as a function of frequency of discrete time signal, ω. The Fourier series of discrete time signal is called ***Discrete Time Fourier Series*** (***DTFS***). The frequency components are also called frequency spectrum of the discrete time signal.

The Fourier representation of periodic discrete time signals has been extended to nonperiodic signals by letting the fundamental period N to infinity, and this Fourier method of representing nonperiodic discrete time signals as a function of frequency of discrete time signal, ω is called Fourier transform of discrete time signals or ***Discrete Time Fourier Transform*** (***DTFT***). The Fourier representation of discrete time signal is also known as frequency domain representation. In general, the Fourier series representation can be obtained only for periodic discrete time signals, but the Fourier transform technique can be applied to both periodic and nonperiodic signals to obtain the frequency domain representation of the discrete time signals.

The Fourier representation of discrete time signals can be used to perform frequency domain analysis of discrete time signals, in which we can study the various frequency components present in the signal, magnitude and phase of various frequency components. The graphical plots of magnitude and phase as a function of frequency are also drawn. The plot of magnitude versus frequency is called ***magnitude spectrum*** and the plot of phase versus frequency is called ***phase spectrum***. In general, these plots are called ***frequency spectrum***.

4.2 Fourier Series of Discrete Time Signals (Discrete Time Fourier Series)

The Fourier series (or ***Discrete Time Fourier Series***, DTFS) of discrete time periodic signal x(n) with periodicity N is defined as,

$$\boxed{x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{N-1} c_k\, e^{j\omega_0 kn} = \sum_{k=0}^{N-1} c_k\, e^{j\omega_k n}} \qquad .....(4.1)$$

where, c_k = Fourier coefficients; $\quad w_0 = \frac{2\pi}{N}$ = Fundamental frequency of x(n)

$\omega_k = \omega_0 k = \frac{2\pi k}{N}$ = k^{th} harmonic frequency of x(n)

$c_k\, e^{j\omega_k n}$ = k^{th} harmonic component of x(n)

The Fourier coefficients, c_k for k = 0, 1, 2, ..., N-1 can be evaluated using equation (4.2).

$$\boxed{c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \;;\; \text{for } k = 0, 1, 2, ..., N-1} \qquad .....(4.2)$$

The ***Fourier coefficient*** $\boldsymbol{c_k}$ represents the amplitude and phase associated with the k^{th} frequency component. Hence we can say that the fourier coefficients provide the description of x(n) in the frequency domain.

Proof :

Consider the Fourier series representation of the discrete time signal x(n).

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}}$$

Let us replace k by p.

$$\therefore\; x(n) = \sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}$$

Let us multiply the above equation by $e^{\frac{-j2\pi kn}{N}}$ on both sides.

$$x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

On evaluating the above equation for n = 0 to N – 1 and summing up the values we get,

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1}\sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

Let us interchange the order of summation in the right-hand side of the above equation and rearrange as shown below.

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} c_p \sum_{n=0}^{N-1} e^{\frac{j2\pi(p-k)n}{N}}$$

When p = k the right-hand side of the above equation reduces to c_k N.

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = c_k N$$

$$\therefore\; c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Note : The sum over one period of the values of a periodic complex exponential is zero, unless that complex exponential is a constant.

$$\therefore \sum_{n=0}^{N-1} e^{\frac{j2\pi(p-k)n}{N}} = N\;;\; (p-k) = 0, \pm N, \pm 2N, ...$$
$$= 0\;;\; (p-k) \neq N$$

Difference Between Continuous Time and Discrete Time Fourier Series

1. The frequency range of continuous time signal is $-\infty$ to $+\infty$, and so it has infinite frequency spectrum.

2. The frequency range of discrete time signal is 0 to 2π (or $-\pi$ to $+\pi$) and so it has finite frequency spectrum. A discrete time signal with fundamental period N will have N frequency components whose frequencies are,

$$\boxed{\omega_k = \frac{2\pi k}{N} \quad ; \text{ for } k = 0, 1, 2, ..., N-1}$$

4.2.1 Frequency Spectrum of Periodic Discrete Time Signals

Let, x(n) be a periodic discrete time signal. Now, the Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}}$$

where, c_k is the Fourier coefficient of k^{th} harmonic component.

The Fourier coefficient, c_k is a complex quantity and so it can be expressed in the polar form as shown below.

$$c_k = |c_k| \angle c_k \quad ; \quad \text{for } k = 0, 1, 2, 3, ...N-1$$

where, $|c_k|$ = Magnitude of c_k ; $\angle c_k$ = Phase of c_k

The term, $|c_k|$ represents the magnitude of k^{th} harmonic component and the term $\angle c_k$ represents the phase of the k^{th} harmonic component.

The plot of harmonic magnitude / phase of a discrete time signal versus "k" (or harmonic frequency ω_k) is called ***frequency spectrum***. The plot of harmonic magnitude versus "k" (or ω_k) is called ***magnitude spectrum*** and the plot of harmonic phase versus "k" (or ω_k) is called ***phase spectrum***.

The Fourier coefficients are periodic with period N.

$$\therefore c_{k+N} = c_k$$

Since Fourier coefficients are periodic, the frequency spectrum is also periodic, with period N.

Proof :

Consider the Fourier coefficient c_k of the discrete time signal x(n).

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Now, the Fourier coefficient c_{k+N} is given by,

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi (k+N)\, n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\left(\frac{-j2\pi kn}{N} + \frac{-j2\pi Nn}{N}\right)}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi\, k\, n}{N}}\, e^{-j2\pi\, n} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi\, k\, n}{N}} = c_k$$

For integer values of n, $e^{-j2\pi n} = 1$

For a periodic discrete time signal with period N, there are N Fourier coefficients denoted as c_0, c_1, c_2, ... c_{N-1}, and so the N-number of Fourier coefficients can be expressed as a sequence consisting of N values.

Fourier coefficients, $c_k = \{c_0, c_1, c_2, c_3, ...c_{N-1}\}$

Magnitude spectrum, $|c_k| = \{|c_0|, |c_1|, |c_2|, |c_3|, ...|c_{N-1}|\}$

Phase spectrum, $\angle c_k = \{\angle c_0, \angle c_1, \angle c_2, \angle c_3, ...\angle c_{N-1}\}$

4.2.2 Properties of Discrete Time Fourier Series

The properties of discrete time Fourier series coefficients are listed in table 4.1. The proof of these properties are left as exercise to the readers.

Table 4.1 : Properties of Discrete Time Fourier Series Coefficients

Note : c_k are Fourier series coefficients of x(n) and d_k are Fourier series coefficients of y(n).

Property	Discrete time periodic signal	Fourier series coefficients
Linearity	$A\,x(n) + B\,y(n)$	$A\,c_k + B\,d_k$
Time shifting	$x(n-m)$	$c_k\,e^{\frac{j2\pi km}{N}}$
Frequency shifting	$e^{\frac{j2\pi nm}{N}}\,x(n)$	c_{k-m}
Conjugation	$x^*(n)$	c^*_{-k}
Time reversal	$x(-n)$	c_{-k}
Time scaling	$x(\frac{n}{m})$; for n multiple of m (periodic with period mN)	$\frac{1}{m}\,c_k$
Multiplication	$x(n)\,y(n)$	$\sum_{m=0}^{N-1} c_m\,d_{k-m}$
Circular convolution	$\sum_{m=0}^{N-1} x(m)\,y((n-m))_N$	$N\,c_K\,d_K$
Symmetry of real signals	x(n) is real	$c_k = c^*_{-k}$ $\lvert c_k\rvert = \lvert c_{-k}\rvert$ $\angle c_k = -\angle c_{-k}$ $\mathrm{Re}\{c_k\} = \mathrm{Re}\{c_{-k}\}$ $\mathrm{Im}\{c_k\} = -\mathrm{Im}\{c_{-k}\}$
Real and even	x(n) is real and even	c_k are real and even
Real and odd	x(n) is real and odd	c_k are imaginary and odd
Parseval's relation	Average power P of x(n) is defined as, $P = \frac{1}{N}\sum_{n=0}^{N-1} \lvert x(n)\rvert^2$	Average power P in terms of Fourier series coefficients is, $P = \sum_{k=0}^{N-1} \lvert c_k\rvert^2$
	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x(n)\rvert^2 = \sum_{k=0}^{N-1} \lvert c_k\rvert^2$	

Note : The average power in the signal is the sum of the powers of the individual frequency components. The sequence $|c_k|^2$ for k = 0, 1, 2,....., (N - 1) is the distribution of power as a function of frequency and so it is called the power density spectrum (or) power spectral density of the periodic signal.

Example 4.1

Determine the Fourier series representation of the following discrete time signals.

a) $x(n) = 2\sin\frac{\sqrt{3}}{2}\pi n$ **b)** $x(n) = 3\cos\frac{\pi n}{4}$ **c)** $x(n) = e^{\frac{j5\pi n}{2}}$

Solution

a) Given that, $\mathbf{x(n) = 2\sin\frac{\sqrt{3}}{2}\pi n}$

Test for Periodicity

Let, $x(n+N) = 2\sin\frac{\sqrt{3}}{2}\pi(n+N) = 2\sin\left(\frac{\sqrt{3}}{2}\pi n + \frac{\sqrt{3}}{2}\pi N\right)$

For periodicity $\frac{\sqrt{3}}{2}\pi N$ should be equal to integral multiple of $2p$.

Let, $\frac{\sqrt{3}}{2}\pi N = M ´ 2p$; where M and N are integers. $\Rightarrow$ $N = \frac{4}{\sqrt{3}}M$

Here N cannot be an integer for any integer value of M and so x(n) will not be periodic.

Fourier Series

Here x(n) is nonperiodic signal and so Fourier series does not exists.

---

b) Given that, $\mathbf{x(n) = 3\cos\frac{\pi n}{4}}$

Test for Periodicity

Let, $x(n+N) = 3\cos\frac{\pi}{4}(n+N) = 3\cos\left(\frac{\pi n}{4} + \frac{\pi N}{4}\right)$

For periodicity $\frac{\pi N}{4}$ should be integral multiple of $2p$.

Let, $\frac{\pi N}{4} = 2\pi \times M$; where M and N are integers Þ $N = 8M$

Here, N is an integer for M = 1, 2, 3, ...

Let M = 1, \ N = 8

Hence x(n) is periodic, with fundamental period N = 8, and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{8} = \frac{\pi}{4}$.

Fourier Series

The Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \text{ for } k = 0, 1, 2, 3, \ldots, N-1$$

Here, $N = 8$ and $x(n) = 3\cos\frac{\pi n}{4}$

$$\therefore\ c_k = \frac{1}{8}\sum_{n=0}^{7} 3\cos\frac{\pi n}{4}\, e^{\frac{-j2\pi kn}{8}} \quad ; \text{ for } k = 0, 1, 2, 3, 4, 5, 6, 7$$

$$= \frac{3}{8}\sum_{n=0}^{7} \cos\frac{\pi n}{4}\, e^{\frac{-j\pi kn}{4}}$$

$$= \frac{3}{8}\left[\begin{array}{l}(\cos 0)e^0 + \left(\cos\frac{\pi}{4}\right)e^{-j\frac{\pi k}{4}} + \left(\cos\frac{2\pi}{4}\right)e^{-j\frac{2\pi k}{4}} + \left(\cos\frac{3\pi}{4}\right)e^{-j\frac{3\pi k}{4}} + \left(\cos\frac{4\pi}{4}\right)e^{-j\frac{4\pi k}{4}} \\ + \left(\cos\frac{5\pi}{4}\right)e^{-j\frac{5\pi k}{4}} + \left(\cos\frac{6\pi}{4}\right)e^{-j\frac{6\pi k}{4}} + \left(\cos\frac{7\pi}{4}\right)e^{-j\frac{7\pi k}{4}}\end{array}\right]$$

$$= \frac{3}{8}\left[1 + \frac{1}{\sqrt{2}}e^{-j\frac{\pi k}{4}} + 0 - \frac{1}{\sqrt{2}}e^{-j\frac{3\pi k}{4}} - e^{-j\pi k} - \frac{1}{\sqrt{2}}e^{-j\frac{5\pi k}{4}} + 0 + \frac{1}{\sqrt{2}}e^{-j\frac{7\pi k}{4}}\right]$$

$$\therefore c_k = \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}e^{-j\frac{\pi k}{4}} - \frac{1}{\sqrt{2}}e^{-j\frac{3\pi k}{4}} - e^{-j\pi k} - \frac{1}{\sqrt{2}}e^{\left(-j\frac{8\pi k}{4}+j\frac{3\pi k}{4}\right)} + \frac{1}{\sqrt{2}}e^{\left(-j\frac{8\pi k}{4}+j\frac{\pi k}{4}\right)}\right]$$

$$= \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}e^{-j\frac{\pi k}{4}} - \frac{1}{\sqrt{2}}e^{-j\frac{3\pi k}{4}} - e^{-j\pi k} - \frac{1}{\sqrt{2}}e^{-j2\pi k}\,e^{j\frac{3\pi k}{4}} + \frac{1}{\sqrt{2}}e^{-j2\pi k}\,e^{j\frac{\pi k}{4}}\right] \qquad \boxed{e^{x+y} = e^x\,e^y}$$

$$= \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}e^{-j\frac{\pi k}{4}} - \frac{1}{\sqrt{2}}e^{-j\frac{3\pi k}{4}} - e^{-j\pi k} - \frac{1}{\sqrt{2}}e^{j\frac{3\pi k}{4}} + \frac{1}{\sqrt{2}}e^{j\frac{\pi k}{4}}\right]$$

For integer k,
$e^{-j2\pi k} = \cos 2\pi k - j\sin 2\pi k$
$= 1 - j0 = 1$

$$= \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}\left(e^{j\frac{\pi k}{4}} + e^{-j\frac{\pi k}{4}}\right) - \frac{1}{\sqrt{2}}\left(e^{j\frac{3\pi k}{4}} + e^{-j\frac{3\pi k}{4}}\right) - e^{-j\pi k}\right]$$

$$= \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}\,2\cos\frac{\pi k}{4} - \frac{1}{\sqrt{2}}\,2\cos\frac{3\pi k}{4} - \left(\cos\pi k - j\sin\pi k\right)\right]$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

For integer k, $\sin \pi k = 0$

$$= \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi k}{4} - \sqrt{2}\cos\frac{3\pi k}{4} - \cos\pi k\right]$$

When k = 0 ; $c_k = c_0 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 0}{4} - \sqrt{2}\cos\frac{3\pi\times 0}{4} - \cos\pi\times 0\right] = \frac{3}{8}\times 0 = 0$

When k = 1 ; $c_k = c_1 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 1}{4} - \sqrt{2}\cos\frac{3\pi\times 1}{4} - \cos\pi\times 1\right] = \frac{3}{8}\times 4 = 1.5$

When k = 2 ; $c_k = c_2 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 2}{4} - \sqrt{2}\cos\frac{3\pi\times 2}{4} - \cos\pi\times 2\right] = \frac{3}{8}\times 0 = 0$

When k = 3 ; $c_k = c_3 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 3}{4} - \sqrt{2}\cos\frac{3\pi\times 3}{4} - \cos\pi\times 3\right] = \frac{3}{8}\times 0 = 0$

When k = 4 ; $c_k = c_4 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 4}{4} - \sqrt{2}\cos\frac{3\pi\times 4}{4} - \cos\pi\times 4\right] = \frac{3}{8}\times 0 = 0$

When k = 5 ; $c_k = c_5 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 5}{4} - \sqrt{2}\cos\frac{3\pi\times 5}{4} - \cos\pi\times 5\right] = \frac{3}{8}\times 0 = 0$

When k = 6 ; $c_k = c_6 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 6}{4} - \sqrt{2}\cos\frac{3\pi\times 6}{4} - \cos\pi\times 6\right] = \frac{3}{8}\times 0 = 0$

When k = 7 ; $c_k = c_7 = \frac{3}{8}\left[1+\sqrt{2}\cos\frac{\pi\times 7}{4} - \sqrt{2}\cos\frac{3\pi\times 7}{4} - \cos\pi\times 7\right] = \frac{3}{8}\times 4 = 1.5$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\,e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{7} c_k\,e^{\frac{j2\pi kn}{8}} = \sum_{k=0}^{7} c_k\,e^{\frac{j\pi kn}{4}}$$

$$= c_0 + c_1\,e^{\frac{j\pi n}{4}} + c_2\,e^{\frac{j2\pi n}{4}} + c_3\,e^{\frac{j3\pi n}{4}} + c_4\,e^{\frac{j4\pi n}{4}} + c_5\,e^{\frac{j5\pi n}{4}} + c_6\,e^{\frac{j6\pi n}{4}} + c_7\,e^{\frac{j7\pi n}{4}}$$

$$= 0 + 1.5\,e^{\frac{j\pi n}{4}} + 0 + 0 + 0 + 0 + 0 + 1.5\,e^{\frac{j7\pi n}{4}} = 1.5\,e^{j\omega_0 n} + 1.5\,e^{j7\omega_0 n}\ ; \ \text{where } \omega_0 = \frac{\pi}{4}$$

---

c) Given that, $x(n) = e^{\frac{j5\pi n}{2}}$

Test for Periodicity

Let, $x(n+N) = e^{\frac{j5\pi(n+N)}{2}} = e^{\left(\frac{j5\pi n}{2}+\frac{j5\pi N}{2}\right)}$

For periodicity $\frac{5\pi N}{2}$ should be integral multiple of 2π.

Let, $\frac{5\pi N}{2} = 2\pi \times M \quad \Rightarrow \quad N = \frac{4}{5}M$

Here, N is integer for M = 5, 10, 15, ...

Let, M = 5, \ N = 4

Here, x(n) is periodic with fundamental period N = 4, and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$

Fourier Series

The Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0, 1, 2, 3, \ldots, N-1$$

Here, N = 4 and $x(n) = e^{\frac{j5\pi n}{2}}$

$$\therefore\ c_k = \frac{1}{4}\sum_{n=0}^{3} e^{\frac{j5\pi n}{2}}\, e^{\frac{-j2\pi kn}{4}} \quad ; \quad \text{for } k = 0, 1, 2, 3$$

$$= \frac{1}{4}\sum_{n=0}^{3} e^{\frac{j\pi n(5-k)}{2}} = \frac{1}{4}\left[e^0 + e^{\frac{j\pi(5-k)}{2}} + e^{\frac{j2\pi(5-k)}{2}} + e^{\frac{j3\pi(5-k)}{2}}\right]$$

$$= \frac{1}{4}\left[1 + e^{\frac{j\pi(5-k)}{2}} + e^{j\pi(5-k)} + e^{\frac{j3\pi(5-k)}{2}}\right]$$

$$= \frac{1}{4}\left[1 + \cos\frac{\pi(5-k)}{2} + j\sin\frac{\pi(5-k)}{2} + \cos\pi(5-k) + j\sin\pi(5-k) + \cos\frac{3\pi(5-k)}{2} + j\sin\frac{3\pi(5-k)}{2}\right]$$

When k = 0; $c_k = c_0 = \frac{1}{4}\left[1 + \cos\frac{5\pi}{2} + j\sin\frac{5\pi}{2} + \cos 5\pi + j\sin 5\pi + \cos\frac{15\pi}{2} + j\sin\frac{15\pi}{2}\right]$

$$= \frac{1}{4}\left[1 + 0 + j - 1 + j0 + 0 - j\right] = 0$$

When k = 1; $c_k = c_1 = \frac{1}{4}\left[1 + \cos 2\pi + j\sin 2\pi + \cos 4\pi + j\sin 4\pi + \cos 6\pi + j\sin 6\pi\right]$

$$= \frac{1}{4}\left[1 + 1 + j0 + 1 + j0 + 1 + j0\right] = 1$$

When k = 2; $c_k = c_2 = \frac{1}{4}\left[1 + \cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2} + \cos 3\pi + j\sin 3\pi + \cos\frac{9\pi}{2} + j\sin\frac{9\pi}{2}\right]$

$$= \frac{1}{4}\left[1 + 0 - j - 1 + j0 + 0 + j\right] = 0$$

When k = 3; $c_k = c_3 = \frac{1}{4}\left[1 + \cos\pi + j\sin\pi + \cos 2\pi + j\sin 2\pi + \cos 3\pi + j\sin 3\pi\right]$

$$= \frac{1}{4}\left[1 - 1 + j0 + 1 + j0 - 1 + j0\right] = 0$$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{3} c_k\, e^{\frac{j2\pi kn}{4}} = \sum_{k=0}^{3} c_k\, e^{\frac{j\pi kn}{2}}$$

$$= c_0 + c_1 e^{\frac{j\pi n}{2}} + c_2 e^{j\pi n} + c_3 e^{\frac{j3\pi n}{2}} = 0 + e^{\frac{j\pi n}{2}} + 0 + 0 = e^{\frac{j\pi n}{2}} = e^{j\omega_0 n}$$

Note : $x(n) = e^{\frac{j5\pi n}{2}} = e^{j\left(\frac{4\pi n}{2} + \frac{\pi n}{2}\right)} = e^{j2\pi n}\, e^{\frac{j\pi n}{2}} = e^{\frac{j\pi n}{2}} = e^{j\omega_0 n}$

$\therefore$ *The given signal itself is in the Fourier series form.*

Example 4.2

Determine the Fourier series representation of the following discrete time signal and sketch the frequency spectrum.

$$x(n) = \{....., 1, 2, -3, \underset{\uparrow}{1}, 2, -3, 1, 2, -3, ...\}$$

Solution

Given that, $x(n) = \{....., 1, 2, -3, \underset{\uparrow}{1}, 2, -3, 1, 2, -3, ...\}$

Here the three samples 1, 2, –3 repeat again and again.

Therefore, x(n) is periodic with periodicity of N = 3, and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{3}$.

Let, x(n) = {1, 2, –3} (considering one period). Now, the Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \frac{1}{3}\sum_{n=0}^{2} x(n)\, e^{\frac{-j2\pi kn}{3}}$$

$$= \frac{1}{3}\left[x(0) + x(1)\, e^{\frac{-j2\pi k}{3}} + x(2)\, e^{\frac{-j4\pi k}{3}}\right] = \frac{1}{3}\left[1 + 2\, e^{\frac{-j2\pi k}{3}} - 3e^{\frac{-j4\pi k}{3}}\right]$$

When $k = 0;\ c_k = c_0 = \frac{1}{3}[1 + 2 - 3] = 0$

When $k = 1;\ c_k = c_1 = \frac{1}{3}\left[1 + 2\, e^{\frac{-j2\pi}{3}} - 3e^{\frac{-j4\pi}{3}}\right]$

$$= \frac{1}{3}\left[1 + 2\cos\frac{2\pi}{3} - j2\sin\frac{2\pi}{3} - 3\cos\frac{4\pi}{3} + 3j\sin\frac{4\pi}{3}\right]$$

$$= \frac{1}{3}\left[1 - 2\times\frac{1}{2} - j2\times\frac{\sqrt{3}}{2} + 3\times\frac{1}{2} - 3j\times\frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{3}\left[\frac{3}{2} - j\frac{5\sqrt{3}}{2}\right] = \frac{1}{2} - j\frac{5\sqrt{3}}{6} = 0.5 - j1.443 \qquad \boxed{\frac{1.24}{\pi}\times\pi = 0.395\pi}$$

$$= 1.527\angle -1.24 \text{ rad} = 1.527\angle -0.395\pi = 1.527\, e^{-0.395\pi}$$

When $k = 2;\ c_k = c_2 = \frac{1}{3}\left[1 + 2\, e^{\frac{-j4\pi}{3}} - 3e^{\frac{-j8\pi}{3}}\right]$

$$= \frac{1}{3}\left[1 + 2\cos\frac{4\pi}{3} - j2\sin\frac{4\pi}{3} - 3\cos\frac{8\pi}{3} + 3j\sin\frac{8\pi}{3}\right]$$

$$= \frac{1}{3}\left[1 - 2\times\frac{1}{2} + j2\times\frac{\sqrt{3}}{2} + 3\times\frac{1}{2} + 3j\times\frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{3}\left[\frac{3}{2} + j\frac{5\sqrt{3}}{2}\right] = \frac{1}{2} + j\frac{5\sqrt{3}}{6} = 0.5 + j1.443 \qquad \boxed{\frac{1.24}{\pi}\times\pi = 0.395\pi}$$

$$= 1.527\angle 1.24 \text{ rad} = 1.527\angle 0.395\pi = 1.527\, e^{j0.395\pi}$$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{2} c_k\, e^{\frac{j2\pi kn}{3}} = c_0 + c_1\, e^{\frac{j2\pi n}{3}} + c_2\, e^{\frac{j4\pi n}{3}}$$

$$= 0 + 1.527\, e^{-j0.395\pi}\, e^{\frac{j2\pi n}{3}} + 1.527\, e^{j0.395\pi}\, e^{\frac{j4\pi n}{3}}$$

$$= 1.527\, e^{-j0.395\pi}\, e^{j\omega_0 n} + 1.527\, e^{j0.395\pi}\, e^{j2\omega_0 n}$$

Frequency Spectrum

The frequency spectrum has two components : Magnitude spectrum and Phase spectrum.

The magnitude spectrum is obtained from magnitude of c_k and phase spectrum is obtained from phase of c_k.

Here, $c_k = \{c_0, \ c_1, \ c_2\} = \{0, \ \ 1.527 \angle -0.395\pi, \ \ 1.527 \angle 0.395\pi\}$

$\therefore$ Magnitude spectrum, $|c_k| = \{0, \ 1.527, \ 1.527\}$

Phase spectrum, $\angle c_k = \{0, \ -0.395\pi, \ 0.395\pi\}$

The sketch of magnitude and phase spectrum are shown in fig 1.

Here both the spectrum are periodic with period, N = 3.

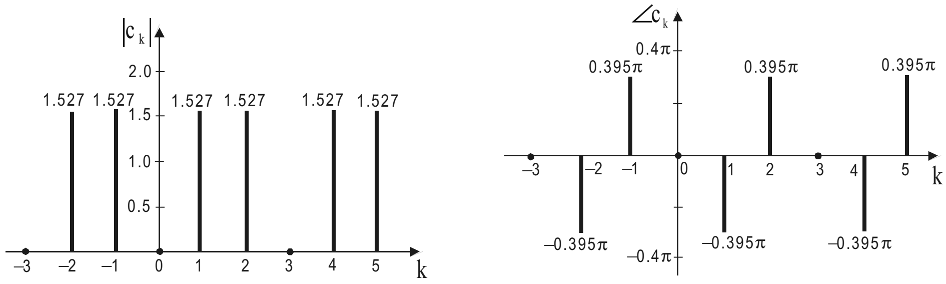


Fig 1a : Magnitude spectrum. *Fig 1b : Phase spectrum.*

Fig 1 : Frequency spectrum.

4.3 Fourier Transform of Discrete Time Signals (Discrete Time Fourier Transform)

4.3.1 Development of Discrete Time Fourier Transform From Discrete Time Fourier Series

Let $\tilde{x}(n)$ be a periodic sequence with period N. If the period N tends to infinity then the periodic sequence $\tilde{x}(n)$ will become a nonperiodic sequence x(n).

$$\therefore \ x(n) = \underset{N \to \infty}{Lt} \ \tilde{x}(n)$$

Let c_k be Fourier coefficients of $\tilde{x}(n)$.

$$\therefore c_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}(n) \, e^{\frac{-j2\pi kn}{N}} \qquad \Rightarrow \qquad Nc_k = \sum_{n=0}^{N-1} \tilde{x}(n) \, e^{\frac{-j2\pi kn}{N}}$$

Since $\tilde{x}(n)$ is periodic, for even values of N, the summation index in the above equation can be changed from $n = -\left(\frac{N}{2} - 1\right)$ to $+\frac{N}{2}$. (For odd values of N, the summation index is $n = -\frac{N}{2}$ to $+\frac{N}{2}$).

$$\therefore Nc_k = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}} \tilde{x}(n) \, e^{\frac{-j2\pi kn}{N}} = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}} \tilde{x}(n) \, e^{-j\omega_k n} \qquad .....(4.3)$$

$$\text{where, } \ \omega_k = \frac{2\pi k}{N}$$

Let us define Nc_k as a function of $e^{j\omega_k}$.

$$\therefore\ X(e^{j\omega_k}) = Nc_k \qquad .....(4.4)$$

Now, using equation (4.3), the equation (4.4) can be expressed as shown below.

$$X(e^{j\omega_k}) = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}} \tilde{x}(n)\, e^{-j\omega_k n} \qquad .....(4.5)$$

Let, $N \rightarrow \infty$, in equation (4.5).

Now, $\tilde{x}(n) \rightarrow x(n)$, $w_k \rightarrow w$, and the summation index become $-\infty$ to $+\infty$.

Therefore, the equation (4.5) can be written as shown below.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(4.6)$$

The equation (4.6) is called Fourier transform of x(n), which is used to represent nonperiodic discrete time signal (as a function of frequency,w) in frequency domain.

Consider the Fourier series representation of $\tilde{x}(n)$ given below.

$$\tilde{x}(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}}$$

Let us multiply and divide the above equation by $N/2\pi$.

$$\tilde{x}(n) = \frac{N}{2\pi} \times \frac{2\pi}{N} \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \frac{N}{2\pi} \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} \frac{2\pi}{N}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} Nc_k\, e^{\frac{j2\pi kn}{N}} \frac{2\pi}{N} \qquad \boxed{\omega_k = \frac{2\pi k}{N}} \quad \boxed{\text{Using equation (4.4).}}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} X(e^{j\omega_k})\, e^{j\omega_k n} \frac{2\pi}{N} \qquad .....(4.7)$$

Let, $N \rightarrow \infty$, in equation (4.7).

Now, $\tilde{x}(n) \rightarrow x(n)$, $w_k \rightarrow w$, $2\pi / N \rightarrow dw$, and summation becomes integral with limits 0 to 2π

Therefore, the equation (4.7) can be written as shown below.

$$x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad .....(4.8)$$

The equation (4.8) is called inverse Fourier transform of x(n), which is used to extract the discrete time signal from its frequency domain representation.

Since equation (4.6) extracts the frequency components of discrete time signal, the transformation using equation (4.6) is also called ***analysis*** of discrete time signal x(n). Since equation (4.8) integrates or combines the frequency components of discrete time signal, the inverse transformation using equation (4.8) is also called ***synthesis*** of discrete time signal x(n).

4.3.2 Definition of Discrete Time Fourier Transform

The Fourier transform (FT) of discrete-time signals is called ***Discrete Time Fourier Transform*** (i.e., DTFT). But for convenience the DTFT is also referred as FT in this book.

Let, x(n) = Discrete time signal

$X(e^{jw})$ = Fourier transform of x(n)

The Fourier transform of a finite energy discrete time signal, x(n) is defined as,

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}}$$

Symbolically the Fourier transform of x(n) is denoted as,

$\mathcal{F}\{x(n)\}$

where, $\mathcal{F}$ is the operator that represents Fourier transform.

$$\boxed{\therefore\ X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}}$$

The Fourier transform of a signal is said to exist if it can be expressed in a valid functional form. Since the computation of Fourier transform involves summing infinite number of terms, the Fourier transform exists only for the signals that are absolutely summable, i.e., given a signal x(n), the $X(e^{jw})$ exists only when,

$$\sum_{n=-\infty}^{+\infty} |x(n)| < \infty$$

4.3.3 Frequency Spectrum of Discrete Time Signal

The Fourier transform $X(e^{jw})$ of a signal x(n) represents the frequency content of x(n). We can say that, by taking Fourier transform, the signal x(n) is decomposed into its frequency components. Hence $X(e^{jw})$ is also called ***frequency spectrum*** of discrete time signal or ***signal spectrum***.

Magnitude and Phase Spectrum

The $X(e^{jw})$ is a complex valued function of w, and so it can be expressed in rectangular form as,

$X(e^{jw}) = X_r(e^{jw}) + jX_i(e^{jw})$

where, $X_r(e^{jw})$ = Real part of $X(e^{jw})$

$X_i(e^{jw})$ = Imaginary part of $X(e^{jw})$

The polar form of $X(e^{jw})$ is,

$X(e^{jw}) = |X(e^{jw})|\ Ð\ X(e^{jw})$

where, $|X(e^{jw})|$ = Magnitude spectrum

$ÐX(e^{jw})$ = Phase spectrum

The ***magnitude spectrum*** is defined as,

$$\boxed{|X(e^{jw})|^2 = X(e^{jw})\, X^*(e^{jw}) \quad \text{or} \quad |X(e^{j\omega})| = \sqrt{X(e^{j\omega})\, X^*(e^{j\omega})}}$$

where, $X^*(e^{jw})$ is complex conjugate of $X(e^{jw})$

Alternatively, $|X(e^{j\omega})|^2 = X(e^{j\omega})\ X^*(e^{j\omega})$

$$= \left[X_r(e^{j\omega}) + jX_i(e^{j\omega})\right]\left[X_r(e^{j\omega}) - jX_i(e^{j\omega})\right] = X_r^2(e^{j\omega}) + X_i^2(e^{j\omega})$$

$$\boxed{\therefore\ |X(e^{j\omega})| = \sqrt{X_r^2(e^{j\omega}) + X_i^2(e^{j\omega})}}$$

The ***phase spectrum*** is defined as,

$$\boxed{\angle X(e^{j\omega}) = \mathrm{Arg}[X(e^{j\omega})] = \tan^{-1}\left[\frac{X_i(e^{j\omega})}{X_r(e^{j\omega})}\right]}$$

4.3.4 Inverse Discrete Time Fourier Transform

Let, $x(n)$ = Discrete time signal

$X(e^{jw})$ = Fourier transform of $x(n)$

The ***inverse discrete time Fourier transform*** of $X(e^{jw})$ is defined as,

$$\boxed{x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad ; \text{ for } \quad n = -\infty \text{ to } +\infty} \qquad .....(4.9)$$

Symbolically the inverse Fourier transform can be expressed as, $\mathcal{F}^{-1}\{X(e^{jw})\}$, where, $\mathcal{F}^{-1}$ is the operator that represents the inverse Fourier transform.

$$\boxed{\therefore\ x(n) = \mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \ ; \ \text{for} \quad n = -\infty \text{ to } +\infty}$$

Since $X(e^{jw})$ is periodic with period 2p, the limits of integral in the above definition of inverse Fourier transform can be either "-p to +p", or "0 to 2p", or "any interval of 2p".

We also refer to $x(n)$ and $X(e^{jw})$ as a Fourier transform pair and this relation is expressed as,

$$\boxed{x(n) \xrightleftharpoons[\mathcal{F}^{-1}]{\mathcal{F}} X(e^{jw})}$$

Alternate Method for Inverse Fourier Transform

The integral solution of equation (4.9)for the inverse Fourier transform is useful for analytic purpose, but sometimes it will be difficult to evaluate for typical functional forms of $X(e^{jw})$. An alternate and more useful method of determining the values of $x(n)$ follows directly from the definition of the Fourier transform.

Consider the definition of Fourier transform of $x(n)$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

Let us expand the above equation of $X(e^{jw})$ as shown below.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$$= ..... + x(-2)\, e^{j2\omega} + x(-1)\, e^{j\omega} + x(0)\, e^{0}$$

$$+ x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + ... \qquad .....(4.10.1)$$

Let us express the given function of $X(e^{jw})$ as a power series of e^{-jw} by long division as shown below.

$$X(e^{j\omega}) = ... + b_2\, e^{j2\omega} + b_1\, e^{j\omega} + a_0\, e^{0} + a_1\, e^{-j\omega} + a_2\, e^{-j2\omega} + ... \qquad .....(4.10.2)$$

On comparing the equations (4.10.1) and (4.10.2) we can say that the samples of signal $x(n)$ are simply the coefficients of e^{-jwn}.

4.3.5 Comparison of Fourier Transform of Discrete and Continuous Time Signals

1. The Fourier transform of a continuous time signal consists of a spectrum with a frequency range - ¥ to + ¥. But the Fourier transform of a discrete time signal is unique in the frequency range - p to + p (or equivalently 0 to 2p). Also Fourier transform of discrete time signal is periodic with period 2p. Hence the frequency range for any discrete-time signal is limited to -p to p (or 0 to 2p) and any frequency outside this interval has an equivalent frequency within this interval.

2. Since the continuous time signal is continuous in time the Fourier transform of continuous time signal involves integration but the Fourier transform of discrete time signal involves summation because the signal is discrete.

4.4 Properties of Discrete Time Fourier Transform

1. Linearity property

The linearity property of Fourier transform states that the Fourier transform of a linear weighted combination of two or more signals is equal to the similar linear weighted combination of the Fourier transform of the individual signals.

Let, $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$ and $\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$ then by linearity property,

$\mathcal{F}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$; where a_1 and a_2 are constants.

Proof :

By the definition of Fourier transform,

$$X_1(e^{j\omega}) = \mathcal{F}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} \qquad .....(4.11)$$

$$X_2(e^{j\omega}) = \mathcal{F}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n} \qquad .....(4.12)$$

$$\mathcal{F}\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[a_1 x_1(n) + a_2 x_2(n)\right] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} \left[a_1 x_1(n)\, e^{-j\omega n} + a_2 x_2(n)\, e^{-j\omega n}\right]$$

$$= \sum_{n=-\infty}^{+\infty} a_1 x_1(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{+\infty} a_2 x_2(n)\, e^{-j\omega n}$$

$$= a_1 \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} + a_2 \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n}$$

$$= a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

Using equations (4.11) and (4.12)

2. Periodicity

Let, $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $X(e^{j\omega})$ is periodic with period 2π.

$\therefore\ X(e^{j(\omega + 2\pi m)}) = X(e^{j\omega})$; where m is an integer

Proof :

$$X(e^{j(\omega + 2\pi m)}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j(\omega + 2\pi m)n}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}\, e^{-j2\pi mn}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = X(e^{j\omega})$$

Since m and n are integers, $e^{-j2\pi mn} = 1$

3. Time shifting or Fourier transform of delayed signal

Let, $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $\mathcal{F}\{x(n-m)\} = e^{-j\omega m}\, X(e^{j\omega})$

Also, $\mathcal{F}\{x(n+m)\} = e^{j\omega m} X(e^{j\omega})$

This relation means that if a signal is shifted in time domain by m samples, its magnitude spectrum remains unchanged. However, the phase spectrum is changed by an amount $\pm\omega m$. This result can be explained if we recall that the frequency content of a signal depends only on its shape. Mathematically, we can say that delaying by m units in time domain is equivalent to multiplying the spectrum by $e^{-j\omega m}$ in the frequency domain.

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(4.13)$$

$$\therefore\ \mathcal{F}\{x(n-m)\} = \sum_{n=-\infty}^{+\infty} x(n-m)\, e^{-j\omega n}$$

Let, n − m = p, \ n = p + m
when n ® -¥, p ® -¥
when n ® +¥, p ® +¥

$$= \sum_{p=-\infty}^{+\infty} x(p)\, e^{-j\omega(m+p)}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, e^{-j\omega m}\, e^{-j\omega p}$$

$$= e^{-j\omega m} \sum_{p=-\infty}^{+\infty} x(p)\, e^{-j\omega p} = e^{-j\omega m} \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

Let, $p \rightarrow n$

$$= e^{-j\omega m}\, X(e^{j\omega})$$

Using equation (4.13)

4. Time reversal

Let, $\mathcal{F}\{x(n)\} = X(e^{jw})$, then $\mathcal{F}\{x(-n)\} = X(e^{-jw})$

This means that if a signal is folded about the origin in time, its magnitude spectrum remains unchanged and the phase spectrum undergoes a change in sign (phase reversal).

Proof :

By the definition of Fourier transform,

$$\mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(4.14)$$

$$\mathcal{F}\{x(-n)\} = \sum_{n=-\infty}^{+\infty} x(-n)\, e^{-j\omega n} = \sum_{p=-\infty}^{+\infty} x(p)\, e^{j\omega p}$$

Let, p = −n
when n ® -¥, p ® +¥
when n ® +¥, p ® -¥

$$= \sum_{p=-\infty}^{+\infty} x(p)\, (e^{-j\omega})^{-p} \qquad .....(4.15)$$

$$= X(e^{-j\omega})$$

The equation (4.15) is similar to the form of equation (4.14)

5. Conjugation

If, $\mathcal{F}\{x(n)\} = X\,(e^{jw})$

then, $\mathcal{F}\{x^*(n)\} = X^*(e^{-jw})$

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$$\mathcal{F}\{x^*(n)\} = \sum_{n=-\infty}^{+\infty} x^*(n)\, e^{-j\omega n}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x(n)\, (e^{-j\omega})^{-n}\right]^* = \left[X(e^{-j\omega})\right]^*$$

$$= X^*(e^{-j\omega})$$

6. Frequency shifting

Let, $\mathcal{F}\{x(n)\} = X(e^{jw})$, then $\mathcal{F}\{e^{j\omega_0 n}\,x(n)\} = X(e^{j(\omega - \omega_0)})$

According to this property, multiplication of a sequence x(n) by $e^{j\omega_0 n}$ is equivalent to a frequency translation of the spectrum $X(e^{jw})$ by w_0.

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(4.16)$$

$$\therefore\ \mathcal{F}\{e^{j\omega_0 n}\, x(n)\} = \sum_{n=-\infty}^{+\infty} e^{j\omega_0 n}\, x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j(\omega - \omega_o)n} \qquad .....(4.17)$$

$$= X(e^{j(\omega - \omega_o)})$$

The equation (4.17) is similar to the form of equation (4.16)

7. Fourier transform of the product of two signals

Let, $\mathcal{F}\{x_1(n)\} = X_1(e^{jw})$

$\mathcal{F}\{x_2(n)\} = X_2(e^{jw})$

$$\text{Now, } \mathcal{F}\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega - \lambda)})\, d\lambda \qquad .....(4.18)$$

The equation (4.18) is convolution of $X_1(e^{jw})$ and $X_2(e^{jw})$

This relation is the dual of time domain convolution. In other words, the Fourier transform of the product of two discrete time signals is equivalent to the convolution of their Fourier transform. [On the other hand, the Fourier transform of the convolution of two discrete time signals is equivalent to the product of their Fourier transform.]

Proof :

Let, $x_2(n)\, x_1(n) = x_3(n)$

$$\text{Now, } \mathcal{F}\{x_2(n)\, x_1(n)\} = \mathcal{F}\{x_3(n)\} = \sum_{n=-\infty}^{+\infty} x_3(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x_2(n)\, x_1(n)\, e^{-j\omega n} \qquad .....(4.19)$$

By the definition of inverse Fourier transform we get,

$$x_1(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Let, w = 1

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, e^{j\lambda n}\, d\lambda \qquad .....(4.20)$$

On substituting for $x_1(n)$ from equation (4.20) in equation (4.19) we get,

$$\mathcal{F}\{x_1(n)\, x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n) \left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, e^{j\lambda n}\, d\lambda\right] e^{-j\omega n}$$

On interchanging the order of summation and integration in the above equation we get,

$$\mathcal{F}\{x_1(n)\,x_2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left[\sum_{n=-\infty}^{+\infty} x_2(n)\,e^{-j(\omega-\lambda)n}\right] X_1(e^{j\lambda})\,d\lambda$$

The term in the paranthesis in the above equation is similar to the definition of fourier transform of $x_2(n)$ but at a frequency argument of $(w - l)$.

$$\therefore\ \mathcal{F}\{x_1(n)\,x_2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_2(e^{j(\omega-\lambda)})\,X_1(e^{j\lambda})\,d\lambda$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\,X_2(e^{j(\omega-\lambda)})\,d\lambda$$

8. Differentiation in frequency domain

If, $\mathcal{F}\{x(n)\} = X(e^{jw})$

then, $\mathcal{F}\{n\,x(n)\} = j\dfrac{d}{d\omega}X(e^{j\omega})$

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n} \quad .....(4.21)$$

$$\mathcal{F}\{n\,x(n)\} = \sum_{n=-\infty}^{+\infty} n\,x(n)\,e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} n\,x(n)\,j\times(-j)\,e^{-j\omega n}$$

Multiply by j and -j
j ´ (–j) = 1

$$= j\sum_{n=-\infty}^{+\infty} x(n)\,[-jn\,e^{-j\omega n}]$$

$$= j\sum_{n=-\infty}^{+\infty} x(n)\left[\frac{d}{d\omega}e^{-j\omega n}\right]$$

$\dfrac{d}{d\omega}e^{-j\omega n} = -jn\,e^{-j\omega n}$

$$= j\,\frac{d}{d\omega}\left[\sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n}\right]$$

Interchanging summation and differentiation

$$= j\,\frac{d}{d\omega}X(e^{j\omega})$$

Using equation (4.21)

9. Convolution theorem

If, $\mathcal{F}\{x_1(n)\} = X_1(e^{jw})$

and, $\mathcal{F}\{x_2(n)\} = X_2(e^{jw})$

then, $\mathcal{F}\{x_1(n) * x_2(n)\} = X_1(e^{jw})\,X_2(e^{jw})$

$$\text{where,}\quad x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\,x_2(n-m) \quad .....(4.22)$$

The Fourier transform of the convolution of $x_1(n)$ and $x_2(n)$ is equal to the product of $X_1(e^{jw})$ and $X_2(e^{jw})$. It means that if we convolve two signals in time domain, it is equivalent to multiplying their spectra in frequency domain.

Proof :

By the definition of Fourier transform,

$$X_1(e^{j\omega}) = \mathcal{F}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} \qquad .....(4.23)$$

$$X_2(e^{j\omega}) = \mathcal{F}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n} \qquad .....(4.24)$$

$$\mathcal{F}\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[x_1(n) * x_2(n)\right] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} \left[\sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\right] e^{-j\omega n}$$

Using equation (4.22)

$$= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\, e^{-j\omega n}\, e^{-j\omega m}\, e^{j\omega m}$$

Multiply by e^{-jwm} and e^{jwm}
$e^{-jwm} \times e^{jwm} = 1$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, e^{-j\omega m} \sum_{n=-\infty}^{+\infty} x_2(n-m)\, e^{-j\omega(n-m)}$$

Let, $n - m = p$
when $n \rightarrow -\infty,\ p \rightarrow -\infty$
when $n \rightarrow +\infty,\ p \rightarrow +\infty$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, e^{-j\omega m} \sum_{p=-\infty}^{+\infty} x_2(p)\, e^{-j\omega p}$$

Let $m = n$, in first summation
Let $p = n$, in second summation

$$= \left[\sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n}\right]\left[\sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n}\right]$$

Using equations (4.23) and (4.24)

$$= X_1(e^{j\omega})\, X_2(e^{j\omega})$$

10. Correlation

If, $\mathcal{F}\{x(n)\} = X(e^{jw})$ and $\mathcal{F}\{y(n)\} = Y(e^{jw})$

then, $\mathcal{F}\{r_{xy}(m)\} = X(e^{jw})\, Y(e^{-jw})$

$$\text{where, } r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) \qquad .....(4.25)$$

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(4.26)$$

$$Y(e^{j\omega}) = \mathcal{F}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\, e^{-j\omega n} \qquad .....(4.27)$$

$$\mathcal{F}\{r_{xy}(m)\} = \sum_{m=-\infty}^{+\infty} r_{xy}(m)\, e^{-j\omega m} = \sum_{m=-\infty}^{+\infty} \left[\sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)\right] e^{-j\omega m}$$

Using equation (4.25)

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)\, e^{-j\omega m}\, e^{-j\omega n}\, e^{j\omega n}$$

Multiply by e^{-jwn} and e^{jwn}
$e^{-jwn} \times e^{jwn} = 1$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \sum_{m=-\infty}^{+\infty} y(n-m)\, e^{j\omega(n-m)}$$

Let, $n - m = p \quad \therefore m = n - p$
when $m \rightarrow -\infty,\ p \rightarrow +\infty$,
when $m \rightarrow +\infty,\ p \rightarrow -\infty$.

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \sum_{p=-\infty}^{+\infty} y(p)\, e^{j\omega p}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}\right]\left[\sum_{p=-\infty}^{+\infty} y(p)\, (e^{-j\omega})^{-p}\right]$$

Using equations (4.26) and (4.27)

$$= X(e^{j\omega})\, Y(e^{-j\omega})$$

11. Parseval's relation

If, $\mathcal{F}\{x_1(n)\} = X_1(e^{jw})$ and, $\mathcal{F}\{x_2(n)\} = X_2(e^{jw})$,

then the Parseval's relation states that,

$$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\; d\omega \qquad .....(4.28)$$

When $x_1(n) = x_2(n) = x(n)$, then Parseval's relation can be written as,

$$\sum_{n=-\infty}^{+\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \left|X(e^{j\omega})\right|^2\; d\omega$$

$x(n)\, x^*(n) = |x(n)|^2$

$X(e^{jw})\, X^*(e^{jw}) = |X(e^{jw})|^2$

The above equation is also called ***energy density spectrum*** of the signal x(n).

Proof :

Let, $\mathcal{F}\{x_1(n)\} = X_1(e^{jw})$ and $\mathcal{F}\{x_2(n)\} = X_2(e^{jw})$

Now, by definition of Fourier transform,

$$\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} \qquad .....(4.29)$$

Now, by definition of inverse Fourier transform,

$$x_2(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_2(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad .....(4.30)$$

Consider left-hand side of Parseval's relation [equation (4.28)],

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\; d\omega$$

In the above expression, Let us substitute for $X_1(e^{jw})$ from equation (4.29),

$$\therefore\ \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\; d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left[\sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n}\right] X_2^*(e^{j\omega})\; d\omega$$

Interchanging summation and integration

$$= \sum_{n=-\infty}^{+\infty} x_1(n)\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_2^*(e^{j\omega})\, e^{-j\omega n}\; d\omega\right]$$

Using equation (4.30)

$$= \sum_{n=-\infty}^{+\infty} x_1(n)\left[\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_2(e^{j\omega})\, e^{j\omega n}\; d\omega\right]^*$$

$$= \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)$$

Table 4.2 : Properties of Discrete Time Fourier Transform

Note : $X(e^{jw}) = \mathcal{F}\{x(n)\}$; $X_1(e^{jw}) = \mathcal{F}\{x_1(n)\}$; $X_2(e^{jw}) = \mathcal{F}\{x_2(n)\}$; $Y(e^{jw}) = F\{y(n)\}$

Property	Discrete time signal	Fourier transform
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(e^{jw}) + a_2 X_2(e^{jw})$
Periodicity	$x(n)$	$X(e^{jw+2pm}) = X(e^{jw})$
Time shifting	$x(n-m)$	$e^{-jwm} X(e^{jw})$
Time reversal	$x(-n)$	$X(e^{-jw})$
Conjugation	$x^*(n)$	$X^*(e^{-jw})$
Frequency shifting	$e^{j\omega_0 n}\ x(n)$	$X(e^{j(\omega - \omega_0)})$
Multiplication	$x_1(n)\, x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega-\lambda)})\, d\lambda$
Differentiation in frequency domain	$n\,x(n)$	$j\frac{d}{d\omega} X(e^{j\omega})$
Convolution	$x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)$	$X_1(e^{jw})\, X_2(e^{jw})$
Correlation	$r_{xy}(m) = \sum_{m=-\infty}^{+\infty} x(n)\, y(n-m)$	$X(e^{jw})\, Y(e^{-jw})$
Symmetry of real signals	x(n) is real	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $Re\{X(e^{j\omega})\} = Re\{X(e^{-j\omega})\}$ $Im\{X(e^{j\omega})\} = -Im\{X(e^{-j\omega})\}$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert,\ \angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
Symmetry of real and even signal	x(n) is real and even	$X(e^{jw})$ is real and even
Symmetry of real and odd signal	x(n) is real and odd	$X(e^{jw})$ is imaginary and odd
Parseval's relation	$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)$	$\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\, d\omega$
Parseval's relation	Energy in time domain, $E = \sum_{n=-\infty}^{+\infty} \lvert x(n)\rvert^2$	Energy in frequency domain, $E = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \lvert X(e^{j\omega})\rvert^2\, d\omega$
	$\sum_{n=-\infty}^{+\infty} \lvert x(n)\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2\, d\omega$	

Note : The term $\left|X(e^{jw})\right|^2$ *represents the distribution of energy as a function of frequency and so it is called energy density spectrum or energy spectral density.*

4.5 Discrete Time Fourier Transform of Periodic Discrete Time Signals

The Fourier transform of any periodic discrete time signal can be obtained from the knowledge of Fourier transform of periodic discrete time signal $e^{j\omega_0 n}$, with period N.

The Fourier transform of continuous time periodic signal is a train of impulses. Similarly, the Fourier transform of discrete time periodic signal is also a train of impulses, but the impulse train should be periodic. Therefore, the Fourier transform of $e^{j\omega_0 n}$ will be in the form of periodic impulse train with period 2π as shown in equation (4.31).

Let, $g(n) = e^{j\omega_0 n}$

$$\therefore G(e^{j\omega}) = \mathcal{F}\{g(n)\} = \mathcal{F}\{e^{j\omega_0 n}\} = \sum_{m=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi m) \qquad .....(4.31)$$

where, $\omega_0 = \dfrac{2\pi}{N}$ = Fundamental frequency of g(n).

In equation (4.31), $\delta(\omega)$ is an impulse function of ω and ω_0 lie in the range $-\pi$ to $+\pi$.

The equation (4.31) can be proved by taking inverse Fourier transform of $G(e^{j\omega})$ as shown below.

Proof :

$$G(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi m)$$

By the definition of inverse Fourier transform,

$$g(n) = \mathcal{F}^{-1}\{G(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\omega})\; e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sum_{m=-\infty}^{+\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi m) e^{j\omega n}\, d\omega$$

$$= \int_{-\pi}^{+\pi} \delta(\omega - \omega_0)\; e^{j\omega n}\, d\omega = e^{j\omega n}\Big|_{\omega=\omega_0} = e^{j\omega_0 n}$$

Note : Here the integral limit is $-\pi$ to $+\pi$, and in this range there is only one impulse located at ω_0.

Consider the Fourier series representation of periodic discrete time signal x(n), shown below.

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{j\omega_k n}$$

$$\text{where, } c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \;;\; \text{for } k = 0, 1, 2, ..., (N-1) \qquad .....(4.32)$$

$$\omega_k = \frac{2\pi k}{N}$$

On comparing g(n) and x(n), we can say that the Fourier transform of x(n) can be obtained from its Fourier series representation, as shown below.

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \mathcal{F}\left\{\sum_{k=0}^{N-1} c_k\; e^{j\omega_k n}\right\} = \sum_{k=-\infty}^{+\infty} c_k\; 2\pi\; \delta(\omega - \omega_k) \qquad .....(4.33)$$

The equation (4.33) can be used to compute Fourier transform of any periodic discrete time signal x(n), and the Fourier transform consists of train of impulses located at the harmonic frequencies of x(n).

Table 4.3 : Some Common Discrete Time Fourier Transform Pairs

x(t)	**x(n)**	**$X(e^{jw})$**	
		with positive power of e^{jw}	**with negative power of e^{jw}**
	$d(n)$	1	1
	$d(n-n_0)$	$\frac{1}{e^{j\omega n_0}}$	$e^{-j\omega n_0}$
	$u(n)$	$\frac{e^{j\omega}}{e^{j\omega}-1}+\sum_{m=-\infty}^{+\infty}\pi\,\delta(\omega-2\pi m)$	$\frac{1}{1-e^{-j\omega}}+\sum_{m=-\infty}^{+\infty}\pi\,\delta(\omega-2\pi m)$
	$a^n\,u(n)$	$\frac{e^{j\omega}}{e^{j\omega}-a}$	$\frac{1}{1-a\,e^{-j\omega}}$
	$n\,a^n\;u(n)$	$\frac{a\,e^{j\omega}}{(e^{j\omega}-a)^2}$	$\frac{a\,e^{-j\omega}}{(1-a\,e^{-j\omega})^2}$
	$n^2\,a^n\;u(n)$	$\frac{a\,e^{j\omega}\,(e^{j\omega}+a)}{(e^{j\omega}-a)^3}$	$\frac{a\,e^{-j\omega}\,(1+a\,e^{-j\omega})}{(1-a\,e^{-j\omega})^3}$
$e^{-at}u(t)$	$e^{-anT}u(nT)$	$\frac{e^{j\omega}}{e^{j\omega}-e^{-aT}}$	$\frac{1}{1-e^{-j\omega}e^{-aT}}$
	1	$2\pi\sum_{m=-\infty}^{+\infty}\delta(\omega-2\pi m)$	
	$a^{\lvert n\rvert}$	$\frac{1-a^2}{1-2a\cos\omega+a^2}$	
	$\sum_{m=-\infty}^{+\infty}\delta(n-mN)$	$\frac{2\pi}{N}\sum_{m=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi m}{N}\right)$	
$e^{j\Omega_0 t}$	$e^{j\Omega_0 nt}=e^{j\omega_0 n}$ where, $\omega_0=\Omega_0 T$	$2\pi\sum_{m=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi m)$	
$\sin\Omega_0 t$	$\sin\Omega_0 nT$ $=\sin\omega_0 n$ where, $\omega_0=\Omega_0 T$	$\frac{\pi}{j}\sum_{m=-\infty}^{+\infty}\left[\delta(\omega-\omega_0-2\pi m)-\delta(\omega+\omega_0-2\pi m)\right]$	
$\cos\Omega_0 t$	$\cos\Omega_0 nT$ $=\cos\omega_0 n$ where, $\omega_0=\Omega_0 T$	$\pi\sum_{m=-\infty}^{+\infty}\left[\delta(\omega-\omega_0-2\pi m)+\delta(\omega+\omega_0-2\pi m)\right]$	

4.6 Analysis of LTI Discrete Time System Using Discrete Time Fourier Transform

4.6.1 Transfer Function of LTI Discrete Time System in Frequency Domain

The ratio of Fourier transform of output and the Fourier transform of input is called ***transfer function*** of LTI discrete time system in frequency domain.

Let, $x(n)$ = Input to the discrete time system

$y(n)$ = Output of the discrete time system

\ $X(e^{jw})$ = Fourier transform of $x(n)$

$Y(e^{jw})$ = Fourier transform of $y(n)$

$$\text{Now, Transfer function} = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \qquad .....(4.34)$$

The transfer function of an LTI discrete time system in frequency domain can be obtained from the difference equation governing the input-output relation of the LTI discrete time system given below, [refer Chapter 2, equation (2.17)].

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

On taking Fourier transform of above equation and rearranging the resultant equation as a ratio of $Y(e^{jw})$ and $X(e^{jw})$, the transfer function of LTI discrete time system in frequency domain is obtained.

Impulse Response and Transfer Function

Let, $x(n)$ = Input of an LTI discrete time system

$y(n)$ = Output / Response of the LTI discrete time system for the input $x(n)$

$h(n)$ = Impulse response (i.e., response for impulse input)

Now, the response $y(n)$ of the discrete time system is given by convolution of input and impulse response, [refer Chapter 2, equation (2.33)].

$$\therefore\; y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) \qquad .....(4.35)$$

Let, $\mathcal{F}\{y(n)\} = Y(e^{jw})$; $\mathcal{F}\{x(n)\} = X(e^{jw})$; $\mathcal{F}\{h(n)\} = H(e^{jw})$

Now by convolution theorem of Fourier transform,

$$\mathcal{F}\{x(n) * h(n)\} = X(e^{jw})\, H(e^{jw}) \qquad .....(4.36)$$

Using equation (4.35), the equation (4.36) can be written as,

$$\mathcal{F}\{y(n)\} = X(e^{jw})\, H(e^{jw})$$

$$\backslash\; Y(e^{jw}) = X(e^{jw})\, H(e^{jw})$$

$$\boxed{\therefore\; H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}} \qquad .....(4.37)$$

From equations (4.34) and (4.37) we can say that the ***transfer function*** of a discrete time system in frequency domain is also given by discrete time Fourier transform of impulse response.

4.6.2 Response of LTI Discrete Time System using Discrete Time Fourier Transform

Consider the transfer function of LTI discrete time system.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Now, response in frequency domain, $Y(e^{j\omega}) = X(e^{j\omega})\ H(e^{j\omega})$ .....(4.38)

On taking inverse Fourier transform of equation (4.38) we get,

$$y(n) = \mathcal{F}^{-1}\{X(e^{j\omega})\ H(e^{j\omega})\} \qquad .....(4.39)$$

From the equation (4.39) we can say that the output y(n) is given by the inverse Fourier transform of the product of $X(e^{j\omega})$ and $H(e^{j\omega})$.

Since the transfer function is defined with zero initial conditions, the response obtained by using equation (4.39) is the forced response or steady state response of discrete time system.

4.6.3 Frequency Response of LTI Discrete Time System

The output y(n) of LTI system is given by convolution of h(n) and x(n).

$$y(n) = x(n) * h(n) = h(n) * x(n) = \sum_{m=-\infty}^{+\infty} h(m)\ x(n-m) \qquad .....(4.40)$$

$\boxed{Ae^{j\omega n} = A(\cos\omega n + j\sin\omega n)}$

Consider a special class of input (sinusoidal input),

$$x(n) = A\ e^{j\omega n}\ ; \quad -\infty < n < \infty \qquad .....(4.41)$$

where, A = Amplitude

ω = Arbitrary frequency in the interval $-\pi$ to $+\pi$.

$$\therefore\ x(n-m) = Ae^{j\omega(n-m)} \qquad .....(4.42)$$

On substituting for x(n–m) from equation (4.42) in equation (4.40) we get,

$$y(n) = \sum_{m=-\infty}^{+\infty} h(m)\ A\ e^{j\omega(n-m)} = \sum_{m=-\infty}^{+\infty} h(m)\ A\ e^{j\omega n}\ e^{-j\omega m}$$

$$= A\ e^{j\omega n} \sum_{m=-\infty}^{+\infty} h(m)\ e^{-j\omega m} \qquad .....(4.43)$$

By the definition of Fourier transform,

$\boxed{\text{Replace n by m.}}$

$$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\ e^{-j\omega n} = \sum_{m=-\infty}^{+\infty} h(m)\ e^{-j\omega m} \qquad .....(4.44)$$

Using equations (4.41) and (4.44), the equation (4.43) can be written as,

$$y(n) = x(n)\ H(e^{j\omega}) \qquad .....(4.45)$$

From equation (4.45), we can say that if a complex sinusoidal signal is given as input signal to an LTI system, then the output is also a sinusoidal signal of the same frequency modified by $H(e^{j\omega})$. Hence $H(e^{j\omega})$ is called the ***frequency response*** of the system. An LTI system is characterized in the frequency domain by its frequency response.

The function $H(e^{j\omega})$ is a complex quantity. Therefore, $H(e^{j\omega})$ produces a change in the amplitude and phase of the input signal.

Let us express $H(e^{j\omega})$ as magnitude function and phase function.

$$\therefore \; H(e^{j\omega}) = |H(e^{j\omega})| \; \angle H(e^{j\omega})$$

where, $|H(e^{j\omega})| =$ Magnitude function

$\angle H(e^{j\omega}) =$ Phase function

The sketch of magnitude function and phase function with respect to ω will give the frequency response graphically.

Let, $H(e^{j\omega}) = H_r(e^{j\omega}) + jH_i(e^{j\omega})$

where, $H_r(e^{j\omega}) =$ Real part of $H(e^{j\omega})$

$H_i(e^{j\omega}) =$ Imaginary part of $H(e^{j\omega})$

The ***magnitude function*** is defined as,

$$|H(e^{j\omega})|^2 = H(e^{j\omega})\, H^*(e^{j\omega}) = [H_r(e^{j\omega}) + jH_i(e^{j\omega})]\,[H_r(e^{j\omega}) - jH_i(e^{j\omega})]$$

where, $H^*(e^{j\omega})$ is complex conjugate of $H(e^{j\omega})$

$$\therefore \; |H(e^{j\omega})|^2 = H_r^2(e^{j\omega}) + H_i^2(e^{j\omega}) \quad \Rightarrow \quad |H(e^{j\omega})| = \sqrt{H_r^2(e^{j\omega}) + H_i^2(e^{j\omega})}$$

The ***phase function*** is defined as,

$$\angle H(e^{j\omega}) = \text{Arg}[H(e^{j\omega})] = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$$

The drawback in frequency response analysis using Fourier transform is that the frequency response is a continuous function of ω and so it cannot be processed by digital systems. This drawback is overcome in Discrete Fourier Transform (DFT) discussed in Chapter 5.

From equation (4.37) we can say that the frequency response $H(e^{j\omega})$ of an LTI system is same as transfer function in frequency domain and so, the frequency response is also given by the ratio of Fourier transform of output to Fourier transform of input.

$$\text{i.e., Frequency response, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \qquad .....(4.46)$$

Properties of Frequency Response

1. The frequency response is periodic function of ω with a period of 2π
2. If h(n) is real, then the magnitude of $H(e^{j\omega})$ is symmetric and phase of $H(e^{j\omega})$ is antisymmetric over the interval $0 \le \omega \le 2\pi$.
3. If h(n) is complex, then the real part of $H(e^{j\omega})$ is symmetric and the imaginary part of $H(e^{j\omega})$ is antisymmetric over the interval $0 \le \omega \le 2\pi$.
4. The impulse response h(n) is discrete, whereas the frequency response $H(e^{j\omega})$ is continuous function of ω.

4.6.4 Frequency Response of First-Order Discrete Time System

A first-order discrete time system is characterized by the difference equation,

$$y(n) = x(n) + a\, y(n-1) \qquad .....(4.47)$$

On taking Fourier transform of equation(4.47) we get,

$$Y(e^{jw}) = X(e^{jw}) + a\, e^{-jw}\, Y(e^{jw}) \quad \Rightarrow \quad Y(e^{jw}) - a\, e^{-jw}\, Y(e^{jw}) = X(e^{jw})$$

$$\therefore\ Y(e^{jw})\,[1 - a\, e^{-jw}] = X(e^{jw}) \quad \Rightarrow \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - a\, e^{-j\omega}} \qquad .....(4.48)$$

The equation(4.48) is the frequency response of first-order system. The frequency response can be expressed graphically as two functions: Magnitude function and Phase function.

The magnitude function of $H(e^{jw})$ is defined as,

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})\, H^*(e^{j\omega}) = \frac{1}{(1 - a\, e^{-j\omega})}\ \frac{1}{(1 - a\, e^{j\omega})} = \frac{1}{1 - a\, e^{j\omega} - a\, e^{-j\omega} + a^2\, e^{-j\omega}\, e^{j\omega}}$$

$$= \frac{1}{1 - a(e^{j\omega} + e^{-j\omega}) + a^2} = \frac{1}{1 - 2a\cos\omega + a^2} \qquad .....(4.49)$$

$$\therefore \left|H(e^{j\omega})\right| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

The phase function of $H(e^{jw})$ is defined as,

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]; \text{ where } H_r(e^{j\omega}) \text{ is real part and } H_i(e^{j\omega}) \text{ is imaginary part.}$$

To find the real part and imaginary part of $H(e^{jw})$, multiply the numerator and denominator of $H(e^{jw})$ [equation (4.48)], by the complex conjugate of the denominator as shown below.

$$H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}} \times \frac{1 - a\, e^{+j\omega}}{1 - a\, e^{+j\omega}} = \frac{1 - a\, e^{j\omega}}{1 - 2a\cos\omega + a^2} = \frac{1 - a(\cos\omega + j\sin\omega)}{1 - 2a\cos\omega + a^2}$$

$$= \frac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2} + j\, \frac{-a\sin\omega}{1 - 2a\cos\omega + a^2}$$

Using equation (4.49)
$e^{j\omega} = \cos\omega + j\sin\omega$

$$\therefore\ H_r(e^{j\omega}) = \frac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2} \quad \text{and} \quad H_i(e^{j\omega}) = \frac{-a\sin\omega}{1 - 2a\cos\omega + a^2}$$

The phase function, $\angle H(e^{j\omega}) = \tan^{-1}\left[\dfrac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\dfrac{-a\sin\omega}{1 - a\cos\omega}\right]$

The Magnitude and Phase responses are calculated for a = 0.5, 0.8, –0.5 and –0.8 and tabulated in Table-4.4. Using the calculated values, the $|H(e^{jw})|$ and $\angle H(e^{jw})$ are sketched graphically for a = 0.5, 0.8, –0.5 and –0.8 in fig 4.1, 4.2, 4.3 and 4.4 respectively. From the plots it is inferred that the first-order system behaves as a lowpass filter when "a" is in the range of "0 < a < 1" and behaves as a highpass filter when "a" is in the range of "–1 < a < 0".

Table 4.4 : Frequency Response of First-Order Discrete Time System

$$\left|H(e^{j\omega})\right| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) = \left[\frac{1}{\pi}\tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right)\right]\pi$$

w	a = 0.5		a = 0.8		a = –0.5		a = –0.8	
	$\|H(e^{jw})\|$	Ð $H(e^{jw})$	$\|H(e^{jw})\|$	Ð $H(e^{jw})$	$\|H(e^{jw})\|$	Ð $H(e^{jw})$	$\|H(e^{jw})\|$	Ð$H(e^{jw})$
$\frac{-8\pi}{8} = -\pi$	0.667	0	0.556	0	2	0	5	0
$\frac{-7\pi}{8}$	0.678	0.04p	0.566	0.06p	1.751	–0.11p	2.486	–0.28p
$\frac{-6\pi}{8}$	0.715	0.08p	0.601	0.11p	1.357	–0.16p	1.402	–0.29p
$\frac{-5\pi}{8}$	0.783	0.12p	0.666	0.16p	1.074	–0.17p	0.986	–0.26p
$\frac{-4\pi}{8} = \frac{-\pi}{2}$	0.894	0.15p	0.781	0.21p	0.894	–0.15p	0.781	–0.21p
$\frac{-3\pi}{8}$	1.074	0.17p	0.986	0.26p	0.783	–0.12p	0.666	–0.16p
$\frac{-2\pi}{8}$	1.357	0.16p	1.402	0.29p	0.715	–0.08p	0.601	–0.11p
$\frac{-\pi}{8}$	1.751	0.11p	2.486	0.28p	0.678	–0.04p	0.566	–0.06p
0	2	0	5	0	0.667	0	0.556	0
$\frac{\pi}{8}$	1.751	–0.11p	2.486	–0.28p	0.678	0.04p	0.566	0.06p
$\frac{2\pi}{8}$	1.357	–0.16p	1.402	–0.29p	0.715	0.08p	0.601	0.11p
$\frac{3\pi}{8}$	1.074	–0.17p	0.986	–0.26p	0.783	0.12p	0.666	0.16p
$\frac{4\pi}{8} = \frac{\pi}{2}$	0.894	–0.15p	0.781	–0.21p	0.894	0.15p	0.781	0.21p
$\frac{5\pi}{8}$	0.783	–0.12p	0.666	–0.16p	1.074	0.17p	0.986	0.26p
$\frac{6\pi}{8}$	0.715	–0.08p	0.601	–0.11p	1.357	0.16p	1.402	0.29p
$\frac{7\pi}{8}$	0.678	–0.04p	0.566	–0.06p	1.751	0.11p	2.486	0.28p
$\frac{8\pi}{8} = \pi$	0.667	0	0.556	0	2	0	5	0

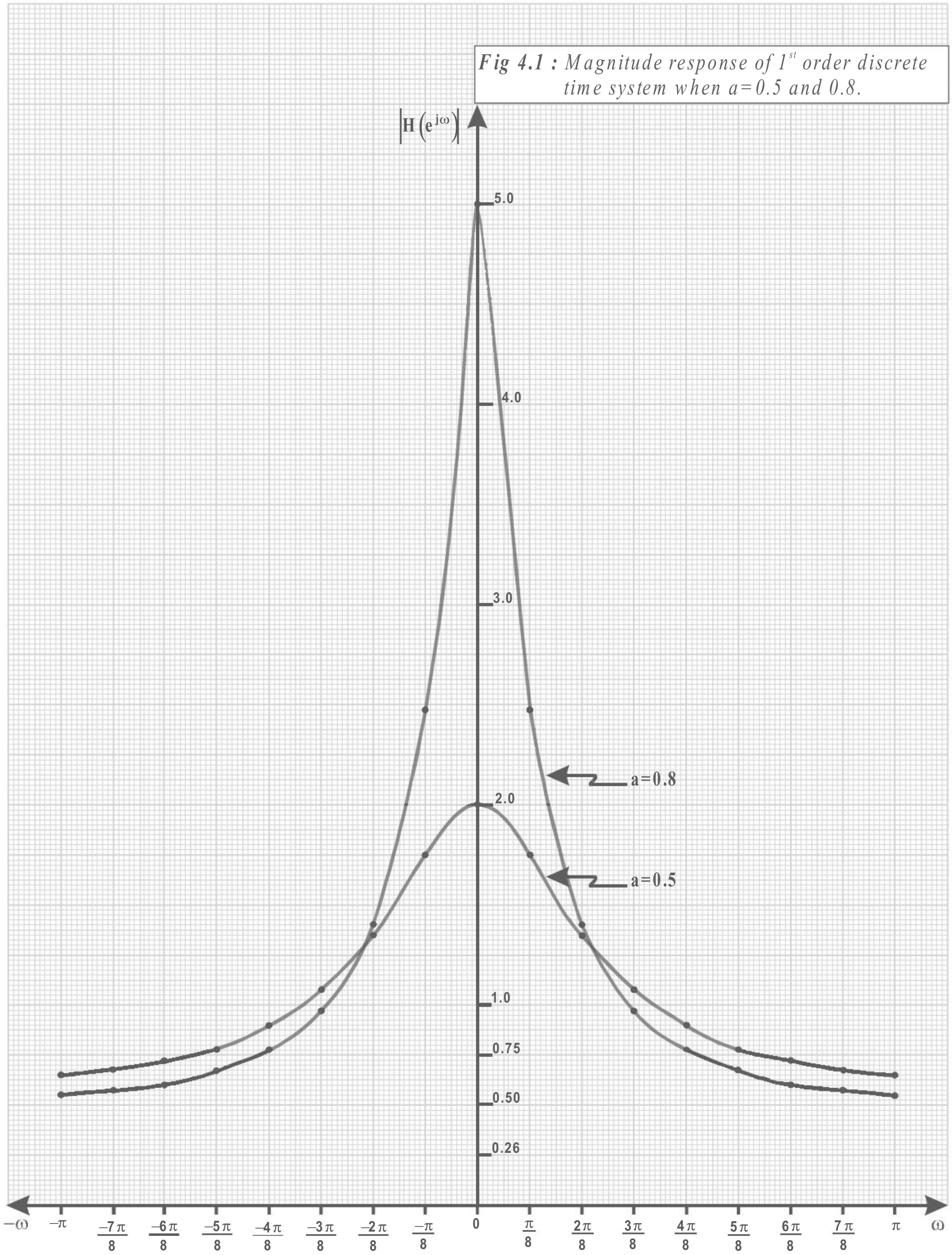
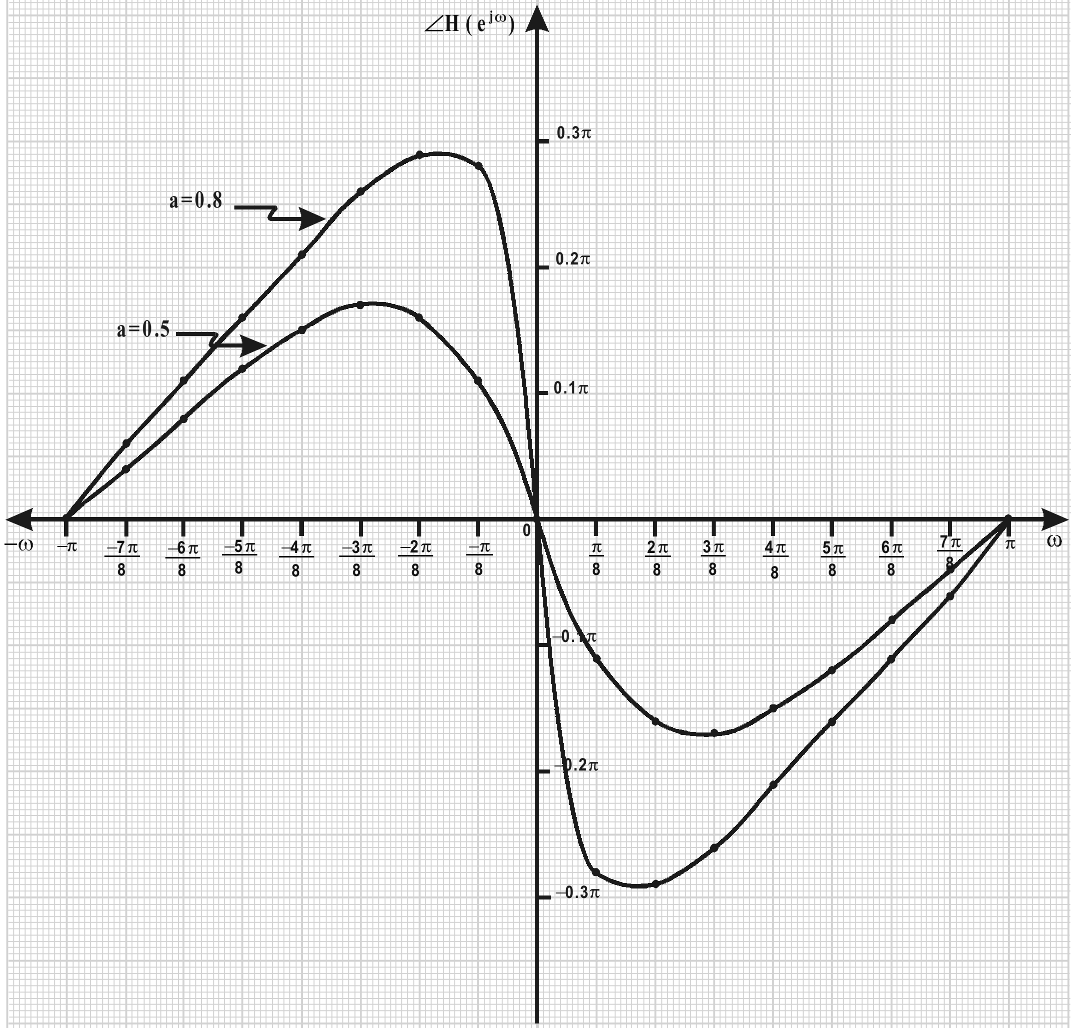


Fig 4.1 : *Magnitude response of 1st order discrete time system when a=0.5 and 0.8.*

Fig 4.2 : *Phase response of 1st order discrete time system when a=0.5 and a= 0.8*



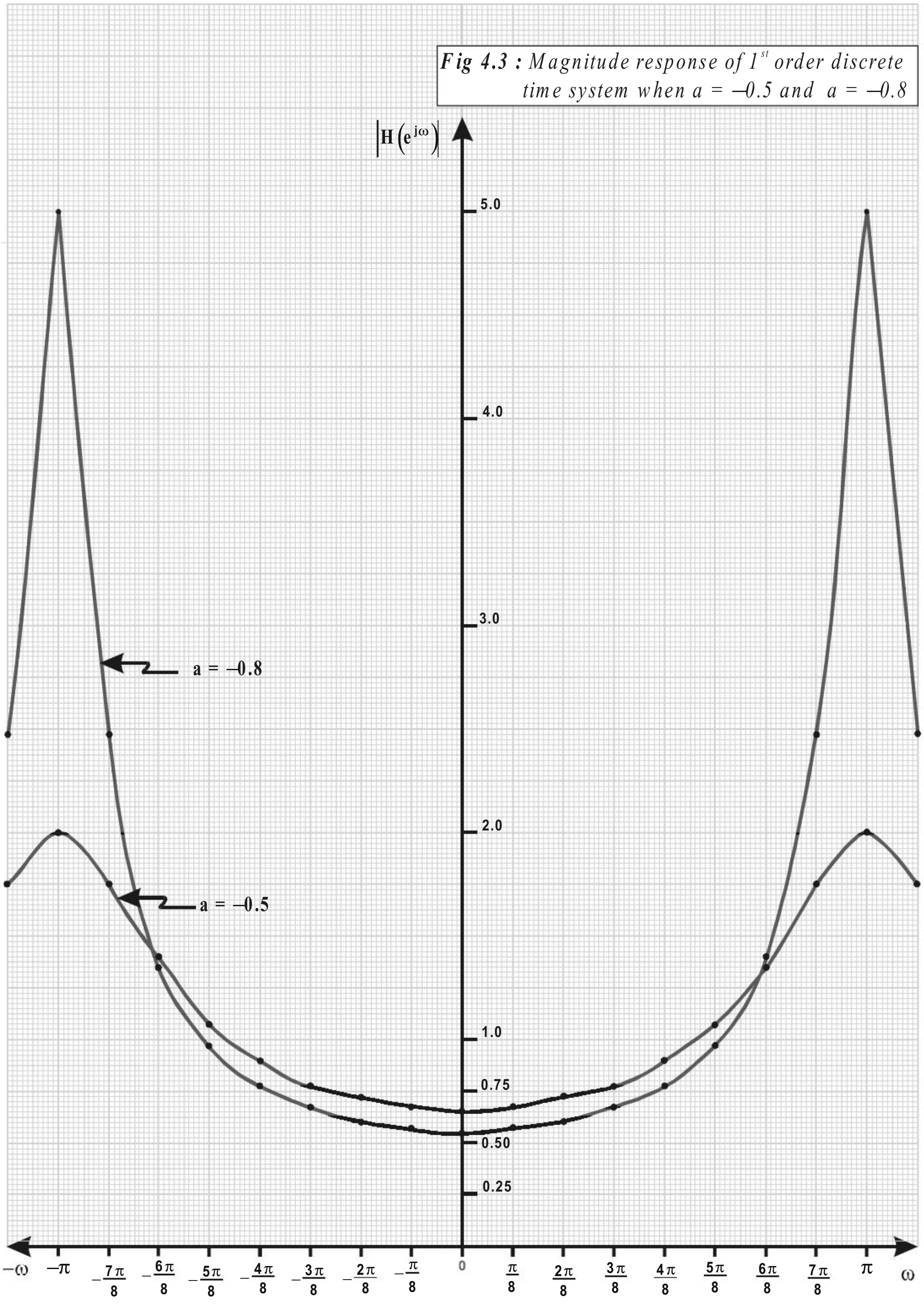
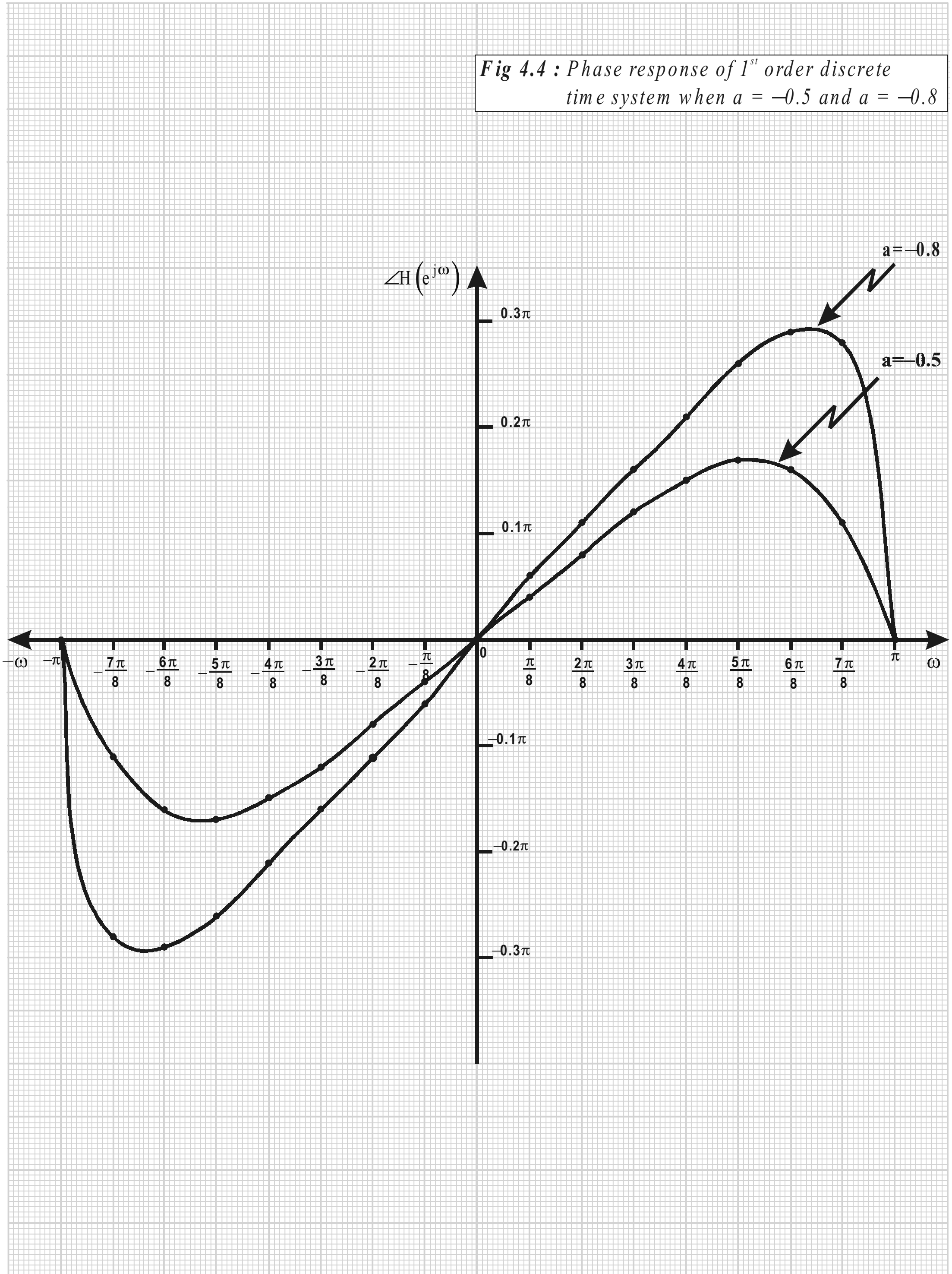


Fig 4.3 : Magnitude response of 1st order discrete time system when a = –0.5 and a = –0.8
|H(e^{jω})|
5.0
4.0
3.0
2.0
1.0
0.75
0.50
0.25
a = –0.8
a = –0.5
–ω
–π
– 7π/8
– 6π/8
– 5π/8
– 4π/8
– 3π/8
– 2π/8
– π/8
0
π/8
2π/8
3π/8
4π/8
5π/8
6π/8
7π/8
π
ω

Fig 4.4 : *Phase response of 1st order discrete time system when a = –0.5 and a = –0.8*



4.6.5 Frequency Response of Second-Order Discrete Time System

A second order discrete time system is characterized by the difference equation.

$$y(n) = 2r\cos w_0\, y(n-1) - r^2\, y(n-2) + x(n) - r\cos w_0\, x(n-1)$$

Let $a = -r\cos w_0$; $a = -2r\cos w_0$; $b = r^2$

$$\backslash \quad y(n) = -a\, y(n-1) - b\, y(n-2) + x(n) + a\, x(n-1) \qquad .....(4.50)$$

On taking Fourier transform of the equation (4.50) we get,

$$Y(e^{j\omega}) = -\alpha\, e^{-j\omega}\, Y(e^{j\omega}) - \beta\, e^{-j2\omega}\, Y(e^{j\omega}) + X(e^{j\omega}) + a\, e^{-j\omega}\, X(e^{j\omega})$$

$$Y(e^{j\omega}) + \alpha\, e^{-j\omega}\, Y(e^{j\omega}) + \beta\, e^{-j2\omega}\, Y(e^{j\omega}) = X(e^{j\omega}) + a\, e^{-j\omega}\, X(e^{j\omega})$$

$$Y(e^{j\omega})\left[1 + \alpha\, e^{-j\omega} + \beta\, e^{-j2\omega}\right] = X(e^{j\omega})\left[1 + a\, e^{-j\omega}\right]$$

$$\therefore\ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + a\, e^{-j\omega}}{1 + \alpha\, e^{-j\omega} + \beta\, e^{-j2\omega}} \qquad .....(4.51)$$

The equation (4.51) is the frequency response of second-order system.The frequency response can be expressed graphically as two functions: Magnitude function and Phase function.

The magnitude function of $H(e^{jw})$ is defined as,

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})\, H^*(e^{j\omega}) = \frac{1 + a\, e^{-j\omega}}{1 + \alpha e^{-j\omega} + \beta\, e^{-j2\omega}}\ \frac{1 + a\, e^{+j\omega}}{1 + \alpha\, e^{+j\omega} + \beta\, e^{+j2\omega}}$$

$$= \frac{1 + a\, e^{j\omega} + a\, e^{-j\omega} + a^2}{1 + \alpha\, e^{j\omega} + \beta\, e^{j2\omega} + \alpha\, e^{-j\omega} + \alpha^2 + \alpha\,\beta\, e^{j\omega} + \beta\, e^{-j2\omega} + \alpha\,\beta\, e^{-j\omega} + \beta^2}$$

$$= \frac{1 + a(e^{j\omega} + e^{-j\omega}) + a^2}{1 + \alpha^2 + \beta^2 + \alpha\beta(e^{j\omega} + e^{-j\omega}) + \beta(e^{j2\omega} + e^{-j2\omega}) + \alpha\,(e^{j\omega} + e^{-j\omega})}$$

$$= \frac{1 + 2a\cos\omega + a^2}{1 + \alpha^2 + \beta^2 + 2\alpha\beta\cos\omega + 2\beta\cos 2\omega + 2\alpha\cos\omega} \qquad .....(4.52)$$

$$\therefore \text{ Magnitude function, } \left|H(e^{j\omega})\right| = \left[\frac{1 + a^2 + 2a\cos\omega}{1 + \alpha^2 + \beta^2 + 2\alpha(1+\beta)\cos\omega + 2\beta\cos 2\omega}\right]^{\frac{1}{2}}$$

The phase function of $H(e^{jw})$ is defined as,

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$$; where $H_r(e^{j\omega})$ is real part and $H_i(e^{j\omega})$ is imaginary part.

To find the real part and imaginary part of $H(e^{jw})$, multiply the numerator and denominator of $H(e^{jw})$ [equation (4.51)], by the complex conjugate of the denominator as shown below.

$$\therefore\ H(\omega) = \frac{1 + a\, e^{-j\omega}}{1 + \alpha\, e^{-j\omega} + \beta\, e^{-j2\omega}}\ \frac{1 + \alpha\, e^{j\omega} + \beta\, e^{j2\omega}}{1 + \alpha\, e^{j\omega} + \beta\, e^{j2\omega}}$$

$\boxed{\text{Using equation (4.52)}}$

$$= \frac{1 + \alpha\, e^{j\omega} + \beta\, e^{j2\omega} + a\, e^{-j\omega} + a\alpha + a\beta\, e^{j\omega}}{1 + \alpha^2 + \beta^2 + 2\alpha(1+\beta)\cos\omega + 2\beta\cos 2\omega}$$

$$H(e^{j\omega}) = \frac{1 + a\alpha + ae^{-j\omega} + (a\beta + \alpha)\, e^{j\omega} + \beta\, e^{j2\omega}}{1 + \alpha^2 + \beta^2 + 2\alpha\,(1+\beta)\cos\omega + 2\beta\cos 2\omega} \qquad \boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$= \frac{1 + a\alpha + a(\cos\omega - j\sin\omega) + (a\beta + \alpha)\,(\cos\omega + j\sin\omega) + \beta(\cos 2\omega + j\sin 2\omega)}{1 + \alpha^2 + \beta^2 + 2\alpha\,(1+\beta)\cos\omega + 2\beta\cos 2\omega}$$

$$\text{Real part, } H_r(e^{j\omega}) = \frac{1 + a\alpha + (a + a\beta + \alpha)\cos\omega + \beta\cos 2\omega}{1 + \alpha^2 + \beta^2 + 2\alpha(1+\beta)\cos\omega + 2\beta\cos 2\omega}$$

$$\text{Imaginary part, } H_i(e^{j\omega}) = \frac{(a\beta + \alpha - a)\sin\omega + \beta\sin 2\omega}{1 + \alpha^2 + \beta^2 + 2\alpha(1+\beta)\cos\omega + 2\beta\cos 2\omega}$$

$$\therefore \text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\frac{(a\beta + \alpha - a)\sin\omega + \beta\sin 2\omega}{1 + a\alpha + (a + a\beta + \alpha)\cos\omega + \beta\cos 2\omega}$$

The magnitude and phase response are calculated for r = 0.5 and 0.9 and $\omega_0 = \pi/4$, and tabulated in table 4.5. Using the calculated values, the $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$ are sketched graphically for r = 0.5 and 0.8 and $\omega_0 = \pi/4$ as shown in fig 4.5. From the plots it can be inferred that the second-order system behaves as a resonant filter (or bandpass filter). The magnitude response shows a sharp peak close to the frequency $\omega = \omega_0 = \pi/4$, which is called resonant frequency.

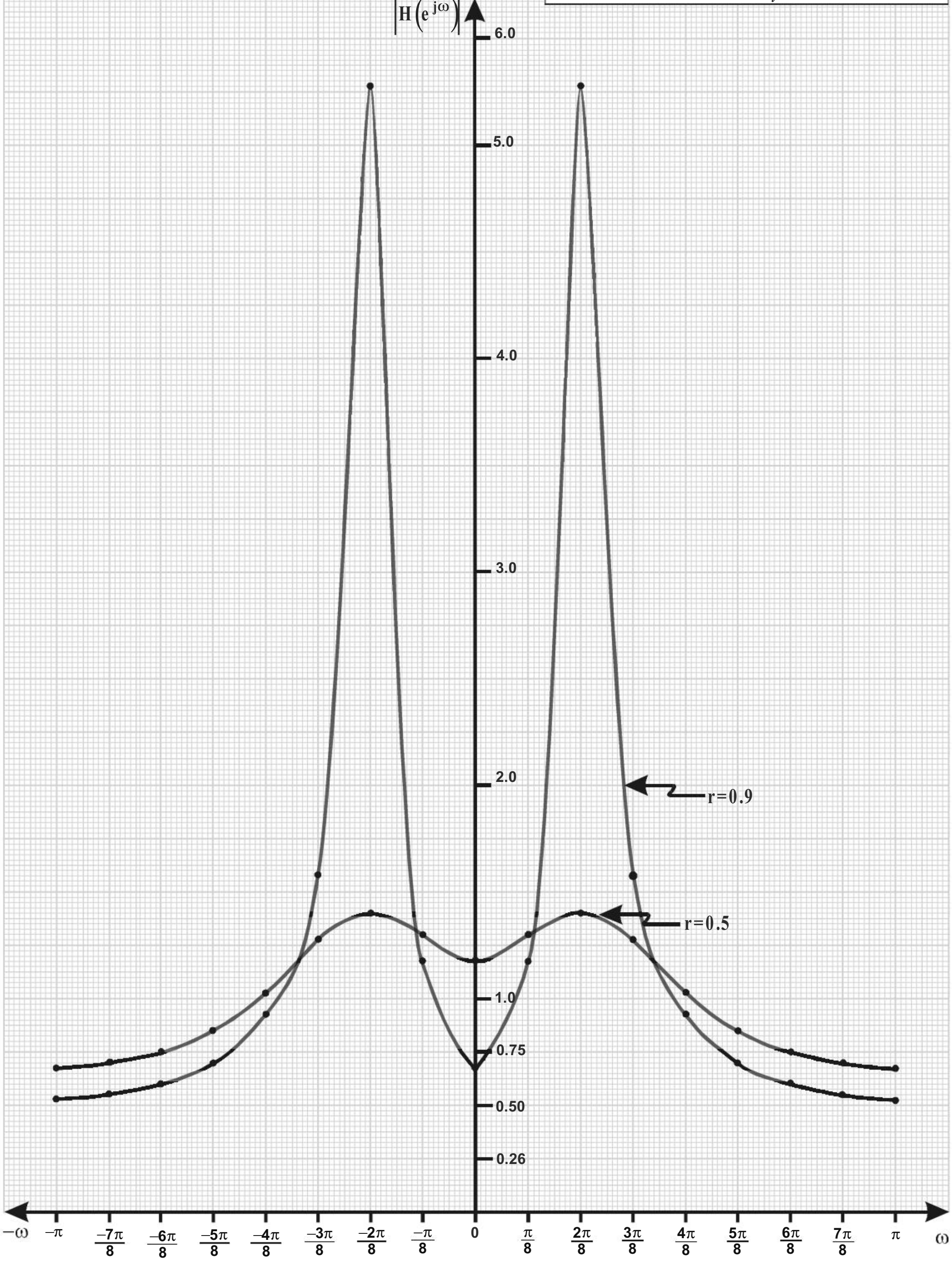
Table 4.5 : Frequency Response of Second-Order Discrete Time System

$\left\|H(e^{j\omega})\right\| = \left(\dfrac{1 + a^2 + 2a\cos\omega}{1 + \alpha^2 + \beta^2 + 2\alpha(1+\beta)\cos\omega + 2\beta\cos 2\omega}\right)^{1/2}$ $\angle H(e^{j\omega}) = \tan^{-1}\left(\dfrac{(a\beta + \alpha - a)\sin\omega + \beta\sin 2\omega}{1 + a\alpha + (a + a\beta + \alpha)\cos\omega + \beta\cos 2\omega}\right) = \left[\dfrac{1}{\pi}\tan^{-1}\left(\dfrac{(a\beta + \alpha - a)\sin\omega + \beta\sin 2\omega}{1 + a\alpha + (a + a\beta + \alpha)\cos\omega + \beta\cos 2\omega}\right)\right]\pi$	
Case - i	
$r = 0.5,\ \omega_0 = \dfrac{\pi}{4}$ $\therefore\ a = -r\cos\omega_0 = -0.5\cos\dfrac{\pi}{4} = -0.3536$ $\alpha = -2r\cos\omega_0 = -2\times 0.5\cos\dfrac{\pi}{4} = -0.7071$ $\beta = r^2 = 0.5^2 = 0.25$	$\left\|H(e^{j\omega})\right\| = \left(\dfrac{1.125 - 0.7072\cos\omega}{1.5625 - 1.7678\cos\omega + 0.5\cos 2\omega}\right)^{1/2}$ $\angle H(e^{j\omega}) = \left[\dfrac{1}{\pi}\tan^{-1}\left(\dfrac{-0.4419\sin\omega + 0.25\sin 2\omega}{1.25 - 1.1491\cos\omega + 0.25\cos 2\omega}\right)\right]\pi$
Case - ii	
$r = 0.9,\ \omega_0 = \dfrac{\pi}{4}$ $\therefore\ a = -r\cos\omega_0 = -0.9\cos\dfrac{\pi}{4} = -0.6364$ $\alpha = -2r\cos\omega_0 = -2\times 0.9\cos\dfrac{\pi}{4} = -1.2728$ $\beta = r^2 = 0.9^2 = 0.81$	$\left\|H(e^{j\omega})\right\| = \left(\dfrac{1.405 - 1.2728\cos\omega}{3.2761 - 4.6075\cos\omega + 1.62\cos 2\omega}\right)^{1/2}$ $\angle H(e^{j\omega}) = \left[\dfrac{1}{\pi}\tan^{-1}\left(\dfrac{-1.1519\sin\omega + 0.81\sin 2\omega}{1.81 - 2.4247\cos\omega + 0.81\cos 2\omega}\right)\right]\pi$

Table 4.5 : Continued...

w	r = 0.5		r = 0.9	
	$\|H(e^{jw})\|$	Ð $H(e^{jw})$	$\|H(e^{jw})\|$	Ð $H(e^{jw})$
$\frac{-8\pi}{8} = -\pi$	0.69	0	0.53	0
$\frac{-7\pi}{8}$	0.71	0.04p	0.55	0.07p
$\frac{-6\pi}{8}$	0.76	0.08p	0.59	0.14p
$\frac{-5\pi}{8}$	0.86	0.12p	0.7	0.21p
$\frac{-4\pi}{8} = \frac{-\pi}{2}$	1.03	0.13p	0.92	0.27p
$\frac{-3\pi}{8}$	1.27	0.11p	1.58	0.32p
$\frac{-2\pi}{8}$	1.41	0.05p	5.28	0.02p
$\frac{-\pi}{8}$	1.29	−0.01p	1.18	–0.24p
0	1.19	0	0.68	0
$\frac{\pi}{8}$	1.29	0.01p	1.18	0.24p
$\frac{2\pi}{8}$	1.41	–0.05p	5.28	–0.02p
$\frac{3\pi}{8}$	1.27	–0.11p	1.58	–0.32p
$\frac{4\pi}{8} = \frac{\pi}{2}$	1.03	–0.13p	0.92	–0.27p
$\frac{5\pi}{8}$	0.86	–0.12p	0.7	–0.21p
$\frac{6\pi}{8}$	0.76	–0.08p	0.59	–0.14p
$\frac{7\pi}{8}$	0.71	–0.04p	0.55	–0.07p
$\frac{8\pi}{8} = \pi$	0.69	0	0.53	0

Fig 4.5 : Magnitude response of 2^{nd} order discrete time system.



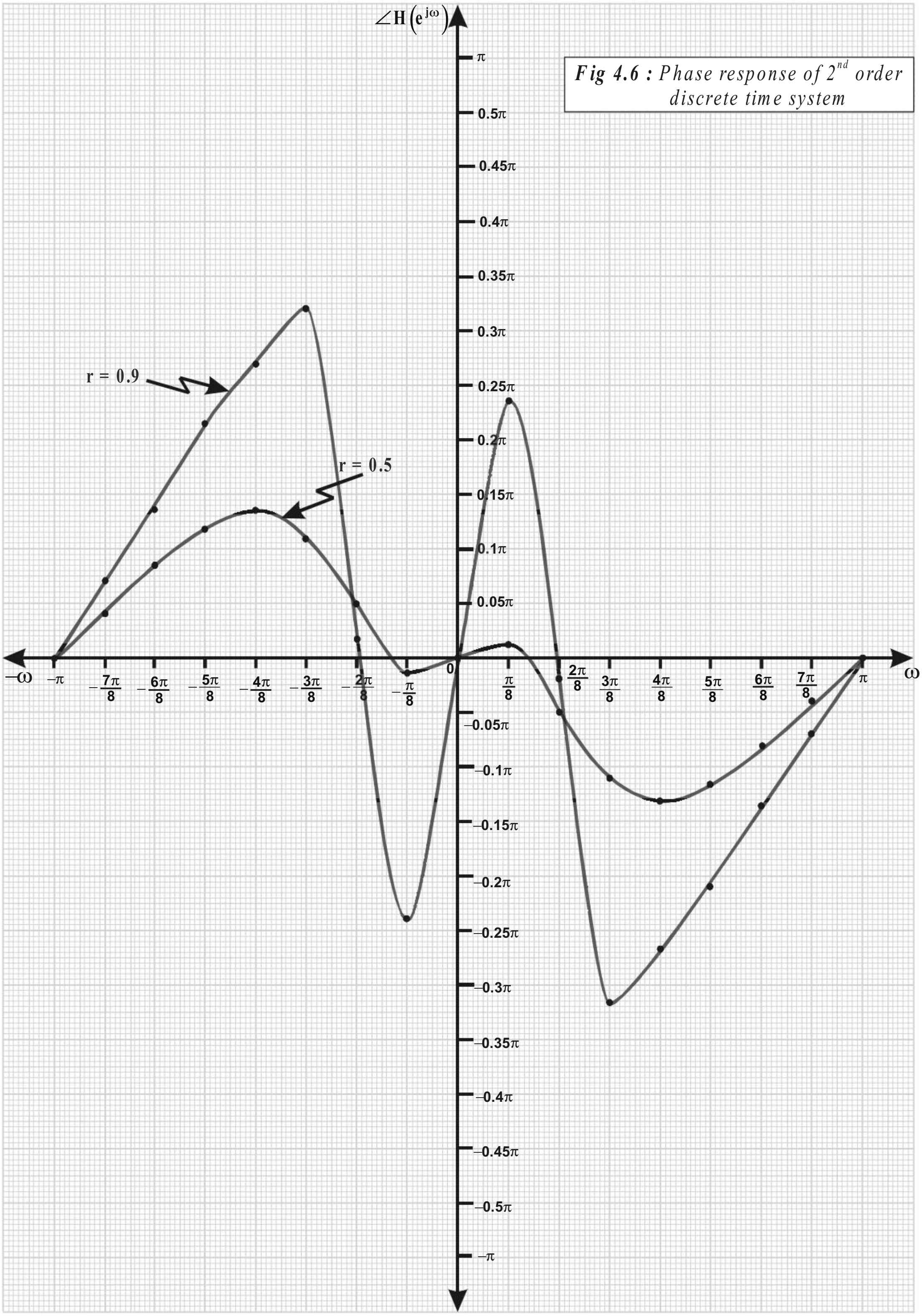


Fig 4.6 : Phase response of 2nd order discrete time system

4.7 Aliasing in Frequency Spectrum Due to Sampling

Let x(t) be an analog signal and X(jW) be Fourier transform of x(t).

Now by definition of continuous time inverse Fourier transform,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega \qquad .....(4.53)$$

Let x(nT) be a discrete time signal obtained by sampling x(t) with sampling period, T.

$$\therefore x(nT) = x(t)\big|_{t = nT}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega\Big|_{t = nT}$$

Using equation (4.53)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega nT}\, d\Omega$$

Expressing the integration as summation of infinite number of integrals.

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{\frac{(2m-1)\pi}{T}}^{\frac{(2m+1)\pi}{T}} X\left(j\left(\Omega + \frac{2\pi m}{T}\right)\right) e^{j\left(\Omega + \frac{2\pi m}{T}\right)nT}\, d\Omega$$

$X(j\Omega)$ in the interval $\frac{(2m-1)\pi}{T}$ to $\frac{(2m+1)\pi}{T}$ is identical with $X(j\Omega)$ in the interval $-\frac{\pi}{T}$ to $+\frac{\pi}{T}$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{-\pi/T}^{+\pi/T} X\left(j\left(\Omega + \frac{2\pi m}{T}\right)\right) e^{j\Omega nT}\, e^{j2\pi mn} d\Omega$$

Since m and n are integers $e^{j2pmn} = 1$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{-\pi/T}^{+\pi/T} X\left(j\left(\frac{\omega}{T} + \frac{2\pi m}{T}\right)\right) e^{j\omega n}\, d\omega$$

The relation between analog and digital frequency is $\Omega = \frac{\omega}{T}$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\frac{1}{T}\int_{-\pi}^{\pi} X\left(j\left(\frac{\omega}{T} + \frac{2\pi m}{T}\right)\right) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{1}{T}\sum_{m=-\infty}^{+\infty} X\left(j\left(\frac{\omega}{T} + \frac{2\pi m}{T}\right)\right) e^{j\omega n}\, d\omega \qquad .....(4.54)$$

By the definition of inverse Fourier transform of a discrete time signal, the x(nT) can be written as,

$$x(nT) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad .....(4.55)$$

On comparing equations (4.54) and (4.55) we can write,

$$X(e^{j\omega}) = \frac{1}{T}\sum_{m=-\infty}^{+\infty} X\left(j\left(\frac{\omega}{T} + \frac{2\pi m}{T}\right)\right) \qquad .....(4.56)$$

$$= \frac{1}{T}\sum_{m=-\infty}^{+\infty} X\left(j\left(\Omega + \frac{2\pi m}{T}\right)\right) \qquad .....(4.57)$$

In equation (4.57) if X(jW) is the original spectrum of analog signal, then $X\left(j\left(\Omega + \frac{2\pi m}{T}\right)\right)$ is the frequency shifted version of X(jW), shifted by $\frac{2\pi m}{T}$. In equation (4.57) the term $\frac{1}{T}$ will scale the amplitude of the spectrum $X\left(j\left(\Omega + \frac{2\pi m}{T}\right)\right)$ by a factor $\frac{1}{T}$.

Therefore from equation (4.57) we can say that $X(e^{jw})$ is sum of frequency shifted and amplitude scaled version of $X(jW)$. In general we can say that the *frequency spectrum of a discrete time signal obtained by sampling continuous time signal will be sum of frequency shifted and amplitude scaled spectrum of continuous time signal*. This concept is illustrated in fig 4.7.

The frequency W of a continuous time signal can be converted to frequency w of a discrete time signal by choosing the transformation, w = WT, where T is the sampling time, $1/T = F_s$ is the sampling cyclic frequency, and $2pF_s = W_s$ is the radian sampling frequency.

In this transformation, the radian frequency w of sampled version of discrete time signal is unique in the interval -p to +p, and the cyclic frequency f of sampled version of discrete time signal is unique in the interval -1/2 to +1/2.

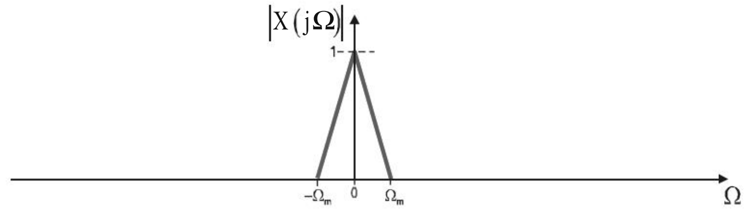


Fig 4.7a : *Spectrum of a continuous time signal x(t), with maximum frequency* Ω_m.

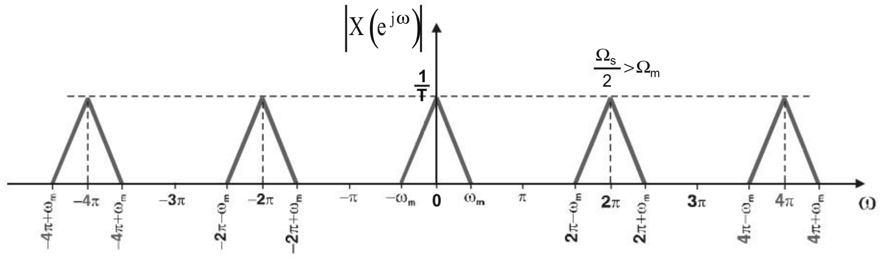


Fig 4.7b : *Spectrum of sampled version of x(t), with* $\Omega_s/2 > \Omega_m$

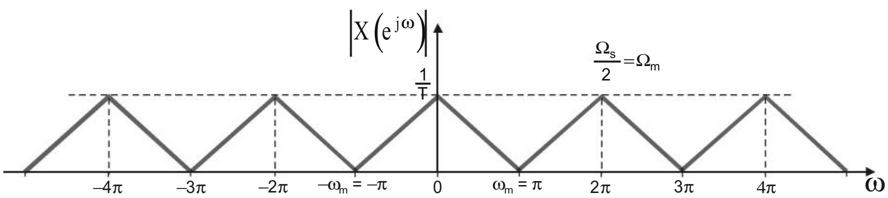


Fig 4.7c : *Spectrum of sampled version of x(t), with* $\Omega_s/2 = \Omega_m$.

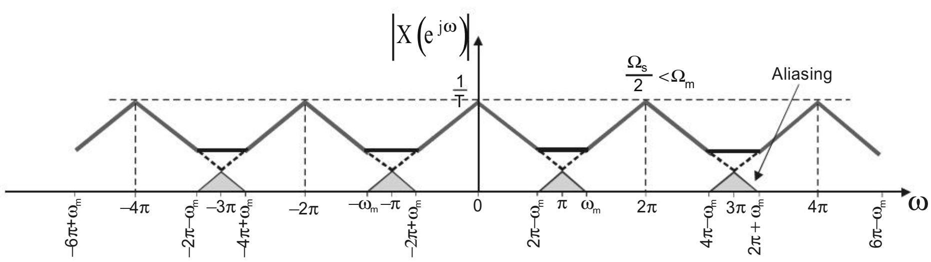


Fig 4.7d : *Spectrum of sampled version of x(t), with* $\Omega_s/2 < \Omega_m$.

Fig 4.7 : *Spectrum of a continuous time signal and its sampled version, sampled at various sampling rates.*

The maximum frequency in the spectrum shown in fig 4.7a is Ω_m. Let ω_m be the corresponding maximum frequency of the sampled version of the discrete time signal when the spectrum of fig 4.7a is sampled at a frequency of $\Omega_s/2$. If Ω_m is equal to $\Omega_s/2$, then the corresponding value of ω_m is given by,

$$\omega_m = \Omega_m T = \frac{\Omega_s}{2} T = \frac{2\pi F_s}{2} T = \frac{\pi}{T} T = \pi$$

From the above equation we can say that if Ω_m is less than $\Omega_s/2$, then corresponding ω_m will be less than π and if Ω_m is greater than $\Omega_s/2$ then corresponding ω_m will be greater than π. From fig 4.7b and fig 4.7c it is observed that, as long as Ω_m is less than $\Omega_s/2$, then corresponding ω_m is less than or equal to π and so there is no overlapping of the components of frequency spectrum.

From fig 4.7d it is observed, when Ω_m is greater than $\Omega_s/2$, then corresponding ω_m will be greater than π and so the components of frequency spectrum overlaps. Due to overlap of frequency spectrum, the high frequency components get the identity of low frequency components. This phenomenon is called ***aliasing***. Due to aliasing the information shifts from one band of frequency to another band of frequency.

Therefore in order to avoid aliasing, $\Omega_s/2$, should be greater than or equal to Ω_m.

Since, $\Omega_m = 2\pi F_m$ and $\Omega_s = 2\pi F_s$, to avoid aliasing, $2\pi F_s/2 > 2\pi F_m$

$$\therefore \; F_s > 2F_m \qquad .....(4.58)$$

Therefore, *in order to avoid aliasing the sampling frequency F_s should be greater than twice the maximum frequency of continuous time signal F_m* .

4.7.1 Signal Reconstruction (Recovery of Continuous Time Signal)

In the above discussion it is observed that, if the sampling frequency $F_s > 2F_m$, then the spectrum $X(e^{j\omega})$ of the sampled continuous time signal will have aliased components of the spectrum $X(j\Omega)$ of original continuous time signal. The aliasing of spectral components prevents the recovery of original signal x(t) from the sampled signal x(n).

When the spectrum of sampled signal has no aliasing then it is possible to recover the original signal from the sampled signal. When there is no aliasing, the spectrum $X(e^{j\omega})$ can be passed through a low pass filter with cut-off frequency, ω_s/π. Now the equation of spectrum $X(e^{j\omega})$ [equation 4.57] can be written as shown below.

$$X(e^{j\omega}) = \frac{1}{T} X(j\Omega) \quad \Rightarrow \quad X(j\Omega) = T\, X(e^{j\omega}) \qquad .....(4.59)$$

On taking inverse Fourier transform of $X(j\Omega)$ we get x(t). Hence by definition of inverse Fourier transform of continuous time signal we get,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}\int_{-\pi/T}^{+\pi/T} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

Because $X(j\Omega)$ is zero outside the interval $-\pi/T$ to π/T

$$= \frac{1}{2\pi}\int_{-\pi/T}^{+\pi/T} T\, X(e^{j\omega})\, e^{j\Omega t}\, d\Omega$$

Substituting for $X(j\Omega)$ from equation (4.59).

$$= \frac{1}{2\pi}\int_{-\pi/T}^{+\pi/T} T \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j\omega n}\, e^{j\Omega t}\, d\Omega$$

Using the definition of Fourier transform of discrete time signal.

$$x(t) = \frac{1}{2\pi}\int_{-\pi/T}^{+\pi/T} T \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j\Omega Tn}\, e^{j\Omega t}\, d\Omega = \frac{T}{2\pi}\sum_{n=-\infty}^{+\infty} x(nT)\int_{-\pi/T}^{+\pi/T} e^{j\Omega(t-nT)}\, d\Omega$$

$$= \frac{T}{2\pi}\sum_{n=-\infty}^{+\infty} x(nT)\left[\frac{e^{j\Omega(t-nT)}}{j(t-nT)}\right]_{-\pi/T}^{+\pi/T} = \frac{T}{2\pi}\sum_{n=-\infty}^{+\infty} x(nT)\left[\frac{e^{j(\pi/T)(t-nT)}}{j(t-nT)} - \frac{e^{j(-\pi/T)(t-nT)}}{j(t-nT)}\right]$$

$$= \sum_{n=-\infty}^{+\infty} x(nT)\,\frac{1}{(\pi/T)(t-nT)}\left[\frac{e^{j(\pi/T)(t-nT)} - e^{-j(\pi/T)(t-nT)}}{2j}\right]$$

$$= \sum_{n=-\infty}^{+\infty} x(nT)\,\frac{\sin((\pi/T)(t-nT))}{(\pi/T)(t-nT)} \qquad .....(4.60)$$

The equation (4.60) can be used to reconstruct the original continuous time signal x(t) from its samples and the equation (4.60) is also called ***ideal interpolation formula***.

The concepts discussed above are summarized as sampling theorem given below.

Sampling Theorem : *A bandlimited continuous time signal with maximum frequency F_m hertz can be fully recovered from its samples provided that the sampling frequency F_s is greater than or equal to two times the maximum frequency F_m, (i.e., $F_s \geq 2F_m$).*

4.7.2 Sampling of Bandpass Signal

A continuous time signal is called ***bandpass signal*** if its frequency spectrum lies in a narrow band of frequencies. Let the lower and upper value of this narrow band of frequency be F_1 and F_2 respectively. Now the ***bandwidth***, "B = F_1 - F_2". Let F_c be a frequency corresponding to centre of bandwidth. The frequency spectrum of some of the bandpass signals are shown in fig 4.8.

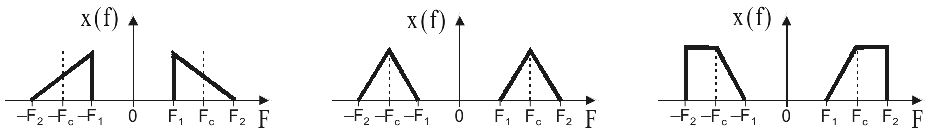


Fig 4.8 : *Sample frequency spectrum of continuous time bandpass signals.*

The maximum frequency in the bandpass signal is F_2. According to sampling theorem, to avoid aliasing the bandpass signal has to be sampled at a sampling frequency greater than $2F_2$. When F_2 happens to be a very high frequency, then sampling rate will be very high. In order to avoid high sampling rates the bandpass signals can be shifted in frequency to an equivalent lowpass signal and the equivalent lowpass signal can be sampled at a lower rate.

A bandpass signal can be shifted in frequency by an amount F_c to convert the signal to an equivalent lowpass signal, and when the upper cutoff frequency F_2 is an integer multiple of bandwidth B, then the equivalent lowpass signal can be sampled at a rate of 2B samples per second.When the upper cutoff frequency F_2 is not an integer multiple of bandwidth B, then the sampling rate has to be slightly increased and go upto 4B.

In general, the bandpass signals with a bandwidth of B Hz can be sampled at a rate of 2B to 4B Hz.

4.8 Relation Between Z-Transform and Discrete Time Fourier Transform

The Z-transform of a discrete time signal x(n) is defined as,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} \qquad \text{..... (4.61)}$$

where, z is a complex variable (or number).

The Fourier transform of a discrete time signal x(n) is given by,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad \text{..... (4.62)}$$

From equation (4.61) and (4.62) we can say that if we replace **z** by $\mathbf{e^{jw}}$ in the Z-transform of x(n) we get Fourier transform of x(n).

The X(z) can be viewed as a unique representation of the signal x(n) in the complex z-plane. In z-plane, the point $z = e^{jw}$, represents a point with unit magnitude and having a phase of w. The range of frequency of discrete time signal w is 0 to 2p. Hence we can say that, the points on unit circle in z-plane are given by $z = e^{jw}$, when w is varied from 0 to 2p. Therefore the Fourier transform of a discrete time signal x(n) can be obtained by evaluating the Z- transform on a circle of unit radius as shown in equation (4.63).

$$\therefore\ X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\Bigg|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \text{.....(4.63)}$$

It is important to note that X(z) exists for $z = e^{jw}$ if unit circle is included in ROC of X(z). Therefore the Fourier transform can be obtained from Z-transform by evaluating X(z) at $z = e^{jw}$, if and only if ROC of X(z) includes the unit circle. Fourier transform of some of the common signals that can be obtained from Z-transform are listed in table 4.6.

Table 4.6 : Some Common Z-transform and Fourier Transform Pairs

x(t)	x(n)	X(z)	$X(e^{jw})$
	$\delta(n)$	1	1
	$a^n u(n)$; $\lvert a\rvert < 1$	$\frac{z}{z-a}$	$\frac{e^{j\omega}}{e^{j\omega}-a}$
	$n\, a^n\, u(n)$; $\lvert a\rvert < 1$	$\frac{az}{(z-a)^2}$	$\frac{a\,e^{j\omega}}{(e^{j\omega}-a)^2}$
	$n^2 a^n\, u(n)$; $\lvert a\rvert < 1$	$\frac{az\,(z+a)}{(z-a)^3}$	$\frac{a\,e^{j\omega}\,(e^{j\omega}+a)}{(e^{j\omega}-a)^3}$
$e^{-at}u(t)$	$e^{-anT}u(nT)$; $\lvert e^{-aT}\rvert < 1$	$\frac{z}{z-e^{-aT}}$	$\frac{e^{j\omega}}{e^{j\omega}-e^{-aT}}$
$te^{-at}u(t)$	$nTe^{-anT}u(nT)$; $\lvert e^{-aT}\rvert < 1$	$\frac{z\,T\,e^{-aT}}{(z-e^{-aT})^2}$	$\frac{e^{j\omega}\,T\,e^{-aT}}{(e^{j\omega}-e^{-aT})^2}$

Example 4.3

Find the Fourier transform of x(n), where $x(n) = 1 \;;\; 0 \le n \le 5$
$= 0 \;;\;$ otherwise

Solution

By the definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{5} x(n)\, e^{-j\omega n} = \frac{1-e^{-j6\omega}}{1-e^{-j\omega}}$$

Using finite geometric series sum formula,

$$\sum_{n=0}^{N} C^n = \frac{1-C^{N+1}}{1-C}$$

$$= \frac{1-e^{\frac{-j6\omega}{2}}\,e^{\frac{-j6\omega}{2}}}{1-e^{\frac{-j\omega}{2}}\,e^{\frac{-j\omega}{2}}} = \frac{\left(e^{\frac{j6\omega}{2}}-e^{\frac{-j6\omega}{2}}\right)e^{\frac{-j6\omega}{2}}}{\left(e^{\frac{j\omega}{2}}-e^{\frac{-j\omega}{2}}\right)e^{\frac{-j\omega}{2}}}$$

$$= \left(\frac{2j\sin\frac{6\omega}{2}}{2j\sin\frac{\omega}{2}}\right)e^{\frac{-j6\omega}{2}+\frac{j\omega}{2}} = \frac{\sin\frac{6\omega}{2}}{\sin\frac{\omega}{2}}\,e^{\frac{-j5\omega}{2}} = \frac{\sin 3\omega}{\sin\frac{\omega}{2}}\,e^{\frac{-j5\omega}{2}}$$

$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

Example 4.4

Determine the Fourier transform of the signal $x(n) = a^{|n|} \;;\; -1 < a < 1$

Solution

The signal x(n) can be expressed as sum of two signals $x_1(n)$ and $x_2(n)$ as shown below.

$\therefore\; x(n) = x_1(n) + x_2(n)$

where, $x_1(n) = a^n \;;\; n \ge 0$ and $x_2(n) = a^{-n} \;;\; n < 0$
$= 0 \;;\; n < 0$ $\qquad = 0 \;;\; n \ge 0$

Let, $X_1(e^{jw})$ = Fourier transform of $x_1(n)$ and $X_2(e^{jw})$ = Fourier transform of $x_2(n)$.

By definition of Fourier transform,

$$X_1(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} = \sum_{n=0}^{+\infty} a^n\, e^{-j\omega n} = \sum_{n=0}^{+\infty} (ae^{-j\omega})^n = \frac{1}{1-a\,e^{-j\omega}}$$

Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

$$(a\,e^{j\omega})^0 = 1$$

By definition of Fourier transform,

$$X_2(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n}\, e^{-j\omega n} = \sum_{n=-\infty}^{-1} (a\,e^{j\omega})^{-n} = \sum_{n=1}^{+\infty} (ae^{j\omega})^n = \sum_{n=0}^{+\infty} (ae^{j\omega})^n - 1$$

$$= \frac{1}{1-a\,e^{j\omega}} - 1 = \frac{1-1+ae^{j\omega}}{1-a\,e^{j\omega}} = \frac{a\,e^{j\omega}}{1-a\,e^{j\omega}}$$

Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

Let $X(e^{jw})$ = Fourier transform of x(n).

By property of linearity,

$$X(e^{j\omega}) = X_1(e^{j\omega}) + X_2(e^{j\omega}) = \frac{1}{1-a\,e^{-j\omega}} + \frac{a\,e^{j\omega}}{1-a\,e^{j\omega}}$$

$$= \frac{1-a\,e^{j\omega}+a\,e^{j\omega}(1-a\,e^{-j\omega})}{(1-a\,e^{-j\omega})\,(1-a\,e^{j\omega})} = \frac{1-a\,e^{j\omega}+a\,e^{j\omega}-a^2}{1-a\,e^{-j\omega}-a\,e^{j\omega}+a^2}$$

$$= \frac{1-a^2}{1-2a\,\cos\omega + a^2}$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

Example 4.5

Find $X(e^{j\omega})$, if $x(n) = \frac{1}{2}\left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{5}\right)^n\right] u(n)$

Solution

Given that, $x(n) = \frac{1}{2}\left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{5}\right)^n\right] u(n)$; for all n

$$\therefore\ x(n) = \frac{1}{2}\left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{5}\right)^n\right] = \frac{1}{2}\left(\frac{1}{3}\right)^n + \frac{1}{2}\left(\frac{1}{5}\right)^n \quad ; \text{ for } n \geq 0$$

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{+\infty} \left[\frac{1}{2}\left(\frac{1}{3}\right)^n + \frac{1}{2}\left(\frac{1}{5}\right)^n\right] e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n e^{-j\omega n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n e^{-j\omega n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{3}\, e^{-j\omega}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{5}\, e^{-j\omega}\right)^n$$

$$= \frac{1}{2}\ \frac{1}{1-\frac{1}{3}\, e^{-j\omega}} + \frac{1}{2}\ \frac{1}{1-\frac{1}{5}\, e^{-j\omega}}$$

Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

$$= \frac{1}{2}\left[\frac{1-\frac{1}{5}\, e^{-j\omega} + 1 - \frac{1}{3}\, e^{-j\omega}}{\left(1-\frac{1}{3}\, e^{-j\omega}\right)\left(1-\frac{1}{5}\, e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{2-0.53\, e^{-j\omega}}{\left(1-\frac{1}{5}\, e^{-j\omega} - \frac{1}{3}\, e^{-j\omega} + \frac{1}{15}\, e^{-j2\omega}\right)}\right] = \frac{1-0.265\, e^{-j\omega}}{1-0.53\, e^{-j\omega} + 0.067\, e^{-j2\omega}}$$

Example 4.6

Compute the Fourier transform and sketch the magnitude and phase function of causal three sample sequence given by,

$$x(n) = \frac{1}{3} \ ; \ 0 \leq n \leq 2$$
$$= 0 \ ; \ \text{else}$$

Solution

Let, $X(e^{jw})$ be Fourier transform of x(n).

Now by definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{2} x(n)\, e^{-j\omega n}$$

$$= x(0)\, e^0 + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} = \frac{1}{3} + \frac{1}{3}\, e^{-j\omega} + \frac{1}{3}\, e^{-j2\omega}$$

$$= \frac{1}{3} + \frac{1}{3}(\cos\omega - j\sin\omega) + \frac{1}{3}(\cos 2\omega - j\sin 2\omega)$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$= \frac{1}{3}(1 + \cos\omega + \cos 2\omega) - j\frac{1}{3}(\sin\omega + \sin 2\omega)$$

The $X(e^{jw})$ is evaluated for various values of w and tabulated in table 1. The magnitude and phase of $X(e^{jw})$ for various values of w are also listed in table 1. Using the values listed in table 1, the magnitude and phase function are sketched as shown in fig 1 and fig 2 respectively.

Table 1 : Frequency Response of the System

w	$X(e^{jw})$	$\lvert X(e^{jw})\rvert$	$\angle X(e^{jw})$ in rad
0	$1 + j0 \quad = 1\angle 0$	1	0
$\frac{\pi}{8}$	$0.877 - j0.363 = 0.949\angle -0.392 = 0.949\angle -0.125\pi$	0.949	-0.125π
$\frac{2\pi}{8}$	$0.569 - j0.569 = 0.805\angle -0.785 = 0.805\angle -0.25\pi$	0.805	-0.25π
$\frac{3\pi}{8}$	$0.225 - j0.544 = 0.587\angle -1.179 = 0.587\angle -0.375\pi$	0.587	-0.375π
$\frac{4\pi}{8} = \frac{\pi}{2}$	$0 - j0.333 \quad = 0.333\angle -1.571 = 0.333\angle -0.5\pi$	0.333	-0.5π
$\frac{5\pi}{8}$	$-0.03 - j0.072 = 0.078\angle -1.966 = 0.078\angle -0.625\pi$	0.078	-0.625π
$\frac{6\pi}{8}$	$0.098 - j0.098 = 0.139\angle -0.785 = 0.139\angle -0.25\pi$	0.139	-0.25π
$\frac{7\pi}{8}$	$0.261 + j0.108 = 0.282\angle 0.392 = 0.282\angle 0.125\pi$	0.282	0.125π
$\frac{8\pi}{8} = \pi$	$0.333 + j0 \quad = 0.333\angle 0 \quad = 0.333\angle 0$	0.333	0
$\frac{9\pi}{8}$	$0.261 - j0.108 = 0.282\angle 0.392 = 0.282\angle -0.125\pi$	0.282	-0.125π
$\frac{10\pi}{8}$	$0.098 + j0.098 = 0.139\angle 0.785 = 0.139\angle 0.25\pi$	0.139	0.25π
$\frac{11\pi}{8}$	$-0.03 + j0.072 = 0.078\angle 1.966 = 0.078\angle 0.625\pi$	0.078	0.625π
$\frac{12\pi}{8} = \frac{3\pi}{2}$	$0 + j0.333 \quad = 0.333\angle 1.571 = 0.333\angle 0.5\pi$	0.333	0.5π
$\frac{13\pi}{8}$	$0.225 + j0.544 = 0.589\angle 1.179 = 0.589\angle 0.375\pi$	0.589	0.375π
$\frac{14\pi}{8}$	$0.569 + j0.569 = 0.805\angle 0.785 = 0.805\angle 0.25\pi$	0.805	0.25π
$\frac{15\pi}{8}$	$0.877 + j0.363 = 0.949\angle 0.392 = 0.949\angle 0.125\pi$	0.949	0.125π
$\frac{16\pi}{8} = 2\pi$	$1 + j0 \quad = 1\angle 0$	1	0

Note : The function $X(e^{jw})$ is calculated using complex mode of calculator. The magnitude and phase are calculated using rectangular to polar conversion technique.

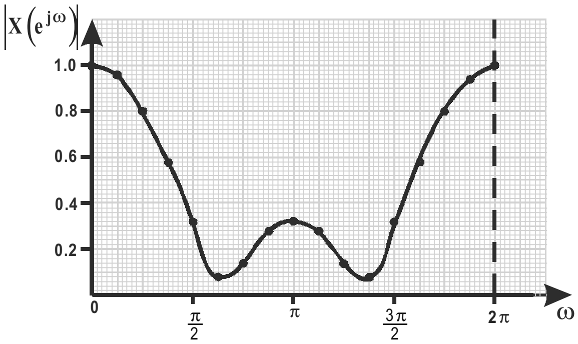


Fig 1 : Magnitude of X (e $^{j\omega}$).

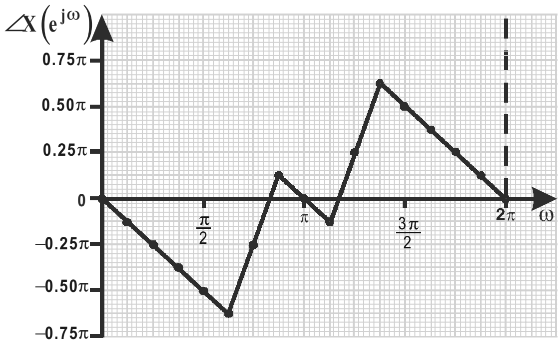


Fig 2 : Phase of X (e $^{j\omega}$).

Example 4.7

Find the convolution of the sequences, $x_1(n) = x_2(n) = \{1, \underset{\uparrow}{3}, 5\}$

Solution

$$X_1(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} = \sum_{n=-1}^{+1} x_1(n) e^{-j\omega n} = e^{j\omega} + 3 + 5\,e^{-j\omega}$$

Since, $x_1(n) = x_2(n)$, $\quad X_2(e^{jw}) = X_1(e^{jw}) = e^{jw} + 3 + 5\, e^{-jw}$

Let, $x(n) = x_1(n) * x_2(n)$, and $X(e^{jw}) = \mathcal{F}\{x(n)\} = \mathcal{F}\{x_1(n) * x_2(n)\}$

By convolution property of Fourier transform.

$$\mathcal{F}\{x_1(n) * x_2(n)\} = X_1(e^{jw})\, X_2(e^{jw})$$

$$\therefore\ X(e^{j\omega}) = X_1(e^{j\omega})\, X_2(e^{j\omega}) = (e^{j\omega} + 3 + 5\,e^{-j\omega})\,(e^{j\omega} + 3 + 5\,e^{-j\omega})$$

$$= e^{j2\omega} + 3\,e^{j\omega} + 5 + 3\,e^{j\omega} + 9 + 15\,e^{-j\omega} + 5 + 15\,e^{-j\omega} + 25\,e^{-j2\omega}$$

$$= e^{j2\omega} + 6e^{j\omega} + 19 + 30\,e^{-j\omega} + 25\,e^{-j2\omega} \qquad .....(1)$$

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = ...\ x(-2)\, e^{j2\omega} + x(-1)\, e^{j\omega} + x(0)$$

$$+\, x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + ... \qquad .....(2)$$

On comparing the coefficient of e^{jwn} in the two equations [equations (1) and (2)] of $X(e^{jw})$ we get,

$$x(n) = \{1,\ 6,\ \underset{\uparrow}{19},\ 30,\ 25\}$$

Example 4.8

If $H(e^{j\omega}) = \dfrac{1}{5}(1 + 3\cos\omega)$, find h(n).

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Solution

Given that, $H(e^{j\omega}) = \dfrac{1}{5}(1 + 3\cos\omega) = \dfrac{1}{5} + \dfrac{3}{5}\,\dfrac{e^{j\omega} + e^{-j\omega}}{2} = \dfrac{3}{10}e^{j\omega} + \dfrac{1}{5} + \dfrac{3}{10}e^{-j\omega}$

$$= 0.3\,e^{j\omega} + 0.2 + 0.3e^{-j\omega} \qquad .....(1)$$

Let, h(n) = Inverse Fourier transform of $H(e^{jw})$.

By definition of Fourier transform we get,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = ...+h(-2)\,e^{j2\omega} + h(-1)\,e^{j\omega} + h(0) + h(1)\,e^{-j\omega} + h(2)\,e^{-j2\omega}+..... \quad .....(2)$$

On comparing the two expressions [equations (1) and (2)] for $H(e^{jw})$, we can say that the samples of h(n) are the coefficients of e^{jwn}. Hence by inspection we can write,

$$h(-1) = 0.3 \;\; ; \;\; h(0) = 0.2 \;\; ; \;\; h(1) = 0.3 \;\; ; \;\; \text{and } \; h(n) = 0, \text{ for } n < -1 \text{ and } \; n > 1$$

$$\therefore \; h(n) = \{0.3, \; \underset{\uparrow}{0.2}, \; 0.3\}$$

Example 4.9

Find the inverse Fourier transform of the frequency response of first order system, $H(e^{jw}) = (1 - a\,e^{-jw})^{-1}$.

Solution

Given that, $H(e^{j\omega}) = (1 - a\,e^{-j\omega})^{-1} = \dfrac{1}{1 - a\,e^{-j\omega}}$

Using Taylor series expansion, the above equation of $H(e^{jw})$ can be expanded as shown below.

$$H(e^{j\omega}) = 1 + a\,e^{-j\omega} + a^2\,e^{-j2\omega} + ...+ a^k\,e^{-jk\omega} + ... \quad .....(1)$$

Let, h(n) = Inverse Fourier transform of $H(e^{jw})$.

By definition of Fourier transform we get,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n}$$

$$= .....+h(-2)\,e^{j2\omega} + h(-1)\,e^{j\omega} + h(0) + h(1)\,e^{-j\omega} + h(2)\,e^{-j2\omega} + ... \quad .....(2)$$

On comparing the two expressions for $H(e^{jw})$ [equation (1) and (2)] we can say that the samples of h(n) are the coefficients of e^{-jwn}.

$$\therefore \; h(n) = \left\{ \underset{\uparrow}{1}, \; a, \; a^2, ..., \; a^k, ... \right\}$$

$$h(n) = \begin{cases} a^n & ; \; n \geq 0 \\ 0 & ; \; n < 0 \end{cases} \qquad \Rightarrow \qquad h(n) = a^n u(n)$$

Example 4.10

Determine the output sequence from the output spectrum $Y(e^{jw})$, where $Y(e^{j\omega}) = \dfrac{1}{2}\,\dfrac{e^{j\omega} + 1 + e^{-j\omega}}{1 - a\,e^{-j\omega}}$

Solution

The output sequence y(n) is obtained by taking inverse Fourier transform of $Y(e^{jw})$.

$$Y(e^{j\omega}) = \frac{1}{2}\,\frac{e^{j\omega} + 1 + e^{-j\omega}}{1 - a\,e^{-j\omega}} = \frac{1}{2}\left[\frac{e^{j\omega}}{1 - a\,e^{-j\omega}} + \frac{1}{1 - a\,e^{-j\omega}} + \frac{e^{-j\omega}}{1 - a\,e^{-j\omega}}\right]$$

$$Y(e^{j\omega}) = \frac{1}{2}\left[Y_1(e^{j\omega}) + Y_2(e^{j\omega}) + Y_3(e^{j\omega})\right]$$

$$\text{where, } \; Y_1(e^{j\omega}) = \frac{e^{j\omega}}{1 - a\,e^{-j\omega}} \;\; ; \;\; Y_2(e^{j\omega}) = \frac{1}{1 - a\,e^{-j\omega}} \;\; \text{and } \; Y_3(e^{j\omega}) = \frac{e^{-j\omega}}{1 - a\,e^{-j\omega}}$$

Let, $y_1(n) = \mathcal{F}^{-1}\{Y_1(e^{jw})\}$; $y_2(n) = \mathcal{F}^{-1}\{Y_2(e^{jw})\}$; $y_3(n) = \mathcal{F}^{-1}\{Y_3(e^{jw})\}$

By Taylor's series expansion we get,

$$\boxed{\begin{aligned} u(n) &= 1 \text{ for } n \geq 0 \\ &= 0 \text{ for } n < 0 \end{aligned}}$$

$$Y_2(e^{j\omega}) = \frac{1}{1 - a\,e^{-j\omega}} = 1 + a\,e^{-j\omega} + a^2 e^{-j2\omega} + a^3\,e^{-j3\omega} + ...$$

$$= \sum_{n=0}^{+\infty} a^n\,e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} a^n\,u(n)\,e^{-j\omega n} \quad .....(1)$$

By definition of Fourier transform we can write,

$$Y_2(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y_2(n)\, e^{-j\omega n} \qquad .....(2)$$

By comparing equations (1) and (2) we can write,

<u>Shifting property:</u> If $\mathcal{F}\{x(n)\} = X(e^{jw})$ then, $\mathcal{F}\{x(n \pm m\} = e^{\pm j\omega m}\, X\left(e^{j\omega}\right)$

$$y_2(n) = a^n\, u(n)$$

$$\text{Here, } Y_1(e^{j\omega}) = \frac{e^{j\omega}}{1 - a\, e^{-j\omega}} = e^{j\omega}\, Y_2(e^{j\omega})$$

$$\therefore\ y_1(n) = a^{(n+1)}\, u(n+1)$$

Using shifting property.

$$\text{Here, } Y_3(e^{j\omega}) = \frac{e^{-j\omega}}{1 - a\, e^{-j\omega}} = e^{-j\omega}\, Y_2(e^{j\omega})$$

$$\therefore\ y_3(n) = a^{(n-1)}\, u(n-1)$$

Using shifting property.

Let, y(n) = Inverse Fourier transform of $Y(e^{jw})$.

$$\therefore\ y(n) = \mathcal{F}^{-1}\{Y(e^{j\omega})\} = \mathcal{F}^{-1}\left\{\frac{1}{2}\left[Y_1(e^{j\omega}) + Y_2(e^{j\omega}) + Y_3(e^{j\omega})\right]\right\}$$

$$= \frac{1}{2}\left[\mathcal{F}^{-1}\{Y_1(\omega)\} + \mathcal{F}^{-1}\{Y_2(\omega)\} + \mathcal{F}^{-1}\{Y_3(\omega)\}\right]$$

$$= \frac{1}{2}\left[y_1(n) + y_2(n) + y_3(n)\right]$$

$$= \frac{1}{2}\left[a^{(n+1)}\, u(n+1) + a^n\, u(n) + a^{(n-1)}\, u(n-1)\right]$$

Example 4.11

If $X(e^{jw}) = e^{-j3w}$; $|w| \le 1$

$\qquad = 0$; $1 < |w| \le \pi$, Find x(n) and plot.

Solution

The x(n) is obtained by taking inverse Fourier transform of $X(e^{jw})$.

By definition of inverse Fourier transform,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-1}^{+1} e^{-j3\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{+1} e^{j\omega(n-3)}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-3)}}{j(n-3)}\right]_{-1}^{1} = \frac{1}{j2\pi(n-3)}\left[e^{j(n-3)} - e^{-j(n-3)}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{\pi(n-3)}\left[\frac{e^{j(n-3)} - e^{-j(n-3)}}{2j}\right] = \frac{1}{\pi(n-3)}\sin(n-3)$$

$$= \frac{\sin(n-3)}{\pi(n-3)} \quad ; \quad \text{for all } n \quad , \quad \text{except } n = 3.$$

L' Hospital rule $\underset{\theta \to 0}{\text{Lt}}\ \frac{\sin\theta}{\theta} = 1$

$$\text{When } n = 3,\ \ x(n) = \underset{(n-3)\to 0}{\text{Lt}}\ \frac{\sin(n-3)}{\pi(n-3)} = \frac{1}{\pi}\ \underset{(n-3)\to 0}{\text{Lt}}\ \frac{\sin(n-3)}{(n-3)} = \frac{1}{\pi}$$

The signal x(n) is an infinite duration signal and can be evaluated for all integer values of n in the range $n = -\infty$ to $+\infty$. Here x(n) is evaluated for n = –2 to + 8 and plotted.

$$x(n) = \frac{\sin(n-3)}{\pi(n-3)}$$

Note : Evaluate sin(n – 3) by keeping calculator in radians mode.

When $n=-2$; $x(-2)=\dfrac{\sin(-2-3)}{\pi(-2-3)}=-0.061$

When $n=-1$; $x(-1)=\dfrac{\sin(-1-3)}{\pi(-1-3)}=-0.06$

When $n=0$; $x(0)=\dfrac{\sin(0-3)}{\pi(0-3)}=0.015$

When $n=1$; $x(1)=\dfrac{\sin(1-3)}{\pi(1-3)}=0.145$

When $n=2$; $x(2)=\dfrac{\sin(2-3)}{\pi(2-3)}=0.268$

When $n=3$; $x(3)=\dfrac{1}{\pi}=0.318=0.318$

When $n=4$; $x(4)=\dfrac{\sin(4-3)}{\pi(4-3)}=0.268$

When $n=5$; $x(5)=\dfrac{\sin(5-3)}{\pi(5-3)}=0.145$

When $n=6$; $x(6)=\dfrac{\sin(6-3)}{\pi(6-3)}=0.015$

When $n=7$; $x(7)=\dfrac{\sin(7-3)}{\pi(7-3)}=-0.06$

When $n=8$; $x(8)=\dfrac{\sin(8-3)}{\pi(8-3)}=-0.061$

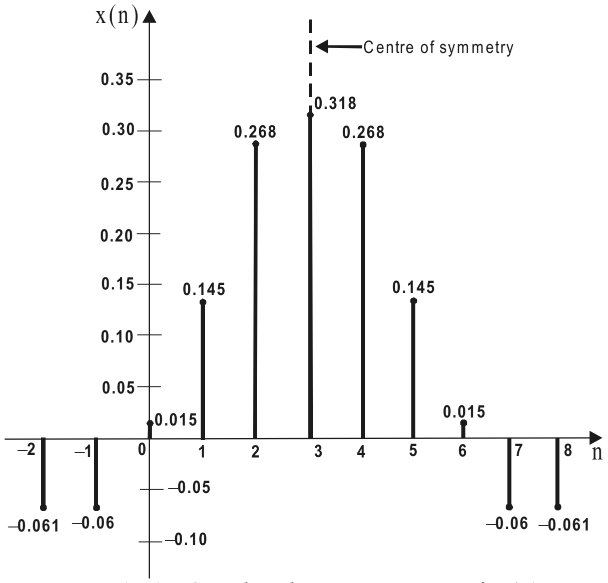


Fig 1 : Graphical representation of x(n).

Here x(n) is a symmetrical signal with centre of symmetry at n = 3.

Example 4.12

Find x(n), if $X(e^{j\omega})=\dfrac{1}{1-\frac{1}{8}e^{-j\omega}}$

Solution

Given that, $X(e^{j\omega})=\dfrac{1}{1-\frac{1}{8}e^{-j\omega}}$

By Taylor's series expansion we can write,

$$X(e^{j\omega})=\frac{1}{1-\frac{1}{8}e^{-j\omega}}=1+\frac{1}{8}e^{-j\omega}+\left(\frac{1}{8}e^{-j\omega}\right)^2+\left(\frac{1}{8}e^{-j\omega}\right)^3+\ldots..$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{8}e^{-j\omega}\right)^n=\sum_{n=0}^{\infty}\left(\frac{1}{8}\right)^n e^{-j\omega n} \qquad \ldots..(1)$$

Using infinite geometric series sum formula,

$$\sum_{n=0}^{\infty}C^n=\frac{1}{1-C}$$

By definition of Fourier transform we can write,

$$X(e^{j\omega})=\sum_{n=0}^{\infty}x(n)\,e^{-j\omega n}\ ;\ \text{for } n\ge 0 \qquad \ldots..(2)$$

On comparing equations (1) and (2) we get,

$$x(n)=\left(\frac{1}{8}\right)^n\ ;\ \text{for } n\ge 0$$

$$\therefore\ x(n)=\left(\frac{1}{8}\right)^n u(n)\ ;\ \text{for all } n$$

Example 4.13

If H(e^jw) = 1 ; |w| £ 1

= 0 ; 1< |w| £ p, Find the impulse response h(n), and plot.

Solution

The impulse response h(n) can be obtained by taking inverse Fourier transform of H(e^jw).

By definition of inverse Fourier transform,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-1}^{+1} 1\times e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-1}^{+1}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{j2\pi n}\left[e^{jn} - e^{-jn}\right] = \frac{1}{\pi n}\left[\frac{e^{jn} - e^{-jn}}{2j}\right] = \frac{2\sin n}{\pi n} \quad ; \quad \text{for all } n \;,\; \text{except when } n = 0$$

When n = 0; h(n) can be evaluated using L' Hospital rule.

$$\boxed{\text{L' Hospital rule} \quad \underset{\theta\to 0}{\mathrm{Lt}}\ \frac{\sin\theta}{\theta} = 1}$$

$$\text{When } n = 0;\ h(n) = \underset{n\to 0}{\mathrm{Lt}}\ \frac{\sin n}{\pi n} = \frac{1}{\pi}\underset{n\to 0}{\mathrm{Lt}}\ \frac{\sin n}{n} = \frac{1}{\pi}$$

$$\therefore \text{ Impulse response, } h(n) = \frac{1}{\pi} \quad ; \text{ when } n = 0$$

$$= \frac{\sin n}{\pi n} \quad ; \text{ when } n \neq 0$$

The impulse response is an infinite duration signal and can be evaluated for all integer values of n in the range n = –¥ to +¥. Here h(n) is evaluated for n = –5 to +5 and plotted.

$$\text{When } n = -5\ ;\ h(-5) = \frac{\sin(-5)}{\pi(-5)} = -0.061$$

$$\text{When } n = -4\ ;\ h(-4) = \frac{\sin(-4)}{\pi(-4)} = -0.06$$

$$\text{When } n = -3\ ;\ h(-3) = \frac{\sin(-3)}{\pi(-3)} = 0.015$$

$$\text{When } n = -2\ ;\ h(-2) = \frac{\sin(-2)}{\pi(-2)} = 0.145$$

$$\text{When } n = -1\ ;\ h(-1) = \frac{\sin(-1)}{\pi(-1)} = 0.268$$

$$\text{When } n = 0\ ;\ h(0) = \frac{1}{\pi} = 0.318 = 0.318$$

$$\text{When } n = 1\ ;\ h(1) = \frac{\sin(1)}{\pi(1)} = 0.268$$

$$\text{When } n = 2\ ;\ h(2) = \frac{\sin(2)}{\pi(2)} = 0.145$$

$$\text{When } n = 3\ ;\ h(3) = \frac{\sin(3)}{\pi(3)} = 0.015$$

$$\text{When } n = 4\ ;\ h(4) = \frac{\sin(4)}{\pi(4)} = -0.06$$

$$\text{When } n = 5\ ;\ h(5) = \frac{\sin(5)}{\pi(5)} = -0.061$$

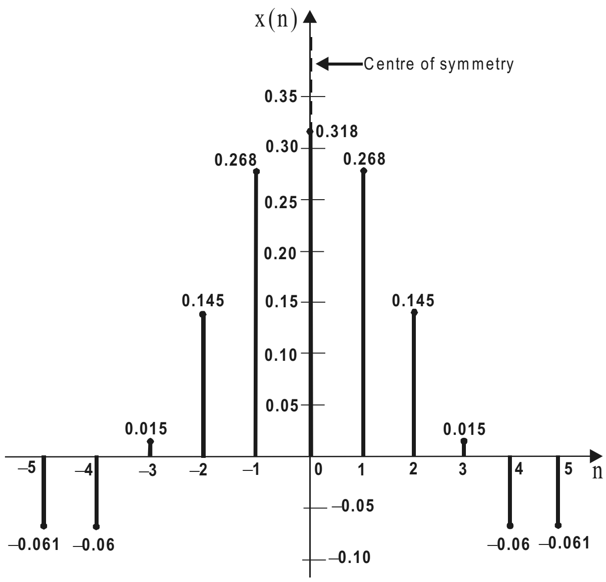


Fig 1 : Graphical representation of h(n).

Here h(n) is a symmetrical signal with centre of symmetry at n = 0.

Example 4.14

Find the transfer function of the second order recursive filter in frequency domain whose impulse response is $h(n) = r^n \sin(w_0 n)\, u(n)$ for all n.

Solution

The transfer function of a system is the Fourier transform of impulse response.

By definition of Fourier transform,

$$u(n) = 1 \text{ for } n \geq 0 \\ = 0 \text{ for } n < 0$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{n=0}^{+\infty} r^n \sin\omega_0 n \;\; e^{-j\omega n}$$

$$= \sum_{n=0}^{+\infty} r^n \left[\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right] e^{-j\omega n} = \frac{1}{2j}\sum_{n=0}^{+\infty}\left[r^n e^{j\omega_0 n}\, e^{-j\omega n} - r^n e^{-j\omega_0 n}\, e^{-j\omega n}\right]$$

$$= \frac{1}{2j}\sum_{n=0}^{+\infty}\left[r\, e^{j\omega_0}\, e^{-j\omega}\right]^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left[r\, e^{-j\omega_0}\, e^{-j\omega}\right]^n$$

For $|r| < 1$, we can apply the infinite geometric series sum formula to give,

$$H(e^{j\omega}) = \frac{1}{2j}\,\frac{1}{1 - r\, e^{j\omega_0}\, e^{-j\omega}} - \frac{1}{2j}\,\frac{1}{1 - r\, e^{-j\omega_0}\, e^{-j\omega}} = \frac{1}{2j}\left[\frac{1 - r\, e^{-j\omega_0}\, e^{-j\omega} - 1 + r\, e^{j\omega_0}\, e^{-j\omega}}{\left(1 - r\, e^{j\omega_0}\, e^{-j\omega}\right)\left(1 - r\, e^{-j\omega_0}\, e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2j}\,\frac{r\left(-e^{-j\omega_0} + e^{j\omega_0}\right)e^{-j\omega}}{1 - r\, e^{-j\omega_0}\, e^{-j\omega} - r\, e^{j\omega_0}\, e^{-j\omega} + r^2\, e^{-j2\omega}} = \frac{1}{2j}\,\frac{r\left(e^{j\omega_0} - e^{-j\omega_0}\right)e^{-j\omega}}{1 - r\left(e^{j\omega_0} + e^{-j\omega_0}\right)e^{-j\omega} + r^2\, e^{-j2\omega}}$$

$$= \frac{1}{2j}\,\frac{r\, 2j\, \sin\omega_0\, e^{-j\omega}}{1 - r\, 2\, \cos\omega_0\, e^{-j\omega} + r^2\, e^{-j2\omega}} = \frac{r \sin\omega_0\, e^{-j\omega}}{1 - 2r\cos\omega_0\, e^{-j\omega} + r^2\, e^{-j2\omega}}$$

Example 4.15

Find the output spectrum of an LTI system, if input $x(n) = \frac{2}{3}$; $-1 \leq n \leq 1$

$= 0$; else

and the impulse response $h(n) = a^n$; $n \geq 0$

$= 0$; else

Solution

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=-1}^{1} x(n)\, e^{-j\omega n} = x(-1)\, e^{j\omega} + x(0) + x(1)\, e^{-j\omega}$$

$$= \frac{2}{3}e^{j\omega} + \frac{2}{3} + \frac{2}{3}\, e^{-j\omega} = \frac{2}{3} + \frac{2}{3}(e^{j\omega} + e^{-j\omega}) = \frac{2}{3} + \frac{2}{3}(2\cos\omega)$$

$$= \frac{2}{3}(1 + 2\cos\omega)$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{n=0}^{\infty} a^n\, e^{-j\omega n} = \sum_{n=0}^{\infty} (a\, e^{-j\omega})^n = \frac{1}{1 - a\, e^{-j\omega}}$$

Using infinite geometric series sum formula,

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$$

The output spectrum $Y(e^{jw})$ is given by,

$$Y(e^{j\omega}) = X(e^{j\omega}) \times H(e^{j\omega}) = \frac{2}{3}(1 + 2\cos\omega) \times \frac{1}{1 - a\, e^{-j\omega}} = \frac{2(1 + 2\cos\omega)}{3(1 - a\, e^{-j\omega})}$$

Example 4.16

The impulse response of an LTI system is h(n) = {1, 2, 2, 1}. Find the response of the system for the input x(n) = {1, 2, 3, 4}

Solution

The response y(n) of the system is given by convolution of x(n) and h(n).

$$\therefore\ y(n) = x(n) * h(n) \qquad \text{..... (1)}$$

By convolution theorem of Fourier transform we get,

$$\mathcal{F}\{x(n) * h(n)\} = X(e^{jw})\,H(e^{jw}) \qquad \text{..... (2)}$$

From equations (1) and (2) we can write,

$$\mathcal{F}\{y(n)\} = X(e^{jw})\,H(e^{jw})$$

Let, $\mathcal{F}\{y(n)\} = Y(e^{jw})$; $\therefore\ Y(e^{jw}) = X(e^{jw})\,H(e^{jw})$

$$\therefore\ y(n) = \mathcal{F}^{-1}\{Y(e^{jw})\} = \mathcal{F}^{-1}\{X(e^{jw})\,H(e^{jw})\}$$

By definition of Fourier transform, we can write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n} = \sum_{n=0}^{3} x(n)\,e^{-j\omega n}$$

$$= x(0)\,e^{0} + x(1)\,e^{-j\omega} + x(2)\,e^{-j2\omega} + x(3)\,e^{-j3\omega}$$

$$= 1 + 2\,e^{-j\omega} + 3\,e^{-j2\omega} + 4\,e^{-j3\omega}$$

By definition of Fourier transform we can write,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\,e^{-j\omega n} = \sum_{n=0}^{3} h(n)\,e^{-j\omega n}$$

$$= h(0)\,e^{0} + h(1)\,e^{-j\omega} + h(2)\,e^{-j2\omega} + h(3)\,e^{-j3\omega}$$

$$= 1 + 2e^{-j\omega} + 2\,e^{-j2\omega} + e^{-j3\omega}$$

$$X(e^{jw})\,H(e^{jw}) = (1 + 2\,e^{-jw} + 3\,e^{-j2w} + 4e^{-j3w})\,(1 + 2\,e^{-jw} + 2e^{-j2w} + e^{-j3w})$$

$$= 1 + 2\,e^{-jw} + 2e^{-j2w} + e^{-j3w}$$
$$+ 2\,e^{-jw} + 4\,e^{-j2w} + 4\,e^{-j3w} + 2\,e^{-j4w}$$
$$+ 3\,e^{-j2w} + 6\,e^{-j3w} + 6\,e^{-j4w} + 3\,e^{-j5w}$$
$$+ 4e^{-j3w} + 8\,e^{-j4w} + 8e^{-j5w} + 4e^{-j6w}$$

$$\therefore\ Y(e^{jw}) = 1 + 4\,e^{-jw} + 9\,e^{-j2w} + 15\,e^{-j3w} + 16\,e^{-j4w} + 11\,e^{-j5w} + 4e^{-j6w} \qquad \text{..... (3)}$$

By definition of Fourier transform we get,

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y(n)\,e^{-j\omega n}$$

$$= \ldots..\ y(0)\,e^{0} + y(1)\,e^{-jw} + y(2)\,e^{-j2w} + y(3)\,e^{-j3w} + y(4)\,e^{-j4w} + y(5)\,e^{-j5w} + y(6)\,e^{-j6w} + \ldots.. \qquad \text{..... (4)}$$

On comparing equations (3) and (4) we get,

$$y(n) = \{\underset{\uparrow}{1},\ 4,\ 9,\ 15,\ 16,\ 11,\ 4\}$$

Example 4.17

Determine the impulse response and frequency response of the LTI system defined by,
y(n) = x(n) + b y(n – 1).

Solution

a) Impulse Response

The impulse response h(n) is given by inverse $\mathcal{Z}$-transform of H(z), where, $H(z) = \dfrac{Y(z)}{X(z)}$.

Given that, $y(n) = x(n) + b\,y(n-1)$. .....(1)

On taking Ƶ-transform of equation (1) we get,

$Y(z) = X(z) + b\,z^{-1}\,Y(z)$ Þ $Y(z) - b\,z^{-1}\,Y(z) = X(z)$ Þ $Y(z)\,(1 - b\,z^{-1}) = X(z)$

$$\therefore\ H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - b\,z^{-1}} \qquad .....(2)$$

On taking inverse Ƶ-transform of equation (2) we get,

$h(n) = Ƶ^{-1}\,\{H(z)\} = b^n\,u(n)$

$$\boxed{Ƶ\left\{a^n\,u(n)\right\} = \frac{1}{1 - az^{-1}}}$$

The impulse response, $h(n) = b^n\,u(n)$, for all n.

b) Frequency Response

The frequency response $H(e^{jw})$ is obtained by evaluating H(z) when $z = e^{jw}$.

$$\therefore\ \text{Frequency response, } H(e^{j\omega}) = H(z)\Big|_{z\,=\,e^{j\omega}} = \frac{1}{1 - bz^{-1}}\Bigg|_{z\,=\,e^{j\omega}} = \frac{1}{1 - be^{-j\omega}}$$

The magnitude function of $H(e^{jw})$ is defined as,

$$\left|\,H(e^{j\omega})\,\right| = \sqrt{H(e^{j\omega})\,H^{*}(e^{j\omega})}\ ,\ \text{where } H^{*}(e^{j\omega}) = \text{Conjugate of } H(e^{j\omega}).$$

$$\therefore\ \text{Magnitude function, } \left|H(e^{j\omega})\right| = \left[\frac{1}{1 - b\,e^{-j\omega}} \times \frac{1}{1 - b\,e^{j\omega}}\right]^{\frac{1}{2}} = \left[\frac{1}{1 - b\,e^{j\omega} - b\,e^{-j\omega} + b^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1 + b^2 - b\left(e^{j\omega} + e^{-j\omega}\right)}\right]^{\frac{1}{2}} \quad \frac{1}{\left(1 + b^2 - 2b\cos\omega\right)^{\frac{1}{2}}}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\text{The phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$$

where, $H_i(e^{jw})$ = Imaginary part of $H(e^{jw})$ and $H_r(e^{jw})$ = Real part of $H(e^{jw})$

To separate the real parts and imaginary parts of $H(e^{jw})$, multiply the numerator and denominator by the complex conjugate of the denominator.

$$\therefore\ H(e^{j\omega}) = \frac{1}{1 - be^{-j\omega}} \times \frac{1 - b\,e^{j\omega}}{1 - b\,e^{j\omega}} = \frac{1 - b\,e^{j\omega}}{1 - b\,e^{j\omega} - b\,e^{-j\omega} + b^2}$$

$$= \frac{1 - b(\cos\omega + j\sin\omega)}{1 + b^2 - b\left(e^{j\omega} + e^{-j\omega}\right)} = \frac{1 - b\cos\omega\ - jb\sin\omega}{1 + b^2 - 2b\cos\omega}$$

$$= \frac{1 - b\cos\omega}{1 + b^2 - 2b\cos\omega} - j\,\frac{b\sin\omega}{1 + b^2 - 2b\cos\omega}$$

$$\therefore\ H_i(e^{j\omega}) = \frac{-b\sin\omega}{1 + b^2 - 2b\cos\omega} \ \text{ and } H_r(e^{j\omega}) = \frac{1 - b\cos\omega}{1 + b^2 - 2b\cos\omega}$$

$$\text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})} = \tan^{-1}\left[\frac{-b\ \sin\omega}{1 - b\ \cos\omega}\right]$$

Example 4.18

The impulse response of an LTI system is given by, $h(n) = 0.8^n\,u(n)$. Find the frequency response.

Solution

The frequency response $H(e^{jw})$ is obtained by taking Fourier transform of the impulse response h(n).

Given that, impulse response, $h(n) = 0.8^n\,u(n)$ for all n.

On taking Fourier transform we get,

$$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\ e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} 0.8^n u(n) e^{-j\omega n} = \sum_{n=0}^{\infty} 0.8^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(0.8e^{-j\omega}\right)^n$$

u(n) = 1; n ³ 0   = 0; n < 0

$$= \frac{1}{1-0.8\ e^{-j\omega}}$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$ when $\lvert C\rvert < 1$

The frequency response has two functions: Magnitude function and phase function,

The magnitude function is defined as,

$$\text{Magnitude function, } \left|H(e^{j\omega})\right| = \sqrt{H(e^{j\omega})\ H^*(e^{j\omega})} = \sqrt{\frac{1}{1-0.8\,e^{-j\omega}} \times \frac{1}{1-0.8\,e^{j\omega}}}$$

$$= \frac{1}{\sqrt{1-0.8\,e^{j\omega}-0.8e^{-j\omega}+0.8^2}} = \frac{1}{\sqrt{1-0.8(e^{j\omega}+e^{-j\omega})+0.64}}$$

$$= \frac{1}{\sqrt{1.64-0.8(2\cos\omega)}} = \frac{1}{\sqrt{1.64-1.6\ \cos\omega}} \qquad .....(1)$$

The phase function can be determined by separating the real and imaginary part of $H(e^{jw})$. To separate the real and imaginary parts of $H(e^{jw})$, multiply the numerator and denominator by complex conjugate of the denominator.

Using equation (1)

$$\therefore\ H(e^{j\omega}) = \frac{1}{1-0.8\ e^{-j\omega}} \times \frac{1-0.8\ e^{j\omega}}{1-0.8\ e^{j\omega}} = \frac{1-0.8\ e^{j\omega}}{1.64-1.6\ \cos\omega}$$

$$= \frac{1-0.8(\cos\omega + j\sin\omega)}{1.64-1.6\ \cos\omega} = \frac{1-0.8\cos\omega}{1.64-1.6\ \cos\omega} - \frac{j0.8\sin\omega}{1.64-1.6\ \cos\omega}$$

$$\therefore\ H_i(e^{j\omega}) = \frac{-0.8\sin\omega}{1.64-1.6\ \cos\omega}$$

$$H_r(e^{j\omega}) = \frac{1-0.8\cos\omega}{1.64-1.6\ \cos\omega}$$

$$\therefore\ \frac{H_i(e^{j\omega})}{H_r(e^{j\omega})} = \frac{\dfrac{-0.8\sin\omega}{1.64-1.6\cos\omega}}{\dfrac{1-0.8\cos\omega}{1.64-1.6\cos\omega}} = \frac{-0.8\sin\omega}{1-0.8\cos\omega} \qquad .....(2)$$

The phase function is defined as,

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{-0.8\ \sin\omega}{1-0.8\ \cos\omega}\right]$$

Using equation (2)

Example 4.19

A system has impulse response h(n) given by, $h(n) = -\ 0.25\ \delta(n+1) + 0.5\ \delta(n) - 0.75\ \delta(n-1)$.

a) Is the system BIBO stable? b) Is the system causal? Justify your answer. c) Find the frequency response.

Solution

We know that, d(n) = 1 ; when n = 0
= 0 ; when n ¹ 0

Let us evaluate h(n) for different values of n.

When $n = -2$; $h(n) = h(-2) = -0.25\,\delta(-1) + 0.5\,\delta(-2) - 0.75\,\delta(-3) = 0 + 0 + 0 = 0$

When $n = -1$; $h(n) = h(-1) = -0.25\,\delta(0) + 0.5\,\delta(-1) - 0.75\,\delta(-2) = -0.25 + 0 + 0 = -0.25$

When $n = 0$; $h(n) = h(0) = -0.25\,\delta(1) + 0.5\,\delta(0) - 0.75\,\delta(-1) = 0 + 0.5 + 0 = 0.5$

When $n = 1$; $h(n) = h(1) = -0.25\,\delta(2) + 0.5\,\delta(1) - 0.75\,\delta(0) = 0 + 0 - 0.75 = -0.75$

When $n = 2$; $h(n) = h(2) = -0.25\,\delta(3) + 0.5\,\delta(2) - 0.75\,\delta(1) = 0 + 0 + 0 = 0$

From the above analysis, we can infer that $h(n) = 0$ for $n < -1$ and $n > 1$, and $h(n)$ ¹ 0 only for $n = -1, 0, 1$.

Here, $h(-1) = -0.25$, $h(0) = 0.5$, $h(1) = -0.75$

$$\therefore \text{ Impulse response, } h(n) = \{-0.25,\ \underset{\uparrow}{0.5},\ -0.75\}$$

a) Check for Stability

For stability of a system, $\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$

$$\sum_{n=-\infty}^{+\infty} |h(n)| = |h(-1)| + |h(0)| + |h(1)| = 0.25 + 0.5 + 0.75 = 1.5$$

Since $\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$, the system is BIBO stable.

b) Check for Causality

In a causal system the present output should depend only on present and past inputs or outputs, and should not depend on future inputs or outputs. In the given system, the response h(n) depends on the future input d(n+1). Hence the system is noncausal.

c) Frequency Response

The frequency response, $H(e^{jw})$ is given by the Fourier transform of h(n).

By definition of Fourier transform,

$$\text{The frequency response, } H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = h(-1)\, e^{j\omega} + h(0) + h(1)\, e^{-j\omega}$$

$$= -0.25e^{j\omega} + 0.5 - 0.75\, e^{-j\omega}$$

$$= -0.25(\cos\omega + j\sin\omega) + 0.5 - 0.75(\cos\omega - j\sin\omega)$$

$$= -0.25\cos\omega - j0.25\sin\omega + 0.5 - 0.75\cos\omega + j0.75\sin\omega$$

$$= 0.5 - \cos\omega + j0.5\sin\omega$$

The frequency response is complex function of w.

$$\therefore \text{ Magnitude function, } |H(e^{j\omega})| = \sqrt{H_i^2(e^{j\omega}) + H_r^2(e^{j\omega})} = \sqrt{(0.5 - \cos\omega)^2 + (0.5\sin\omega)^2}$$

$$\text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{0.5\sin\omega}{0.5 - \cos\omega}\right]$$

Example 4.20

A causal system is represented by the following difference equation.

$$y(n) + \frac{1}{4}\, y(n-1) = x(n) + \frac{1}{2}\, x(n-1)$$

Find the system transfer function H(z), the impulse response and frequency response of the system.

Solution

a) System Transfer Function

$$\text{The system transfer function, } H(z) = \frac{Y(z)}{X(z)}$$

Given that, $y(n)+\frac{1}{4}y(n-1)=x(n)+\frac{1}{2}\,x(n-1)$

Let, $\mathcal{Z}\{y(n)\}=Y(z)$, $\quad\therefore\ \mathcal{Z}\{y(n-1)\}=z^{-1}\,Y(z)$

Let, $\mathcal{Z}\{x(n)\}=X(z)$, $\quad\therefore\ \mathcal{Z}\{x(n-1)\}=z^{-1}\,X(z)$

On taking $\mathcal{Z}$-transform of the difference equation governing the system we get,

$$Y(z)+\frac{1}{4}z^{-1}Y(z)=X(z)+\frac{1}{2}z^{-1}\,X(z)$$

$$Y(z)\left(1+\frac{1}{4}z^{-1}\right)=X(z)\left(1+\frac{1}{2}z^{-1}\right)$$

$$\therefore\ \text{System transfer function, } H(z)=\frac{Y(z)}{X(z)}=\frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}}$$

b) Impulse Response

The impulse response h(n) is given by inverse $\mathcal{Z}$-transform of H(z).

$$H(z)=\frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}}=\frac{1}{1+\frac{1}{4}z^{-1}}+\frac{1}{2}\,\frac{z^{-1}}{1+\frac{1}{4}z^{-1}}$$

$$=\frac{1}{1-\left(-\frac{1}{4}\right)z^{-1}}+\frac{1}{2}\,\frac{z^{-1}}{1-\left(-\frac{1}{4}\right)z^{-1}}$$

$$\boxed{\mathcal{Z}\{a^n u(n)\}=\frac{1}{1-az^{-1}}}$$

On taking inverse $\mathcal{Z}$-transform of H(z) we get,

$$\boxed{\text{If } \mathcal{Z}\{x(n)\}=X(z) \text{ then by time shifting property } \mathcal{Z}\{x(n-1)\}=z^{-1}X(z)}$$

$$\text{Impulse response, } h(n)=\left(-\frac{1}{4}\right)^n u(n)+\frac{1}{2}\left(-\frac{1}{4}\right)^{(n-1)}u(n-1)$$

c) Frequency Response

The frequency response $H(e^{jw})$ is the Fourier transform of h(n), or $H(e^{jw})$ is obtained by evaluating H(z) at $z=e^{jw}$, or $H(e^{jw})$ is given by $\frac{Y(e^{j\omega})}{X(e^{j\omega})}$.

Method 1

By definition of Fourier transform,

$$\boxed{\begin{array}{l}\text{Let, } n-1=m\\ \quad\therefore\ n=m+1\\ \text{When } n=-\infty,\ m=-\infty\\ \text{When } n=+\infty,\ m=+\infty\end{array}}$$

$$H(e^{j\omega})=\mathcal{F}\{h(n)\}=\sum_{n=-\infty}^{+\infty}h(n)\,e^{-j\omega n}$$

$$=\sum_{n=-\infty}^{+\infty}\left[\left(-\frac{1}{4}\right)^n u(n)+\frac{1}{2}\left(-\frac{1}{4}\right)^{n-1}u(n-1)\right]e^{-j\omega n}$$

$$=\sum_{n=-\infty}^{+\infty}\left(-\frac{1}{4}\right)^n u(n)\,e^{-j\omega n}+\frac{1}{2}\sum_{n=-\infty}^{+\infty}\left(-\frac{1}{4}\right)^{n-1}u(n-1)e^{-j\omega n}$$

$$=\sum_{n=-\infty}^{+\infty}\left(-\frac{1}{4}\right)^n u(n)e^{-j\omega n}+\frac{1}{2}\sum_{m=-\infty}^{+\infty}\left(-\frac{1}{4}\right)^m u(m)e^{-j\omega(m+1)}$$

$$\therefore H(e^{j\omega}) = \sum_{n=0}^{+\infty}\left(-\frac{1}{4}\right)^n e^{-j\omega n} + \frac{1}{2}\sum_{m=0}^{+\infty}\left(-\frac{1}{4}\right)^m e^{-j\omega m}\,e^{-j\omega} = \sum_{n=0}^{+\infty}\left(-\frac{1}{4}e^{-j\omega}\right)^n + \frac{e^{-j\omega}}{2}\sum_{m=0}^{+\infty}\left(-\frac{1}{4}e^{-j\omega}\right)^m$$

$$= \frac{1}{1-\left(-\frac{1}{4}e^{-j\omega}\right)} + \frac{e^{-j\omega}}{2}\;\frac{1}{1-\left(-\frac{1}{4}e^{-j\omega}\right)}$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$

$$= \frac{1}{1+\frac{1}{4}e^{-j\omega}} + \frac{\frac{1}{2}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}} = \frac{1+\frac{1}{2}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}}$$

Method 2

The frequency response, $H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \left.\frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{4}z^{-1}}\right|_{z=e^{j\omega}} = \frac{1+\frac{1}{2}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}}$

Method 3

Given that, $y(n) + \frac{1}{4}y(n-1) = x(n) + \frac{1}{2}x(n-1)$

On taking Fourier transform,

$$Y(e^{j\omega}) + \frac{1}{4}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) + \frac{1}{2}e^{-j\omega}X(e^{j\omega}) \quad\Rightarrow\quad Y(e^{j\omega})\left[1+\frac{1}{4}e^{-j\omega}\right] = X(e^{j\omega})\left[1+\frac{1}{2}e^{-j\omega}\right]$$

$$\therefore \text{ Frequency response, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+\frac{1}{2}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}}$$

Magnitude and Phase Function

Magnitude function, $\left|H(e^{j\omega})\right| = \left[H(e^{j\omega})\,H^*(e^{j\omega})\right]^{\frac{1}{2}}$; where $H^*(e^{j\omega})$ is conjugate of $H(e^{j\omega})$

$$= \left[\frac{1+\frac{1}{2}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}} \times \frac{1+\frac{1}{2}e^{j\omega}}{1+\frac{1}{4}e^{j\omega}}\right]^{\frac{1}{2}} = \left[\frac{1+\frac{1}{2}e^{j\omega}+\frac{1}{2}e^{-j\omega}+\frac{1}{4}}{1+\frac{1}{4}e^{j\omega}+\frac{1}{4}e^{-j\omega}+\frac{1}{16}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1+\frac{1}{2}(e^{j\omega}+e^{-j\omega})+\frac{1}{4}}{1+\frac{1}{4}(e^{j\omega}+e^{-j\omega})+\frac{1}{16}}\right]^{\frac{1}{2}} = \left[\frac{\frac{5}{4}+\cos\omega}{\frac{17}{16}+\frac{1}{2}\cos\omega}\right]^{\frac{1}{2}}$$

$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$

The phase function of $H(e^{jw})$ is defined as, $\angle H(e^{j\omega}) = \tan^{-1}\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}$

where, $H_i(e^{jw})$ = Imaginary part of $H(e^{jw})$ and $H_r(e^{jw})$ = Real part of $H(e^{jw})$.

In order to separate the real part and imaginary parts of $H(e^{jw})$, multiply the numerator and denominator of $H(e^{jw})$ by the conjugate of denominator of $H(e^{jw})$.

$$\therefore H(e^{j\omega}) = \frac{1+\frac{1}{2}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}} \times \frac{1+\frac{1}{4}e^{j\omega}}{1+\frac{1}{4}e^{j\omega}} = \frac{1+\frac{1}{4}e^{j\omega}+\frac{1}{2}e^{-j\omega}+\frac{1}{8}}{1+\frac{1}{4}e^{j\omega}+\frac{1}{4}e^{-j\omega}+\frac{1}{16}}$$

$$= \frac{\frac{9}{8}+\frac{1}{4}(\cos\omega+j\sin\omega)+\frac{1}{2}(\cos\omega-j\sin\omega)}{\frac{17}{16}+\frac{1}{4}(e^{j\omega}+e^{-j\omega})}$$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

$$= \frac{\frac{9}{8}+\frac{1}{4}\cos\omega+\frac{1}{2}\cos\omega+j\frac{1}{4}\sin\omega-j\frac{1}{2}\sin\omega}{\frac{17}{16}+\frac{1}{2}\cos\omega}$$

$$= \frac{\frac{9}{8}+\frac{3}{4}\cos\omega}{\frac{17}{16}+\frac{1}{2}\cos\omega} + \frac{j\left(-\frac{1}{4}\sin\omega\right)}{\frac{17}{16}+\frac{1}{2}\cos\omega}$$

$$\therefore\ H_r(e^{j\omega}) = \frac{\frac{9}{8}+\frac{3}{4}\cos\omega}{\frac{17}{16}+\frac{1}{2}\cos\omega} \quad \text{and} \quad H_i(e^{j\omega}) = \frac{-\frac{1}{4}\sin\omega}{\frac{17}{16}+\frac{1}{2}\cos\omega}$$

$$\text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})} = \tan^{-1}\left[\frac{-\frac{1}{4}\sin\omega}{\frac{9}{8}+\frac{3}{4}\cos\omega}\right] = \tan^{-1}\left[\frac{-2\sin\omega}{9+6\cos\omega}\right]$$

Example 4.21

Find the frequency response of the LTI system, governed by the difference equation,

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = x(n)$$

Solution

Let, $\mathcal{F}\{x(n)\} = X(e^{jw})$, $\mathcal{F}\{y(n)\} = Y(e^{jw})$, \ $\mathcal{F}\{y(n-k)\} = e^{-jwk}\,Y(e^{jw})$

Given that, $y(n) + a_1\, y(n-1) + a_2\, y(n-2) = x(n)$

On taking Fourier transform we get,

$$Y(e^{jw}) + a_1 e^{-jw} Y(e^{jw}) + a_2 e^{-j2w} Y(e^{jw}) = X(e^{jw}) \quad Þ \quad (1 + a_1 e^{-jw} + a_2 e^{-j2w})Y(e^{jw}) = X(e^{jw})$$

$$\therefore\ \text{Frequency response, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1+a_1 e^{-j\omega}+a_2 e^{-j2\omega}}$$

The magnitude function of $H(e^{jw})$ is defined as,

$$|H(e^{jw})| = [H(e^{jw})\,H^*(e^{jw})]^{1/2} \quad ; \text{ where } H^*(e^{jw}) \text{ is conjugate of } H(e^{jw})$$

$$\therefore\ |H(e^{j\omega})| = \left[\frac{1}{1+a_1 e^{-j\omega}+a_2 e^{-j2\omega}} \times \frac{1}{1+a_1 e^{j\omega}+a_2 e^{j2\omega}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1+a_1 e^{j\omega}+a_2 e^{j2\omega}+a_1 e^{-j\omega}+a_1^2+a_1a_2e^{j\omega}+a_2 e^{-j2\omega}+a_1a_2e^{-j\omega}+a_2^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1+a_1^2+a_2^2+a_1(e^{j\omega}+e^{-j\omega})+a_2(e^{j2\omega}+e^{-j2\omega})+a_1a_2(e^{j\omega}+e^{-j\omega})}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1+a_1^2+a_2^2+2a_1\cos\omega+2a_2\cos 2\omega+2a_1a_2\cos\omega}\right]^{\frac{1}{2}} \qquad \boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

$$= \left[\frac{1}{1+a_1^2+a_2^2+2a_1(a_2+1)\cos\omega+2a_2\cos 2\omega}\right]^{\frac{1}{2}} \qquad .....(1)$$

The Phase function of $H(e^{j\omega})$ is defined as, $\angle H(e^{j\omega}) = \tan^{-1}\dfrac{H_i(e^{j\omega})}{H_r(e^{j\omega})}$

where, $H_i(e^{jw})$ = Imaginary part of $H(e^{jw})$ and $H_r(e^{jw})$ = Real part of $H(e^{jw})$

To separate the real and imaginary parts, multiply the numerator and denominator of $H(e^{jw})$ by the conjugate of the denominator of $H(e^{jw})$.

$$\therefore H(e^{j\omega}) = \frac{1}{1+a_1\,e^{-j\omega}+a_2\,e^{-j2\omega}} \times \frac{1+a_1\,e^{j\omega}+a_2\,e^{j2\omega}}{1+a_1\,e^{j\omega}+a_2\,e^{j2\omega}} \qquad .....(2)$$

Using equation (1), the equation (2) can be written as,

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

$$H(e^{j\omega}) = \frac{1+a_1\,e^{j\omega}+a_2\,e^{j2\omega}}{1+a_1^2+a_2^2+2a_1(a_2+1)\cos\omega+2a_2\cos 2\omega}$$

$$= \frac{1+a_1(\cos\omega+j\sin\omega)+a_2(\cos 2\omega+j\sin 2\omega)}{1+a_1^2+a_2^2+2a_1(a_2+1)\cos\omega+2a_2\cos 2\omega}$$

$$= \frac{1+a_1\cos\omega+a_2\cos 2\omega}{1+a_1^2+a_2^2+2a_1(a_2+1)\cos\omega+2a_2\cos 2\omega}$$

$$+ j\frac{a_1\sin\omega+a_2\sin 2\omega}{1+a_1^2+a_2^2+2a_1(a_2+1)\cos\omega+2a_2\cos 2\omega}$$

$$\therefore\ H_r(e^{j\omega}) = \frac{1+a_1\cos\omega+a_2\cos 2\omega}{1+a_1^2+a_2^2+2a_1(a_2+1)\cos\omega+2a_2\cos 2\omega}$$

$$H_i(\omega) = \frac{a_1\sin\omega+a_2\sin 2\omega}{1+a_1^2+a_2^2+2a_1(a_2+1)\cos\omega+2a_2\cos 2\omega}$$

The phase function, $\angle H(e^{j\omega}) = \tan^{-1}\dfrac{H_i(e^{j\omega})}{H_r(e^{j\omega})} = \tan^{-1}\left[\dfrac{a_1\sin\omega+a_2\sin 2\omega}{1+a_1\cos\omega+a_2\cos 2\omega}\right]$

Example 4.22

The impulse response of an LTI system is given by $h(n) = r^n\cos(w_0 n)\,u(n)$. Find the frequency response of the system.

Solution

The frequency response $H(e^{jw})$ is obtained by taking Fourier transform of h(n).

By definition of Fourier transform,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\,e^{-j\omega n} = \sum_{n=0}^{+\infty} r^n\cos\omega_0 n\;e^{-j\omega n}$$

$$= \sum_{n=0}^{+\infty} r^n\left[\frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2}\right]e^{-j\omega n} = \frac{1}{2}\sum_{n=0}^{+\infty}\left[r^n\,e^{j\omega_0 n}\,e^{-j\omega n}+r^n\,e^{-j\omega_0 n}\,e^{-j\omega n}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{+\infty}\left[r\,e^{j\omega_0}\,e^{-j\omega}\right]^n + \frac{1}{2}\sum_{n=0}^{\infty}\left[r\,e^{-j\omega_0}\,e^{-j\omega}\right]^n$$

For |r| < 1, we can apply the infinite geometric series sum formula to give,

$$H(e^{j\omega}) = \frac{1}{2}\,\frac{1}{1 - r\,e^{j\omega_0}\,e^{-j\omega}} + \frac{1}{2}\,\frac{1}{1 - r\,e^{-j\omega_0}\,e^{-j\omega}} = \frac{1}{2}\left[\frac{1 - r\,e^{-j\omega_0}\,e^{-j\omega} + 1 - r\,e^{j\omega_0}\,e^{-j\omega}}{\left(1 - r\,e^{j\omega_0}\,e^{-j\omega}\right)\left(1 - r\,e^{-j\omega_0}\,e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2}\,\frac{2 - r\,e^{-j\omega}\left(e^{-j\omega_0} + e^{j\omega_0}\right)}{1 - r\,e^{-j\omega_0}\,e^{-j\omega} - r\,e^{j\omega_0}\,e^{-j\omega} + r^2\,e^{-j2\omega}} = \frac{1}{2}\,\frac{2 - r\,e^{-j\omega}\left(e^{j\omega_0} + e^{-j\omega_0}\right)}{1 - r\,e^{-j\omega}\left(e^{j\omega_0} + e^{-j\omega_0}\right) + r^2\,e^{-j2\omega}}$$

$$= \frac{1}{2}\,\frac{2 - r\,e^{-j\omega}\,2\cos\omega_0}{1 - r\,e^{-j\omega}\,2\cos\omega_0 + r^2\,e^{-j2\omega}} = \frac{1 - r\cos\omega_0\,e^{-j\omega}}{1 - 2r\cos\omega_0\,e^{-j\omega} + r^2\,e^{-j2\omega}}$$

Frequency response, $H(e^{j\omega}) = \dfrac{1 - r\cos\omega_0\,e^{-j\omega}}{1 - 2r\cos\omega_0\,e^{-j\omega} + r^2\,e^{-j2\omega}}$

Let, $-r\cos w_0 = a$; $-2r\cos w_0 = a$; and $r^2 = b$. $\therefore\ H(e^{j\omega}) = \dfrac{1 + a\,e^{-j\omega}}{1 + \alpha\,e^{-j\omega} + \beta\,e^{-j2\omega}}$

The function $H(e^{jw})$ is same as frequency response of standard second order system. Hence refer section 4.6.5.

Example 4.23

An LTI system is described by the difference equation, y(n) = ay(n – 1) + bx(n). Find the impulse response, magnitude function and phase function. Solve b, if $|H(e^{jw})| = 1$. Sketch the magnitude and phase response for a = 0.7.

Solution

a) To Find Impulse Response

Let, $\mathcal{Z}\{x(n)\} = X(z)$, $\mathcal{Z}\{y(n)\} = Y(z)$, $\therefore\ \mathcal{Z}\{y(n-1)\} = z^{-1}\,Y(z)$.

Given that, y(n) = ay(n – 1) + bx(n).

On taking $\mathcal{Z}$-transform we get,

$$Y(z) = az^{-1}Y(z) + b\,X(z) \quad\Rightarrow\quad Y(z) - az^{-1}Y(z) = b\,X(z) \quad\Rightarrow\quad (1 - az^{-1})\,Y(z) = b\,X(z)$$

$$\therefore\ H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1 - az^{-1}}$$

$$\boxed{\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - az^{-1}}}$$

The impulse response is obtained by taking inverse $\mathcal{Z}$-transform of H(z).

$$\therefore\ \text{Impulse response}, h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{b}{1 - a\,z^{-1}}\right\} = b\,\mathcal{Z}^{-1}\left\{\frac{1}{1 - a\,z^{-1}}\right\} = b\,a^n\,u(n);\ \text{for all } n$$

$$\text{or}\quad h(n) = b\,a^n; \quad \text{for } n \geq 0$$

b) To Find Frequency Response

The frequency response $H(e^{jw})$ is obtained by evaluating H(z) at, $z = e^{jw}$.

$$\therefore\ \text{Frequency response},\ H(e^{j\omega}) = H(z)\big|_{z = e^{j\omega}} = \frac{b}{1 - az^{-1}}\bigg|_{z = e^{j\omega}} = \frac{b}{1 - a\,e^{-j\omega}}$$

$$\text{Magnitude function},\ |H(e^{j\omega})| = \left[H(e^{j\omega}) \times H^*(e^{j\omega})\right]^{\frac{1}{2}} = \left[\frac{b}{1 - a\,e^{-j\omega}} \times \frac{b}{1 - a\,e^{j\omega}}\right]^{\frac{1}{2}} = \left[\frac{b^2}{1 - ae^{j\omega} - ae^{-j\omega} + a^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{b^2}{1 + a^2 - a\left(e^{j\omega} + e^{-j\omega}\right)}\right]^{\frac{1}{2}} = \left[\frac{b^2}{1 + a^2 - 2a\cos\omega}\right]^{\frac{1}{2}} = \frac{b}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

The phase function is defined as,

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]; \quad \text{where, } H_i(e^{jw}) \text{ and } H_r(e^{jw}) \text{ are imaginary and real parts of } H(e^{jw}).$$

To separate real and imaginary parts of $H(e^{jw})$, multiply the numerator and denominator of $H(e^{jw})$ by the complex conjugate of the denominator.

$$\therefore\ H(e^{j\omega}) = \frac{b}{1-a\,e^{-j\omega}} \times \frac{1-a\,e^{j\omega}}{1-a\,e^{j\omega}} = \frac{b-ab\,e^{j\omega}}{1-a\,e^{j\omega}-a\,e^{-j\omega}+a^2} = \frac{b-ab(\cos\omega+j\sin\omega)}{1+a^2-a(e^{j\omega}+e^{-j\omega})}$$

$$= \frac{b-ab\cos\omega - jab\sin\omega}{1+a^2-2a\cos\omega} = \frac{b(1-a\cos\omega)}{1+a^2-2a\cos\omega} + j\frac{-ab\sin\omega}{1+a^2-2a\cos\omega}$$

$$\therefore\ H_r(e^{j\omega}) = \frac{b(1-a\cos\omega)}{1+a^2-2a\cos\omega} \quad \text{and } H_i(e^{j\omega}) = \frac{-ab\sin\omega}{1+a^2-2a\cos\omega}$$

$$\text{Phase function,}\quad \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{-ab\sin\omega}{b(1-a\cos\omega)}\right] = \tan^{-1}\left[\frac{-a\sin\omega}{1-a\cos\omega}\right]$$

c) To Evaluate b and Sketch Frequency Response

Given that, $|H(e^{jw})| = 1$

$$\therefore\ \frac{b}{\sqrt{1+a^2-2a\cos\omega}} = 1 \qquad \text{or} \qquad b = \sqrt{1+a^2-2a\cos\omega}$$

$$\text{When } a = 0.7,\ \angle H(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1-a\cos\omega}\right) = \tan^{-1}\left(\frac{-0.7\sin\omega}{1-0.7\cos\omega}\right)$$

The phase function is periodic in the range –p to +p. Hence the phase function is evaluated for various values of w in the range –p to +p.

$$\text{When } \omega = \frac{-4\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.7\sin(-\pi)}{1-0.7\cos(-\pi)} = 0$$

$$\text{When } \omega = \frac{-3\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.7\sin\left(\frac{-3\pi}{4}\right)}{1-0.7\cos\left(\frac{-3\pi}{4}\right)} = 0.32 = \frac{0.32}{\pi}\times\pi = 0.1\pi \text{ rad}$$

$$\text{When } \omega = \frac{-2\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.7\sin\left(\frac{-\pi}{2}\right)}{1-0.7\cos\left(\frac{-\pi}{2}\right)} = 0.61 = \frac{0.61}{\pi}\times\pi = 0.19\pi \text{ rad}$$

$$\text{When } \omega = \frac{-\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.7\sin\left(\frac{-\pi}{4}\right)}{1-0.7\cos\left(\frac{-\pi}{4}\right)} = 0.775 = \frac{0.775}{\pi}\times\pi = 0.25\pi \text{ rad}$$

$$\text{When } \omega = 0;\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.7\sin(0)}{1-0.7\cos(0)} = 0$$

$$\text{When } \omega = \frac{\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.7\sin\left(\frac{\pi}{4}\right)}{1-0.7\cos\left(\frac{\pi}{4}\right)} = -0.775 = \frac{-0.775}{\pi}\times\pi = -0.25\,\pi \text{ rad}$$

$$\text{When } \omega = \frac{2\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.7\sin\left(\frac{\pi}{2}\right)}{1-0.7\cos\left(\frac{\pi}{2}\right)} = -0.61 = \frac{-0.61}{\pi}\times\pi = -0.19\,\pi \text{ rad}$$

$$\text{When } \omega = \frac{3\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.7\sin\left(\frac{3\pi}{4}\right)}{1-0.7\cos\left(\frac{3\pi}{4}\right)} = -0.32 = \frac{-0.32}{\pi}\times\pi = -0.1\pi \text{ rad}$$

$$\text{When } \omega = \frac{4\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.7\sin\pi}{1-0.7\cos\pi} = 0$$

The phase function of fig 2 is sketched using the above calculated values. The magnitude function is a straight line, passing through "1" as shown in fig 1.

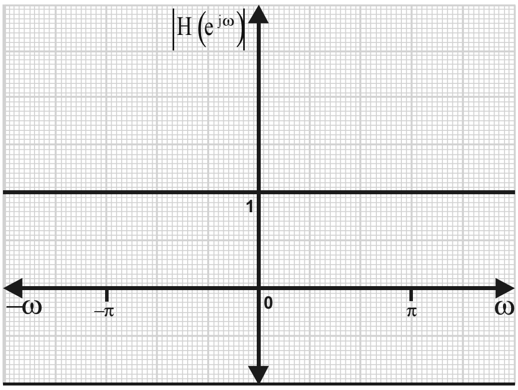


Fig 1 : Magnitude function.

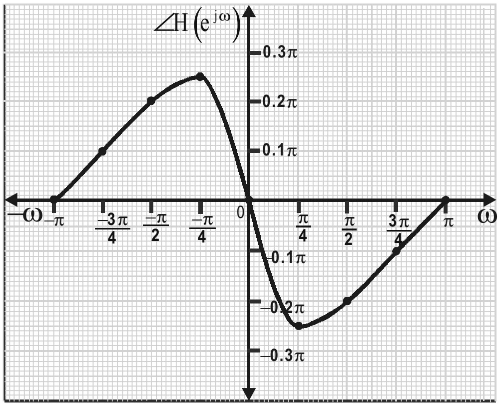


Fig 2 : Phase function.

Example 4.24

Determine the frequency response of an LTI system governed by the difference equation,

$$y(n) = x(n) + 0.81\ x(n-1) + 0.81\ x(n-2) - 0.45\ y(n-2)$$

Solution

Let, $\mathcal{F}\{y(n)\} = Y(e^{jw})$ \ $\mathcal{F}\{y(n-k)\} = e^{-jwk}\, Y(e^{jw})$

Let, $\mathcal{F}\{x(n)\} = X(e^{jw})$ \ $\mathcal{F}\{x(n-k)\} = e^{-jwk}\, X(e^{jw})$

Given that, $y(n) = x(n) + 0.81\ x(n-1) + 0.81\ x(n-2) - 0.45\ y(n-2)$

On taking Fourier transform we get,

$$Y(e^{jw}) = X(e^{jw}) + 0.81\ e^{-jw}\, X(e^{jw}) + 0.81\ e^{-j2w}\, X(e^{jw}) - 0.45\ e^{-j2w}\ Y(e^{jw})$$

$$Y(e^{jw}) + 0.45e^{-j2w} Y(e^{jw}) = X(e^{jw}) + 0.81\ e^{-jw}\, X(e^{jw}) + 0.81\ e^{-j2w}\ X(e^{jw})$$

$$(1+0.45\ e^{-j2w})\ Y(e^{jw}) = (1+0.81e^{-jw} + 0.81e^{-j2w})\ X(e^{jw})$$

$$\therefore\quad \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+0.81\ e^{-j\omega} + 0.81\ e^{-j2\omega}}{1+0.45\ e^{-j2\omega}}$$

The frequency response, $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{1+0.81\ e^{-j\omega} + 0.81\ e^{-j2\omega}}{1+0.45\ e^{-j2\omega}}$

Magnitude function, $|H(e^{j\omega})| = \left[H(e^{j\omega})\ H^*(e^{j\omega})\right]^{\frac{1}{2}}$

$$= \left[\frac{1+0.81\ e^{-j\omega} + 0.81\ e^{-j2\omega}}{1+0.45\ e^{-j2\omega}} \times \frac{1+0.81\ e^{j\omega} + 0.81\ e^{j2\omega}}{1+0.45\ e^{j2\omega}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{\begin{array}{r}1+0.81\ e^{j\omega} + 0.81\ e^{j2\omega} + 0.81\ e^{-j\omega} + 0.81^2 + 0.81^2\ e^{j\omega} \\ + 0.81\, e^{-j2\omega} + 0.81^2\ e^{-j\omega} + 0.81^2\end{array}}{1+0.45\ e^{j2\omega} + 0.45\ e^{-j2\omega} + 0.45^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2.31 + 0.81(e^{j\omega} + e^{-j\omega}) + 0.66(e^{j\omega} + e^{-j\omega}) + 0.81(e^{j2\omega} + e^{-j2\omega})}{1.2 + 0.45(e^{j2\omega} + e^{-j2\omega})}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2.31 + 1.62\ \cos\omega + 1.32\ \cos\omega + 1.62\ \cos 2\omega}{1.2 + 0.9\ \cos 2\omega}\right]^{\frac{1}{2}}$$

$$\therefore\ |H(e^{j\omega})| = \left[\frac{2.31 + 2.94\ \cos\omega + 1.62\ \cos 2\omega}{1.2 + 0.9\cos 2\omega}\right]^{\frac{1}{2}} \qquad .....(1)$$

Phase function, $\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$; where, $H_i(e^{jw})$ = Imaginary part and $H_r(e^{jw})$ = Real part

To separate real part and imaginary parts of $H(e^{jw})$, multiply the numerator and denominator of $H(e^{jw})$ by the complex conjugate of $H(e^{jw})$.

$$\therefore\ H(e^{j\omega}) = \frac{1 + 0.81\ e^{-j\omega} + 0.81\ e^{-j2\omega}}{1 + 0.45\ e^{-j2\omega}} \times \frac{1 + 0.45\ e^{j2\omega}}{1 + 0.45\ e^{j2\omega}} \qquad ..... (2)$$

$$= \frac{(1 + 0.81\ e^{-j\omega} + 0.81\ e^{-j2\omega})\ (1 + 0.45\ e^{j2\omega})}{1.2 + 0.9\ \cos 2\omega} \qquad \boxed{\text{Using equation (1)}}$$

$$= \frac{1 + 0.45\ e^{j2\omega} + 0.81\ e^{-j\omega} + 0.36\ e^{j\omega} + 0.81\ e^{-j2\omega} + 0.36}{1.2 + 0.9\cos 2\omega}$$

$$= \frac{\begin{array}{c}1.36 + 0.45(\cos 2\omega + j\sin 2\omega) + 0.81(\cos\omega - j\sin\omega)\\ + 0.36(\cos\omega + j\sin\omega) + 0.81(\cos 2\omega - j\sin 2\omega)\end{array}}{1.2 + 0.9\ \cos 2\omega}$$

$$\therefore\ H_r(e^{j\omega}) = \frac{1.36 + 0.45\ \cos 2\omega + 0.81\ \cos\omega + 0.36\ \cos\omega + 0.81\ \cos 2\omega}{1.2 + 0.9\ \cos 2\omega}$$

$$= \frac{1.36 + 1.17\ \cos\omega + 1.26\ \cos 2\omega}{1.2\ +\ 0.9\ \cos 2\omega}$$

$$H_i(e^{j\omega}) = \frac{0.45\ \sin 2\omega - 0.81\ \sin\omega + 0.36\ \sin\omega - 0.81\ \sin 2\omega}{1.2 + 0.9\ \cos 2\omega}$$

$$= \frac{-0.\ 45\sin\omega\ - 0.36\sin 2\omega}{1.2 + 0.9\ \cos 2\omega}$$

$$\text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{-0.45\ \sin\omega - 0.36\ \sin 2\omega}{1.36 + 1.17\ \cos\omega + 1.26\ \cos 2\omega}\right]$$

Example 4.25

The impulse response of system is $h(n) = 1\ ;\ 0 \le n \le (N-1)$

$= 0\ ;$ otherwise

Find the transfer function and frequency response.

> Using finite geometric series sum formula
> $$\sum_{n=0}^{N-1} C^n = \frac{1 - C^N}{1 - C}$$

Solution

The transfer function H(z) is obtained by taking Z-transform of the impulse response,

$$\therefore\ \text{Transfer function, } H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{\infty} h(n)\ z^{-n} = \sum_{n=0}^{N-1} z^{-n} = \frac{1 - \left(z^{-1}\right)^N}{1 - z^{-1}} = \frac{1 - z^{-N}}{1 - z^{-1}}$$

The frequency response $H(e^{jw})$ is obtained by evaluating H(z) at $z = e^{jw}$.

$$\therefore\ \text{Frequency response, } H(e^{j\omega}) = H(z)\big|_{z = e^{j\omega}} = \left.\frac{1 - z^{-N}}{1 - z^{-1}}\right|_{z = e^{j\omega}} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$\text{Magnitude function, } |H(e^{j\omega})| = \left[H(e^{j\omega})\ H^*(e^{j\omega})\right]^{\frac{1}{2}}$$

$$= \left[\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \times \frac{1 - e^{j\omega N}}{1 - e^{j\omega}}\right]^{\frac{1}{2}} = \left[\frac{1 - e^{j\omega N} - e^{-j\omega N} + 1}{1 - e^{j\omega} - e^{-j\omega} + 1}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2\ -\ (e^{j\omega N} + e^{-j\omega N})}{2\ -\ (e^{j\omega} + e^{-j\omega})}\right]^{\frac{1}{2}} = \left[\frac{2\ -\ 2\ \cos\omega N}{2\ -\ 2\ \cos\omega}\right]^{\frac{1}{2}} = \left[\frac{1\ -\ \cos\omega N}{1\ -\ \cos\omega}\right]^{\frac{1}{2}}$$

In order to determine the phase function, the real and imaginary part of $H(e^{j\omega})$ has to be separated.

$$\therefore\ H(e^{j\omega}) = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \times \frac{1-e^{j\omega}}{1-e^{j\omega}}$$

$$= \frac{1-e^{j\omega}-e^{-j\omega N}+e^{-j\omega N}\,e^{j\omega}}{1-e^{j\omega}-e^{-j\omega}+1} = \frac{1-e^{j\omega}-e^{-j\omega N}+e^{-j\omega(N-1)}}{2-(e^{j\omega}+e^{-j\omega})}$$

$$= \frac{1-(\cos\omega + j\sin\omega)-(\cos\omega N - j\sin\omega N)+(\cos\omega(N-1) - j\sin\omega(N-1))}{2-2\cos\omega}$$

Real part, $$H_r(e^{j\omega}) = \frac{1-\cos\omega-\cos\omega N+\cos\omega(N-1)}{2-2\cos\omega}$$

Imaginary part, $$H_i(e^{j\omega}) = \frac{-\sin\omega+\sin\omega N-\sin\omega(N-1)}{2-2\cos\omega}$$

$$\therefore\ \text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{-\sin\omega+\sin\omega N-\sin\omega(N-1)}{1-\cos\omega-\cos\omega N+\cos\omega(N-1)}\right]$$

Example 4.26

Consider the analog signal, $x_a(t) = 2\cos 2000\pi t + 5\sin 4000\pi t + 12\cos 12000\pi t$.

a) Determine the Nyquist sampling rate.

b) If the analog signal is sampled at $F_s = 5000$ Hz, determine the discrete time signal obtained by sampling.

Solution

a) To Find Nyquist Sampling Rate

The given analog signal can be written as shown below.

$$x_a(t) = 2\cos 2000\pi t + 5\sin 4000\pi t - 12\cos 12000\pi t = 2\cos 2\pi F_1 t + 5\sin 2\pi F_2 t - 12\cos 2\pi F_3 t$$

where, $2\pi F_1 = 2000\pi \Rightarrow F_1 = 1000\,\text{Hz}$

$2\pi F_2 = 4000\pi \Rightarrow F_2 = 2000\,\text{Hz}$

$2\pi F_3 = 12000\pi \Rightarrow F_3 = 6000\,\text{Hz}$

The maximum analog frequency in the given signal, F_{max} is 6000 Hz. The Nyquist sampling rate is twice that of this maximum analog frequency.

$\therefore$ Nyquist sampling rate, $F_s = 2F_{max} = 2 \times 6000 = 12000$ Hz

In order to avoid aliasing the sampling frequency, F_s should be greater than or equal to Nyquist rate.

b) To Determine the Discrete Time Signal Sampled at 5000 Hz

Let $x_a(nT)$ be the discrete time signal obtained by sampling the given analog signal.

$$\therefore\ x_a(nT) = x_a(t)\big|_{t=nT} = x_a(t)\Big|_{t=\frac{n}{F_s}} = 2\cos\frac{2000\pi n}{F_s} + 5\sin\frac{4000\pi n}{F_s} + 12\cos\frac{12000\pi n}{F_s}$$

$$= 2\cos\frac{2000\pi n}{5000} + 5\sin\frac{4000\pi n}{5000} + 12\cos\frac{12000\pi n}{5000}$$

$$= 2\cos\frac{2\pi n}{5} + 5\sin\frac{4\pi n}{5} + 12\cos\frac{12\pi n}{5} = 2\cos\frac{2\pi n}{5} + 5\sin\frac{4\pi n}{5} + 12\cos\left(\frac{2\pi n}{5} + \frac{10\pi n}{5}\right)$$

$$= 2\cos\frac{2\pi n}{5} + 5\sin\frac{4\pi n}{5} + 12\cos\left(\frac{2\pi n}{5} + 2\pi n\right) = 2\cos\frac{2\pi n}{5} + 5\sin\frac{4\pi n}{5} + 12\cos\frac{2\pi n}{5}$$

$$= 14\cos\frac{2\pi n}{5} + 5\sin\frac{4\pi n}{5}$$

For integer n, $\cos(\theta + 2\pi n) = \cos\theta$

Comment : When sampled at 5000 Hz, the component $12\cos 12000\pi t$ is an alias of the component $2\cos 2000\pi t$.

4.9 Summary of Important Concepts

1. A periodic discrete time signal with a fundamental period N can be decomposed into N harmonically related frequency components.
2. The Fourier series representation can be obtained only for periodic discrete time signals.
3. The Fourier transform technique can be applied to both periodic and nonperiodic discrete time signals.
4. The Fourier coefficients of periodic discrete time signal with period N is also periodic with period N.
5. The Fourier coefficient c_k represents the amplitude and phase associated with the k^{th} frequency component.
6. The frequency range of discrete time signal is 0 to 2π (or $-\pi$ to $+\pi$) and so it has finite frequency spectrum.
7. The plot of harmonic magnitude / phase of a discrete time signal versus "k" (or harmonic frequency ω_k) is called Frequency spectrum.
8. The plot of harmonic magnitude versus "k" (or ω_k) is called magnitude spectrum.
9. The plot of harmonic phase versus "k" (or ω_k) is called phase spectrum.
10. The sequence $|c_k|^2$ for k = 0, 1, 2,....., (N - 1) is called the power density spectrum (or) power spectral density of the periodic signal.
11. The Fourier transform is also called analysis of discrete time signal x(n).
12. The inverse Fourier transform is also called synthesis of discrete time signal x(n).
13. The Fourier transform exists only for the discrete time signals that are absolutely summable.
14. The Fourier transform of a signal is also called signal spectrum.
15. The Fourier transform of a discrete time signal is periodic with period 2π.
16. The Fourier transform of any periodic discrete time signal consists of train of impulses located at harmonic frequencies of the signal.
17. The ratio of Fourier transform of output and input of an LTI discrete time system is called transfer function of the LTI discrete time system in frequency domain.
18. The frequency domain transfer function is also given by Fourier transform of impulse response.
19. The Fourier transform of impulse response is called frequency response of the system.
20 The frequency response of discrete time system is periodic continuous function of ω with period 2π
21. The first order discrete time system behaves as either lowpass filter or highpass filter.
22. The second order discrete time system behaves as a resonant filter or bandpass filter.
23. The frequency spectrum of a discrete time signal obtained by sampling continuous time signal will be sum of frequency shifted and amplitude scaled spectrum of continuous time signal.
24. The frequency Ω of a continuous time signal can be converted to frequency ω of a discrete time signal by choosing the transformation, $\omega = \Omega T$, where T is the sampling time.
25. The overlap of frequency spectrum is called aliasing.
26. Due to aliasing the information shifts from one band of frequency to another band of frequency.
27. In order to avoid aliasing, the sampling frequency F_s should be greater than or equal to twice the maximum frequency F_m of continuous time signal.
28. When the spectrum of sampled signal has no aliasing then it is possible to recover the original signal from the sampled signal.
29. The bandpass signals with a bandwidth of B Hz can be sampled at a rate of 2B to 4B Hz.
30. The Fourier transform of a discrete time signal can be obtained by evaluating the $\mathcal{Z}$-transform on a circle of unit radius provided the ROC of $\mathcal{Z}$-transform includes unit circle.

4.10 Short Questions and Answers

Q4.1 ***Find Fourier coefficients of x(n), where*** $x(n) = \sum_{k=-\infty}^{+\infty} \delta(n-3k)$.

Solution

Given signal is a periodic impulse signal with impulses located at n = 3k, for integer values of k.

Let, one period of the given signal be $x_1(n)$.

Now, $x_1(n) = \{1, 0, 0\}$, with period N = 3, and with fundamental frequency, w_0=2p/3.

The Fourier coefficient c_k is given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-jk2\pi n}{N}} = \frac{1}{3}\sum_{n=0}^{2} x_1(n)\, e^{\frac{-jk2\pi n}{3}} = \frac{1}{3}\,x(0) + 0 + 0 = \frac{1}{3}\ ; \quad \text{for all k.}$$

Q4.2 ***Determine the discrete time Fourier series of*** $x(n) = \cos^2\left(\frac{\pi}{6}n\right)$.

Solution

Given that, $x(n) = \cos^2\left(\frac{\pi}{6}n\right)$. Let us check, whether the given signal is periodic.

$$x(n+N) = \cos^2\frac{\pi}{6}(n+N) = \left(\cos\left(\frac{\pi n}{6} + \frac{\pi N}{6}\right)\right)^2$$

Since $\cos(\theta + 2\pi M) = \cos\theta$, For periodicity, $\frac{\pi N}{6}$ should be an integral multiple of 2π.

Let, $\frac{\pi N}{6} = M \times 2\pi$, where M and N are integers. $\Rightarrow$ N = 12 M, Let M = 1, $\therefore$ N = 12.

$\therefore$ x(n) is periodic with fundamental period, N = 12 and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{12} = \frac{\pi}{6}$

The Fourier series of x(n) can be obtained from Euler's formula as shown below.

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$x(n) = \cos^2\left(\frac{\pi}{6}n\right) = \left[\cos\left(\frac{\pi n}{6}\right)\right]^2 = \left[\frac{e^{\frac{j\pi n}{6}} + e^{-\frac{j\pi n}{6}}}{2}\right]^2 = \left[\frac{e^{\frac{j\pi n}{6}}}{2} + \frac{e^{-\frac{j\pi n}{6}}}{2}\right]^2$$

$$= \frac{1}{4}e^{\frac{j2\pi n}{6}} + \frac{1}{4}e^{\frac{-j2\pi n}{6}} + \frac{1}{2} = \frac{1}{4}e^{\frac{-j2\pi n}{6}} + \frac{1}{2} + \frac{1}{4}e^{\frac{j2\pi n}{6}}$$

$$= \frac{1}{4}e^{-j2\omega_0 n} + \frac{1}{2} + \frac{1}{4}e^{j2\omega_0 n} \quad ; \text{ where } \omega_0 = \frac{\pi}{6}$$

Q4.3 ***Find the Fourier transform of x(n) = { 2, 1, 2 }.***

Solution

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{2} x(n)\, e^{-j\omega n} = x(0)\, e^0 + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega}$$

$$= 2 + e^{-j\omega} + 2\, e^{-j2\omega} = 2\, e^{-j\omega}\,(e^{j\omega} + e^{-j\omega}) + e^{-j\omega}$$

$$= 4\cos\omega\, e^{-j\omega} + e^{-j\omega} = (1 + 4\cos\omega)\, e^{-j\omega}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Q4.4 ***Determine the Fourier transform of x(n) = u(n) – u(n–N).***

Solution

x(n) can be expressed as, x(n) = 1 ; for n = 0 to N–1.

By definition of Fourier transform,

Using finite geometric series sum formula $\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{N-1} 1 \times e^{-j\omega n} = \sum_{n=0}^{N-1} \left(e^{-j\omega}\right)^n = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$= \frac{1-\left(e^{\frac{-j\omega N}{2}}\, e^{\frac{-j\omega N}{2}}\right)}{1-\left(e^{\frac{-j\omega}{2}}\, e^{\frac{-j\omega}{2}}\right)} = \frac{e^{\frac{-j\omega N}{2}}\left[e^{\frac{j\omega N}{2}} - e^{\frac{-j\omega N}{2}}\right]}{e^{\frac{-j\omega}{2}}\left[e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right]} = e^{-j\omega\left(\frac{N}{2}-\frac{1}{2}\right)}\left[\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}\right] = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}\right]$$

Q4.5 ***Find the Fourier transform of , x(n) = –aⁿ u(–n –1), where |a| < 1.***

Solution

By definition of Fourier transform,

Using finite geometric series sum formula $\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}$

when n = 0; $a^{-n} e^{jwn} = 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{-1} -a^n\, e^{-j\omega n} = \sum_{n=1}^{\infty} -a^{-n}\, e^{j\omega n} = 1-\sum_{n=0}^{\infty} a^{-n}\, e^{j\omega n} = 1-\sum_{n=0}^{\infty}\left(a^{-1}e^{j\omega}\right)^n$$

$$= 1-\frac{1}{1-a^{-1}e^{j\omega}} = 1-\frac{a}{a-e^{j\omega}} = \frac{a-e^{j\omega}-a}{a-e^{j\omega}} = \frac{-e^{j\omega}}{a-e^{j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-a}$$

Q4.6 ***Find the discrete time Fourier transform of the signal, $x(n) = (0.2)^n\, u(n) + (0.2)^{-n}\, u(-n-1)$.***

Solution

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\; e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (0.2)^n\; u(n)\; e^{-j\omega n} + \sum_{n=-\infty}^{\infty} (0.2)^{-n}\; u(-n-1)\; e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.2\, e^{-j\omega})^n + \sum_{n=-\infty}^{-1} (\,0.2\, e^{j\omega})^{-n} = \sum_{n=0}^{\infty} (0.2\, e^{-j\omega})^n + \sum_{n=1}^{\infty} (0.2\, e^{j\omega})^n$$

when n = 0; $(0.2e^{jw})^n = 1$

$$= \sum_{n=0}^{\infty} (0.2\, e^{-j\omega})^n + \sum_{n=0}^{\infty} (0.2\, e^{j\omega})^n - 1 = \frac{1}{1-0.2\, e^{-j\omega}} + \frac{1}{1-0.2\, e^{j\omega}} - 1$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$ when $|C| < 1$

$$= \frac{1-0.2\, e^{j\omega} + 1-0.2\, e^{-j\omega} - (1-0.2\, e^{-j\omega})\,(1-0.2\, e^{j\omega})}{(1-0.2\, e^{-j\omega})\,(1-0.2\, e^{j\omega})}$$

$$= \frac{1-0.2\, e^{j\omega} + 1-0.2\, e^{-j\omega} - (1-0.2\, e^{j\omega} - 0.2\, e^{-j\omega} + 0.04)}{1-0.2\, e^{j\omega} - 0.2\, e^{-j\omega} + 0.04}$$

$$= \frac{1-0.04}{1-0.2\,(e^{j\omega}+e^{-j\omega})+0.04} = \frac{0.96}{1.04\; - \;0.4\cos\,\omega}$$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

Q4.7 ***Determine the energy density spectrum of a discrete time signal , $x(n) = a^n\, u(n)$ for $-1 < a < 1$.***

Solution

By definition of Fourier transform,

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$ when $|C| < 1$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{\infty} a^n\, e^{-j\omega n} = \sum_{n=0}^{\infty}\left(a\, e^{-j\omega}\right)^n = \frac{1}{1-a\, e^{-j\omega}}$$

Now the energy density spectrum is,

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\left|X(e^{j\omega})\right|^2 = X(e^{j\omega})\ X^*(e^{j\omega}) = \frac{1}{1 - a\,e^{-j\omega}} \times \frac{1}{1 - a\,e^{j\omega}}$$

$$= \frac{1}{1 - a\,e^{j\omega} - a\,e^{-j\omega} + a^2} = \frac{1}{1 - a\,(e^{j\omega} + e^{-j\omega}) + a^2} = \frac{1}{1 - 2a\cos\omega + a^2}$$

Q4.8 ***Find the inverse Fourier transform of the rectangular pulse spectrum defined as,***

$$X(e^{j\omega}) = 1 \ ; \ |\omega| \le W$$
$$= 0 \ ; \ W \le |\omega| \le \pi$$

Solution

By definition inverse Fourier transform,

$$\boxed{\sin\theta = \frac{e^{j\theta} + e^{-j\theta}}{2j}} \qquad \boxed{\frac{\sin\theta}{\theta} = \operatorname{sinc}\theta}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-W}^{W} = \frac{1}{2\pi}\left[\frac{e^{jWn}}{jn} - \frac{e^{-jWn}}{jn}\right] = \frac{1}{\pi n}\left[\frac{e^{jWn} - e^{-jWn}}{2j}\right]$$

$$= \frac{\sin Wn}{\pi n} = \frac{W}{\pi}\,\frac{\sin Wn}{Wn} = \frac{W}{\pi}\operatorname{sinc} Wn$$

Q4.9 ***Determine the inverse Fourier transform of*** $X(e^{j\omega}) = 2\pi\,\delta(\omega - \omega_0)$, $\left|\omega_0\right| \le \pi$.

Solution

The inverse Fourier transform of $X(e^{jw})$ is,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\ e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\ \delta(\omega - \omega_0)\ e^{j\omega n}\ d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega - \omega_0)\ e^{j\omega n}\ d\omega = e^{j\omega n}\Big|_{\omega=\omega_0} = e^{j\omega_0 n}$$

Note : Here the integral limit is -p to +p, and in this range there is only one impulse located at w_0.

Q4.10 ***A causal discrete time LTI system has a system function*** $H(z) = \dfrac{1 - 2a\,z^{-1}}{2b + z^{-1}}$ ***. Here 'a' is real and*** $|a| < 1$***. Find the value of 'b' so that the frequency response*** $H(e^{jw})$ ***of the system satisfies the condition*** $|H(e^{jw})| = 1$ ***for all w.***

Solution

Given that, $H(z) = \dfrac{1 - 2a\,z^{-1}}{2b + z^{-1}}$

The frequency response of the system can be obtained by putting, $z = e^{jw}$ in H(z).

$$\therefore H(e^{j\omega}) == H(z)\big|_{z=e^{j\omega}} = \frac{1 - 2a\,e^{-j\omega}}{2b + e^{-j\omega}}$$

Here, $\left|H(e^{j\omega})\right| = 1$; $\quad\therefore \left|\dfrac{1 - 2a\,e^{-j\omega}}{2b + e^{-j\omega}}\right| = 1 \quad \Rightarrow \left|1 - 2a\,e^{-j\omega}\right| = \left|2b + e^{-j\omega}\right|$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$\therefore \quad \left|1 - 2a\cos\omega + j\ 2a\sin\omega\right| = \left|2b + \cos\omega - j\sin\omega\right|$$

$$\left(1 - 2a\cos\omega\right)^2 + \left(2a\sin\omega\right)^2 = \left(2b + \cos\omega\right)^2 + \left(\sin\omega\right)^2$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$1 + 4a^2\cos^2\omega - 4a\cos\omega + 4a^2\sin^2\omega = 4b^2 + 4b\cos\omega + \cos^2\omega + \sin^2\omega$$

$$1 + 4a^2 - 4a\cos\omega = 4b^2 + 4b\cos\omega + 1$$

The above equation is true, when $b = -a$.

Hence to satisfy the condition $\left|H(e^{j\omega})\right| = 1$ for all w, $b = -a$.

Q4.11 ***Determine the sampling period for the signal X(jW) = U (jW + jW$_0$) – U (jW – jW$_0$), to sample without aliasing.***

Solution

The frequency spectrum of the given signal can be plotted as shown in fig Q4.11.

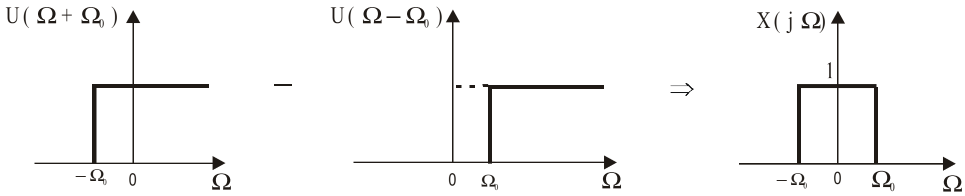


Fig Q4.11.

From the frequency spectrum of fig Q4.11, it is observed that the maximum frequency, W$_{max}$ is,

$$\Omega_{max} = \Omega_0 \;; \qquad \therefore\; 2\pi\, F_{max} = \Omega_0 \qquad \Rightarrow \qquad F_{max} = \frac{\Omega_0}{2\pi}$$

$$\therefore \text{Sampling frequency, } F_s \;\geq\; 2\,F_{max} \;\;\Rightarrow\;\; \text{Sampling period, } T \;\leq\; \frac{1}{F_s}$$

$$\therefore \text{Minimum sampling period, } T \;=\; \frac{1}{F_s} = \frac{1}{2F_{max}} \;=\; \frac{\pi}{\Omega_0}$$

$$\therefore \text{In order to avoid aliasing the sampling period T should be less than } \frac{\pi}{\Omega_0} \left(\text{i.e., } T < \frac{\pi}{\Omega_0} \right).$$

Q4.12 ***Determine the Nyquist sampling frequency and Nyquist interval for the signal, $x(t) = \left[\dfrac{\sin 200\,\pi\, t}{\pi\, t}\right]^2$.***

Solution

$$x(t) = \left[\frac{\sin 200\,\pi\, t}{\pi\, t}\right]^2 \;=\; \frac{1}{\pi^2\, t^2}\, \sin^2(200\,\pi\, t) = \frac{1}{\pi^2\, t^2}\, \frac{\left[1 - \cos\, 2(200\pi\, t)\right]}{2}$$

$$= \frac{1}{2\pi^2\, t^2}\left[1 - \cos\, 400\pi\, t\right] = \frac{1}{2\pi^2\, t^2} \;-\; \frac{\cos\, 400\pi t}{2\pi^2\, t^2}$$

$$\boxed{\sin^2\theta = \frac{1 - \cos\, 2\theta}{2}}$$

On comparing the cosine component with standard cosine wave "A cosW t" we get,

W = 400p Þ 2pF = 400p Þ F = 200 Hz

From the above analysis it is observed that, the maximum frequency in the signal F_{max} = 200 Hz.

$\therefore$ Nyquist rate = 2 F_{max} = 2×200 = 400 Hz

$$\text{Nyquist interval} \;=\; \frac{1}{\text{Nyquist rate}} = \frac{1}{400} \;=\; 2.5 \text{ ms}$$

Q4.13 ***A signal x(t) whose spectrum is shown in fig Q4.13.1 is sampled at a rate of 300 samples / sec. Sketch the spectrum of the sampled discrete time signal.***

Solution

From the spectrum shown in fig Q4.13.1 it is observed that the maximum frequency, F_m in the signal is 100 Hz. Given that, Sampling frequency, F_s is 300 Hz, which is greater than 2 F_m, and so the signal is sampled without aliasing.

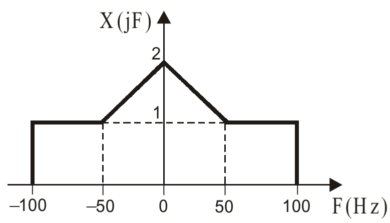


Fig Q4.13.1.

Frequency "f" of sampled discrete time signal corresponding to any frequency "F" of continuous time signal is given by, $f = F / F_s$.

The magnitude of the spectrum of discrete time signal will be scaled by 1/T, where $T = 1/F_s$. The frequency spectrum of a discrete time signal will be periodic with periodicity of - 0.5 to + 0.5. (Refer Chapter-2, Section 2.3). Therefore the frequency spectrum of sampled discrete time signal will be as shown in fig Q4.13.2.

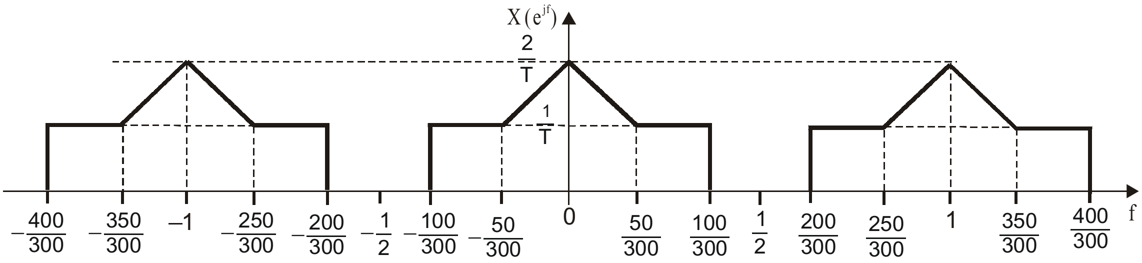


Fig Q4.13.2.

Q4.14 If the spectrum shown in fig Q4.13.1 is sampled at a rate of 100 samples / sec. Sketch the spectrum of the sampled discrete time signal.

Since the sampling frequency is less than $2F_m$, the spectrum of the sampled signal will have aliasing as shown in fig Q4.14.1.

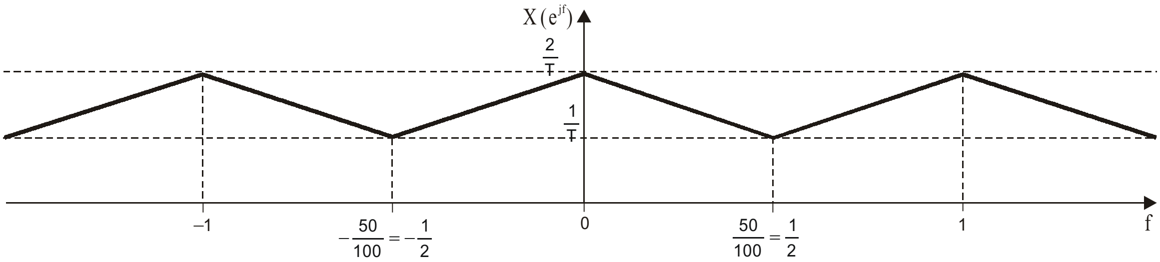


Fig Q4.14.1.

Q4.15 Consider the sampling of the bandpass signal whose frequency spectrum is shown in fig Q4.15. Determine the minimum sampling rate F_s to avoid aliasing.

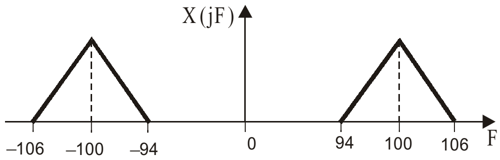


Fig Q4.15.

Solution

The given signal is a bandpass signal. The bandwidth, B = 106 – 94 = 12 Hz.

Here the upper cutoff frequency (106 Hz) is not an integer multiple of bandwidth, B. Hence the minimum sampling rate should be 4B, in order to avoid aliasing.

$\therefore$ Minimum sampling rate = $4 \times B = 4 \times 12 = 48\text{Hz}$

4.11 MATLAB Programs

Program 4.1

Write a MATLAB program to find Fourier coefficients of the discrete time signal x(n)={1,2,-1}, and sketch the magnitude and phase spectrum.

```
% Program to find Fourier coefficients of x(n)={1,2,-1}
% and to sketch the magnitude and phase spectrum

clear all
N=3; i=sqrt(-1);
x0=1; x1=2; x2=-1;
Ck=[];
for k=0:1:11
C=(1/N)*(x0+(x1*(exp(-i*2*pi*k/N)))+(x2*(exp(-i*4*pi*k/N))));
Ck=[Ck,C];
end

k = 0:1:11;
Ck                         %print the Fourier coefficients Ck
Mag_of_Ck = abs(Ck)        %evaluate and print the magnitude of Fourier
%coefficients
Pha_of_Ck = angle(Ck)      %evaluate and print the phase of Fourier
%coefficients

subplot(2,1,1), stem(k,Mag_of_Ck);
xlabel('k'), ylabel('Magnitude of Ck');
subplot(2,1,2), stem(k,Pha_of_Ck);
xlabel('k'), ylabel('Phase of Ck in rad.');
```

OUTPUT

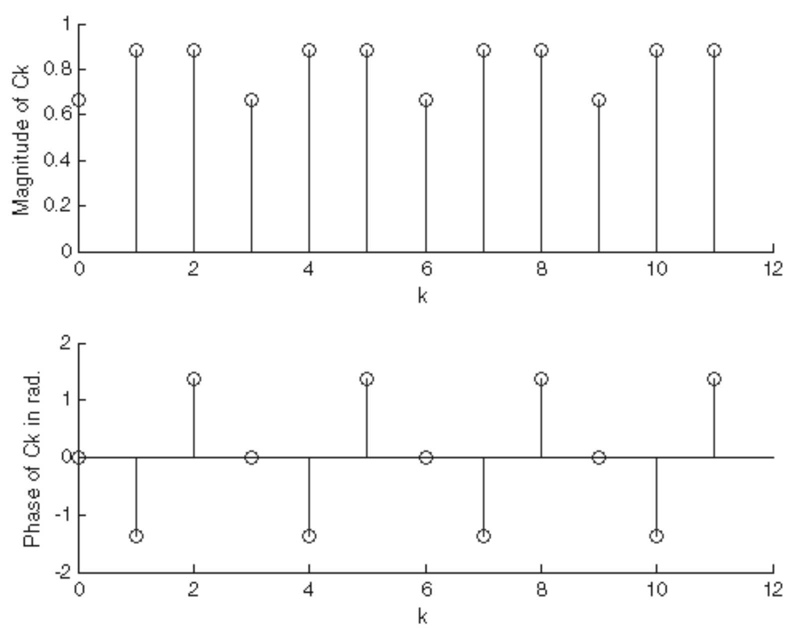


***Fig P4.1** : Magnitude and phase spectrum of program 4.1.*

```
Ck =
   Columns 1 through 7
    0.6667                0.1667 - 0.8660i    0.1667 + 0.8660i    0.6667
    0.1667 - 0.8660i      0.1667 + 0.8660i    0.6667

   Columns 8 through 12
    0.1667 - 0.8660i    0.1667 + 0.8660i    0.6667                0.1667 - 0.8660i
    0.1667 + 0.8660i

Mag_of_Ck =
     0.6667     0.8819     0.8819     0.6667     0.8819     0.8819     0.6667     0.8819
     0.8819     0.6667     0.8819     0.8819


Pha_of_Ck =
          0    -1.3807     1.3807          0    -1.3807     1.3807          0    -1.3807
     1.3807          0    -1.3807     1.3807
```

The magnitude and phase spectrum of program 4.1 are shown in fig P4.1.

Program 4.2

Write a MATLAB program to sketch the magnitude and phase spectrum of discrete time systems represented by the following transfer functions.

a) $H(e^{jw})=(1-e^{-j3w})/3(1-e^{-jw})$ b) $H(e^{jw})=2e^{-jw/2}\cos(w/2)$

c) $H(e^{jw})=2e^{-jw/2}\sin(w/2)$

```
% Program to sketch the magnitude and phase spectrum
% of the given discrete time systems

clear all

MagH1=[]; MagH2=[]; MagH3=[]; PhaH1=[]; PhaH2=[]; PhaH3=[]; w1=[];

for w=-2*pi:0.01:2*pi
H1=(1/3)*(1-exp(-3*i*w))/(1-exp(-i*w));
H2=2*(exp(-i*w/2))*(cos(w/2));
H3=2*(exp(-i*w/2))*(sin(w/2));

H1_M=abs(H1);     H2_M=abs(H2);    H3_M=abs(H3);
H1_P=angle(H1); H2_P=angle(H2); H3_P=angle(H3);

MagH1=[MagH1,H1_M];    %store the magnitude as an array
MagH2=[MagH2,H2_M];
MagH3=[MagH3,H3_M];

PhaH1=[PhaH1,H1_P];    %store the phase as an array
PhaH2=[PhaH2,H2_P];
PhaH3=[PhaH3,H3_P];

w1=[w1,w];                %store the frequency as an array
end

subplot(3,2,1),plot(w1,MagH1);
xlabel('w in rad.'),ylabel('Mag. of H1');
subplot(3,2,2),plot(w1,PhaH1);
xlabel('w in rad.'),ylabel('Pha. of H1');

subplot(3,2,3),plot(w1,MagH2);
xlabel('w in rad.'),ylabel('Mag. of H2');
subplot(3,2,4),plot(w1,PhaH2);
```

```
xlabel('w in rad.'),ylabel('Pha. of H2');

subplot(3,2,5),plot(w1,MagH3);
xlabel('w in rad.'),ylabel('Mag. of H3');
subplot(3,2,6),plot(w1,PhaH3);
xlabel('w in rad.'),ylabel('Pha. of H3');
```

OUTPUT

The magnitude and phase spectrum of program 4.2 are shown in fig P4.2.

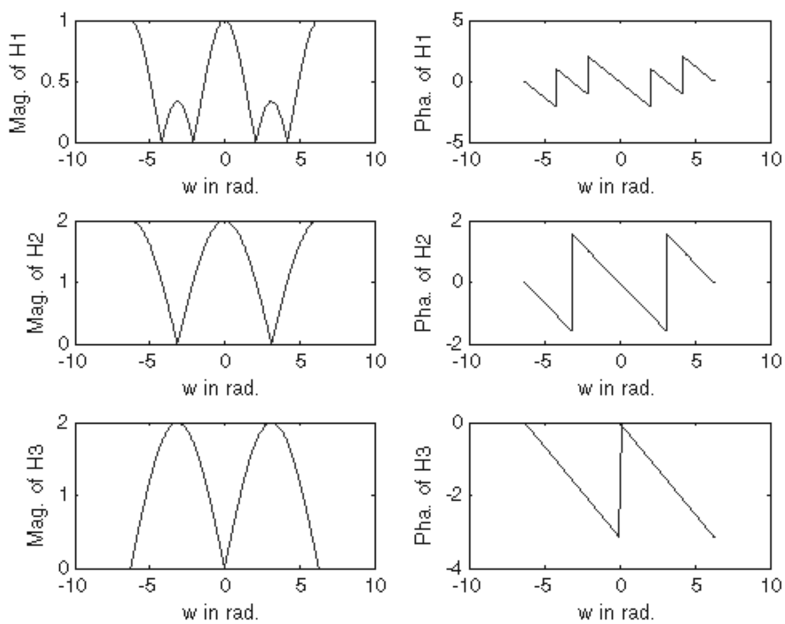


***Fig P4.2** : Magnitude and phase spectrum of program 4.2.*

Program 4.3

Write a MATLAB program to sketch the frequency response of the first-order discrete time system governed by the transfer function,

$H(e^{jw})=1/(1-ae^{-jw})$ for a=0.5 and a=-0.5.

```
% Program to sketch frequency response of first-order discrete        %
  time system

clear all

j=sqrt(-1);w=[];Mag_H1=[];Pha_H1=[];Mag_H2=[];Pha_H2=[];

for w1=-pi:0.01:pi
     H1 = 1/(1-0.5*exp(-j*w1));
     H2 = 1/(1+0.5*exp(-j*w1));
     H1_M = abs(H1);
     H2_M = abs(H2);
```

```
        H1_P = angle(H1);
        H2_P = angle(H2);
        Mag_H1=[Mag_H1, H1_M];
        Mag_H2=[Mag_H2, H2_M];
        Pha_H1=[Pha_H1,H1_P];
        Pha_H2=[Pha_H2,H2_P];
        w=[w,w1];
end

subplot(2,2,1),plot(w,Mag_H1);
xlabel('w in rad.'),ylabel('Magnitude of H1(jw)');
subplot(2,2,2),plot(w,Mag_H2);
xlabel('w in rad.'),ylabel('Magnitude of H2(jw)');
subplot(2,2,3),plot(w,Pha_H1);
xlabel('w in rad.'),ylabel('Phase of H1(jw) in rad.');
subplot(2,2,4),plot(w,Pha_H2);
xlabel('w in rad.'),ylabel('Phase of H2(jw) in rad.');
```

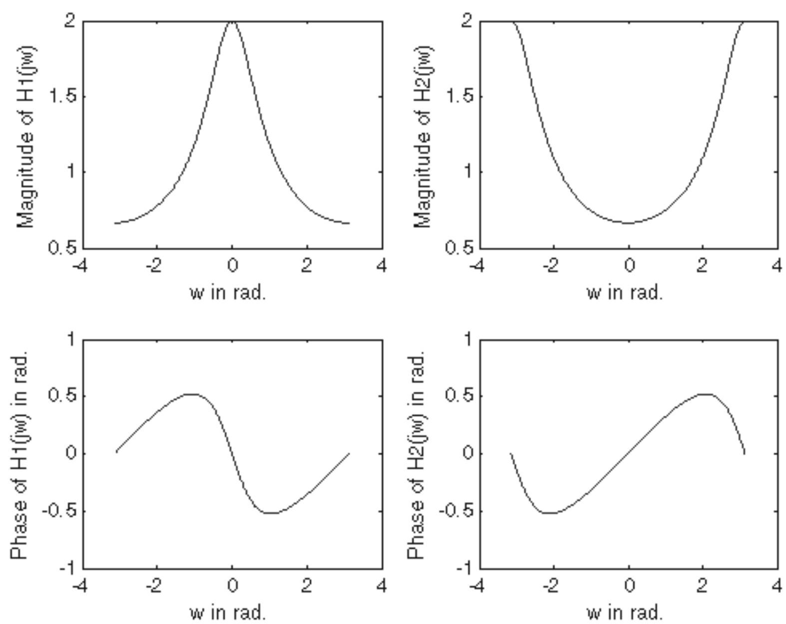


***Fig P4.3** : Magnitude and phase spectrum of first-order discrete time system.*

OUTPUT

The frequency response consists of two parts : Magnitude spectrum and Phase spectrum. The magnitude and phase spectrum of first-order discrete time system for a=0.5 and for a=-0.5 are shown in fig P4.3.

Program 4.4

Write a MATLAB program to sketch the frequency response of the second-order discrete time system governed by the transfer function,

$H(e^{jw})=(1+ae^{-jw})/(1+ae^{-jw}+be^{-j2w})$

where, $a=-r\cos w_0$; $a=2a$; $b=r^2$; $r=0.9$; $w_0=p/2$.

```
% Program to sketch frequency response of second-order
% discrete time system

clear all

j=sqrt(-1);w=[];Mag_H=[];Pha_H=[];
r=0.9; wo=pi/2;
a=(-1*r*cos(wo));
alpha=2*a;
Beta=r^2;

for w1=-pi:0.01:pi
     Num_of_H=(1+a*exp(-j*w1));
     Den_of_H=(1+((alpha)*exp(-j*w1))+((Beta)*exp(-j*2*w1)));
     H=Num_of_H / Den_of_H;
     H_M=abs(H);
     H_P=angle(H);
     Mag_H=[Mag_H,H_M];
     Pha_H=[Pha_H,H_P];
     w=[w,w1];
end
subplot(2,1,1),plot(w,Mag_H);
xlabel('w in radians'),ylabel('Magnitude of H(jw)');
subplot(2,1,2),plot(w,Pha_H);
xlabel('w in radians'),ylabel('Phase of H(jw)');
```

OUTPUT

The frequency response consists of two parts : Magnitude spectrum and Phase spectrum. The magnitude and phase spectrum of the given second-order discrete time system are shown in fig P4.4.

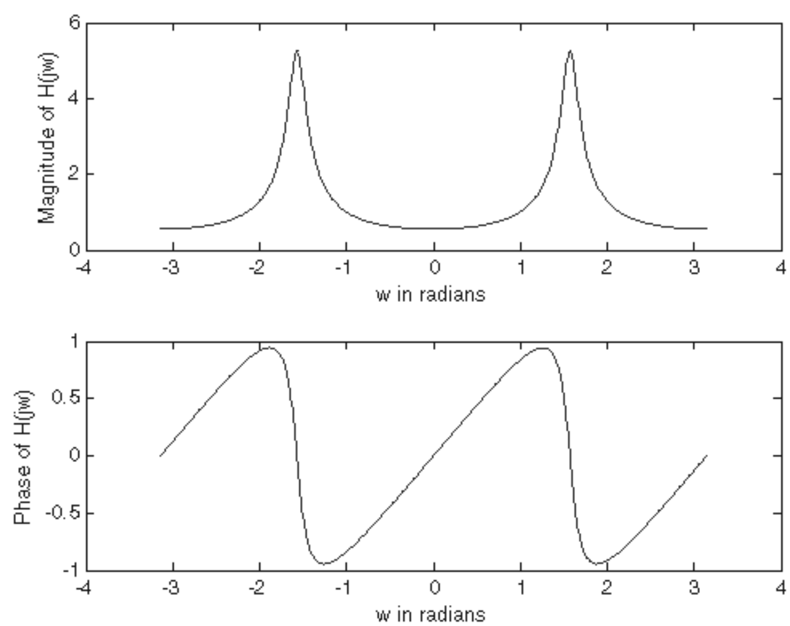


Fig P4.4 *: Magnitude and phase spectrum of second-order discrete time system.*

4.12 Exercises

I. Fill in the blanks with appropriate words

1. The Fourier transform of continuous time signal involves integration, whereas the Fourier transform of discrete time signal involves _______.
2. In Fourier transform of a real signal, the magnitude function is symmetric and phase function is _______.
3. The _______ operation of x(n) with h(n) is equal to the product $X(e^{jw})\ H(e^{jw})$.
4. The Fourier transform of product of two time domain signals is equivalent to _______ of their Fourier transforms.
5. The Fourier transform of the impulse response of an LTI system is called _______.
6. The Fourier transform of the discrete signal can be obtained by evaluating the Z-transform along _______.
7. A second-order LTI system will behave as a _______ filter.
8. A first-order LTI system will behave as a _______ filter.
9. A bandlimited signal with maximum frequency F_m can be fully recovered from its samples if sampled at a frequency greater than or equal to _______.
10. The sampling rate for a bandpass signal with bandwidth "B" is _______.

Answers

1. summation	4. convolution	7. bandpass	9. 2 F_m
2. antisymmetric	5. frequency response	8. lowpass or highpass	10. 2B to 4B
3. convolution	6. unit circle		

II. State whether the following statements are True/False

1. The discrete time Fourier series exists only for periodic discrete time signal.
2. The convergence of the discrete time Fourier series is exact at every point.
3. The Fourier coefficients of a discrete time signal is nonperiodic.
4. The Fourier transform exists only for signals that are absolutely summable.
5. The Fourier transform of discrete signal is a discrete function of w.
6. Fourier transform of an even signal is purely real and Fourier transform of an odd signal is purely imaginary.
7. The frequency response is periodic with a periodicity of 2p.
8. When the impulse response is complex, the real part of frequency response is symmetric and imaginary part is antisymmetric.
9. Convolving two signals in time domain is equivalent to multiplying their spectra in frequency domain.
10. Multiplication of a sequence x(n) by $e^{j\omega_0 n}$ is same as frequency translation of the spectrum $X(e^{jw})$ by w_0.
11. Impulse response h(n) is discrete, whereas frequency response $H(e^{jw})$ is continuous function of w.
12. The second-order system can be designed to behave as either low pass or high pass filter.
13. The spectrum of sampled version of a discrete time signal is sum of frequency shifted and amplitude scaled version of original spectrum of continuous time signal, X(jW).
14. If a discrete time signal is shifted in time by 'n_0' samples, then its magnitude spectrum shifts by wn_0.
15. The Fourier transform can be obtained from Z-transform only if ROC of X(z) includes unit circle.

Answers

1. True	4. True	7. True	10. True	13. True
2. True	5. False	8. True	11. True	14. False
3. False	6. True	9. True	12. False	15. True

III. Choose the right answer for the following questions

1. *The Fourier coefficients of x(n) is,* $c_k = \{3, \ \ 2+j, \ \ 1, \ \ 2-j\}$. *The value of x(7) is,*

   a) 1 b) 0 c) 2 - j d) 2 + j

2. *For a periodic discrete time signal x(n), the Fourier coefficient* $c_1 = -1 + j4.5$. *The value of* c_{1+N} *will be,*

   a) $-1-j\,4.5$ b) -1 c) $j4.5$ d) $-1+j\,4.5$

3. *The Fourier coefficients of x(n) is* c_k, *then Fourier coefficients of* $x^*(n)$ *is,*

   a) c^*_k b) c^*_{-k} c) c_{-k} d) c_k

4. *The average power of x(n) in terms of Fourier series coefficient* c_k *is,*

   a) $\sum\limits_{k=0}^{\infty} |c_k|^2$ b) $\frac{1}{N}\sum\limits_{k=0}^{\infty} |c_k|^2$ c) $\frac{1}{N}\sum\limits_{k=0}^{N-1} |c_k|^2$ d) $\sum\limits_{k=0}^{N-1} |c_k|^2$

5. *The Fourier transform of x(n) = 1, for all 'n' is,*

   a) $2\pi \sum\limits_{m=-\infty}^{+\infty}\delta(\omega-2\pi m)$ b) $\pi \sum\limits_{m=-\infty}^{+\infty}\delta(\omega-2\pi m)$ c) $2\pi \sum\limits_{m=-\infty}^{+\infty}\delta(\omega-m)$ d) $2\pi \sum\limits_{m=-\infty}^{+\infty}\delta(\omega-\pi m)$

6. *If* $\mathcal{F}\{x(n)\} = X(e^{jw})$, *then* $\mathcal{F}\{x(n-3)\}$ *will be,*

   a) $e^{-j3w}X(e^{-jw})$ b) $e^{j3w}X(e^{-jw})$ c) $e^{-j3w}X(e^{jw})$ d) $e^{j3w}X(e^{jw})$

7. *If a signal is folded about the origin in time then its,*

   a) magnitude spectrum undergoes change in sign b) phase spectrum undergoes change in sign

   c) magnitude remains unchanged d) both c and b

8. *The Fourier transform of correlation sequence of two discrete time signals* $x_1(n)$ *and* $x_2(n)$ *is given by,*

   a) $X_1(e^{jw})\,X_2(e^{jw})$ b) $X_1(e^{jw})\,X_2(e^{-jw})$ c) $X_1(e^{-jw})\,X_2(e^{-jw})$ d) none of the above

9. *If h(n) is real, then magnitude of* $H(e^{jw})$ *is ______ and phase of* $H(e^{jw})$ *is _____.*

   a) symmetric, antisymmetric b) antisymmetric, symmetric

   c) symmetric, symmetric d) antisymmetric, antisymmetric

10. *The second order LTI discrete time system behaves as,*

    a) low pass filter b) high pass filter c) resonant filter d) all pass filter

11. *The ideal interpolation formula is used to,*

    a) obtain frequency spectrum of discrete time signal b) sample continuous time signal

    c) reconstruct original continuous time signal d) remove aliasing

12. *If* $X(j\Omega)$ *is frequency spectrum of a continuous time signal then, the frequency spectrum of sampled version of the signal* $X(e^{jw})$ *is, (where* $\omega = \Omega T$*),*

    a) $\frac{1}{T}\sum\limits_{m=-\infty}^{+\infty} X\left(j\left(\frac{\omega}{T}+\frac{2\pi m}{T}\right)\right)$ b) $\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} X(j\tfrac{\omega}{T})\,e^{j\omega nT}\,d\omega$ c) $\frac{1}{T}\sum\limits_{m=-\infty}^{+\infty} X\left(j\left(\omega T+\frac{2\pi m}{T}\right)\right)$ d) $\frac{1}{T}\sum\limits_{m=-\infty}^{+\infty} X\left(j\left(\frac{m\omega}{T}\right)\right)$

13. A bandlimited continuous time signal with maximum frequency F_m, sampled at a frequency F_s, can be fully recovered from its samples, provided that,

a) F_s ³ $2F_m$ **b)** $F_s = 2F_m$ **c)** F_m ³ $2F_s$ **d)** $F_s = F_m$

14. If Z-transform of x(n) includes unit circle in its ROC, then the Fourier transform of x(n) can be expressed as,

a) $\left.\sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\right|_{z=e^{-j\omega}}$ **b)** $\left.\sum_{n=0}^{\infty} x(n)\, z^{-jn}\right|_{z=e^{-\omega}}$ **c)** $\left.\sum_{n=-\infty}^{\infty} x(n)\, z^{n}\right|_{z=\omega}$ **d)** $\left.\sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\right|_{z=e^{j\omega}}$

15. Let x(n) is real and $x(n) = x_e(n) + x_o(n)$. If $A(e^{jw})$ is Fourier transform of $x_e(n)$ and if $B(e^{jw})$ is Fourier transform of $x_o(n)$, then Fourier transform of x(n) is,

a) $A(e^{jw}) + B(e^{jw})$ **b)** $A(e^{-jw}) + j\,B(e^{-jw})$ **c)** $A(e^{jw}) - jB(e^{jw})$ **d)** $A(e^{-jw}) - j\,B(e^{-jw})$

16. If a continuous time signal x(t) has a nyquist rate of W_0, then nyquist rate for the continuous time signal $x^2(t)$ is,

a) $\frac{\Omega_0}{2}$ **b)** $2W_0$ **c)** $\frac{\Omega_0}{4}$ **d)** W_0

17. If the bandwidth of a bandpass signal x(t) is 2F, then the minimum sampling rate for bandpass signal must be,

a) 2F samples/sec **b)** 4F samples/sec **c)** $\frac{F}{2}$ samples/sec **d)** $\frac{F}{4}$ samples/sec

18. If $X(e^{jw}) = e^{-jw}$ for –p £ w £ p, then the discrete time signal x(n) is,

a) $\frac{\sin 2\pi(n-1)}{2\pi(n-1)}$ **b)** $\sin\pi(n-1)$ **c)** $\frac{\sin\pi(n-1)}{\pi(n-1)}$ **d)** $\frac{\sin\pi(2n-1)}{\pi(2n-1)}$

19. The discrete time Fourier transform of the signal, $x(n) = 0.5^{(n-1)}\, u(n-1)$ is,

a) $\frac{e^{-j\omega}}{1-0.5e^{-j\omega}}$ **b)** $e^{-j\omega}\left(1-0.5e^{-j\omega}\right)$ **c)** $\frac{0.5e^{-j\omega}}{1-0.5e^{-j\omega}}$ **d)** $\frac{0.5e^{j\omega}}{1-0.5e^{-j\omega}}$

20. The Fourier transform of, $x(n) = (0.8)^n$; $n = 0, \pm 1, \pm 2, \ldots\ldots$ is,

a) does not exist **b)** $\frac{1}{1-0.8e^{-j\omega}}$ **c)** $\frac{0.8}{1-0.8e^{-j\omega}}$ **d)** $\frac{0.8e^{-j\omega}}{1-0.8e^{-j\omega}}$

Answers

1. c	5. a	9. a	13. a	17. b
2. d	6. c	10. c	14. d	18. c
3. b	7. d	11. c	15. a	19. a
4. d	8. b	12. a	16. b	20. a

IV. Answer the following questions

1. Define Fourier series of a periodic discrete time signal.
2. Define Fourier coefficients of a periodic discrete time signal.
3. Write any two properties of Fourier series coefficients of discrete time signal.
4. Define the frequency spectrum of a periodic discrete time signal in terms of Fourier series coefficients.
5. Write the differences between Fourier series of a discrete time signal and continuous time signal.
6. Define Fourier transform of a discrete time signal.
7. State and prove any two properties of Fourier transform.
8. State and prove the time delay property of Fourier transform.
9. Give the significance of Parseval's relation.
10. Define inverse Fourier transform.
11. Write the differences between Fourier transform of discrete time signal and continuous time signal.
12. Define the frequency spectrum of a discrete time signal in terms of Fourier transform.
13. Write a short note on Fourier transform of periodic discrete time signal.
14. Write the properties of frequency response of an LTI system.
15. What is frequency response of an LTI system?
16. What is the relation between Fourier transform and $\mathcal{Z}$-transform?
17. What is aliasing of frequency spectrum?
18. Explain how a bandlimited signal can be sampled without aliasing?
19. What is ideal interpolation formula? What is its significance?
20. Write a short note on sampling of bandpass signals.

V. Solve the following problems

E4.1 Determine the Fourier series representation of the following discrete time signals.

a) $x(n) = 4\cos\sqrt{8}\,\pi n$

b) $x(n) = \{.....4,\ 3,\ 2,\ 1,\ \underset{\uparrow}{4},\ 3,\ 2,\ 1,\ 4,\ 3,\ 2,\ 1,.....\}$

c) $x(n) = 9\,e^{\frac{j5\pi n}{2}}$

d) $x(n) = 4\sin\frac{2\pi n}{3}$

e) $x(n) = \cos\frac{\pi n}{3} + \sin\frac{\pi n}{5}$

E4.2 Determine the Fourier transform of the following signals.

a) $x(n) = 3\cos\frac{2\pi}{5}n$

b) $x(n) = \{\underset{\uparrow}{-3},\ 4,\ -1,\ 2\}$

c) $x(n) = (-1)^n \quad ; \quad 0 \le n \le 7$
$\quad\;\; = 0 \quad ; \quad \text{otherwise}$

d) $x(n) = 0.5\left[\left(\frac{1}{0.4}\right)^n - \left(\frac{1}{0.8}\right)^n\right]u(n)$

E4.3 Determine the convolution of the following sequences, using Fourier transform.

a) $x_1(n) = \{2,\ \underset{\uparrow}{-2},\ 2\},\ x_2(n) = \{-2,\ \underset{\uparrow}{2},\ -2\}$

b) $x_1(n) = \{\underset{\uparrow}{-2},\ -1,\ 0\},\ x_2(n) = \{\underset{\uparrow}{-3},\ 5,\ -7\}$

E4.4 ***Determine the inverse Fourier transform of the following functions of w.***

a) $X(e^{j\omega}) = 2j\omega$

b) $X(e^{j\omega}) = \dfrac{1}{\left(1-ae^{-j\omega}\right)^2}\ ;\quad |a|<1$

c) $Y(e^{j\omega}) = \dfrac{1+\frac{1}{7}e^{-j\omega}}{1-\frac{1}{7}e^{-j\omega}}$

d) $H(e^{j\omega}) = \dfrac{1}{\left(1-\frac{1}{6}e^{-j\omega}\right)\left(1-\frac{1}{5}e^{-j\omega}\right)}$

E4.5 ***a) A causal discrete time system is described by the equation,*** $y(n) - \frac{5}{14}y(n-1) - \frac{1}{14}y(n-2) = x(n)$ ***,***

where x(n) and y(n) are input and output of the system. Find the impulse response h(n), frequency response $H(e^{jw})$***, magnitude function and phase function of the system.***

b) Consider an LTI system described by, $y(n) - \frac{1}{5}y(n-1) = x(n) + \frac{1}{5}x(n-1)$

(i) Determine the frequency response $H(e^{jw})$ of the system.

(ii) Find the impulse response h(n) of the system.

(ii) Determine the response y(n) for the input, $x(n)=\cos\frac{\pi n}{2}$.

E4.6 ***A discrete LTI system is described by a difference equation, y(n) = x(n) – x(n – 1). Determine the frequency response*** $H(e^{jw})$***, impulse response h(n). Sketch the magnitude function and phase function.***

E4.7 ***Sketch the magnitude and phase function of the discrete time LTI system described by the equation y(n) = x(n) + x(n – 1).***

E4.8 ***The impulse response of a system is,*** $h(n) = \frac{1}{0.2}\delta(n+2) + \frac{1}{0.4}\delta(n+1) + \frac{1}{0.3}\delta(n) + \frac{1}{0.4}\delta(n-1)$

(i) Is the system BIBO stable? (ii) Is the system causal? (iii) Find the frequency response.

E4.9 ***The impulse response of an LTI system is*** $h(n) = \{-2,\ \ -1,\ \ 1,\ \ -2\}$ ***. Find the response of the system for the input*** $x(n) = \{2,\ \ 2,\ \ 4,\ \ 1\}$ ***, using convolution property of Fourier transform.***

E4.10 ***A causal system is represented by the following difference equation,***

$$y(n) - 0.2\,y(n-1) = x(n) - 0.6\,x(n-1).$$

Find the system transfer function H(z), impulse response and frequency response of the system. Also determine the magnitude and phase function.

Answers

E4.1 **a)** x(n) is nonperiodic.

b) $x(n) = \frac{5}{2} + \frac{1}{\sqrt{2}}\,e^{-j0.248\pi}\,e^{j\omega_0 n} + \frac{1}{2}e^{j2\omega_0 n} + \frac{1}{\sqrt{2}}\,e^{j0.248\pi}\,e^{j3\omega_0 n}\ ;\ \omega_0 = \frac{\pi}{2}$

c) $x(n) = 9\,e^{j\omega_0 n}\ ;\ \omega_0 = \frac{\pi}{2}$

d) $x(n) = 2e^{-j\frac{\pi}{2}}\,e^{j\omega_0 n} + 2e^{j\frac{\pi}{2}}\,e^{j\omega_0 n}\ ;\ \omega_0 = \frac{2\pi}{3}$

e) $x(n) = \frac{1}{2}e^{-j5\omega_0 n} + \frac{1}{2}e^{j\frac{\pi}{2}}\,e^{-j3\omega_0 n} + \frac{1}{2}e^{-j\frac{\pi}{2}}\,e^{j3\omega_0 n} + \frac{1}{2}e^{j5\omega_0 n}\ ;\ \boxed{\omega_0 = \frac{\pi}{15}}$

(or) $x(n) = \frac{1}{2}e^{-j\frac{\pi}{2}}\,e^{j3\omega_0 n} + \frac{1}{2}e^{j5\omega_0 n} + \frac{1}{2}e^{j25\omega_0 n} + \frac{1}{2}e^{j\frac{\pi}{2}}e^{j27\omega_0 n}\ ;\ \boxed{\omega_0 = \frac{\pi}{15}}$

E4.2

a) $X(e^{j\omega}) = 3\pi \sum_{m=-\infty}^{+\infty} \left[\delta(\omega - \frac{2\pi}{5} - 2\pi m) + \delta(\omega + \frac{2\pi}{5} - 2\pi m)\right]$ **b)** $X(e^{j\omega}) = -3 + 4e^{-j\omega} - e^{-j2\omega} + 2e^{-j3\omega}$

c) $X(e^{j\omega}) = \dfrac{\sin 4\omega}{\cos(\omega/2)} e^{j\left(\frac{\pi - 7\omega}{2}\right)}$ **d)** $X(e^{j\omega}) = \dfrac{0.625e^{-j\omega}}{1 - 3.75e^{-j\omega} + 3.125e^{-j2\omega}}$

E4.3

a) $x(n) = \{-4,\ 8,\ \underset{\uparrow}{-12},\ -8,\ -4\}$ **b)** $x(n) = \{\underset{\uparrow}{6},\ -7,\ 9,\ 7\}$

E4.4

a) $x(n) = \dfrac{2\cos \pi n}{n}$ **b)** $x(n) = (n+1)\, a^n u(n); \quad |a| < 1$

c) $y(n) = \left(\frac{1}{7}\right)^n \left[u(n) + u(n-1)\right]$ **d)** $h(n) = \left[6\left(\frac{1}{5}\right)^n - 5\left(\frac{1}{6}\right)^n\right] u(n)$

E4.5

a) $h(n) = \left[\frac{7}{9}\left(\frac{1}{2}\right)^n + \frac{2}{9}\left(\frac{-1}{7}\right)^n\right] u(n)$; $H(e^{j\omega}) = \dfrac{1}{\left(1 - \frac{1}{2} e^{-j\omega}\right)\left(1 + \frac{1}{7} e^{-j\omega}\right)}$

$$\left|H(e^{j\omega})\right| = \left[\frac{1}{1.13 - 0.664\cos\omega - 0.14\cos 2\omega}\right]^{\frac{1}{2}} ; \quad \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{-5\sin\omega - \sin 2\omega}{14 - 5\cos\omega - \cos 2\omega}\right]$$

b) (i) $H(e^{j\omega}) = \dfrac{1 + \frac{1}{5}e^{-j\omega}}{1 - \frac{1}{5}e^{-j\omega}}$; $h(n) = \left(\frac{1}{5}\right)^n u(n) + \frac{1}{5}\left(\frac{1}{5}\right)^{n-1} u(n-1)$

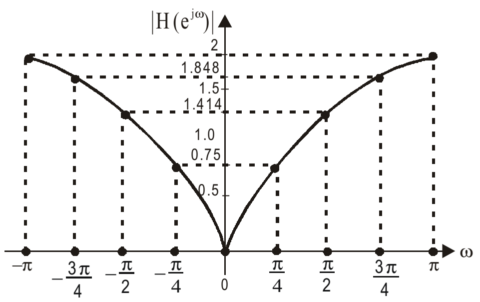
(ii) $y(n) = \cos\left(\frac{n\pi}{2} - \frac{\pi}{8}\right)$

E4.6

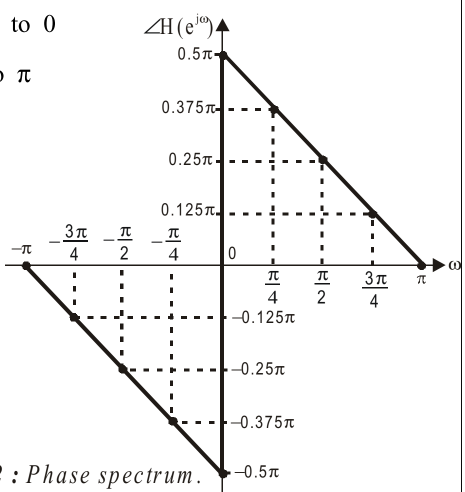
$H(e^{j\omega}) = 2\sin\frac{\omega}{2}\, e^{j\left(\frac{\pi - \omega}{2}\right)}$; $h(n) = \delta(n) - \delta(n-1)$

$\left|H(e^{j\omega})\right| = \left|2\sin\left(\frac{\omega}{2}\right)\right|$; $\angle H(e^{j\omega}) = -\left(\frac{\pi + \omega}{2}\right)$; for $\omega = -\pi$ to 0

$= \frac{\pi - \omega}{2}$; for $\omega = 0$ to π



FigE4.6.1 : Magnitude spectrum.



FigE4.6.2 : Phase spectrum.

E4.7

$$H(e^{j\omega}) = 2\cos\left(\frac{\omega}{2}\right) e^{-j\frac{\omega}{2}} \;; \qquad \left|H(e^{j\omega})\right| = 2\cos\left(\frac{\omega}{2}\right)$$

$$\angle H(e^{j\omega}) = -\frac{\omega}{2}$$

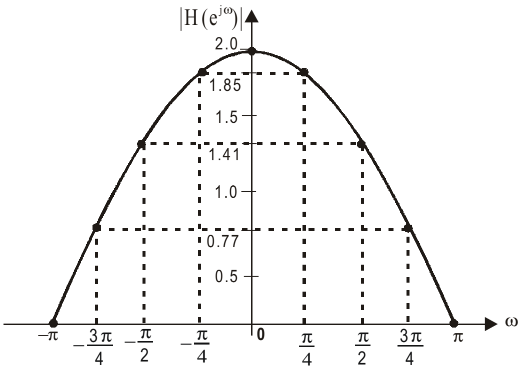


Fig E4.7.1 : Magnitude spectrum.

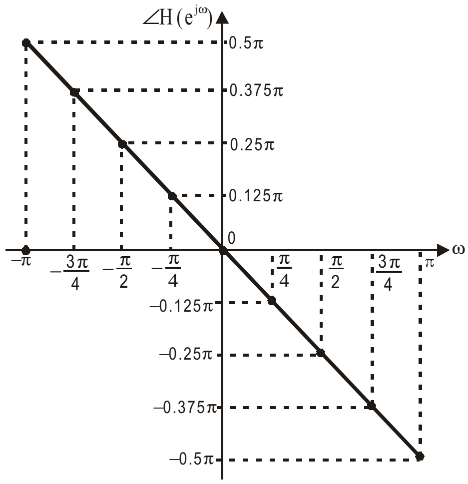


Fig E4.7.2 : Phase spectrum.

E4.8 $h(n) = \left\{\frac{1}{0.2},\ \frac{1}{0.4},\ \underset{\uparrow}{\frac{1}{0.3}},\ \frac{1}{0.4}\right\}$; **(i)** The system is stable ; **(ii)** The system is noncausal.

(iii) $H(e^{j\omega}) = \frac{1}{0.12}\left[(0.4 + 0.6\cos\omega + 0.6\cos 2\omega + j0.6\sin 2\omega)\right]$

E4.9 $y(n) = \{\underset{\uparrow}{-4},\ -6,\ -8,\ -8,\ -1,\ -7,\ -2\}$

E4.10

$$H(z) = \frac{1-0.6z^{-1}}{1-0.2z^{-1}} \;; \quad H(e^{j\omega}) = \frac{1-0.6e^{-j\omega}}{1-0.2e^{-j\omega}} \;; \quad h(n) = 0.2^n u(n) - 0.6\,(0.2)^{n-1}\,u(n-1)$$

$$\left|H(e^{j\omega})\right| = \sqrt{\frac{1.36-1.2\cos\omega}{1.04-0.4\cos\omega}} \;; \qquad \angle H(e^{j\omega}) = \tan^{-1}\left(\frac{0.4\sin\omega}{1.12-0.8\cos\omega}\right)$$

Solution for Exercise Problems

E4.1. Determine the Fourier series representation of the following discrete time signals.

a) $x(n) = 4\cos\sqrt{8}\pi n$

Solution

$$x(n+N) = 4\cos\sqrt{8}\pi(n+N) = 4\cos\left(\sqrt{8}\pi n + \sqrt{8}\pi N\right)$$

For periodicity, $\sqrt{8}\pi N$ should be integral multiple of 2π.

Let, $\sqrt{8}\pi N = 2\pi \times M \quad \Rightarrow \quad N = \dfrac{2\pi}{\sqrt{8}\pi} \times M$

Here, N cannot be an integer, for any integer value of M. Hence x(n) will not be periodic.

So Fourier series does not exist.

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b) $x(n) = \{.... 4, 3, 2, 1, \underset{\uparrow}{4}, 3, 2, 1, 4, 3, 2, 1 .....\}$

Solution

Given that, $\{.... 4, 3, 2, 1, \underset{\uparrow}{4}, 3, 2, 1, 4, 3, 2, 1....\}$

Here, x(n) is periodic with period, $N = 4$ samples.

$$\therefore\ N = 4, \qquad \therefore\ \omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Let, $x(n) = \{4, 3, 2, 1\}$

Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} = \frac{1}{4}\sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi kn}{4}}$$

$$= \frac{1}{4}\left[x(0) + x(1)\, e^{-j\frac{2\pi k}{4}} + x(2)\, e^{-j\frac{4\pi k}{4}} + x(3)\, e^{-j\frac{6\pi k}{4}}\right]$$

$$= \frac{1}{4}\left[4 + 3e^{-j\frac{\pi k}{2}} + 2e^{-j\pi k} + e^{-j\frac{3\pi k}{2}}\right]$$

When, $k = 0$

$$c_k = c_0 = \frac{1}{4}\left[4 + 3e^0 + 2e^0 + e^0\right] = \frac{1}{4}\left[4+3+2+1\right] = \frac{10}{4} = \frac{5}{2}$$

When, $k = 1$

$$c_k = c_1 = \frac{1}{4}\left[4 + 3\, e^{-j\frac{\pi}{2}} + 2\, e^{-j\pi} + e^{-j\frac{3\pi}{2}}\right]$$

$$= \frac{1}{4}\left[4 + 3\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) + 2\left(\cos\pi - j\sin\pi\right) + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$$

$$= \frac{1}{4}\left[4 + 3(0-j) + 2(-1-j0) + 0 - (-j)\right] = \frac{1}{4}\left[4 - 3j - 2 + j\right]$$

$$= \frac{1}{4}\left[2 - 2j\right] = \frac{1}{2} - j\frac{1}{2} = 0.5 - j0.5 = 0.707\ \angle -0.78$$

$$= 0.707\ \angle -0.248\pi = \frac{1}{\sqrt{2}}\ e^{-j0.248\pi}$$

$\dfrac{0.78\pi}{\pi} \times \pi = 0.248\pi$
$e^{-j\pi k} = -1$; for k odd $= +1$; for k even

When, $k = 2$

$$c_k = c_2 = \frac{1}{4}\left[4 + 3\, e^{-j\pi} + 2\, e^{-j2\pi} + e^{-j3\pi}\right]$$

$$= \frac{1}{4}\left[4 + 3\,(-1) + 2\,(1) + 1\,(-1)\right] = \frac{1}{4}\left[4 - 3 + 2 - 1\right] = \frac{2}{4} = \frac{1}{2}$$

When, k = 3

$$c_k = c_3 = \frac{1}{4}\left[4 + 3\,e^{-j\frac{3\pi}{2}} + 2\,e^{-j3\pi} + e^{-j\frac{9\pi}{2}}\right] = \frac{1}{4}\left[4 + 3\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 2(\cos 3\pi - \sin 3\pi) + \cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right]$$

$$= \frac{1}{4}\left[4 + 3(0 + j) + 2(-1 - j0) + 0 - j\right]$$

$$= \frac{1}{4}\left[4 + 3j - 2 - j\right] = \frac{2 + 2j}{4} = 0.5 + j0.5 = 0.707\angle 0.78 = 0.707\angle 0.248\pi$$

$$\boxed{\frac{0.78}{\pi} \times \pi = 0.248\pi}$$

$$= \frac{1}{\sqrt{2}}\,e^{j0.248\pi}$$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\,e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{3} c_k\,e^{\frac{j2\pi kn}{4}} = \sum_{k=0}^{3} c_k\,e^{j\frac{\pi kn}{2}} = \sum_{k=0}^{3} c_k\,e^{j\omega_0 kn} = c_0 + c_1\,e^{j\omega_0 n} + c_2\,e^{j2\omega_0 n} + c_3\,e^{j3\omega_0 n}$$

$$= \frac{5}{2} + \frac{1}{\sqrt{2}}\,e^{-j0.248\pi}\,e^{j\omega_0 n} + \frac{1}{2}\,e^{j2\omega_0 n} + \frac{1}{\sqrt{2}}\,e^{j0.248\pi}e^{j3\omega_0 n} \qquad \boxed{\omega_0 = \frac{\pi}{2}}$$

---

c) $x(n) = 9\,e^{j\frac{5\pi n}{2}}$

Solution

Test for periodicity

$$x(n+N) = 9e^{j\frac{5\pi(n+N)}{2}} = 9e^{\left(j\frac{5\pi n}{2} + j\frac{5\pi N}{2}\right)}$$

Let, $\dfrac{5\pi N}{2} = 2\pi \times M$

$$\therefore\ N = \frac{2\pi \times 2}{5\pi} \times M = \frac{4}{5} \times M$$

Here 'N' is an integer for M = 5, 10, 15,....

Let, M = 5

\ N = 4

Here x(n) is periodic with fundamental period, N = 4.

$$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{2}$$

Fourier series

The Fourier coefficients 'c_k' are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\,e^{-j\frac{2\pi kn}{N}} \quad ; \quad \text{for, } k = 0, 1, 2, 3, \ldots\ldots (N-1)$$

Here, N = 4, $x(n) = 9\,e^{j\frac{5\pi n}{2}}$

$$\therefore\ c_k = \frac{1}{4}\sum_{n=0}^{3} 9\,e^{j\frac{5\pi n}{2}}\,e^{-j\frac{2\pi kn}{4}} \quad ; \quad \text{for, } k = 0, 1, 2, 3$$

$$= \frac{9}{4}\sum_{n=0}^{3} e^{j\pi n\left(\frac{5-k}{2}\right)} = \frac{9}{4}\left[e^0 + e^{j\pi\left(\frac{5-k}{2}\right)} + e^{j2\pi\left(\frac{5-k}{2}\right)} + e^{j3\pi\left(\frac{5-k}{2}\right)}\right]$$

$$= \frac{9}{4}\left[1 + e^{j\pi\left(\frac{5-k}{2}\right)} + e^{j\pi(5-k)} + e^{j3\pi\left(\frac{5-k}{2}\right)}\right]$$

$$= \frac{9}{4}\left[1 + \cos\frac{\pi(5-k)}{2} + j\sin\frac{\pi(5-k)}{2} + \cos\pi(5-k) + j\sin\pi(5-k) + \cos\frac{3\pi(5-k)}{2} + j\sin\frac{3\pi(5-k)}{2}\right]$$

When k = 0

$$c_k = c_0 = \frac{9}{4}\left[1 + \cos\frac{5\pi}{2} + j\sin\frac{5\pi}{2} + \cos 5\pi + j\sin 5\pi + \cos\frac{15\pi}{2} + j\sin\frac{15\pi}{2}\right]$$

$$= \frac{9}{4}\left[1 + 0 + j - 1 + j0 + 0 - j\right] = \frac{9}{4} \times 0 = 0$$

When k = 1

$$c_k = c_1 = \frac{9}{4}\left[1 + \cos 2\pi + j\sin 2\pi + \cos 4\pi + j\sin 4\pi + \cos 6\pi + j\sin 6\pi\right]$$

$$= \frac{9}{4}\left[1 + 1 + j0 + 1 + j0 + 1 + j0\right] = \frac{9}{4} \times 4 = 9$$

When $k = 2$

$$c_k = c_2 = \frac{9}{4}\left[1+\cos\frac{3\pi}{2}+j\sin\frac{3\pi}{2}+\cos 3\pi+j\sin 3\pi+\cos\frac{9\pi}{2}+j\sin\frac{9\pi}{2}\right]$$

$$= \frac{9}{4}\left[1+0-j-1+j0+0+j\right] = \frac{9}{4}\times 0 = 0$$

When $k = 3$

$$c_k = c_3 = \frac{9}{4}\left[1+\cos\pi+j\sin\pi+\cos 2\pi+j\sin 2\pi+\cos 3\pi+j\sin 3\pi\right]$$

$$= \frac{9}{4}\left[1-1+j0+1+j0-1+j0\right] = \frac{9}{4}\times 0 = 0$$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{3} c_k\, e^{\frac{j2\pi kn}{4}} = \sum_{k=0}^{3} c_k\, e^{\frac{j\pi kn}{2}} = \sum_{k=0}^{3} c_k\, e^{j\omega_0 kn}$$

$$= c_0 + c_1\, e^{j\omega_0 n} + c_2\, e^{j2\omega_0 n} + c_3\, e^{j3\omega_0 n} = 0 + 9e^{j\omega_0 n} + 0 + 0 \qquad \boxed{\omega_0 = \frac{\pi}{2}}$$

$$x(n) = 9\, e^{j\omega_0 n}$$

---

d) $x(n) = 4\sin\frac{2\pi n}{3}$

Solution

Test for periodicity

$$x(n+N) = 4\sin\frac{2\pi}{3}(n+N) = 4\sin\left(\frac{2\pi n}{3}+\frac{2\pi N}{3}\right)$$

Let, $\frac{2\pi N}{3} = 2\pi\times M \quad\Rightarrow\quad N = \frac{6\pi}{2\pi}\times M = 3M$

Here, N is an integer for $M = 1, 2, 3, ...$

Let, $M = 1, \quad \therefore\ N = 3$

$\therefore$ x(n) is periodic with fundamental period $N = 3$.

$\therefore\ N = 3$

Fourier series

The Fourier coefficients 'c_k' are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} \quad ; \quad \text{for, } k = 0, 1, 2, 3, ... (N-1)$$

Here, $N = 3,\ x(n) = 4\sin\frac{2\pi n}{3}$

$$\therefore\ c_k = \frac{1}{3}\sum_{n=0}^{2} 4\sin\frac{2\pi n}{3}\, e^{-j\frac{2\pi kn}{3}} \quad ; \quad \text{for } k = 0, 1, 2$$

$$= \frac{4}{3}\left[\sum_{n=0}^{2}\sin\frac{2\pi n}{3}\left(\cos\frac{2\pi kn}{3} - j\sin\frac{2\pi kn}{3}\right)\right]$$

$$= \frac{4}{3}\left[\sin 0\,(\cos 0 - j\sin 0) + \sin\frac{2\pi}{3}\left(\cos\frac{2\pi k}{3} - j\sin\frac{2\pi k}{3}\right) + \sin\frac{4\pi}{3}\left(\cos\frac{4\pi k}{3} - j\sin\frac{4\pi k}{3}\right)\right]$$

$$\therefore\ c_k = \frac{4}{3}\left[0.866\left(\cos\frac{2\pi k}{3} - j\sin\frac{2\pi k}{3}\right) - 0.866\left(\cos\frac{4\pi k}{3} - j\sin\frac{4\pi k}{3}\right)\right]$$

When, $k = 0$,

$$c_k = c_0 = \frac{4}{3}\left[0.866\,(\cos 0 - j\sin 0) - 0.866\,(\cos 0 - j\sin 0)\right]$$

$$= \frac{4}{3}\left[0.866\,(1-j0) - 0.866\,(1-j0)\right] = 0$$

When, $k = 1$,

$$c_k = c_1 = \frac{4}{3}\left[0.866\left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right) - 0.866\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right)\right]$$

$$= \frac{4}{3}\left[0.866(-0.5 - j0.866) - 0.866\,(-0.5 + j0.866)\right]$$

$$\therefore\ c_1 = \frac{4}{3}\left[0.866\ (-0.5 - j0.866 + 0.5 - j0.866)\right] = \frac{4}{3}\left[0.866 \times (-j1.732)\right] = -2j = 2\ \angle -\frac{\pi}{2} = 2e^{-j\frac{\pi}{2}}$$

When, k = 2,

$$c_k = c_2 = \frac{4}{3}\left[0.866\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}\right) - 0.866\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3}\right)\right] = \frac{4}{3}\left[0.866(-0.5 + j0.866) - 0.866\ (-0.5 - j0.866)\right]$$

$$= \frac{4}{3}\left[0.866\ (-0.5 + j0.866 + 0.5 + j0.866)\right] = \frac{4}{3}\left[0.866 \times (j1.732)\right] = j2 = 2\ \angle\frac{\pi}{2} = 2e^{j\frac{\pi}{2}}$$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\ e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{2} c_k\ e^{\frac{j2\pi kn}{3}} = \sum_{k=0}^{2} c_k\ e^{j\omega_0 kn}$$

$$= c_0 + c_1\, e^{j\omega_0 n} + c_2\, e^{j2\omega_0 n} = 2\ e^{-j\frac{\pi}{2}}\ e^{j\omega_0 n} + 2\ e^{j\frac{\pi}{2}}\ e^{j\omega_0 n} \qquad \boxed{\omega_0 = \frac{2\pi}{3}}$$

---

e) $x(n) = \cos\dfrac{\pi n}{3} + \sin\dfrac{\pi n}{5}$

Solution

Test for periodicity

Compare sine and cosine terms with standard form.

Let, $\cos\ 2\pi f_1 n = \cos\dfrac{\pi n}{3}$

$$2f_1 = \frac{1}{3} \quad\Rightarrow\quad f_1 = \frac{1}{6};\quad N_1 = \frac{1}{f_1} = 6$$

Let, $\sin\ 2\pi f_2 n = \sin\dfrac{\pi n}{5}$

$$2f_2 = \frac{1}{5} \quad\Rightarrow\quad f_2 = \frac{1}{10}\ ;\quad N_2 = \frac{1}{f_2} = 10$$

Here, fundamental period is LCM (Least Common Multiple) of N_1 and N_2. The LCM of 6 and 10 is 30.

\ Fundamental period, N = 30.

$$\therefore\ \omega_0 = \frac{2\pi}{N} = \frac{2\pi}{30} = \frac{\pi}{15}$$

Fourier series

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}} \qquad \boxed{\sin\theta = \frac{e^{j\theta} + e^{-j\theta}}{2j}}$$

$$x(n) = \cos\frac{\pi n}{3} + \sin\frac{\pi n}{5} = \frac{e^{j\frac{\pi n}{3}} + e^{-j\frac{\pi n}{3}}}{2} + \frac{e^{j\frac{\pi n}{5}} - e^{-j\frac{\pi n}{5}}}{2j} = \frac{1}{2}\ e^{j\frac{\pi n}{3}} + \frac{1}{2}\ e^{-j\frac{\pi n}{3}} + \frac{1}{2j}e^{j\frac{\pi n}{5}} - \frac{1}{2j}\ e^{-j\frac{\pi n}{5}} \quad .....(1)$$

$$= \frac{1}{2}\ e^{j\frac{\pi n}{3}\times\frac{5}{5}} + \frac{1}{2}\ e^{-j\frac{\pi n}{3}\times\frac{5}{5}} - j\frac{1}{2}\ e^{j\frac{\pi n}{5}\times\frac{3}{3}} + j\frac{1}{2}\ e^{-j\frac{\pi n}{5}\times\frac{3}{3}}$$

$$= \frac{1}{2}\ e^{j5\frac{\pi}{15}n} + \frac{1}{2}\ e^{-j5\frac{\pi}{15}n} + e^{-j\frac{\pi}{2}}\frac{1}{2}\ e^{j3\frac{\pi}{15}n} + e^{j\frac{\pi}{2}}\frac{1}{2}\ e^{-j3\frac{\pi}{15}n}$$

$$= \frac{1}{2}\ e^{-j5\omega_0 n} + \frac{1}{2}\ e^{j\frac{\pi}{2}}\ e^{-j3\omega_0 n} + \frac{1}{2}\ e^{-j\frac{\pi}{2}}\ e^{j3\omega_0 n} + \frac{1}{2}\ e^{j5\omega_0 n} \qquad \boxed{e^{\pm j\frac{\pi}{2}} = \pm j}$$

Here, $c_{-5} = \dfrac{1}{2},\quad c_{-3} = \dfrac{1}{2}e^{j\frac{\pi}{2}},\quad c_3 = \dfrac{1}{2}e^{-j\frac{\pi}{2}},\quad c_5 = \dfrac{1}{2},\quad c_k = 0$ for other k. $\boxed{\omega_0 = \dfrac{\pi}{15}}$

Alternatively, the Fourier series can be expressed as shown below.

Consider equation (1). $\boxed{\text{For integer n, } e^{j2\pi n} = 1}$

$$x(n) = \frac{1}{2}\ e^{j\frac{\pi n}{3}} + \frac{1}{2}\ e^{-j\frac{\pi n}{3}} + \frac{1}{2j}e^{j\frac{\pi n}{5}} - \frac{1}{2j}\ e^{-j\frac{\pi n}{5}} = \frac{1}{2}\ e^{j\frac{\pi n}{3}} + \frac{1}{2}\ e^{-j\frac{\pi n}{3}}\ e^{j2\pi n} - j\frac{1}{2}e^{j\frac{\pi n}{5}} + j\frac{1}{2}\ e^{-j\frac{\pi n}{5}}\ e^{j2\pi n}$$

$$= \frac{1}{2}\ e^{j\frac{\pi n}{3}} + \frac{1}{2}\ e^{-j\frac{\pi n}{3}+j2\pi n} + \frac{1}{2}e^{-j\frac{\pi}{2}}\ e^{j\frac{\pi n}{5}} + \frac{1}{2}\ e^{j\frac{\pi}{2}}\ e^{-j\frac{\pi n}{5}+2\pi n} = \frac{1}{2}\ e^{j\frac{\pi n}{3}} + \frac{1}{2}\ e^{j\frac{5\pi n}{3}} + \frac{1}{2}e^{-j\frac{\pi}{2}}\ e^{j\frac{\pi n}{5}} + \frac{1}{2}\ e^{j\frac{\pi}{2}}\ e^{j\frac{9\pi n}{5}}$$

$$= \frac{1}{2}\ e^{j\frac{\pi n}{3}\times\frac{5}{5}} + \frac{1}{2}\ e^{j\frac{5\pi n}{3}\times\frac{5}{5}} + \frac{1}{2}e^{-j\frac{\pi}{2}}\ e^{j\frac{\pi n}{5}\times\frac{3}{3}} + \frac{1}{2}\ e^{j\frac{\pi}{2}}\ e^{j\frac{9\pi n}{5}\times\frac{3}{3}}$$

$$= \frac{1}{2}\ e^{j\frac{5\pi}{15}n} + \frac{1}{2}\ e^{j25\frac{\pi}{15}n} + \frac{1}{2}e^{-j\frac{\pi}{2}}\ e^{j3\frac{\pi}{15}n} + \frac{1}{2}\ e^{j\frac{\pi}{2}}\ e^{j27\frac{\pi}{15}n} = \frac{1}{2}\ e^{j5\omega_0 n} + \frac{1}{2}e^{j25\omega_0 n} + \frac{1}{2}e^{\frac{-j\pi}{2}}\ e^{j3\omega_0 n} + \frac{1}{2}e^{\frac{j\pi}{2}}\ e^{j27\omega_0 n}$$

$$= \frac{1}{2}\ e^{-j\frac{\pi}{2}}\ e^{j3\omega_0 n} + \frac{1}{2}\ e^{ej5\omega_0 n} + \frac{1}{2}e^{j25\omega_0 n} + \frac{1}{2}\ e^{j\frac{\pi}{2}}\ e^{j27\omega_0 n} \qquad \boxed{\omega_0 = \frac{\pi}{15}}$$

Here, $c_3 = \dfrac{1}{2}e^{-j\frac{\pi}{2}},\quad c_5 = \dfrac{1}{2},\quad c_{25} = \dfrac{1}{2},\quad c_{27} = \dfrac{1}{2}e^{j\frac{\pi}{2}},\quad c_k = 0$ for other k.

E4.2. Determine the Fourier transform of the following signals.

a) $x(n) = 3\cos\dfrac{2\pi}{5}n$

Solution

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \mathcal{F}\left\{3\cos\frac{2\pi}{5}n\right\} = \mathcal{F}\left\{3\,\frac{e^{j\frac{2\pi}{5}n} - e^{-j\frac{2\pi}{5}n}}{2}\right\}$$

$$= \frac{3}{2}\,\mathcal{F}\left\{e^{j\frac{2\pi}{5}n}\right\} - \frac{3}{2}\,\mathcal{F}\left\{e^{-j\frac{2\pi}{5}n}\right\}$$

$$= \frac{3}{2}\sum_{m=-\infty}^{\infty} 2\pi\delta\left(\omega - \frac{2\pi}{5} - 2\pi m\right) - \frac{3}{2}\sum_{m=-\infty}^{\infty} 2\pi\delta\left(\omega + \frac{2\pi}{5} - 2\pi m\right)$$

$$= 3\pi\sum_{m=-\infty}^{\infty}\left[\delta\left(\omega - \frac{2\pi}{5} - 2\pi m\right) + \delta\left(\omega + \frac{2\pi}{5} - 2\pi m\right)\right]$$

$$\boxed{\mathcal{F}\left\{e^{j\omega_0 n}\right\} = \sum_{m=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi m)}$$

---

b) $x(n) = \{\underset{\uparrow}{-3},\ 4,\ -1,\ 2\}$

Solution

By definition,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n} = \sum_{n=0}^{3} x(n)\,e^{-j\omega n}$$

$$= x(0)e^{0} + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + x(3)e^{-j3\omega} = -3\times 1 + 4\times e^{-j\omega} - 1\times e^{-j2\omega} + 2e^{-j3\omega}$$

$$= -3 + 4e^{-j\omega} - e^{-j2\omega} + 2e^{-j3\omega}$$

---

c) $x(n) = \begin{cases} (-1)^n\ ; & 0 \le n \le 7 \\ 0\ ; & otherwise, \end{cases}$

Solution

By definition,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n} = \sum_{n=0}^{7} (-1)^n\,e^{-j\omega n} = \sum_{n=0}^{7} (-e^{-j\omega})^n$$

$$= \frac{1-(-e^{-j\omega})^8}{1-(-e^{-j\omega})} = \frac{1-(-1)^8(e^{-j\omega})^8}{1-(-e^{-j\omega})} = \frac{1-e^{-j8\omega}}{1+e^{-j\omega}}$$

$$= \frac{1-e^{\frac{-j8\omega}{2}}\times e^{\frac{-j8\omega}{2}}}{1+e^{\frac{-j\omega}{2}}\times e^{\frac{-j\omega}{2}}} = \frac{e^{\frac{j8\omega}{2}}\times e^{\frac{-j8\omega}{2}} - e^{\frac{-j8\omega}{2}}\times e^{\frac{-j8\omega}{2}}}{e^{\frac{j\omega}{2}}\times e^{\frac{-j\omega}{2}} + e^{\frac{-j\omega}{2}}\times e^{\frac{-j\omega}{2}}}$$

$$X(e^{j\omega}) = \frac{e^{\frac{-j8\omega}{2}}\left[e^{j4\omega} - e^{-j4\omega}\right]}{e^{\frac{-j\omega}{2}}\left[e^{\frac{j\omega}{2}} + e^{\frac{-j\omega}{2}}\right]} = e^{\frac{-j8\omega}{2}}\times e^{\frac{j\omega}{2}}\left(\frac{2j\sin 4\omega}{2\cos\left(\frac{\omega}{2}\right)}\right) = e^{\frac{-j7\omega}{2}}\,j\,\frac{\sin 4\omega}{\cos\left(\frac{\omega}{2}\right)}$$

$$= e^{\frac{-j7\omega}{2}}\,e^{\frac{j\pi}{2}}\,\frac{\sin 4\omega}{\cos\left(\frac{\omega}{2}\right)} = \frac{\sin 4\omega}{\cos\left(\frac{\omega}{2}\right)}\,e^{j\left(\frac{\pi-7\omega}{2}\right)}$$

---

d) $x(n) = 0.5\left[\left(\dfrac{1}{0.4}\right)^n - \left(\dfrac{1}{0.8}\right)^n\right]u(n)$

Solution

Given that,

$$x(n) = 0.5\left[\left(\frac{1}{0.4}\right)^n - \left(\frac{1}{0.8}\right)^n\right]u(n)$$

$$= 0.5\left(\frac{1}{0.4}\right)^n - (0.5)\left(\frac{1}{0.8}\right)^n \quad ; \quad \text{for } n \ge 0$$

By definition,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=0}^{\infty}\left[0.5\left(\frac{1}{0.4}\right)^n - 0.5\left(\frac{1}{0.8}\right)^n\right]e^{-j\omega n} = 0.5\sum_{n=0}^{\infty}\left(\frac{1}{0.4}\right)^n e^{-j\omega n} - 0.5\sum_{n=0}^{\infty}\left(\frac{1}{0.8}\right)^n e^{-j\omega n}$$

$$= 0.5\,\frac{1}{1-\frac{1}{0.4}e^{-j\omega}} - 0.5\,\frac{1}{1-\frac{1}{0.8}e^{-j\omega}} = 0.5\left[\frac{\left(1-\frac{1}{0.8}e^{-j\omega}\right) - \left(1-\frac{1}{0.4}e^{-j\omega}\right)}{\left(1-\frac{1}{0.4}e^{-j\omega}\right)\left(1-\frac{1}{0.8}e^{-j\omega}\right)}\right]$$

$$= 0.5\left[\frac{1.25e^{-j\omega}}{1-\frac{1}{0.8}e^{-j\omega} - \frac{1}{0.4}e^{-j\omega} + \frac{1}{0.32}e^{-j2\omega}}\right] = 0.5\left[\frac{1.25e^{-j\omega}}{1-3.75e^{-j\omega}+3.125e^{-j2\omega}}\right]$$

$$= \frac{0.625e^{-j\omega}}{1-3.75e^{-j\omega} + 3.125e^{-j2\omega}}$$

E4.3. Determine the convolution of the following sequences using Fourier transform.

a) $x_1(n) = \{2, \underset{\uparrow}{-2}, 2\}, \quad x_2(n) = \{-2, \underset{\uparrow}{2}, -2\}$

Solution

$$X_1(e^{j\omega}) = \sum_{n=-1}^{1} x_1(n)e^{-j\omega n} = x_1(-1)e^{j\omega} + x_1(0)e^0 + x_1(1)e^{-j\omega} = 2e^{j\omega} - 2 + 2e^{-j\omega}$$

$$X_2(e^{j\omega}) = \sum_{n=-1}^{1} x_2(n)e^{-j\omega n} = x_2(-1)e^{j\omega} + x_2(0)e^0 + x_2(1)e^{-j\omega} = -2e^{j\omega} + 2 - 2e^{-j\omega}$$

Using convolution property of Fourier transform.

$$X(e^{j\omega}) = \mathcal{F}\{x_1(n) * x_2(n)\} = X_1(e^{j\omega})\,X_2(e^{j\omega}) = \left(2e^{j\omega} - 2 + 2e^{-j\omega}\right)\left(-2e^{j\omega} + 2 - 2e^{-j\omega}\right)$$

$$= -4e^{j2\omega} + 4e^{j\omega} - 4 + 4e^{j\omega} - 4 + 4e^{-j\omega} - 4 + 4e^{-j\omega} - 4e^{-j2\omega} = -4e^{j2\omega} + 8e^{j\omega} - 12 - 8e^{-j\omega} - 4e^{-j2\omega} \quad .....(1)$$

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n} = ...x(-2)e^{j2\omega} + x(-1)e^{j\omega} + x(0)e^0 + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + ... \quad .....(2)$$

On comparing equations (1) and (2) we get,

$$x(n) = \{-4, 8, \underset{\uparrow}{-12}, -8, -4\}$$

b) $x_1(n) = \{\underset{\uparrow}{-2}, -1, 0\}, \quad x_2(n) = \{\underset{\uparrow}{-3}, 5, -7\}$

Solution

$$X_1(e^{j\omega}) = \sum_{n=0}^{2} x_1(n)e^{-j\omega n} = x_1(0)e^0 + x_1(1)e^{-j\omega} + x_1(2)e^{-j2\omega} = -2\times e^0 + \left(-1\times e^{-j\omega}\right) + 0 = -2 - e^{-j\omega}$$

$$X_2(e^{j\omega}) = \sum_{n=0}^{2} x_2(n)e^{-j\omega n} = x_2(0)e^0 + x_2(1)e^{-j\omega} + x_2(2)e^{-j2\omega} = \left(-3\times e^0\right) + \left(5\times e^{-j\omega}\right) + \left(-7\times e^{-j2\omega}\right) = -3 + 5e^{-j\omega} - 7e^{-j2\omega}$$

$$X(e^{j\omega}) = F\{x_1(n) * x_2(n)\} = X_1(e^{j\omega})\,X_2(e^{j\omega}) = \left(-2 - e^{-j\omega}\right)\left(-3 + 5e^{-j\omega} - 7e^{-j2\omega}\right)$$

$$= 6 - 10e^{-j\omega} + 14e^{-j2\omega} + 3e^{-j\omega} - 5e^{-j2\omega} + 7e^{-j3\omega} = 6 - 7e^{-j\omega} + 9e^{-j2\omega} + 7e^{-j3\omega} \quad .....(1)$$

Using convolution property of Fourier transform.

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n} = ...x(0)e^0 + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + x(3)e^{-j3\omega} + ... \quad .....(2)$$

On comparing equations (1) and (2) we get,

$$x(n) = \{\underset{\uparrow}{6}, -7, 9, 7\}$$

E4.4. Determine the inverse Fourier trasnform of the following function of w.

a) $X(e^{j\omega}) = 2j\omega$

Solution

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2j\omega \times e^{j\omega n} \times d\omega = \frac{j2}{2\pi}\int_{-\pi}^{\pi} \omega \times e^{j\omega n} \times d\omega = \frac{j}{\pi}\left[\frac{\omega \times e^{j\omega n}}{jn} - \int \frac{e^{j\omega n}}{jn} d\omega\right]_{-\pi}^{\pi}$$

$u = \omega$
$du = d\omega$
$dv = e^{j\omega n}\, d\omega$
$v = \dfrac{e^{j\omega n}}{jn}$
$\int u dv = uv - \int v du$

$$= \frac{j}{\pi}\left[\frac{\omega e^{j\omega n}}{jn} - \frac{e^{j\omega n}}{j^2 n^2}\right]_{-\pi}^{\pi} = \frac{j}{\pi}\left[\frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{j}{\pi}\left[\frac{\pi e^{j\pi n}}{jn} + \frac{e^{j\pi n}}{n^2} - \frac{(-\pi)e^{-j\pi n}}{jn} - \frac{e^{-j\pi n}}{n^2}\right]$$

$$= \frac{j}{\pi}\left[\frac{\pi}{jn}\left(e^{j\pi n} + e^{-j\pi n}\right) + \frac{\left(e^{j\pi n} - e^{-j\pi n}\right)}{n^2}\right]$$

$$= \frac{j}{\pi}\left[\frac{\pi}{jn}(2\cos \pi n) + \frac{2j(\sin \pi n)}{n^2}\right] = \frac{j}{\pi}\left[\frac{\pi}{jn}(2\cos \pi n)\right] = \frac{2\cos \pi n}{n}$$

for $n =$ integer,
$\sin \pi n = 0$

b) $X(e^{j\omega}) = \dfrac{1}{\left(1 - a e^{-j\omega}\right)^2}$, $|a| < 1$

Solution

By convolution property,

$$X(e^{j\omega}) = \underbrace{\frac{1}{\left(1 - a e^{-j\omega}\right)}}_{X_1(e^{j\omega})} \underbrace{\frac{1}{\left(1 - a e^{-j\omega}\right)}}_{X_2(e^{j\omega})} \quad , \quad |a| < 1$$

Now, $X_1(e^{j\omega}) = \dfrac{1}{1 - a e^{-j\omega}} = 1 + a\, e^{-j\omega} + a^2\, e^{-j2\omega} + a^3\, e^{-j3\omega} + ...$

$$= \sum_{n=0}^{\infty} a^n\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n} \quad \Rightarrow \quad x_1(n) = a^n u(n)$$

$\therefore\ x_1(n) = x_2(n) = a^n u(n)$

Here, $X(e^{j\omega}) = X_1(e^{j\omega})\ X_2(e^{j\omega})$

$$\therefore\ x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\ x_2(n-k) = \sum_{k=0}^{n} a^k u(k)\, a^{n-k} u(n-k) = \sum_{k=0}^{n} a^k\, a^n\, a^{-k} = a^n \sum_{k=0}^{n} a^{k-k}$$

$$= a^n \sum_{k=0}^{n} 1 = a^n\,(n+1)\ ; \quad n \ge 0$$

$x(n) = (n+1)a^n u(n)$

c) $Y(e^{j\omega}) = \dfrac{1 + \frac{1}{7}e^{-j\omega}}{1 - \frac{1}{7}e^{-j\omega}}$

Solution

$$Y(e^{j\omega}) = \frac{1}{1 - \frac{1}{7}e^{-j\omega}} + \frac{1}{7}e^{-j\omega}\left(\frac{1}{1 - \frac{1}{7}e^{-j\omega}}\right)$$

$\mathcal{F}\{a^n u(n)\} = \dfrac{1}{1 - a e^{-j\omega}}$
if $\mathcal{F}\{x(n)\} = X(e^{j\omega})$ then $\mathcal{F}\{x(n-1)\} = e^{-j\omega} X(e^{j\omega})$

On taking inverse Fourier transform we get,

$$y(n) = \left(\frac{1}{7}\right)^n u(n) + \frac{1}{7}\left(\frac{1}{7}\right)^{n-1} u(n-1) = \left(\frac{1}{7}\right)^n u(n) + \left(\frac{1}{7}\right)^n u(n-1) = \left(\frac{1}{7}\right)^n \left[u(n) + u(n-1)\right]$$

d) $H(e^{j\omega}) = \dfrac{1}{\left(1-\frac{1}{6}e^{-j\omega}\right)\left(1-\frac{1}{5}e^{-j\omega}\right)}$

Solution

Using partial fraction,

$$H(e^{j\omega}) = \frac{A}{\left(1-\frac{1}{6}e^{-j\omega}\right)} + \frac{B}{\left(1-\frac{1}{5}e^{-j\omega}\right)}$$

$$A = \cancel{\left(1-\frac{1}{6}e^{-j\omega}\right)} \times \frac{1}{\cancel{\left(1-\frac{1}{6}e^{-j\omega}\right)}\left(1-\frac{1}{5}e^{-j\omega}\right)}\Bigg|_{e^{-j\omega}=6} = \frac{1}{1-\frac{1}{5}\times 6} = \frac{1}{1-\frac{6}{5}} = \frac{1}{-\frac{1}{5}} = -5$$

$$B = \cancel{\left(1-\frac{1}{5}e^{-j\omega}\right)} \times \frac{1}{\left(1-\frac{1}{6}e^{-j\omega}\right)\cancel{\left(1-\frac{1}{5}e^{-j\omega}\right)}}\Bigg|_{e^{-j\omega}=5} = \frac{1}{1-\frac{1}{6}\times 5} = \frac{1}{1-\frac{5}{6}} = \frac{1}{\frac{1}{6}} = 6$$

$$\therefore\ H(e^{j\omega}) = \frac{-5}{1-\frac{1}{6}e^{-j\omega}} + \frac{6}{1-\frac{1}{5}e^{-j\omega}} = 6\left[\frac{1}{1-\frac{1}{5}e^{-j\omega}}\right] - 5\left[\frac{1}{1-\frac{1}{6}e^{-j\omega}}\right]$$

On taking Inverse Fourier transform,

$$h(n) = 6\left(\frac{1}{5}\right)^n u(n) - 5\left(\frac{1}{6}\right)^n u(n) = \left[6\left(\frac{1}{5}\right)^n - 5\left(\frac{1}{6}\right)^n\right] u(n)$$

E4.5. a) ***A causal discrete time system is described by the equation, $y(n) - \frac{5}{14}y(n-1) - \frac{1}{14}y(n-2) = x(n)$, where $x(n)$ and $y(n)$ are input and output of the system. Find the impulse response $h(n)$, frequency response $H(e^{jw})$, magnitude function and phase function of the system.***

Solution

Given, $y(n) - \frac{5}{14}\,y(n-1) - \frac{1}{14}\,y(n-2) = x(n)$

On taking Fourier transform of above equation we get,

$$Y(e^{j\omega}) - \frac{5}{14}\,e^{-j\omega}\,Y(e^{j\omega}) - \frac{1}{14}e^{-j2\omega}\,Y(e^{j\omega}) = X(e^{j\omega})$$

$$\Rightarrow \left[1-\frac{5}{14}e^{-j\omega} - \frac{1}{14}e^{-j2\omega}\right] Y(e^{j\omega}) = X(e^{j\omega})$$

$$\therefore\ \text{Frequency response, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{5}{14}e^{-j\omega} - \frac{1}{14}e^{-j2\omega}}$$

$$= \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+\frac{1}{7}e^{-j\omega}\right)}$$

Let, $1-\frac{5}{14}e^{-j\omega} - \frac{1}{14}e^{-j2\omega} = 0$

$\therefore\ e^{-j2\omega}\left(e^{j2\omega} - \frac{5}{14}e^{j\omega} - \frac{1}{14}\right) = 0$

Let, $e^{j\omega} = x$

$\therefore\ x^2 - \frac{5}{14}x - \frac{1}{14} = 0$

The roots of quadratic

$x^2 - \frac{5}{14}x - \frac{1}{14} = 0$ are,

$$x = \frac{\frac{5}{14} \pm \sqrt{\left(\frac{5}{14}\right)^2 + \frac{4}{14}}}{2}$$

$$= \frac{1}{2}\left(\frac{5}{14} \pm \frac{9}{14}\right) = \frac{1}{2},\ -\frac{1}{7}$$

Using partial fraction,

$$H(e^{j\omega}) = \frac{A}{\left(1-\frac{1}{2}e^{-j\omega}\right)} + \frac{B}{\left(1+\frac{1}{7}e^{-j\omega}\right)}$$

$$A = \cancel{\left(1-\frac{1}{2}e^{-j\omega}\right)} \frac{1}{\cancel{\left(1-\frac{1}{2}e^{-j\omega}\right)}\left(1+\frac{1}{7}e^{-j\omega}\right)}\Bigg|_{e^{-j\omega}=2} = \frac{1}{1+\frac{1}{7}\times 2} = \frac{1}{1+\frac{2}{7}} = \frac{1}{\frac{9}{7}} = \frac{7}{9}$$

$$B = \cancel{\left(1+\frac{1}{7}e^{-j\omega}\right)} \frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\cancel{\left(1+\frac{1}{7}e^{-j\omega}\right)}}\Bigg|_{e^{-j\omega}=-7} = \frac{1}{1-\frac{1}{2}\times -7} = \frac{1}{1+\frac{7}{2}} = \frac{1}{\frac{9}{2}} = \frac{2}{9}$$

$$\therefore\ H(e^{j\omega}) = \frac{\frac{7}{9}}{\left(1-\frac{1}{2}\,e^{-j\omega}\right)} + \frac{\frac{2}{9}}{\left(1+\frac{1}{7}\,e^{-j\omega}\right)} = \frac{7}{9}\ \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{2}{9}\ \frac{1}{1-\left(-\frac{1}{7}\right)e^{-j\omega}}$$

Impulse response, $h(n) = \mathcal{F}^{-1}\{H(e^{j\omega})\} = \left[\frac{7}{9}\left(\frac{1}{2}\right)^n + \frac{2}{9}\left(\frac{-1}{7}\right)^n\right] u(n)$

Magnitude and phase function :-

$$H(e^{j\omega}) = \frac{1}{1 - \frac{5}{14}e^{j\omega} - \frac{1}{14}e^{-j2\omega}}$$

The magnitude function is,

$$\left|H(e^{j\omega})\right| = \left[\frac{1}{\left(1 - \frac{5}{14}e^{-j\omega} - \frac{1}{14}e^{-j2\omega}\right)} \times \frac{1}{\left(1 - \frac{5}{14}e^{j\omega} - \frac{1}{14}e^{j2\omega}\right)}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1 - \frac{5}{14}e^{j\omega} - \frac{1}{14}e^{j2\omega} - \frac{5}{14}e^{-j\omega} + \frac{5^2}{14^2} + \frac{5}{14^2}e^{j\omega} - \frac{1}{14}e^{-j2\omega} + \frac{5}{14^2}e^{-j\omega} + \frac{1}{14^2}}\right]^{\frac{1}{2}}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{1}{\left(1 + \frac{5^2}{14^2} + \frac{1}{14^2}\right) + \left(-\frac{5}{14} + \frac{5}{14^2}\right)\left(e^{j\omega} + e^{-j\omega}\right) - \frac{1}{14}\left(e^{j2\omega} + e^{-j2\omega}\right)}$$

$$= \left[\frac{1}{1.13 - 0.664\cos\omega - 0.14\cos 2\omega}\right]^{\frac{1}{2}} \qquad .....(1)$$

The phase function is,

$$\angle H(e^{j\omega}) = \tan^{-1}\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}$$

$$\text{Let, } H(e^{j\omega}) = \frac{1}{\left(1 - \frac{5}{14}e^{-j\omega} - \frac{1}{14}e^{-j2\omega}\right)} \times \frac{\left(1 - \frac{5}{14}e^{j\omega} - \frac{1}{14}e^{j2\omega}\right)}{\left(1 - \frac{5}{14}e^{j\omega} - \frac{1}{14}e^{j2\omega}\right)} = \frac{1 - \frac{5}{14}(\cos\omega + j\sin\omega) - \frac{1}{14}(\cos 2\omega + j\sin 2\omega)}{1.13 - 0.664\cos\omega - 0.14\cos 2\omega}$$

$$= \frac{1 - \frac{5}{14}\cos\omega - \frac{1}{14}\cos 2\omega}{1.13 - 0.664\cos\omega - 0.14\cos 2\omega} + \frac{-j\left(\frac{5\sin\omega}{14} + \frac{\sin 2\omega}{14}\right)}{1.13 - 0.664\cos\omega - 0.14\cos 2\omega}$$

$$\boxed{\text{Using equation (1)}}$$

$$\therefore \quad \angle H(e^{j\omega}) = \tan^{-1}\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})} = \tan^{-1}\left[\frac{-5\sin\omega - \sin 2\omega}{14 - 5\cos\omega - \cos 2\omega}\right]$$

b) Consider an LTI system described by,

$$y(n) - \frac{1}{5}y(n-1) = x(n) + \frac{1}{5}x(n-1)$$

i) Determine the frequency response $H(e^{jw})$ of the system.

ii) Find the impulse response h(n) of the system.

iii) Determine its response y(n) for the input, $x(n) = \cos\frac{\pi}{2}n$

Solution

i) Frequency response

Given that, $y(n) - \frac{1}{5}y(n-1) = x(n) + \frac{1}{5}x(n-1)$

Taking Fourier transform,

$$Y(e^{j\omega}) - \frac{1}{5}e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) + \frac{1}{5}e^{-j\omega}X(e^{j\omega}) \Rightarrow Y(e^{j\omega})\left[1 - \frac{1}{5}e^{-j\omega}\right] = X(e^{j\omega})\left[1 + \frac{1}{5}e^{-j\omega}\right]$$

$$\therefore \text{ Frequency response, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + \frac{1}{5}e^{-j\omega}}{1 - \frac{1}{5}e^{-j\omega}}$$

ii) Impulse response

$$\text{Let, } H(e^{j\omega}) = \frac{1+\frac{1}{5}e^{-j\omega}}{1-\frac{1}{5}e^{-j\omega}} = \left[\frac{1}{1-\frac{1}{5}e^{-j\omega}}\right] + \frac{1}{5}e^{-j\omega}\left(\frac{1}{1-\frac{1}{5}e^{-j\omega}}\right)$$

By taking inverse Fourier transform we get,

$$\text{Impulse response, } h(n) = \mathcal{F}^{-1}\left\{H(e^{j\omega})\right\} = \left(\frac{1}{5}\right)^n u(n) + \frac{1}{5}\left(\frac{1}{5}\right)^{n-1} u(n-1)$$

iii) Response for given input

$$\text{Given that, } x(n) = \cos\frac{\pi}{2}n$$

$$\therefore\ X(e^{j\omega}) = \mathcal{F}\left\{x(n)\right\} = \mathcal{F}\left\{\cos\frac{\pi}{2}n\right\} = \pi\sum_{m=-\infty}^{+\infty}\left[\delta\left(\omega-\frac{\pi}{2}-2\pi m\right)+\delta\left(\omega+\frac{\pi}{2}-2\pi m\right)\right]$$

We know that,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \quad\Rightarrow\quad Y(e^{j\omega}) = H(e^{j\omega})\ X(e^{j\omega})$$

$$\therefore Y(e^{j\omega}) = H(e^{j\omega})\ X(e^{j\omega}) = \frac{1+\frac{1}{5}e^{-j\omega}}{1-\frac{1}{5}e^{-j\omega}}\ \pi\sum_{m=-\infty}^{+\infty}\left[\delta\left(\omega-\frac{\pi}{2}-2\pi m\right)+\delta\left(\omega+\frac{\pi}{2}-2\pi m\right)\right]$$

$$= \frac{\pi\left(5+e^{-j\omega}\right)}{\left(5-e^{-j\omega}\right)}\sum_{m=-\infty}^{+\infty}\left[\delta\left(\omega-\frac{\pi}{2}-2\pi m\right)+\delta\left(\omega+\frac{\pi}{2}-2\pi m\right)\right]$$

The response, y(n) is obtained by taking inverse Fourier transform of $Y(e^{jw})$.

By definition of inverse Fourier transform,

$$\text{Response, } y(n) = \mathcal{F}^{-1}\left\{Y(e^{j\omega})\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})\ e^{j\omega n}\ d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\pi\left(5+e^{-j\omega}\right)}{5-e^{-j\omega}}\sum_{m=-\infty}^{+\infty}\left[\delta\left(\omega-\frac{\pi}{2}-2\pi m\right)+\delta\left(\omega+\frac{\pi}{2}-2\pi m\right)\right]e^{j\omega n}\ d\omega$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\frac{5+e^{-j\omega}}{5-e^{-j\omega}}\left[\delta\left(\omega-\frac{\pi}{2}\right)+\delta\left(\omega+\frac{\pi}{2}\right)\right]e^{j\omega n}\ d\omega$$

In the interval $\omega = -\frac{\pi}{2}$ to $+\frac{\pi}{2}$ there are only two impluses at $\omega = -\frac{\pi}{2}$ and $\omega = +\frac{\pi}{2}$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\frac{\left(5+e^{-j\omega}\right)e^{j\omega n}}{5-e^{-j\omega}}\ \delta\left(\omega-\frac{\pi}{2}\right)d\omega + \frac{1}{2}\int_{-\pi}^{\pi}\frac{\left(5+e^{-j\omega}\right)e^{j\omega n}}{5-e^{-j\omega}}\ \delta\left(\omega+\frac{\pi}{2}\right)d\omega$$

$$= \frac{1}{2}\left.\frac{\left(5+e^{-j\omega}\right)e^{j\omega n}}{5-e^{-j\omega}}\right|_{\omega=-\frac{\pi}{2}} + \frac{1}{2}\left.\frac{\left(5+e^{-j\omega}\right)e^{j\omega n}}{5-e^{-j\omega}}\right|_{\omega=\frac{\pi}{2}}$$

The impulse $\delta\left(\omega-\frac{\pi}{2}\right)$ is nonzero only at $\omega = -\frac{\pi}{2}$

The impulse $\delta\left(\omega+\frac{\pi}{2}\right)$ is nonzero only at $\omega = \frac{\pi}{2}$

$$= \frac{1}{2}\ \frac{\left(5+e^{j\frac{\pi}{2}}\right)e^{-j\frac{n\pi}{2}}}{5-e^{j\frac{\pi}{2}}} + \frac{1}{2}\ \frac{\left(5+e^{-j\frac{\pi}{2}}\right)e^{j\frac{n\pi}{2}}}{5-e^{-j\frac{\pi}{2}}}$$

$$= \frac{1}{2}\ \frac{5+j}{5-j}\ e^{-j\frac{n\pi}{2}} + \frac{1}{2}\ \frac{5-j}{5+j}\ e^{j\frac{n\pi}{2}}$$

$e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\ \sin\frac{\pi}{2} = 0 + j$

$e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\ \sin\frac{\pi}{2} = 0 - j$

$$= \frac{1}{2}\ \frac{5.099\ \angle 0.197}{5.099\ \angle -0.197}\ e^{-jn\frac{\pi}{2}} + \frac{1}{2}\ \frac{5.099\ \angle -0.197}{5.099\ \angle 0.197}\ e^{jn\frac{\pi}{2}}$$

$$= 0.5\ \angle 0.394\ e^{-jn\frac{\pi}{2}} + 0.5\ \angle -0.394\ e^{jn\frac{\pi}{2}}$$

$0.394 = \frac{0.394}{\pi}\times\pi = 0.125\pi = \frac{\pi}{8}$

$$= 0.5\ \angle 0.125\pi\ e^{-jn\frac{\pi}{2}} + 0.5\ \angle -0.125\pi\ e^{jn\frac{\pi}{2}}$$

$$= 0.5\ e^{j0.125\pi}\ e^{-jn\frac{\pi}{2}} + 0.5\ e^{-j0.125\pi}\ e^{jn\frac{\pi}{2}}$$

$$\therefore\ y(n) = 0.5\, e^{j\frac{\pi}{8}}\, e^{-jn\frac{\pi}{2}} + 0.5\, e^{-j\frac{\pi}{8}}\, e^{jn\frac{\pi}{2}}$$

$$= 0.5\, e^{-j\left(\frac{n\pi}{2}-\frac{\pi}{8}\right)} + 0.5\, e^{j\left(\frac{n\pi}{2}-\frac{\pi}{8}\right)} = 0.5\left[e^{j\left(\frac{n\pi}{2}-\frac{\pi}{8}\right)} + e^{-j\left(\frac{n\pi}{2}-\frac{\pi}{8}\right)}\right]$$

$$= 0.5 \times 2\cos\left(\frac{n\pi}{2}-\frac{\pi}{8}\right) = \cos\left(\frac{n\pi}{2}-\frac{\pi}{8}\right)$$

E4.6. ***A discrete LTI system is described by a difference equation, y(n) = x(n) – x(n –1). Determine the frequency response H(e^{jw}), impulse response h(n). Sketch the magnitude function and the phase function.***

Solution

i) Frequency response

Given, $y(n) = x(n) - x(n-1)$

By taking Fourier transform,

$$Y(e^{j\omega}) = X(e^{j\omega}) - e^{-j\omega}\, X(e^{-j\omega}) \ \Rightarrow\ Y(e^{j\omega}) = X(e^{j\omega})\left[1 - e^{-j\omega}\right]$$

$$\therefore\ \text{Frequency response, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 - e^{-j\omega} = 1 - e^{-j\frac{\omega}{2}}\, e^{-j\frac{\omega}{2}} \qquad \boxed{e^{-j\frac{\omega}{2}}\, e^{j\frac{\omega}{2}} = 1}$$

$$= e^{-j\frac{\omega}{2}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right] = e^{-j\frac{\omega}{2}}\, 2j\sin\left(\frac{\omega}{2}\right)$$

$$= 2\sin\frac{\omega}{2}\, e^{j\frac{\pi}{2}}\, e^{-j\frac{\omega}{2}} \qquad \boxed{j = e^{j\frac{\pi}{2}}}$$

$$= 2\sin\frac{\omega}{2}\, e^{j\left(\frac{\pi-\omega}{2}\right)}$$

ii) Impulse response

Given, $y(n) = x(n) - x(n-1)$

When $x(n) = \delta(n)$ = impulse input,

$y(n) = h(n)$ = impulse response

$\therefore$ Impulse response, $h(n) = \delta(n) - \delta(n-1)$

iii) Magnitude and phase function

Frequency response, $H(e^{j\omega}) = 2\sin\frac{\omega}{2}\, e^{j\left(\frac{\pi-\omega}{2}\right)}$

Here, $\sin\frac{\omega}{2}$ is negative for negative frequency.

$$\therefore\ \text{When } \omega = -\pi \text{ to } 0\ ;\quad H(e^{j\omega}) = -\left|2\sin\frac{\omega}{2}\right| e^{j\frac{\pi-\omega}{2}}$$

$$= e^{-j\pi}\left|2\sin\frac{\omega}{2}\right| e^{j\frac{\pi-\omega}{2}} \qquad \boxed{e^{-j\pi} = -1}$$

$$= \left|2\sin\frac{\omega}{2}\right| e^{-j\left(\frac{\pi+\omega}{2}\right)}$$

$$\text{When } \omega = 0 \text{ to } +\pi\ ;\quad H(e^{j\omega}) = 2\sin\frac{\omega}{2}\, e^{j\frac{\pi-\omega}{2}}$$

$$\therefore\ \text{Magnitude function, } \left|H(e^{j\omega})\right| = \left|2\sin\frac{\omega}{2}\right|$$

$$\text{Phase function, } \angle H(e^{j\omega}) = -\left(\frac{\pi+\omega}{2}\right);\quad \text{for } \omega = -\pi \text{ to } 0$$

$$= \frac{\pi-\omega}{2}\ ;\quad \text{for } \omega = 0 \text{ to } \pi$$

The magnitude and phase of H(e^{jw}) are calculated for various values of w and listed in the following table. Using the tabulated values, the magnitude and phase spectrum are sketched as shown in fig 1 and fig 2.

Table : Magnitude and phase of H(e^jw) for various values of w.

w	|H(e^jw)|	Ð H(e^jw)
$-\frac{4\pi}{4}$	2	0
$-\frac{3\pi}{4}$	1.848	–0.125p
$-\frac{2\pi}{4}$	1.414	–0.25p
$-\frac{\pi}{4}$	0.765	–0.375p
0	0	± 0.5p
$\frac{\pi}{4}$	0.765	0.375p
$\frac{2\pi}{4}$	1.414	0.25p
$\frac{3\pi}{4}$	1.848	0.125p
$\frac{4\pi}{4}$	2	0

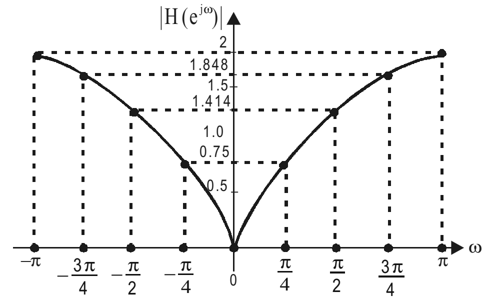


Fig1 : Magnitude spectrum.

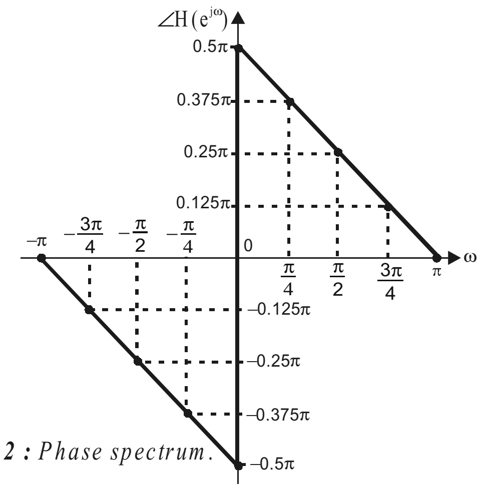


Fig 2 : Phase spectrum.

E4.7. Sketch the magnitude and phase function of the discrete time LTI system described by the equation,

$$y(n) = x(n) + x(n-1).$$

Solution

Given that, $y(n) = x(n) + x(n-1)$

On taking Fourier transform of above equation we get,

$$Y(e^{j\omega}) = X(e^{j\omega}) + e^{-j\omega}\, X(e^{-j\omega}) \quad \Rightarrow \quad Y(e^{j\omega}) = X(e^{j\omega})\left(1+e^{-j\omega}\right) \qquad \boxed{e^{-j\frac{\omega}{2}}\, e^{j\frac{\omega}{2}} = 1}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + e^{-j\omega} = 1 + e^{-j\frac{\omega}{2}}\, e^{-j\frac{\omega}{2}} = \left[e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right] e^{-j\frac{\omega}{2}} = 2\cos\frac{\omega}{2}\, e^{-j\frac{\omega}{2}}$$

\ Magnitude function, $\left|H(e^{j\omega})\right| = 2\cos\frac{\omega}{2}$

Phase function, $\angle H(e^{j\omega}) = \frac{-\omega}{2}$

The magnitude and phase of H(e^jw) are calculated for various values of w and listed in the following table. Using the tabulated values, the magnitude and phase spectrum are sketched as shown in fig 1 and fig 2.

w	|H(e^jw)|	Ð H(e^jw)
$-\frac{4\pi}{4}$	0	0.5p
$-\frac{3\pi}{4}$	0.77	0.375p
$-\frac{2\pi}{4}$	1.41	0.25p
$-\frac{\pi}{4}$	1.85	0.125p
0	2	0
$\frac{\pi}{4}$	1.85	– 0.125p
$\frac{2\pi}{4}$	1.41	– 0.25p
$\frac{3\pi}{4}$	0.77	– 0.375p
$\frac{4\pi}{4}$	0	– 0.5p

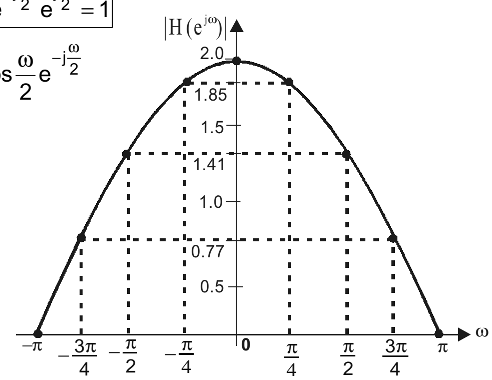


Fig 1 : Magnitude spectrum.

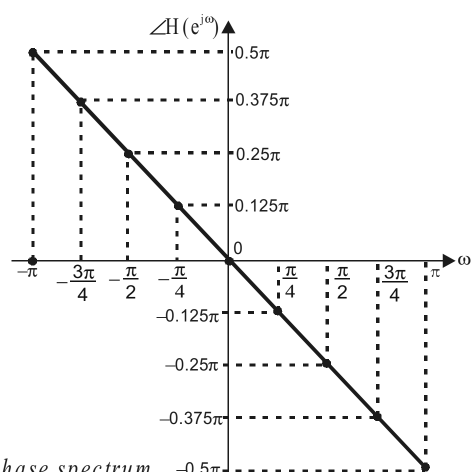


Fig 2 : Phase spectrum.

E4.8. The impulse response of a system is,

$$h(n) = \frac{1}{0.2}\delta(n+2) + \frac{1}{0.4}\delta(n+1) + \frac{1}{0.3}\delta(n) + \frac{1}{0.4}\delta(n-1)$$

i) Is the system BIBO stable?

ii) Is the system causal?

iii) Find the frequency response.

Solution

Given that,

$$h(n) = \frac{1}{0.2}\delta(n+2) + \frac{1}{0.4}\delta(n+1) + \frac{1}{0.3}\delta(n) + \frac{1}{0.4}\delta(n-1)$$

We know, $\delta(n) = 1$; when $n = 0$

$= 0$; when $n \neq 0$

When, $n=-3$; $h(-3) = \frac{1}{0.2}\delta(-1) + \frac{1}{0.4}\delta(-2) + \frac{1}{0.3}\delta(-3) + \frac{1}{0.4}\delta(-4) = 0 + 0 + 0 + 0 = 0$

When, $n=-2$; $h(-2) = \frac{1}{0.2}\delta(0) + \frac{1}{0.4}\delta(-1) + \frac{1}{0.3}\delta(-2) + \frac{1}{0.4}\delta(-3) = \frac{1}{0.2} + 0 + 0 + 0 = \frac{1}{0.2}$

When, $n=-1$; $h(-1) = \frac{1}{0.2}\delta(1) + \frac{1}{0.4}\delta(0) + \frac{1}{0.3}\delta(-1) + \frac{1}{0.4}\delta(-2) = 0 + \frac{1}{0.4} + 0 + 0 = \frac{1}{0.4}$

When, $n=0$; $h(0) = \frac{1}{0.2}\delta(2) + \frac{1}{0.4}\delta(1) + \frac{1}{0.3}\delta(0) + \frac{1}{0.4}\delta(-1) = 0 + 0 + \frac{1}{0.3} + 0 = \frac{1}{0.3}$

When, $n=1$; $h(1) = \frac{1}{0.2}\delta(3) + \frac{1}{0.4}\delta(2) + \frac{1}{0.3}\delta(1) + \frac{1}{0.4}\delta(0) = 0 + 0 + 0 + \frac{1}{0.4} = \frac{1}{0.4}$

When, $n=2$; $h(2) = \frac{1}{0.2}\delta(4) + \frac{1}{0.4}\delta(3) + \frac{1}{0.3}\delta(2) + \frac{1}{0.4}\delta(1) = 0 + 0 + 0 + 0 = 0$

Here, h(n) is zero for $n \leq -3$ and $n \geq 2$, and nonzero in the interval $n = -2$ to $+1$.

$\therefore$ Impulse response,

$$h(n) = \left\{\frac{1}{0.2}, \frac{1}{0.4}, \underset{\uparrow}{\frac{1}{0.3}}, \frac{1}{0.4}\right\}$$

i) Check for stability

For stable system, $\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$

$$\sum_{n=-\infty}^{+\infty} |h(n)| = |h(-2)| + |h(-1)| + |h(0)| + |h(1)|$$

$$= \frac{1}{0.2} + \frac{1}{0.4} + \frac{1}{0.3} + \frac{1}{0.4} = 13.33 = \text{Constant}$$

Hence the system is stable.

ii) Causality

For any value of 'n', the impulse response h(n) depends on future input $\delta(n+2)$ and $\delta(n+1)$.

Hence the system is noncausal.

iii) Frequency response, $H(e^{jw})$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n) e^{-j\omega n}$$

$$= h(-2)e^{j2\omega} + h(-1)e^{j\omega} + h(0) + h(1)e^{-j\omega} = \frac{1}{0.2}e^{j2\omega} + \frac{1}{0.4}e^{j\omega} + \frac{1}{0.3} + \frac{1}{0.4}e^{-j\omega}$$

$$= \frac{0.6e^{j2\omega} + 0.3e^{j\omega} + 0.4 + 0.3e^{-j\omega}}{0.4 \times 0.3}$$

$$= \frac{1}{0.12}\left[0.6(\cos 2\omega + j\sin 2\omega) + 0.3\left(e^{j\omega} + e^{-j\omega}\right) + 0.4\right]$$

$$= \frac{1}{0.12}\left[0.6\cos 2\omega + j0.6\sin 2\omega + 0.6\cos\omega + 0.4\right]$$

$$= \frac{1}{0.12}\left[(0.4 + 0.6\cos\omega + 0.6\cos 2\omega + j0.6\sin 2\omega)\right]$$

E4.9. The impulse response of an LTI system is h(n) = {–2, –1, 1, –2}.

Find the response of the system for the input x(n) = {2, 2, 4, 1}, using convolution property of Fourier transform.

Solution

Response, $y(n) = x(n) * h(n)$

On taking Fourier transform we get,

$$\mathcal{F}\{y(n)\} = \mathcal{F}\{x(n) * h(n)\}$$

$\therefore\ Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$ $\boxed{\text{Using convolution porperty.}}$

By definition of Fourier transform,

$$H(e^{j\omega}) = \sum_{n=0}^{3} h(n)e^{-j\omega n} = h(0) + h(1)e^{-j\omega} + h(2)e^{-j2\omega} + h(3)e^{-j3\omega} = -2 - e^{-j\omega} + e^{-j2\omega} - 2e^{-j3\omega}$$

$$X(e^{j\omega}) = \sum_{n=0}^{3} x(n)e^{-j\omega n} = x(0) + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + x(3)e^{-j3\omega} = 2 + 2e^{-j\omega} + 4e^{-j2\omega} + e^{-j3\omega}$$

$$\begin{aligned} X(e^{j\omega})\, H(e^{j\omega}) &= \left(2 + 2e^{-j\omega} + 4e^{-j2\omega} + e^{-j3\omega}\right)\left(-2 - e^{-j\omega} + e^{-j2\omega} - 2e^{-j3\omega}\right) \\ &= -4 - 2e^{-j\omega} + 2e^{-j2\omega} - 4e^{-j3\omega} \\ &\quad - 4e^{-j\omega} - 2e^{-j2\omega} + 2e^{-j3\omega} - 4e^{-j4\omega} \\ &\quad - 8e^{-j2\omega} - 4e^{-j3\omega} + 4e^{-j4\omega} - 8e^{-j5\omega} \\ &\quad - 2e^{-j3\omega} - e^{-j4\omega} + e^{-j5\omega} - 2e^{-j6\omega} \end{aligned}$$

$$\therefore\ Y(e^{j\omega}) = -4 - 6e^{-j\omega} - 8e^{-j2\omega} - 8e^{-j3\omega} - e^{-j4\omega} - 7e^{-j5\omega} - 2e^{-j6\omega} \qquad .....(1)$$

By definition of Fourier transform,

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y(n)(e^{-j\omega n}) = ...y(0) + y(1)e^{-j\omega} + y(2)e^{-j2\omega} + y(3)e^{-j3\omega} + y(4)e^{-j4\omega} + y(5)e^{-j5\omega} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$$\therefore\ \text{Response, } y(n) = \{\underset{\uparrow}{-4},\ -6,\ -8,\ -8,\ -1,\ -7,\ -2\}$$

E4.10. A causal system is represented by the following difference equation,

y(n) – 0.2 y(n – 1) = x(n) – 0.6 x(n –1).

Find the system transfer function H(z), impulse response and frequency response of the system. Also determine the magnitude and phase function.

Solution

i) System transfer function

Given that, $y(n) - 0.2\, y(n-1) = x(n) - 0.6\, x(n-1)$

On taking $\mathcal{Z}$ - transform,

$$Y(z) - 0.2z^{-1}Y(z) = X(z) - 0.6X(z)z^{-1}$$

$$Y(z)\left[1 - 0.2z^{-1}\right] = \left[1 - 0.6z^{-1}\right]X(z) \Rightarrow \frac{Y(z)}{X(z)} = \frac{1 - 0.6z^{-1}}{1 - 0.2z^{-1}}$$

$$\therefore\ \text{System transfer function, } H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 0.6z^{-1}}{1 - 0.2z^{-1}}$$

ii) Impulse response

$$H(z) = \frac{1 - 0.6z^{-1}}{1 - 0.2z^{-1}} = \frac{1}{1 - 0.2z^{-1}} - 0.6\,\frac{z^{-1}}{1 - 0.2z^{-1}}$$

Impulse response, $h(n) = \mathcal{Z}^{-1}\{H(z)\}$

$$\begin{aligned} &= \mathcal{Z}^{-1}\left\{\frac{1}{1 - 0.2z^{-1}} - 0.6\,\frac{z^{-1}}{1 - 0.2z^{-1}}\right\} \\ &= \mathcal{Z}^{-1}\left\{\frac{1}{1 - 0.2z^{-1}}\right\} - 0.6\,\mathcal{Z}^{-1}\left\{\frac{z^{-1}}{1 - 0.2z^{-1}}\right\} \\ &= 0.2^n u(n) - 0.6(0.2)^{n-1}u(n-1) \end{aligned}$$

$\mathcal{Z}\{a^n u(n)\} = \dfrac{1}{1 - az^{-1}}$
if $\mathcal{Z}\{x(n)\} = X(z)$ then by shifting property $\mathcal{Z}\{x(n-1)\} = z^{-1}X(z)$

iii) Frequency response

Frequency response, $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

$$\therefore \; H(e^{j\omega}) = \frac{1-0.6\,e^{-j\omega}}{1-0.2\,e^{-j\omega}}$$

$$\left|H(e^{j\omega})\right| = \left[H(e^{j\omega})\;H^{*}(e^{j^{*}})\right]^{\frac{1}{2}} = \left[\frac{1-0.6\,e^{-j\omega}}{1-0.2\,e^{-j\omega}} \times \frac{1-0.6\,e^{j\omega}}{1-0.2\,e^{j\omega}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1-0.6e^{j\omega}-0.6\,e^{-j\omega}+0.6^{2}}{1-0.2e^{j\omega}-0.2\,e^{-j\omega}+0.2^{2}}\right]^{\frac{1}{2}} = \left[\frac{1-0.6\left(e^{j\omega}+e^{-j\omega}\right)+0.36}{1-0.2\left(e^{j\omega}+e^{-j\omega}\right)+0.04}\right]^{\frac{1}{2}}$$

$$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

$$= \sqrt{\frac{1.36-1.2\cos\omega}{1.04-0.4\cos\omega}} \qquad .....(1)$$

Let, $$H(e^{j\omega}) = \frac{1-0.6\,e^{-j\omega}}{1-0.2\,e^{-j\omega}} \times \frac{1-0.2\,e^{j\omega}}{1-0.2\,e^{j\omega}} = \frac{1-0.2e^{j\omega}-0.6e^{-j\omega}+0.12}{1.04-0.4\cos\omega}$$

$\boxed{\text{Using equation (1)}}$

$$= \frac{1.12-0.2(\cos\omega + j\,\sin\omega)-0.6(\cos\omega - j\,\sin\omega)}{1.04-0.4\;\cos\omega}$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$= \frac{1.12-0.8\cos\omega}{1.04-0.4\cos\omega} + \frac{j0.4\sin\omega}{1.04-0.4\cos\omega}$$

$$\therefore \; \angle H(e^{j\omega}) = \tan^{-1}\left(\frac{H_i}{H_r}\right) = \tan^{-1}\left(\frac{0.4\sin\omega}{1.12-0.8\cos\omega}\right)$$

Chapter 5

Discrete Fourier Transform(DFT) and Fast Fourier Transform(FFT)

5.1 Introduction

The ***Discrete Time Fourier Transform*** (DTFT) discussed in Chapter 4, provides a method to represent a discrete time signal in frequency domain and to perform frequency analysis of discrete time signal.

The drawback in DTFT is that the frequency domain representation of a discrete time signal obtained using DTFT will be a continuous function of w and so it cannot be processed by digital system. The discrete Fourier transform (DFT) has been developed to convert a continuous function of w to a discrete function of w, so that frequency analysis of discrete time signals can be performed on a digital system.

Basically, the DFT of a discrete time signal is obtained by sampling the DTFT of the signal at N uniform frequency intervals and the number of samples (i.e., value of N) should be sufficient to avoid aliasing of frequency spectrum. The samples of DTFT are represented as a function of integer k, and so the DFT is a sequence consisting of N complex numbers represented as X(k) for $k = 0,1,2,3, ... (N - 1)$.

Since X(k) is a sequence consisting of complex numbers, the magnitude and phase of each sample can be computed and listed as magnitude sequence and phase sequence respectively. The graphical plots of magnitude and phase as a function of k are also drawn.

The plot of magnitude versus k is called ***magnitude spectrum*** and the plot of phase versus k is called ***phase spectrum***. In general, these plots are called ***frequency spectrum***.

The drawback in ***DFT*** is that the computation of each sample of DFT involves a large number of calculations and when large number of samples are required, the number of calculations will further increase. In order to overcome this drawback, a number of methods or algorithms have been developed to reduce the number of calculations. The various methods developed to compute DFT with reduced number of calculations are collectively called ***Fast Fourier Transform*** (FFT).

5.2 Discrete Fourier Transform (DFT) of Discrete Time Signal

5.2.1 Development of DFT from DTFT

The frequency domain representation of a discrete time signal obtained using discrete time Fourier transform (DTFT) will be a continuous and periodic function of ω, with periodicity of 2π. In order to obtain discrete function of ω, the DTFT can be sampled at sufficient number of frequency intervals.

Let $X(e^{j\omega})$ be discrete time Fourier transform of the discrete time signal x(n). The discrete Fourier transform (DFT) of x(n) is obtained by sampling one period of the discrete time Fourier transform $X(e^{j\omega})$ at a finite number of frequency points.

The frequency domain sampling is conventionally performed at N equally spaced frequency points in the period, $0 \le \omega \le 2\pi$. The sampling frequency points are denoted as ω_k and they are given by,

$$\omega_k = \frac{2\pi k}{N} \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, N-1$$

Now, the DFT is a sequence consisting of N-samples of DTFT. Let the samples are denoted by X(k) for k = 0, 1, 2, ... N-1. Therefore, the sampling of $X(e^{j\omega})$ is mathematically expressed as,

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, N-1 \qquad \ldots\ldots(5.1)$$

The DFT sequence starts at k = 0, corresponding to $\omega = 0$ but does not include k = N, corresponding to $\omega = 2\pi$, (since the sample at $\omega = 0$ is same as the sample at $\omega = 2\pi$). Generally, the DFT is defined along with number of samples and is called ***N-point DFT***. The number of samples N for a finite duration sequence x(n) of length L should be such that, $N \ge L$, in order to avoid aliasing of frequency spectrum.

The sampling of Fourier transform of a sequence to get DFT is shown in example 5.1. To calculate DFT of a sequence it is not necessary to compute Fourier transform, since the DFT can be directly computed using the definition of DFT as given by equation (5.2).

5.2.2 Definition of Discrete Fourier Transform (DFT)

Let, x(n) = Discrete time signal of length L

X(k) = DFT of x(n)

Now, the ***N-point DFT*** of x(n), where $N \ge L$, is defined as,

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0,1,2,\ldots\ldots, N-1} \qquad \ldots\ldots(5.2)$$

Symbolically, the N-point DFT of x(n) can be expressed as,

$\mathcal{DFT}\{x(n)\}$

where, $\mathcal{DFT}$ is the operator that represents discrete Fourier transform.

$$\boxed{\therefore\ \mathcal{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, N-1}$$

Since X(k) is a sequence consisting of N-complex numbers for k = 0, 1, 2, ... N-1, the DFT of x(n) can be expressed as a sequence as shown below.

$$X(k) = \{X(0),\ X(1),\ X(2), \ldots\ldots\ldots\ldots\ldots\ X(N-1)\}$$

5.2.3 Frequency Spectrum Using DFT

The X(k) is a discrete function of frequency of discrete time signal w, and so it is also called discrete frequency spectrum (or signal spectrum) of the discrete time signal x(n).

The X(k) is a complex valued function of k and so it can be expressed in rectangular form as,

$$X(k) = X_r(k) + jX_i(k)$$

where, $X_r(k)$ = Real part of X(k)

$X_i(k)$ = Imaginary part of X(k)

Now the ***Magnitude function*** (or ***Magnitude spectrum***) |X(k)| is defined as,

$$\boxed{|X(k)|^2 = X(k)\,X^*(k) \quad \text{or} \quad |X(k)| = \sqrt{X(k)\,X^*(k)}}$$

where $X^*(k)$ is complex conjugate of X(k)

Alternatively, $|X(k)|^2 = X(k)\;X^*(k) = \left[X_r(k) + jX_i(k)\right]\left[X_r(k) - jX_i(k)\right]$

$$= X_r^2(k) + X_i^2(k)$$

$$\therefore\ |X(k)| = \sqrt{X_r^2(k) + X_i^2(k)}$$

The ***Phase function*** (or ***Phase spectrum***) ÐX(k) is defined as,

$$\boxed{\angle X(k) = \text{Arg}[X(k)] = \tan^{-1}\left[\frac{X_i(k)}{X_r(k)}\right]}$$

Since X(k) is a sequence consisting of N-complex numbers for k = 0, 1, 2, ... N - 1, the magnitude and phase spectrum of X(k) can be expressed as a sequence as shown below.

Magnitude sequence, $|X(k)| = \left\{|X(0)|,\quad |X(1)|,\quad |X(2)|,................\quad |X(N-1)|\right\}$

Phase sequence, $\angle X(k) = \left\{\angle X(0),\quad \angle X(1),\quad \angle X(2),................\quad \angle X(N-1)\right\}$

The magnitude and phase sequence can be sketched graphically as a function of k.

The plot of samples of magnitude sequence versus k is called ***magnitude spectrum*** and the plot of samples of phase sequence versus k is called ***phase spectrum***. In general, these plots are called ***frequency spectrum***.

5.2.4 Inverse DFT

Let, x(n) = Discrete time signal

X(k) = N-point DFT of x(n)

The ***inverse DFT*** of the sequence X(k) of length N is defined as,

$$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,e^{\frac{j2\pi kn}{N}} \quad ; \quad \text{for } n = 0, 1, \ldots.., N-1} \qquad .....(5.3)$$

Symbolically the inverse DFT of x(n) can be expressed as,

$\mathcal{DFT}^{-1}\{X(k)\}$

where, $\mathcal{DFT}^{-1}$ is the operator that represents inverse DFT.

$$\boxed{\mathcal{DFT}^{-1}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,e^{\frac{j2\pi kn}{N}} \quad ; \quad \text{for } n = 0, 1, \ldots\ldots, N-1}$$

We also refer to x(n) and X(k) as a DFT pair and this relation is expressed as,

$$\boxed{x(n) \underset{\mathcal{DFT}^{-1}}{\overset{\mathcal{DFT}}{\longleftrightarrow}} X(k)}$$

5.3 Properties of DFT

1. Linearity

The linearity property of DFT states that the DFT of a linear weighted combination of two or more signals is equal to similar linear weighted combination of the DFT of individual signals.

Let, $\mathcal{DFT}\{x_1(n)\} = X_1(k)$ and $\mathcal{DFT}\{x_2(n)\} = X_2(k)$. Then by linearity property,

$\mathcal{DFT}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(k) + a_2 X_2(k)$, where a_1 and a_2 are constants.

Proof :

By definition of discrete Fourier transform,

$$X_1(k) = \mathcal{DFT}\{x_1(n)\} = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi kn}{N}} \quad .....(5.4)$$

$$X_2(k) = \mathcal{DFT}\{x_2(n)\} = \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi kn}{N}} \quad .....(5.5)$$

$$\mathcal{DFT}\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=0}^{N-1} \left[a_1 x_1(n) + a_2 x_2(n)\right] e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} \left[a_1 x_1(n) e^{\frac{-j2\pi kn}{N}} + a_2 x_2(n)\, e^{\frac{-j2\pi kn}{N}}\right]$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi kn}{N}} + a_2 \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$= a_1 X_1(k) + a_2 X_2(k) \qquad \boxed{\text{Using equations (5.4) and (5.5).}}$$

2. Periodicity

If a sequence x(n) is periodic with periodicity of N samples then N-point DFT, X(k) is also periodic with a periodicity of N samples.

Hence, if x(n) and X(k) are N point DFT pair then,

$x(n + N) = x(n)$; for all n

$X(k + N) = X(k)$; for all k

Proof :

By definition of DFT, the $(k + N)^{th}$ coefficient of X(k) is given by,

$$X(k + N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n\,(k+N)}{N}} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n\, k}{N}}\, e^{\frac{-j2\pi n\, N}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n\, k}{N}}\, e^{-j2\pi n} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n\, k}{N}} \qquad \boxed{\text{for integer n, } e^{-j2\pi n} = 1}$$

$$= X(k) \qquad \boxed{\text{Using definition of DFT.}}$$

3. Circular time shift

The circular time shift property of DFT says that if a discrete time signal is circularly shifted in time by m units then its DFT is multiplied by $e^{\frac{-j2\pi km}{N}}$.

$$\text{i.e., if, } \mathcal{DFT}\{x(n)\} = X(k), \text{ then } \mathcal{DFT}\{x((n-m))_N\} = X(k)\, e^{-j\frac{2\pi km}{N}}$$

Proof :

$$\mathcal{DFT}\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi k(p+m)}{N}} \qquad \boxed{\text{Let, } p = n-m, \ \therefore\ n = p+m}$$

$$= \sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi kp}{N}}\, e^{\frac{-j2\pi km}{N}}$$

$$= \left[\sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi kp}{N}}\right] e^{\frac{-j2\pi km}{N}}$$

$$= X(k)\, e^{\frac{-j2\pi km}{N}} \qquad \boxed{\text{Using definition of DFT.}}$$

4. Time reversal

The time reversal property of DFT says that reversing the N-point sequence in time is equivalent to reversing the DFT sequence.

$$\text{i.e., if, } \mathcal{DFT}\{x(n)\} = X(k), \text{ then } \mathcal{DFT}\{x(N-n)\} = X(N-k).$$

Proof :

$$\mathcal{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi k(N-m)}{N}} \qquad \boxed{\text{Let, } m = N-n, \ \therefore\ n = N-m}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi kN}{N}}\, e^{\frac{j2\pi km}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{-j2\pi k} \qquad \boxed{\text{Since k is an integer, } e^{-j2pk} = 1.}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{-j2\pi m} \qquad \boxed{\text{Since m is an integer, } e^{-j2pm} = 1.}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{\frac{-j2\pi mN}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi m(N-k)}{N}}$$

$$= X(N-k) \qquad \boxed{\text{Using definition of DFT.}}$$

5. Conjugation

Let x(n) be a complex N-point discrete sequence and $x^*(n)$ be its conjugate sequence.

Now if, $\mathcal{DFT}\{x(n)\} = X(k)$, then $\mathcal{DFT}\{x^*(n)\} = X^*(N-k)$.

Proof :

$$\mathcal{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{\frac{-j2\pi kn}{N}} = \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\, e^{-j2\pi n}\right]^* = \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\, e^{\frac{-j2\pi nN}{N}}\right]^* \qquad \boxed{e^{-j2pn} = 1}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(N-k)}{N}}\right]^* = [X(N-k)]^* = X^*(N-k) \qquad \boxed{\text{Using definition of DFT.}}$$

6. Circular frequency shift

The circular frequency shift property of DFT says that if a discrete time signal is multiplied by $e^{\frac{j2\pi mn}{N}}$ its DFT is circularly shifted by m units.

$$\text{i.e., if, } \mathcal{DFT}\{x(n)\} = X(k) \quad \text{then } \mathcal{DFT}\left\{x(n)\, e^{\frac{j2\pi mn}{N}}\right\} = X((k-m))_N$$

Proof :

$$\mathcal{DFT}\left\{x(n)\, e^{\frac{j2\pi mn}{N}}\right\} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi mn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi(k-m)n}{N}}$$

$$= X((k-m))_N \qquad \boxed{\text{Using definition of DFT.}}$$

7. Multiplication

The multiplication property of DFT says that the DFT of product of two discrete time sequences is equivalent to circular convolution of the DFTs of the individual sequences scaled by a factor 1/N.

$$\text{i.e., if, } \mathcal{DFT}\{x(n)\} = X(k), \text{ then } \mathcal{DFT}\{x_1(n)\, x_2(n)\} = \frac{1}{N}\left[X_1(k) * X_2(k)\right]$$

Proof :

By definition of inverse DFT,
$$x_1(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, e^{\frac{j2\pi kn}{N}} = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\, e^{\frac{j2\pi mn}{N}} \qquad \boxed{\text{Let, } k=m} \quad .....(5.6)$$

By definition of DFT,

$$\mathcal{DFT}\{x_1(n)\, x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)\, x_2(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\, e^{\frac{j2\pi mn}{N}}\right] x_2(n)\, e^{\frac{-j2\pi kn}{N}} \qquad \boxed{\text{Using equation (5.6).}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi kn}{N}}\, e^{\frac{j2\pi mn}{N}}\right] \qquad \boxed{\text{Rearranging the order of summation.}}$$

$$\boxed{\text{Using definition of DFT.}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi(k-m)n}{N}}\right] = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\; X_2((k-m))_N$$

$$= \frac{1}{N}\left[X_1(k) \circledast X_2(k)\right] \qquad \boxed{\text{Using definition of circular convolution.}}$$

8. Circular convolution

The circular convolution of two N-point sequences $x_1(n)$ and $x_2(n)$ is defined as,

$$x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

Refer equation (2.57) of Chapter 2.

The convolution property of DFT says that, the DFT of circular convolution of two sequences is equivalent to product of their individual DFTs.

Let, $\mathcal{DFT}\{x_1(n)\} = X_1(k)$ and $\mathcal{DFT}\{x_2(n)\} = X_2(k)$, then by convolution property,

$$\mathcal{DFT}\{x_1(n) \circledast x_2(n)\} = X_1(k)\, X_2(k)$$

Proof :

Let, $x_1(n)$ and $x_2(n)$ be N-point sequences. Now by definition of DFT,

Let, n = m

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi nk}{N}} = \sum_{n=0}^{N-1} x_1(m)\, e^{\frac{-j2\pi nk}{N}} \quad ; \quad k = 0, 1, 2, \ldots\ldots N-1 \qquad \ldots\ldots(5.7)$$

Let, n = p

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi nk}{N}} = \sum_{p=0}^{N-1} x_2(p)\, e^{\frac{-j2\pi pk}{N}} \quad ; \quad k = 0, 1, 2, \ldots\ldots N-1 \qquad \ldots\ldots(5.8)$$

Consider the product $X_1(k)\, X_2(k)$. The inverse DFT of the product is given by,

$$\mathcal{DFT}^{-1}\{X_1(k)\, X_2(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2(k)\, e^{\frac{j2\pi nk}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x_1(m)\, e^{\frac{-j2\pi mk}{N}}\right]\left[\sum_{p=0}^{N-1} x_2(p)\, e^{\frac{-j2\pi pk}{N}}\right] e^{\frac{j2\pi nk}{N}}$$

Using equations (5.7) and (5.8).

$$= \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{p=0}^{N-1} x_2(p) \sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}} \qquad \ldots\ldots(5.9)$$

Consider the summation $\sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}}$ in equation (5.9).

Let, $n - m - p = qN$, where q is an integer.

Since q is an integer, $e^{j2pq} = 1$.

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}} = \sum_{k=0}^{N-1} e^{\frac{j2\pi kqN}{N}} = \sum_{k=0}^{N-1} \left(e^{j2\pi q}\right)^k = \sum_{k=0}^{N-1} 1^k = N \qquad \ldots\ldots(5.10)$$

Consider the summation $\sum_{p=0}^{N-1} x_2(p)$ in equation (5.9).

Since, $n - m - p = qN$, $p = n - m - qN$

$$\therefore \sum_{p=0}^{N-1} x_2(p) = \sum_{m=0}^{N-1} x_2(n-m-qN) = \sum_{m=0}^{N-1} x_2(n-m,\ \text{mod } N) = \sum_{m=0}^{N-1} x_2((n-m))_N \qquad \ldots\ldots(5.11)$$

Using equations (5.10) and (5.11), the equation (5.9) can be written as shown below.

$$\mathcal{DFT}^{-1}\{X_1(k)\, X_2(k)\} = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{m=0}^{N-1} x_2((n-m))_N\, N = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

$$= x_1(n) \circledast x_2(n)$$

Using definition of circular convolution.

$$\therefore X_1(k)\, X_2(k) = \mathcal{DFT}\{x_1(n) \circledast x_2(n)\}$$

9. Circular correlation

The circular correlation of two sequences x(n) and y(n) is defined as,

$$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N$$

Refer equation (2.70) of Chapter 2.

Let, $\mathcal{DFT}\{x(n)\} = X(k)$ and $\mathcal{DFT}\{y(n)\} = Y(k)$, then by correlation property,

$$\mathcal{DFT}\{\bar{r}_{xy}(m)\} = \mathcal{DFT}\left\{\sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N\right\} = X(k)\, Y^*(k)$$

Proof :

Let, x(n) and y(n) be N - point sequences. Now by definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} \quad ; \quad k = 0, 1, 2, \ldots\ldots N-1 \qquad \ldots\ldots(5.12)$$

Let, n = p

$$Y(k) = \sum_{n=0}^{N-1} y(n)\, e^{\frac{-j2\pi nk}{N}} = \sum_{p=0}^{N-1} y(p)\, e^{\frac{-j2\pi pk}{N}} \quad ; \quad k = 0, 1, 2, \ldots\ldots N-1 \qquad \ldots\ldots(5.13)$$

Consider the product $X(k)Y^*(k)$. The inverse DFT of the product is given by,

Let, n = m

$$\mathcal{DFT}^{-1}\{X(k)\, Y^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, Y^*(k)\, e^{\frac{j2\pi nk}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, Y^*(k)\, e^{\frac{j2\pi mk}{N}}$$

Using equations (5.12) and (5.13).

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}\right]\left[\sum_{p=0}^{N-1} y(p)\, e^{\frac{-j2\pi pk}{N}}\right]^* e^{\frac{j2\pi mk}{N}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n) \sum_{p=0}^{N-1} y^*(p) \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n+p)}{N}} \qquad \ldots\ldots(5.14)$$

Consider the summation $\sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n+p)}{N}}$ in equation (5.14).

Let, $m - n + p = qN$, where q is an integer.

Since q is an integer, $e^{j2pq} = 1$.

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n+p)}{N}} = \sum_{k=0}^{N-1} e^{\frac{j2\pi kqN}{N}} = \sum_{k=0}^{N-1}\left(e^{j2\pi q}\right)^k = \sum_{k=0}^{N-1} 1^k = N \qquad \ldots\ldots(5.15)$$

Consider the summation $\sum_{p=0}^{N-1} y^*(p)$ in equation (5.14).

.....(5.16)

Since, $m - n + p = qN$, $p = n - m + qN$

$$\therefore \sum_{p=0}^{N-1} y^*(p) = \sum_{n=0}^{N-1} y^*(n-m+qN) = \sum_{n=0}^{N-1} y^*(n-m,\ \text{mod } N) = \sum_{n=0}^{N-1} y^*((n-m))_N$$

Using equations (5.15) and (5.16), the equation (5.14) can be written as shown below.

$$\mathcal{DFT}^{-1}\{X(k)\, Y^*(k)\} = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \sum_{n=0}^{N-1} y^*((n-m))_N\, N$$

$$= \sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N = \bar{r}_{xy}(m)$$

$$\therefore X(k)\, Y^*(k) = \mathcal{DFT}\left\{\bar{r}_{xy}(m)\right\}$$

Using definition of circular correlation.

10. Parseval's relation

Let $\mathcal{DFT}\{x_1(n)\} = X_1(k)$ and $\mathcal{DFT}\{x_2(n)\} = X_2(k)$. Then by Parseval's relation,

$$\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2^*(k)$$

Proof :

Let, $x_1(n)$ and $x_2(n)$ be N-point sequences.

Now by definition of DFT, $X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi nk}{N}}$ .....(5.17)

Now by definition of inverse DFT, $x_2(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_2(k)\, e^{\frac{j2\pi nk}{N}}$ .....(5.18)

Consider the right-hand side term of Parseval's relation. Using equation (5.17).

$$\frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2^*(k) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi nk}{N}}\right] X_2^*(k)$$

$$= \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X_2^*(k)\, e^{\frac{-j2\pi nk}{N}}\right] = \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X_2(k)\, e^{\frac{j2\pi nk}{N}}\right]^*$$

$$= \sum_{n=0}^{N-1} x_1(n)\, x_2^*(n)$$ Using equation (5.18).

5.4 Relation Between DFT and Ƶ-Transform

The Ƶ-transform of N-point sequence x(n) is given by,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

Let us evaluate X(z) at N equally spaced points on unit circle, i.e., at $z = e^{\frac{j2\pi k}{N}}$

Note : Since, $\left|e^{\frac{j2\pi k}{N}}\right| = 1$ *and* $\angle e^{\frac{j2\pi k}{N}} = \frac{2\pi k}{N}$,

the term, $z = e^{\frac{j2\pi k}{N}}$, *for* $k = 0, 1, 2, 3 ..... N-1$
represents N equally spaced points on unit circle in z-plane.

$$\therefore\ X(z)\Big|_{z = e^{\frac{j2\pi k}{N}}} = \sum_{n=0}^{N-1} x(n) z^{-n}\Big|_{z = e^{\frac{j2\pi k}{N}}} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad .....(5.19)$$

By the definition of N-point DFT we get,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad .....(5.20)$$

From equations (5.19) and (5.20) we can say that,

$$X(k) = X(z)\Big|_{z=e^{\frac{j2\pi k}{N}}} \qquad .....(5.21)$$

From equation (5.21), we can conclude that the N-point DFT of a finite duration sequence can be obtained from the Ƶ-transform of the sequence, by evaluating the Ƶ-transform of the sequence at N equally spaced points around the unit circle. Since the evaluation is performed on unit circle the ROC of X(z) should include unit circle.

Table 5.1 : Properties of Discrete Fourier Transform (DFT)

Note : $X(k) = \mathcal{DFT}\{x(n)\}$; $X_1(k) = \mathcal{DFT}\{x_1(n)\}$; $X_2(k) = \mathcal{DFT}\{x_2(n)\}$; $Y(k) = \mathcal{DFT}\{y(n)\}$

Property	Discrete time signal	Discrete Fourier Transform
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Periodicity	$x(n + N) = x(n)$	$X(k + N) = X(k)$
Circular time shift	$x((n - m))_N$	$X(k)e^{\frac{-j2\pi km}{N}}$
Time reversal	$x(N - n)$	$X(N - k)$
Conjugation	$x^*(n)$	$X^*(N - k)$
Circular frequency shift	$x(n)\,e^{\frac{j2\pi mn}{N}}$	$X((k - m))_N$
Multiplication	$x_1(n)\,x_2(n)$	$\frac{1}{N}\left[X_1(k) \circledast X_2(k)\right]$
Circular convolution	$x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)\,x_2((n-m))_N$	$X_1(k)\,X_2(k)$
Circular correlation	$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\,y^*((n-m))_N$	$X(k)\,Y^*(k)$
Symmetry of real signals	x(n) is real	$X(k) = X^*(N-k)$ $X_r(k) = X_r(N-k)$ $X_i(k) = -X_i(N-k)$ $\lvert X(k)\rvert = \lvert X(N-k)\rvert$ $\angle X(k) = -\angle X(N-k)$
Symmetry of real and even signal	x(n) is real and even $x(n) = x(N - n)$	$X(k) = X_r(k)$ and $X_i(k) = 0$
Symmetry of real and odd signal	x(n) is real and odd $x(n) = -x(N - n)$	$X(k) = jX_i(k)$ and $X_r(k) = 0$
Parseval's relation	$\sum_{n=0}^{N-1} x_1(n)\,x_2^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\,X_2^*(k)$

5.5 Analysis of LTI Discrete Time Systems Using DFT

In Chapter 4, Section 4.6, it is shown that Fourier transform is an useful tool for the analysis of discrete time systems in frequency domain. But the drawback in Fourier transform is that it is a continuous function of w and so it will not be useful for digital processing of signals and systems. Hence DFT is proposed, therefore the analysis of discrete time systems in frequency domain can be conveniently performed using DFT for digital processing of signals and systems.

Discrete Frequency Spectrum

In general the DFT of a signal gives the discrete frequency spectrum of a signal.

Let x(n) and X(k) be a DFT pair.

Now, $X(k)$ = Discrete frequency spectrum of discrete time signal.

$|X(k)|$ = Magnitude spectrum of discrete time signal.

$\angle X(k)$ = Phase spectrum of discrete time signal.

In particular, the DFT of impulse response, h(n) of a discrete time system gives discrete frequency response or frequency spectrum of the discrete time system.

Let h(n) and H(k) be a DFT pair.

Now, $H(k)$ = Discrete frequency spectrum of discrete time system.

$|H(k)|$ = Magnitude spectrum of discrete time system.

$\angle H(k)$ = Phase spectrum of discrete time system.

Response of LTI Discrete Time System Using DFT

The response of an LTI discrete time system is given by linear convolution of input and impulse response of the system.

Let, $x(n)$ = Input to an LTI system

$h(n)$ = Impulse response of the LTI system

Now, the response or output of the system y(n) is given by,

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

$$\text{where,} \quad x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) \qquad .....(5.22)$$

The DFT supports only circular convolution and so, the linear convolution of equation (5.22) has to be computed via circular convolution. If x(n) is N_1-point sequence and h(n) is N_2-point sequence then linear convolution x(n) and h(n) will generate y(n) of size $N_1 + N_2 - 1$. Therefore in order to perform linear convolution via circular convolution the x(n) and h(n) should be converted to $N_1 + N_2 - 1$ point sequences by appending zeros. Now the circular convolution of $N_1 + N_2 - 1$ point sequences x(n) and h(n) will give same result as that of linear convolution.

Let, x(n) be N_1-point sequence and h(n) be N_2-point sequence.

Let us convert x(n) and h(n) to N_1+N_2-1 point sequences.

Let, $Y(k) = N_1 + N_2 - 1$ point DFT of y(n)

$X(k) = N_1 + N_2 - 1$ point DFT of x(n)

$H(k) = N_1 + N_2 - 1$ point DFT of h(n)

Now by circular convolution theorem of DFT,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\, H(k)$$

On taking inverse DFT of the above equation we get,

$$x(n) \circledast h(n) = \mathcal{DFT}^{-1}\{X(k)\, H(k)\}$$

Since, $x(n) \circledast h(n) = y(n)$, the above equation can be written as,

$$y(n) = \mathcal{DFT}^{-1}\{X(k)\, H(k)\} \qquad .....(5.23)$$

From the equation (5.23), we can say that the output y(n) is given by the inverse DFT of the product of X(k) and H(k). Hence to determine the response of an LTI discrete time system, first find $N_1 + N_2 - 1$ point DFT of input x(n) to get X(k) and $N_1 + N_2 - 1$ point DFT of impulse response h(n) to get H(k), then take inverse DFT of the product X(k) H(k).

Example 5.1

Compute 4-point DFT and 8-point DFT of causal three sample sequence given by,

$$x(n) = \frac{1}{3} \quad ; \quad 0 \le n \le 2$$
$$= 0 \quad ; \quad \text{else}$$

Show that DFT coefficients are samples of Fourier transform of x(n), (Refer example 4.6 of Chapter 4 for Fourier transform).

Solution

By the definition of N-point DFT, the k[th] complex coefficient of X(k), for 0 £ k £ N – 1, is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

a) 4-point DFT (\ N = 4)

$$X(k) = \sum_{n=0}^{4-1} x(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{2} x(n)\, e^{\frac{-j\pi kn}{2}} = x(0)\, e^{0} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{-j\pi k}$$

$\boxed{e^{\pm jq} = \cos q \pm j\sin q}$

$$= \frac{1}{3} + \frac{1}{3} e^{\frac{-j\pi k}{2}} + \frac{1}{3} e^{-j\pi k} = \frac{1}{3}\left[1 + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + \cos \pi k - j\sin \pi k\right]$$

For 4-point DFT, X(k) has to be evaluated for k = 0, 1, 2, 3.

$$\text{When } k = 0\ ;\ X(0) = \frac{1}{3}[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0]$$
$$= \frac{1}{3}(1 + 1 - j0 + 1 - j0) = 1 = 1\angle 0$$
$$\text{When } k = 1\ ;\ X(1) = \frac{1}{3}\left[1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos \pi - j\sin \pi\right]$$
$$= \frac{1}{3}(1 + 0 - j - 1 - j0) = -j\frac{1}{3} = \frac{1}{3}\angle -\pi/2 = 0.333\angle -0.5\pi$$
$$\text{When } k = 2\ ;\ X(2) = \frac{1}{3}[1 + \cos \pi - j\sin \pi + \cos 2\pi - j\sin 2\pi]$$
$$= \frac{1}{3}(1 - 1 - j0 + 1 - j0) = \frac{1}{3} = 0.333\angle 0$$
$$\text{When } k = 3\ ;\ X(3) = \frac{1}{3}\left[1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi\right]$$
$$= \frac{1}{3}(1 + 0 + j - 1 - j0) = j\frac{1}{3} = \frac{1}{3}\angle \pi/2 = 0.333\angle 0.5\pi$$

\ The 4-point DFT sequence X(k) is given by,

$$X(k) = \{\ 1\angle 0,\quad 0.333\angle -0.5\pi,\quad 0.333\angle 0,\quad 0.333\angle 0.5\pi\ \}$$
$$\therefore \text{ Magnitude Function, } |X(k)| = \{\ 1,\quad 0.333,\quad 0.333,\quad 0.333\ \}$$
$$\text{Phase Function, } \angle X(k) = \{0,\quad -0.5\pi,\quad 0,\quad 0.5\pi\ \}$$

Phase angles are in radians.

b) 8-point DFT (\ N = 8)

$$X(k) = \sum_{n=0}^{8-1} x(n)\, e^{\frac{-j2\pi kn}{8}} = \sum_{n=0}^{2} x(n)\, e^{\frac{-j\pi kn}{4}} = x(0)\, e^{0} + x(1)\, e^{\frac{-j\pi k}{4}} + x(2)\, e^{\frac{-j\pi k}{2}}$$

$\boxed{e^{\pm j q} = \cos q \pm j\sin q}$

$$= \frac{1}{3} + \frac{1}{3} e^{\frac{-j\pi k}{4}} + \frac{1}{3} e^{\frac{-j\pi k}{2}} = \frac{1}{3}\left[1 + \cos\frac{\pi k}{4} - j\sin\frac{\pi k}{4} + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right]$$

For 8-point DFT, X(k) has to be evaluated for k = 0, 1, 2, 3, 4, 5, 6, 7.

When k = 0 ; $X(0) = \frac{1}{3}[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0]$

$= \frac{1}{3}(1 + 1 - j0 + 1 - j0) = 1 = 1\angle 0$

When k = 1 ; $X(1) = \frac{1}{3}\left[1 + \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right]$

$= 0.333\ (1 + 0.707 - j0.707 + 0 - j1)$

$= 0.568 - j0.568 = 0.803\angle -0.785 = 0.803\angle -0.25\pi$

$\boxed{\frac{0.785}{\pi} \times \pi = 0.25\pi}$

When k = 2 ; $X(2) = \frac{1}{3}\left[1 + \cos\frac{2\pi}{4} - j\sin\frac{2\pi}{4} + \cos\frac{2\pi}{2} - j\sin\frac{2\pi}{2}\right]$

$= 0.333\ (1 + 0 - j1 - 1 - j0)$

$= -\,j0.333 = 0.333\angle -\pi/2 = 0.333\angle -0.5\pi$

When k = 3 ; $X(3) = \frac{1}{3}\left[1 + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$

$= 0.333\ (1 - 0.707 - j0.707 + 0 + j1)$

$= 0.098 + j0.098 = 0.139\angle 0.785 = 0.139\angle 0.25\pi$

When k = 4 ; $X(4) = \frac{1}{3}\left[1 + \cos\frac{4\pi}{4} - j\sin\frac{4\pi}{4} + \cos\frac{4\pi}{2} - j\sin\frac{4\pi}{2}\right]$

$= 0.333\ (1 - 1 - j0 + 1 - j0) = 0.333 = 0.333\angle 0$

When k = 5 ; $X(5) = \frac{1}{3}\left[1 + \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} + \cos\frac{5\pi}{2} - j\sin\frac{5\pi}{2}\right]$

$= 0.333\ (1 - 0.707 + j0.707 + 0 - j1)$

$= 0.098 - j0.098 = 0.139\angle -0.785 = 0.139\angle -0.25\pi$

When k = 6 ; $X(6) = \frac{1}{3}\left[1 + \cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4} + \cos\frac{6\pi}{2} - j\sin\frac{6\pi}{2}\right]$

$= 0.333\ (1 + 0 + j1 - 1 - j0)$

$= j0.333 = 0.333\angle \pi/2 = 0.333\angle 0.5\pi$

When k = 7 ; $X(7) = \frac{1}{3}\left[1 + \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4} + \cos\frac{7\pi}{2} - j\sin\frac{7\pi}{2}\right]$

$= 0.333\ (1 + 0.707 + j0.707 + 0 + j1)$

$= 0.568 + j0.568 = 0.803\angle 0.785 = 0.803\angle 0.25\pi$

Phase angles are in radians.

\ The 8-point DFT sequence X(k) is given by,

$X(k) = \{1\angle 0,\ 0.803\angle -0.25\pi,\ 0.333\angle -0.5\pi,\ 0.139\angle 0.25\pi,\ 0.333\angle 0,\ 0.139\angle -0.25\pi,$
$0.333\angle 0.5\pi,\ 0.803\angle 0.25\pi\}$

$\therefore$ Magnitude Function, $|X(k)| = \{\ 1,\ 0.803,\ 0.333,\ 0.139,\ 0.333,\ 0.139,\ 0.333,\ 0.803\ \}$

Phase Function, $\angle X(k) = \{\ 0,\ -0.25\pi,\ -0.5\pi,\ 0.25\pi,\ 0,\ -0.25\pi,\ 0.5\pi,\ 0.25\pi\ \}$

The magnitude spectrum of X(k) are shown in fig 1, 2 and 3 for N = 4, N = 8, and N = 16 respectively. The curve shown in dotted line is the sketch of magnitude function of $X(e^{jw})$ for w in the range 0 to 2p. Here it is observed that the magnitude of DFT coefficients are samples of magnitude function of $X(e^{jw})$, (Refer example 4.6 for the magnitude function of $X(e^{jw})$).

The phase spectrum of X(k) are shown in fig 4, 5 and 6 for N = 4, N = 8, and N = 16 respectively. The curve shown in dotted line is the sketch of phase function of $X(e^{jw})$ for w in the range 0 to 2p. Here it is observed that the phase of the DFT coefficients are samples of phase function of $X(e^{jw})$, (Refer example 4.6 for the phase function of $X(e^{jw})$).

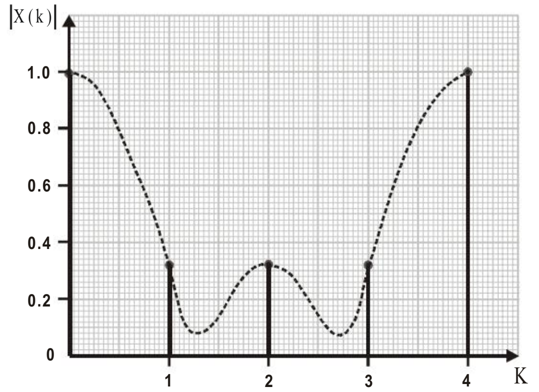


Fig 1 : Magnitude spectrum of X(k) for N=4.

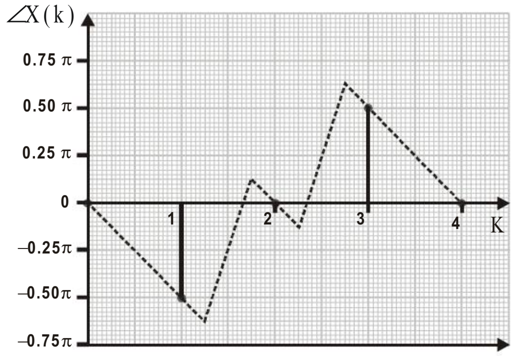


Fig 4 : Phase spectrum of X(k) for N=4.

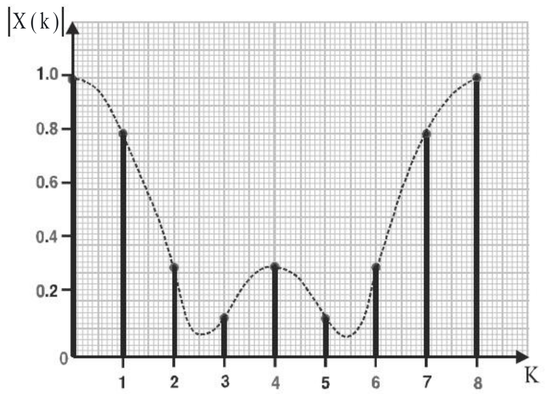


Fig 2 : Magnitude spectrum of X(k) for N=8.

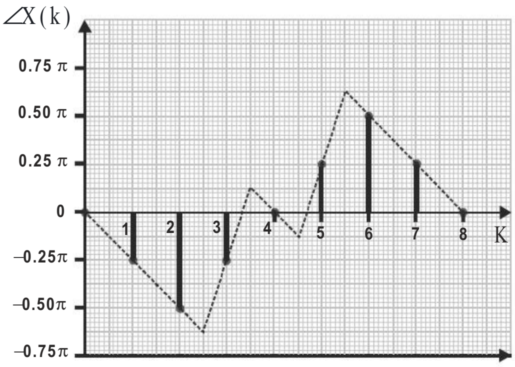


Fig 5 : Phase spectrum of X(k) for N=8.

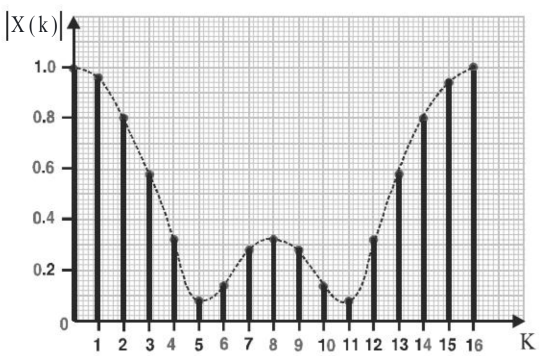


Fig 3 : Magnitude spectrum of X(k) for N=16.

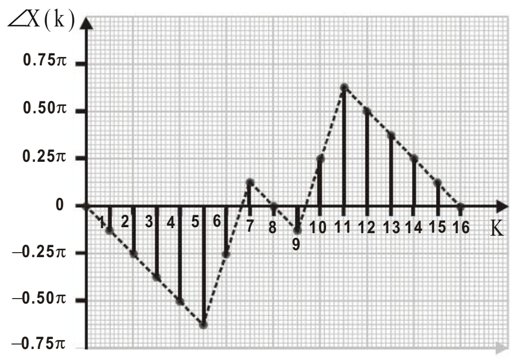


Fig 6 : Phase spectrum of X(k) for N=16.

Example 5.2

Compute the DFT of the sequence, x(n) = {0, 1, 2, 1}. Sketch the magnitude and phase spectrum.

Solution

The given signal x(n) is 4-point signal and so, let us compute 4-point DFT.

By the definition of DFT, the 4-point DFT is given by, $\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

$$X(k) = \sum_{n=0}^{4-1} x(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi kn}{2}}$$

$$= x(0)\,e^{0} + x(1)\,e^{\frac{-j\pi k}{2}} + x(2)\,e^{-j\pi k} + x(3)\,e^{\frac{-j3\pi k}{2}} = 0 + e^{\frac{-j\pi k}{2}} + 2\,e^{-j\pi k} + e^{\frac{-j3\pi k}{2}}$$

$$= \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + 2(\cos \pi k - j\sin \pi k) + \cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}$$

$$= \left(\cos\frac{\pi k}{2} + 2\cos\pi k + \cos\frac{3\pi k}{2}\right) - j\left(\sin\frac{\pi k}{2} + \sin\frac{3\pi k}{2}\right) \qquad \boxed{\sin\pi k = 0 \text{ for integer } k}$$

When k = 0 ; $X(0) = (\cos 0 + 2\cos 0 + \cos 0) - j(\sin 0 + \sin 0)$

$= (1 + 2 + 1) - j(0 + 0) = 4 = 4\angle 0$

When k = 1; $X(1) = \left(\cos\frac{\pi}{2} + 2\cos\pi + \cos\frac{3\pi}{2}\right) - j\left(\sin\frac{\pi}{2} + \sin\frac{3\pi}{2}\right)$

$= (0 - 2 + 0) - j(1 - 1) = -2 = 2\angle 180^\circ = 2\angle\pi$

When k = 2; $X(2) = (\cos\pi + 2\cos 2\pi + \cos 3\pi) - j(\sin\pi + \sin 3\pi)$

$= (-1 + 2 - 1) - j(0 + 0) = 0$

When k = 3; $X(3) = \left(\cos\frac{3\pi}{2} + 2\cos 3\pi + \cos\frac{9\pi}{2}\right) - j\left(\sin\frac{3\pi}{2} + \sin\frac{9\pi}{2}\right)$

$= (0 - 2 + 0) - j(-1 + 1) = -2 = 2\angle 180^\circ = 2\angle\pi$

$\therefore X(k) = \{4\angle 0,\ 2\angle\pi,\ 0,\ 2\angle\pi\}$

Magnitude Spectrum, $|X(k)| = \{4,\ 2,\ 0,\ 2\}$

Phase Spectrum, $\angle X(k) = \{0,\ \pi,\ 0,\ \pi\}$

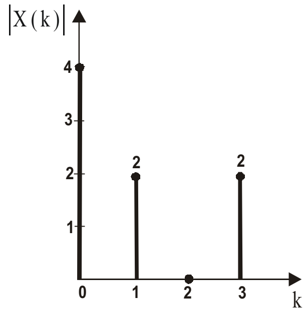


Fig 1 : Magnitude Spectrum.

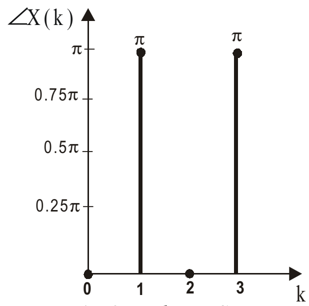


Fig 2 : Phase Spectrum.

Example 5.3

Compute circular convolution of the following two sequences using DFT.

$x_1(n) = \{ \underline{0}, 1, 0, 1 \}$ and $x_2(n) = \{ \underline{1}, 2, 1, 2 \}$

Solution

Given that, $x_1(n) = \{ 0, 1, 0, 1 \}$. The 4-point DFT of $x_1(n)$ is,

$$\mathcal{DFT}\{x_1(n)\} = X_1(k) = \sum_{n=0}^{4-1} x_1(n)\, e^{\frac{-j2\pi nk}{4}} = \sum_{n=0}^{3} x_1(n)\, e^{\frac{-j\pi nk}{2}} \quad ; \quad k = 0, 1, 2, 3$$

$$= x_1(0)\, e^0 + x_1(1)\, e^{-j\frac{\pi k}{2}} + x_1(2)\, e^{-j\pi k} + x_1(3)\, e^{-j\frac{3\pi k}{2}}$$

$$= 0 + e^{-j\frac{\pi k}{2}} + 0 + e^{-j\frac{3\pi k}{2}} = e^{-j\frac{\pi k}{2}} + e^{-j\frac{3\pi k}{2}}$$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

When $k = 0$; $X_1(0) = e^0 + e^0 = 1 + 1 = 2$

When $k = 1$; $X_1(1) = e^{\frac{-j\pi}{2}} + e^{\frac{-j3\pi}{2}} = -j + j = 0$

When $k = 2$; $X_1(2) = e^{-j\pi} + e^{-j3\pi} = -1 - 1 = -2$

When $k = 3$; $X_1(3) = e^{\frac{-j3\pi}{2}} + e^{\frac{-j9\pi}{2}} = j - j = 0$

Given that, $x_2(n) = \{ 1, 2, 1, 2 \}$. The 4-point DFT of $x_2(n)$ is,

$$\mathcal{DFT}\{x_2(n)\} = X_2(k) = \sum_{n=0}^{4-1} x_2(n)\, e^{\frac{-j2\pi nk}{4}} = \sum_{n=0}^{3} x_2(n)\, e^{\frac{-j\pi nk}{2}} \quad ; \quad k = 0, 1, 2, 3$$

$$= x_2(0)\, e^0 + x_2(1)\, e^{\frac{-j\pi k}{2}} + x_2(2)\, e^{-j\pi k} + x_2(3)\, e^{\frac{-j3\pi k}{2}}$$

$$= 1 + 2\, e^{\frac{-j\pi k}{2}} + e^{-j\pi k} + 2\, e^{\frac{-j3\pi k}{2}}$$

When $k = 0$; $X_2(0) = 1 + 2e^0 + e^0 + 2e^0 = 1 + 2 + 1 + 2 = 6$

When $k = 1$; $X_2(1) = 1 + 2\, e^{\frac{-j\pi}{2}} + e^{-j\pi} + 2\, e^{\frac{-j3\pi}{2}} = 1 - 2j - 1 + 2j = 0$

When $k = 2$; $X_2(2) = 1 + 2\, e^{-j\pi} + e^{-j2\pi} + 2\, e^{-j3\pi} = 1 - 2 + 1 - 2 = -2$

When $k = 3$; $X_2(3) = 1 + 2\, e^{\frac{-j3\pi}{2}} + e^{-j3\pi} + 2\, e^{\frac{-j9\pi}{2}} = 1 + 2j - 1 - 2j = 0$

$$X_1(k) = \begin{cases} 2 \; ; & k = 0 \\ 0 \; ; & k = 1 \\ -2 \; ; & k = 2 \\ 0 \; ; & k = 3 \end{cases} \qquad X_2(k) = \begin{cases} 6 \; ; & k = 0 \\ 0 \; ; & k = 1 \\ -2 \; ; & k = 2 \\ 0 \; ; & k = 3 \end{cases}$$

Let, $X_3(k)$ be the product of $X_1(k)$ and $X_2(k)$.

$\therefore X_3(k) = X_1(k)\, X_2(k)$

When $k = 0$; $X_3(0) = X_1(0) \times X_2(0) = 2 \times 6 = 12$

When $k = 1$; $X_3(1) = X_1(1) \times X_2(1) = 0 \times 0 = 0$

When $k = 2$; $X_3(2) = X_1(2) \times X_2(2) = -2 \times -2 = 4$

When $k = 3$; $X_3(3) = X_1(3) \times X_2(3) = 0 \times 0 = 0$

$$\therefore X_3(k) = \{12, \; 0, \; 4, \; 0\}$$

By circular convolution theorem of DFT, we get,

$$\mathcal{DFT}\{x_1(n) \circledast x_2(n)\} = X_1(k)\,X_2(k) \quad \Rightarrow \quad x_1(n) \circledast x_2(n) = \mathcal{DFT}^{-1}\{X_1(k)\,X_2(k)\} = \mathcal{DFT}^{-1}\{X_3(k)\}$$

Let $x_3(n)$ be the 4-point sequence obtained by taking inverse DFT of $X_3(k)$.

$$\mathcal{DFT}^{-1}\{X_3(k)\} = x_3(n) = \frac{1}{4}\sum_{k=0}^{4-1} X_3(k)\, e^{\frac{j2\pi nk}{4}} = \frac{1}{4}\sum_{k=0}^{3} X_3(k)\, e^{\frac{j\pi nk}{2}} \quad ; \quad n = 0, 1, 2, 3$$

$$= \frac{1}{4}\left[X_3(0)\, e^0 + X_3(1)\, e^{\frac{j\pi n}{2}} + X_3(2)\, e^{j\pi n} + X_3(3)\, e^{\frac{j3\pi n}{2}}\right]$$

$\boxed{\begin{array}{l}\sin \pi n = 0 \\ \text{for integer n}\end{array}}$

$$= \frac{1}{4}\left[12 + 0 + 4e^{j\pi n} + 0\right] = 3 + e^{j\pi n} = 3 + \cos \pi n + j \sin \pi n = 3 + \cos \pi n$$

When $n = 0$; $x_3(0) = 3 + \cos 0 = 3 + 1 = 4$

When $n = 1$; $x_3(1) = 3 + \cos \pi = 3 - 1 = 2$

When $n = 2$; $x_3(2) = 3 + \cos 2\pi = 3 + 1 = 4$

When $n = 3$; $x_3(3) = 3 + \cos 3\pi = 3 - 1 = 2$

$$\therefore \; x_1(n) \circledast x_2(n) = x_3(n) = \{\underset{\uparrow}{4}, 2, 4, 2\}$$

Example 5.4

Compute linear and circular convolution of the following two sequences using DFT.

$$x(n) = \{\underset{\uparrow}{1}, 2\} \quad \text{and} \quad h(n) = \{\underset{\uparrow}{2}, 1\}$$

Solution

Linear Convolution by DFT

The linear convolution of x(n) and h(n) will produce a 3 sample sequence. To avoid time aliasing let us convert the 2 sample input sequences into 3-sample sequences by padding with zeros.

$$\therefore x(n) = \{\underset{\uparrow}{1}, 2, 0\} \quad \text{and} \quad h(n) = \{\underset{\uparrow}{2}, 1, 0\}$$

By the definition of N-point DFT, the three point DFT of x(n) is,

$$X(k) = \sum_{n=0}^{3-1} x(n)\, e^{\frac{-j2\pi kn}{3}} = x(0)\, e^0 + x(1)\, e^{\frac{-j2\pi k}{3}} + x(2)\, e^{\frac{-j4\pi k}{3}} = 1 + 2\, e^{\frac{-j2\pi k}{3}}$$

When $k = 0$; $X(0) = 1 + 2e^0 = 1 + 2 = 3$

When $k = 1$; $X(1) = 1 + 2\, e^{\frac{-j2\pi}{3}} = 1 + 2(-0.5 - j0.866) = -j1.732$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

When $k = 2$; $X(2) = 1 + 2\, e^{\frac{-j4\pi}{3}} = 1 + 2(-0.5 + j0.866) = j1.732$

By the definition of N-point DFT, the three point DFT of h(n) is,

$$H(k) = \sum_{n=0}^{3-1} h(n)\, e^{\frac{-j2\pi kn}{3}} = h(0)\, e^0 + h(1)\, e^{\frac{-j2\pi k}{3}} + h(2)\, e^{\frac{-j4\pi k}{3}} = 2 + e^{\frac{-j2\pi k}{3}}$$

When $k = 0$; $H(0) = 2 + e^0 = 2 + 1 = 3$

When $k = 1$; $H(1) = 2 + e^{\frac{-j2\pi}{3}} = 2 - 0.5 - j0.866 = 1.5 - j0.866$

When $k = 2$; $H(2) = 2 + e^{\frac{-j4\pi}{3}} = 2 - 0.5 + j0.866 = 1.5 + j0.866$

Let, Y(k) = X(k) H(k) ; for k = 0, 1, 2

When k = 0 ; Y(0) = X(0) H(0) = 3 ´ 3 = 9

When k = 1 ; Y(1) = X(1) H(1) = (– j1.732) ´ (1.5–j0.866) = –1.5 – j2.598

When k = 2 ; Y(2) = X(2) H(2) = (j1.732) ´ (1.5+j0.866) = –1.5 + j2.598

\ Y(k) = {9, –1.5 – j2.598, –1.5 + j2.598}

The sequence y(n) is obtained from inverse DFT of Y(k). By definition of inverse DFT,

$$y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{\frac{j2\pi kn}{N}} \quad ; \quad \text{for } n = 0, 1, 2, \ldots\ldots, N-1$$

$$\therefore\ y(n) = \frac{1}{3}\sum_{k=0}^{2} Y(k)\, e^{\frac{j2\pi kn}{3}}$$

$$= \frac{1}{3}\left[Y(0)\, e^{0} + Y(1)\, e^{\frac{j2\pi n}{3}} + Y(2)\, e^{\frac{j4\pi n}{3}}\right] \quad ; \quad \text{for } n = 0, 1, 2$$

$$= \frac{1}{3}\left[9 + (-1.5 - j2.598)\, e^{\frac{j2\pi n}{3}} + (-1.5 + j2.598)\, e^{\frac{j4\pi n}{3}}\right]$$

$$= 3 + (-0.5 - j0.866)\, e^{\frac{j2\pi n}{3}} + (-0.5 + j0.866)\, e^{\frac{j4\pi n}{3}}$$

When $n = 0$; $y(0) = 3 + (-0.5 - j0.866)e^{0} + (-0.5 + j0.866)e^{0}$

$= 3 - 0.5 - j0.866 - 0.5 + j0.866 = 2$

e±jq = cosq ± jsinq

When $n = 1$; $y(1) = 3 + (-0.5 - j0.866)e^{\frac{j2\pi}{3}} + (-0.5 + j0.866)e^{\frac{j4\pi}{3}}$

$= 3 + (-0.5 - j0.866)(-0.5 + j0.866) + (-0.5 + j0.866)(-0.5 - j0.866)$

$= 3 + (0.5^2 + 0.866^2) + (0.5^2 + 0.866^2) = 3 + 1 + 1 = 5$

When $n = 2$; $y(2) = 3 + (-0.5 - j0.866)e^{\frac{j4\pi}{3}} + (-0.5 + j0.866)e^{\frac{j8\pi}{3}}$

$= 3 + (-0.5 - j0.866)(-0.5 - j0.866) + (-0.5 + j0.866)(-0.5 + j0.866)$

$= 3 + (-0.5 - j0.866)^2 + (-0.5 + j0.866)^2$

$= 3 - 0.5 + j0.866 - 0.5 - j0.866 = 2$

$$\therefore\ x(n) * h(n) = y(n) = \{\underset{\uparrow}{2}, 5, 2\}$$

Circular Convolution by DFT

The given sequences are 2-point sequences. Hence 2-point DFT of the sequences are obtained as follows.

The 2-point DFT of x(n) is given by,

$$X(k) = \sum_{n=0}^{2-1} x(n)\, e^{\frac{-j2\pi kn}{2}} = x(0)\, e^{0} + x(1)\, e^{-j\pi k} = 1 + 2\, e^{-j\pi k} \quad ; \text{ for } k = 0,1$$

When k = 0; X(0) = 1 + 2 e⁰ = 1 + 2 = 3

When k = 1; X(1) = 1 + 2 e-jp = 1 – 2 = -1

$$\therefore\ X(k) = \{3, -1\}$$

The 2-point DFT of h(n) is given by,

$$H(k) = \sum_{n=0}^{2-1} h(n)\, e^{\frac{-j2\pi kn}{2}} = h(0)\, e^{0} + h(1)\, e^{-j\pi k} = 2 + e^{-j\pi k} \quad ; \quad \text{for } k = 0, 1$$

When k = 0; $H(0) = 2 + e^0 = 2+1=3$

When k = 1; $H(1) = 2 + e^{-j\pi} = 2 - 1 = 1$

$$\therefore \quad H(k) = \{\underset{-}{3}, 1\}$$

Let the product of X(k) and H(k) be equal to Y(k).

$$\therefore \; Y(k) = X(k)\,H(k) \quad ; \quad \text{for } k = 0,\ 1$$

When k = 0 ; $Y(0) = X(0)\,H(0) = 3\times 3 = 9$

When k = 1 ; $Y(1) = X(1)\,H(1) = -1\times 1 = -1$

$$\therefore \; Y(k) = \{\underset{\uparrow}{9}, \; -1\}$$

The sequence y(n) is obtained from inverse DFT of Y(k). By the definition of inverse DFT,

$$y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{\frac{j2\pi kn}{N}} \quad ; \text{ for } n = 0,\ 1,\ 2,\ \ldots\ldots,\ N-1$$

Here, N = 2

$$\therefore \quad y(n) = \frac{1}{2}\sum_{k=0}^{1} Y(k)\, e^{\frac{j2\pi kn}{2}} = \frac{1}{2}\left[Y(0) + Y(1)\, e^{j\pi n}\right] = \frac{1}{2}\left[9 - e^{j\pi n}\right] = 4.5 - 0.5\,e^{j\pi n}$$

When n = 0; $y(0) = 4.5 - 0.5\,e^0 = 4.5 - 0.5 = 4$

When n = 1; $y(1) = 4.5 - 0.5\,e^{j\pi} = 4.5 + 0.5 = 5$ $\boxed{e^{j\pi} = -1}$

$$\therefore \quad x(n) \circledast h(n) = y(n) = \{\underset{-}{4}, 5\}$$

5.6 Fast Fourier Transform (FFT)

The **F**ast **F**ourier **T**ransform (**FFT**) is a method (or algorithm) for computing the discrete Fourier transform (DFT) with reduced number of calculations. The computational efficiency is achieved if we adopt a divide and conquer approach. This approach is based on the decomposition of an N-point DFT into successively smaller DFTs. This basic approach leads to a family of an efficient computational algorithms known collectively as FFT algorithms.

Radix-r FFT

In an N-point sequence, if N can be expressed as $N = r^m$, then the sequence can be decimated into r-point sequences. For each r-point sequence, r-point DFT can be computed. From the results of r-point DFT, the r^2-point DFTs are computed. From the results of r^2-point DFTs, the r^3-point DFTs are computed and so on, until we get r^m point DFT. This FFT algorithm is called radix-**r** FFT. In computing N-point DFT by this method the number of stages of computation will be **m** times.

Radix-2 FFT

For radix-2 FFT, the value of N should be such that, $N = 2^m$, so that the N-point sequence is decimated into 2-point sequences and the 2-point DFT for each decimated sequence is computed. From the results of 2-point DFTs, the 4-point DFTs can be computed. From the results of 4-point DFTs, the 8-point DFTs can be computed and so on, until we get N-point DFT.

Number of Calculations in N-point DFT

Let, X(k) be N-point DFT of an L-point discrete time sequence x(n), where N ³ L. Now, the N-point DFT is a sequence consisting of N-complex numbers. Each complex number of the sequence is calculated using the following equation (equation 5.2).

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0,1,2,.......,N-1$$

$$= x(0)\,e^0 + x(1)\,e^{\frac{-j2\pi k}{N}} + x(2)\,e^{\frac{-j4\pi k}{N}} + x(3)\,e^{\frac{-j6\pi k}{N}} + ... + x(N-1)\,e^{\frac{-j2(N-1)\pi k}{N}}$$

$$\therefore\; X(k) = \underbrace{\underbrace{x(0)e^0}_{\substack{\text{Complex}\\\text{multiplication}}} + \underbrace{x(1)e^{\frac{-j2\pi k}{N}}}_{\substack{\text{Complex}\\\text{multiplication}}} + \underbrace{x(2)e^{\frac{-j4\pi k}{N}}}_{\substack{\text{Complex}\\\text{multiplication}}} + \underbrace{x(3)e^{\frac{-j6\pi k}{N}}}_{\substack{\text{Complex}\\\text{multiplication}}} + ... + \underbrace{x(N-1)e^{\frac{-j2(N-1)\pi k}{N}}}_{\substack{\text{Complex}\\\text{multiplication}}}}_{N-1 \text{ Complex additions}}$$

From the above equation we can say that,

The number of calculations to calculate X(k) for one value of k are,

N number of complex multiplications and

N – 1 number of complex additions.

The X(k) is a sequence consisting of N complex numbers.

Therefore, the number of calculations to calculate all the N complex numbers of the X(k) are,

N ´ N = $\mathbf{N^2}$ number of complex multiplications and

N ´ (N – 1) = N(N – 1) number of complex additions.

Hence, in direct computation of N-point DFT, the total number of complex additions are N(N – 1) and total number of complex multiplications are N^2.

Number of Calculations in Radix-2 FFT

In radix-2 FFT, $N = 2^m$, and so there will be m stages of computations, where $m = \log_2 N$, with each stage having N/2 butterflies. (Refer section 5.7.2 and 5.8.2).

The number of calculations in one butterfly are,

1 number of Complex multiplication and

2 number of Complex additions.

There are $\frac{N}{2}$ butterflies in each stage.

Therefore, number of calculations in one stage are,

$\mathbf{\frac{N}{2} \times 1 = \frac{N}{2}}$ complex multiplications and

$\mathbf{\frac{N}{2} \times 2 = N}$ complex additions.

The N-point DFT involves m stages of computations. Therefore, the number of calculations for m stages are,

$\mathbf{m \times \frac{N}{2} = \log_2 N \times \frac{N}{2} = \frac{N}{2} \log_2 N}$ complex multiplications and

$\mathbf{m \times N = \log_2 N \times N = N \log_2 N}$ complex additions.

Hence, in radix-2 FFT, the total number of complex additions are reduced to $N\log_2 N$ and total number of complex multiplications are reduced to $(N/2) \log_2 N$.

The table 5.2 presents a comparison of the number of complex multiplications and additions in radix-2 FFT and in direct computation of DFT. From the table it can be observed that for large values of N, the percentage reduction in calculations is also very large.

$\boxed{\log_2 2^m = m}$ $\boxed{\log_y x = \dfrac{\log_{10} x}{\log_{10} y}}$

Table 5.2 : Comparison of Number of Computation in Direct DFT and FFT

Number of points N	Direct Computation		Radix-2 FFT	
	Complex additions N(N–1)	**Complex Multiplications N^2**	**Complex additions $N\log_2 N$**	**Complex Multiplications $(N/2)\log_2 N$**
4 (= 2^2)	12	16	4 ´ $\log_2 2^2$=4 ´ 2 = 8	$\frac{4}{2} \times \log_2 2^2 = \frac{4}{2} \times 2 = 4$
8 (= 2^3)	56	64	8 ´ $\log_2 2^3$ = 8 ´ 3 = 24	$\frac{8}{2} \times \log_2 2^3 = \frac{8}{2} \times 3 = 12$
16 (= 2^4)	240	256	16 ´ $\log_2 2^4$ = 16 ´ 4 = 64	$\frac{16}{2} \times \log_2 2^4 = \frac{16}{2} \times 4 = 32$
32 (= 2^5)	992	1,024	32 ´ $\log_2 2^5$ = 32 ´ 5 = 160	$\frac{32}{2} \times \log_2 2^5 = \frac{32}{2} \times 5 = 80$
64 (= 2^6)	4,032	4,096	64 ´ $\log_2 2^6$ = 64 ´ 6 = 384	$\frac{64}{2} \times \log_2 2^6 = \frac{64}{2} \times 6 = 192$
128 (= 2^7)	16,256	16,384	128 ´ $\log_2 2^7$ = 128 ´ 7 = 896	$\frac{128}{2} \times \log_2 2^7 = \frac{128}{2} \times 7 = 448$

Phase or Twiddle Factor

By the definition of DFT, the N-point DFT is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} \quad ; \text{ for } k = 0, 1, 2, \ldots\ldots, N-1 \qquad \ldots\ldots(5.24)$$

To simplify the notation it is desirable to define the complex valued phase factor W_N (also called as twiddle factor) which is an N^{th} root of unity as,

$$W_N = e^{\frac{-j2\pi}{N}}$$

Here, W represents a complex number 1Ð –2p. Hence the phase or argument of W is –2p. Therefore, when a number is multiplied by W, only its phase changes by –2p but magnitude remains same.

$$\boxed{\therefore\ W = e^{-j2\pi}}$$

The phase value –2p of W can be multiplied by any integer and it is represented as prefix in W. For example multiplying –2p by k can be represented as W^k.

$$\boxed{\therefore\ e^{-j2\pi \times k} \quad \Rightarrow \quad W^k}$$

The phase value –2p of W can be divided by any integer and it is represented as suffix in W. For example dividing –2p by N can be represented as W_N.

$$\therefore\ e^{-j2\pi \div N} = e^{-j2\pi \times \frac{1}{N}} \Rightarrow W_N$$

$$\therefore\ e^{\frac{-j2\pi nk}{N}} = \left(e^{-j2\pi}\right)^{\frac{nk}{N}} = W_N^{nk} \qquad .....(5.25)$$

Using equation (5.25) the equation (5.24) can be written as,

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad ; \text{ for } k = 0, 1, 2, ..., N-1} \qquad .....(5.26)$$

The equation (5.26) is the definition of N-point DFT using phase factor, and this equation is popularly used in FFT.

5.7 Decimation in Time (DIT) Radix-2 FFT

The N-point DFT of a sequence x(n) converts the time domain N-point sequence x(n) to a frequency domain N-point sequence X(k). In **D**ecimation **I**n **T**ime (DIT) algorithm, the time domain sequence x(n) is decimated and smaller point DFTs are performed. The results of smaller point DFTs are combined to get the result of N-point DFT.

In DIT radix-2 FFT, the time domain sequence is decimated into 2-point sequences. For each two point sequence, the two point DFT is computed. The results of 2-point DFTs are used to compute 4-point DFTs. A pair of 2-point DFT results are used to compute one 4-point DFT. The results of 4-point DFTs are used to compute 8-point DFTs. A pair of 4-point DFT results are used to compute one 8-point DFT. This process is continued until we get N-point DFT.

In general we can say that, in decimation in time algorithm, the N-point DFT can be realized from two numbers of N/2 point DFTs, the N/2 point DFT can be realized from two numbers of N/4 point DFTs, and so on.

Let, x(n) be N-sample sequence. We can decimate x(n) into two sequences of N/2 samples. Let the two sequences be $f_1(n)$ and $f_2(n)$. Let $f_1(n)$ consists of even numbered samples of x(n) and $f_2(n)$ consists of odd numbered samples of x(n).

$$\therefore\ f_1(n) = x(2n) \quad ; \text{ for } n = 0, 1, 2, 3\ ....., \frac{N}{2}-1$$

$$f_2(n) = x(2n+1) \quad ; \text{ for } n = 0, 1, 2, 3\ ....., \frac{N}{2}-1$$

Let, X(k) = N-point DFT of x(n)

$F_1(k)$ = N/2 point DFT of $f_1(n)$

$F_2(k)$ = N/2 point DFT of $f_2(n)$

By definition of DFT the N/2 point DFT of $f_1(n)$ and $f_2(n)$ are given by,

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{N/2}^{kn} \quad ; \quad F_2(k) = \sum_{n=0}^{\frac{N}{2}-1} f_2(n)\, W_{N/2}^{kn}$$

Now, N-point DFT X(k), in terms of N/2 point DFTs $F_1(k)$ and $F_2(k)$ is given by,

$$X(k) = F_1(k) + W_N^k\ F_2(k), \quad \text{where, } k = 0, 1, 2, ..., N-1 \qquad .....(5.27)$$

The proof of equation (5.27) is given below.

Proof :

when $n \rightarrow 2n$, even numbered samples of x(n) are selected. when $n \rightarrow 2n+1$, odd numbered samples of x(n) are selected.

By definition of DFT, the N-point DFT of x(n) is,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \sum_{n=\text{even}} x(n)\, W_N^{kn} + \sum_{n=\text{odd}} x(n)\, W_N^{kn} \quad ; \; k = 0, 1, 2 \,.....,N-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{k(2n)} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_N^{k(2n+1)} \qquad .....(5.28)$$

The phase factors in equation (5.28) can be modified as shown below.

$$W_N^{k(2n)} = \left(e^{-j2\pi}\right)^{\frac{k(2n)}{N}} = \left(e^{-j2\pi}\right)^{\frac{kn}{N/2}} = W_{N/2}^{kn} \qquad .....(5.29)$$

$$W_N^{k(2n+1)} = \left(e^{-j2\pi}\right)^{\frac{k(2n+1)}{N}} = \left(e^{-j2\pi}\right)^{\frac{k2n}{N}} \left(e^{-j2\pi}\right)^{\frac{k}{N}} = \left(e^{-j2\pi}\right)^{\frac{kn}{N/2}} \left(e^{-j2\pi}\right)^{\frac{k}{N}} = W_{N/2}^{kn}\, W_N^{k} \qquad .....(5.30)$$

Using equations (5.29) and (5.30), the equation (5.28) can be written as,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_{N/2}^{kn}\, W_N^{k}$$

$x(2n) = f_1(n)$ and $x(2n+1) = f_2(n)$

$$= \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{N/2}^{kn} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} f_2(n)\, W_{N/2}^{kn} \qquad .....(5.31)$$

By definition of DFT the N/2 point DFT of $f_1(n)$ and $f_2(n)$ are given by,

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{N/2}^{kn} \quad \text{and} \quad F_2(n) = \sum_{n=0}^{\frac{N}{2}-1} f_2(n)\, W_{N/2}^{kn} \qquad .....(5.32)$$

Using equation (5.32) in equation(5.31) we get,

$$X(k) = F_1(k) + W_N^{k}\, F_2(k)\,, \qquad \text{where } k = 0, 1, 2, ..., N-1$$

Having performed the decimation in time once, we can repeat the process for each of the sequences $f_1(n)$ and $f_2(n)$. Thus $f_1(n)$ would result in the two N/4 point sequences and $f_2(n)$ would result in another two N/4 point sequences.

Let the decimated N/4 point sequences of $f_1(n)$ be $v_{11}(n)$ and $v_{12}(n)$.

$$\therefore\; v_{11}(n) = f_1(2n) \quad ; \; \text{for } n = 0, 1, 2, ..., \frac{N}{4}-1$$

$$v_{12}(n) = f_1(2n+1) \quad ; \; \text{for } n = 0, 1, 2, ..., \frac{N}{4}-1$$

Let the decimated N/4 point sequences of $f_2(n)$ be $v_{21}(n)$ and $v_{22}(n)$.

$$\therefore\; v_{21}(n) = f_2(2n) \quad ; \; \text{for } n = 0, 1, 2\,, ..., \frac{N}{4}-1$$

$$v_{22}(n) = f_2(2n+1) \quad ; \; \text{for } n = 0, 1, 2, ..., \frac{N}{4}-1$$

Let, $V_{11}(k)$ = N/4 point DFT of $v_{11}(n)$; $\quad V_{21}(k)$ = N/4 point DFT of $v_{21}(n)$

$V_{12}(k)$ = N/4 point DFT of $v_{12}(n)$; $\quad V_{22}(k)$ = N/4 point DFT of $v_{22}(n)$

Then like earlier analysis we can show that,

$$F_1(k) = V_{11}(k) + W_{N/2}^{k} V_{12}(k) \;;\; \text{for } k = 0, 1, 2, \ldots, \frac{N}{2} - 1 \qquad \ldots(5.33)$$

$$F_2(k) = V_{21}(k) + W_{N/2}^{k} V_{22}(k) \;;\; \text{for } k = 0, 1, 2, \ldots, \frac{N}{2} - 1 \qquad \ldots(5.34)$$

Hence the N/2 point DFTs are obtained from the results of N/4 point DFTs.

The decimation of the data sequence can be repeated again and again until the resulting sequences are reduced to 2-point sequences.

5.7.1 8-Point DFT Using Radix-2 DIT FFT

The input sequence is 8-point sequence. Therefore, $N = 8 = 2^3 = r^m$. Here, $r = 2$ and $m = 3$.

Therefore, the computation of 8-point DFT using radix-2 FFT, involves three stages of computation. The given 8-point sequence is decimated to 2-point sequences. For each 2-point sequence, the 2-point DFT is computed. From the results of 2-point DFT, the 4-point DFT can be computed. From the results of 4-point DFT, the 8-point DFT can be computed.

Let the given sequence be x(0), x(1), x(2), x(3), x(4),x(5), x(6), x(7), which consists of 8 samples. The 8-samples should be decimated into sequences of 2-samples. Before decimation they are arranged in bit reversed order, as shown in table 5.3.

Table 5.3

Normal order		Bit reversed order	
x(0)	x(000)	x(0)	x(000)
x(1)	x(001)	x(4)	x(100)
x(2)	x(010)	x(2)	x(010)
x(3)	x(011)	x(6)	x(110)
x(4)	x(100)	x(1)	x(001)
x(5)	x(101)	x(5)	x(101)
x(6)	x(110)	x(3)	x(011)
x(7)	x(111)	x(7)	x(111)

The x(n) in bit reversed order is decimated into 4 numbers of 2-point sequences as shown below.

Sequence-1 : {x(0), x(4)}

Sequence-2 : {x(2), x(6)}

Sequence-3 : {x(1), x(5)}

Sequence-4 : {x(3), x(7)}

Using the decimated sequences as input the 8-point DFT is computed.The fig 5.1 shows the three stages of computation of an 8-point DFT.

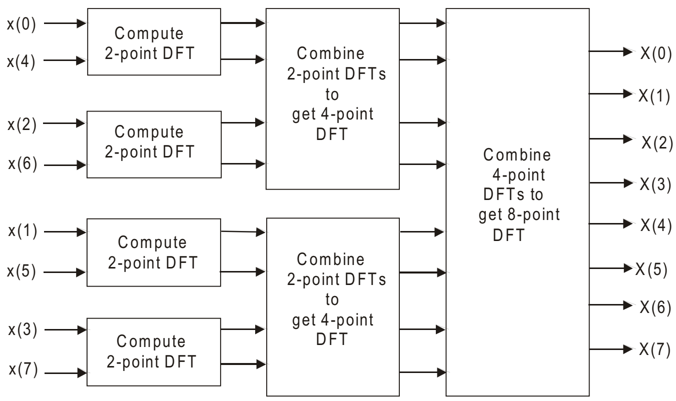


Fig 5.1. Three stages of computations in 8-point DFT.

Let us examine the 8-point DFT of an 8-point sequence in detail. The 8-point sequence is decimated into 4-point sequences and 2-point sequences as shown below.

Let $x(n)$ = 8-point sequence

$f_1(n), f_2(n)$ = 4-point sequences obtained from $x(n)$

$v_{11}(n), v_{12}(n)$ = 2-point sequences obtained from $f_1(n)$

$v_{21}(n), v_{22}(n)$ = 2-point sequences obtained from $f_2(n)$.

The relations between the samples of various sequences are given below.

$v_{11}(0) = f_1(0) = x(0)$	$v_{21}(0) = f_2(0) = x(1)$
$v_{11}(1) = f_1(2) = x(4)$	$v_{21}(1) = f_2(2) = x(5)$
$v_{12}(0) = f_1(1) = x(2)$	$v_{22}(0) = f_2(1) = x(3)$
$v_{12}(1) = f_1(3) = x(6)$	$v_{22}(1) = f_2(3) = x(7)$

First Stage Computation

In the first stage of computation the two point DFTs of the 2-point sequences are computed.

Let, $V_{11}(k) = \mathcal{DFT}\{v_{11}(n)\}$.

Using equation (5.26), the 2-point DFT of $v_{11}(n)$ is given by,

$$V_{11}(k) = \sum_{n=0,1} v_{11}(n)\, W_2^{nk} \quad ; \text{for } k = 0, 1$$

When $k = 0$; $V_{11}(k) = V_{11}(0) = v_{11}(0)\, W_2^0 + v_{11}(1)\, W_2^0 = v_{11}(0) + v_{11}(1) = x(0) + x(4)$

When $k = 1$; $V_{11}(k) = V_{11}(1) = v_{11}(0)\, W_2^0 + v_{11}(1)\, W_2^1 = v_{11}(0) - W_2^0 v_{11}(1) = x(0) - W_2^0 x(4)$

$\boxed{W_2^0 = e^{\frac{j2\pi \times 0}{2}} = e^0 = 1}$ $\boxed{W_2^1 = e^{\frac{-j2\pi \times 1}{2}} = e^{-j\pi} = (\cos\pi - j\sin\pi) = -1 = -1 \times W_2^0 = -W_2^0}$

Let, $V_{12}(k) = \mathcal{DFT}\{v_{12}(n)\}$.

Using equation (5.26), the 2-point DFT of $v_{12}(n)$ is given by,

$$V_{12}(k) = \sum_{n=0,1} v_{12}(n)\, W_2^{nk} \quad ; \text{for } k = 0, 1$$

When $k = 0$; $V_{12}(k) = V_{12}(0) = v_{12}(0)\, W_2^0 + v_{12}(1)\, W_2^0 = v_{12}(0) + v_{12}(1) = x(2) + x(6)$

When $k = 1$; $V_{12}(k) = V_{12}(1) = v_{12}(0)\, W_2^0 + v_{12}(1)\, W_2^1 = v_{12}(0) - W_2^0 v_{12}(1) = x(2) - W_2^0 x(6)$

Let, $V_{21}(k) = \mathcal{DFT}\{v_{21}(n)\}$.

Using equation (5.26), the 2-point DFT of $v_{21}(n)$ is given by,

$$V_{21}(k) = \sum_{n=0,1} v_{21}(n)\, W_2^{nk} \quad ; \text{ for } k = 0, 1$$

When k = 0; $V_{21}(k) = V_{21}(0) = v_{21}(0)\, W_2^0 + v_{21}(1)\, W_2^0 = v_{21}(0) + v_{21}(1) \quad = x(1) + x(5)$

When k = 1; $V_{21}(k) = V_{21}(1) = v_{21}(0)\, W_2^0 + v_{21}(1)\, W_2^1 = v_{21}(0) - W_2^0 v_{21}(1) \; = x(1) - W_2^0 x(5)$

Let, $V_{22}(k) = \mathcal{DFT}\{v_{22}(n)\}$.

Using equation (5.26), the 2-point DFT of $v_{22}(n)$ is given by,

$$V_{22}(k) = \sum_{n=0,1} v_{22}(n)\, W_2^{nk} \quad ; \text{ for } k = 0, 1.$$

When k = 0; $V_{22}(k) = V_{22}(0) = v_{22}(0)\, W_2^0 + v_{22}(1)\, W_2^0 = v_{22}(0) + v_{22}(1) \quad = x(3) + x(7)$

When k = 1; $V_{22}(k) = V_{22}(1) = v_{22}(0)\, W_2^0 + v_{22}(1)\, W_2^1 = v_{22}(0) - W_2^0 v_{22}(1) = x(3) - W_2^0 x(7)$

Second Stage Computation

In the second stage of computation the 4-point DFTs are computed using the results of first stage as input. Let, $F_1(k) = \mathcal{DFT}\{f_1(n)\}$. The 4-point DFT of $f_1(n)$ can be computed using equation (5.33).

$$\therefore\; F_1(k) = V_{11}(k) + W_4^k\, V_{12}(k) \; ; \text{ for } k = 0, 1, 2, 3.$$

$V_{11}(k)$ and $V_{12}(k)$ are periodic with periodicity of 2 samples.
$\therefore V_{11}(k+2) = V_{11}(k)$
$V_{12}(k+2) = V_{12}(k)$

When k = 0; $F_1(k) = F_1(0) = V_{11}(0) + W_4^0\, V_{12}(0)$

When k = 1; $F_1(k) = F_1(1) = V_{11}(1) + W_4^1\, V_{12}(1)$

When k = 2; $F_1(k) = F_1(2) = V_{11}(2) + W_4^2\, V_{12}(2) = V_{11}(0) - W_4^0\, V_{12}(0)$

When k = 3; $F_1(k) = F_1(3) = V_{11}(3) + W_4^3\, V_{12}(3) = V_{11}(1) - W_4^1\, V_{12}(1)$

$$\boxed{W_4^2 = e^{-j2\pi \times \frac{2}{4}} = e^{-j\pi} = (\cos\pi - j\sin\pi) = -1 = -1 \times W_4^0 = -W_4^0}$$

$$\boxed{W_4^3 = e^{-j2\pi \times \frac{3}{4}} = e^{-j2\pi \times \frac{2}{4}}\, e^{-j2\pi \times \frac{1}{4}} = e^{-j\pi}\, e^{-j2\pi \times \frac{1}{4}} = (\cos\pi - j\sin\pi)\, W_4^1 = -1 \times W_4^1 = -W_4^1}$$

Let, $F_2(k) = \mathcal{DFT}\{f_2(n)\}$. The 4-point DFT of $f_2(n)$ can be computed using equation (5.34).

$$\therefore\; F_2(k) = V_{21}(k) + W_4^k\, V_{22}(k) \; ; \text{ for } k = 0, 1, 2, 3.$$

$V_{21}(k)$ and $V_{22}(k)$ are periodic with periodicity of 2 samples.
$\therefore V_{21}(k+2) = V_{21}(k)$
$V_{22}(k+2) = V_{22}(k)$

When k = 0; $F_2(k) = F_2(0) = V_{21}(0) + W_4^0\, V_{22}(0)$

When k = 1; $F_2(k) = F_2(1) = V_{21}(1) + W_4^1\, V_{22}(1)$

When k = 2; $F_2(k) = F_2(2) = V_{21}(2) + W_4^2\, V_{22}(2) = V_{21}(0) - W_4^0\, V_{22}(0)$

When k = 3; $F_2(k) = F_2(3) = V_{21}(3) + W_4^3\, V_{22}(3) = V_{21}(1) - W_4^1\, V_{22}(1)$

Third Stage Computation

In the third stage of computation the 8-point DFTs are computed using the results of second stage as inputs.

Let, $X(k) = \mathcal{DFT}\{X(n)\}$. The 8-point DFT of x(n) can be computed using equation (5.27).

$\therefore\ X(k) = F_1(k) + W_8^k F_2(k)$; for k = 0, 1, 2, 3, 4, 5, 6, 7

$F_1(k)$ and $F_2(k)$ are periodic with periodicity of 4 samples.
$\therefore F_1(k+4) = F_1(k)$
$F_2(k+4) = F_2(k)$

When k = 0; $X(k) = X(0) = F_1(0) + W_8^0 F_2(0)$

When k = 1; $X(k) = X(1) = F_1(1) + W_8^1 F_2(1)$

When k = 2; $X(k) = X(2) = F_1(2) + W_8^2 F_2(2)$

When k = 3; $X(k) = X(3) = F_1(3) + W_8^3 F_2(3)$

When k = 4; $X(k) = X(4) = F_1(4) + W_8^4 F_2(4) = F_1(0) - W_8^0 F_2(0)$

When k = 5; $X(k) = X(5) = F_1(5) + W_8^5 F_2(5) = F_1(1) - W_8^1 F_2(1)$

When k = 6; $X(k) = X(6) = F_1(6) + W_8^6 F_2(6) = F_1(2) - W_8^2 F_2(2)$

When k = 7; $X(k) = X(7) = F_1(7) + W_8^7 F_2(7) = F_1(3) - W_8^3 F_2(3)$

$$W_8^4 = e^{-j2\pi\times\frac{4}{8}} = e^{-j\pi} = (\cos\pi - j\sin\pi) = -1$$

$\boxed{W_8^4 = W_8^4 \times W_8^0 = -W_8^0}$ $\boxed{W_8^5 = W_8^4 \times W_8^1 = -W_8^1}$ $\boxed{W_8^6 = W_8^4 \times W_8^2 = -W_8^2}$ $\boxed{W_8^7 = W_8^4 \times W_8^3 = -W_8^3}$

5.7.2 Flow Graph for 8-Point DFT using Radix-2 DIT FFT

If we observe the basic computation performed at every stage of radix-2 DIT FFT in previous section, we can arrive at the following conclusion.

1. In each computation two complex numbers "a" and "b" are considered.
2. The complex number "b" is multiplied by a phase factor "W_N^k".
3. The product "bW_N^k" is added to complex number "a" to form new complex number "A".
4. The product "bW_N^k" is subtracted from complex number "a" to form new complex number "B".

The above basic computation can be expressed by a signal flow graph shown in Fig 5.2. (For detailed discussion on signal flow graph, refer Chapter 2, Section 2.6.2).

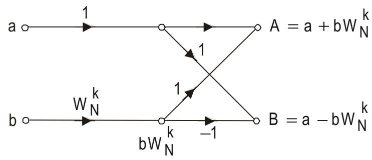


Fig 5.2 : Basic butterfly or flow graph of DIT radix-2 FFT.

The signal flow graph is also called ***butterfly diagram*** since it resembles a butterfly. In radix-2 FFT, N/2 butterflies per stage are required to represent the computational process. The butterfly diagram used to compute the 8-point DFT via radix-2 DIT FFT can be arrived as shown below, using the computations shown in previous section.

The sequence x(n) is arranged in bit reversed order and then decimated into two sample sequences as shown below.

x(0)	x(2)	x(1)	x(3)
x(4)	x(6)	x(5)	x(7)

Flow Graph or (Butterfly Diagram) for First Stage of Computation

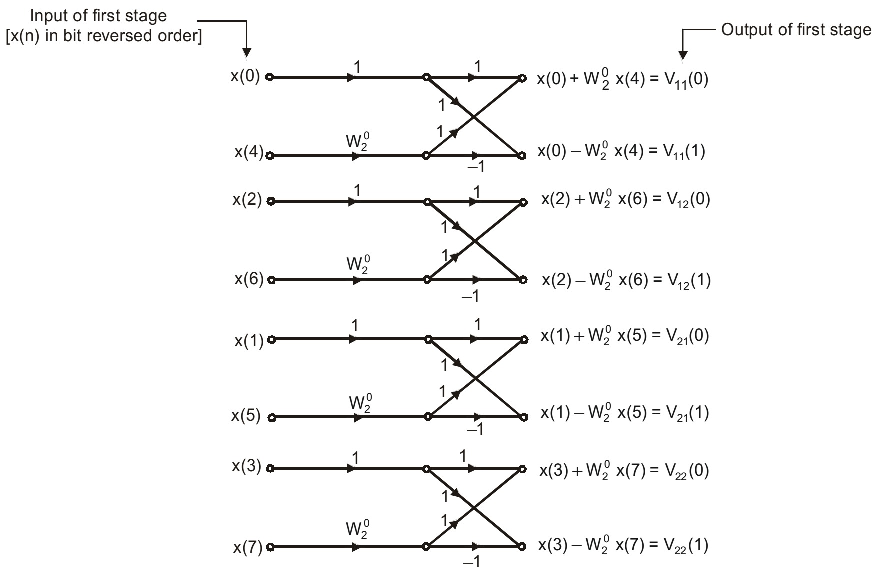


Fig 5.3 : *First stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIT FFT.*

Flow Graph (or Butterfly Diagram) for Second Stage of Computation

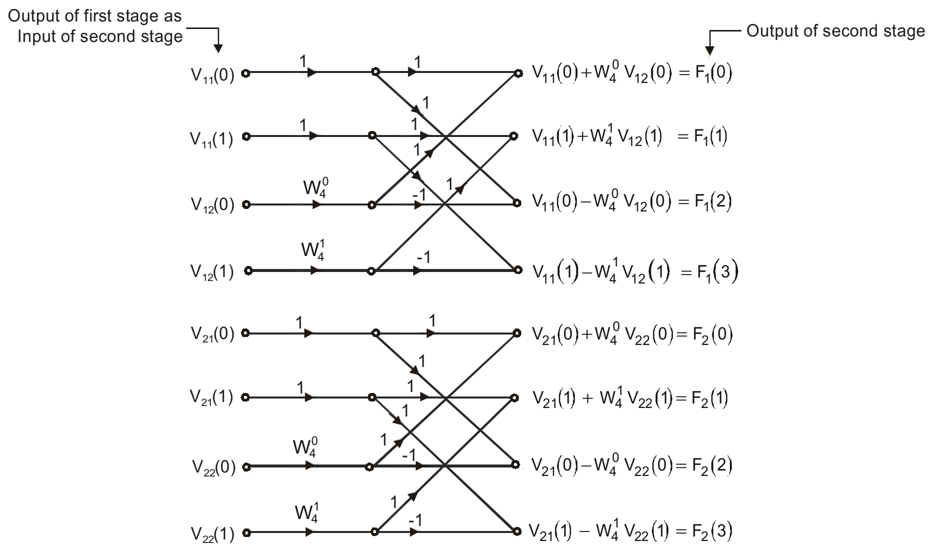


Fig 5.4 : *Second stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIT FFT.*

Flow Graph (or Butterfly Diagram) for Third Stage of Computation

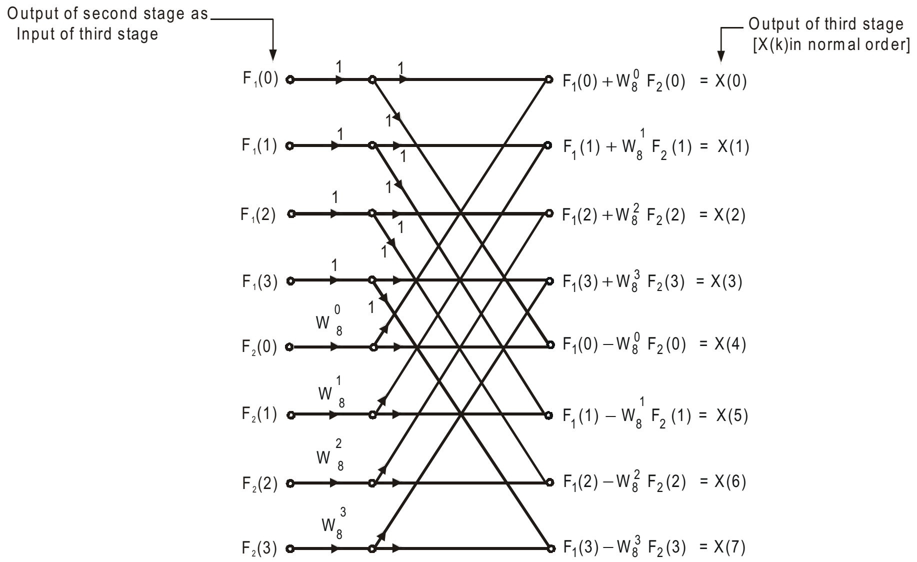


Fig 5.5 : Third stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIT FFT.

The Combined Flow Graph (or Butterfly Diagram) of All the Three Stages of Computation

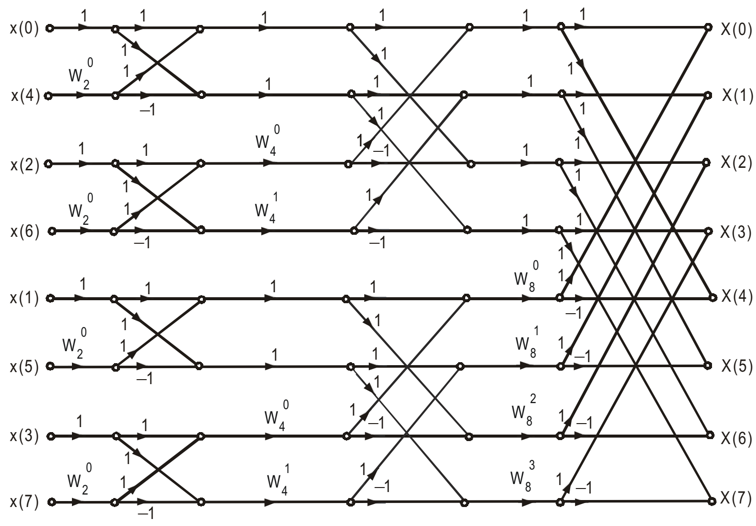


Fig 5.6 : The flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIT FFT.

5.8 Decimation in Frequency (DIF) Radix-2 FFT

In decimation in frequency algorithm the frequency domain sequence X(k) is decimated, (but in decimation in time algorithm, the time domain sequence x(n) is decimated).

In this algorithm, the N-point time domain sequence is converted to two numbers of N/2 point sequences. Then each N/2 point sequence is converted to two numbers of N/4 point sequences.

Thus we get 4 numbers of N/4 point sequences. This process is continued until we get N/2 numbers of 2-point sequences. Finally the 2-point DFT of each 2-point sequence is computed. The 2-point DFTs of N/2 numbers of 2-point sequences will give N samples, which is the N-point DFT of the time domain sequence.

Here the equations for forming N/2 point sequences, N/4 point sequences, etc., are obtained by decimation of frequency domain sequences. Hence this method is called DIF. For example the N-point frequency domain sequence X(k) can be decimated to two numbers of N/2 point frequency domain sequences $G_1(k)$ and $G_2(k)$. The $G_1(k)$ and $G_2(k)$ defines new time domain sequences $g_1(n)$ and $g_2(n)$ respectively, whose samples are obtained from x(n).

It can be shown that the N-point DFT of x(n) can be realized from two numbers of N/2 point DFTs. The N/2 point DFTs can be realized from two numbers of N/4 point DFTs and so on. The decimation continues upto 2-point DFTs.

Let x(n) and X(k) be N-point DFT pair.

Let $G_1(k)$ and $G_2(k)$ be two numbers of N/2 point sequences obtained by the decimation of X(k).

Let $G_1(k)$ be N/2 point DFT of $g_1(n)$, and $G_2(k)$ be N/2 point DFT of $g_2(n)$.

Now, the N-point DFT X(k) can be obtained from the two numbers of N/2 point DFTs $G_1(k)$ and $G_2(k)$, as shown below.

$$X(k)\big|_{k=\text{even}} = G_1(k)$$

$$X(k)\big|_{k=\text{odd}} = G_2(k)$$

Proof :

By definition of DFT, the N-point DFT of x(n) is,

$$W_N^{\frac{kN}{2}} = e^{-j\frac{2\pi}{N}\frac{kN}{2}} = e^{-j\pi k} = \left(e^{-j\pi}\right)^k = (-1)^k$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\,W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\,W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{k\left(n+\frac{N}{2}\right)} = \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn}\,W_N^{\frac{kN}{2}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n)\,W_N^{kn} + (-1)^k\,x\left(n+\frac{N}{2}\right)W_N^{kn}\right] = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^k\,x\left(n+\frac{N}{2}\right)\right]W_N^{kn}$$

Let us split X(k) into even and odd numbered samples.

$$X(k)\big|_{k=\text{even}} = X(2k)\ ;\ \text{for } k = 0, 1, 2, \ldots, \frac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^{2k}\,x\left(n+\frac{N}{2}\right)\right]W_N^{2kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + x\left(n+\frac{N}{2}\right)\right]W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g_1(n)\,W_{N/2}^{kn} = G_1(k)$$

$$g_1(n) = x(n) + x\left(n+\frac{N}{2}\right)\ ;\ \text{for } n = 0, 1, 2, \ldots \frac{N}{2}-1$$

$G_1(k)$ is $\frac{N}{2}$ point DFT of $g_1(n)$.

$$\therefore G_1(k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n)\,W_{N/2}^{kn}\ ;\ \text{for } k = 0, 1, 2, \ldots \frac{N}{2}-1$$

$$X(k)\Big|_{k=\text{odd}} = X(2k+1) \quad ; \text{ for } k = 0,\ 1,\ 2,\ \ldots..,\ \frac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^{(2k+1)}\, x\left(n+\frac{N}{2}\right)\right] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{2kn}\, W_N^{n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{n}\, W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g_2(n)\ W_{N/2}^{kn} = G_2(k)$$

$$g_2(n) = \left(x(n) - x\left(n+\frac{N}{2}\right)\right) W_N^{n};\ \text{for } n = 0,\ 1,\ 2,\ \ldots..,\ \frac{N}{2}-1$$

$G_2(k)$ is $\frac{N}{2}$ point DFT of $g_2(n)$.

$$\therefore G_2(k) = \sum_{n=0}^{\frac{N}{2}-1} g_2(n)\, W_{N/2}^{kn}\ ;\ \text{for } k = 0,\ 1,\ 2,\ \ldots..,\ \frac{N}{2}-1$$

In the next stage of decimation the N/2 point frequency domain sequence $G_1(k)$ is decimated into two numbers of N/4 point sequences $D_{11}(k)$ and $D_{12}(k)$, and $G_2(k)$ is decimated into two numbers of N/4 point sequences $D_{21}(k)$ and $D_{22}(k)$.

Let $D_{11}(k)$ and $D_{12}(k)$ be two numbers of N/4 point sequences obtained by the decimation of $G_1(k)$.

Let $D_{11}(k)$ be N/4 point DFT of $d_{11}(n)$, and $D_{12}(k)$ be N/4 point DFT of $d_{12}(n)$.

Let $D_{21}(k)$ and $D_{22}(k)$ be two numbers of N/4 point sequences obtained by the decimation of $G_2(k)$.

Let $D_{21}(k)$ be N/4 point DFT of $d_{21}(n)$, and $D_{22}(k)$ be N/4 point DFT of $d_{22}(n)$.

Now, N/2 point DFTs can be obtained from two numbers of N/4 point DFTs as shown below.

$$G_1(k)\Big|_{k=\text{even}} = D_{11}(k)$$

$$G_1(k)\Big|_{k=\text{odd}} = D_{12}(k)$$

$$G_2(k)\Big|_{k=\text{even}} = D_{21}(k)$$

$$G_2(k)\Big|_{k=\text{odd}} = D_{22}(k)$$

Proof :

By definition of DFT, the N/2 point DFT of $G_1(k)$ is,

$$G_1(k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n)\, W_{N/2}^{kn} = \sum_{n=0}^{\frac{N}{4}-1} g_1(n)\, W_{N/2}^{kn} + \sum_{n=\frac{N}{4}}^{\frac{N}{2}-1} g_1(n)\, W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g_1(n)\, W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right) W_{N/2}^{k\left(n+\frac{N}{4}\right)} = \sum_{n=0}^{\frac{N}{4}-1} g_1(n)\, W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right) W_{N/2}^{kn}\, W_{N/2}^{k\frac{N}{4}}$$

$$= \sum_{n=0}^{\frac{N}{4}-1}\left[g_1(n) + W_{N/2}^{\frac{kn}{4}}\, g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{kn}$$

$$W_N^{\frac{kN}{2}} = e^{-j\frac{2\pi}{N}\frac{kN}{2}} = e^{-j\pi k} = \left(e^{-j\pi}\right)^k = (-1)^k$$

$$= \sum_{n=0}^{\frac{N}{4}-1}\left[g_1(n) + (-1)^k\, g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{kn}$$

Let us split $G_1(k)$ into even and odd numbered samples.

$$G_1(k)\big|_{k\,=\,\text{even}} = G_1(2k)\ ;\ \text{for}\ \ k = 0,\ 1,\ 2,\ \ldots..,\ \frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1}\left[g_1(n) + (-1)^{2k}\ g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{2kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1}\left[g_1(n) + g_1\left(n+\frac{N}{4}\right)\right] W_{N/4}^{kn} = \sum_{n=0}^{\frac{N}{4}-1} d_{11}(n)\ W_{N/4}^{kn} = D_{11}(k)$$

$$d_{11}(n) = g_1(n) + g_1\left(n+\frac{N}{4}\right)$$

$D_{11}(k)$ is $\frac{N}{4}$ point DFT of $d_{11}(n)$.

$$\therefore D_{11}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{11}(n)\ W_{N/4}^{kn}$$

$$d_{12}(n) = \left[g_1(n) - g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{n}$$

$D_{12}(k)$ is $\frac{N}{4}$ point DFT of $d_{12}(n)$.

$$\therefore D_{12}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{12}(n)\ W_{N/4}^{kn}$$

$$G_1(k)\big|_{k\,=\,\text{odd}} = G_1(2k+1)\ \ ;\ \text{for}\ k = 0,\ 1,\ 2,\ldots..,\ \frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1}\left[g_1(n)\ +\ (-1)^{(2k+1)}\ g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1}\left[g_1(n)\ -\ g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{n}\ W_{N/4}^{kn} = \sum_{n=0}^{\frac{N}{4}-1} d_{12}(n)\ W_{N/4}^{kn}\ = D_{12}(k)$$

Similarly the N/2 point sequence G_2 (k) can be decimated into two numbers of N/4 point sequences.

$$d_{21}(n) = g_2(n) + g_2\left(n+\frac{N}{4}\right)$$

$D_{21}(k)$ is $\frac{N}{4}$ point DFT of $d_{21}(n)$.

$$\therefore D_{21}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{21}(n)\ W_{N/4}^{kn}$$

$$G_2(k)\big|_{k\,=\,\text{even}} = G_2(2k)\ ;\ \text{for}\ \ k = 0,\ 1,\ 2,\ \ldots..,\frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1} d_{21}(n)\ W_{N/4}^{kn} = D_{21}(k)$$

$$d_{22}(n) = \left[g_2(n)\ -\ g_2\left(n+\frac{N}{4}\right)\right] W_{N/2}^{n}$$

$D_{22}(k)$ is $\frac{N}{4}$ point DFT of $d_{22}(n)$.

$$\therefore D_{22}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{22}(n)\ W_{N/4}^{kn}$$

$$G_2(k)\big|_{k\,=\,\text{odd}} = G_2(2k+1)\ \ ;\ \text{for}\ k = 0,\ 1,\ 2,\ \ldots..,\ \frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1} d_{22}(n)\ W_{N/4}^{kn} = D_{22}(k)$$

The decimation of the frequency domain sequence can be continued until the resulting sequence are reduced to 2-point sequences. The entire process of decimation involves, m stages of decimation where $m = \log_2 N$. The computation of the N-point DFT via the decimation in frequency FFT algorithm requires $(N/2)\log_2 N$ complex multiplications and $N \log_2 N$ complex additions. (i.e., the total number of computations remains same in both DIT and DIF).

5.8.1 8-point DFT Using Radix-2 DIF FFT

The DIF computations for an eight sequence is discussed in detail in this section. Let x(n) be an 8-point sequence. Therefore $N = 8 = 2^3 = r^m$. Here, r = 2 and m = 3. Therefore, the computation of 8-point DFT using radix-2 FFT involves three stages of computation.

The samples of x(n) are,

x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)

First Stage Computation

In the first stage of computation, two numbers of 4-point sequences $g_1(n)$ and $g_2(n)$ are obtained from x(n) as shown below.

$$g_1(n) = \left[x(n)\ +\ x\left(n+\frac{N}{2}\right)\right] = \left[x(n)\ +\ x(n+4)\right]\ \ ;\text{for}\ n\ =\ 0,\ 1,\ 2,\ 3$$

When $n = 0$; $g_1(n) = g_1(0) = x(0) + x(4)$

When $n = 1$; $g_1(n) = g_1(1) = x(1) + x(5)$

When $n = 2$; $g_1(n) = g_1(2) = x(2) + x(6)$

When $n = 3$; $g_1(n) = g_1(3) = x(3) + x(7)$

$$g_2(n) = \left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^n = \left[x(n) - x(n+4)\right] W_8^n \; ; \text{ for } n = 0, 1, 2, 3$$

When $n = 0$; $g_2(n) = g_2(0) = [x(0) - x(4)]\, W_8^0$

When $n = 1$; $g_2(n) = g_2(1) = [x(1) - x(5)]\, W_8^1$

When $n = 2$; $g_2(n) = g_2(2) = [x(2) - x(6)]\, W_8^2$

When $n = 3$; $g_2(n) = g_2(3) = [x(3) - x(7)]\, W_8^3$

Second Stage Computation

In the second stage of computation, 2 numbers of 2-point sequences $d_{11}(n)$ and $d_{12}(n)$ are generated from the samples of $g_1(n)$, and another 2 numbers of 2-point sequences $d_{21}(n)$ and $d_{22}(n)$ are generated from the samples of $g_2(n)$, as shown below.

$$d_{11}(n) = g_1(n) + g_1(n + N/4) = g_1(n) + g_1(n+2) \; ; \text{ for } n = 0, 1$$

When $n = 0$; $d_{11}(n) = d_{11}(0) = g_1(0) + g_1(2)$

When $n = 1$; $d_{11}(n) = d_{11}(1) = g_1(1) + g_1(3)$

$$d_{12}(n) = \left[g_1(n) - g_1(n + N/4)\right] W_{N/2}^n = \left[g_1(n) - g_1(n-2)\right] W_4^n \; ; \text{ for } n = 0, 1$$

When $n = 0$; $d_{12}(n) = d_{12}(0) = [g_1(0) - g_1(2)]\, W_4^0$

When $n = 1$; $d_{12}(n) = d_{12}(1) = [g_1(1) - g_1(3)]\, W_4^1$

$$d_{21}(n) = g_2(n) + g_2(n + N/4) = g_2(n) + g_2(n+2) \; ; \text{ for } n = 0, 1$$

When $n = 0$; $d_{21}(n) = d_{21}(0) = [g_2(0) + g_2(2)]$

When $n = 1$; $d_{21}(n) = d_{21}(1) = [g_2(1) + g_2(3)]$

$$d_{22}(n) = \left[g_2(n) - g_2(n + N/4)\right] W_{N/2}^n = \left[g_2(n) - g_2(n+2)\right] W_4^n \; ; \text{ for } n = 0, 1$$

When $n = 0$; $d_{22}(n) = d_{22}(0) = [g_2(0) - g_2(2)]\, W_4^0$

When $n = 1$; $d_{22}(n) = d_{22}(1) = [g_2(1) - g_2(3)]\, W_4^1$

Third Stage Computation

In the third stage of computation, 2-point DFTs of the 2-point sequences $d_{11}(n)$, $d_{12}(n)$, $d_{21}(n)$ and $d_{22}(n)$ are computed.

The 2-point DFT of the 2-point sequence $d_{11}(n)$ is computed as shown below.

$$\mathcal{DFT}\{d_{11}(n)\} = D_{11}(k) = \sum_{n=0}^{1} d_{11}(n)\, W_2^{kn} \; ; \text{ for } k = 0, 1$$

When $k = 0;\ D_{11}(0) = \sum_{n=0}^{1} d_{11}(n)\, W_2^0 = d_{11}(0) + d_{11}(1)$ $\boxed{W_2^0 = 1}$

When $k = 1\ ;\ D_{11}(1) = \sum_{n=0}^{1} d_{11}(n)\, W_2^n = d_{11}(0)\, W_2^0 + d_{11}(1)\, W_2^1$ $\boxed{W_2^1 = -1 = -1 \times W_2^0}$

$$= d_{11}(0)\, W_2^0 + d_{11}(1)\, W_2^1\, W_2^0 = \left[d_{11}(0) - d_{11}(1)\right] W_2^0$$

Similarly the 2-point DFTs of the 2-point sequences $d_{12}(n)$, $d_{21}(n)$ and $d_{22}(n)$ are computed and the results are given below.

$$D_{11}(0) = d_{11}(0) + d_{11}(1)$$
$$D_{11}(1) = \left[d_{11}(0) - d_{11}(1)\right] W_2^0$$
$$D_{12}(0) = d_{12}(0) + d_{12}(1)$$
$$D_{12}(1) = \left[d_{12}(0) - d_{12}(1)\right] W_2^0$$
$$D_{21}(0) = d_{21}(0) + d_{21}(1)$$
$$D_{21}(1) = \left[d_{21}(0) - d_{21}(1)\right] W_2^0$$
$$D_{22}(0) = d_{22}(0) + d_{22}(1)$$
$$D_{22}(1) = \left[d_{22}(0) - d_{22}(1)\right] W_2^0$$

Combining the Three Stages of Computation

The final output $D_{ij}(k)$ gives the X(k). The relation can be obtained as shown below.

$X(2k) = G_1(k)\ ;\ k = 0,1,2,3$ \ $X(0) = G_1(0)$ $X(2) = G_1(1)$ $X(4) = G_1(2)$ $X(6) = G_1(3)$	$X(2k+1) = G_2(k)\ ;\ k = 0, 1, 2, 3$ \ $X(1) = G_2(0)$ $X(3) = G_2(1)$ $X(5) = G_2(2)$ $X(7) = G_2(3)$
$G_1(2k) = D_{11}(k)\ ;\ k = 0, 1$ $\therefore\ G_1(0) = D_{11}(0)$ $G_1(2) = D_{11}(1)$	$G_1(2k+1) = D_{12}(k)\ ;\ k = 0, 1$ $\therefore\ G_1(1) = D_{12}(0)$ $G_1(3) = D_{11}(1)$
$G_2(2k) = D_{21}(k)\ ;\ k = 0, 1$ $\therefore\ G_2(0) = D_{21}(0)$ $G_2(2) = D_{21}(1)$	$G_2(2k+1) = D_{22}(k)\ ;\ k = 0, 1$ $\therefore\ G_2(1) = D_{22}(0)$ $G_2(3) = D_{22}(1)$

From above relations we get,

$D_{11}(0) = G_1(0) = X(0)$	$D_{21}(0) = G_2(0) = X(1)$
$D_{11}(1) = G_1(2) = X(4)$	$D_{21}(1) = G_2(2) = X(5)$
$D_{12}(0) = G_1(1) = X(2)$	$D_{22}(0) = G_2(1) = X(3)$
$D_{12}(1) = G_1(3) = X(6)$	$D_{22}(1) = G_2(3) = X(7)$

From the above we observe that the output is in bit reversed order. In radix-2 DIF FFT, the input is in normal order the output will be in bit reversed order.

5.8.2 Flow Graph For 8-point DFT using Radix-2 DIF FFT

If we observe the basic computation performed at every stage of radix-2 DIF FFT in previous section, we can arrive at the following conclusion.

1. In each computation two complex numbers "a" and "b" are considered.
2. The sum of the two complex numbers is computed which forms a new complex number "A".
3. Then subtract complex number "b" from "a" to get the term "a-b". The difference term "a-b" is multiplied with the phase factor or twiddle factor "W_N^k" to form a new complex number "B".

The above basic computation can be expressed by a signal flow graph shown in Fig 5.7. (For detailed discussion on signal flow graph, refer Chapter 2, Section 2.6.2).

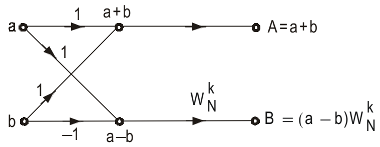


Fig 5.7 : Basic butterfly or flow graph of DIF radix-2 FFT.

The signal flow graph is also called ***butterfly diagram*** since it resembles a butterfly. In radix-2 FFT, N/2 butterflies per stage are required to represent the computational process. The butterfly diagram used to compute the 8-point DFT via radix-2 DIF FFT can be arrived as shown below, using the computations shown in previous section.

Flow Graph (or Butterfly Diagram) for First Stage of Computation

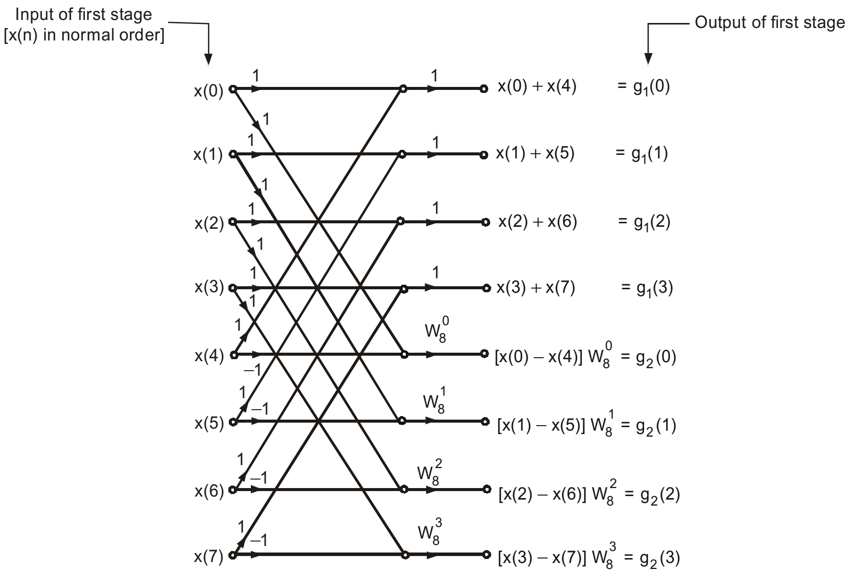


Fig 5.8 : First stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIF FFT.

Flowgraph or Butterfly Diagram for Second Stage of Computation

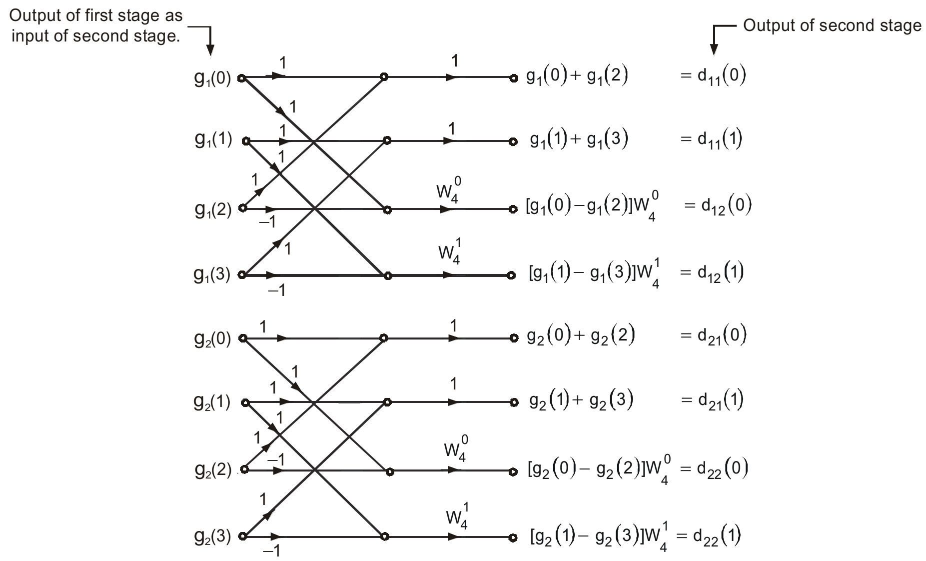


Fig 5.9 : *Second stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIF FFT.*

Flow Graph(or Butterfly Diagram) for Third Stage of Computation

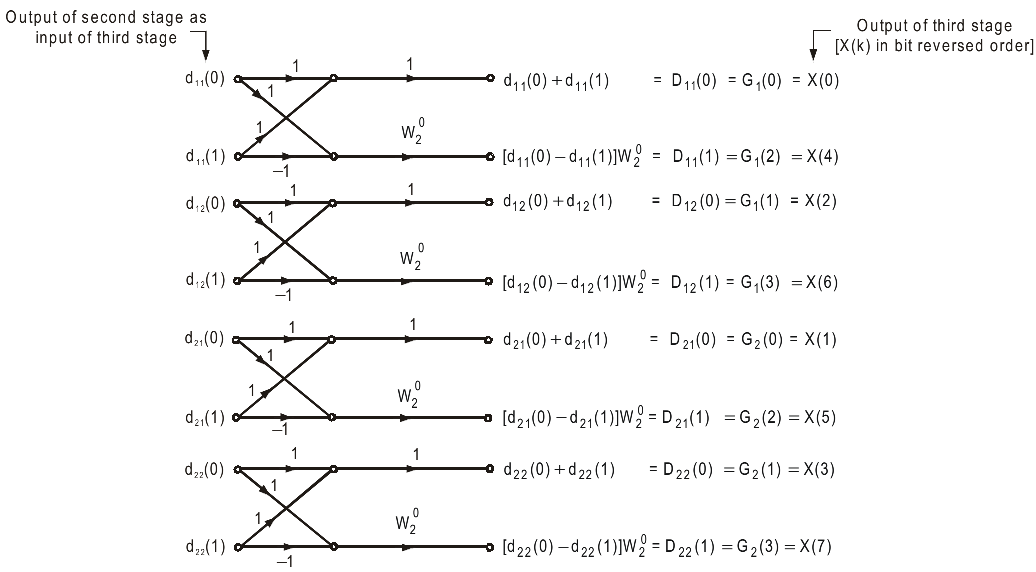


Fig 5.10 : *Third stage of flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIF FFT.*

The Combined Flow Graph (or Butterfly Diagram) of All the Three Stages of Computation

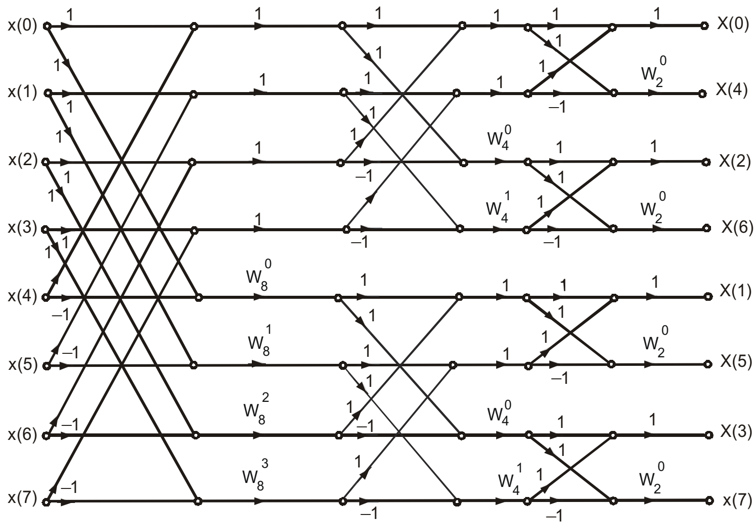


Fig 5.11 : The flow graph (or butterfly diagram) for 8-point DFT via radix-2 DIF FFT.

5.8.3 Comparison of DIT and DIF Radix-2 FFT

Differences in DIT and DIF

- In DIT the time domain sequence is decimated, whereas in DIF the frequency domain sequence is decimated.
- In DIT the input should be in bit-reversed order and the output will be in normal order. For DIF the reverse is true, i.e., input is normal order, while output is bit reversed.
- Considering the butterfly diagram, in DIT the complex multiplication takes place before the add-subtract operation, whereas in DIF the complex multiplication takes place after the add-subtract operation.

Similarities in DIT and DIF

- For both the algorithms the value of N should be such that, $N = 2^m$, and there will be m stages of butterfly computations, with N/2 butterfly per stage.
- Both algorithms involve same number of operations. The total number of complex additions are $N\log_2 N$ and total number of complex multiplications are $(N/2)\log_2 N$.
- Both algorithms require bit reversal at some place during computation.

5.9 Computation of Inverse DFT Using FFT

Let, x(n) and X(k) be N-point DFT pair.

Now by the definition of inverse DFT,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi nk}{N}} \quad ; \text{ for } n = 0, 1, 2, \ldots\ldots, N-1$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left(e^{\frac{-j2\pi nk}{N}}\right)^{*} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left(W_N^{nk}\right)^{*} = \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k)\left(W_N^{nk}\right)^{*}\right] \qquad .....(5.35)$$

In equation (5.35), the expression inside the bracket is similar to that of DFT computation of a sequence, with following differences.

1. The summation index is k instead of n.
2. The input sequence is X(k) instead of x(n).
3. The phase factors are conjugate of the phase factor used for DFT.

Hence, in order to compute inverse DFT of X(k), the FFT algorithm can be used by taking the conjugate of phase factors. Also from equation (5.35) it is observed that the output of FFT computation should be divided by N to get x(n).

The following procedure can be followed to compute inverse DFT using FFT algorithm.

1. Take N-point frequency domain sequence X(k) as input sequence.
2. Compute FFT by using conjugate of phase factors.
3. Divide the output sequence obtained in FFT computation by N, to get the sequence x(n).

Thus a single FFT algorithm can be used for evaluation of both DFT and inverse DFT.

Example 5.5

An 8-point sequence is given by x(n) = {2, 1, 2, 1, 1, 2, 1, 2}. Compute 8-point DFT of x(n) by **a)** radix-2 DIT-FFT and **b)** radix-2 DIF-FFT. Also sketch the magnitude and phase spectrum.

Solution

a) 8-point DFT by Radix-2 DIT-FFT

The given sequence is first arranged in the bit reversed order.

The sequence x(n) in normal order	The sequence x(n) in bit reversed order
x(0) = 2	x(0) = 2
x(1) = 1	x(4) = 1
x(2) = 2	x(2) = 2
x(3) = 1	x(6) = 1
x(4) = 1	x(1) = 1
x(5) = 2	x(5) = 2
x(6) = 1	x(3) = 1
x(7) = 2	x(7) = 2

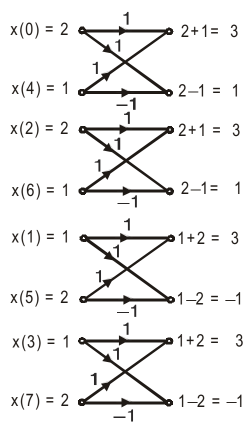


Fig 1 : Butterfly diagram for first stage of radix-2 DIT FFT.

The 8-point DFT by radix-2 FFT involve 3 stages of computation with 4-butterfly computations in each stage. The sequence rearranged in the bit reversed order forms the input to the first stage. For other stages of computation the output of previous stage will be the input for current stage.

First stage computation

The input sequence of first stage computation = { 2, 1, 2, 1, 1, 2, 1, 2}

The butterfly computations of first stage are shown in fig 1.

> The phase factor involved in first stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

The output sequence of first stage of computation = { 3, 1, 3, 1, 3, –1, 3, –1 }

Second stage computation

The input sequence to second stage computation = { 3, 1, 3, 1, 3, –1, 3, –1 }

The phase factors involved in second stage computation are W_4^0 and W_4^1.

The butterfly computations of second stage are shown in fig 2.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

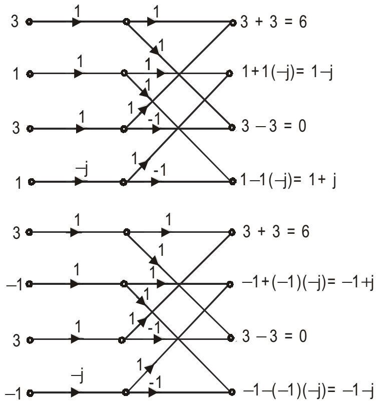


Fig 2 : Butterfly diagram for second stage of radix-2 DIT FFT.

The output sequence of second stage of computation = {6, 1–j, 0, 1+j, 6, –1+j, 0, –1–j}

Third stage computation

The input sequence to third stage computation = {6, 1–j, 0, 1+j, 6, –1+j, 0, –1–j}

The phase factors involved in third stage computation are W_8^0, W_8^1, W_8^2 and W_8^3.

The butterfly computations of third stage are shown in fig 3.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

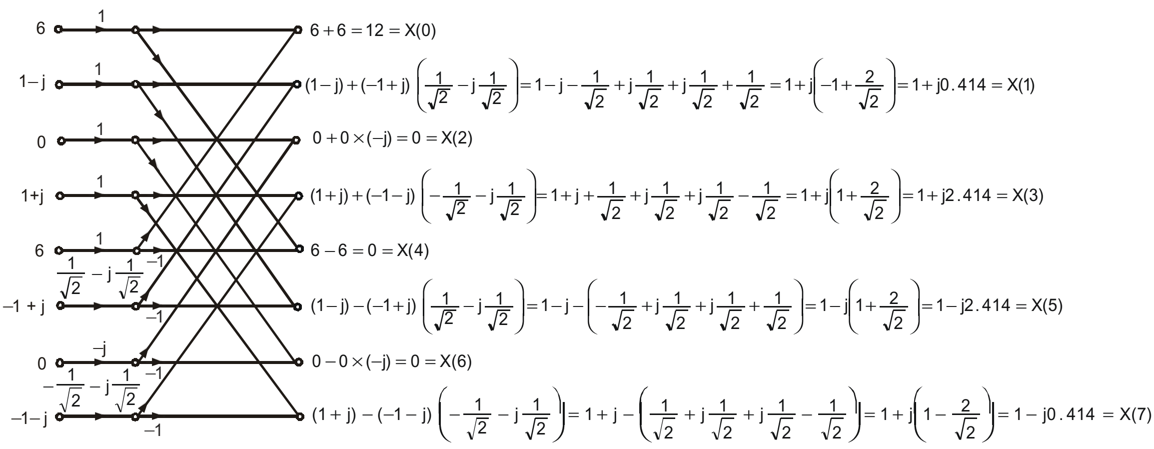


Fig 3 : Butterfly diagram for third stage of radix-2 DIT FFT of X(k) .

$$\left.\begin{array}{r}\text{The output sequence of third}\\ \text{stage of computation}\end{array}\right\} = \{12,\ 1+j0.414,\ 0,\ 1+j2.414,\ 0,\ 1-j2.414,\ 0,\ 1-j0.414\}$$

The output sequence of third stage of computation is the 8-point DFT of the given sequence in normal order.

$$\boxed{\therefore\ \mathcal{DFT}\{x(n)\} = X(k) = \{12,\ 1+j0.414,\ 0,\ 1+j2.414,\ 0,\ 1-j2.414,\ 0,\ 1-j0.414\}}$$

---

b) 8-point DFT by Radix-2 DIF-FFT

For 8-point DFT by radix-2 FFT we require 3-stages of computation with 4-butterfly computation in each stage. The given sequence is the input to first stage. For other stages of computations, the output of previous stage will be the input for current stage.

First stage computation

The input sequence for first stage of computation = { 2, 1, 2, 1, 1, 2, 1, 2 }

The phase factors involved in first stage computation are W_8^0, W_8^1, W_8^2 and W_8^3.

The butterfly computations of first stage are shown in fig 4.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j\frac{\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j\frac{3\pi}{4}} = \cos\left(-\frac{3\pi}{4}\right) + j\sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

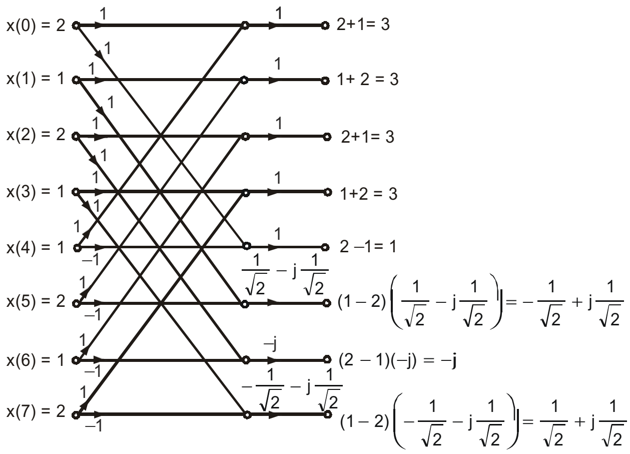


Fig 4 : Butterfly diagram for first stage of radix-2 DIF FFT.

$$\left.\begin{array}{r}\text{The output sequence of first}\\ \text{stage of computation}\end{array}\right. = \left\{3,\ 3,\ 3,\ 3,\ 1,\ -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}},\ -j,\ \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right\}$$

Second stage computation

The input sequence for second stage of computation $= \left\{3,\ 3,\ 3,\ 3,\ 1,\ -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}},\ -j,\ \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right\}$

The phase factors involved in second stage computation are W_4^0 and W_4^1.

The butterfly computations of second stage are shown in fig 5.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

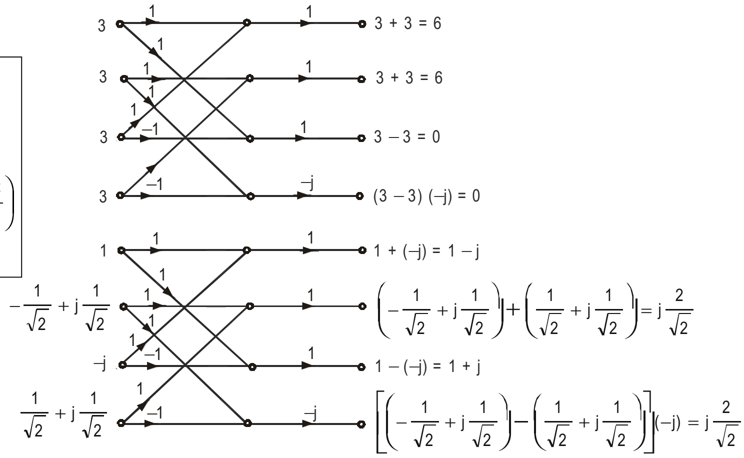


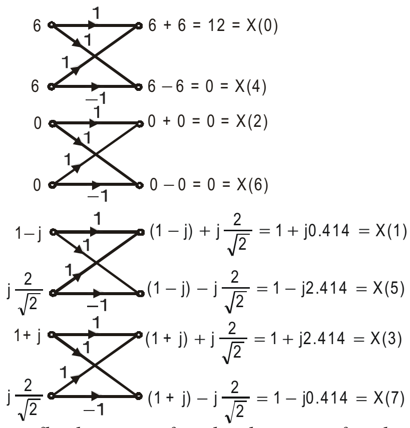
Fig 5 : Butterfly diagram for second stage of radix-2 DIF FFT.

The output sequence of second stage of computation $= \left\{6,\ 6,\ 0,\ 0,\ 1-j,\ j\frac{2}{\sqrt{2}},\ 1+j,\ j\frac{2}{\sqrt{2}}\right\}$

Third stage computation

The input sequence to third stage of computation $= \left\{6,\ 6,\ 0,\ 0,\ 1-j,\ j\frac{2}{\sqrt{2}},\ 1+j,\ j\frac{2}{\sqrt{2}}\right\}$

The butterfly computations of third stage are shown in fig 6.



The phase factor involved in third stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

Fig 6 : Butterfly diagram for third stage of radix-2 DIF FFT.

The output sequence of third stage of computation = { 12, 0, 0, 0, 1+j0.414, 1–j2.414, 1+j2.414, 1–j0.414 }

The output sequence of third stage of computation is the 8-point DFT of the given sequence in bit reversed order.

In DIF-FFT algorithm the input to first stage is in normal order and the output of third stage will be in the bit reversed order. Hence the actual result is obtained by arranging the output sequence of third stage in normal order as shown below.

The sequence X(k) in bit reversed order	The sequence X(k) in normal order
$X(0) = 12$	$X(0) = 12$
$X(4) = 0$	$X(1) = 1 + j0.414$
$X(2) = 0$	$X(2) = 0$
$X(6) = 0$	$X(3) = 1 + j2.414$
$X(1) = 1 + j0.414$	$X(4) = 0$
$X(5) = 1 - j2.414$	$X(5) = 1 - j2.414$
$X(3) = 1 + j2.414$	$X(6) = 0$
$X(7) = 1 - j0.414$	$X(7) = 1 - j0.414$

\ *DFT* $\{x(n)\} = X(k) = \{12,\ 1+j0.414,\ 0,\ 1+j2.414,\ 0,\ 1-j2.414,\ 0,\ 1-j0.414\}$

Magnitude and phase specturm

Each element of the sequence X(k) is a complex number and they are expressed in rectangular coordinates. If they are converted to polar coordinates then the magnitude and phase of each element can be obtained.

Note : The rectangular to polar conversion can be obtained by using R ® P conversion in calculator.

$$X(k) = \{12,\ 1+j0.414,\ 0,\ 1+j2.414,\ 0,\ 1-j2.414,\ 0,\ 1-j0.414\}$$

$$= \{12\angle 0°,\ 1.08\angle 22°,\ 0\angle 0°,\ 2.61\angle 67°,\ 0\angle 0°,\ 2.61\angle -67°,\ 0\angle 0°,\ 1.08\angle -22°\}$$

$$= \left\{ \begin{array}{c} 12\angle 0,\ \ 1.08\angle 22° \times \dfrac{\pi}{180°},\ \ 0\angle 0,\ \ 2.61\angle 67° \times \dfrac{\pi}{180°},\ \ 0\angle 0, \\ 2.61\angle -67° \times \dfrac{\pi}{180°},\ \ 0\angle 0,\ 1.08\angle -22° \times \dfrac{\pi}{180°} \end{array} \right\}$$

$$= \{12\angle 0,\ 1.08\angle 0.12\pi,\ 0\angle 0,\ 2.61\angle 0.37\pi,\ 0\angle 0,\ 2.61\angle -0.37\pi,\ 0\angle 0,\ 1.08\angle -0.12\pi\}$$

$$\therefore |X(k)| = \{12,\ 1.08,\ 0,\ 2.61,\ 0,\ 2.61,\ 0,\ 1.08\}$$

$$\angle X(k) = \{0,\ 0.12\pi,\ 0,\ 0.37\pi,\ 0,\ -0.37\pi,\ 0,\ -0.12\pi\}$$

The magnitude specturm is the plot of the magnitude of each sample of X(k) as a function of k as shown in fig 7. The phase spectrum is the plot of phase of X(k) as a function of k as shown in fig 8.

When N-point DFT is performed on a sequence x(n) then the DFT sequence X(k) will have a periodicity of N. Hence in this example the magnitude and phase specturm will have a periodicity of 8 as shown in fig 7 and fig 8.

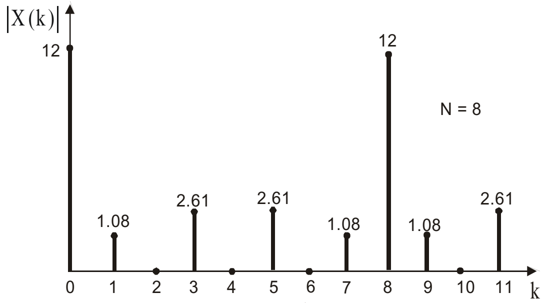


Fig 7 : Magnitude spectrum.

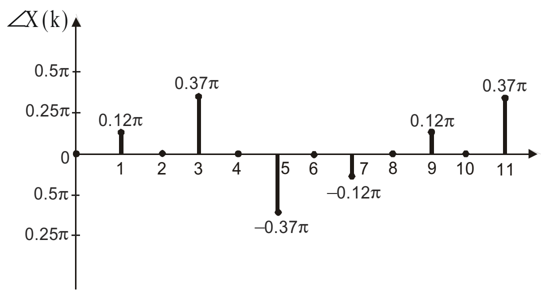


Fig 8: Phase spectrum.

Example 5.6

In an LTI system the input x(n) = {1, 2, 3} and the impulse response h(n) = {–1, –1}. Determine the response of the LTI system by radix-2 DIT FFT.

Solution

The response y(n) of LTI system is given by linear convolution of input x(n) and impulse response h(n).

$\therefore$ Response or Output, $y(n) = x(n) * h(n)$

The DFT (or FFT) supports only circular convolution. Hence to get the result of linear convolution from circular convolution, the sequences x(n) and h(n) should be converted to the size of y(n) by appending with zeros and circular convolution of x(n) and h(n) is performed.

The length of x(n) is 3 and h(n) is 2. Hence the length of y(n) is 3 + 2 – 1 = 4. Therefore given sequences x(n) and h(n) are converted to 4 point sequences by appending zeros.

$\therefore$ x(n) = {1, 2, 3, 0} and h(n) = {–1, –1, 0, 0}

Now the response y(n) is given by, $y(n) = x(n) \circledast h(n)$.

Let, $\mathcal{DFT}\{x(n)\} = X(k)$, $\mathcal{DFT}\{h(n)\} = H(k)$, $\mathcal{DFT}\{y(n)\} = Y(k)$.

By convolution theorem of DFT we get,

$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\,H(k)$

$\therefore$ $y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \mathcal{DFT}^{-1}\{X(k)\,H(k)\}$

The various steps in computing y(n) are,

Step - 1 : Determine X(k) using radix-2 DIT algorithm.

Step - 2 : Determine H(k) using radix-2 DIT algorithm.

Step - 3 : Determine the product X(k)H(k).

Step - 4 : Take inverse DFT of the product X(k)H(k) using radix-2 DIT algorithm.

Step-1: To determine X(k)

Since x(n) is a 4-point sequence, we have to compute 4-point DFT. The 4-point DFT by radix-2 FFT consists of two stages of computations with 2-butterflies in each stage. The given sequence x(n), is first arranged in bit reversed order as shown in table.

x(n) Normal order	x(n) Bit reversed order
x(0) = 1	x(0) = 1
x(1) = 2	x(2) = 3
x(2) = 3	x(1) = 2
x(3) = 0	x(3) = 0

The sequence arranged in bit reversed order forms the input sequence to first stage computation.

First stage computation

Input sequence to first stage = { 1, 3, 2, 0 }. The butterfly computations of first stage are shown in fig1.

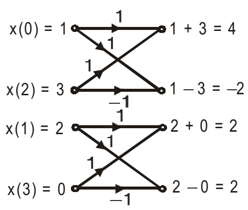


Fig 1 : Butterfly diagram for first stage of radix-2 DIT FFT.

The phase factor involved in first stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

Output sequence of first stage of computation = { 4, -2, 2, 2 }

Second stage computation

Input sequence to second stage computation = { 4, -2, 2, 2 }

The phase factors involved in second stage computation are W_4^0 and W_4^1.

The butterfly computations of second stage are shown in fig 2.

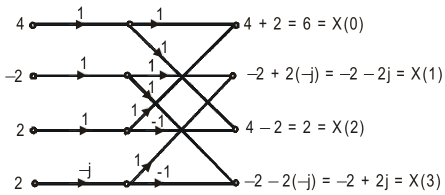


Fig 2 : Butterfly diagram for second stage of radix-2 DIT FFT.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

Output sequence of second stage computation = { 6, -2–2j, 2, -2+2j }

The output sequence of second stage of computation is the 4-point DFT of x(n).

\ X(k) = DFT{x(n)} = {6, -2–2j, 2, -2+2j }

Step - 2: To determine H(k)

Since h(n) is a 4-point sequence, we have to compute 4-point DFT. The 4-point DFT by radix-2 FFT consists of two stages of computations with 2-butterflies in each stage. The sequence h(n) is first arranged in bit reversed order as shown in table.

h(n) Normal order	h(n) Bit reversed order
h(0) = –1	h(0) = –1
h(1) = –1	h(2) = 0
h(2) = 0	h(1) = –1
h(3) = 0	h(3) = 0

The sequence in bit reversed order forms the input sequence to first stage computation.

First stage computation

Input sequence of first stage = { –1, 0, –1, 0 }. The butterfly computations of first stage are shown in fig 3.

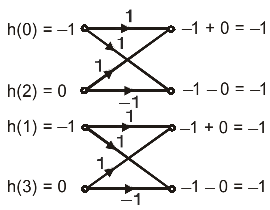


Fig 3 : Butterfly diagram for first stage of radix-2 DIT FFT.

The phase factor involved in first stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

Output sequence of first stage computation = { –1, –1, –1, –1 }

Second stage computation

Input sequence to second stage computation = { –1, –1, –1, –1 }

The phase factors involved are W_4^0 and W_4^1.

The butterfly computations of second stage are shown in fig 4.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

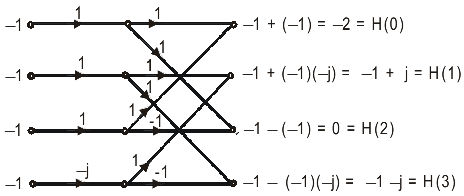


Fig 4 : Butterfly diagram for second stage of radix-2 DIT FFT of H(k).

Output sequence of second stage computation = { –2, –1 + j, 0, –1 –j }

The output sequence of second stage computation is the 4-point DFT of h(n).

$$\boxed{\therefore\ H(k) = \mathcal{DFT}\{h(n)\} = \{-2,\ -1+j,\ 0,\ -1-j\}}$$

Step 3 : To determine the product X(k)H(k)

Let the product, X(k)H(k) = Y(k); for k = 0, 1, 2, 3.

when k = 0; Y(0) = X(0) × H(0) = 6 × (–2) = –12

when k = 1; Y(1) = X(1) × H(1) = (–2–2j) × (–1 + j) = 4

when k = 2; Y(2) = X(2) × H(2) = 2 × 0 = 0

when k = 3; Y(3) = X(3) × H(3) = (–2+2j) × (–1–j) = 4

$$\boxed{\therefore\ Y(k) = \{-12,\ 4,\ 0,\ 4\}}$$

Step - 4: To determine inverse DFT of Y(k)

The 4-point inverse DFT of Y(k) can be computed using radix-2 DIT FFT by taking conjugate of the phase factors and then dividing the output sequence of FFT by 4.

Y(k) = { –12, 4, 0, 4}

The 4-point inverse DFT of Y(k) using radix-2 DIT FFT involves two stages of computations with 2-butterflies in each stage. The sequence Y(k) is arranged in bit reversed order as shown in the table.

Y(k) Normal order	**Y(k) Bit reversed order**
Y(0) = –12	Y(0) = –12
Y(1) = 4	Y(2) = 0
Y(2) = 0	Y(1) = 4
Y(3) = 4	Y(3) = 4

The sequence arranged in bit reversed order forms the input sequence to first stage computation.

First stage computation

Input sequence to first stage = { –12, 0, 4, 4 }. The butterfly computations of first stage are shown in fig 5.

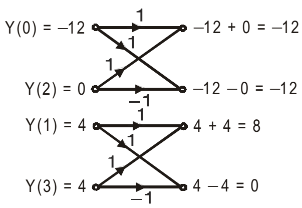


Fig 5 : Butterfly diagram for first stage of inverse DFT of Y(k).

The phase factor involved in first stage of computation is $\left(W_2^0\right)^*$. Since, $\left(W_2^0\right)^* = 1$, it is not considered for computation.

The output sequence of first stage computation = { – 12, – 12, 8, 0 }

Second stage computation

Input sequence to second stage computation = { – 12, – 12, 8, 0 }

The phase factors involved are $(W_4^0)^*$ and $(W_4^1)^*$.

The butterfly computation of second stage is shown in fig 6.

$$\left(W_4^0\right)^* = e^{j2\pi \times \frac{0}{4}} = 1$$

$$\left(W_4^1\right)^* = e^{j2\pi \times \frac{1}{4}} = e^{j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)$$

$$= j$$

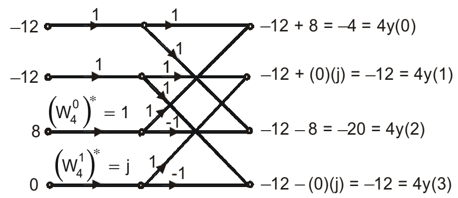


***Fig 6 :** Butterfly diagram for second stage of inverse DFT of Y(k).*

The output sequence of second stage computation = { – 4, – 12, – 20, – 12 }

The sequence y(n) is obtained by dividing each sample of output sequence of second stage by 4.

$\boxed{\therefore \text{ The response of the LTI system, } y(n) = \{-1,\ -3,\ -5,\ -3\}}$

Example 5.7

Determine the response of LTI system when the input sequence x(n) = {–1, 2, 2, 2, –1} by radix 2 DIT FFT. The impulse response of the system is h(n) = {–1, 1, –1, 1}.

Solution

The response of an LTI system is given by linear convolution of input x(n) and impulse response h(n).

∴ Response or Output, y(n) = x(n) ∗ h(n).

The DFT (or FFT) supports only circular convolution. Hence to get the result of linear convolution from circular convolution, the sequence x(n) and h(n) should be converted to the size of y(n), by appending with zeros, and then circular convolution of x(n) and h(n) is performed.

The length of x(n) = 5, and h(n) = 4. Hence the length of y(n) is 5 + 4 – 1 = 8.

Therefore x(n) and h(n) are converted into 8-point sequence by appending zeros.

∴ x(n) = { –1, 2, 2, 2, –1, 0, 0, 0 } and h(n) = { –1, 1, –1, 1, 0, 0, 0, 0 }

Now, the response y(n) is given by, y(n) = x(n) ⊛ h(n).

Let, $\mathcal{DFT}$ {x(n)} = X(k), $\mathcal{DFT}$ {h(n)} = H(k), $\mathcal{DFT}$ {y(n)} = Y(k).

By convolution theorem of DFT we get,

$\mathcal{DFT}$ {x(n) ⊛ h(n)} = X(k) H(k)

∴ y(n) = $\mathcal{DFT}^{-1}$ {Y(k)} = $\mathcal{DFT}^{-1}$ {X(k) H(k)}

The various steps in computing y(n) are,

Step - 1 : Determine X(k) using radix-2 DIT algorithm.
Step - 2 : Determine H(k) using radix-2 DIT algorithm.
Step - 3 : Determine the product X(k)H(k).
Step - 4 : Take inverse DFT of the product X(k)H(k) using radix-2 DIT algorithm.

Step-1 : To determine X(k)

Since x(n) is an 8 point sequence, we have to compute 8-point DFT.

The 8-point DFT by radix-2 FFT algorithm consists of 3 stages of computations with 4 butterflies in each stage.

The given sequence x(n) is arranged in bit reversed order as shown in the following table.

x(n) Normal order	x(n) Bit reversed order
x(0) = –1	x(0) = –1
x(1) = 2	x(4) = –1
x(2) = 2	x(2) = 2
x(3) = 2	x(6) = 0
x(4) = –1	x(1) = 2
x(5) = 0	x(5) = 0
x(6) = 0	x(3) = 2
x(7) = 0	x(7) = 0

The sequence arranged in bit-reversed order forms the input sequence to the first stage computation.

First stage computation

Input sequence to first stage = { –1, –1, 2, 0, 2, 0, 2, 0 }.

The butterfly computation of first stage is shown in fig 1.

The phase factor involved in first stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

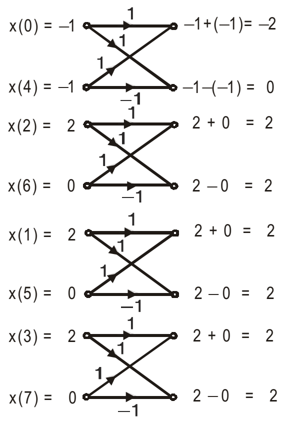


***Fig 1 :** Butterfly diagram for first stage of radix-2 DIT FFT of X(k).*

Output sequence of first stage of computation = { –2, 0, 2, 2, 2, 2, 2, 2 }

Second stage computation

The input sequence to second stage of computation = { –2, 0, 2, 2, 2, 2, 2, 2 }

Phase factors involved in second stage are W_4^0 and W_4^1.

The butterfly computation of second stage is shown in fig 2.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

Output sequence of second stage of computation = { 0, –2j, –4, 2j, 4, 2 –2 j, 0, 2 + 2j }

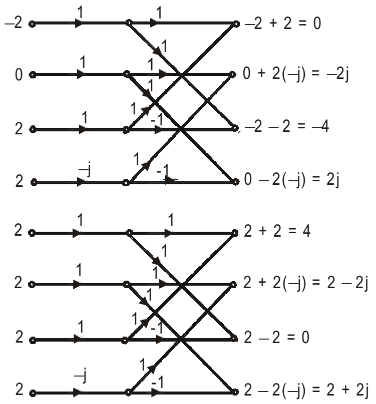


Fig 2 : Butterfly diagram for second stage of radix-2 DIT FFT of X(k).

Third stage computation

Input sequence to third stage computation = { 0, –2j, –4, 2j, 4, 2 – 2j, 0, 2 + 2j }.

Phase factors involved are W_8^0, W_8^1, W_8^2 and W_8^3.

The butterfly computation of third stage is shown in fig 3.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.707 - j0.707$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = -0.707 - j0.707$$

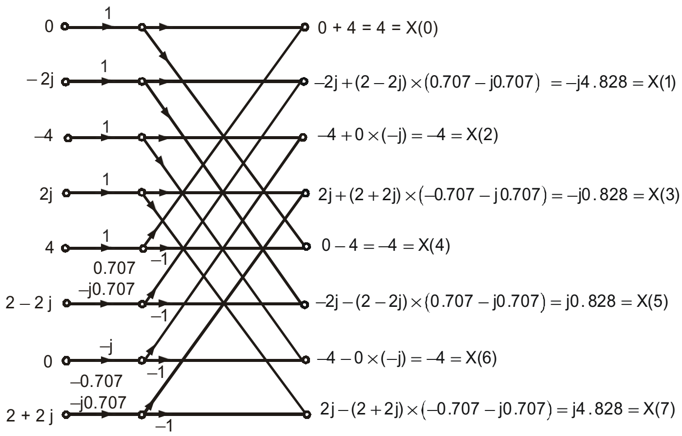


Fig 3 : Butterfly diagram for third stage of radix-2 DIT FFT of X(k) .

$$\left.\begin{array}{r}\text{Output sequence of third}\\ \text{stage of computation}\end{array}\right\} = \{4,\ -j4.828,\ -4,\ -j0.828,\ -4,\ j0.828,\ -4,\ j4.828\}$$

$$\therefore\ \mathcal{DFT}\{x(n)\} = X(k) = \{4,\ -j4.828,\ -4,\ -j0.828,\ -4,\ j0.828,\ -4,\ j4.828\}$$

Step 2 : To determine H(k)

Since h(n) is an 8-point sequence, we have to compute 8-point DFT. The 8-point DFT by radix-2 FFT consists of three stages of computations with four butterflies in each stage.

The sequence h(n) is first arranged in bit reversed order as shown in the following table .

h(n) Normal order	h(n) Bit reversed order
h(0) = –1	h(0) = –1
h(1) = 1	h(4) = 0
h(2) = –1	h(2) = –1
h(3) = 1	h(6) = 0
h(4) = 0	h(1) = 1
h(5) = 0	h(5) = 0
h(6) = 0	h(3) = 1
h(7) = 0	h(7) = 0

The sequence arranged in bit reversed order forms the input sequence to the first stage.

First stage computation

Input sequence to first stage computation = { –1, 0, –1, 0, 1, 0, 1, 0 }

The butterfly computations of first stage is shown in fig 4.

Output sequence of first stage of computation = { –1, –1, –1, –1, 1, 1, 1, 1 }

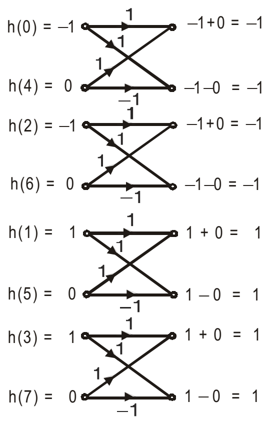


Fig 4 : Butterfly diagram for first stage of radix-2 DIT FFT of H(k).

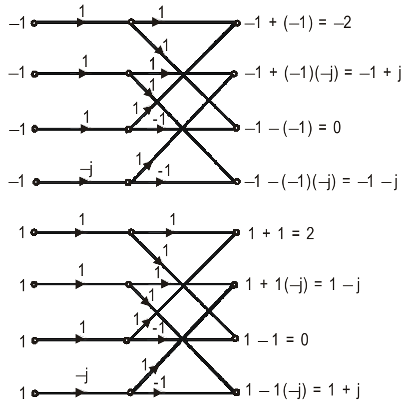
The phase factor involved in first stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

Second stage computation

Input sequence to second stage of computation = { –1, –1, –1, –1, 1, 1, 1, 1 }

Phase factors involved in second stage are W_4^0 and W_4^1.

The butterfly computations of second stage are shown in fig 5.



$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

***Fig 5:** Butterfly diagram for second stage of radix-2 DIT FFT of H(k).*

Output sequence of second stage of computation $= \{-2,\ -1+j,\ 0,\ -1-j,\ 2,\ 1-j,\ 0,\ 1+j\}$

Third stage computation

Input sequence to third stage computation $= \{-2,\ -1+j,\ 0,\ -1-j,\ 2,\ 1-j,\ 0,\ 1+j\}$

Phase factors involved in third stage computations are W_8^0, W_8^1, W_8^2, and W_8^3.

The butterfly computations of third stage are shown in fig 6.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.707 - j0.707$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = -0.707 - j0.707$$

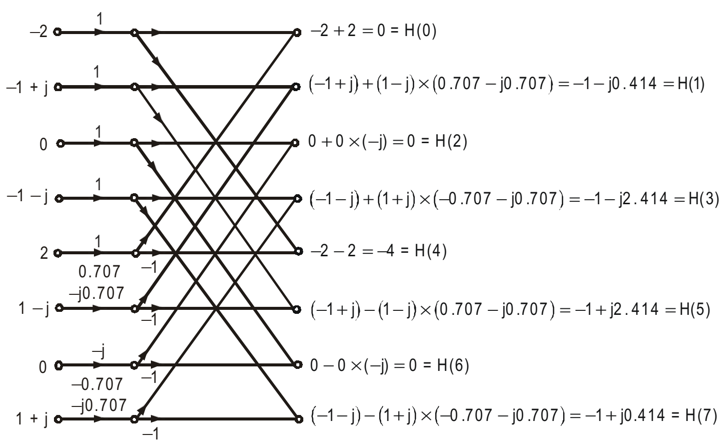


***Fig 6 :** Butterfly diagram for third stage of radix-2 DIT FFT of H(k) .*

$$\left.\begin{array}{r}\text{Output sequence of third}\\ \text{stage computation}\end{array}\right\} = \{0,\ -1-j0.414,\ 0,\ -1-j2.414,\ -4,\ -1+j2.414,\ 0,\ -1+j0.414\}$$

The output sequence of third stage computation is the 8-point DFT of h(n).

$$\boxed{\therefore\ \mathcal{DFT}\{h(n)\} = H(k) = \{0,\ -1-j0.414,\ 0,\ -1-j2.414,\ -4,\ -1+j2.414,\ 0,\ -1+j0.414\}}$$

Step 3 : To determine the product X(k)H(k)

Let the product of X(k)H(k) = Y(k); for k = 0, 1, 2, 3, 4, 5, 6, 7

$$\therefore\ Y(k) = X(k)H(k)$$

When k = 0; $Y(0) = X(0)\,H(0) = 4 \times 0 = 0$

When k = 1; $Y(1) = X(1)\,H(1) = -j4.828 \times [-1-j0.414] = -2+j4.828$

When k = 2; $Y(2) = X(2)\,H(2) = -4 \times 0 = 0$

When k = 3; $Y(3) = X(3)\,H(3) = -j0.828 \times [-1-j2.414] = -2+j0.828$

When k = 4; $Y(4) = X(4)\,H(4) = -4 \times -4 = 16$

When k = 5; $Y(5) = X(5)\,H(5) = j0.828 \times [-1+j2.414] = -2-j0.828$

When k = 6; $Y(6) = X(6)\,H(6) = -4 \times 0 = 0$

When k = 7; $Y(7) = X(7)\,H(7) = j4.828 \times [-1+j0.414] = -2-j4.828$

$$\boxed{\therefore\ Y(k) = \{0,\ -2+j4.828,\ 0,\ -2+j0.828,\ 16,\ -2-j0.828,\ 0,\ -2-j4.828\}}$$

Step - 4: To determine inverse DFT of Y(k)

The 8-point inverse DFT of Y(k) can be computed using radix-2 DIT FFT by taking conjugate of the phase factors and then dividing the output sequence of FFT by 8.

The 8-point inverse DFT of Y(k) using radix-2 DIT FFT involves three stages of computations with 4-butterflies in each stage. The sequence Y(k) is arranged in bit reversed order as shown in the following table.

The sequence arranged in bit reversed order forms the input sequence to first stage computation.

Y(k) Normal order	**Y(k) Bit reversed order**
$Y(0) = 0$	$Y(0) = 0$
$Y(1) = -2+j4.828$	$Y(4) = 16$
$Y(2) = 0$	$Y(2) = 0$
$Y(3) = -2+j0.828$	$Y(6) = 0$
$Y(4) = 16$	$Y(1) = -2+j4.828$
$Y(5) = -2-j0.828$	$Y(5) = -2-j0.828$
$Y(6) = 0$	$Y(3) = -2+j0.828$
$Y(7) = -2-j4.828$	$Y(7) = -2-j4.828$

First stage computation

$$\text{Input sequence of first stage} = \left\{\begin{array}{l}0,\ 16,\ 0,\ 0,\ -2+j4.828,\ -2+j0.828,\\ -2+j0.828,\ -2-j4.828\end{array}\right\}$$

The butterfly computations of first stage are shown in fig 7.

The phase factor involved in first stage of computation is $\left(W_2^0\right)^*$.

Since, $\left(W_2^0\right)^* = e^{j2\pi\times\frac{0}{4}} = e^0 = 1$, it is not considered for computation.

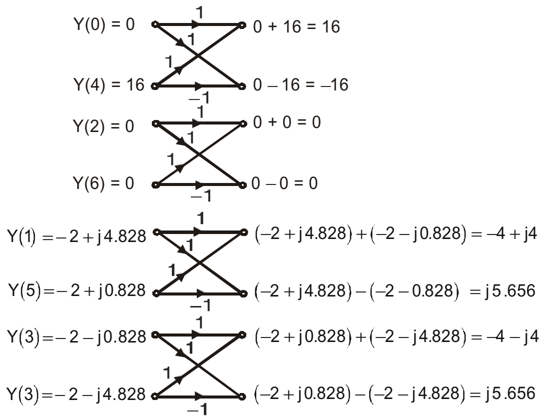


Fig 7 : Butterfly diagram for first stage of inverse DFT of Y(k).

Output sequence of first stage $= \{16,\ -16,\ \ 0,\ \ 0,\ \ -4+j4,\ \ j5.656,\ \ -4-j4,\ \ j5.656\}$

Second stage computation

Input sequence of second stage $= \{16,\ -16,\ \ 0,\ \ 0,\ \ -4+j4,\ \ j5.656,\ \ -4-j4,\ \ j5.656\}$

The butterfly computation of second stage is shown in fig 8.

The phase factors involved are $(W_4^0)^*$ and $(W_4^1)^*$.

$$\left(W_4^0\right)^* = e^{j2\pi\times\frac{0}{4}} = e^0 = 1$$

$$\left(W_4^1\right)^* = e^{j2\pi\times\frac{1}{4}} = e^{j\times\frac{\pi}{2}}$$

$$= \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)$$

$$= j$$

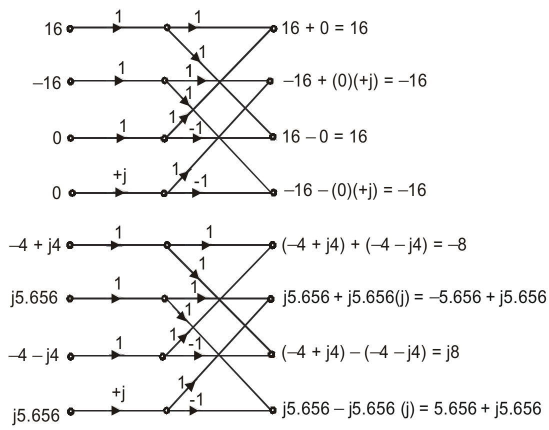


Fig 8: Butterfly diagram for second stage of inverse DFT of Y(k)

$$\left.\begin{array}{c}\text{Output sequence of second}\\ \text{stage computation}\end{array}\right\} = \{16,\ -16,\ 16,-16,\ -8,\ -5.656+j5.656,\ j8,\ 5.656+j5.656\}$$

Third stage computation

$$\left.\begin{array}{c}\text{Input sequence of third}\\ \text{stage computation}\end{array}\right\} = \{16,\ -16,\ 16,-16,\ -8,\ -5.656+j5.656,\ j8,\ 5.656+j5.656\}$$

The butterfly computation of third stage is shown in fig 9.

The phase factors involved are $(W_8^0)^*$, $(W_8^1)^*$, $(W_8^2)^*$ and $(W_8^3)^*$.

$$(W_8^0)^* = e^{j2\pi \times \frac{0}{8}} = 1$$

$$(W_8^1)^* = e^{j2\pi \times \frac{1}{8}} = e^{j \times \frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} = 0.707 + j0.707$$

$$(W_8^2)^* = e^{j2\pi \times \frac{2}{8}} = e^{j \times \frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j$$

$$(W_8^3)^* = e^{j2\pi \times \frac{3}{8}} = e^{j \times \frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} = -0.707 + j0.707$$

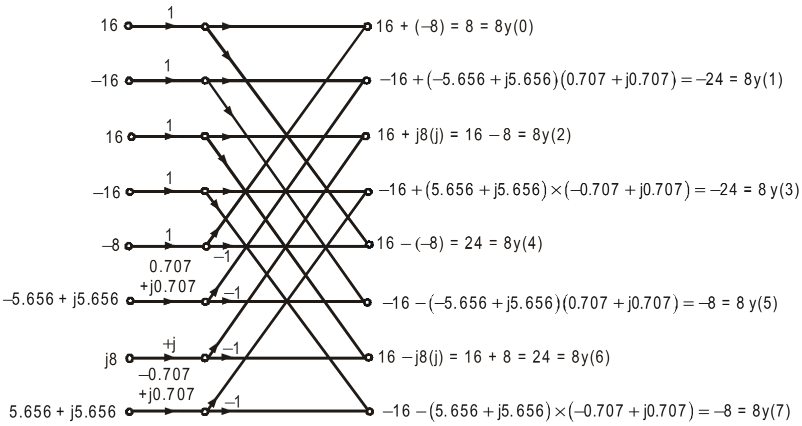


Fig 9 : Butterfly diagram for third stage of inverse DFT of Y(k).

Output sequence of third stage computation = { 8, –24, 8, –24, 24, –8, 24, –8 }

The sequence y(n) is obtained by dividing each sample of output sequence of third stage by 8.

$\therefore$ **The response of the LTI system, y(n) = { 1, –3, 1, –3, 3, –1, 3, –1 }**

5.10 Summary of Important Concepts

1. The drawback in DTFT is that the frequency domain representation of a discrete time signal obtained using DTFT will be a continuous function of ω.
2. The DFT has been developed to convert a continuous function of ω to a discrete function of ω.
3. The DFT of a discrete time signal can be obtained by sampling the DTFT of the signal.
4. The sampling of the DTFT is conventionally performed at N equally spaced frequency points in the period, $0 \le \omega \le 2\pi$.
5. DFT sequence starts at k = 0, corresponding to $\omega = 0$ but does not include k = N, corresponding to $\omega = 2\pi$.
6. The DFT is defined along with number of samples and is called N-point DFT.
7. The number of samples N for a finite duration sequence x(n) of length L should be such that, $N \ge L$, in order to avoid aliasing of frequency spectrum.
8. The X(k) is also called discrete frequency spectrum (or signal spectrum) of the discrete time signal x(n).
9. The plot of samples of magnitude sequence versus k is called magnitude spectrum.
10. The plot of samples of phase sequence versus k is called phase spectrum.
11. The DFT sequence X(k) is periodic with periodicity of N samples.

12. The DFT of circular convolution of two sequences is equivalent to product of their individual DFTs.
13. The N-point DFT of a finite duration sequence can be obtained from the $\mathcal{Z}$-transform of the sequence, by evaluating the $\mathcal{Z}$-transform at N equally spaced points around the unit circle.
14. The DFT supports only circular convolution and so, the linear convolution using DFT has to be computed via circular convolution.
15. The FFT is a method (or algorithm) for computing the DFT with reduced number of calculations.
16. In N-point DFT by radix-r FFT, the number of stages of computation will be "m" times, where $m = \log_r N$.
17. In direct computation of N-point DFT, the total number of complex additions are N(N–1) and total number of complex multiplications are N^2.
18. In computation of N-point DFT via radix-2 FFT, the total number of complex additions are $N\log_2 N$ and total number of complex multiplications are $(N/2)\log_2 N$.
19. The complex valued phase factor or twiddle factor W_N is defined as, $W_N = e^{\frac{-j2\pi}{N}}$.
20. The term W in phase factor represents a complex number $1\angle -2\pi$.
21. The multiplication by k of the phase value -2π of W can be represented as W^k.
22. The division by N of the phase value -2π of W can be represented as W_N.
23. In DIT the time domain sequence is decimated, whereas in DIF the frequency domain sequence is decimated.
24. In radix-2 FFT algorithm, the N-point DFT can be realized from two numbers of N/2 point DFTs, the N/2 point DFT can be realized from two numbers of N/4 points DFTs, and so on.
25. In radix-2 FFT, N/2 butterflies per stage are required to represent the computational process.
26. In radix-2 DIT FFT, the input should be in bit reversed order and the output will be in normal order.
27. In radix-2 DIF FFT, the input should be in normal order and the output will be in bit reversed order.
28. In butterfly computation of DIT, the multiplication of phase factor takes place before the add-subtract operation.
29. In butterfly computation of DIF, the multiplication of phase factor takes place after the add-subtract operation.
30. In FFT, the phase factor for computing inverse DFT will be conjugate of phase factors for computing DFT.

5.11 Short Questions and Answers

Q5.1 Calculate the DFT of the sequence, x(n) = {1, 1, –2, –2}.

Solution

The N-point DFT of x(n) is given by,

$$\mathcal{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}} \quad ; \text{ for } k = 0,\ 1,\ 2, \ldots\ldots N-1$$

Since x(n) is a 4-point sequence, we can take 4-point DFT.

$$\therefore\ X(k) = \sum_{n=0}^{3} x(n)e^{-j\frac{2\pi nk}{4}} = x(0)e^{0} + x(1)e^{-j\frac{\pi k}{2}} + x(2)e^{-j\pi k} + x(3)e^{-j\frac{3\pi k}{2}}$$

$$= 1 + e^{-j\frac{\pi k}{2}} - 2e^{-j\pi k} - 2e^{-j\frac{3\pi k}{2}} \quad ; \text{ for } k = 0,\ 1,\ 2,\ 3$$

Q5.2 ***Find the DFT of the sequence x(n) = {1, 1, 0, 0}. Also find magnitude and phase sequence.***

Solution

The N-point DFT of x(n) is given by,

$$\mathcal{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}} \quad ; \text{ for } k = 0, 1, 2, \ldots, N-1$$

Since x(n) is a 4-point sequence, we can take 4-point DFT.

$$\therefore\ X(k) = \sum_{n=0}^{3} x(n)e^{-j\frac{2\pi nk}{4}} = x(0)e^0 + x(1)e^{-j\frac{\pi k}{2}} + x(2)e^{-j\pi k} + x(3)e^{-j\frac{3\pi k}{2}}$$

$$= 1 + e^{-j\frac{\pi k}{2}} + 0 + 0 = e^{-j\frac{\pi k}{4}}\left(e^{j\frac{\pi k}{4}} + e^{-j\frac{\pi k}{4}}\right)$$

$$\boxed{e^{j\theta}\,e^{-j\theta} = 1}$$

$$= e^{-j\frac{\pi k}{4}}\, 2\cos\left(\frac{\pi k}{4}\right) = 2\cos\left(\frac{\pi k}{4}\right)e^{-j\frac{\pi k}{4}} \quad ; \text{ for } k = 0, 1, 2, 3$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\therefore\ |X(k)| = 2\cos\left(\frac{\pi k}{4}\right) \text{ and } \angle X(k) = -\frac{\pi k}{4} \quad ; \text{ for } k = 0, 1, 2, 3$$

Q5.3 ***Compute the DFT of the sequence x(n) = (–1)ⁿ for the period N = 16.***

Solution

Given that, $x(n) = (-1)^n = \{\ldots\ldots 1, -1, 1, -1, 1, -1, \ldots\ldots\ldots\}$. On evaluating the sequence for all values of n, it can be observed that x(n) is periodic with periodicity of 2 samples. The DFT of x(n) has to be computed for the period N = 16. Let us consider the 16-sample of the infinite sequence from n = 0 to n = 15.

$$\boxed{\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}}$$

$$\boxed{e^{j\theta} \times e^{-j\theta} = 1}$$

$$\boxed{e^{-j\theta} = \cos\theta - j\sin\theta}$$

The 16-point DFT of x(n) is given by,

$$X(k) = \sum_{n=0}^{15} x(n)e^{-j\frac{2\pi nk}{16}} = \sum_{n=0}^{15} (-1)^n \times e^{-j\frac{\pi nk}{8}} = \sum_{n=0}^{15}\left(-e^{-j\frac{\pi k}{8}}\right)^n$$

$$= \frac{1-\left(-e^{-j\frac{\pi k}{8}}\right)^{16}}{1-\left(-e^{-j\frac{\pi k}{8}}\right)} = \frac{1 - e^{-j\frac{\pi k 16}{8}}}{1 + e^{-j\frac{\pi k}{8}}} = \frac{1 - e^{-j2\pi k}}{1 + e^{-j\frac{\pi k}{8}}}$$

$$= \frac{1-(\cos 2\pi k - j\sin 2\pi k)}{e^{-j\frac{\pi k}{16}}\left(e^{j\frac{\pi k}{16}} + e^{-j\frac{\pi k}{16}}\right)} = \frac{1-\cos 2\pi k}{e^{-j\frac{\pi k}{16}}\, 2\cos\frac{\pi k}{16}}$$

For interger k, $\sin 2pk = 0$.

$$= \frac{1-\cos 2\pi k}{2\cos\frac{\pi k}{16}}\, e^{j\frac{\pi k}{16}} \quad ; \text{ for } k = 0, 1, 2, 3, \ldots\ldots.15$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Q5.4 ***Find the inverse DFT of Y(k) = {1, 0, 1, 0}.***

Solution

The inverse DFT of the sequence Y(k) of length 4 is given by,

$$\mathcal{DFT}^{-1}\{Y(k)\} = y(n) = \frac{1}{4}\sum_{k=0}^{3} Y(k)e^{j\frac{2\pi kn}{4}} \;;\; \text{for } n = 0,\ 1,\ 2,\ 3$$

$$\therefore\ y(n) = \frac{1}{4}\left[Y(0)e^{0} + Y(1)e^{j\frac{\pi n}{2}} + Y(2)e^{j\pi n} + Y(3)e^{j\frac{3\pi n}{2}}\right]$$

For interger n, sinpn = 0.

$$= \frac{1}{4}\left[1+0+e^{j\pi n}+0\right] = \frac{1}{4}\left[1+\cos\pi n + j\sin\pi n\right] = 0.25(1+\cos\pi n) \;;\; \text{for } n = 0,\ 1,\ 2,\ 3$$

When n = 0; y(0) = 0.25 (1 + cos 0) = 0.5

When n = 1; y(1) = 0.25 (1 + cos p) = 0

When n = 2; y(2) = 0.25 (1 + cos 2p) = 0.5

When n = 3; y(3) = 0.25 (1 + cos 3p) = 0

\ y(n) = {0.5, 0, 0.5, 0}

Q5.5 ***Calculate the percentage saving in calculations in a 512-point radix-2 FFT, when compared to direct DFT.***

Solution

Direct computation of DFT

Number of complex additions = N(N – 1) = 512 ´ (512 – 1) = 2,61,632

Number of complex multiplications = N^2 = 512^2 = 2,62,144

Radix-2 FFT

Number of complex additions = $N\log_2 N$ = 512 ´ $\log_2 512$

= 512 ´ $\log_2 2^9$ = 512 ´ 9 = 4,608

$$\text{Number of complex multiplications} = \frac{N}{2}\log_2 N = \frac{512}{2}\times\log_2 512$$

$$= \frac{512}{2}\times\log_2 2^9 = \frac{512}{2}\times 9 = 2304$$

Percentage Saving

$$\text{Percentage saving in additions} = 100 - \frac{\text{Number of additions in radix-2 FFT}}{\text{Number of additions in direct DFT}}\times 100$$

$$= 100 - \frac{4{,}608}{2{,}61{,}632}\times 100 = 98.2\%$$

$$\text{Percentage saving in multiplications} = 100 - \frac{\text{Number of multiplications in radix-2 FFT}}{\text{Number of multiplications in direct DFT}}\times 100$$

$$= 100 - \frac{2{,}304}{2{,}62{,}144}\times 100 = 99.1\%$$

Q5.6 ***Arrange the 8-point sequence, x(n) = {1, 2, 3, 4, –1, –2, –3, –4} in bit reversed order.***

The x(n) in normal order = {1, 2, 3, 4, –1, –2, –3, –4}

The x(n) in bit reversed order = {1, –1, 3, –3, 2, –2, 4, –4}

Q5.7 Compare the DIT and DIF radix-2 FFT.

DIT raidx-2 FFT	DIF radix-2 FFT
1. The time domain sequence is decimated.	1. The frequency domain sequence is decimated.
2. The input should be in bit reversed order, the output will be in normal order.	2. The input should be in normal order, the output will be in bit reversed order.
3. In each stage of computations, the phase factors are multiplied before add and subtract operations.	3. In each stage of computations, the phase factors are multiplied after add and subtract operations.
4. The value of N should be expressed such that $N = 2^m$ and this algorithm consists of m stages of computations.	4. The value of N should be expressed such that, $N = 2^m$ and this algorithm consists of m stages of computations.
5. Total number of arithmetic operations are $N\log_2 N$ complex additions and $(N/2)\log_2 N$ complex multiplications.	5. Total number of arithmetic operations are $N\log_2 N$ complex additions and $(N/2)\log_2 N$ complex multiplications.

Q5.8 What are direct (or slow) convolution and fast convolution?

The response of an LTI system is given by convolution of input and impulse response.

The computation of the response of the LTI system by convolution sum formula is called slow convolution because it involves very large number of calculations.

The number of calculations in DFT computations can be reduced to a very large extent by FFT algorithms. Hence computation of the response of the LTI system by FFT algorithm is called fast convolution.

Q5.9 Why is FFT needed?

The FFT is needed to compute DFT with reduced number of calculations. The DFT is required for spectrum analysis and filtering operations on the signals using digital computers.

Q5.10 What is bin spacing?

Solution

The N-point DFT of x(n) is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}} = \sum_{n=0}^{N-1} x(n)W_N^{nk}$$

$$\text{where, } W_N^{nk} = \left(e^{-j2\pi}\right)^{\frac{nk}{N}} \text{ is the phase factor or twiddle factor.}$$

The phase factors are equally spaced around the unit circle at frequency increments of F_s/N where F_s is the sampling frequency of the time domain signal. This frequency increment or resolution is called bin spacing. (The X(k) consists of N-numbers of frequency samples whose discrete frequency locations are given by $f_k = kF_s/N$, for k = 0, 1, 2, ... N–1).

5.12 MATLAB Programs

Program 5.1

Write a MATLAB program to perform circular convolution of the discrete time sequences x1(n)={0,1,0,1} and x1(n)={1,2,1,2} using DFT.

```
% Program to perform Circular Convolution via DFT

clear all
clc

N = 4;              % declare the value of N
x1 = [0,1,0,1];     % declare the input sequences
x2 = [1,2,1,2];

disp('The 4-point DFT of  x1(n) is,');
X1 = fft(x1,N)      % compute 4-point DFT of x1(n)

disp('The 4-point DFT of  x2(n) is,');
X2 = fft(x2,N)      % compute 4-point DFT of x2(n)

disp('The product of DFTs is,');
X1X2 = X1.*X2       % product of DFTs

disp('Circular convolution of x1(n) and x2(n) is,');
X3 = ifft(X1X2)     % perform IDFT to get result of circular convolution
```

OUTPUT

```
The 4-point DFT of  x1(n) is,
X1 =
     2     0    -2     0

The 4-point DFT of  x2(n) is,
X2 =
     6     0    -2     0

The product of DFTs is,
X1X2 =
    12     0     4     0

Circular convolution of x1(n) and x2(n) is,
X3 =
     4     2     4     2
```

Note : Verify the above result with example 5.3.

Program 5.2

Write a MATLAB program to perform 16-point DFT of the discrete time sequence x(n)={1/3,1/3,1/3} and sketch the magnitude and phase spectrum.

```
% program to find DFT and frequency spectrum

clear all
clc

N = 16;                % specify the length of the DFT
j = sqrt(-1);

xn = zeros (1,N);      % initialize input sequence as zeros
```

```
xn(1) = 1/3;           %let given sequence be first three samples
xn(2) = 1/3;
xn(3) = 1/3;
Xk = zeros (1,N);      %initialize output sequence as zeros

for k = 0:1:N-1        % compute DFT
     for n = 0:1:N-1
      Xk(k+1) = Xk(k+1)+xn(n+1)*exp(-j*2*pi*k*n/N);
     end
end

disp ('The DFT sequence is,'); Xk
disp ('The Magnitude sequence is,');MagXk = abs(Xk)
disp ('The Phase sequence is,');PhaXk = angle(Xk)

Wk=0:1:N-1;            %specify a discrete frequency vector

subplot(2,1,1)
stem(Wk,MagXk);
title('Magnitude spectrum')
xlabel('k'); ylabel('MagXk')

subplot(2,1,2)
stem(Wk,PhaXk);
title('Phase spectrum')
xlabel('k'); ylabel('PhaXk')
```

OUTPUT

```
The DFT sequence is,
Xk =
   Columns 1 through 7
    1.0000        0.8770 - 0.3633i   0.5690 - 0.5690i     0.2252 - 0.5437i
    0 - 0.3333i  -0.0299 - 0.0723i   0.0976 + 0.0976i

  Columns 8 through 14

    0.2611 + 0.1081i   0.3333  + 0.0000i 0.2611 - 0.1081i   0.0976 - 0.0976i
   -0.0299 + 0.0723i   -0.0000  + 0.3333i  0.2252 + 0.5437i

  Columns 15 through 16
    0.5690 + 0.5690i    0.8770 + 0.3633i

The Magnitude sequence is,
MagXk =
  Columns 1 through 12
    1.0000   0.9493   0.8047   0.5885   0.3333    0.0782    0.1381   0.2826
    0.3333   0.2826   0.1381   0.0782

  Columns 13 through 16
    0.3333   0.5885   0.8047   0.9493

The Phase sequence is,
PhaXk =
   Columns 1 through 12
    0         -0.3927    -0.7854    -1.1781    -1.5708   -1.9635   0.7854   0.3927
    0.0000    -0.3927    -0.7854     1.9635

   Columns 13 through 16
    1.5708    1.1781    0.7854     0.3927
```

Note : Verify the above results with example 4.6 and example 5.1.

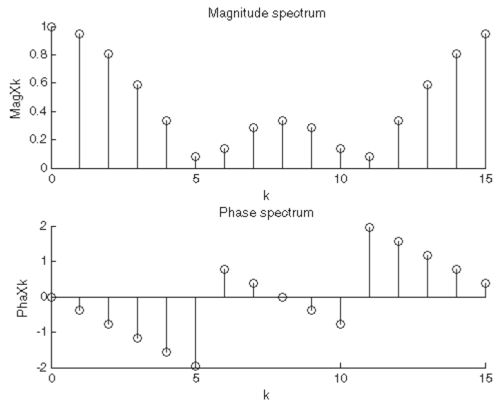


***Fig P5.2** : Magnitude and phase spectrum of program 5.2.*

The magnitude and phase spectrum of program 5.2 are shown in fig P5.2.

Program 5.3

Write a MATLAB program to perform 8-point DFT of the discrete time sequence x(n)={2,1,2,1,1,2,1,2} and sketch the magnitude and phase spectrum.

```
% program to find DFT and frequency spectrum

clear all
clc
N = 8;                        % specify the length of the DFT
j=sqrt(-1);
xn = [2,1,2,1,1,2,1,2];       % input sequence
Xk = zeros (1,N);             % initialize output sequence as zeros

for k = 0:1:N-1               % compute DFT
     for n = 0:1:N-1
      Xk(k+1) = Xk(k+1)+xn(n+1)*exp(-j*2*pi*k*n/N);
     end
end

disp ('The DFT sequence is,'); Xk
disp ('The Magnitude sequence is,');MagXk = abs(Xk)
disp ('The Phase sequence is,');PhaXk = angle(Xk)

Wk=0:1:N-1;                   % specify a discrete frequency vector
subplot(2,1,1)
stem(Wk,MagXk);
title('Magnitude spectrum')
xlabel('k'); ylabel('MagXk')
subplot(2,1,2)
stem(Wk,PhaXk);
title('Phase spectrum')
xlabel('k'); ylabel('PhaXk')
```

OUTPUT

```
The DFT sequence is,
Xk =
    12.0000       1.0000 + 0.4142i  -0.0000 - 0.0000i   1.0000 + 2.4142i
    0 - 0.0000i  1.0000 - 2.4142i  -0.0000 - 0.0000i   1.0000 - 0.4142i
```

```
The Magnitude sequence is,
MagXk =
    12.0000   1.0824    0.0000    2.6131    0.0000    2.6131    0.0000    1.0824

The Phase sequence is,
PhaXk =
    0         0.3927    -2.3201   1.1781    -1.5708   -1.1781   -2.9644   -0.3927
```

Note : Verify the above results with example 5.5.

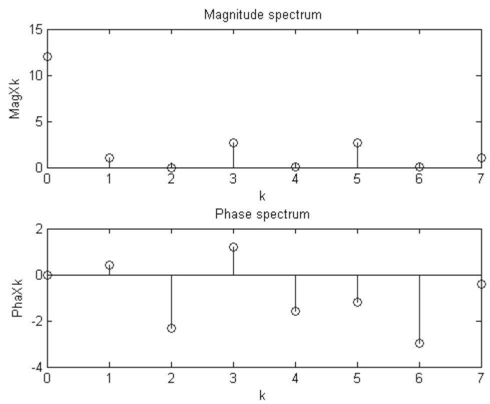


***Fig P5.3** : Magnitude and phase spectrum of program 5.3.*

The magnitude and phase spectrum of program 5.3 are shown in fig P5.3.

Program 5.4

Write a MATLAB program to perform inverse DFT. Take the frequency domain output sequence of program 5.3 as input.

```
% program to compute N-point inverse DFT

clear all
clc
N = 8;                    % declare the length of the inverse DFT
j=sqrt(-1);
                          % Xk is input sequence
xk = [12, 1+j*0.4142,0,1+j*2.4142,0,1-j*2.4142,0,1-j*0.4142];
xn = zeros (1,N); %initialize output sequence as zeros

for n= 0:1:N-1            % compute inverse DFT
for k = 0:1:N-1
xn(n+1) = xn(n+1)+(Xk(k+1)*exp(j*2*pi*n*k/N))/N;
end
end
disp('The inverse DFT sequence is,' ); xn
```

OUTPUT

```
The inverse DFT sequence is,
xn =
   2.0000 + 0.0000i   1.0000 + 0.0000i   2.0000 - 0.0000i    1.0000 + 0.0000i
   1.0000 + 0.0000i   2.0000 - 0.0000i   1.0000 + 0.0000i    2.0000 - 0.0000i
```

Program 5.5

Write a MATLAB program to perform 4-point DFT of the discrete time sequence x(n)={1,1,2,3} using function FFT and sketch the magnitude and phase spectrum.

Also perform inverse DFT on the frequency domain sequence using function IFFT, to extract the time domain sequence.

```
% program to demonstrate DFT and inverse DFT Computation using FFT

clear all
clc

N = 4;                     % specify the value of N
xn = [1,1,2,3];            % input Sequence

disp('DFT of the sequence xn is, ')
Xk = fft(xn,N)             % compute N-point DFT of input

disp('The magnitude sequence is, ')
MagXk = abs(Xk)            % compute magnitude spectrum

disp('The phase sequence is, ')
PhaXk = angle(Xk)          % compute phase spectrum

disp('inverse DFT of the sequence Xk is, ')
Xn = ifft(Xk)              % compute inverse DFT

n = 0:1:N-1;               % declare a discrete time vector
Wk = 0:1:N-1;              % declare a discrete frequency vector

subplot(2,2,1)             % Plot the input sequence
stem(n,xn)
title(' Input sequence')
xlabel('n'); ylabel('xn')

subplot(2,2,2)
stem(n,Xn)
title('inverse DFT sequence')       % Plot the inverse DFT sequence
xlabel('n'); ylabel('Xn')

subplot(2,2,3)             % Plot the magnitude spectrum
stem(Wk,MagXk)
title('Magnitude spectrum')
xlabel('k'); ylabel('MagXk')

subplot(2,2,4)             % Plot the frequency spectrum
stem(Wk,PhaXk)
title('Phase spectrum')
xlabel('k'); ylabel('PhaXk')
```

OUTPUT

```
DFT of the sequence xn is,
Xk =
   7.0000      -1.0000 + 2.0000i      -1.0000        -1.0000 - 2.0000i

The magnitude sequence is,
MagXk =
    7.0000     2.2361       1.0000       2.2361
```

```
The phase sequence is,
PhaXk =
     0    2.0344    3.1416   -2.0344

inverse DFT of the sequence Xk is,
Xn =
     1       1        2        3
```

The input sequence, inverse DFT sequence, magnitude spectrum, and phase spectrum of program 5.5 are shown in fig P5.5.

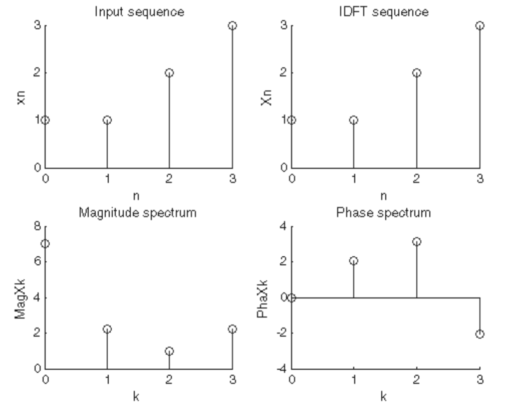


Fig P5.5 : Iuput sequence, Magnitude spectrum and phase spectrum of program 5.5.

5.13 Exercises

I. Fill in the blanks with appropriate words

1. In an N-point DFT of a finite duration sequence x(n) of length L, the value of N should be such that_______.
2. The N-point DFT of a L-point sequence will have a periodicity of _______.
3. The convolution property of DFT says that DFT{x(n)⊛ h(n)} = _______.
4. The N-point DFT of a sequence is given by Z-transform of the sequence at N equally spaced points around the _______ in z-plane.
5. The convolution by FFT is called _______.
6. The convolution using convolution sum formula is called _______.
7. Appending zeros to a sequence in order to increase its length is called _______.
8. In DFT computation using radix-2 FFT, the value of N should be such that _______.
9. The number of complex additions and multiplications in radix-2 FFT are _______ and _______ respectively.
10. The number of complex additions and multiplications in direct DFT are _______ and _______ respectively.
11. In 8-point DFT by radix-2 FFT there are _______ stages of computations with _______ butterflies per stage.
12. In _______ butterfly diagram the _______ is multiplied after add-subtract operations.

Answers

1. N ³ L	4. unit circle	7. zero padding	10. N(N–1), N^2
2. N-samples	5. fast convolution	8. N = 2^m	11. four, four
3. X(k) H(k)	6. slow convolution	9. $N\log_2N$, $(N/2)\log_2N$	12. DIF, phase factor

II State whether the following statements are True/False

1. The DFT of a sequence is a continuous function of w.
2. The DFT of a signal can be obtained by sampling one period of Fourier transform of the signal.
3. In sampling $X(e^{jw})$, the value of sample at w = 0 is same as the value of sample at w=2p.
4. The DFT of even sequence is purely imaginary and DFT of odd sequence is purely real.
5. In a DFT of real sequence, the real component is even and imaginary component is odd.
6. The multiplication of the DFTs of the two sequences is equal to the DFT of the linear convolution of two sequences.
7. The DFT supports only circular convolution.
8. In FFT algorithm the N-point DFT is decomposed into successively smaller DFTs.
9. In N-point DFT using radix-2 FFT, the decimation is performed m times, where $m=\log_2 N$.
10. Both DIT and DIF algorithms involves same number of computations.
11. Bit reversing is required for both DIT and DIF algorithms.

Answers

1. False	3. True	5. True	7. True	9. True	11. True
2. True	4. False	6. False	8. True	10. True	

III. Choose the right answer for the following questions

1. In N-point DFT of L-point sequence, the value of N to avoid aliasing in frequency spectrum is,

a) N „ L **b)** N £ L

c) N ‡ L **d)** N = L

2. The inverse DFT of x(n) can be expressed as,

a) $x(n) = \frac{1}{N} \sum_{k=0}^{N} X(k)\, e^{-\frac{j2pkn}{N}}$ **b)** $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2pkn}{N}}$

c) $x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(n)\, e^{-\frac{j2pkn}{N}}$ **d)** $x(n) = N \sum_{n=0}^{N-1} X(k)\, e^{-\frac{j2pkn}{N}}$

3. If DFT{x(n)} = X(k), then DFT{$x(n+m)_N$}

a) $X(k)\, e^{\frac{-j2pkm}{N}}$ **b)** $X(k)\, e^{\frac{-j2pk}{mN}}$

c) $X(k)\, e^{\frac{j2pkm}{N}}$ **d)** $X(k)\, e^{\frac{j2pk}{mN}}$

4. The DFT of product of two discrete time sequences $x_1(n)$ and $x_2(n)$ is equivalent to,

a) $\frac{1}{N}\left[X_1(k) \circledast X_2(k)\right]$ **b)** $\frac{1}{N}\left[X_1(k)\, X_2(k)\right]$

c) $\frac{1}{N}\left[X_1(k) \circledast X_2^*(k)\right]$ **d)** $X_1(k) \circledast X_2(k)$

5. By correlation property, the DFT of circular correlation of two sequences x(n) and y(n) is,

a) $X(k)Y^*(k)$ **b)** $X(k)\circledast Y(k)$

c) $X(k)\circledast Y^*(k)$ **d)** $X(k)\,Y(k)$

6. The N-point DFT of a finite duration sequence can be obtained as,

a) $X(k) = X(z)\Big|_{z = e^{\frac{j2pn}{N}}}$ **b)** $X(k) = X(z)\Big|_{z = e^{\frac{j2pk}{N}}}$

c) $X(k) = X(z)\Big|_{z = e^{-\frac{j2pkn}{N}}}$ **d)** $X(k) = X(z)\Big|_{z = e^{\frac{j2pkn}{N}}}$

7. In an N-point sequence, if N = 16, the total number of complex additions and multiplications using Radix-2 FFT are,

a) 64 and 80 **b)** 80 and 64

c) 64 and 32 **d)** 24 and 12

8. The complex valued phase factor/twiddle factor, W_N can be represented as,

a) e^{-j2pN} **b)** $e^{-\frac{j2p}{N}}$

c) e^{-j2p} **d)** e^{-j2pkN}

9. The phase factors are multiplied before the add and subtract operations in,

a) DIT radix-2 FFT **b)** DIF radix-2 FFT

c) inverse DFT **d)** both a and c

10. If X(k) consists of N-number of frequency samples, then its discrete frequency locations are given by,

a) $f_k = \frac{kF_s}{N}$ **b)** $f_k = \frac{F_s}{N}$

c) $f_k = \frac{kN}{F_s}$ **d)** $f_k = N$

Answers

1. c	3. c	5. a	7. c	9. a
2. b	4. a	6. b	8. b	10. a

IV. Answer the Following questions

1. Define DFT of a discrete time sequence.
2. Define inverse DFT.
3. What is the relation between DTFT and DFT?
4. What is the drawback in Fourier transform and how is it overcome?
5. List any four properties of DFT.
6. State and prove the shifting property of DFT.
7. What is FFT?
8. What is radix-2 FFT?
9. How many multiplications and additions are involved in radix-2 FFT?
10. What is DIT radix-2 FFT?
11. What is phase factor or twiddle factor?
12. Draw and explain the basic butterfly diagram or flow graph of DIT radix-2 FFT.
13. What are the phase factors involved in the third stage of computation in the 8-point DIT radix-2 FFT?
14. What is DIF radix-2 FFT?
15. Draw and explain the basic butterfly diagram or flow graph of DIF radix-2 FFT.
16. What are the phase factors involved in first stage of computation in 8-point DIF radix-2 FFT?
17. How will you compute inverse DFT using radix-2 FFT algorithm?
18. What is magnitude and phase spectrum?

V. Solve the Following Problems

E5.1 ***Compute 4-point DFT and 8-point DFT of causal sequence given by,*** $x(n) = \frac{1}{8}$; $0 \le n \le 3$

$= 0$; *else*

E5.2 ***Compute DFT of the sequence,*** $x(n) = \{0, 2, 3, -1\}$ ***. Sketch the magnitude and phase spectrum.***

E5.3 ***Compute DFT of the sequence,*** $x(n) = \{1, 3, 3, 3\}$ ***. Sketch the magnitude and phase spectrum.***

E5.4 ***Compute circular convolution of the following sequences using DFT.***

$x_1(n) = \{\underset{\uparrow}{-1}, 2, -2, -1\}$ ***and*** $x_2(n) = \{\underset{\uparrow}{1}, -2, -1, -2\}$.

E5.5 ***Compute linear and circular convolution of the following sequences using DFT.***

$x(n) = \{1, 0.2, -1\}$, $h(n) = \{1, -1, 0.2\}$.

E5.6 ***Compute 8-point DFT of the discrete time signal,*** $x(n) = \{1, 2, 1, 2, 1, 3, 1, 3\}$,

a) using radix-2 DIT FFT and ***b)*** using radix-2 DIF FFT.

Also sketch the magnitude and phase spectrum.

E5.7 ***In an LTI system the input, $x(n) = \{1, 2, 1\}$ and the impulse response, $h(n) = \{1, 3\}$.***
Determine the response of LTI system by radix-2 DIT FFT.

E5.8 ***Compute the DFT and plot the magnitude and phase spectrum of the discrete time sequence, $x(n) = \{4, 4, 0, 2\}$, and verify the result using the inverse DFT.***

E5.9 ***Determine the response of LTI system when the input sequence, $x(n) = \{-2, -1, -1, 0, 2\}$ by radix 2 DIT FFT. The impulse response of the system is, $h(n) = \{1, -1, -1, 1\}$.***

Answers

E5.1 4 - point DFT: $X(k) = \{0.5,\ 0,\ 0,\ 0\}$

$$\text{8 - point DFT: } X(k) = \left\{ \begin{array}{r} 0.5–0,\ 0.326– - 0.374p,\ 0,\ 0.135– - 0.125p, \\ 0,\ 0.135–0.125p,\ 0,\ 0.326–0.374p \end{array} \right\}$$

E5.2 $X(k) = \{4–0,\ 4.243– - 0.75p,\ 2–0,\ 4.243–0.75p\}$

$|X(k)| = \{4,\ 4.243,\ 2,\ 4.243\}$

$– X(k) = \{0,\ - 0.75p,\ 0,\ 0.75p\}$

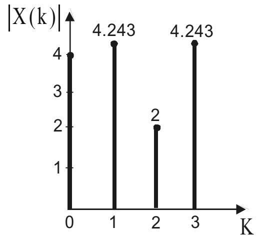


Fig E5.2.1 : *Magnitude spectrum.*

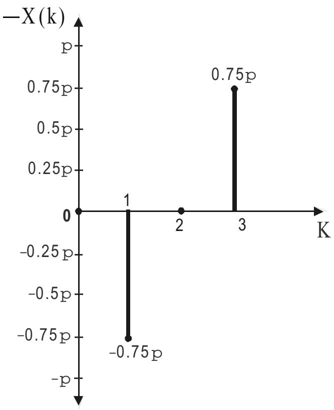


Fig E5.2.2 : *Phase spectrum.*

E5.3 $X(k) = \{10–0,\ 2–p,\ 2–p,\ 2–p\}$

$|X(k)| = \{10,\ 2,\ 2,\ 2\}$

$–X(k) = \{0, p,\ p,\ p\}$

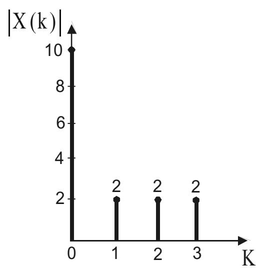


Fig E5.3.1 : *Magnitude spectrum.*

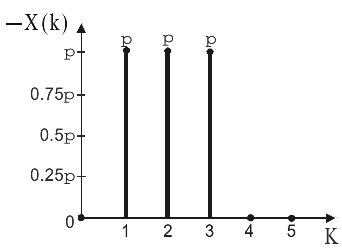


Fig E5.3.2 : *Phase spectrum.*

E5.4 $x_1(n) \circledast x_2(n) = \{-1,\ 9,\ -3,\ 3\}$

E5.5 $x(n) * h(n) = \{1,\ -0.8,\ -1,\ 1.04,\ -0.2\}$

$x(n) \circledast h(n) = \{2.04,\ -1,\ -1\}$

E5.6 $X(k) = \{14,\ j1.414,\ 0,\ j1.414,\ -6,\ -j1.414,\ 0,\ -j1.414\}$

$= \{14,\ 1.414–0.5p,\ 0,\ 1.414–0.5p,\ 6–p,\ 1.414–-0.5p,\ 0,\ 1.414–-0.5p\}$

$|X(k)| = \{14,\ 1.414,\ 0,\ 1.414,\ 6,\ 1.414,\ 0,\ 1.414\}$

$–X(k) = \{0,\ 0.5p,\ 0,\ 0.5p,\ p,\ -0.5p,\ 0,\ -0.5p\}$

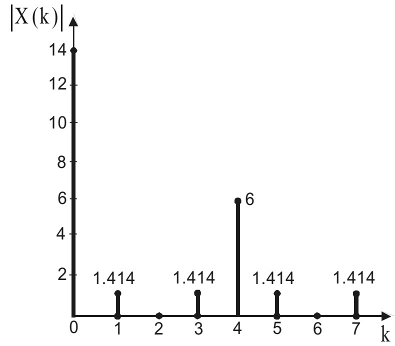


Fig E5.6.1 : *Magnitude spectrum.*

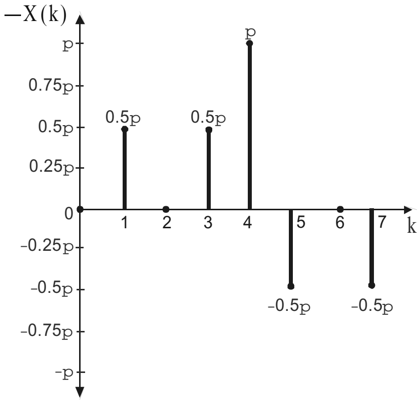


Fig E5.6.2 : *Phase spectrum.*

E5.7 $y(n) = \{1,\ 5,\ 7,\ 3\}$

E5.8 $X(k) = \{10,\ 4-j2,\ -2,\ 4+j2\}$

$= \{10–0,\ 4.472–-0.15p,\ 2–p,\ 4.472–0.15p\}$

$|X(k)| = \{10,\ 4.472,\ 2,\ 4.472\}$

$–X(k) = \{0,\ -0.15p,\ p,\ 0.15p\}$

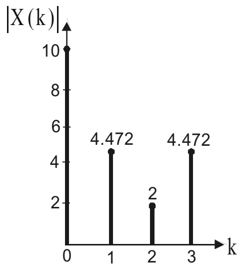


Fig E5.8.1 : *Magnitude spectrum.*

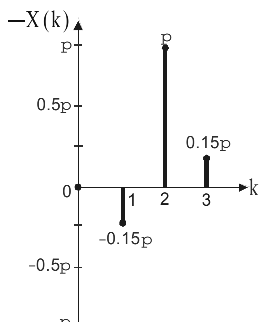


Fig E5.8.2 : *Phase spectrum.*

E5.9 $y(n) = \{-2,\ 1,\ 2,\ 0,\ 2,\ -3,\ -2,\ 1\}$

Solution for Exercise Problems

E5.1. Compute 4-point DFT and 8-point DFT of causal sequence given by,

a) $x(n) = \frac{1}{8}\ ; \quad 0 \le n \le 3$

$\quad = 0 \qquad else$

Solution

By definition,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

Case (i) : 4-point DFT, (\ N = 4)

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi kn}{4}} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi}{2}kn} = x(0)e^{0} + x(1)e^{-j\frac{\pi}{2}k} + x(2)e^{-j\pi k} + x(3)e^{-j\frac{3\pi}{2}k}$$

$$= \frac{1}{8} + \frac{1}{8}e^{-j\frac{\pi}{2}k} + \frac{1}{8}e^{-j\pi k} + \frac{1}{8}e^{-j\frac{3\pi}{2}k} = \frac{1}{8}\left[1 + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + \cos\pi k - j\sin\pi k + \cos\frac{3\pi}{2}k - j\sin\frac{3\pi}{2}k\right]$$

For 4-point DFT, the X(k) has to be evaluated for, k = 0, 1, 2, 3.

When $k = 0\ ;\ X(0) = \frac{1}{8}\left[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0\right]$

$$= \frac{1}{8}\left[1 + 1 - j0 + 1 - j0 + 1 - j0\right] = \frac{4}{8} = 0.5$$

When $k = 1\ ;\ X(1) = \frac{1}{8}\left[1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$

$$= \frac{1}{8}\left[1 + 0 - j - 1 - j0 + 0 + j\right] = 0$$

When $k = 2\ ;\ X(2) = \frac{1}{8}\left[1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi + \cos 3\pi - j\sin 3\pi\right]$

$$= \frac{1}{8}\left[1 - 1 - j0 + 1 - j0 - 1 - j0\right] = 0$$

When $k = 3\ ;\ X(3) = \frac{1}{8}\left[1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi + \cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right]$

$$= \frac{1}{8}\left[1 + 0 + j - 1 - j0 + 0 - j\right] = 0$$

$$\therefore\ X(k) = \{0.5,\ 0,\ 0,\ 0\}$$

Case (ii) : 8-point DFT, (\ N = 8)

$$X(k) = \sum_{n=0}^{7} x(n) e^{-j\frac{2\pi kn}{8}} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi kn}{4}} = x(0)e^{0} + x(1)e^{-j\frac{\pi}{4}k} + x(2)e^{-j\frac{\pi}{2}k} + x(3)e^{-j\frac{3\pi k}{4}}$$

$$= \frac{1}{8}\left[1 + \cos\frac{\pi k}{4} - j\sin\frac{\pi k}{4} + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + \cos\frac{3\pi k}{4} - j\sin\frac{3\pi k}{4}\right]$$

When $k = 0\ ;\ X(0) = \frac{1}{8}\left[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0\right] = \frac{1}{8}\left[1 + 1 - j0 + 1 - j0 + 1 - j0\right] = \frac{4}{8} = 0.5\angle 0$

When $k = 1\ ;\ X(1) = \frac{1}{8}\left[1 + \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}\right]$

$$= \frac{1}{8}\left[1 + 0.707 - j0.707 + 0 - j - 0.707 - j0.707\right]$$

$$= \frac{1}{8}\left[1 - j2.414\right] = 0.125 - j0.302 = 0.326\ \angle -1.177 = 0.326\ \angle -0.374\pi$$

$$\boxed{\frac{1.177}{\pi} \times \pi = 0.374\pi}$$

When $k = 2$; $X(2) = \frac{1}{8}\left[1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$

$= \frac{1}{8}\left[1 + 0 - j - 1 - j0 + 0 + j\right] = 0$

When $k = 3$; $X(3) = \frac{1}{8}\left[1 + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos\frac{9\pi}{4} - j\sin\frac{9\pi}{4}\right]$

$= \frac{1}{8}\left[1 - 0.707 - j0.707 + 0 + j + 0.707 - j0.707\right]$

$= \frac{1}{8}\left[1 - j0.414\right] = 0.125 - j0.052 = 0.135 \angle -0.394 = 0.135 \angle -0.125\pi$

$$\boxed{\frac{0.394}{\pi} \times \pi = 0.125\pi}$$

When $k = 4$; $X(4) = \frac{1}{8}\left[1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi + \cos 3\pi - j\sin 3\pi\right]$

$= \frac{1}{8}\left[1 - 1 - j0 + 1 - j0 - 1 + j0\right] = 0$

When $k = 5$; $X(5) = \frac{1}{8}\left[1 + \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} + \cos\frac{5\pi}{2} - j\sin\frac{5\pi}{2} + \cos\frac{15\pi}{4} - j\sin\frac{15\pi}{4}\right]$

$= \frac{1}{8}\left[1 - 0.707 + j0.707 + 0 - j + 0.707 + j0.707\right]$

$= \frac{1}{8}\left[1 + j0.414\right] = 0.125 + j0.052 = 0.135 \angle 0.394 = 0.135 \angle 0.125\pi$

When $k = 6$; $X(6) = \frac{1}{8}\left[1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi + \cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right]$

$= \frac{1}{8}\left[1 + 0 + j - 1 - j0 + 0 - j\right] = 0$

When $k = 7$; $X(7) = \frac{1}{8}\left[1 + \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4} + \cos\frac{7\pi}{2} - j\sin\frac{7\pi}{2} + \cos\frac{21\pi}{4} - j\sin\frac{21\pi}{4}\right]$

$= \frac{1}{8}\left[1 + 0.707 + j0.707 + j - 0.707 + j0.707\right] = \frac{1}{8}\left[1 + j2.414\right]$

$= 0.125 + j0.302 = 0.326 \angle 1.177 = 0.326 \angle 0.374\pi$

$$\boxed{\frac{1.177}{\pi} \times \pi = 0.374\pi}$$

The 8-point DFT sequence is given by,

$$X(k) = \{0.5 \angle 0,\ \ 0.326 \angle -0.374\pi,\ \ 0,\ \ 0.135 \angle -0.125\pi,\ \ 0,\ \ 0.135 \angle 0.125\pi,\ \ 0,\ \ 0.326 \angle 0.374\pi\}$$

$$\therefore\ \ |X(k)| = \{0.5,\ \ 0.326,\ \ 0,\ \ 0.135,\ \ 0,\ \ 0.135,\ \ 0,\ \ 0.326\}$$

$$\angle X(k) = \{0,\ \ -0.374\pi,\ \ 0,\ \ -0.125\pi,\ \ 0,\ \ 0.125\pi,\ \ 0,\ \ 0.374\pi\}$$

E5.2. Compute DFT of the sequence, x(n) = {0, 2, 3, –1}. Sketch the magnitude and phase spectrum.

Solution

By definition, the 4 point DFT is given by,

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi kn}{4}} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi}{2}kn}$$

$$= x(0) + x(1) e^{-j\frac{\pi}{2}k} + x(2) e^{-j\pi k} + x(3) e^{-j\frac{3\pi}{2}k}$$

$$= 0 + 2\left[\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right] + 3\left[\cos\pi k - j\sin\pi k\right] - \left(\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right)$$

$$= 2\cos\frac{\pi k}{2} - j2\sin\frac{\pi k}{2} + 3\cos\pi k - j3\sin\pi k - \cos\frac{3\pi k}{2} + j\sin\frac{3\pi k}{2}$$

When $k = 0$; $X(0) = 2\cos 0 - j2\sin 0 + 3\cos 0 - j3\sin 0 - \cos 0 + j\sin 0$

$= 2 - j0 + 3 - j0 - 1 + j0 = 4 = 4 \angle 0$

When $k = 1$; $X(1) = 2\cos\frac{\pi}{2} - j2\sin\frac{\pi}{2} + 3\cos\pi - j3\sin\pi - \cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}$

$= 0 - j2 - 3 - j0 - 0 - j = -3 - 3j = 4.243 \angle -2.356 = 4.243 \angle -0.75\pi$

$$\boxed{\frac{2.356}{\pi} \times \pi = 0.75\pi}$$

When $k = 2$; $X(2) = 2\cos\pi - j2\sin\pi + 3\cos 2\pi - j3\sin 2\pi - \cos 3\pi + j\sin 3\pi$

$= -2 - j0 + 3 - j0 + 1 + j0 = 2 = 2 \angle 0$

When $k = 3$; $X(3) = 2\cos\frac{3\pi}{2} - j2\sin\frac{3\pi}{2} + 3\cos 3\pi - j3\sin 3\pi - \cos\frac{9\pi}{2} + j\sin\frac{9\pi}{2}$

$= 0 + j2 - 3 - j0 - 0 + j = -3 + 3j = 4.243 \angle 2.356 = 4.243 \angle 0.75\pi$

$$\therefore \quad X(k) = \{4\angle 0,\ 4.243\angle -0.75\pi,\ 2\angle 0,\ 4.243\angle 0.75\pi\}$$

$$|X(k)| = \{4,\ 4.243,\ 2,\ 4.243\}$$

$$\angle X(k) = \{0,\ -0.75\pi,\ 0,\ 0.75\pi\}$$

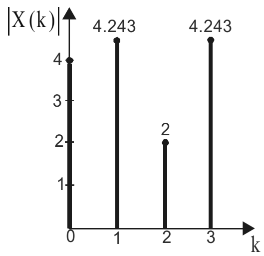


Fig 1 : *Magnitude spectrum.*

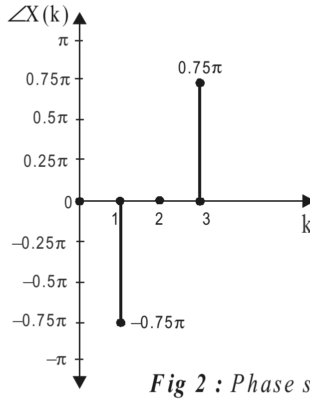


Fig 2 : *Phase spectrum.*

E5.3. Compute DFT of the sequence, $x(n) = \{1, 3, 3, 3\}$.

Sketch the magnitude and phase spectrum.

Solution

$$x(n) = \{\underset{\uparrow}{1},\ 3,\ 3,\ 3\}$$

By definition, the 4-point DFT is,

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi kn}{4}} = x(0) + x(1)e^{-j\frac{2\pi k}{4}} + x(2)e^{-j\frac{4\pi k}{4}} + x(3)e^{-j\frac{6\pi k}{4}}$$

$$= 1 + 3\left[\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right] + 3[\cos\pi k - j\sin\pi k] + 3\left[\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right]$$

For integer k, $\sin \pi k = 0$.

$$= 1 + 3\cos\frac{\pi k}{2} - j3\sin\frac{\pi k}{2} + 3\cos\pi k + 3\cos\frac{3\pi k}{2} - j3\sin\frac{3\pi k}{2}$$

When $k = 0$; $X(0) = 1 + 3\cos 0 - j3\sin 0 + 3\cos 0 + 3\cos 0 - j\sin 0$

$$= 1 + 3 - j0 + 3 + 3 - j0 = 10 = 10\angle 0$$

When $k = 1$; $X(1) = 1 + 3\cos\frac{\pi}{2} - j3\sin\frac{\pi}{2} + 3\cos\pi + 3\cos\frac{3\pi}{2} - j3\sin\frac{3\pi}{2}$

$$= 1 + 0 - j3 - 3 + 0 + j3 = -2 = 2\angle\pi$$

When $k = 2$; $X(2) = 1 + 3\cos\pi - j3\sin\pi + 3\cos 2\pi + 3\cos 3\pi - j3\sin 3\pi$

$$= 1 - 3 - j0 + 3 - 3 - j0 = -2 = 2\angle\pi$$

When $k = 3$; $X(3) = 1 + 3\cos\frac{3\pi}{2} - j3\sin\frac{3\pi}{2} + 3\cos 3\pi + 3\cos\frac{9\pi}{2} - j3\sin\frac{9\pi}{2}$

$$= 1 + 0 + j3 - 3 + 0 - j3 = -2 = 2\angle\pi$$

$$\therefore \quad X(k) = \{10\angle 0,\ 2\angle\pi,\ 2\angle\pi,\ 2\angle\pi\}$$

$$|X(k)| = \{10,\ 2,\ 2,\ 2\}$$

$$\angle X(k) = \{0,\ \pi,\ \pi,\ \pi\}$$

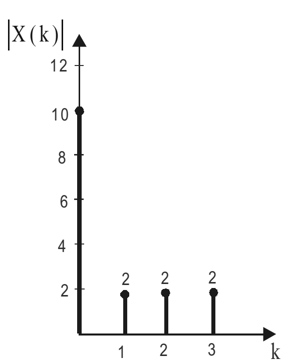


Fig 1 : *Magnitude spectrum.*

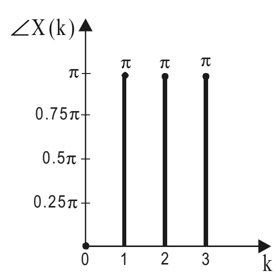


Fig 2 : *Phase spectrum.*

E5.4. Compute circular convolution of the following sequences using DFT.

$$x_1(n) = \{\underset{-}{-1},\ 2,\ -2,\ -1\} \quad \text{and} \quad x_2(n) = \{\underset{-}{1},\ -2,\ -1,\ -2\}$$

Solution

Given that, $x_1(n) = \{\underset{\uparrow}{-1},\ 2,\ -2,\ -1\}$

The 4-point DFT of $x_1(n)$ is,

$$X_1(k) = \sum_{n=0}^{4-1} x_1(n) e^{-j\frac{2\pi nk}{4}} = \sum_{n=0}^{3} x_1(n) e^{-j\frac{\pi nk}{2}} \quad ; \ k = 0,\ 1,\ 2,\ 3$$

$$= x_1(0)e^0 + x_1(1)e^{-j\frac{\pi k}{2}} + x_1(2)e^{-j\pi k} + x_1(3)e^{-j\frac{3\pi k}{2}}$$

$$= -1 + 2e^{-j\frac{\pi k}{2}} - 2e^{-j\pi k} - e^{-j\frac{3\pi k}{2}}$$

$$= -1 + 2\left(\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right) - 2(\cos\pi k - j\sin\pi k) - \left(\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right)$$

$$= -1 + 2\frac{\cos\pi k}{2} - j2\sin\frac{\pi k}{2} - 2\cos\pi k - \cos\frac{3\pi k}{2} + j\sin\frac{3\pi k}{2}$$

For integer k, $\sin \pi k = 0$.

When $k = 0$; $X(0) = -1 + 2\cos\frac{\pi\times 0}{2} - j2\sin\frac{\pi\times 0}{2} - 2\cos\pi\times 0 - \cos\frac{3\pi\times 0}{2} + j\sin\frac{3\pi\times 0}{2} = -2$

When $k = 1$; $X(1) = -1 + 2\cos\frac{\pi\times 1}{2} - j2\sin\frac{\pi\times 1}{2} - 2\cos\pi\times 1 - \cos\frac{3\pi\times 1}{2} + j\sin\frac{3\pi\times 1}{2} = 1 - j3$

When $k = 2$; $X(2) = -1 + 2\cos\frac{\pi\times 2}{2} - j2\sin\frac{\pi\times 2}{2} - 2\cos\pi\times 2 - \cos\frac{3\pi\times 2}{2} + j\sin\frac{3\pi\times 2}{2} = -4$

When $k = 3$; $X(3) = -1 + 2\cos\frac{\pi\times 3}{2} - j2\sin\frac{\pi\times 3}{2} - 2\cos\pi\times 3 - \cos\frac{3\pi\times 3}{2} + j\sin\frac{3\pi\times 3}{2} = 1 + j3$

$\therefore\ X_1(k) = \{-2,\ 1 - j3,\ -4,\ 1 + j3\}$

Given that, $x_2(n) = \{\underset{\uparrow}{1},\ -2,\ -1,\ -2\}$

The 4-point DFT of $x_2(n)$ is,

$$X_2(k) = \sum_{n=0}^{4-1} x_2(n) e^{-j\frac{2\pi nk}{4}} = \sum_{n=0}^{3} x_2(n) e^{-j\frac{\pi nk}{2}} \quad ; \ k = 0,\ 1,\ 2,\ 3$$

$$= x_2(0)e^0 + x_2(1)e^{-j\frac{\pi k}{2}} + x_2(2)e^{-j\pi k} + x_2(3)e^{-j\frac{3\pi k}{2}}$$

$$= 1 - 2e^{-j\frac{\pi k}{2}} - e^{-j\pi k} - 2e^{-j\frac{3\pi k}{2}}$$

$$= 1 - 2\left(\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right) - (\cos\pi k - j\sin\pi k) - 2\left(\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right)$$

$$= 1 - 2\cos\frac{\pi k}{2} + j2\sin\frac{\pi k}{2} - \cos\pi k - 2\cos\frac{3\pi k}{2} + j2\sin\frac{3\pi k}{2}$$

For integer k, $\sin \pi k = 0$.

When $k = 0$; $X(0) = 1 - 2\cos\frac{\pi\times 0}{2} + j2\sin\frac{\pi\times 0}{2} - \cos\pi\times 0 - 2\cos\frac{3\pi\times 0}{2} + j2\sin\frac{3\pi\times 0}{2} = -4$

When $k = 1$; $X(1) = 1 - 2\cos\frac{\pi\times 1}{2} + j2\sin\frac{\pi\times 1}{2} - \cos\pi\times 1 - 2\cos\frac{3\pi\times 1}{2} + j2\sin\frac{3\pi\times 1}{2} = 2$

When $k = 2$; $X(2) = 1 - 2\cos\frac{\pi\times 2}{2} + j2\sin\frac{\pi\times 2}{2} - \cos\pi\times 2 - 2\cos\frac{3\pi\times 2}{2} + j2\sin\frac{3\pi\times 2}{2} = 4$

When $k = 3$; $X(3) = 1 - 2\cos\frac{\pi\times 3}{2} + j2\sin\frac{\pi\times 3}{2} - \cos\pi\times 3 - 2\cos\frac{3\pi\times 3}{2} + j2\sin\frac{3\pi\times 3}{2} = 2$

$\therefore\ X_2(k) = \{-4,\ 2,\ 4,\ 2\}$

Let, $X_3(k) = X_1(k)\, X_2(k)$

When $k = 0$; $X_3(0) = X_1(0) \times X_2(0) = -2 \times -4 = 8$

When $k = 1$; $X_3(1) = X_1(1) \times X_2(1) = (1 - j3) \times 2 = 2 - j6$

When $k = 2$; $X_3(2) = X_1(2) \times X_2(2) = -4 \times 4 = -16$

When $k = 3$; $X_2(3) = X_1(3) \times X_2(3) = (1 + j3) \times 2 = 2 + j6$

By convolution theorem of DFT,

$$\mathcal{DFT}\{x_1(n) \circledast x_2(n)\} = X_1(k)X_2(k)$$

$$\therefore \quad x_1(n) \circledast x_2(n) = \mathcal{DFT}^{-1}\{X_1(k)X_2(k)\} = \mathcal{DFT}^{-1}\{X_3(k)\} = x_3(n)$$

By definition of inverse DFT,

$$x_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)e^{j\frac{2\pi nk}{N}} = \frac{1}{4}\sum_{k=0}^{3} X_3(k)e^{j\frac{2\pi nk}{4}} \quad ; \; n = 0,\ 1,\ 2,\ 3$$

$$= \frac{1}{4}\left[X_3(0)e^0 + X_3(1)e^{j\frac{2\pi n}{4}} + X_3(2)e^{j\frac{4\pi n}{4}} + X_3(3)e^{j\frac{6\pi n}{4}}\right]$$

$$= \frac{1}{4}\times 8 + \frac{1}{4}(2-j6)\left(\cos\frac{\pi n}{2} + j\sin\frac{\pi n}{2}\right) - \frac{1}{4}\times 16(\cos\pi n + j\sin\pi n) + \frac{1}{4}(2+j6)\left(\cos\frac{3\pi n}{2} + j\sin\frac{3\pi n}{2}\right)$$

$$= 2 + \left(\frac{2-j6}{4}\right)\left(\cos\frac{\pi n}{2} + j\sin\frac{\pi n}{2}\right) - 4\cos\pi n + \frac{2+j6}{4}\left(\cos\frac{3\pi n}{2} + j\sin\frac{3\pi n}{2}\right)$$

For integer n, $\sin n\pi = 0$.

When $n = 0$; $x_3(0) = 2 + \dfrac{(2-j6)}{4}\left(\cos\dfrac{\pi\times 0}{2} + j\sin\dfrac{\pi\times 0}{2}\right) - 4\cos\pi\times 0 + \dfrac{2+j6}{4}\left(\cos\dfrac{3\pi\times 0}{2} + j\sin\dfrac{3\pi\times 0}{2}\right) = -1$

When $n = 1$; $x_3(1) = 2 + \dfrac{(2-j6)}{4}\left(\cos\dfrac{\pi\times 1}{2} + j\sin\dfrac{\pi\times 1}{2}\right) - 4\cos\pi\times 1 + \dfrac{2+j6}{4}\left(\cos\dfrac{3\pi\times 1}{2} + j\sin\dfrac{3\pi\times 1}{2}\right) = 9$

When $n = 2$; $x_3(2) = 2 + \dfrac{(2-j6)}{4}\left(\cos\dfrac{\pi\times 2}{2} + j\sin\dfrac{\pi\times 2}{2}\right) - 4\cos\pi\times 2 + \dfrac{2+j6}{4}\left(\cos\dfrac{3\pi\times 2}{2} + j\sin\dfrac{3\pi\times 2}{2}\right) = -3$

When $n = 3$; $x_3(3) = 2 + \dfrac{(2-j6)}{4}\left(\cos\dfrac{\pi\times 3}{2} + j\sin\dfrac{\pi\times 3}{2}\right) - 4\cos\pi\times 3 + \dfrac{2+j6}{4}\left(\cos\dfrac{3\pi\times 3}{2} + j\sin\dfrac{3\pi\times 3}{2}\right) = 3$

$$\therefore \quad x_3(n) = x_1(n) \circledast x_2(n) = \{-1,\ 9,\ -3,\ 3\}$$

E5.5. Compute linear and circular convolution of the following sequences using DFT.

$$x(n) = \{1,\ 0.2,\ -1\}, \qquad h(n) = \{1,\ -1,\ 0.2\}$$

Solution

Linear Convolution

Given that, x(n) = {1, 0.2, –1}

h(n) = {1, –1, 0.2}

The given sequences are 3-point sequences.

Therefore the length of output sequence of linear convolution will be 3 + 3 – 1 = 5.

Hence convert x(n) and h(n) to 5-point sequences by appending zero.

\ x(n) = {1, 0.2, –1, 0, 0}

h(n) = {1, –1, 0.2, 0, 0}

By definition, the 5-point DFT of x(n) is,

$$X(k) = \sum_{n=0}^{4} x(n)e^{-j\frac{2\pi kn}{5}} = \sum_{n=0}^{2} x(n)e^{-j\frac{2\pi kn}{5}} \quad ; \; k = 0,\ 1,\ 2,\ 3,\ 4$$

$$= x(0)e^0 + x(1)e^{-j\frac{2\pi k}{5}} + x(2)e^{-j\frac{4\pi k}{5}}$$

$$= 1 + 0.2\left(\cos\frac{2\pi k}{5} - j\sin\frac{2\pi k}{5}\right) - \left(\cos\frac{4\pi k}{5} - j\sin\frac{4\pi k}{5}\right)$$

$$= 1 + 0.2\cos\frac{2\pi k}{5} - j0.2\sin\frac{2\pi k}{5} - \cos\frac{4\pi k}{5} + j\sin\frac{4\pi k}{5}$$

When $k = 0$; $X(0) = 1 + 0.2\cos\dfrac{2\pi\times 0}{5} - j0.2\sin\dfrac{2\pi\times 0}{5} - \cos\dfrac{4\pi\times 0}{5} + j\sin\dfrac{4\pi\times 0}{5}$

$= 0.2 + j0$

When $k = 1$; $X(1) = 1 + 0.2\cos\dfrac{2\pi\times 1}{5} - j0.2\sin\dfrac{2\pi\times 1}{5} - \cos\dfrac{4\pi\times 1}{5} + j\sin\dfrac{4\pi\times 1}{5}$

$= 1.871 + j0.398$

When $k = 2$; $X(2) = 1 + 0.2\cos\frac{2\pi\times2}{5} - j0.2\sin\frac{2\pi\times2}{5} - \cos\frac{4\pi\times2}{5} + j\sin\frac{4\pi\times2}{5}$

$= 0.529 - j1.069$

When $k = 3$; $X(3) = 1 + 0.2\cos\frac{2\pi\times3}{5} - j0.2\sin\frac{2\pi\times3}{5} - \cos\frac{4\pi\times3}{5} + j\sin\frac{4\pi\times3}{5}$

$= 0.529 + j1.069$

When $k = 4$; $X(4) = 1 + 0.2\cos\frac{2\pi\times4}{5} - j0.2\sin\frac{2\pi\times4}{5} - \cos\frac{4\pi\times4}{5} + j\sin\frac{4\pi\times4}{5}$

$= 1.871 - j0.398$

$$\therefore\ X(k) = \{0.2 + j0,\ 1.871 + j0.398,\ 0.529 - j1.069,\ 0.529 + j1.069,\ 1.871 - j0.398\}$$

By definition, the 5-point DFT of h(n) is,

$$H(k) = \sum_{n=0}^{4} h(n)\,e^{-j\frac{2\pi kn}{5}} = \sum_{n=0}^{2} h(n)\,e^{-j\frac{2\pi kn}{5}}\ ;\quad k = 0,\ 1,\ 2,\ 3,\ 4$$

$$= h(0)e^0 + h(1)e^{-j\frac{2\pi k}{5}} + h(2)e^{-j\frac{4\pi k}{5}}$$

$$= 1 - \left(\cos\frac{2\pi k}{5} - j\sin\frac{2\pi k}{5}\right) + 0.2\left(\cos\frac{4\pi k}{5} - j\sin\frac{4\pi k}{5}\right)$$

$$= 1 - \cos\frac{2\pi k}{5} + j\sin\frac{2\pi k}{5} + 0.2\cos\frac{4\pi k}{5} - j0.2\sin\frac{4\pi k}{5}$$

When $k = 0$; $H(0) = 1 - \cos\frac{2\pi\times0}{5} + j\sin\frac{2\pi\times0}{5} + 0.2\cos\frac{4\pi\times0}{5} - j0.2\sin\frac{4\pi\times0}{5}$

$= 0.2 + j0$

When $k = 1$; $H(1) = 1 - \cos\frac{2\pi\times1}{5} + j\sin\frac{2\pi\times1}{5} + 0.2\cos\frac{4\pi\times1}{5} - j0.2\sin\frac{4\pi\times1}{5}$

$= 0.529 + j0.834$

When $k = 2$; $H(2) = 1 - \cos\frac{2\pi\times2}{5} + j\sin\frac{2\pi\times2}{5} + 0.2\cos\frac{4\pi\times2}{5} - j0.2\sin\frac{4\pi\times2}{5}$

$= 1.871 + j0.778$

When $k = 3$; $H(3) = 1 - \cos\frac{2\pi\times3}{5} + j\sin\frac{2\pi\times3}{5} + 0.2\cos\frac{4\pi\times3}{5} - j0.2\sin\frac{4\pi\times3}{5}$

$= 1.871 - j0.778$

When $k = 4$; $H(4) = 1 - \cos\frac{2\pi\times4}{5} + j\sin\frac{2\pi\times4}{5} + 0.2\cos\frac{4\pi\times4}{5} - j0.2\sin\frac{4\pi\times4}{5}$

$= 0.529 - j0.834$

$$\therefore\ H(k) = \{0.2 + j0,\ 0.529 + j0.834,\ 1.871 + j0.778,\ 1.871 - j0.778,\ 0.529 - j0.834\}$$

Let, $Y(k) = X(k)\ H(k)$ for, $k = 0,\ 1,\ 2,\ 3,\ 4.$

When, $k = 0$; $Y(0) = X(0)\ H(0) = [0.2 + j0] \times [0.2 + j0] = 0.04 + j0$

When, $k = 1$; $Y(1) = X(1)\ H(1) = [1.871 + j0.398] \times [0.529 + j0.834] = 0.658 + j1.771$

When, $k = 2$; $Y(2) = X(2)\ H(2) = [0.529 - j1.069] \times [1.871 + j0.778] = 1.821 - j1.589$

When, $k = 3$; $Y(3) = X(3)\ H(3) = [0.529 + j1.069] \times [1.871 - j0.778] = 1.821 + j1.589$

When, $k = 4$; $Y(4) = X(4)\ H(4) = [1.871 - j0.398] \times [0.529 - j0.834] = 0.658 - j1.771$

$$\therefore\ Y(k) = \{0.04 + j0,\ 0.658 + j1.771,\ 1.821 - j1.589,\ 1.821 + j1.589,\ 0.658 - j1.771\}$$

By convolution property of DFT,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\ H(k)$$

$$\therefore\ x(n) \circledast h(n) = \mathcal{DFT}^{-1}\{X(k)\ H(k)\} = \mathcal{DFT}^{-1}\{Y(k)\} = y(n)$$

By definition of inverse DFT,

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)e^{j\frac{2\pi nk}{N}} = \frac{1}{5}\sum_{k=0}^{4} Y(k)e^{j\frac{2\pi nk}{5}} \quad ; \quad n = 0,\ 1,\ 2,\ 3,\ 4$$

$$= \frac{1}{5}Y(0)e^{0} + \frac{1}{5}Y(1)e^{j\frac{2\pi n}{5}} + \frac{1}{5}Y(2)e^{j\frac{4\pi n}{5}} + \frac{1}{5}Y(3)e^{j\frac{6\pi n}{5}} + \frac{1}{5}Y(4)e^{j\frac{8\pi n}{5}}$$

$$= \frac{1}{5}\times 0.04 + \frac{1}{5}\ (0.658 + j1.771)\left(\cos\frac{2\pi n}{5} + j\sin\frac{2\pi n}{5}\right) + \frac{1}{5}\ (1.821 - j1.589)\left(\cos\frac{4\pi n}{5} + j\sin\frac{4\pi n}{5}\right)$$

$$+ \frac{1}{5}\ (1.821 + j1.589)\left(\cos\frac{6\pi n}{5} + j\sin\frac{6\pi n}{5}\right) + \frac{1}{5}\ (0.658 - j1.771)\left(\cos\frac{8\pi n}{5} + j\sin\frac{8\pi n}{5}\right)$$

$$= 0.008 + (0.132 + j0.354)\ (\cos 0.4\pi n + j\sin 0.4\pi n) + (0.364 - j0.318)\ (\cos 0.8\pi n + j\sin 0.8\pi n)$$

$$+ (0.364 + j0.318)\ (\cos 1.2\pi n + j\sin 1.2\pi n) + (0.132 - j0.354)\ (\cos 1.6\pi n + j\sin 1.6\pi n)$$

When $n = 0$; $y(0) = \left[0.008 + (0.132 + j0.354)\ (\cos 0 + j\sin 0) + (0.364 - j0.318)\ (\cos 0 + j\sin 0)\right]$

$$+\left[(0.364 + j0.318)\ (\cos 0 + j\sin 0) + (0.132 - j0.354)\ (\cos 0 + j\sin 0)\right]$$

$$= [0.504 + j0.036] + [0.496 - j0.036]$$

$$= 1$$

When $n = 1$; $y(1) = \left[0.008 + (0.132 + j0.354)\ (\cos 0.4\pi + j\sin 0.4\pi) + (0.364 - j0.318)\ (\cos 0.8\pi + j\sin 0.8\pi)\right]$

$$+ \left[(0.364 + j0.318)\ (\cos 1.2\pi + j\sin 1.2\pi) + (0.132 - j0.354)\ (\cos 1.6\pi + j\sin 1.6\pi)\right]$$

$$= [-0.395 + j0.706] + [0.403 - j0.706]$$

$$= -0.798 \approx -0.8$$

When $n = 2$; $y(2) = \left[0.008 + (0.132 + j0.354)\ (\cos 0.8\pi + j\sin 0.8\pi) + (0.364 - j0.318)\ (\cos 1.6\pi + j\sin 1.6\pi)\right]$

$$+\left[(0.364 + j0.318)\ (\cos 2.4\pi + j\sin 2.4\pi) + (0.132 - j0.354)\ (\cos 3.2\pi + j\sin 3.2\pi)\right]$$

$$= [-0.476 - j0.624] + [0.505 + j0.653]$$

$$= -0.981 + j0.029 \approx -1$$

When $n = 3$; $y(3) = \left[0.008 + (0.132 + j0.354)\ (\cos 1.2\pi + j\sin 1.2\pi) + (0.364 - j0.318)\ (\cos 2.4\pi + j\sin 2.4\pi)\right]$

$$+\left[(0.364 + j0.318)\ (\cos 3.6\pi + j\sin 3.6\pi) + (0.132 - j0.354)\ (\cos 4.8\pi + j\sin 4.8\pi)\right]$$

$$= [0.524 - j0.116] + [0.516 - j0.116]$$

$$= 1.04$$

When $n = 4$; $y(4) = \left[0.008 + (0.132 + j0.354)\ (\cos 1.6\pi + j\sin 1.6\pi) + (0.364 - j0.318)\ (\cos 3.2\pi + j\sin 3.2\pi)\right]$

$$+\left[(0.364 + j0.318)\ (\cos 4.8\pi + j\sin 4.8\pi) + (0.132 - j0.354)\ (\cos 6.4\pi + j\sin 6.4\pi)\right]$$

$$= [-0.096 + j0.027] + [0.104 - j0.027]$$

$$= -0.2$$

$$\therefore \quad y(n) = x(n) * h(n) = \{1,\ -0.8,\ -1,\ 1.04,\ -0.2\}$$

Circular Convolution

Given that, $x(n) = \{1, 0.2, -1\}$

The 3-point DFT of $x(n)$ is,

$$X(k) = \sum_{n=0}^{3-1} x(n)e^{-j\frac{2\pi nk}{3}} = \sum_{n=0}^{2} x(n)e^{-j\frac{2\pi nk}{3}} \quad ; \quad k = 0,\ 1,\ 2$$

$$= x(0)e^{0} + x(1)e^{-j\frac{2\pi k}{3}} + x(2)e^{-j\frac{4\pi k}{3}}$$

$$= 1 + 0.2\left(\cos\frac{2\pi k}{3} - j\sin\frac{2\pi k}{3}\right) - \left(\cos\frac{4\pi k}{3} - j\sin\frac{4\pi k}{3}\right)$$

$$= 1 + 0.2\ \cos\frac{2\pi k}{3} - j0.2\ \sin\frac{2\pi k}{3} - \cos\frac{4\pi k}{3} + j\sin\frac{4\pi k}{3}$$

When $k = 0$; $X(0) = 1 + 0.2\cos 0 - j\,0.2\sin 0 - \cos 0 + j\sin 0 = 0.2$

When $k = 1$; $X(1) = 1 + 0.2\cos\frac{2\pi}{3} - j0.2\sin\frac{2\pi}{3} - \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} = 1.4 - j1.039$

When $k = 2$; $X(2) = 1 + 0.2\cos\frac{4\pi}{3} - j0.2\sin\frac{4\pi}{3} - \cos\frac{8\pi}{3} + j\sin\frac{8\pi}{3} = 1.4 + j1.039$

$\therefore\ X(k) = \{0.2,\ 1.4 - j1.039,\ 1.4 + j1.039\}$

Given that, h(n) = {1, –1, 0.2}

The 3-point DFT of h(n) is,

$$H(k) = \sum_{n=0}^{3-1} h(n)e^{-j\frac{2\pi nk}{3}} = \sum_{n=0}^{2} h(n)e^{-j\frac{2\pi nk}{3}} \quad ; \quad k = 0,\ 1,\ 2$$

$$= h(0)e^{0} + h(1)e^{-j\frac{2\pi k}{3}} + h(2)e^{-j\frac{4\pi k}{3}}$$

$$= 1 - \left(\cos\frac{2\pi k}{3} - j\sin\frac{2\pi k}{3}\right) + 0.2\left(\cos\frac{4\pi k}{3} - j\sin\frac{4\pi k}{3}\right)$$

$$= 1 - \cos\frac{2\pi k}{3} + j\sin\frac{2\pi k}{3} + 0.2\cos\frac{4\pi k}{3} - j0.2\sin\frac{4\pi k}{3}$$

When $k = 0$; $H(0) = 1 - \cos 0 - j\sin 0 + 0.2\cos 0 - j0.2\sin 0 = 0.2$

When $k = 1$; $H(1) = 1 - \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} + 0.2\cos\frac{4\pi}{3} - j0.2\sin\frac{4\pi}{3} = 1.4 + j1.039$

When $k = 2$; $H(2) = 1 - \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} + 0.2\cos\frac{8\pi}{3} - j0.2\sin\frac{8\pi}{3} = 1.4 - j1.039$

$\therefore\ H(k) = \{0.2,\ 1.4 + j1.039,\ 1.4 - j1.039\}$

Let, Y(k) = X(k) H(k)

When, $k = 0$; $Y(0) = X(0)\ H(0) = 0.2 \times 0.2 = 0.04$

When, $k = 1$; $Y(1) = X(1)\ H(1) = [1.4 - j1.039] \times [1.4 + j1.039] = 3.04$

When, $k = 2$; $Y(2) = X(2)\ H(2) = [1.4 + j1.039] \times [1.4 - j1.039] = 3.04$

By convolution property of DFT,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\ H(k)$$

$$\therefore\ x(n) \circledast h(n) = \mathcal{DFT}^{-1}\{X(k)\ H(k)\} = \mathcal{DFT}^{-1}\{Y(k)\} = y(n)$$

By definition of inverse DFT,

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)e^{j\frac{2\pi nk}{N}} = \frac{1}{3}\sum_{k=0}^{2} Y(k)e^{j\frac{2\pi nk}{3}} \quad ; \quad n = 0,\ 1,\ 2$$

$$= \frac{1}{3}Y(0)e^{0} + \frac{1}{3}Y(1)e^{j\frac{2\pi n}{3}} + \frac{1}{3}Y(2)e^{j\frac{4\pi n}{3}}$$

$$= \frac{0.04}{3} + \frac{3.04}{3}\left(\cos\frac{2\pi n}{3} + j\sin\frac{2\pi n}{3}\right) + \frac{3.04}{3}\left(\cos\frac{4\pi n}{3} + j\sin\frac{4\pi n}{3}\right)$$

$$= 0.013 + 1.013\cos\frac{2\pi n}{3} + j1.013\sin\frac{2\pi n}{3} + 1.013\cos\frac{4\pi n}{3} + j1.013\sin\frac{4\pi n}{3}$$

When $n = 0$; $y(0) = 0.013 + 1.013\cos 0 + j1.013\sin 0 + 1.013\cos 0 + j\sin 0 = 2.039 \approx 2.04$

When $n = 1$; $y(1) = 0.013 + 1.013\cos\frac{2\pi}{3} + j1.013\sin\frac{2\pi}{3} + 1.013\cos\frac{4\pi}{3} + j1.013\sin\frac{4\pi}{3} = -1$

When $n = 2$; $y(2) = 0.013 + 1.013\cos\frac{4\pi}{3} + j1.013\sin\frac{4\pi}{3} + 1.013\cos\frac{8\pi}{3} + j1.013\sin\frac{8\pi}{3} = -1$

$\therefore\ y(n) = x(n) \circledast h(n) = \{2.04,\ -1,\ -1\}$

E5.6. Compute 8-point DFT of the discrete time signal, x(n) = {1, 2, 1, 2, 1, 3, 1, 3}

(a) using radix-2 DIT FFT and (b) using radix-2 DIF FFT.

Also sketch the magnitude and phase spectrum.

Solution

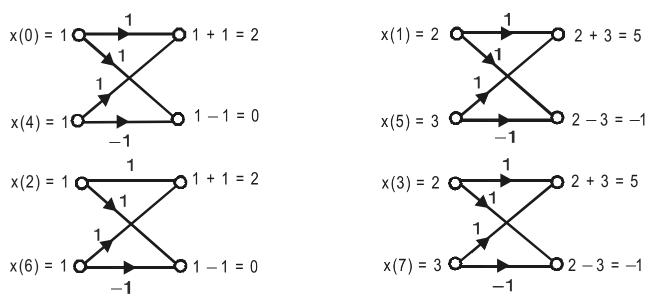
I. The 8-point DFT by radix-2 DIT FFT

The given sequence is first arranged in the bit reversed order.

Normal order	Bit reversed order
x(0) = 1	x(0) = 1
x(1) = 2	x(4) = 1
x(2) = 1	x(2) = 1
x(3) = 2	x(6) = 1
x(4) = 1	x(1) = 2
x(5) = 3	x(5) = 3
x(6) = 1	x(3) = 2
x(7) = 3	x(7) = 3

First stage computation

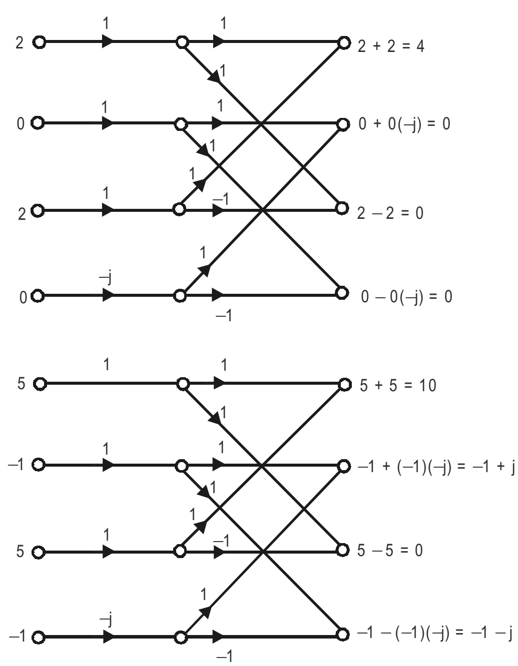
Input = { 1, 1, 1, 1, 2, 3, 2, 3 }



Output of first stage computation = { 2, 0, 2, 0, 5, –1, 5, –1}.

Second stage computation

Input = { 2, 0, 2, 0, 5, –1, 5, –1 }



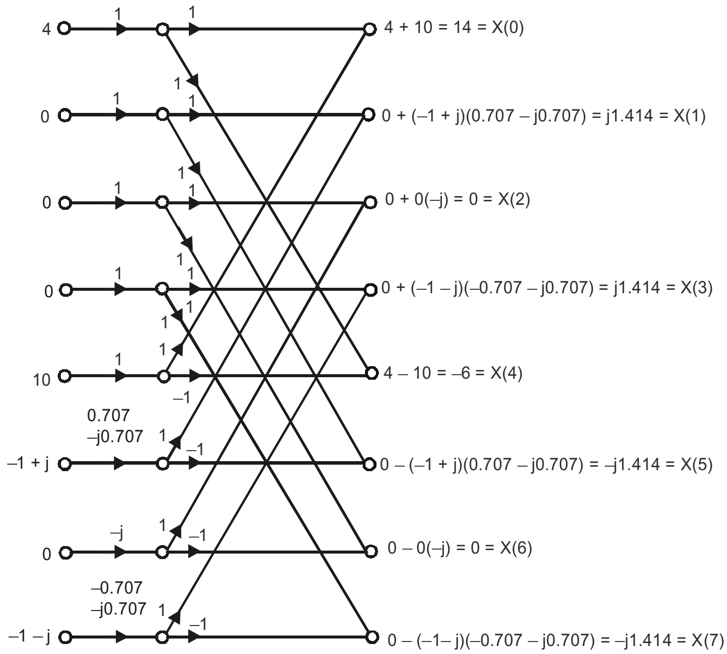
$W_4^0 = 1$

$W_4^1 = -j$

Output of second stage computation = {4, 0, 0, 0, 10, –1 + j, 0, –1 – j }

Third stage computation

Input = {4, 0, 0, 0, 10, –1 + j, 0, –1 – j }



$W_8^0 = 1$
$W_8^1 = 0.707 - j0.707$
$W_8^2 = -j$
$W_8^3 = -0.707 - j0.707$

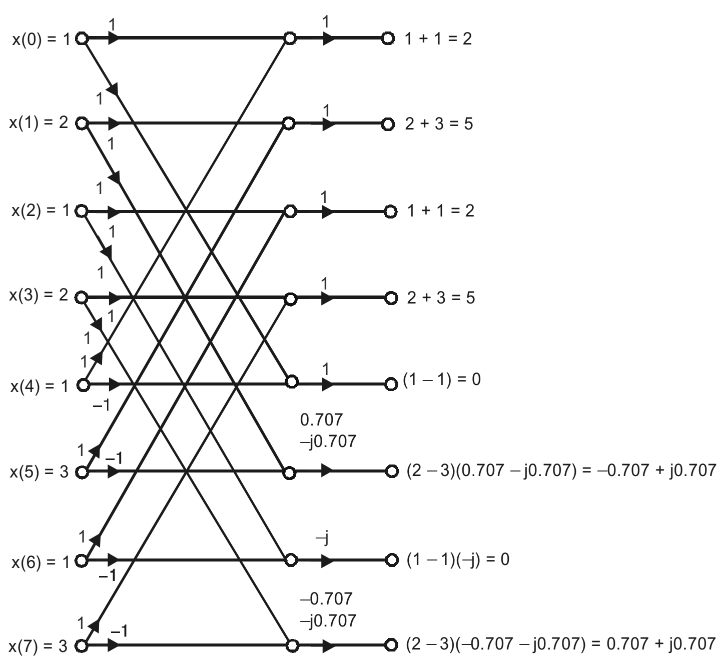
$$\left.\begin{array}{l}\text{Output of third stage}\\ \text{computation}\end{array}\right\} = \{14,\ j1.414,\ 0,\ j1.414,\ -6,\ -j1.414,\ 0,\ -j1.414\}$$

$$\therefore\ X(k) = \mathcal{DFT}\{x(n)\} = \{14,\ j1.414,\ 0,\ j1.414,\ -6,\ -j1.414,\ 0,\ -j1.414\}$$

II. 8-point DFT by radix-2 DIF-FFT

First stage computation

Input sequence = { 1, 2, 1, 2, 1, 3, 1, 3 }



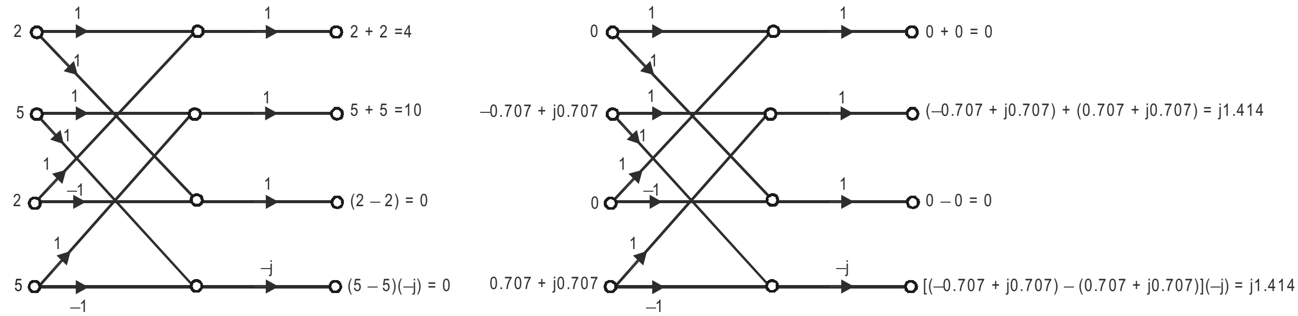
$W_8^0 = 1$
$W_8^1 = 0.707 - j0.707$
$W_8^2 = -j$
$W_8^3 = -0.707 - j0.707$

Output of first stage computation = {2, 5, 2, 5, 0, –0.707 + j0.707, 0, 0.707 + j0.707 }

Second stage computation

$$\boxed{\begin{aligned} W_4^0 &= 1 \\ W_4^1 &= -j \end{aligned}}$$

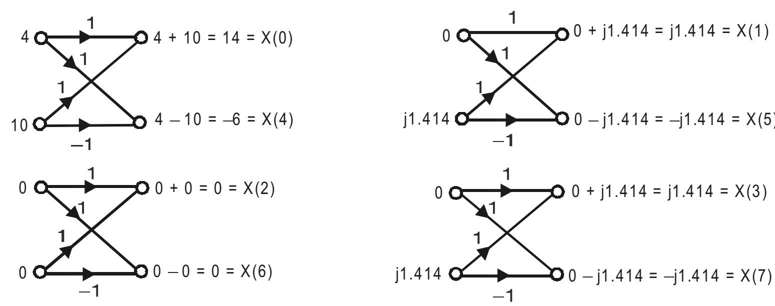
Input sequence = {2, 5, 2, 5, 0, –0.707 + j0.707, 0, 0.707 + j0.707 }



Output of second stage computation = { 4, 10, 0, 0, 0, j1.414, 0, j1.414 }.

Third stage computation

Input sequence = {4, 10, 0, 0, 0, j1.414, 0, j1.414 }.



$\therefore$ X(k) = { 14, j1.414, 0, j1.414, –6, –j1.414, 0, –j1.414 }.

Magnitude and Phase Spectrum

$$\begin{aligned} X(k) &= \{14,\ j1.414,\ 0,\ j1.414,\ -6,\ -j1.414,\ 0,\ -j1.414\} \\ &= \{14,\ 1.414\angle 90^\circ,\ 0,\ 1.414\angle 90^\circ,\ 6\angle 180^\circ,\ 1.414\angle -90^\circ,\ 0,\ 1.414\angle -90^\circ\} \\ &= 14,\ 1.414\angle 0.5\pi,\ 0,\ 1.414\angle 0.5\pi,\ 6\angle\pi,\ 1.414\angle -0.5\pi,\ 0,\ 1.414\angle -0.5\pi \end{aligned}$$

$$|X(k)| = \{14,\ 1.414,\ 0,\ 1.414,\ 6,\ 1.414,\ 0,\ 1.414\}$$

$$\angle X(k) = \{0,\ 0.5\pi,\ 0,\ 0.5\pi,\ \pi,\ -0.5\pi,\ 0,\ -0.5\pi\}$$

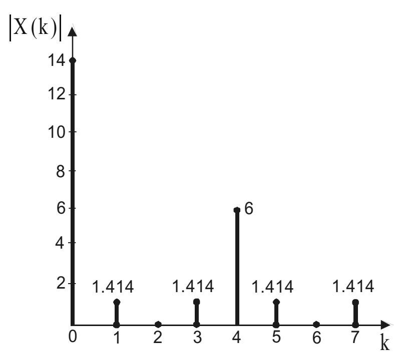


Fig 1 : Magnitude spectrum.

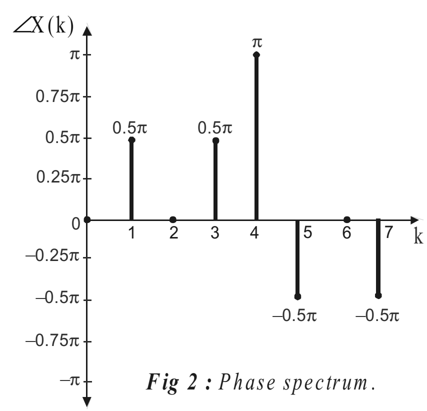


Fig 2 : Phase spectrum.

E5.7. ***In an LTI system the input, x(n) = { 1, 2, 1 } and the impulse response, h(n) = { 1, 3 }. Determine the response of LTI system by radix-2 DIT FFT.***

Solution

Response, y(n) = x(n) ∗ h(n) [linear convolution]

Given that, x(n) = { 1, 2, 1 } , h(n) = { 1, 3 }

Here, the length of x(n) is 3 and length of h(n) is 2.

$\therefore$ Length of y(n) = 3 + 2 –1 = 4

Let us convert x(n) and h(n) to 4-point sequence by appending zeros.

$\therefore$ x(n) = { 1, 2, 1, 0 } , h(n) = {1, 3, 0, 0 }

Now, y(n) = x(n) ⊛ h(n)

On taking DFT,

$$y(n) = \mathcal{DFT}\{x(n) ⊛ h(n)\}$$

Using convolution theorem.

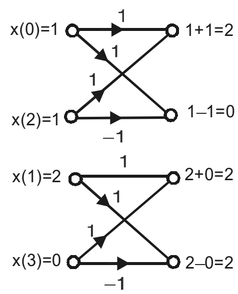
$$\backslash \quad Y(k) = X(k)\, H(k)$$

On taking inverse DFT,

$$y(n) = \mathcal{DFT}^{-1}\{X(k)\, H(k)\} = \mathcal{DFT}^{-1}\{Y(k)\}$$

Step - 1 : Determine X(k)

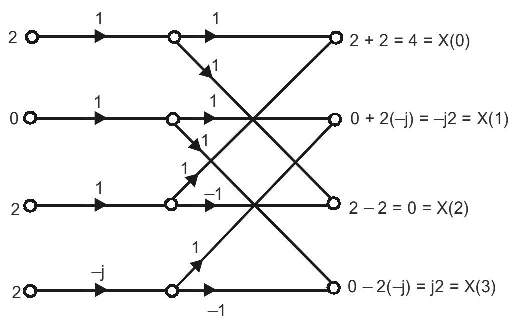
First Stage



x(n) (Normal)	x(n) (Bit reversed)
x(0) = 1	x(0) = 1
x(1) = 2	x(2) = 1
x(2) = 1	x(1) = 2
x(3) = 0	x(3) = 0

Output = {2, 0, 2, 2}

Second Stage

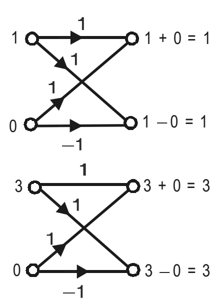


Output sequence ={4, –j2, 0, j2 }

$\backslash$ X(k) = $\mathcal{DFT}$\{x(n)\} = {4, –j2, 0, j2}

Step - 2 : Determine H(k)

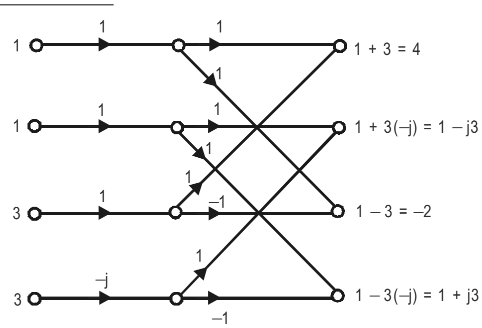
First Stage



h(n) (Normal)	h(n) (Bit reversed)
h(0) = 1	h(0) = 1
h(1) = 3	h(2) = 0
h(2) = 0	h(1) = 3
h(3) = 0	h(3) = 0

Output sequence ={1, 1, 3, 3}

Second Stage



$W_4^0 = 1$

$W_4^1 = -j$

$\backslash$ H(k) = $\mathcal{DFT}$\{h(n)\} = {4, 1-j3, - 2, 1 + j3}

Step - 3 : Determine Y(k)

$$Y(k) = X(k)\ H(k) = Y(k)$$

When $k = 0$; $Y(0) = X(0)\,H(0) = 4 \times 4 = 16$

When $k = 1$; $Y(1) = X(1)\,H(1) = -j2 \times (1 - j3) = -6 - j2$

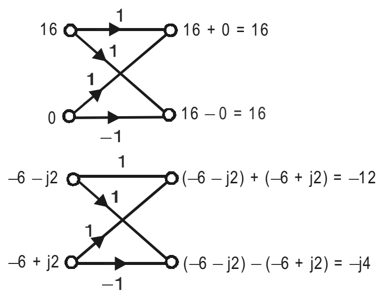
When $k = 2$; $Y(2) = X(2)\,H(2) = 0 \times (-2) = 0$

When $k = 3$; $Y(3) = X(3)\,H(3) = j2 \times (1 + j3) = -6 + j2$

$$\therefore \quad Y(k) = \{16, \quad -6 - j2, \quad 0, \quad -6 + j2\}$$

Step - 4 : Inverse DFT of Y(k)

First Stage

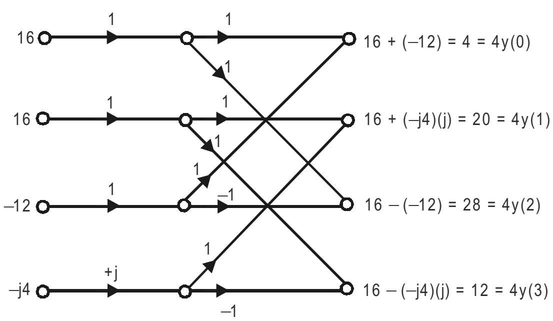


Y(k) (Normal)	**Y(k) (Bit reversed)**
$Y(0) = 16$	$Y(0) = 16$
$Y(1) = -6 - j2$	$Y(2) = 0$
$Y(2) = 0$	$Y(1) = -6 - j2$
$Y(3) = -6 + j2$	$Y(3) = -6 + j2$

$$\left(W_4^0\right)^* = 1$$
$$\left(W_4^1\right)^* = j$$

Output sequence = {16, 16, -12, –j4}

Second Stage



Output = {4, 20, 28, 12}

On dividing each sample of output sequence by 4 we get y(n),

$$y(n) = \{1, \ 5, \ 7, \ 3\}$$

E5.8. Compute the DFT and plot the magnitude and phase spectrum of the discrete time sequence, x(n) = {4, 4, 0, 2}, and verify the result using the inverse DFT.

Solution

I. DFT of x(n)

Given that, $x(n) = \{\underset{\uparrow}{4}, 4, 0, 2\}$

The 4-point DFT of x(n) is,

$$X(k) = \sum_{n=0}^{4-1} x(n)e^{-j\frac{2\pi nk}{4}} = \sum_{n=0}^{3} x(n)e^{-j\frac{\pi nk}{2}} \quad ; \quad \text{for } k = 0, 1, 2, 3$$

$$= x(0)e^0 + x(1)e^{-j\frac{\pi k}{2}} + x(2)e^{-j\pi k} + x(3)e^{-j\frac{3\pi k}{2}}$$

$$= 4 + 4\left(\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right) + 0 + 2\left(\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right)$$

$$= 4 + 4\cos\frac{\pi k}{2} - j4\sin\frac{\pi k}{2} + 2\cos\frac{3\pi k}{2} - j2\sin\frac{3\pi k}{2}$$

When $k = 0$; $X(0) = 4 + 4\cos\frac{\pi \times 0}{2} - j4\sin\frac{\pi \times 0}{2} + 2\cos\frac{3\pi \times 0}{2} - j2\sin\frac{3\pi \times 0}{2}$

$= 10 = 10\angle 0$

$\boxed{\frac{0.464}{\pi} \times \pi = 0.15\pi}$

When $k = 1$; $X(1) = 4 + 4\cos\frac{\pi \times 1}{2} - j4\sin\frac{\pi \times 1}{2} + 2\cos\frac{3\pi \times 1}{2} - j2\sin\frac{3\pi \times 1}{2}$

$= 4 - j2 = 4.472\angle -0.464 = 4.472\angle -0.15\pi$

When $k = 2$; $X(2) = 4 + 4\cos\frac{\pi \times 2}{2} - j4\sin\frac{\pi \times 2}{2} + 2\cos\frac{3\pi \times 2}{2} - j2\sin\frac{3\pi \times 2}{2}$

$= -2 = 2\angle\pi$

When $k = 3$; $X(3) = 4 + 4\cos\frac{\pi \times 3}{2} - j4\sin\frac{\pi \times 3}{2} + 2\cos\frac{3\pi \times 3}{2} - j2\sin\frac{3\pi \times 3}{2}$

$= 4 + j2 = 4.472\angle 0.464 = 4.472\angle 0.15\pi$

$\therefore\ X(k) = \{10,\ 4 - j2,\ -2,\ 4 + j2\}$

$\therefore\ X(k) = \{10\angle 0,\ 4.472\angle -0.15\pi,\ 2\angle\pi,\ 4.472\angle 0.15\pi\}$

Magnitude spectrum, $|X(k)| = \{10,\ 4.472,\ 2,\ 4.472\}$

Phase spectrum, $\angle X(k) = \{0,\ -0.15\pi,\ \pi,\ 0.15\pi\}$

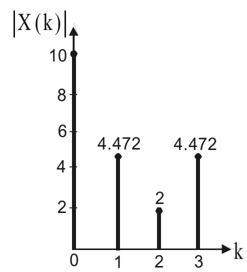


***Fig 1 :** Magnitude spectrum.*

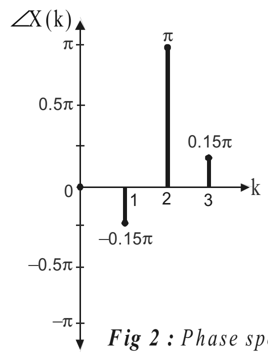


***Fig 2 :** Phase spectrum.*

II. Inverse DFT of X(k)

We know that,

$X(k) = \{10,\ 4 - j2,\ -2,\ 4 + j2\}$

The 4-point inverse DFT of x(n) is,

$$x(n) = \frac{1}{4}\sum_{k=0}^{4-1} X(k)\,e^{j\frac{2\pi nk}{4}} = \frac{1}{4}\sum_{k=0}^{3} X(k)\,e^{j\frac{\pi nk}{2}} \quad ; \quad \text{for } n = 0,\ 1,\ 2,\ 3$$

$$= \frac{1}{4}X(0)e^{0} + \frac{1}{4}X(1)e^{j\frac{\pi n}{2}} + \frac{1}{4}X(2)e^{j\pi n} + \frac{1}{4}X(3)e^{j\frac{3\pi n}{2}}$$

$$= \frac{1}{4} \times 10 + \frac{1}{4}(4 - j2)\left(\cos\frac{\pi n}{2} + j\sin\frac{\pi n}{2}\right) + \frac{1}{4}(-2)(\cos\pi n + j\sin\pi n) + \frac{1}{4}(4 + j2)\left(\cos\frac{3\pi n}{2} + j\sin\frac{3\pi n}{2}\right)$$

For integer k,
sin pn = 0.

$$= 2.5 + (1 - j0.5)\left(\cos\frac{\pi n}{2} + j\sin\frac{\pi n}{2}\right) - 0.5\cos\pi n + (1 + j0.5)\left(\cos\frac{3\pi n}{2} + j\sin\frac{3\pi n}{2}\right)$$

When $n = 0$; $x(0) = 2.5 + (1 - j0.5)\left(\cos\frac{\pi \times 0}{2} + j\sin\frac{\pi \times 0}{2}\right) - 0.5\cos\pi \times 0 + (1 + j0.5)\left(\cos\frac{3\pi \times 0}{2} + j\sin\frac{3\pi \times 0}{2}\right) = 4$

When $n = 1$; $x(1) = 2.5 + (1 - j0.5)\left(\cos\frac{\pi \times 1}{2} + j\sin\frac{\pi \times 1}{2}\right) - 0.5\cos\pi \times 1 + (1 + j0.5)\left(\cos\frac{3\pi \times 1}{2} + j\sin\frac{3\pi \times 1}{2}\right) = 4$

When $n = 2$; $x(2) = 2.5 + (1 - j0.5)\left(\cos\frac{\pi \times 2}{2} + j\sin\frac{\pi \times 2}{2}\right) - 0.5\cos\pi \times 2 + (1 + j0.5)\left(\cos\frac{3\pi \times 2}{2} + j\sin\frac{3\pi \times 2}{2}\right) = 0$

When $n = 3$; $x(3) = 2.5 + (1 - j0.5)\left(\cos\frac{\pi \times 3}{2} + j\sin\frac{\pi \times 3}{2}\right) - 0.5\cos\pi \times 3 + (1 + j0.5)\left(\cos\frac{3\pi \times 3}{2} + j\sin\frac{3\pi \times 3}{2}\right) = 2$

$\therefore\ x(n) = \{4,\ 4,\ 0,\ 2\}$

E5. 9. Determine the response of LTI system when the input sequence x(n) = {–2, –1, –1, 0, 2} by radix-2 DIT FFT. The impulse response of the system is h(n) = {1, –1, –1, 1}.

Solution

Response or Output, y(n) = x(n) * h(n)

Here, Length of x(n) = 5, and h(n) = 4.

\ Length of y(n) = 5 + 4 –1 = 8

Let us convert x(n) and h(n) to 8-point sequences by appending zeros.

\ x(n) = {–2, –1, –1, 0, 2, 0, 0, 0}

h(n) = {1, –1, –1, 1, 0, 0, 0, 0}

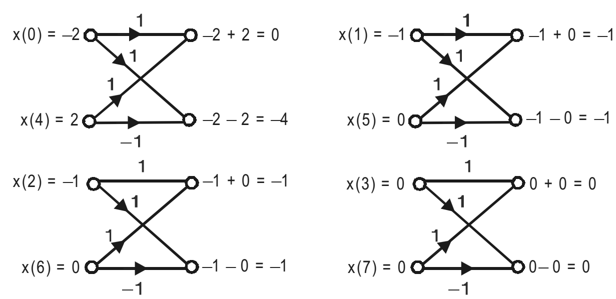
Let, $\mathcal{DFT}$ {x(n)} = X(k) ; $\mathcal{DFT}$ {h(n)} = H(k) ; $\mathcal{DFT}$ {y(n)} = Y(k) and Y(k) = X(k) H(k)

Now, y(n) = $\mathcal{DFT}^{-1}$ { Y(k) } = $\mathcal{DFT}^{-1}$ { X(k) H(k)}

Step - 1 : Find X(k)

First Stage

Input = {–2, 2, –1, 0, –1, 0, 0, 0}



x(n) (Normal)	**x(n) (Bit reversed)**
x(0) = –2	x(0) = –2
x(1) = –1	x(4) = 2
x(2) = –1	x(2) = –1
x(3) = 0	x(6) = 0
x(4) = 2	x(1) = –1
x(5) = 0	x(5) = 0
x(6) = 0	x(3) = 0
x(7) = 0	x(7) = 0

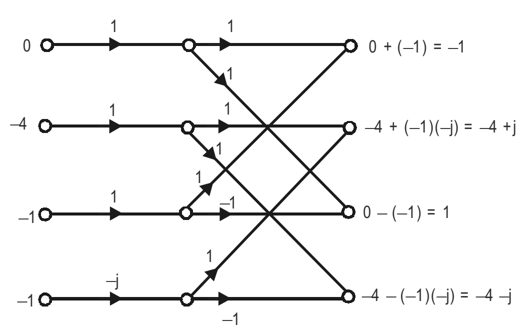
Output = {0, –4, –1, –1, –1, –1, 0, 0}

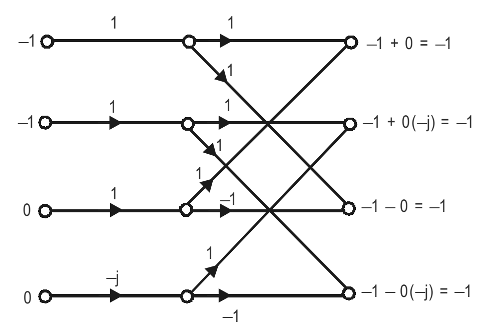
Second Stage

Input = {0, –4, –1, –1, –1, –1, 0, 0}

$W_4^0 = 1$

$W_4^1 = -j$

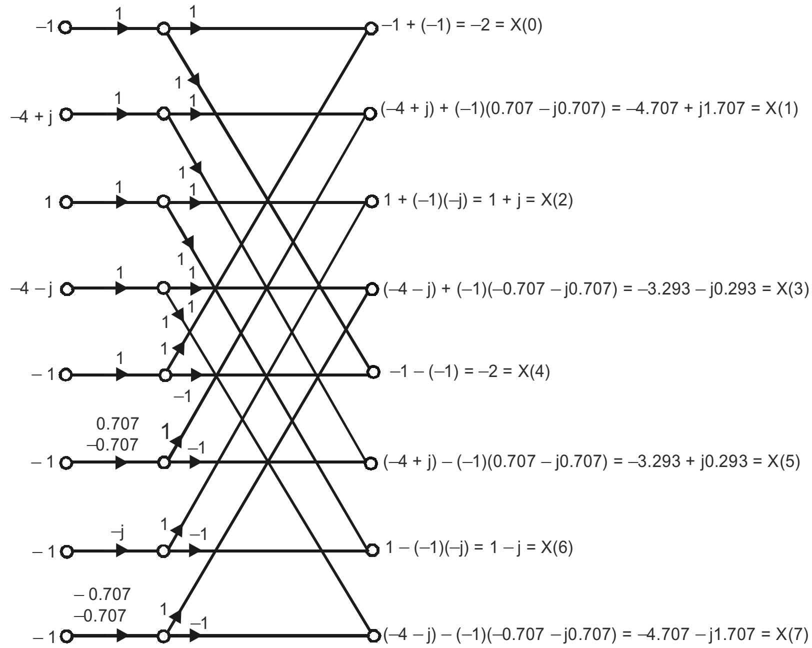




Output = {–1, –4 + j, 1, –4 – j, –1, –1, –1, –1}

Third Stage

Input = {–1, –4 + j, 1, –4 – j, –1, –1, –1, –1}



$W_8^0 = 1$

$W_8^1 = 0.707 - j0.707$

$W_8^2 = -j$

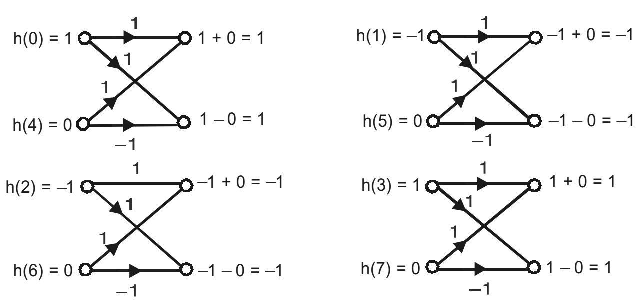
$W_8^3 = -0.707 - j0.707$

\ X(k) {–2, –4.707 + j1.707, 1 + j, –3.293 –j0.293, 0, –3.293 + j0.293, 1 – j, –4.707 – j1.707}

Step - 2 : Find H(k)

First Stage

Input = {1, 0, –1, 0, –1, 0, 1, 0}



x(n) (Normal)	x(n) (Bit reversed)
h(0) = 1	h(0) = 1
h(1) = –1	h(4) = 0
h(2) = –1	h(2) = –1
h(3) = 1	h(6) = 0
h(4) = 0	h(1) = –1
h(5) = 0	h(5) = 0
h(6) = 0	h(3) = 1
h(7) = 0	h(7) = 0

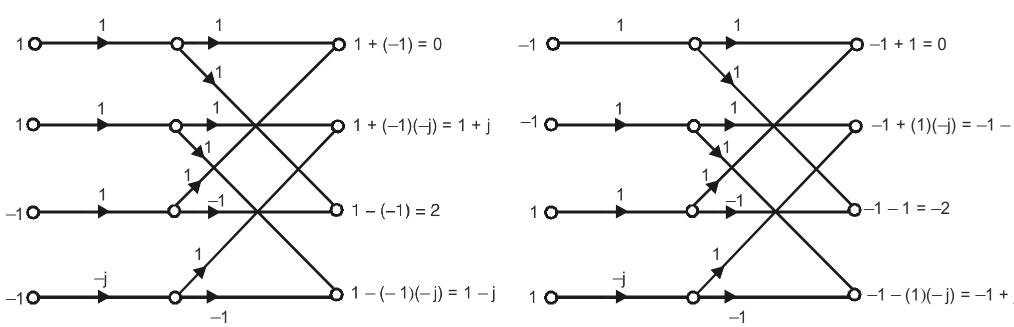
Output = {1, 1, –1, –1, –1, –1, 1, 1}

Second Stage

$W_4^0 = 1$

$W_4^1 = -j$

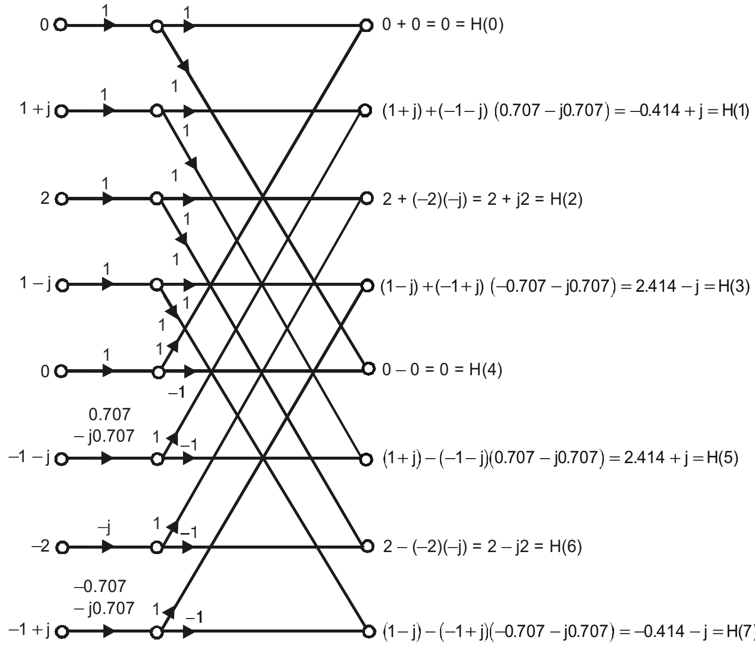
Input = {1, 1, –1, –1, –1, –1, 1, 1}



Output = {0, 1+j, 2, 1–j, 0, –1–j, –2, –1+ j}

Third Stage

Input = {0, 1+j, 2, 1–j, 0, –1–j, –2, –1+ j}



$W_8^0 = 1$
$W_8^1 = 0.707 - j0.707$
$W_8^2 = -j$
$W_8^3 = -0.707 - j0.707$

∴ H(k) = {0, –0.414 + j, 2 + j2, 2.414 – j, 0, 2.414 + j, 2 – j2, –0.414 – j}

Step - 3 : To find the product X(k) H(k)

Let, Y(k) = X(k) H(k)

When k = 0 ; Y(0) = X(0) H(0) = –2 × 0 = 0

When k = 1 ; Y(1) = X(1) H(1) = (–4.707 + j1.707) × (–0.414 + j) = 0.242 –j5.414

When k = 2 ; Y(2) = X(2) H(2) = (1 + j) × (2 + j2) = j4

When k = 3 ; Y(3) = X(3) H(3) = (–3.293 – j0.293) × (2.414 –j) = –8.242 + j2.586

When k = 4 ; Y(4) = X(4) H(4) = 0 × 0 = 0

When k = 5 ; Y(5) = X(5) H(5) = (–3.293 + j0.293) × (2.414 +j) = –8.242 –j2.586

When k = 6 ; Y(6) = X(6) H(6) = (1 – j) × (2 –j2) = –j4

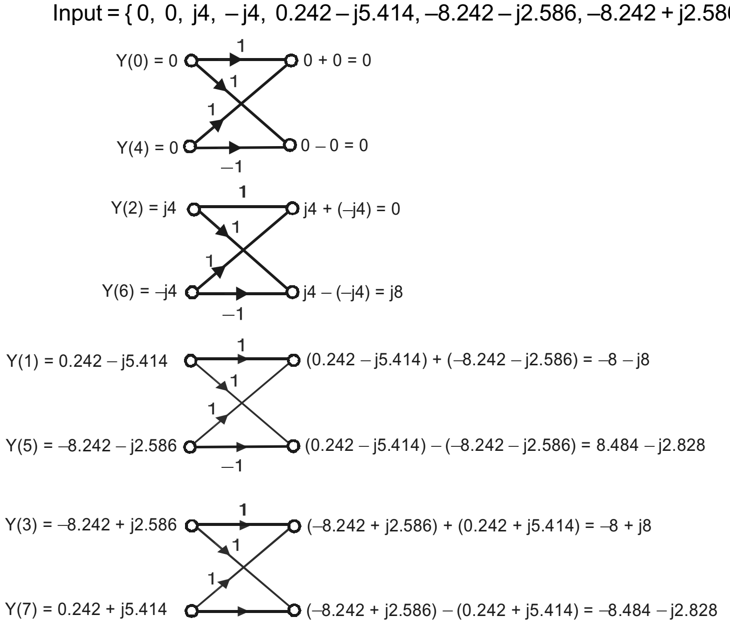
When k = 7 ; Y(7) = X(7) H(7) = (–4.707 – j1.707) × (–0.414–j) = 0.242 + j5.414

∴ Y(k) = { 0, 0.242 – j5.414, j4, –8.242 + j2.586, 0, –8.242 –j2.586, –j4, 0.242 + j5.414}

Step - 4 : To determine inverse DFT of Y(k)

First stage computation

Input = { 0, 0, j4, – j4, 0.242 – j5.414, –8.242 – j2.586, –8.242 + j2.586, 0.242 + j5.414}



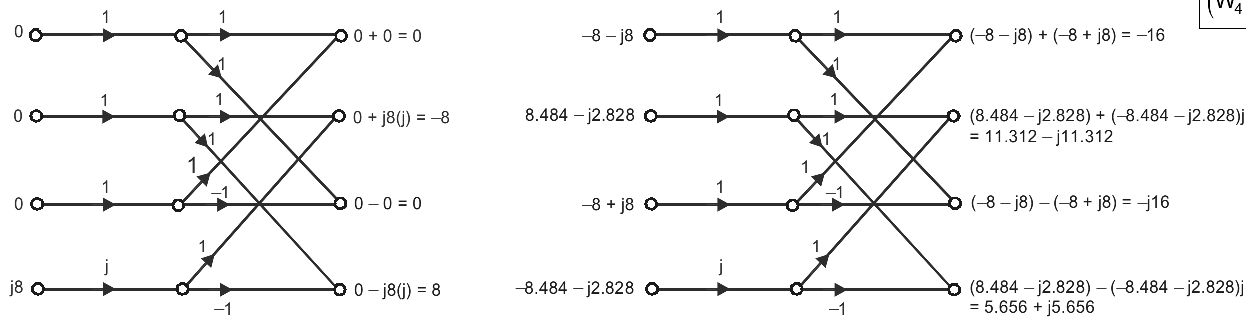
Y(k) (Normal)	**Y(k) (Bit reversed)**
Y(0) = 0	Y(0) = 0
Y(1) = 0.242 – j5.414	Y(4) = 0
Y(2) = j4	Y(2) = j4
Y(3) = –8.242 + j2.586	Y(6) = –j4
Y(4) = 0	Y(1) = 0.242 – j5.414
Y(5) = –8.242 –j2.586	Y(5) = –8.242 – j2.586
Y(6) = –j4	Y(3) = –8.242 + j2.586
Y(7) = 0.242 + j5.414	Y(7) = 0.242 + j5.414

Output = $\{0,\ 0,\ 0,\ j8,\ -8-j8,\ 8.484-j2.828,\ -8+j8,\ -8.484-j2.828\}$

Second stage computation

$$\left(W_4^0\right)^* = 1 \qquad \left(W_4^1\right)^* = j$$

Input $= \{0,\ 0,\ 0,\ j8,\ -8-j8,\ 8.484-j2.828,\ -8+j8,\ -8.484-j2.828\}$

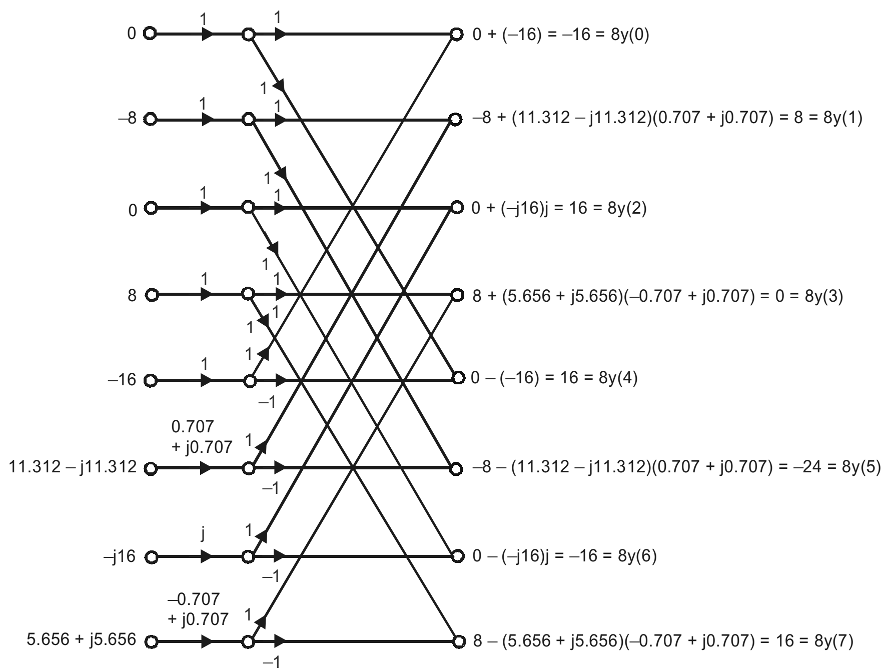


Output $= \{0,\ -8,\ 0,\ 8,\ -16,\ 11.312-j11.312,\ -j16,\ 5.656+j5.656\}$

Third stage computation

Input $= \{0,\ -8,\ 0,\ 8,\ -16,\ 11.312-j11.312,\ -j16,\ 5.656+j5.656\}$

$$\left(W_8^0\right)^* = 1$$
$$\left(W_8^1\right)^* = 0.707 + j0.707$$
$$\left(W_8^2\right)^* = j$$
$$\left(W_8^3\right)^* = -0.707 + j0.707$$



Output $= \{-16,\ 8,\ 16,\ 0,\ 16,\ -24,\ -16,\ 16\}$

The response y(n) is obtained by dividing each sample of output sequence by 8.

$\therefore$ Response, y(n) = {–2, 1, 2, 0, 2, –3, –2, 1}

Chapter 6

FIR Filters

6.1 Introduction

The ***filters*** are frequency selective devices. An LTI system performs a type of discrimination or filtering among the various frequency components at its input. The nature of this filtering action is determined by the frequency response characteristic $H(e^{jw})$, which in turn depends on the choice of the system parameters [e.g., the coefficients a_k and b_k in the difference equation governing the system]. Thus by proper selection of the coefficients, we can design frequency selective filters that pass signals with frequency components in some bands while they attenuate signals that contain frequency components in other frequency bands.

In general, the specification of a digital filter will be desired frequency response, $H_d(e^{jw})$. The desired impulse response, $h_d(n)$ of the digital filter can be obtained by taking inverse Fourier transform of $H_d(e^{jw})$. Now, the $h_d(n)$ will be an infinite duration discrete time signal defined for all values of n in the range $-\infty$ to $+\infty$.

The ***transfer function***, H(z) of the digital filter is obtained by taking $\mathcal{Z}$-transform of impulse response. Since $h_d(n)$ is an infinite duration signal, the transfer function obtained from $h_d(n)$ will have infinite terms, which cannot be realized or implemented in a digital system. Therefore, finite number of samples of $h_d(n)$ are selected to form the impulse response, h(n) of the filter. The transfer function, H(z) is obtained by taking $\mathcal{Z}$-transform of finite sample impulse response, h(n). The filters designed by using finite samples of impulse response are called ***FIR (Finite Impulse Response) filters***.

Various Steps in Designing FIR Filter

i) Choose an ideal (desired) frequency response, $H_d(e^{jw})$.

ii) Take inverse Fourier transform of $H_d(e^{jw})$ to get $h_d(n)$ or sample $H_d(e^{jw})$ at finite number of points (N-point) to get $H(k)$.

iii) If $h_d(n)$ is determined then convert the infinite duration $h_d(n)$ to a finite duration h(n), [usually h(n) is an N-point sequence] or if $H(k)$ is determined then take N-point inverse DFT to get h(n).

iv) Take $\mathcal{Z}$-transform of h(n) to get H(z), where H(z) is the transfer function of the digital filter.

v) Choose a suitable structure and realize the filter.

vi) Verify the design. In order to verify the design, determine the actual frequency response, $H(e^{jw})$ of the filter, by letting $z = e^{jw}$ in H(z) and sketch the magnitude response, $|H(e^{jw})|$.

Advantages of FIR Filters

1. FIR filters with exactly linear phase can be easily designed.
2. Efficient realizations of FIR filter exist as both recursive and nonrecursive structures.
3. FIR filters realized nonrecursively, i.e., by direct convolution are always stable.
4. Roundoff noise, which is inherent in realizations with finite precision arithmetic can easily be made small for nonrecursive realization of FIR filters.

Disadvantages of FIR Filters

1. The duration of the impulse response should be large (i.e., N should be large) to adequately approximate sharp cutoff filter. Hence a large amount of processing is required to realize such filters when realized via slow convolution.
2. The delay of linear phase FIR filters need not always be an integer number of samples. This non-integral delay can lead to problems in some signal processing applications.

6.2 LTI System as Frequency Selective Filters

Let us consider a discrete time signal x(n) with frequency content in a band of frequencies $w_1 < w < w_2$. Let, $X(e^{jw})$ be Fourier transform of x(n). Now, $X(e^{jw})$ will be a bandlimited signal that is,

$$X(e^{jw}) \neq 0 \;;\; w_1 \le w \le w_2$$
$$= 0 \;;\; w < w_1 \text{ and } w > w_2.$$

Let, this signal is passed through an LTI system with frequency response,

$$\begin{aligned} H(e^{j\omega}) &= C e^{-j\alpha\omega} \;; \quad \omega_1 \le \omega \le \omega_2 \\ &= 0 \;; \quad \text{otherwise} \end{aligned} \qquad .....(6.1)$$

where, C and α are positive constants.

Let, h(n) be impulse response of the LTI system, which is obtained by inverse Fourier transform of $H(e^{jw})$. Now, the response, y(n) of LTI system is given by,

$$y(n) = x(n) * h(n)$$

On taking Fourier transform of above equation we get,

$$Y(e^{jw}) = \mathcal{F}\{x(n) * h(n)\}$$ $\boxed{Y(e^{jw}) = \mathcal{F}\{y(n)\}}$

$$\therefore\; Y(e^{jw}) = X(e^{jw})\, H(e^{jw})$$ $\boxed{\text{Using convolution property of Fourier transform.}}$

$$= X(e^{jw})\, C\, e^{-j\alpha w}$$ $\boxed{\text{Using equation (6.1).}}$

$$= C\, X(e^{jw})\, e^{-j\alpha w}$$

On taking inverse Fourier transform of above equation we get,

$$y(n) = C\,\mathcal{F}^{-1}\{X(e^{j\omega})e^{-j\alpha\omega}\}$$

Using shifting property of Fourier transform.
If $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $\mathcal{F}\{x(n-\alpha)\} = X(e^{j\omega})\,e^{-j\alpha\omega}$

$$\therefore\ y(n) = C\,x(n-\alpha) \qquad .....(6.2)$$

From equation (6.2) we can say that the LTI system output is simply a delayed and amplitude scaled version of the input signal. A pure delay is usually tolerable and is not considered as distortion of the signal. Likewise the amplitude scaling. Hence, the LTI system with frequency response defined by equation (6.1) represents an ideal filter (in this example it is bandpass filter).

In general, an LTI system modifies the input spectrum $X(e^{j\omega})$ according to its frequency response $H(e^{j\omega})$ to yield an output signal with spectrum, $Y(e^{j\omega}) = H(e^{j\omega})\,X(e^{j\omega})$. In a sense, $H(e^{j\omega})$ acts as a weighting function or a spectral shaping function to the different frequency components in the input signal. When viewed in this context any LTI system can be considered to be a frequency shaping filter, even though it may not completely block any or all frequency components. Consequently the terms LTI system and filter are synonymous, and are often used interchangeably. The frequency selective filters must be designed to introduce negligible distortion in the signals that pass through them.

The frequency response, $H(e^{j\omega})$ is a complex quantity, hence we can write,

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| \angle H(e^{j\omega}) = C\,e^{-j\alpha\omega} \qquad .....(6.3)$$

where, $|H(e^{j\omega})| = C$ $\quad$ Magnitude

$\angle H(e^{j\omega}) = -\alpha\omega$ $\quad$ Phase

From equation (6.3) we can say that the magnitude of frequency response is constant and its phase is a linear function of frequency. Therefore, if the phase function of frequency response of a filter is linear function of frequency, then the filter is called ***linear phase filter***.

In general, any deviation of the frequency response characteristics of a linear filter from the ideal results in signal distortion. If the filter has a frequency variable magnitude response characteristic in the passband then the filter introduces amplitude distortion. If the phase characteristic is not linear within the desired frequency band, the signal undergoes phase distortion.

In order to examine the linear and nonlinear phase characteristics, two delay functions are defined and they are ***phase delay*** and ***group delay***.

Let, $\angle H(e^{j\omega}) = \theta(\omega)$

$$\text{Phase delay, } \tau_p = -\frac{\theta(\omega)}{\omega} \qquad .....(6.4)$$

$$\text{Group delay, } \tau_g = -\frac{d}{d\omega}\theta(\omega) \qquad .....(6.5)$$

From equation (6.3),

$$\theta(\omega) = -\alpha\omega$$

$$\therefore\ \tau_p = -\frac{\theta(\omega)}{\omega} = -\frac{-\alpha\omega}{\omega} = \alpha$$

$$\tau_g = -\frac{d}{d\omega}\theta(\omega) = -\frac{d}{d\omega}(-\alpha\omega) = \alpha$$

From the above equations it is observed that for a linear-phase filter, the delay is a constant, independent of frequency. Consequently a filter that causes phase distortion has a variable frequency delay and one that has linear phase has a constant delay within the desired frequency range. If the delay is not a constant within the desired frequency range, we say that the filter introduces delay distortion. Thus delay distortion is synonymous with phase distortion.

6.3 Ideal Frequency Response of Linear Phase FIR Filters

The filters are classified according to their frequency response characteristics. The ideal (desired) frequency response $H_d(e^{jw})$ of four major types of filters are given below. The $H_d(e^{jw})$ is periodic, with periodicity of 0 to 2p (or –p to +p). Also any analog frequency W will map (or can be converted) to frequency of digital system w within the range 0 to 2p (or –p to +p). Hence the frequency response of digital filters are defined in the interval 0 to 2p (or –p to +p).

> *Note : For conversion from analog frequency to frequency of digital system refer Impulse invariant transformation or Bilinear transformation in Chapter 7.*

Ideal frequency response of lowpass filter,

$$H_d(e^{j\omega}) = \begin{cases} 0 & ; \text{ for } \omega = -\pi \text{ to } -\omega_c \\ C\,e^{-j\alpha\omega} & ; \text{ for } \omega = -\omega_c \text{ to } +\omega_c \\ 0 & ; \text{ for } \omega = +\omega_c \text{ to } +\pi \end{cases} \qquad .....(6.6)$$

Ideal frequency response of highpass filter,

$$H_d(e^{j\omega}) = \begin{cases} C\,e^{-j\alpha\omega} & ; \text{ for } \omega = -\pi \text{ to } -\omega_c \\ 0 & ; \text{ for } \omega = -\omega_c \text{ to } +\omega_c \\ C\,e^{-j\alpha\omega} & ; \text{ for } \omega = +\omega_c \text{ to } +\pi \end{cases} \qquad .....(6.7)$$

Ideal frequency response of bandpass filter,

$$H_d(e^{j\omega}) = \begin{cases} 0 & ; \text{ for } \omega = -\pi \text{ to } -\omega_{c2} \\ C\,e^{-j\alpha\omega} & ; \text{ for } \omega = -\omega_{c2} \text{ to } -\omega_{c1} \\ 0 & ; \text{ for } \omega = -\omega_{c1} \text{ to } +\omega_{c1} \\ C\,e^{-j\alpha\omega} & ; \text{ for } \omega = +\omega_{c1} \text{ to } +\omega_{c2} \\ 0 & ; \text{ for } \omega = +\omega_{c2} \text{ to } +\pi \end{cases} \qquad .....(6.8)$$

Ideal frequency response of bandstop filter,

$$H_d(e^{j\omega}) = \begin{cases} C\,e^{-j\alpha\omega} & ; \text{ for } \omega = -\pi \text{ to } -\omega_{c2} \\ 0 & ; \text{ for } \omega = -\omega_{c2} \text{ to } -\omega_{c1} \\ C\,e^{-j\alpha\omega} & ; \text{ for } \omega = -\omega_{c1} \text{ to } +\omega_{c1} \\ 0 & ; \text{ for } \omega = +\omega_{c1} \text{ to } +\omega_{c2} \\ C\,e^{-j\alpha\omega} & ; \text{ for } \omega = +\omega_{c2} \text{ to } +\pi \end{cases} \qquad .....(6.9)$$

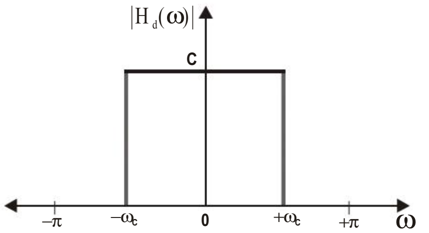


Fig 6.1 : *Magnitude response of ideal lowpass filter.*

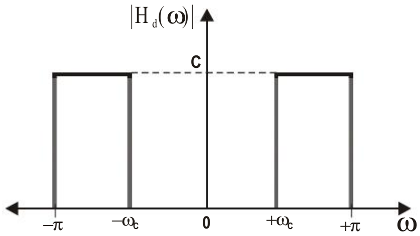


Fig 6.2 : *Magnitude response of ideal highpass filter.*

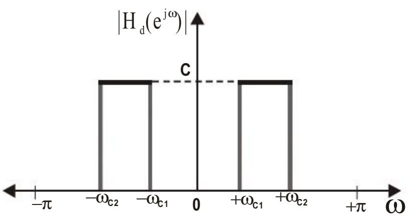


Fig 6.3 : *Magnitude response of ideal bandpass filter.*

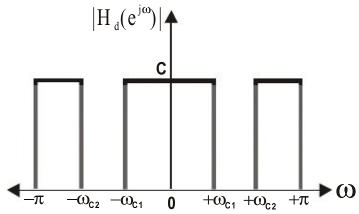


Fig 6.4 : *Magnitude response of ideal bandstop filter.*

The ideal filters are noncausal and hence physically unrealizable for the real time signal processing applications. Causality implies that the frequency response characteristic $H(e^{jw})$ of the filter cannot be zero, except at a finite set of points in frequency. In addition $H(e^{jw})$ cannot have an infinitely sharp cutoff from passband to stopband, that is $H(e^{jw})$ cannot drop from unity to zero abruptly.

In practice it is not necessary to insist that the magnitude $|H(e^{j\omega})|$ be constant in the entire passband of the filters. A small amount of ripple in the passband is usually tolerable. Similarly it is not necessary for the filter response $|H(e^{j\omega})|$ to be zero in the stopband. A small amount of ripple in the stopband is also tolerable.

The transition of the frequency response from passband to stopband defines the ***transition band*** or transition region of the filter. The ***passband edge frequency*** w_p defines the edge of the passband, while the ***stopband edge frequency*** w_s denotes the beginning of the stopband.

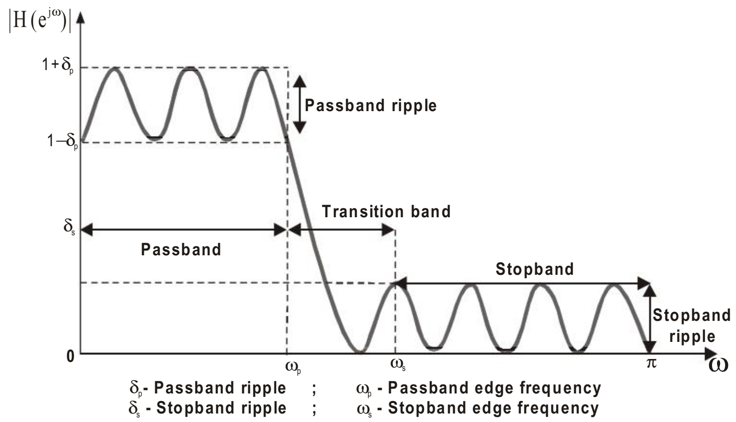


Fig 6.5 : *Magnitude response of a practical lowpass filter.*

6.4 Characteristics of FIR Filters With Linear Phase

Let h(n) be a causal finite duration sequence defined over the interval $0 \le n \le N-1$ and the samples of h(n) be real.

The Fourier transform of h(n) is,

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} \qquad .....(6.10)$$

which is periodic in frequency with period 2π.

$$\therefore\ H(e^{j\omega}) = H(e^{j\omega + 2\pi m})\ ; \quad \text{for} \quad m = 0, \pm 1, \pm 2, ... \qquad .....(6.11)$$

With the restriction that h(n) is real, additional constraints of $H(e^{j\omega})$ are obtained as shown below.

Since $H(e^{j\omega})$ is complex it can be expressed as ***amplitude function, magnitude function*** and ***phase function*** as shown in equation (6.12).

$$H(e^{j\omega}) = \pm \left|H(e^{j\omega})\right| e^{j\angle H(e^{j\omega})} = A(\omega)\, e^{j\theta(\omega)} \qquad .....(6.12)$$

where, $A(\omega) = \pm \left|H(e^{j\omega})\right|$ = Amplitude function

$\theta(\omega) = \angle H(e^{j\omega})$ = Phase function

$\left|H(e^{j\omega})\right|$ = Magnitude function

Note : The magnitude is strictly positive, but the amplitude can be positive or negative.

From the property of Fourier transform when h(n) is real we can say that the magnitude function is a symmetric function and the phase function is an antisymmetric function.

$$\therefore\ |H(e^{j\omega})| = |H(-e^{j\omega})|$$

$$|\theta(\omega)| = -\,|\theta(-\omega)|$$

Linear Phase and Symmetric Impulse Response

For many practical FIR filter, exact linearity of phase is a desired goal. Let us assume that the phase of $H(e^{j\omega})$ is a linear function of ω. Hence $\theta(\omega)$ is directly proportional to ω.

$$\therefore\ \theta(\omega) \propto \omega \quad \text{or} \quad \theta(\omega) = -\alpha\omega \quad ; \quad \text{for} \quad -\pi \le \omega \le +\pi \qquad .....(6.13)$$

where, α is a constant phase delay in samples

From equation (6.10) we get,

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} \qquad .....(6.14)$$

From equation (6.12) and (6.13) we get,

$$H(e^{j\omega}) = \pm\left|H(e^{j\omega})\right| e^{-j\alpha\omega} \qquad .....(6.15)$$

On equating equations (6.14) and (6.15) we get,

$$\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm\left|H(e^{j\omega})\right| e^{-j\alpha\omega} \qquad \boxed{e^{-j\theta} = \cos\theta - j\sin\theta}$$

$$\sum_{n=0}^{N-1} h(n)\left[\cos\omega n - j\sin\omega n\right] = \pm\left|H(e^{j\omega})\right|\left[\cos\alpha\omega - j\sin\alpha\omega\right]$$

On equating the real part and imaginary part of the above equation we get,

$$\pm|H(e^{j\omega})| \cos \alpha\omega = \sum_{n=0}^{N-1} h(n) \cos \omega n \qquad .....(6.16)$$

$$\pm|H(e^{j\omega})| \sin \alpha\omega = \sum_{n=0}^{N-1} h(n) \sin \omega n \qquad .....(6.17)$$

On dividing equation (6.17) by equation (6.16) we get,

$$\frac{\sin \alpha\omega}{\cos \alpha\omega} = \frac{\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n}$$

On cross multiplying the above equation we get,

$$\sin \alpha\omega \sum_{n=0}^{N-1} h(n) \cos \omega n = \cos \alpha\omega \sum_{n=0}^{N-1} h(n) \sin \omega n$$

$$\sum_{n=0}^{N-1} h(n) \sin \alpha\omega \cos \omega n = \sum_{n=0}^{N-1} h(n) \cos \alpha\omega \sin \omega n$$

$$\sum_{n=0}^{N-1} h(n) \left[\sin \alpha\omega \cos \omega n - \cos \alpha\omega \sin \omega n\right] = 0 \qquad \boxed{\sin(A-B) = \sin A \cos B - \cos A \sin B}$$

$$\sum_{n=0}^{N-1} h(n) \sin\left[(\alpha - n)\omega\right] = 0 \qquad .....(6.18)$$

One solution of equation (6.18) exists when,

$$\alpha = \frac{N-1}{2} \quad \text{and} \quad h(n) = h(N-1-n) \quad ; \quad \text{for } 0 \le n \le N-1 \qquad .....(6.19)$$

Proof :

$$\boxed{\alpha = \frac{N-1}{2}} \qquad \boxed{n = N-1-n}$$

$$h(n) \sin\left[(\alpha - n)\omega\right] = h(n) \sin\left[\left(\frac{N-1}{2} - n\right)\omega\right]$$

$$= h(n) \sin\left[\left(\frac{N-1-2n}{2}\right)\omega\right]$$

$$= h(n) \sin\left[\left(\frac{N-1-n-n}{2}\right)\omega\right]$$

$$= h(n) \sin\left[\left(\frac{n-n}{2}\right)\omega\right]$$

$$= h(n) \sin 0$$

$$= 0$$

From the condition, a = (N - 1)/2 we can say that for every value of N there is only one value of phase delay a for which linear phase can be obtained easily.

From the condition h(n) = h(N - 1 - n) we can say that for this value of a, [i.e., a = (N - 1)/2] the h(n) has a special kind of symmetry. The impulse response h(n), when a = (N - 1)/2 and for odd and even values of N are shown in fig 6.6 and 6.7 respectively. It can be observed that the impulse response is symmetric about the centre of the sequence.

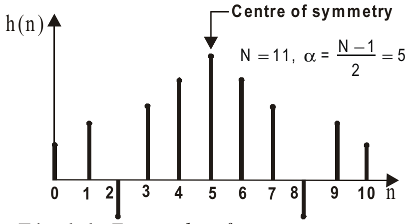


Fig 6.6: *Example of symmetric impulse response for odd N.*

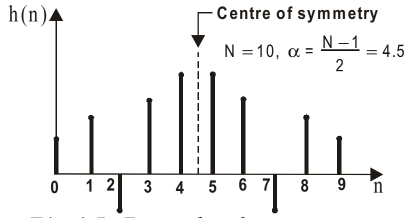


Fig 6.7: *Example of symmetric impulse response for even N.*

Linear Phase and Antisymmetric Impulse Response

The definition of linear phase filter $\theta(\omega) = -\alpha\omega$ requires the filter to have both constant group delay and constant phase delay. If only constant group delay is required an another type of linear phase filter is defined in which the phase of $H(e^{j\omega})$ is a piece-wise linear function of ω. For this case $H(e^{j\omega})$ can be expressed in the Euler form as shown in equation (6.20).

$$H(e^{j\omega}) = \pm\left|H(e^{j\omega})\right| e^{j(\beta - \alpha\omega)} \qquad .....(6.20)$$

When $H(e^{j\omega})$ is expressed in the form of equation (6.20), we can prove that the only possible solution of h(n) exists if,

$$\alpha = \frac{N-1}{2} \;;\quad \beta = \pm\frac{\pi}{2} \text{ and } h(n) = -h(N-1-n) \text{ ; for } 0 \le n \le (N-1) \qquad .....(6.21)$$

The filters that satisfy the three conditions of equation (6.21) have a delay of [(N − 1)/2] samples but their impulse responses are antisymmetric around the centre of the sequence, as opposed to the true linear phase sequences that are symmetric around the centre of the sequence.

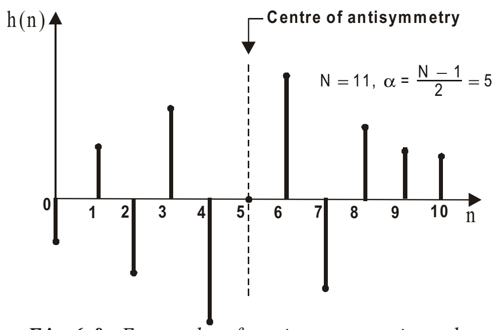


Fig 6.8: *Example of antisymmetry impulse response for odd N.*

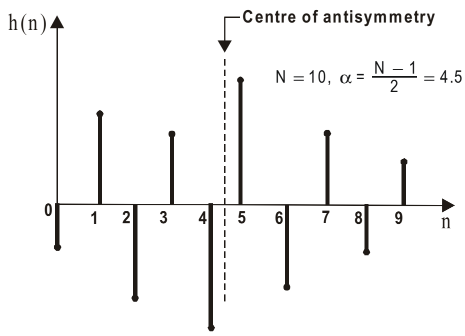


Fig 6.9: *Example of antisymmetry impulse response for even N.*

6.5 Frequency Response of Linear Phase FIR Filters

Depending on the value of N (odd or even) and the type of symmetry of the filter impulse response sequence (symmetric or antisymmetric) there are six possible types of linear phase FIR filters.

The following are the six cases of impulse response for linear phase FIR filters.

Case (i) **:** Symmetric impulse response and N is odd with centre of symmetry at (N – 1)/2.

Case (ii) **:** Symmetric impulse response and N is even with centre of symmetry at (N – 1)/2.

Case (iii) **:** Antisymmetric impulse response and N is odd with centre of antisymmetry at (N – 1)/2.

Case (iv) **:** Antisymmetric impulse response and N is even with centre of antisymmetry at (N – 1)/2.

Case (v) **:** Symmetric impulse response and N is odd with centre of symmetry at n = 0.

Case (vi) **:** Antisymmetric impulse response and N is odd with centre of antisymmetry at n = 0.

The frequency response of the filter is the Fourier transform of the impulse response. If h(n) is impulse response of FIR filter then fourier transform of h(n) is denoted as $H(e^{j\omega})$, which is the frequency response of FIR filter. The $H(e^{j\omega})$ is a complex function of ω and so it can be expressed as, magnitude function $|H(e^{j\omega})|$ and phase function $\angle H(e^{j\omega})$.

Case (i) : Frequency response of linear phase FIR filter when impulse response is symmetric and N is odd with centre of symmetry at (N – 1)/2

The frequency response of linear phase FIR filter when impulse response is symmetric and N is odd with centre of symmetry at (N – 1)/2 is given by,

$$H\left(e^{j\omega}\right)=\left[h\left(\tfrac{N-1}{2}\right)+\sum_{n=1}^{\frac{N-1}{2}}2h\left(\tfrac{N-1}{2}-n\right)\cos\omega n\right]e^{-j\omega\left(\frac{N-1}{2}\right)} \qquad .....(6.22)$$

Let, $H(e^{j\omega})=A(\omega)\,e^{j\theta(\omega)}$ .....(6.23)

where, $A(\omega)$ = Amplitude function

$\theta(\omega)$ = Phase function

On comparing equations (6.22) and (6.23) we get,

$$\text{Amplitude function, } A(\omega) = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}}2\,h\left(\tfrac{N-1}{2}-n\right)\cos\omega n \qquad .....(6.24)$$

$$\text{Phase function, } \theta(\omega)=-\omega\left(\tfrac{N-1}{2}\right)=-\omega\alpha \quad ; \quad \text{where, } \alpha = \tfrac{N-1}{2} \qquad .....(6.25)$$

$$\text{Magnitude function, } \left|H(e^{j\omega})\right|=\left|A(\omega)\right|=\left|h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}}2\,h\left(\tfrac{N-1}{2}-n\right)\cos\omega n\right| \qquad .....(6.26)$$

A typical sketch of symmetric impulse response when N = 9 and its corresponding amplitude function of frequency response are shown in fig 6.10 and fig 6.11 respectively. From these sketches it can be observed that the amplitude function of $H(e^{j\omega})$ is symmetric with $\omega = \pi$, when the impulse response is symmetric and N is odd number.

When impulse response is symmetric and N is odd, the frequency response is non-zero at $\omega = 0$ and $\omega = \pi$, and so this frequency response can be used to design lowpass, highpass, bandpass and bandstop filters.

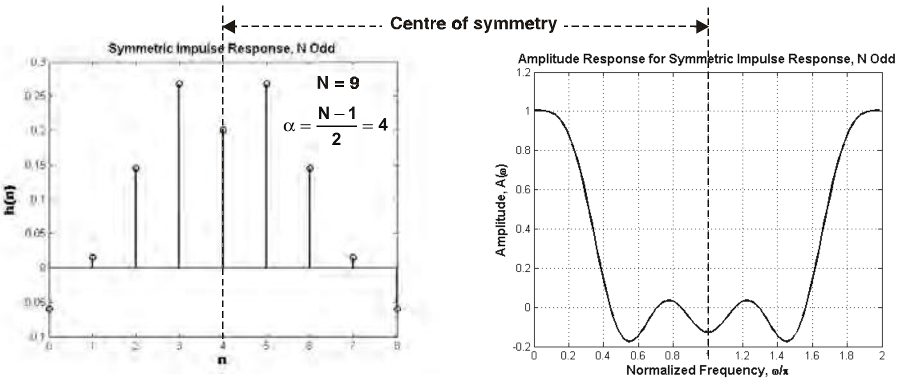


Fig 6.10: *Symmetric impulse response, N= 9.* ***Fig 6.11:*** *Amplitude function of $H(e^{j\omega})$.*

Proof:

The Fourier transform of h(n) is,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

h(n) is defined for $n = 0$ to $N-1$

When N is odd number the symmetric impulse response will have the centre of symmetry at n = (N - 1)/2. Hence $H(e^{jw})$ is expressed as,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\tfrac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\tfrac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Let, $m = N-1-n$; \ $n = N-1-m$
When, $n = \frac{N+1}{2}$; $m = N-1-\frac{N+1}{2} = \frac{N-3}{2}$
When, $n = N-1$; $m = N-1-(N-1) = 0$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\tfrac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

Put, $m = n$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\tfrac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)}$$

For symmetric impulse response, $h(N-1-n) = h(n)$.

$$= \left\{ h\left(\tfrac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{-j\omega n}\, e^{j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1-n)}\, e^{j\omega\left(\frac{N-1}{2}\right)} \right] \right\} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left\{ h\left(\tfrac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left((N-1)-\frac{N-1}{2}-n\right)} \right] \right\} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left\{ h\left(\tfrac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right\} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$= \left\{ h\left(\tfrac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\, 2\cos\left(\left(\tfrac{N-1}{2} - n\right)\omega\right) \right\} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left\{ h\left(\tfrac{N-1}{2}\right) + \sum_{k=1}^{\frac{N-1}{2}} 2\, h\left(\tfrac{N-1}{2} - k\right) \cos\omega k \right\} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

Let, $k = \frac{N-1}{2} - n$; $n = \frac{N-1}{2} - k$
When, $n = 0$; $k = \frac{N-1}{2}$
When, $n = \frac{N-3}{2}$; $k = \frac{N-1}{2} - \frac{N-3}{2} = 1$

$$= \left\{ h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2\, h\left(\tfrac{N-1}{2} - n\right) \cos\omega n \right\} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

Put, $k = n$

Case (ii) : Frequency response of linear phase FIR filter when impulse response is symmetric and N is even with centre of symmetry at (N – 1)/2

The frequency response of linear phase FIR filter when impulse response is symmetric and N is even with centre of symmetry at (N – 1)/2 is given by,

$$H\left(e^{j\omega}\right) = \left[\sum_{n=1}^{\frac{N}{2}} 2h\left(\tfrac{N}{2}-n\right)\cos\left(\omega\left(n-\tfrac{1}{2}\right)\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)} \qquad .....(6.27)$$

Let, $H(e^{jw}) = A(w)\, e^{jq(w)}$ .....(6.28)

where, $A(w)$ = Amplitude function

$q(w)$ = Phase function

On comparing equation (6.27) and (6.28) we get,

$$\text{Amplitude function, } A(\omega) = \sum_{n=1}^{\frac{N}{2}} 2\, h\left(\tfrac{N}{2}-n\right)\cos\left(\omega\left(n-\tfrac{1}{2}\right)\right) \qquad .....(6.29)$$

$$\text{Phase function, } \theta(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha \;\; ; \text{ where, } \alpha = \frac{N-1}{2} \qquad .....(6.30)$$

$$\text{Magnitude function, } \left|H\left(e^{j\omega}\right)\right| = \left|A(\omega)\right| = \left|\sum_{n=1}^{\frac{N}{2}} 2\, h\left(\tfrac{N}{2}-n\right)\cos\left(\omega\left(n-\tfrac{1}{2}\right)\right)\right| \qquad .....(6.31)$$

The sketch of symmetrical impulse response when N = 8 and its corresponding amplitude function of frequency response are shown in fig 6.12 and 6.13 respectively. From these sketches it can be observed that the amplitude function of $H(e^{jw})$ is antisymmetric with w = p, when impulse response is symmetric and N is even number.

When impulse response is symmetric and N is even, the frequency response is non-zero at w = 0 and zero at w = p, and so this frequency response can be used to design lowpass and bandpass filters but cannot be used to design highpass and bandstop filters.

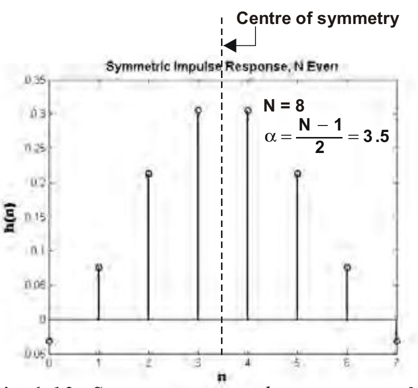


***Fig 6.12:** Symmetric impulse response, N= 8.*

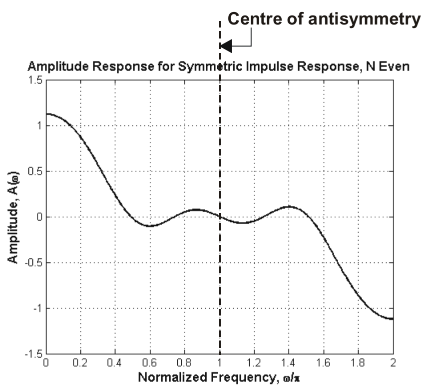


***Fig 6.13:** Amplitude function of $H(e^{j\omega})$.*

Proof:

The Fourier transform of h(n) is,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

h(n) is defined for n = 0 to N – 1

For symmetric impulse response with even number of samples (i.e., when N is even), the centre of symmetry lies between n = (N/2)–1 and n = N/2. Hence $H(e^{jw})$ is expressed as,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Let, $m = N-1-n$; $\therefore\ n = N-1-m$

When, $n = \frac{N}{2}$; $m = N-1-\frac{N}{2} = \frac{N}{2}-1$

When, $n = N-1$; $m = N-1-(N-1) = 0$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

Put, m = n

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega(N-1-n)}$$

For symmetric impulse response, $h(N-1-n) = h(n)$.

$$= \left[\sum_{n=0}^{\frac{N}{2}-1} h(n)\left(e^{-j\omega n}\, e^{j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1-n)}\, e^{j\omega\left(\frac{N-1}{2}\right)}\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left((N-1)-\frac{N-1}{2}-n\right)}\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left[\sum_{n=0}^{\frac{N}{2}-1} h(n)\left(e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left[\sum_{n=0}^{\frac{N}{2}-1} h(n)\, 2\cos\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$= \left[\sum_{n=0}^{\frac{N}{2}-1} 2\,h(n) \cos\left(\omega\left(\frac{N}{2}-n-\frac{1}{2}\right)\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left[\sum_{k=1}^{\frac{N}{2}} 2\,h\left(\frac{N}{2}-k\right) \cos\left(\omega\left(k-\frac{1}{2}\right)\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

Let, $k = \frac{N}{2}-n$; $\therefore\ n = \frac{N}{2}-k$

When, $n = 0$; $k = \frac{N}{2}$

When, $n = \frac{N}{2}-1$; $k = \frac{N}{2}-\left(\frac{N}{2}-1\right) = 1$

$$= \left[\sum_{n=1}^{\frac{N}{2}} 2\,h\left(\frac{N}{2}-n\right) \cos\left(\omega\left(n-\frac{1}{2}\right)\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

Put, k = n

Case (iii) : Frequency response of linear phase FIR filter when impulse response is antisymmetric and N is odd with centre of antisymmetry at (N – 1)/2

The frequency response of linear phase FIR filter when impulse response is antisymmetric and N is odd with centre of antisymmetry at (N – 1)/2 is given by,

$$H\left(e^{j\omega}\right)=\left[\sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2}-n\right)\sin\omega n\right]e^{j\left(\frac{\pi}{2}-\frac{\omega(N-1)}{2}\right)} \quad .....(6.32)$$

Let, $H(e^{j\omega}) = A(\omega)\, e^{j\theta(\omega)}$ .....(6.33)

where, $A(\omega)$ = Amplitude function

$\theta(\omega)$ = Phase function

On comparing equations (6.32) and (6.33) we get,

$$\text{Amplitude function, } A(\omega) = \sum_{n=1}^{\frac{N-1}{2}} 2\, h\left(\tfrac{N-1}{2}-n\right)\sin\omega n \quad .....(6.34)$$

$$\text{Phase function, } \theta(\omega) = \frac{\pi}{2}-\frac{\omega(N-1)}{2} = \beta - \alpha\omega \quad .....(6.35)$$

$$\text{where, } \beta = \frac{\pi}{2} \text{ and } \alpha = \frac{N-1}{2}$$

$$\text{Magnitude function, } \left|H(e^{j\omega})\right| = |A(\omega)| = \left|\sum_{n=1}^{\frac{N-1}{2}} 2\, h\left(\tfrac{N-1}{2}-n\right)\sin\omega n\right| \quad .....(6.36)$$

A typical sketch of antisymmetric impulse response when N = 9 and its corresponding amplitude function of frequency response are shown in fig 6.14 and fig 6.15 respectively. From these sketches it can be observed that the amplitude function is antisymmetric with $\omega = \pi$ when the impulse response is antisymmetric and N is odd number.

The term $e^{j\pi/2}$ makes the frequency response imaginary. Hence this frequency response is suitable for designing Hilbert transformers and differentiators.

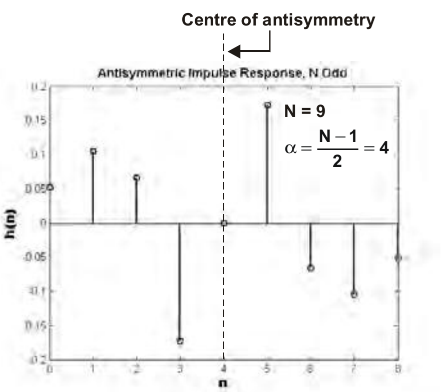


Fig 6.14: Antisymmetric impulse response, N= 9.

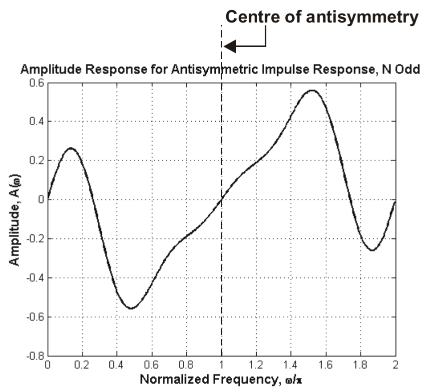


Fig 6.15: Amplitude function of $H(e^{j\omega})$.

Proof:

The Fourier transform of h(n) is,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

h(n) is defined for n = 0 to N – 1

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

The impulse response is antisymmetric with centre of antisymmetry at $n = \frac{N-1}{2}$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

$h\left(\frac{N-1}{2}\right) = 0$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Let, m = N – 1 – n ; \ n = N – 1 – m

When, $n = \frac{N+1}{2}$; $m = N-1-\left(\frac{N+1}{2}\right) = \frac{N-3}{2}$

When n = N – 1 ; m = N – 1 – (N – 1) = 0

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

Put, m = n

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} (-h(n))\, e^{-j\omega(N-1-n)}$$

For antisymmetric impulse response, h(N – 1 – n) = –h(n).

$$= \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \left(e^{-j\omega n}\, e^{j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(N-1-n)}\, e^{j\omega\left(\frac{N-1}{2}\right)} \right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \left(e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left((N-1)-n-\frac{N-1}{2}\right)} \right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \left(e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \left[\sum_{n=0}^{\frac{N-3}{2}} h(n)\, 2j \sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)} = \left[\sum_{n=0}^{\frac{N-3}{2}} 2\, h(n) e^{\frac{j\pi}{2}} \sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$j = e^{j\frac{\pi}{2}}$

$$= \left[\sum_{n=0}^{\frac{N-3}{2}} 2\, h(n) \sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right] e^{j\left(\frac{\pi}{2}-\frac{\omega(N-1)}{2}\right)}$$

Let, $k = \frac{N-1}{2} - n$; $\therefore\ n = \frac{N-1}{2} - k$

When, n = 0 ; $k = \frac{N-1}{2}$

When, $n = \frac{N-3}{2}$; $k = \frac{N-1}{2} - \frac{N-3}{2} = 1$

$$= \left[\sum_{k=1}^{\frac{N-1}{2}} 2\, h\left(\frac{N-1}{2}-k\right) \sin \omega k\right] e^{j\left(\frac{\pi}{2}-\frac{\omega(N-1)}{2}\right)}$$

$$= \left[\sum_{n=1}^{\frac{N-1}{2}} 2\, h\left(\frac{N-1}{2}-n\right) \sin \omega n\right] e^{j\left(\frac{\pi}{2}-\frac{\omega(N-1)}{2}\right)}$$

Put, k = n

Case (iv) : Frequency response of linear phase FIR filter when impulse response is antisymmetric and N is even with centre of antisymmetry at (N – 1)/2

The frequency response of linear phase FIR filter when impulse response is antisymmetric and N is even with centre of antisymmetry at (N – 1)/2 is given by,

$$H\left(e^{j\omega}\right)=\left[\sum_{n=1}^{\frac{N}{2}} 2h\left(\tfrac{N}{2}-n\right)\sin\left(\omega\left(n-\tfrac{1}{2}\right)\right)\right]e^{j\left(\frac{\pi}{2}-\frac{\omega(N-1)}{2}\right)} \qquad .....(6.37)$$

Let, $H(e^{j\omega}) = A(\omega)\, e^{j\theta(\omega)}$ .....(6.38)

where, $A(\omega)$ = Amplitude function

$\theta(\omega)$ = Phase function

On comparing equations (6.37) and (6.38) we get,

$$\text{Amplitude function, } A(\omega) = \sum_{n=1}^{\frac{N}{2}} 2\,h\left(\tfrac{N}{2}-n\right)\sin\left(\omega\left(n-\tfrac{1}{2}\right)\right) \qquad .....(6.39)$$

$$\text{Phase function, } \theta(\omega) = \frac{\pi}{2}-\omega\left(\tfrac{N-1}{2}\right) = \beta-\alpha\omega$$

$$\text{where, } \beta = \frac{\pi}{2} \quad \text{and} \quad \alpha = \frac{N-1}{2} \qquad .....(6.40)$$

$$\text{Magnitude function, } \left|H(e^{j\omega})\right| = \left|A(\omega)\right| = \left|\sum_{n=1}^{\frac{N}{2}} 2\,h\left(\tfrac{N}{2}-n\right)\sin\left(\omega\left(n-\tfrac{1}{2}\right)\right)\right| \qquad .....(6.41)$$

The sketch of antisymmetric impulse response when N = 8 and its corresponding amplitude function of frequency response are shown in fig 6.16 and fig 6.17 respectively. From these sketches it can be observed that the amplitude function of $H(e^{j\omega})$ is symmetric with $\omega = \pi$, when the impulse response is antisymmetric and N is even number.

The term $e^{j\pi/2}$ makes the frequency response imaginary. Hence this frequency response is suitable for designing Hilbert transformers and differentiators.

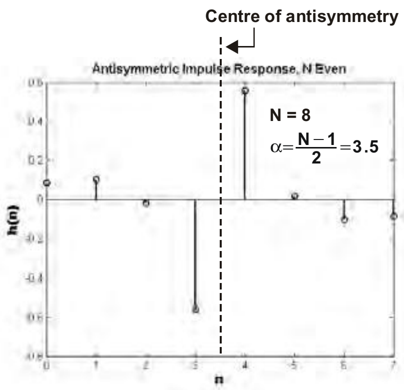


Fig 6.16: *Antisymmetric impulse response, N= 8.*

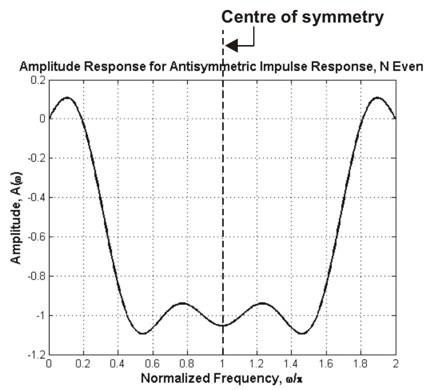


Fig 6.17: *Amplitude function of $H(e^{j\omega})$.*

Proof:

The Fourier transform of h(n) is,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

h(n) is defined only for n = 0 to N – 1

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

The impulse response is antisymmetric with centre of antisymmetry lies between $n = \frac{N-1}{2}$ and $\frac{N}{2}$.

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Let, $m = N-1-n$; $\therefore$ $n = N-1-m$

When, $n = \frac{N}{2}$; $m = N-1-\frac{N}{2} = \frac{N}{2}-1$

When, $n = N-1$; $m = N-1-(N-1) = 0$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

Put, m = n

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} (-h(n)) e^{-j\omega(N-1-n)}$$

For antisymmetric impulse response, $h(N-1-n) = -h(n)$.

$$= \left[\sum_{n=0}^{\frac{N}{2}-1} h(n) \left(e^{-j\omega n}\, e^{j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(N-1-n)}\, e^{j\omega\left(\frac{N-1}{2}\right)}\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left[\sum_{n=0}^{\frac{N}{2}-1} h(n) \left(e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega(N-1)-n-\frac{N-1}{2}}\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left[\sum_{n=0}^{\frac{N}{2}-1} h(n) \left(e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \left[\sum_{n=0}^{\frac{N}{2}-1} h(n)\, 2j \sin\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left[\sum_{n=0}^{\frac{N}{2}-1} h(n)\, 2e^{\frac{j\pi}{2}} \sin\left(\omega\left(\frac{N}{2}-n-\frac{1}{2}\right)\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$j = e^{j\frac{\pi}{2}}$$

$$= \left[\sum_{n=0}^{\frac{N}{2}-1} 2\, h(n) \sin\left(\omega\left(\frac{N}{2}-n-\frac{1}{2}\right)\right)\right] e^{j\left(\frac{\pi}{2}-\frac{\omega(N-1)}{2}\right)}$$

$$= \left[\sum_{k=1}^{\frac{N}{2}} 2\, h\left(\frac{N}{2}-k\right) \sin\left(\omega\left(k-\frac{1}{2}\right)\right)\right] e^{j\left(\frac{\pi}{2}-\frac{\omega(N-1)}{2}\right)}$$

Let, $k = \frac{N}{2}-n$; $\therefore$ $n = \frac{N}{2}-k$

When, $n = 0$; $k = \frac{N}{2}$

When, $n = \frac{N}{2}-1$; $k = \frac{N}{2}-\left(\frac{N}{2}-1\right) = 1$

$$= \left[\sum_{n=1}^{\frac{N}{2}} 2\, h\left(\frac{N}{2}-n\right) \sin\left(\omega\left(n-\frac{1}{2}\right)\right)\right] e^{j\left(\frac{\pi}{2}-\frac{\omega(N-1)}{2}\right)}$$

Put, k = n

Case (v) : Frequency response of linear phase FIR filter when impulse response is symmetric and N is odd with centre of symmetry at n = 0

The frequency response of linear phase FIR filter when impulse response is symmetric and N is odd with centre of symmetry at n = 0, is given by,

$$H\left(e^{j\omega}\right) = h(0) \; + \; \sum_{n=1}^{\frac{N-1}{2}} 2h(n) \; \cos \omega n \qquad .....(6.42)$$

Let, $H(e^{jw}) = A(w)\, e^{jq(w)}$ .....(6.43)

where, $A(w)$ = Amplitude function

$q(w)$ = Phase function

On comparing equations (6.42) and (6.43) we get,

$$\text{Amplitude function, } A(\omega) \; = \; h(0) \; + \; \sum_{n=1}^{\frac{N-1}{2}} 2\, h(n) \; \cos \omega n \qquad .....(6.44)$$

$$\text{Phase function, } \theta(\omega) = 0 \qquad .....(6.45)$$

$$\text{Magnitude function, } \left|H(e^{j\omega})\right| = \left|A(\omega)\right| = \left|h(0) \; + \; \sum_{n=1}^{\frac{N-1}{2}} 2\, h(n) \; \cos \omega n\right| \qquad .....(6.46)$$

A typical sketch of symmetric impulse response when N = 9 and its corresponding amplitude function of frequency response are shown in fig 6.18 and fig 6.19 respectively. From these sketches it can be observed that the amplitude function of $H(e^{jw})$ is symmetric with $w = p$, when the impulse response is symmetric and N is odd number.

When impulse response is symmetric and N is odd, the frequency response is non-zero at $w = 0$ and $w = p$, and so this frequency response can be used to design lowpass, highpass, bandpass and bandstop filters.

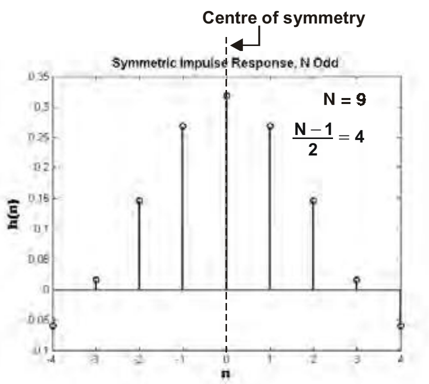


Fig 6.18: *Symmetric impulse response for N= 9.*

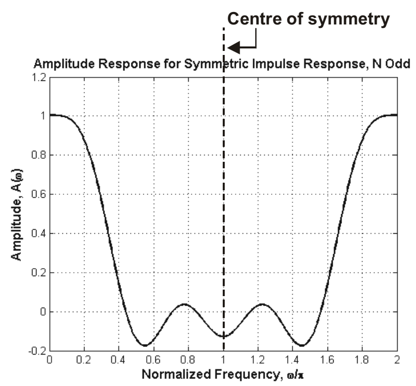


Fig 6.19: *Amplitude function of $H(e^{j\omega})$.*

Proof:

The Fourier transform of h(n) is,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\, e^{-j\omega n}$$

h(n) is defined only for $n = -\frac{N-1}{2}$ to $+\frac{N-1}{2}$.

$$= \sum_{n=-\frac{N-1}{2}}^{-1} h(n)\, e^{-j\omega n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\, e^{-j\omega n}$$

Here, centre of symmetry is n = 0.

$$= \sum_{n=1}^{\frac{N-1}{2}} h(-n)\, e^{j\omega n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\, e^{-j\omega n}$$

Using symmetry condition h(– n) = h(n).

$$= \sum_{n=1}^{\frac{N-1}{2}} h(n)\, e^{j\omega n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\, e^{-j\omega n}$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[e^{j\omega n} + e^{-j\omega n}\right]$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\, 2\cos\omega n$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$$

Case (vi) : Frequency response of linear phase FIR filter when impulse response is antisymmetric and N is odd with centre of antisymmetry at n =0

The frequency response of linear phase FIR filter when impulse response is antisymmetric and N is odd with centre of antisymmetry at n = 0, is given by,

$$H\left(e^{j\omega}\right) = \left[\sum_{n=1}^{\frac{N-1}{2}} 2h(n)\sin\omega n\right] e^{-j\frac{\pi}{2}} \qquad .....(6.47)$$

Let, $H(e^{jw}) = A(w)\, e^{jq(w)}$ .....(6.48)

where, $A(w)$ = Amplitude function

$q(w)$ = Phase function

On comparing equations (6.47) and (6.48) we get,

$$\text{Amplitude function, } A(\omega) = \sum_{n=1}^{\frac{N-1}{2}} 2\, h(n)\sin\omega n \qquad .....(6.49)$$

$$\text{Phase function, } \theta(\omega) = -\frac{\pi}{2} \qquad .....(6.50)$$

$$\text{Magnitude function, } \left|H(e^{j\omega})\right| = |A(\omega)| = \left|\sum_{n=1}^{\frac{N-1}{2}} 2\, h(n)\sin\omega n\right| \qquad .....(6.51)$$

A typical sketch of antisymmetric impulse response when N = 9 and its corresponding amplitude function of frequency response are shown in fig 6.20 and fig 6.21 respectively. From these sketches it can be observed that the amplitude function is antisymmetric with $\omega = \pi$ when the impulse response is antisymmetric and N is an odd number.

The term $e^{j\pi/2}$ makes the frequency response imaginary. Hence this frequency response is suitable for designing Hilbert transformers and differentiators.

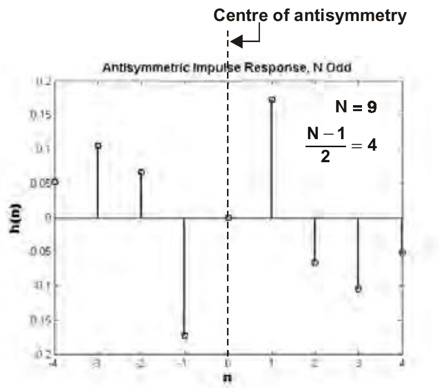


Fig 6.20: Antisymmetric impulse response, N= 9.

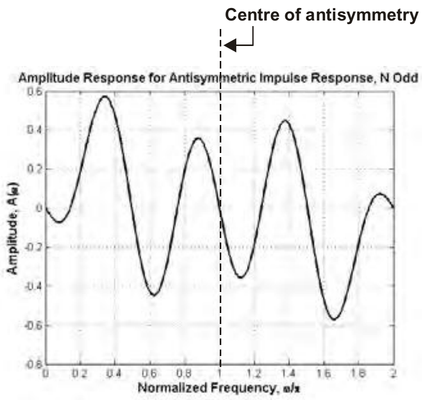


Fig 6.21: Amplitude function of $H(e^{j\omega})$.

Proof:

The Fourier transform of h(n) is,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\, e^{-j\omega n}$$

h(n) is defined only for $n = -\frac{N-1}{2}$ to $+\frac{N-1}{2}$.

$$= \sum_{n=-\frac{N-1}{2}}^{-1} h(n)\, e^{-j\omega n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\, e^{-j\omega n}$$

Here, centre of antisymmetry is n = 0.

$$= \sum_{n=1}^{\frac{N-1}{2}} h(-n)\, e^{j\omega n} + \sum_{n=1}^{\frac{N-1}{2}} h(n)\, e^{-j\omega n}$$

Here, h(0) = 0

$$= \sum_{n=1}^{\frac{N-1}{2}} (-h(n))\, e^{j\omega n} + \sum_{n=1}^{\frac{N-1}{2}} h(n)\, e^{-j\omega n}$$

Using symmetric condition $h(-n) = -h(n)$.

$$= -\sum_{n=1}^{\frac{N-1}{2}} h(n)\left[e^{j\omega n} - e^{-j\omega n}\right] = -\sum_{n=1}^{\frac{N-1}{2}} h(n)\, 2j \sin\omega n$$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$$= \left[\sum_{n=1}^{\frac{N-1}{2}} 2h(n) \sin\omega n\right](-j) = \left[\sum_{n=1}^{\frac{N-1}{2}} 2h(n) \sin\omega n\right] e^{-j\frac{\pi}{2}}$$

$-j = e^{-j\frac{\pi}{2}}$

Table 6.1 : Summary of Frequency Response Characteristics of Linear Phase FIR Filters

Case	h(n) [Impulse Response]	N [Number of samples of h(n)]	Symmetry condition	A(w) [Amplitude function of $H(e^{jw})$]
i	Symmetric	Odd	$h(N-1-n) = h(n)$	Symmetric
ii	Symmetric	Even	$h(N-1-n) = h(n)$	Antisymmetric
iii	Antisymmetric	Odd	$h(N-1-n) = -h(n)$	Antisymmetric
iv	Antisymmetric	Even	$h(N-1-n) = -h(n)$	Symmetric
v	Symmetric	Odd	$h(-n) = h(n)$	Symmetric
vi	Antisymmetric	Odd	$h(-n) = -h(n)$	Antisymmetric

Table 6.2 : Summary of A(w) for Linear Phase FIR Filters

Case	h(n) [Impulse response]	N	Symmetry condition	Magnitude function, $\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
i	Symmetric	Odd	$h(N-1-n) = h(n)$	$\left\lvert h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2\,h\left(\frac{N-1}{2} - n\right) \cos \omega n \right\rvert$
ii	Symmetric	Even	$h(N-1-n) = h(n)$	$\left\lvert \sum_{n=1}^{\frac{N}{2}} 2\,h\left(\frac{N}{2} - n\right) \cos\left(\omega\left(n - \frac{1}{2}\right)\right) \right\rvert$
iii	Antisymmetric	Odd	$h(N-1-n) = -h(n)$	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} 2\,h\left(\frac{N-1}{2} - n\right) \sin \omega n \right\rvert$
iv	Antisymmetric	Even	$h(N-1-n) = -h(n)$	$\left\lvert \sum_{n=1}^{\frac{N}{2}} 2\,h\left(\frac{N}{2} - n\right) \sin\left(\omega\left(n - \frac{1}{2}\right)\right) \right\rvert$
v	Symmetric	Odd	$h(-n) = h(n)$	$\left\lvert h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2\,h(n) \cos \omega n \right\rvert$
vi	Antisymmetric	Odd	$h(-n) = -h(n)$	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} 2\,h(n) \sin \omega n \right\rvert$

6.6 Design Techniques for Linear Phase FIR Filters

There are three well known method of design techniques for linear phase FIR filters.

1. Fourier series method and Window method.
2. Frequency sampling method.
3. Optimal filter design methods.

Design of Linear Phase FIR Filters by Fourier Series Method

The following two concepts leads to the design of FIR filters by Fourier series method.

1. The frequency response of a digital filter is periodic with period equal to $2p$.
2. Any periodic function can be expressed as a linear combination of complex exponentials.

In this method the desired frequency response, $H_d(e^{jw})$ can be converted to a Fourier series representation. Then using this expression the Fourier coefficients are evaluated which is the desired impulse response of the filter, $h_d(n)$. On taking $\mathcal{Z}$-transform of $h_d(n)$ we get $H_d(z)$ which is the transfer function of digital filter.

The $H_d(z)$ obtained from $h_d(n)$ will be a transfer function of unrealizable noncausal digital filter of infinite duration. A finite duration impulse response h(n) can be obtained by truncating the infinite duration impulse response $h_d(n)$ to N-samples. The samples of $h_d(n)$ are selected for $n = -\frac{N-1}{2}$ to $+\frac{N-1}{2}$. Now take $\mathcal{Z}$-transform of h(n) to get H(z) and then multiply H(z) by $z^{-(N-1)/2}$ to get the transfer function of realizable causal digital filter of finite duration.

The abrupt truncation of the Fourier series results in oscillations in the passband and stopband. These oscillations are due to slow convergence of the Fourier series, particularly near the points of discontinuity. This effect is known as the **Gibbs phenomenon.** It can be shown that the undesirable oscillations can be reduced by multiplying the desired impulse response coefficients by an appropriate window function. This leads to the method of FIR filter design using windows.

Design of Linear Phase FIR Filters Using Windows

In this method we begin with the desired frequency response specification $H_d(e^{jw})$ and determine the corresponding unit sample response $h_d(n)$. The $h_d(n)$ is given by inverse Fourier transform of $H_d(e^{jw})$. The unit sample response $h_d(n)$ will be an infinite sequence and must be truncated at some point say at $n = N - 1$ to yield an FIR filter of length N. The truncation is obtained by multiplying $h_d(n)$ by a window sequence w(n). [w(n) is also called window function]. The resultant sequence will be of length N and can be denoted by h(n).

The Fourier transform of h(n) is the frequency response of the filter to be implemented in software or in hardware. The frequency response of the filter is denoted by $H(e^{jw})$. The $\mathcal{Z}$-transform of h(n) will give the filter transfer function H(z). The frequency response of the filter $H(e^{jw})$ depends on the frequency response of the window function.

The desirable characteristics of the frequency response of window function are the following.

1. The width of the main-lobe should be small and it should contain as much of the total energy as possible.
2. The side-lobes should decrease in energy rapidly as w tends to p.

There have been many windows proposed, that approximates the desired characteristics. In the following sections, the Rectangular window, Hanning window, Hamming window, Blackman window and Kaiser window are discussed.

Design of Linear Phase FIR Filters by Frequency Sampling Method

In frequency sampling method of filter design, we begin with the desired frequency response specification $H_d(e^{jw})$ and it is sampled at N-points to generate a sequence H(k). The N-point inverse DFT of the sequence H(k) gives the impulse response of the filter h(n). The Fourier transform of h(n) gives the frequency response, $H(e^{jw})$ and $\mathcal{Z}$-transform of h(n) gives the transfer function H(z) of the filter.

Design of Optimum Equiripple Linear-Phase FIR Filter

The FIR filter design by window and frequency sampling method does not have precise control over the critical frequencies such as w_p (passband edge frequency) and w_s (stopband edge frequency). This drawback can be overcome by using Chebyshev approximation technique. In this method, the weighed approximation error between the desired frequency response and the actual frequency response is spread evenly across the passband and evenly across the stopband of the filter. This results in the reduction of maximum error. The resulting filter have ripples in both the passband and the stopband. This concept of design is called optimum equiripple design criterion.

6.7 Fourier Series Method of FIR Filter Design

The frequency response of a digital filter is periodic, with period equal to 2p. From Fourier series analysis, we know that any periodic function can be expressed as a linear combination of complex exponentials. Therefore, the desired frequency response, $H_d(e^{jw})$ of an FIR digital filter can be represented by the Fourier series as shown in equation (6.52).

$$H_d\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} h_d(n)\, e^{-j\omega n} \qquad .....(6.52)$$

where, the Fourier coefficients $h_d(n)$ are the desired impulse response sequence of the filter.

The samples of $h_d(n)$ can be determined using equation (6.53), which is inverse Fourier transform of $H_d(e^{jw})$.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \qquad .....(6.53)$$

The impulse response obtained from equation (6.53) is an infinite duration sequence. For FIR filters we truncate this infinite impulse response to a finite duration sequence of length N, where N is odd.

$$\therefore\ h(n) = h_d(n)\ ;\ \text{for } n = -\left(\tfrac{N-1}{2}\right) \text{ to } +\left(\tfrac{N-1}{2}\right)$$

Let, $H_N(z) = \mathcal{Z}\{h(n)\}$

By definition of $\mathcal{Z}$-transform,

$$H_N(z) = \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} h(n)\, z^{-n} \qquad .....(6.54)$$

The transfer function of equation (6.54) represents noncausal filter (due to the presence of positive powers of z). Hence the transfer function of equation (6.54) is multiplied by $z^{-(N-1)/2}$.

$$\therefore\ H(z) = z^{-(N-1)/2} H_N(z) = z^{-\frac{N-1}{2}} \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} h(n)\, z^{-n} = z^{-\frac{N-1}{2}} \left[\sum_{n=-\frac{N-1}{2}}^{-1} h(n)\, z^{-n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\, z^{-n}\right]$$

$$= z^{-\frac{N-1}{2}} \left[\sum_{n=1}^{\frac{N-1}{2}} h(-n)\, z^{n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\, z^{-n}\right]$$

> The Fourier coefficients h(n) is symmetric, with n = 0.
> $\therefore\ h(-n) = h(n)$

$$= z^{-\frac{N-1}{2}} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^{n} + z^{-n}\right]\right] \qquad .....(6.55)$$

Hence we see that causality is brought about by multiplying the transfer function by the delay factor a = (N - 1)/2. This modification does not affect the amplitude response of the filter, however the abrupt truncation of the Fourier series results in oscillations in the passband and stopband. These oscillations are due to the slow convergence of the Fourier series, particularly near the points of discontinuity. This effect is known as **Gibbs phenomenon.** The undesirable oscillations can be reduced by multiplying the desired impulse response coefficients by an appropriate window function.

Table 6.3 : Specification and Desired Impulse Response for FIR Filter Design by Fourier Series Method

Type of filter	Specifications	Impulse response
Lowpass	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\omega_c \le \omega \le +\omega_c \\ 0 \ ; \ \text{for} & -\pi \le \omega < -\omega_c \\ 0 \ ; \ \text{for} & \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_c \text{ and } +\omega_c < \omega \le \pi\right]$
Highpass	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\pi \le \omega \le -\omega_c \\ 1 \ ; \ \text{for} & \omega_c \le \omega \le \pi \\ 0 \ ; \ \text{for} & -\omega_c < \omega < +\omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_c < \omega < +\omega_c\right]$
Bandpass	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\omega_{c2} \le \omega \le -\omega_{c1} \\ 1 \ ; \ \text{for} & \omega_{c1} \le \omega \le \omega_{c2} \\ 0 \ ; \ \text{for} & -\pi \le \omega < -\omega_{c2} \\ 0 \ ; \ \text{for} & -\omega_{c1} < \omega < +\omega_{c1} \\ 0 \ ; \ \text{for} & \omega_{c2} < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_{c2} \ ; \ -\omega_{c1} < \omega < +\omega_{c1} \text{ and } +\omega_{c2} < \omega \le \pi\right]$
Bandstop	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\pi \le \omega \le -\omega_{c2} \\ 1 \ ; \ \text{for} & -\omega_{c1} \le \omega \le +\omega_{c1} \\ 1 \ ; \ \text{for} & \omega_{c2} \le \omega \le \pi \\ 0 \ ; \ \text{for} & -\omega_{c2} < \omega < -\omega_{c1} \\ 0 \ ; \ \text{for} & \omega_{c1} < \omega < \omega_{c2} \end{cases}$	$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{-\omega_{c2}} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{\omega_{c2}}^{+\pi} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_{c2} < \omega < -\omega_{c1} \text{ and } +\omega_{c1} < \omega < +\omega_{c2}\right]$

The specifications of lowpass, highpass, bandpass and bandstop filters and their desired impulse response for FIR filter design by Fourier series method are listed in table 6.3.

Procedure for digital FIR filter design by Fourier series method

1. The specifications of digital FIR filter are,

   i) The desired frequency response, $H_d(e^{jw})$.

   ii) The cutoff frequency w_c for lowpass and highpass, and w_{c1} & w_{c2} for bandpass and bandstop filters.

Note: If analog filter cutoff frequency F_c and sampling frequency F_s are specified, then calculate the cutoff frequency of digital filter w_c using the equation, $\omega_c = \dfrac{2\pi F_c}{F_s}$.

   iii) The number of samples of impulse response, N.

2. Determine the desired impulse response, $h_d(n)$ by taking inverse Fourier transform of the desired frequency response, $H_d(e^{jw})$.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

   (For limits of integration in the above equation, refer table 6.3)

3. Calculate N samples of $h_d(n)$ for $n = -(N-1)/2$ to $+(N-1)/2$ and form the impulse response, h(n) of FIR filter.

$$\therefore \text{ Impulse response, } h(n) = h_d(n)\Big|_{n=-\frac{N-1}{2} \text{ to } +\frac{N-1}{2}}$$

   The impulse response is symmetric with n = 0, and so h(–n) = h(n). Hence it is sufficient if we, calculate h(n) for n = 0 to +(N – 1)/2.

4. Take $\mathcal{Z}$-transform of the impulse response to get the noncausal transfer function of FIR filter, $H_N(z)$.

$$\therefore\ H_N(z) = \mathcal{Z}\{h(n)\} = \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\, z^{-n}$$

5. Convert the noncausal transfer function, $H_N(z)$ to causal transfer function, H(z) by multiplying $H_N(z)$ by $z^{-(N-1)/2}$

$$\therefore \text{ Transfer function, } H(z) = z^{-\frac{N-1}{2}} \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\, z^{-n}$$

   Alternatively,

$$\text{Transfer function, } H(z) = z^{-\frac{N-1}{2}} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^{n} + z^{-n}\right] \right]$$

Applying symmetry condition, h(–n) = h(n). Refer equation (6.55).

6. Draw a suitable structure for realization of FIR filter.

Design verification

1. Determine the frequency response, $H(e^{jw})$.

   Method - 1 : Choose a linear phase magnitude function $|H(e^{jw})|$ from table 6.2. Using h(n), obtain an equation for $|H(e^{jw})|$.

   Method - 2 : The frequency response, $|H(e^{jw})|$ can be obtained by replacing z by e^{jw} in the transfer function, H(z).

   $\therefore$ Frequency response, $H\left(e^{j\omega}\right) = H(z)\Big|_{z = e^{j\omega}}$

2. Calculate frequency response for various values of w in the range 0 to p.
3. Calculate the magnitude response, $|H(e^{jw})|$ and sketch the magnitude response to verify the design.

Example 6.1

Design a FIR lowpass filter with cutoff frequency of 1 kHz and sampling frequency of 4 kHz with 11 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.

Solution

Given that, $F_c = 1$ kHz ; $F_s = 4$ kHz

$$\therefore \omega_c = \Omega_c T = \frac{\Omega_c}{F_s} = \frac{2\pi F_c}{F_s} = \frac{2\pi \times 1 \times 10^3}{4 \times 10^3} = 0.5\pi \text{ rad / sample}$$

The desired frequency response $H_d(e^{jw})$ of lowpass filter is,

$$\begin{aligned} H_d(e^{j\omega}) &= 1 \ ; \text{ for } -\omega_c \le \omega \le +\omega_c \\ &= 0 \ ; \text{ for } -\pi \le \omega < -\omega_c \quad \text{and} \quad \omega_c < \omega \le \pi \end{aligned}$$

The desired impulse response $h_d(n)$ of the lowpass filter is,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} 1 \times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{+\omega_c} = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{j\theta}}{2j}}$$

When n = 0, the factor $\frac{\sin\omega_c n}{\pi n}$ becomes 0 / 0, which is indeterminate.

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] = \frac{1}{\pi n}\sin \omega_c n \ ; \text{ for all n, except n = 0.}$$

Using L' Hospital rule, $\displaystyle \mathop{\mathrm{Lt}}_{\theta \to 0} \frac{\sin A\theta}{\theta} = A$

$$\text{When, } n = 0 \ ; \ h_d(n) = h_d(0) = \mathop{\mathrm{Lt}}_{n \to 0} \frac{\sin \omega_c n}{\pi n} = \frac{1}{\pi}\mathop{\mathrm{Lt}}_{n \to 0} \frac{\sin \omega_c n}{n} = \frac{\omega_c}{\pi}$$

The impulse response h(n) of FIR filter is obtained by truncating $h_d(n)$ to 11 samples.

$$\begin{aligned} \therefore\ h(n) = h_d(n) &= \frac{\sin \omega_c n}{n\pi} \ ; \text{ for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}, \text{ except } n = 0 \\ &= \frac{\omega_c}{\pi} \ ; \text{ for } n = 0 \end{aligned}$$

$$\text{Here, } N = 11, \quad \therefore \quad \frac{N-1}{2} = \frac{11-1}{2} = 5$$

Hence, calculate h(n) for n = –5 to +5

Since, the impulse response h(n) satisfies the symmetry condition, h(–n) = h(n), calculate h(n) for n = 0 to 5.

When $n = 0$; $h(0) = \frac{\omega_c}{\pi} = 0.5$

When $n = 1$; $h(1) = \frac{\sin(0.5\pi \times 1)}{\pi \times 1} = 0.3183$

Note : Calculate sin q by keeping the calculator in radian mode.

When $n = 2$; $h(2) = \frac{\sin(0.5\pi \times 2)}{\pi \times 2} = 0$

When $n = 3$; $h(3) = \frac{\sin(0.5\pi \times 3)}{\pi \times 3} = -0.1061$

When $n = 4$; $h(4) = \frac{\sin(0.5\pi \times 4)}{\pi \times 4} = 0$

When $n = 5$; $h(5) = \frac{\sin(0.5\pi \times 5)}{\pi \times 5} = 0.0637$

When $n = -1$; $h(-1) = h(1) = 0.3183$

When $n = -2$; $h(-2) = h(2) = 0$

When $n = -3$; $h(-3) = h(3) = -0.1061$

When $n = -4$; $h(-4) = h(4) = 0$

When $n = -5$; $h(-5) = h(5) = 0.0637$

Using symmetry condition, $h(-n) = h(n)$.

The transfer function H(z) of the digital lowpass filter is given by,

$$H(z) = z^{-\frac{N-1}{2}} \mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}} \left[\sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\, z^{-n} \right] = z^{-5} \sum_{n=-5}^{5} h(n)\, z^{-n}$$

$$= z^{-5} [h(-5)\, z^{5} + h(-4)\, z^{4} + h(-3)\, z^{3} + h(-2)\, z^{2} + h(-1)\, z + h(0)\, z^{0} + h(1) z^{-1} + h(2)\, z^{-2}$$
$$+ h(3)\, z^{-3} + h(4)\, z^{-4} + h(5)\, z^{-5}]$$

$$= z^{-5} [h(5)\, z^{5} + h(4)\, z^{4} + h(3)\, z^{3} + h(2)\, z^{2} + h(1)\, z + h(0)$$
$$+ h(1) z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4} + h(5)\, z^{-5}]$$

Using symmetry condition, $h(-n) = h(n)$.

$$= z^{-5} \Big[h(0) + h(1)\big[z + z^{-1}\big] + h(2)\big[z^{2} + z^{-2}\big] + h(3)\big[z^{3} + z^{-3}\big] + h(4)\big[z^{4} + z^{-4}\big] + h(5)\big[z^{5} + z^{-5}\big] \Big]$$

$$= h(0)\, z^{-5} + h(1)\big[z^{-4} + z^{-6}\big] + h(2)\big[z^{-3} + z^{-7}\big] + h(3)\big[z^{-2} + z^{-8}\big] + h(4)\big[z^{-1} + z^{-9}\big] + h(5)\big[z^{0} + z^{-10}\big]$$

$$= 0.5\, z^{-5} + 0.3183\big[z^{-4} + z^{-6}\big] - 0.1061\big[z^{-2} + z^{-8}\big] + 0.0637\big[1 + z^{-10}\big]$$

$h(2) = 0$
$h(4) = 0$

Structure

Let, $$H(z) = \frac{Y(z)}{X(z)} = 0.5\, z^{-5} + 0.3183\big[z^{-4} + z^{-6}\big] - 0.1061\big[z^{-2} + z^{-8}\big] + 0.0637\big[1 + z^{-10}\big]$$

$$\therefore \quad Y(z) = 0.5\, z^{-5}\, X(z) + 0.3183\big[z^{-4}\, X(z) + z^{-6}\, X(z)\big] - 0.1061\big[z^{-2}\, X(z) + z^{-8}\, X(z)\big]$$
$$+ 0.0637\big[X(z) + z^{-10}\, X(z)\big]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

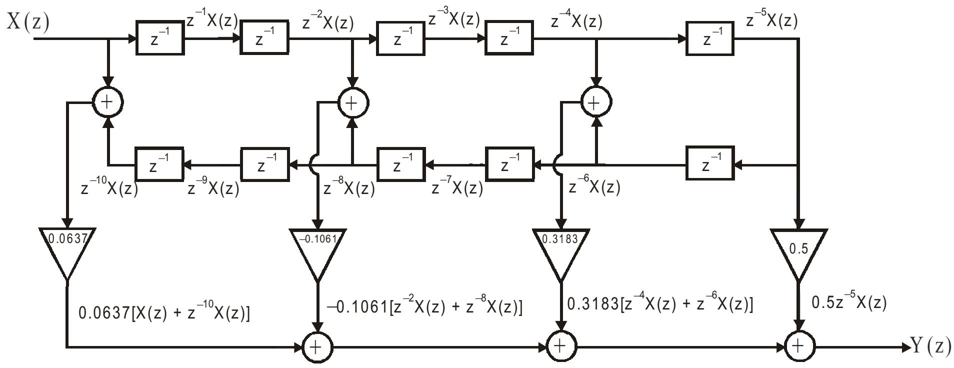


Fig 1 : Linear phase structure of FIR lowpass filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0, the magnitude response, $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where, } A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$$

Refer table 6.2 case (v)

$$= h(0) + \sum_{n=1}^{5} 2h(n)\cos\omega n$$

$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega + 2h(3)\cos 3\omega + 2h(4)\cos 4\omega + 2h(5)\cos 5\omega$$

$$= 0.5 + 2\times 0.3183\cos\omega + 2\times 0\cos 2\omega + 2\times -0.1061\cos 3\omega + 2\times 0\cos 4\omega + 2\times 0.0637\cos 5\omega$$

$$= 0.5 + 0.6366\cos\omega - 0.2122\cos 3\omega + 0.1274\cos 5\omega$$

Using the above equation the amplitude response A(w) and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the tabulated values, the magnitude response is sketched as shown in fig 2.

Table 1: A(w) and $|H(e^{jw})|$ for various values of w.

w	A(w)	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{0\times\pi}{16}$	1.0518	1.0518
$\frac{1\times\pi}{16}$	1.0187	1.0186
$\frac{2\times\pi}{16}$	0.9581	0.9581
$\frac{3\times\pi}{16}$	0.9457	0.9457
$\frac{4\times\pi}{16}$	1.0101	1.0102
$\frac{5\times\pi}{16}$	1.0866	1.0866
$\frac{6\times\pi}{16}$	1.0573	1.0574
$\frac{7\times\pi}{16}$	0.8480	0.8479
$\frac{8\times\pi}{16}$	0.5	0.5

w	A(w)	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{9\times\pi}{16}$	0.1520	0.1520
$\frac{10\times\pi}{16}$	−0.0574	0.0574
$\frac{11\times\pi}{16}$	−0.0866	0.0866
$\frac{12\times\pi}{16}$	−0.0101	0.0101
$\frac{13\times\pi}{16}$	0.0542	0.0542
$\frac{14\times\pi}{16}$	0.0418	0.0418
$\frac{15\times\pi}{16}$	−0.0187	0.0187
$\frac{16\times\pi}{16}$	−0.0518	0.0518

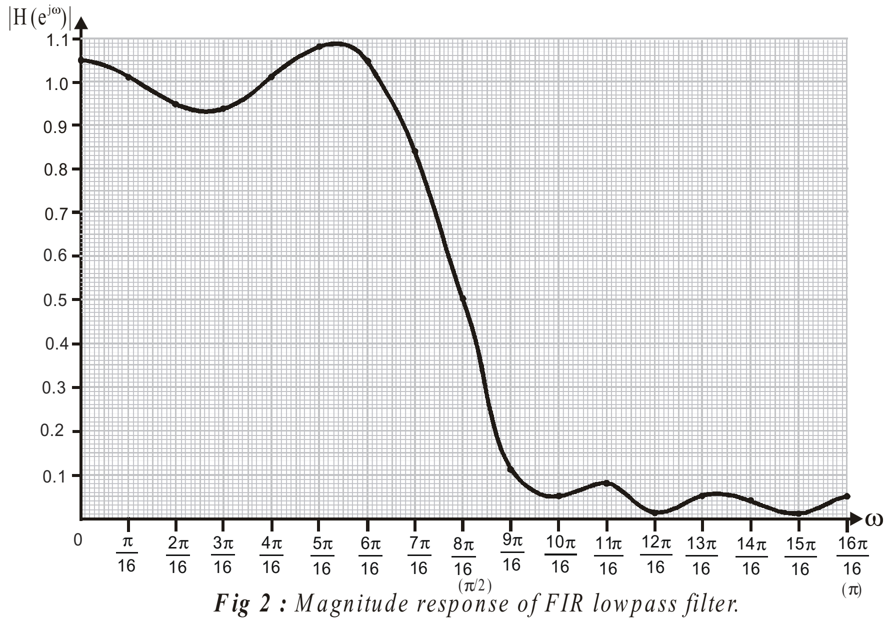


Fig 2 : Magnitude response of FIR lowpass filter.

Alternate Method for Frequency Response

Frequency response, $H\left(e^{j\omega}\right) = H(z)\Big|_{z\,=\,e^{j\omega}}$

$$\therefore\ H\left(e^{j\omega}\right) = 0.5z^{-5} + 0.3183\left[z^{-4} + z^{-6}\right] - 0.1061\left[z^{-2} + z^{-8}\right] + 0.0637\left[1 + z^{-10}\right]\Big|_{z=e^{j\omega}}$$

$$= 0.5e^{-j5\omega} + 0.3183\left[e^{-j4\omega} + e^{-j6\omega}\right] - 0.1061\left[e^{-j2\omega} + e^{-j8\omega}\right] + 0.0637\left[1 + e^{-j10\omega}\right]$$

$$= 0.5\left[\cos 5\omega - j\sin 5\omega\right] + 0.3183\left[\cos 4\omega - j\sin 4\omega + \cos 6\omega - j\sin 6\omega\right]$$

$$- 0.1061\left[\cos 2\omega - j\sin 2\omega + \cos 8\omega - j\sin 8\omega\right] + 0.0637\left[1 + \cos 10\omega - j\sin 10\omega\right]$$

$$= [0.5\cos 5\omega + 0.3183\cos 4\omega + 0.3183\cos 6\omega - 0.1061\cos 2\omega - 0.1061\cos 8\omega + 0.0637 + 0.0637\cos 10\omega]$$

$$+\ j[-0.5\sin 5\omega - 0.3183\sin 4\omega - 0.3183\sin 6\omega + 0.1061\sin 2\omega + 0.1061\sin 8\omega - 0.0637\sin 10\omega]$$

Using the above equation, the frequency response $H(e^{jw})$ and magnitude function $|H(e^{jw})|$ of lowpass filter are calculated for various values of w and listed in table 2. It is observed that the magnitude response obtained by both the methods are same.

Table 2: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	1.0518 + j0	1.0518
$\frac{1\times\pi}{16}$	0.5659 – j0.8470	1.0186
$\frac{2\times\pi}{16}$	–0.3667 – j0.8852	0.9581
$\frac{3\times\pi}{16}$	–0.9276 – j0.1845	0.9457
$\frac{4\times\pi}{16}$	–0.7143 + j0.7143	1.0102
$\frac{5\times\pi}{16}$	0.2120 + j1.0657	1.0866
$\frac{6\times\pi}{16}$	0.9769 + j0.4046	1.0574
$\frac{7\times\pi}{16}$	0.7050 – j0.4711	0.8479
$\frac{8\times\pi}{16}$	0 – j0.5	0.5

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{9\times\pi}{16}$	–0.1264 – j0.0844	0.1520
$\frac{10\times\pi}{16}$	0.0530 – j0.022	0.0574
$\frac{11\times\pi}{16}$	0.0169 – j0.0850	0.0866
$\frac{12\times\pi}{16}$	–0.0071 – j0.0071	0.0100
$\frac{13\times\pi}{16}$	0.0532 – j0.0106	0.0542
$\frac{14\times\pi}{16}$	0.0160 – j0.0386	0.0418
$\frac{15\times\pi}{16}$	0.0104 + 0.0155	0.0186
$\frac{16\times\pi}{16}$	0.0518 + j0	0.0518

Example 6.2

Design a FIR highpass filter with cutoff frequency of 1.5 kHz and sampling frequency of 5 kHz with 7 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.

Solution

Given that, $F_c = 1.5$ kHz ; $F_s = 5$ kHz

$$\omega_c = \Omega_c T = \frac{\Omega_c}{F_s} = \frac{2\pi F_c}{F_s} = \frac{2\pi \times 1.5 \times 10^3}{5 \times 10^3} = 0.6\pi \text{ rad / sample}$$

The desired frequency response $H_d(e^{jw})$ of highpass filter is,

$$H_d(e^{j\omega}) = 1 \;;\; \text{for } -\pi \le \omega \le -\omega_c \text{ and } \omega_c \le \omega \le \pi$$

$$= 0 \;;\; \text{otherwise}$$

The desired impulse response $h_d(n)$ of the highpass filter is,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} 1 \times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} 1 \times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_c}^{\pi} = \frac{1}{2\pi}\left[\frac{e^{-j\omega_c n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_c n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} - \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{\pi n}\left[\sin \pi n - \sin \omega_c n\right] \;;\; \text{for all n, except n = 0.}$$

When n = 0 ; $h_d(n) = h_d(0) = \underset{n\to 0}{\mathrm{Lt}} \left[\frac{\sin \pi n - \sin \omega_c n}{\pi n}\right]$

When n = 0 ; the $h_d(n)$ become 0/0, which is indeterminate.

$$= \underset{n\to 0}{\mathrm{Lt}} \frac{\sin \pi n}{\pi n} - \underset{n\to 0}{\mathrm{Lt}} \frac{\sin \omega_c n}{\pi n}$$

$$= \frac{1}{\pi}\underset{n\to 0}{\mathrm{Lt}} \frac{\sin \pi n}{n} - \frac{1}{\pi}\underset{n\to 0}{\mathrm{Lt}} \frac{\sin \omega_c n}{n}$$

Using L' Hospital rule, $\underset{\theta\to 0}{\mathrm{Lt}} \frac{\sin A\theta}{\theta} = A$

$$= \frac{1}{\pi}\times\pi - \frac{1}{\pi}\times\omega_c = 1 - \frac{\omega_c}{\pi}$$

The impulse response h(n) of FIR filter is obtained by truncating $h_d(n)$ to 7 samples.

$$\therefore\; h(n) = h_d(n) = \frac{\sin \pi n - \sin \omega_c n}{\pi n} = -\frac{\sin \omega_c n}{\pi n} \;;\; \text{for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}, \text{ except } n = 0$$

$$= 1 - \frac{\omega_c}{\pi} \;;\; \text{for } n = 0$$

For any integer n, sin pn = 0

Here, N = 7, $\therefore\; \frac{N-1}{2} = \frac{7-1}{2} = 3$

Hence, calculate h(n) for n= –3 to 3.

Since, the impulse response h(n) satisfies the symmetry condition, h(–n) = h(n), calculate h(n) for n = 0 to 3.

When n = 0 ; $h(0) = 1 - \dfrac{\omega_c}{\pi} = 0.4$

When n = 1 ; $h(1) = -\dfrac{\sin(0.6\pi \times 1)}{\pi \times 1} = -0.3027$

When n = 2 ; $h(2) = -\dfrac{\sin(0.6\pi \times 2)}{\pi \times 2} = 0.0935$

When n = 3 ; $h(3) = -\dfrac{\sin(0.6\pi \times 3)}{\pi \times 3} = 0.0623$

When n = –1 ; $h(-1) = h(1) = -0.3027$

When n = –2 ; $h(-2) = h(2) = 0.0935$

When n = –3 ; $h(-3) = h(3) = 0.0623$

Using symmetry condition, $h(-n) = h(n)$

The transfer function H(z) of the digital highpass filter is given by,

$$H(z) = z^{-\frac{(N-1)}{2}} \mathcal{Z}\{h(n)\} = z^{-\frac{(N-1)}{2}} \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\,z^{-n} = z^{-3} \sum_{n=-3}^{+3} h(n)\,z^{-n}$$

$$= z^{-3}\left[h(-3)\,z^{3} + h(-2)\,z^{2} + h(-1)\,z + h(0)\,z^{0} + h(1)z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3}\right]$$

$$= z^{-3}\left[h(3)\,z^{3} + h(2)\,z^{2} + h(1)\,z + h(0) + h(1)z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3}\right]$$

Using symmetry condition, $h(-n) = h(n)$

$$= z^{-3}\left[h(0) + h(1)\left[z + z^{-1}\right] + h(2)\left[z^{2} + z^{-2}\right] + h(3)\left[z^{3} + z^{-3}\right]\right]$$

$$= h(0)\,z^{-3} + h(1)\left[z^{-2} + z^{-4}\right] + h(2)\left[z^{-1} + z^{-5}\right] + h(3)\left[z^{0} + z^{-6}\right]$$

$$= 0.4\,z^{-3} - 0.3027\left[z^{-2} + z^{-4}\right] + 0.0935\left[z^{-1} + z^{-5}\right] + 0.0623\left[1 + z^{-6}\right]$$

Structure

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = 0.4\,z^{-3} - 0.3027\left[z^{-2} + z^{-4}\right] + 0.0935\left[z^{-1} + z^{-5}\right] + 0.0623\left[1 + z^{-6}\right]$$

$$Y(z) = 0.4z^{-3}X(z) - 0.3027\left[z^{-2}X(z) + z^{-4}X(z)\right] + 0.0935\left[z^{-1}X(z) + z^{-5}X(z)\right] + 0.0623\left[X(z) + z^{-6}X(z)\right]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

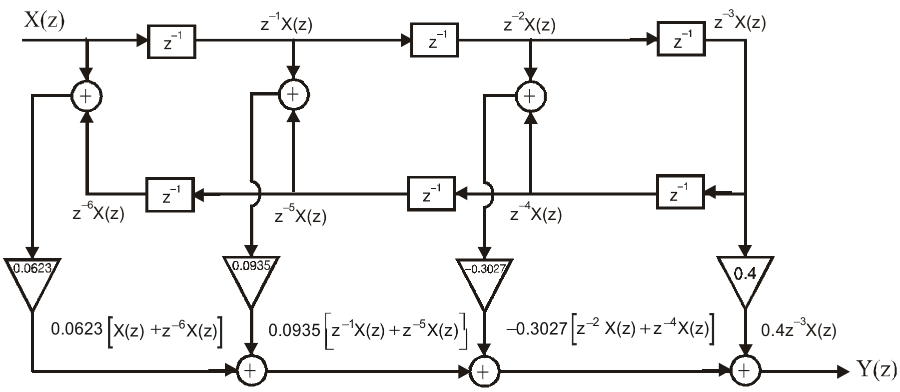


Fig 1 : Linear phase structure of FIR highpass filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0, the magnitude response $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where, } A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n = h(0) + \sum_{n=1}^{3} 2h(n)\cos\omega n$$

Refer table 6.2 case (v)

$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega + 2h(3)\cos 3\omega$$

$$= 0.4 + 2\times -0.3027\cos\omega + 2\times 0.0935\cos 2\omega + 2\times 0.0623\cos 3\omega$$

$$= 0.4 - 0.6054\cos\omega + 0.187\cos 2\omega + 0.1246\cos 3\omega$$

Using the above equation, the amplitude response A(w) and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the tabulated values, the magnitude response is sketched as shown in fig 2.

Table 1: A(w) and $|H(e^{jw})|$ for various values of w.

w	A(w)	$\|H(e^{jw})\| = \|A(w)\|$
$\frac{0\times\pi}{16}$	0.1062	0.1062
$\frac{1\times\pi}{16}$	0.0825	0.0826
$\frac{2\times\pi}{16}$	0.0205	0.0205
$\frac{3\times\pi}{16}$	−0.0561	0.0560
$\frac{4\times\pi}{16}$	−0.1161	0.1161
$\frac{5\times\pi}{16}$	−0.1301	0.1300
$\frac{6\times\pi}{16}$	−0.0790	0.0790
$\frac{7\times\pi}{16}$	0.0399	0.0399
$\frac{8\times\pi}{16}$	0.213	0.213

w	A(w)	$\|H(e^{jw})\| = \|A(w)\|$
$\frac{9\times\pi}{16}$	0.4145	0.4145
$\frac{10\times\pi}{16}$	0.6145	0.6146
$\frac{11\times\pi}{16}$	0.7869	0.7869
$\frac{12\times\pi}{16}$	0.9161	0.9161
$\frac{13\times\pi}{16}$	0.9992	0.9991
$\frac{14\times\pi}{16}$	1.0438	1.0438
$\frac{15\times\pi}{16}$	1.0629	1.0629
$\frac{16\times\pi}{16}$	1.0678	1.0678

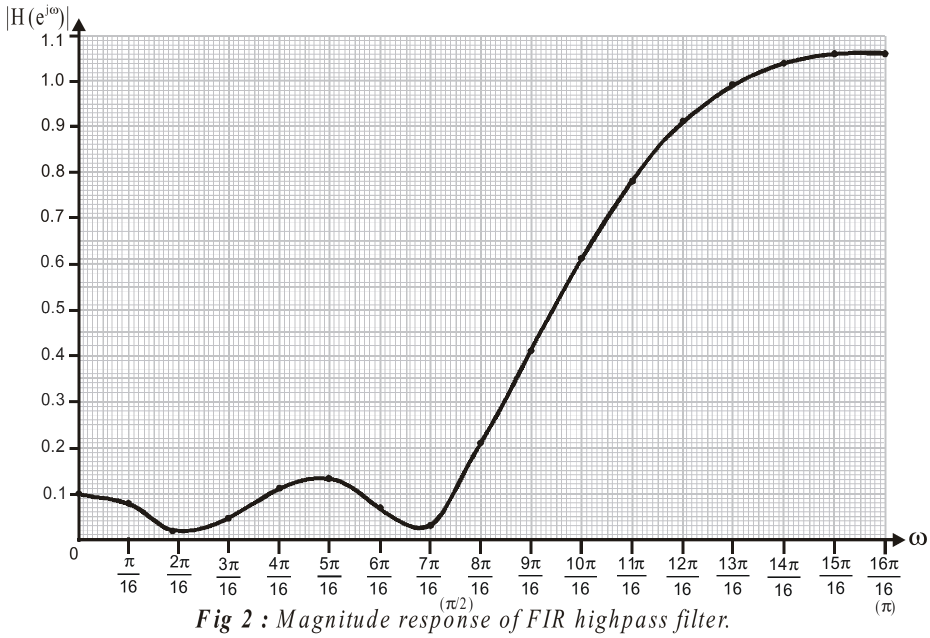


Fig 2 : Magnitude response of FIR highpass filter.

Alternate Method for Frequency Response

Frequency response, $H(e^{j\omega}) = H(z)\Big|_{z = e^{j\omega}}$

$$\therefore\ H(e^{j\omega}) = 0.4\,z^{-3} - 0.3027\left[z^{-2} + z^{-4}\right] + 0.0935\left[z^{-1} + z^{-5}\right] + 0.0623\left[1 + z^{-6}\right]\Big|_{z=e^{j\omega}}$$

$$= 0.4e^{-j3\omega} - 0.3027\left[e^{-j2\omega} + e^{-j4\omega}\right] + 0.0935\left[e^{-j\omega} + e^{-j5\omega}\right] + 0.0623\left[1 + e^{-j6\omega}\right]$$

$$= 0.4\left[\cos3\omega - j\sin3\omega\right] - 0.3027\left[\cos2\omega - j\sin2\omega + \cos4\omega - j\sin4\omega\right]$$

$$+ 0.0935\left[\cos\omega - j\sin\omega + \cos5\omega - j\sin5\omega\right] + 0.0623\left[1 + \cos6\omega - j\sin6\omega\right]$$

$$= [0.4\cos3\omega - 0.3027\cos2\omega - 0.3027\cos4\omega + 0.0935\cos\omega + 0.0935\cos5\omega + 0.0623 + 0.0623\cos6\omega]$$

$$+ j[-0.4\sin3\omega + 0.3027\sin2\omega + 0.3027\sin4\omega - 0.0935\sin\omega - 0.0935\sin5\omega - 0.0623\sin6\omega]$$

Using the above equation, the frequency response $H(e^{jw})$ and magnitude function $|H(e^{jw})|$ of highpass filter are calculated for various values of w and listed in table 2. It is observed that the magnitude response obtained by both the methods are same.

Table 2 : $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	0.1062 + j0	0.1062
$\frac{1\times\pi}{16}$	0.0687 − j0.0459	0.0826
$\frac{2\times\pi}{16}$	0.0078 − j0.0190	0.0205
$\frac{3\times\pi}{16}$	0.0109 − j0.0550	0.0560
$\frac{4\times\pi}{16}$	0.0821 + j0.0821	0.1161
$\frac{5\times\pi}{16}$	0.1276 + j0.0253	0.1300
$\frac{6\times\pi}{16}$	0.0730 − j0.0302	0.0790
$\frac{7\times\pi}{16}$	−0.0221+ j0.0331	0.0397
$\frac{8\times\pi}{16}$	0+ j0.213	0.213

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{9\times\pi}{16}$	0.2303 + j0.3447	0.4145
$\frac{10\times\pi}{16}$	0.5678 + j0.2352	0.6146
$\frac{11\times\pi}{16}$	0.7718 − j0.1535	0.7869
$\frac{12\times\pi}{16}$	0.6478 − j0.6478	0.9161
$\frac{13\times\pi}{16}$	0.1949 − j0.9800	0.9991
$\frac{14\times\pi}{16}$	−0.3994 − j0.9644	1.0438
$\frac{15\times\pi}{16}$	−0.8838 + j0.5905	1.0629
$\frac{16\times\pi}{16}$	−1.0678 + j0	1.0678

Example 6.3

Design an FIR bandpass filter to pass frequencies in the range 1.5 kHz to 3 kHz and sampling frequency of 8 kHz with 7 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.

Solution

Given that, $F_{c1} = 1.5$ kHz ; $F_{c2} = 3$ kHz ; $F_s = 8$ kHz

$$\therefore\ \omega_{c1} = \Omega_{c1}T = \frac{\Omega_{c1}}{F_s} = \frac{2\pi F_{c1}}{F_s} = \frac{2\pi\times1.5\times10^3}{8\times10^3} = 0.375\pi$$

$$\omega_{c2} = \Omega_{c2}T = \frac{\Omega_{c2}}{F_s} = \frac{2\pi F_{c2}}{F_s} = \frac{2\pi\times3\times10^3}{8\times10^3} = 0.75\pi$$

The desired frequency response $H_d(e^{jw})$ of bandpass filter is,

$$H_d(e^{j\omega}) = 1\ ;\ \text{for } -\omega_{c2} \le \omega \le -\omega_{c1} \text{ and } \omega_{c1} \le \omega \le \omega_{c2}$$

$$= 0\ ;\ \text{otherwise}$$

The desired impulse response $h_d(n)$ of the bandpass filter is,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} 1\times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} 1\times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c1}}^{\omega_{c2}} = \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}n}}{jn} - \frac{e^{-j\omega_{c2}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2}n}}{jn} - \frac{e^{j\omega_{c1}n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j} - \frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n} \quad ; \text{ for all } n, \text{ except } n = 0.$$

When $n = 0$, the $h_d(n)$ become 0/0, which is indeterminate.

When $n = 0$; $h_d(n) = h_d(0) = \underset{n\to 0}{Lt}\left[\frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n}\right]$

Using L' Hospital rule, $\underset{\theta\to 0}{Lt}\ \frac{\sin A\theta}{\theta} = A$

$$= \frac{1}{\pi}\left[\underset{n\to 0}{Lt}\ \frac{\sin\omega_{c2}n}{n} - \underset{n\to 0}{Lt}\ \frac{\sin\omega_{c1}n}{n}\right]$$

$$= \frac{1}{\pi}\left(\omega_{c2} - \omega_{c1}\right)$$

The impulse response h(n) of FIR filter is obtained by truncating $h_d(n)$ to 7 samples.

$$\therefore\ h(n) = h_d(n) = \frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n} \quad ; \text{ for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}, \text{ except } n = 0$$

$$= \frac{\omega_{c2} - \omega_{c1}}{\pi} \quad ; \text{ for } n = 0$$

Here, $N = 7$, $\therefore\ \frac{N-1}{2} = \frac{7-1}{2} = 3$

Hence, calculate h(n) for n= –3 to 3.

Since, the impulse response h(n) satisfies the symmetry condition, h(–n) = h(n), calculate h(n) for n = 0 to 3.

When $n = 0$; $h(0) = \frac{\omega_{c2} - \omega_{c1}}{\pi} = \frac{0.75\pi - 0.375\pi}{\pi} = \frac{0.375\pi}{\pi} = 0.375$

When $n = 1$; $h(1) = \frac{\sin(0.75\pi\times 1) - \sin(0.375\pi\times 1)}{\pi\times 1} = -0.069$

When $n = 2$; $h(2) = \frac{\sin(0.75\pi\times 2) - \sin(0.375\pi\times 2)}{\pi\times 2} = -0.2716$

When $n = 3$; $h(3) = \frac{\sin(0.75\pi\times 3) - \sin(0.375\pi\times 3)}{\pi\times 3} = 0.1156$

When $n = -1$; $h(-1) = h(1) = -0.069$

When $n = -2$; $h(-2) = h(2) = -0.2716$

When $n = -3$; $h(-3) = h(3) = 0.1156$

The transfer function H(z) of the digital FIR bandpass filter is given by,

$$H(z) = z^{-\frac{(N-1)}{2}}\,\mathcal{Z}\{h(n)\} = z^{-\frac{(N-1)}{2}} \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\, z^{-n} = z^{-3}\sum_{n=-3}^{+3} h(n)\, z^{-n}$$

$$= z^{-3}\left[h(-3)\,z^{3} + h(-2)\,z^{2} + h(-1)\,z + h(0)\,z^{0} + h(1)z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3}\right]$$

$$\therefore\ H(z) = z^{-3}\left[h(3)\,z^{3} + h(2)\,z^{2} + h(1)\,z + h(0) + h(1)z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3}\right]$$

Using symmetry condition, $h(-n) = h(n)$

$$= z^{-3}\left[h(0) + h(1)\left[z + z^{-1}\right] + h(2)\left[z^{2} + z^{-2}\right] + h(3)\left[z^{3} + z^{-3}\right]\right]$$

$$= h(0)\,z^{-3} + h(1)\left[z^{-2} + z^{-4}\right] + h(2)\left[z^{-1} + z^{-5}\right] + h(3)\left[z^{0} + z^{-6}\right]$$

$$= 0.375\,z^{-3} - 0.069\left[z^{-2} + z^{-4}\right] - 0.2716\left[z^{-1} + z^{-5}\right] + 0.1156\left[1 + z^{-6}\right]$$

Structure

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = 0.375\,z^{-3} - 0.069\left[z^{-2} + z^{-4}\right] - 0.2716\left[z^{-1} + z^{-5}\right] + 0.1156\left[1 + z^{-6}\right]$$

$$Y(z) = 0.375z^{-3}X(z) - 0.069\left[z^{-2}X(z) + z^{-4}X(z)\right] - 0.2716\left[z^{-1}X(z) + z^{-5}X(z)\right] + 0.1156\left[X(z) + z^{-6}X(z)\right]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

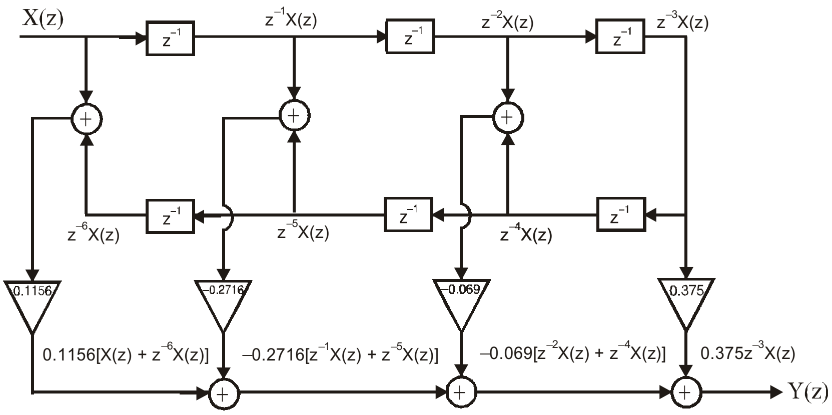


Fig 1: Linear phase structure of FIR bandpass filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0, the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$,

$$\text{where, } A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2\,h(n)\cos\omega n$$

Refer table 6.2 case (v)

$$= h(0) + \sum_{n=1}^{3} 2\,h(n)\cos\omega n$$

$$= h(0) + 2\,h(1)\cos\omega + 2\,h(2)\cos 2\omega + 2\,h(3)\cos 3\omega$$

$$= 0.375 + 2\times -0.069\cos\omega + 2\times -0.2716\cos 2\omega + 2\times 0.1156\cos 3\omega$$

$$= 0.375 - 0.138\cos\omega - 0.5432\cos 2\omega + 0.2312\cos 3\omega$$

Using the above equation, the amplitude response $A(\omega)$ and magnitude function $|H(e^{j\omega})|$ are calculated for various values of ω and listed in table 1. Using the tabulated values, the magnitude response is sketched as shown in fig 2.

Table 1 : A(w) and |H(e^{jw})| for various values of w.

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{0\times\pi}{16}$	–0.075	0.075
$\frac{1\times\pi}{16}$	–0.0699	0.0699
$\frac{2\times\pi}{16}$	–0.0481	0.0481
$\frac{3\times\pi}{16}$	0.0081	0.0081
$\frac{4\times\pi}{16}$	0.1139	0.1138
$\frac{5\times\pi}{16}$	0.2794	0.2793
$\frac{6\times\pi}{16}$	0.4926	0.4925
$\frac{7\times\pi}{16}$	0.7215	0.7215
$\frac{8\times\pi}{16}$	0.9182	0.9182

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{9\times\pi}{16}$	1.0322	1.0321
$\frac{10\times\pi}{16}$	1.0255	1.0254
$\frac{11\times\pi}{16}$	0.8862	0.8862
$\frac{12\times\pi}{16}$	0.6360	0.6359
$\frac{13\times\pi}{16}$	0.3269	0.3269
$\frac{14\times\pi}{16}$	0.0299	0.0298
$\frac{15\times\pi}{16}$	–0.1837	0.1836
$\frac{16\times\pi}{16}$	–0.2614	0.2614

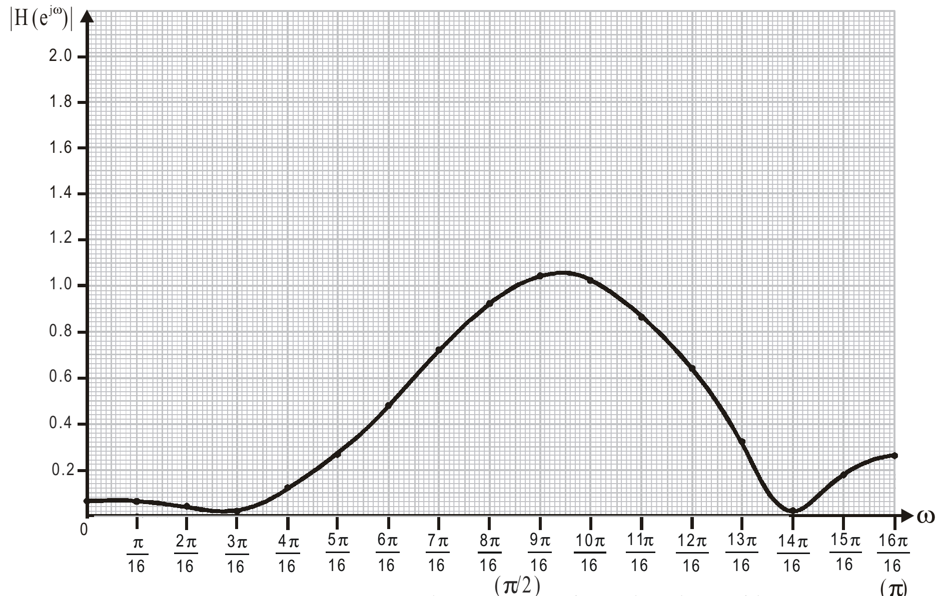


Fig 2 : Magnitude response of FIR bandpass filter.

Alternate Method for Frequency Response

Frequency response, $H\left(e^{j\omega}\right) = H(z)\Big|_{z\,=\,e^{j\omega}}$

$$\therefore\ H\left(e^{j\omega}\right) = 0.375z^{-3} - 0.069\left[z^{-2}+z^{-4}\right] - 0.2716\left[z^{-1}+z^{-5}\right] + 0.1156\left[1+z^{-6}\right]\Big|_{z\,=\,e^{j\omega}}$$

$$= 0.375e^{-j3\omega} - 0.069\left[e^{-j2\omega}+e^{-j4\omega}\right] - 0.2716\left[e^{-j\omega}+e^{-j5\omega}\right] + 0.1156\left[1+e^{-j6\omega}\right]$$

$$= 0.375\left[\cos 3\omega - j\sin 3\omega\right] - 0.069\left[\cos 2\omega - j\sin 2\omega + \cos 4\omega - j\sin 4\omega\right]$$

$$- 0.2716\left[\cos\omega - j\sin\omega + \cos 5\omega - j\sin 5\omega\right] + 0.1156\left[1+\cos 6\omega - j\sin 6\omega\right]$$

$$= [0.375\cos 3\omega - 0.069\cos 2\omega - 0.069\cos 4\omega - 0.2716\cos\omega - 0.2716\cos 5\omega + 0.1156 + 0.1156\cos 6\omega]$$

$$+\ j[-0.375\sin 3\omega + 0.069\sin 2\omega + 0.069\sin 4\omega + 0.2716\sin\omega + 0.2716\sin 5\omega - 0.1156\sin 6\omega]$$

Using the above equation, the frequency response $H(e^{jw})$ and magnitude function $|H(e^{jw})|$ of bandpass filter are calculated for various values of w and listed in table 2. It is observed that the magnitude response obtained by both the methods are same.

Table 2 : $H(e^{jw})$ and $|H(e^{jw})$ for various values of w

w	$H(e^{jw})$	$\|H(e^{jw})\|$
$\frac{0\times\pi}{16}$	–0.075 + j0	0.075
$\frac{1\times\pi}{16}$	–0.0581 + j0.0389	0.0699
$\frac{2\times\pi}{16}$	–0.0184 + j0.0445	0.0481
$\frac{3\times\pi}{16}$	–0.0041 – j0.0071	0.0081
$\frac{4\times\pi}{16}$	–0.0806 – j0.0805	0.1138
$\frac{5\times\pi}{16}$	–0.2740 – j0.0545	0.2793
$\frac{6\times\pi}{16}$	–0.4551 + j0.1885	0.4925
$\frac{7\times\pi}{16}$	–0.4008 + j0.5999	0.7215
$\frac{8\times\pi}{16}$	0 + j0.9182	0.9182

w	$H(e^{jw})$	$\|H(e^{jw})\|$
$\frac{9\times\pi}{16}$	0.5734 + j0.8582	1.0321
$\frac{10\times\pi}{16}$	0.9474 + j0.3924	1.0254
$\frac{11\times\pi}{16}$	0.8692 – j0.1729	0.8862
$\frac{12\times\pi}{16}$	0.4497 – j0.4497	0.6359
$\frac{13\times\pi}{16}$	0.0637 – j0.3207	0.3269
$\frac{14\times\pi}{16}$	–0.0114 – j0.0276	0.0298
$\frac{15\times\pi}{16}$	0.1527 + j0.1020	0.1836
$\frac{16\times\pi}{16}$	0.2614 + j0	0.2614

Example 6.4

Design a FIR bandstop filter to reject frequencies in the range 1.5 kHz to 3 kHz and sampling frequency of 8kHz with 7 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.

Solution

Given that, $F_{c1} = 1.5$ kHz ; $F_{c2} = 3$ kHz ; $F_s = 8$ kHz

$$\therefore \quad \omega_{c1} = \Omega_{c1}T = \frac{\Omega_{c1}}{F_s} = \frac{2\pi F_{c1}}{F_s} = \frac{2\pi \times 1.5 \times 10^3}{8\times 10^3} = 0.375\,\pi$$

$$\omega_{c2} = \Omega_{c2}T = \frac{\Omega_{c2}}{F_s} = \frac{2\pi F_{c2}}{F_s} = \frac{2\pi \times 3 \times 10^3}{8\times 10^3} = 0.75\,\pi$$

The desired frequency response $H_d(e^{jw})$ of bandstop filter is,

$$H_d(e^{j\omega}) = 1 \;;\; -\pi \le \omega \le -\omega_{c2} \;\&\; -\omega_{c1} \le \omega \le \omega_{c1} \;\&\; +\omega_{c2} \le \omega \le \pi$$
$$= 0 \;;\; \text{otherwise}$$

The desired impulse response $h_d(n)$ of the bandstop filter is,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} 1\times e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} 1\times e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{+\omega_{c2}}^{\pi} 1\times e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c1}}^{\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c2}}^{\pi}$$

$$\therefore\ h_d(n) = \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c2}n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c1}n}}{jn} - \frac{e^{-j\omega_{c1}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_{c2}n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} + \frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j} - \frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{\sin\pi n + \sin\omega_{c1}n - \sin\omega_{c2}n}{\pi n} \quad ; \text{ for all n, except } n = 0$$

When $n = 0$, the $h_d(n)$ become 0/0, which is indeterminate.

When $n = 0$; $h_d(n) = h_d(0) = \underset{n\to 0}{\text{Lt}}\left[\frac{\sin\pi n}{\pi n} + \frac{\sin\omega_{c1}n}{\pi n} - \frac{\sin\omega_{c2}n}{\pi n}\right]$

$$= \frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\frac{\sin\pi n}{n} + \frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\frac{\sin\omega_{c1}n}{n} - \frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\frac{\sin\omega_{c2}n}{n}$$

$$\boxed{\text{Using L' Hospital rule, } \underset{\theta\to 0}{\text{Lt}}\frac{\sin A\theta}{\theta} = A}$$

$$= \frac{1}{\pi}\times\pi + \frac{1}{\pi}\times\omega_{c1} - \frac{1}{\pi}\times\omega_{c2}$$

$$= 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)$$

The impulse response h(n) of FIR filter is obtained by truncating $h_d(n)$ to 7 samples.

$$\therefore\ h(n) = h_d(n) = \frac{\sin\omega_{c1}n - \sin\omega_{c2}n}{\pi n} \quad ; \text{ for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}, \text{ except } n = 0$$

$$= 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right) \quad ; \text{ for } n = 0$$

Here, $N = 7$, $\therefore\ \frac{N-1}{2} = \frac{7-1}{2} = 3$

Hence, calculate h(n) for n = –3 to +3.

Since, the impulse response h(n) satisfies the symmetry condition, h(n) = h(–n), calculate h(n) for n = 0 to 3.

When $n = 0$; $h(0) = 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right) = 1 - \left(\frac{0.75\pi - 0.375\pi}{\pi}\right) = 0.625$

When $n = 1$; $h(1) = \frac{\sin(0.375\pi\times 1) - \sin(0.75\pi\times 1)}{\pi\times 1} = 0.069$

When $n = 2$; $h(2) = \frac{\sin(0.375\pi\times 2) - \sin(0.75\pi\times 2)}{\pi\times 2} = 0.2716$

When $n = 3$; $h(3) = \frac{\sin(0.375\pi\times 3) - \sin(0.75\pi\times 3)}{\pi\times 3} = -0.1156$

When $n = -1$; $h(-1) = h(1) = 0.069$

When $n = -2$; $h(-2) = h(2) = 0.2716$

When $n = -3$; $h(-3) = h(3) = -0.1156$

The transfer function H(z) of the digital bandstop filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\,\mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\,z^{-n} = z^{-3} + \sum_{n=-3}^{3} h(n)\,z^{-n}$$

$$= z^{-3}\left[h(-3)\,z^{3} + h(-2)\,z^{2} + h(-1)\,z + h(0)\,z^{0} + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)z^{-3}\right]$$

$$\therefore\ H(z) = z^{-3}\left[h(3)\,z^{3} + h(2)\,z^{2} + h(1)\,z + h(0) + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)z^{-3}\right]$$

Using symmetry condition, $h(-n) = h(n)$

$$= z^{-3}\left[h(0) + h(1)\left[z + z^{-1}\right] + h(2)\left[z^{2} + z^{-2}\right] + h(3)\left[z^{3} + z^{-3}\right]\right]$$

$$= h(0)\,z^{-3} + h(1)\left[z^{-2} + z^{-4}\right] + h(2)\left[z^{-1} + z^{-5}\right] + h(3)\left[z^{0} + z^{-6}\right]$$

$$= 0.625\,z^{-3} + 0.069\left[z^{-2} + z^{-4}\right] + 0.2716\left[z^{-1} + z^{-5}\right] - 0.1156\left[1 + z^{-6}\right]$$

Structure

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)} = 0.625\,z^{-3} + 0.069\left[z^{-2} + z^{-4}\right] + 0.2716\left[z^{-1} + z^{-5}\right] - 0.1156\left[1 + z^{-6}\right]$$

$$\therefore\ Y(z) = 0.625\,z^{-3}\,X(z) + 0.069\left[z^{-2}\,X(z) + z^{-4}\,X(z)\right] + 0.2716\left[z^{-1}\,X(z) + z^{-5}\,X(z)\right] - 0.1156\left[X(z) + z^{-6}\,X(z)\right]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

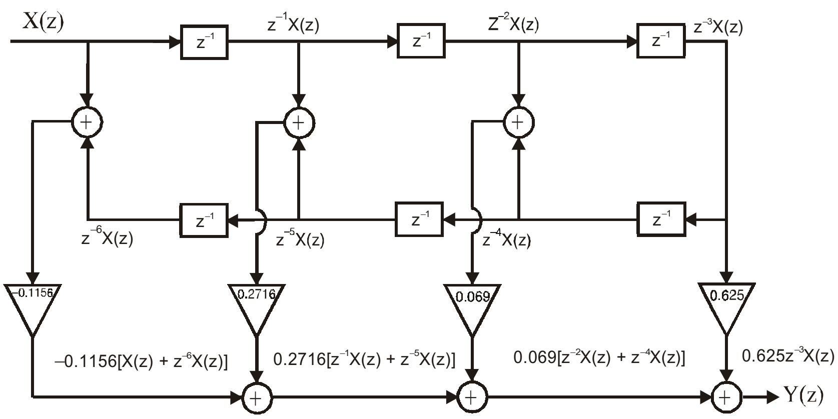


Fig 1 : Linear phase structure of FIR bandstop filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0, the magnitude function $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where,}\quad A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2\,h(n)\cos\omega n$$

Refer table 6.2 case (v)

$$A(\omega) = h(0) + \sum_{n=1}^{3} 2\,h(n)\cos\omega n$$

$$= h(0) + 2\,h(1)\cos\omega + 2\,h(2)\cos 2\omega + 2\,h(3)\cos 3\omega$$

$$= 0.625 + 2\times 0.069\cos\omega + 2\times 0.2716\,\cos 2\omega + 2\times -0.1156\,\cos 3\omega$$

$$= 0.625 + 0.138\,\cos\omega + 0.5432\,\cos 2\omega\ - 0.2312\,\cos 3\omega$$

Using the above equation, the amplitude response $A(w)$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the tabulated values, the magnitude response is sketched as shown in fig 2.

Table 1 : A(w) and $|H(e^{jw})|$ for various values of w.

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert = \lvert Aw \rvert$
$\frac{0\times\pi}{16}$	1.075	1.075
$\frac{1\times\pi}{16}$	1.0699	0.0699
$\frac{2\times\pi}{16}$	1.0481	0.0481
$\frac{3\times\pi}{16}$	0.9927	0.9926
$\frac{4\times\pi}{16}$	0.8860	0.8841
$\frac{5\times\pi}{16}$	0.7205	0.7205
$\frac{6\times\pi}{16}$	0.5073	0.5073
$\frac{7\times\pi}{16}$	0.2785	0.2784
$\frac{8\times\pi}{16}$	0.0818	0.0818

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert = \lvert Aw \rvert$
$\frac{9\times\pi}{16}$	–0.0322	0.0321
$\frac{10\times\pi}{16}$	–0.0255	0.0255
$\frac{11\times\pi}{16}$	0.1137	0.1136
$\frac{12\times\pi}{16}$	0.3639	0.3638
$\frac{13\times\pi}{16}$	0.6730	0.6729
$\frac{14\times\pi}{16}$	0.9700	0.9700
$\frac{15\times\pi}{16}$	1.1837	1.1836
$\frac{16\times\pi}{16}$	1.2614	1.2614

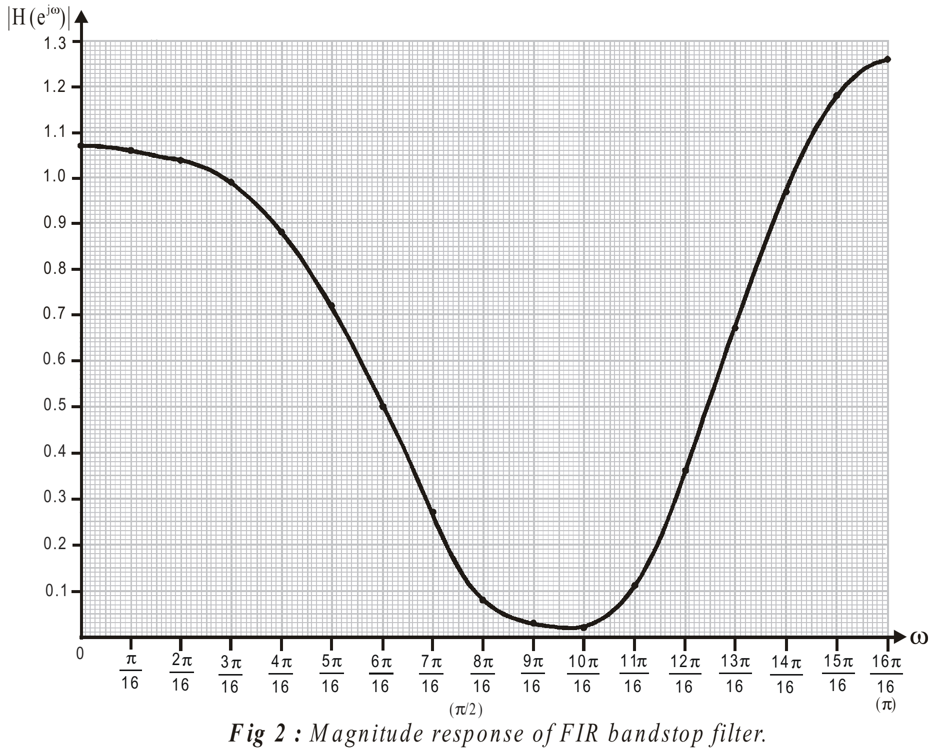


Fig 2 : Magnitude response of FIR bandstop filter.

Alternate Method for Frequency Response

Frequency response, $H(e^{j\omega}) = H(z)\Big|_{z\,=\,e^{j\omega}}$

$$\therefore\ H(e^{j\omega}) = 0.625z^{-3} + 0.069\left[z^{-2} + z^{-4}\right] + 0.2716\left[z^{-1} + z^{-5}\right] - 0.1156\left[1 + z^{-6}\right]\Big|_{z=e^{j\omega}}$$

$$= 0.625e^{-j3\omega} + 0.069\left[e^{-j2\omega} + e^{-j4\omega}\right] + 0.2716\left[e^{-j\omega} + e^{-j5\omega}\right] - 0.1156\left[1 + e^{-j6\omega}\right]$$

$$\therefore\ H\left(e^{j\omega}\right) = 0.625\left[\cos 3\omega - j\sin 3\omega\right] + 0.069\left[\cos 2\omega - j\sin 2\omega + \cos 4\omega - j\sin 4\omega\right]$$

$$+ 0.2716\left[\cos\omega - j\sin\omega + \cos 5\omega - j\sin 5\omega\right] - 0.1156\left[1 + \cos 6\omega - j\sin 6\omega\right]$$

$$= [0.625\cos 3\omega + 0.069\cos 2\omega + 0.069\cos 4\omega + 0.2716\cos\omega + 0.2716\cos 5\omega - 0.1156 - 0.1156\cos 6\omega]$$

$$+ j[-0.625\sin 3\omega - 0.069\sin 2\omega - 0.069\sin 4\omega - 0.2716\sin\omega - 0.2716\sin 5\omega + 0.1156\sin 6\omega]$$

Using the above equation, the frequency response $H(e^{jw})$ and magnitude function $|H(e^{jw})|$ of bandstop filter are calculated for various values of w and listed in table 2. It is observed that the magnitude response obtained by both the methods are same.

Table 2: $H(e^{jw})$ and $|H(e^{jw})|$ for variouse values of w

w	$H(e^{jw})$	$\|H(e^{jw})\|$
$\frac{0\times\pi}{16}$	1.075 + j0	1.075
$\frac{1\times\pi}{16}$	0.8896 – j0.5944	1.0699
$\frac{2\times\pi}{16}$	0.4010 – j0.9683	1.0480
$\frac{3\times\pi}{16}$	–0.1936 – j0.9736	0.9926
$\frac{4\times\pi}{16}$	–0.6265 – j0.6254	0.8841
$\frac{5\times\pi}{16}$	–0.7067 – j0.1405	0.7205
$\frac{6\times\pi}{16}$	–0.4687 + j0.1941	0.5073
$\frac{7\times\pi}{16}$	–0.1547 + j0.2315	0.2784
$\frac{8\times\pi}{16}$	0 + j0.0818	0.0818

w	$H(e^{jw})$	$\|H(e^{jw})\|$
$\frac{9\times\pi}{16}$	–0.0179 – j0.0267	0.0321
$\frac{10\times\pi}{16}$	–0.0235 – j0.0097	0.0255
$\frac{11\times\pi}{16}$	0.1115 – j0.0221	0.1136
$\frac{12\times\pi}{16}$	0.2573 – j0.2573	0.3638
$\frac{13\times\pi}{16}$	0.1313 – j0.6600	0.6729
$\frac{14\times\pi}{16}$	–0.3712 – j0.8962	0.9700
$\frac{15\times\pi}{16}$	–0.9842 – j0.6576	1.1836
$\frac{16\times\pi}{16}$	–1.2614 – j0	1.2614

6.8 Windows

The ***windows*** are finite duration sequences used to modify the impulse response of the FIR filters in order to reduce the ripples in the passband and stopband, and also to achieve the desired transition from passband to stopband.

The FIR filter design starts with desired frequency response, $H_d(e^{jw})$. The desired impulse response, $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{jw})$. The desired impulse response will be an infinite duration sequence. On multiplying finite duration window sequence with infinite duration impulse response, we get a finite duration impulse response with modified samples, which is used to design FIR filter.

The different types of window sequences discussed in this book are,

1. Rectangular window, $w_R(n)$
2. Bartlett or Triangular window, $w_T(n)$
3. Hanning window, $w_C(n)$
4. Hamming window, $w_H(n)$
5. Blackman window, $w_B(n)$
6. Kaiser window, $w_K(n)$

6.8.1 Rectangular Window

The N-point rectangular window, $w_R(n)$ is defined as,

$$\text{Rectangular window, } w_R(n) = 1 \quad ; \quad n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$$
$$= 0 \quad ; \quad \text{other } n \qquad .....(6.56)$$

Alternatively,

$$\text{Rectangular window, } w_R(n) = 1 \quad ; \quad n = 0 \text{ to } N-1$$
$$= 0 \quad ; \quad \text{other } n \qquad .....(6.57)$$

The rectangular window sequence defined by equation (6.56) can be used only for odd values of N, but the window sequence defined by equation (6.57) can be used for both odd and even values of N.

The frequency response or frequency spectrum of rectangular window $W_R(e^{jw})$ is obtained by taking Fourier transform of rectangular window sequence $w_R(n)$.

$$\therefore \; W_R(e^{j\omega}) = \mathcal{F}\{w_R(n)\} = \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} \qquad .....(6.58)$$

Proof :

$$W_R(e^{j\omega}) = \mathcal{F}\{w_R(n)\} = \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega\left(n-\frac{N-1}{2}\right)}$$

$$= \sum_{n=0}^{N-1} e^{-j\omega n}\, e^{j\omega\left(\frac{N-1}{2}\right)} = e^{j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N-1} e^{-j\omega n}$$

Using finite geometric series sum formula,

$$\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = e^{j\omega\left(\frac{N-1}{2}\right)} \frac{e^{\frac{-j\omega N}{2}} e^{\frac{j\omega N}{2}} - e^{\frac{-j\omega N}{2}} e^{\frac{-j\omega N}{2}}}{e^{\frac{-j\omega}{2}} e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}} e^{\frac{-j\omega}{2}}}$$

$$= e^{\frac{j\omega N}{2}} e^{\frac{-j\omega}{2}} \frac{e^{\frac{-j\omega N}{2}}\left(e^{\frac{j\omega N}{2}} - e^{\frac{-j\omega N}{2}}\right)}{e^{\frac{-j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)}$$

$$e^{j\theta} e^{-j\theta} = 1$$

$$= \frac{e^{\frac{j\omega N}{2}} - e^{\frac{-j\omega N}{2}}}{e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}} = \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

The magnitude and log-magnitude response of rectangular window for N = 31 are shown in fig 6.22(c) and (d). The spectrum of $W_R(e^{jw})$ has two features that are important, they are the width of the main-lobe and side-lobe amplitude. The ***main-lobe width*** is defined as the distance between the two points closest to w = 0 where $|W_R(e^{jw})|$ in dB is zero. For the rectangular window the main-lobe width is equal to 4p/N. The maximum side-lobe magnitude for $W_R(e^{jw})$ occurs for the first side-lobe and is equal to approximately -13 dB.

The magnitude response $|H(e^{jw})|$ and log-magnitude response of the lowpass filter designed using rectangular window are shown in fig 6.22(e) and (f). The approximated filter response differs from the ideal desired response in several ways. The sharp transition in the ideal response at $w = w_c$ has been converted into a gradual transition. In the passband a series of overshoots and undershoots occur. In the stopband the ideal desired response is zero, but the FIR filter has a nonzero response called ***leakage***. These features can be explained in terms of the features of the window spectrum.

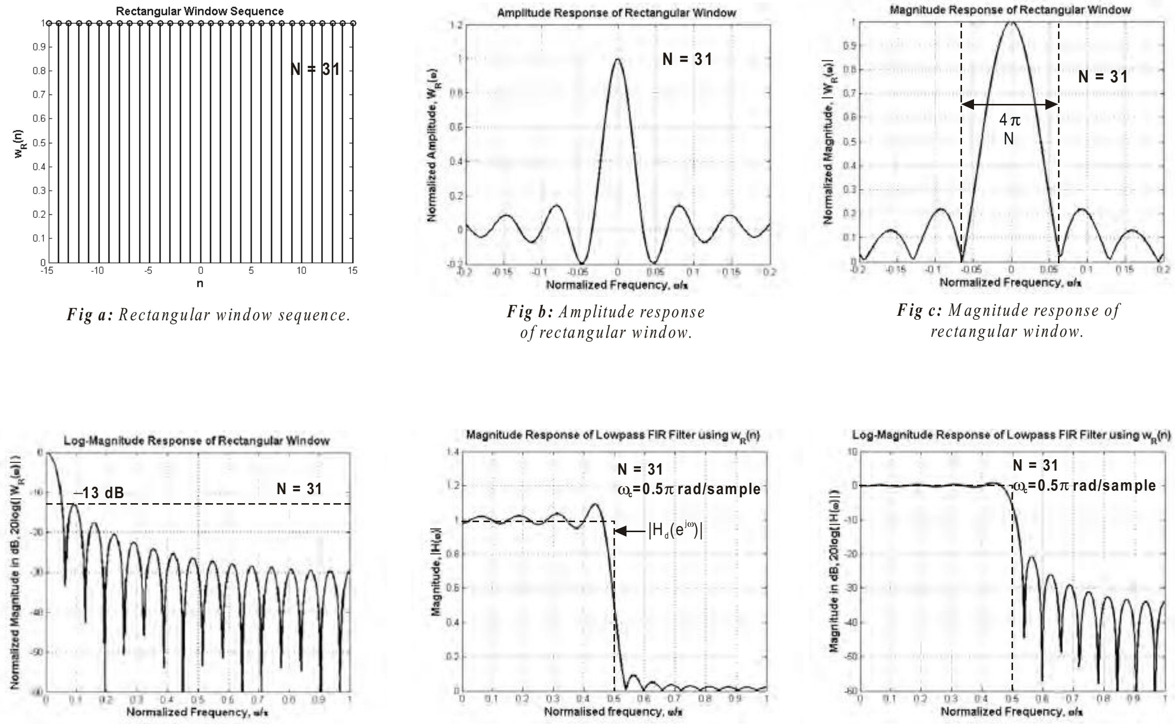


Fig a: *Rectangular window sequence.*

Fig b: *Amplitude response of rectangular window.*

Fig c: *Magnitude response of rectangular window.*

Fig d: *Log-magnitude response of rectangular window.*

Fig e: *Magnitude response of FIR lowpass filter designed using rectangular window.*

Fig f: *Log-magnitude response of FIR lowpass filter designed using rectangular window.*

Fig 6.22: *Rectangular window sequence and its frequency response (when N = 31).*

The main-lobe of $W_R(e^{jw})$ causes the smearing of the desired transfer function features. The discontinuity in $H_d(e^{jw})$ is converted into a gradual transition in $H(e^{jw})$. The width of the transition region is related to the width of the main-lobe of $W_R(e^{jw})$. Since the main-lobe width of $W_R(e^{jw})$ is equal to 4p/N, the size of this transition region can be reduced to any desired size by increasing the size (N), of the window sequence. The increase in N also increases the number of computations necessary to implement the FIR filter.

Since the side-lobes of $W_R(e^{jw})$ extend over a wide frequency range, large magnitude components in $H_d(e^{jw})$ becomes smeared over a wide range of frequencies in $H(e^{jw})$. In the passband this side-lobe effects appear both as overshoots and undershoots to the desired response. In the stopband, these effects appear as a nonzero response. These side-lobe effects do not diminish significantly, but remain almost constant as the duration of rectangular window is increased.

It is observed that whatever be the number of elements of $h_d(n)$ included in the h(n), the magnitudes of the overshoot and leakage will not change significantly, when the rectangular window is used. This result is known as the ***Gibbs phenomenon***, after the American Mathematician Josiak Willard Gibbs of Yale, who first noted this effect.

To reduce these side-lobe effects, we must consider alternate window sequences having spectrum exhibiting smaller side-lobes. We can observe that the side-lobes of the window spectrum $W(e^{jw})$ represent the contribution of the high frequency components in the window sequence. For the rectangular window, these high frequency components are due to the sharp transitions from 0 to 1 at the edges of window sequence. Hence the amplitudes of these high frequency components, (i.e., the side-lobe level) can be reduced by replacing these sharp transitions by more gradual ones. This is the motivation for development of the triangular window, cosine window, etc.

6.8.2 Bartlett or Triangular Window

The triangular window have been chosen such that it has tapered sequences form the middle on either sides. The N-point triangular window, $w_T(n)$ is defined as,

$$\begin{aligned} \text{Triangular window, } w_T(n) &= 1 - \frac{2\,|n|}{N-1} \quad ; \text{ for } -\frac{N-1}{2} \le n \le \frac{N-1}{2} \\ &= 0 \qquad\qquad ; \text{ other } n \end{aligned} \qquad .....(6.59)$$

Alternatively,

$$\begin{aligned} \text{Triangular window, } w_T(n) &= 1 - \frac{2\left|n - \frac{N-1}{2}\right|}{N-1} \quad ; \text{ for } n = 0 \text{ to } N-1 \\ &= 0 \qquad\qquad ; \text{ other } n \end{aligned} \qquad .....(6.60)$$

The triangular window sequence defined by equation (6.59) can be used only for odd values of N, but the window sequence defined by equation (6.60) can be used for both odd and even values of N.

The frequency response or frequency spectrum of triangular window $W_T(e^{jw})$ is obtained by taking Fourier transform of triangular window sequence $w_T(n)$.

$$\therefore \; W_T(e^{j\omega}) = \mathcal{F}\{w_T(n)\} = \left(\frac{\sin\left(\omega\left(\frac{N-1}{4}\right)\right)}{\sin\frac{\omega}{2}} \right)^2 \qquad .....(6.61)$$

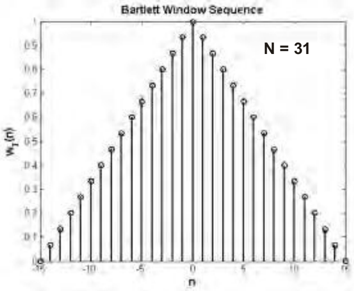


Fig a: Bartlett window sequence.

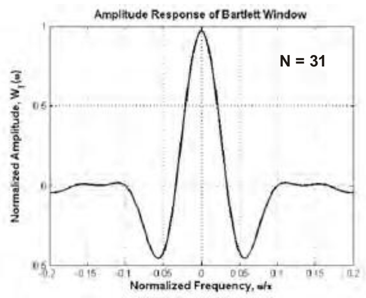


Fig b: Amplitude response of Bartlett window.

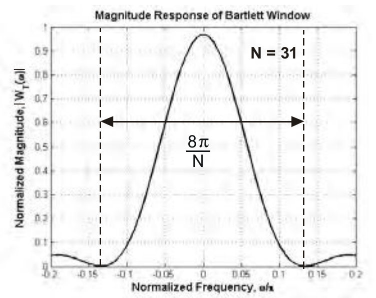


Fig c: Magnitude response of Bartlett window.

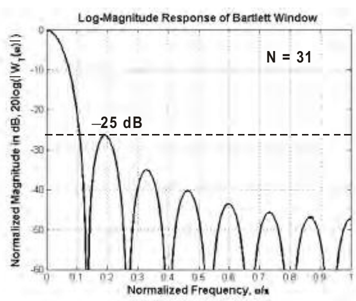


Fig d: Log-magnitude response of Bartlett window.

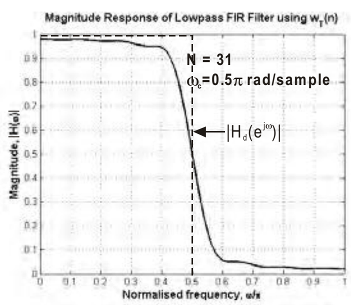


Fig e: Magnitude response of FIR lowpass filter designed using Bartlett window.

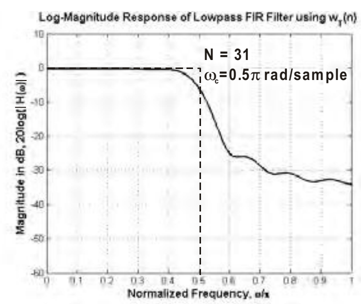


Fig f: Log-magnitude response of FIR lowpass filter designed using Bartlett window.

Fig 6.23 : *Bartlett window sequence and its frequency response (when N = 31).*

The magnitude and log-magnitude response of triangular window for N = 31 are shown in fig 6.23(c) and (d). In log-magnitude response of triangular window the first side-lobe level is smaller than that of the rectangular window, being reduced from –13 to – 25 dB. But the mainlobe width is 8p/N or twice that of the rectangular window having the same duration. This result illustrates that there is a trade off between main-lobe width and sidelobe level.

The magnitude response $|H(e^{jw})|$ and log-magnitude response of the lowpass filter designed using triangular window are shown in fig 6.23(e) and (f). The triangular window produces a smoother magnitude response for FIR filter. The transition from passband to stopband is not as steep as that for FIR filters designed using the rectangular window. In the stopband, the response is smoother, but the attenuation is less than that produced by the rectangular window. Because of these characteristics the triangular window is not usually a good choice.

6.8.3 Raised Cosine Windows

The raised cosine windows are smoother at the ends, but closer to one at the middle. The smoother ends and the broader middle section produces less distortion of $h_d(n)$ around n = 0. It is also called ***generalized Hamming window***.

The N-point raised cosine window $w_{RC}(n)$ is defined as,

$$\begin{aligned} w_{RC}(n) &= a + (1-a)\cos\left(\frac{2\pi n}{N-1}\right) \; ; \; \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2} \\ &= 0 \; ; \; \text{other } n \end{aligned} \qquad .....(6.62)$$

Alternatively,

$$\begin{aligned} w_{RC}(n) &= a - (1-a)\cos\left(\frac{2\pi n}{N-1}\right) \; ; \; \text{for } n = 0 \text{ to } N-1 \\ &= 0 \; ; \; \text{other } n \end{aligned} \qquad .....(6.63)$$

The raised cosine window sequence defined by equation (6.62) can be used only for odd values of N, but the window sequence defined by equation (6.63) can be used for both odd and even values of N.

The frequency response or frequency spectrum of raised cosine window $W_{RC}(e^{jw})$ is obtained by taking Fourier tansform of raised cosine window sequence $w_{RC}(n)$.

$$\therefore\; W_{RC}(e^{j\omega}) = \mathcal{F}\{w_{RC}(n)\} = a\,\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + \frac{1-a}{2}\,\frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} + \frac{1-a}{2}\,\frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)} \qquad .....(6.64)$$

Proof :

$$W_H(e^{j\omega}) = \mathcal{F}\{w_{RC}(n)\} = \sum_{n=-\infty}^{+\infty} w_{RC}(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} \left[a + (1-a)\cos\left(\frac{2\pi n}{N-1}\right)\right] e^{-j\omega n} \qquad \boxed{\text{Using equation (6.62)}}$$

$$= \sum_{n=0}^{N-1} \left[a + (1-a)\cos\left(\frac{2\pi\left(n - \frac{N-1}{2}\right)}{N-1}\right)\right] e^{-j\omega\left(n - \frac{N-1}{2}\right)}$$

$$\therefore\ W_H(e^{j\omega}) = \sum_{n=0}^{N-1}\left[a + (1-a)\cos\left(\frac{2\pi n}{N-1} - \pi\right)\right]e^{-j\omega n}\, e^{j\omega\left(\frac{N-1}{2}\right)}$$

$\boxed{\cos(\theta-\pi) = -\cos\theta}$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\sum_{n=0}^{N-1}\left[a-(1-a)\cos\left(\frac{2\pi n}{N-1}\right)\right]e^{-j\omega n}$$

$$= a e^{j\omega\left(\frac{N-1}{2}\right)}\sum_{n=0}^{N-1}e^{-j\omega n} - (1-a)\, e^{j\omega\left(\frac{N-1}{2}\right)}\sum_{n=0}^{N-1}\cos\left(\frac{2\pi n}{N-1}\right)e^{-j\omega n}$$

$\boxed{\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}}$

$$= a e^{j\omega\left(\frac{N-1}{2}\right)}\sum_{n=0}^{N-1}e^{-j\omega n} - (1-a)\, e^{j\omega\left(\frac{N-1}{2}\right)}\sum_{n=0}^{N-1}\left[\frac{e^{\frac{j2\pi n}{N-1}} + e^{\frac{-j2\pi n}{N-1}}}{2}\right]e^{-j\omega n}$$

$$= a e^{j\omega\left(\frac{N-1}{2}\right)}\sum_{n=0}^{N-1}e^{-j\omega n}$$

$$- \frac{1-a}{2}\, e^{j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{N-1}e^{\frac{j2\pi n}{N-1}}\, e^{-j\omega n} + \sum_{n=0}^{N-1}e^{\frac{-j2\pi n}{N-1}}\, e^{-j\omega n}\right]$$

$$= a e^{j\omega\left(\frac{N-1}{2}\right)}\sum_{n=0}^{N-1}(e^{-j\omega})^n$$

$$- \frac{1-a}{2}\, e^{j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{N-1}\left(e^{-j\left(\omega - \frac{2\pi}{N-1}\right)}\right)^n + \sum_{n=0}^{N-1}\left(e^{-j\left(\omega + \frac{2\pi}{N-1}\right)}\right)^n\right]$$

$$= a e^{j\omega\left(\frac{N-1}{2}\right)}\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$- \frac{1-a}{2}\, e^{j\omega\left(\frac{N-1}{2}\right)}\left[\frac{1 - e^{-j\left(\omega - \frac{2\pi}{N-1}\right)N}}{1 - e^{-j\left(\omega - \frac{2\pi}{N-1}\right)}} + \frac{1 - e^{-j\left(\omega + \frac{2\pi}{N-1}\right)N}}{1 - e^{-j\left(\omega + \frac{2\pi}{N-1}\right)}}\right]$$

Using finite geometric series sum formula, $\displaystyle\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}$

$$= a e^{j\omega\left(\frac{N-1}{2}\right)}\frac{e^{\frac{-j\omega N}{2}}\left(e^{\frac{j\omega N}{2}} - e^{\frac{-j\omega N}{2}}\right)}{e^{\frac{-j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)}$$

$\boxed{e^{j\theta}\, e^{-j\theta} = 1}$

$$- \frac{1-a}{2}\, e^{j\omega\left(\frac{N-1}{2}\right)}\frac{e^{-j\left(\omega - \frac{2\pi}{N-1}\right)\frac{N}{2}}\left(e^{j\left(\omega - \frac{2\pi}{N-1}\right)\frac{N}{2}} - e^{-j\left(\omega - \frac{2\pi}{N-1}\right)\frac{N}{2}}\right)}{e^{-j\left(\omega - \frac{2\pi}{N-1}\right)\frac{1}{2}}\left(e^{j\left(\omega - \frac{2\pi}{N-1}\right)\frac{1}{2}} - e^{-j\left(\omega - \frac{2\pi}{N-1}\right)\frac{1}{2}}\right)}$$

$$- \frac{1-a}{2}\, e^{j\omega\left(\frac{N-1}{2}\right)}\frac{e^{-j\left(\omega + \frac{2\pi}{N-1}\right)\frac{N}{2}}\left(e^{j\left(\omega + \frac{2\pi}{N-1}\right)\frac{N}{2}} - e^{-j\left(\omega + \frac{2\pi}{N-1}\right)\frac{N}{2}}\right)}{e^{-j\left(\omega + \frac{2\pi}{N-1}\right)\frac{1}{2}}\left(e^{j\left(\omega + \frac{2\pi}{N-1}\right)\frac{1}{2}} - e^{-j\left(\omega + \frac{2\pi}{N-1}\right)\frac{1}{2}}\right)}$$

$$\therefore\ W_H\left(e^{j\omega}\right) = a e^{\left[j\omega\left(\frac{N-1}{2}\right) - \frac{j\omega N}{2} + \frac{j\omega}{2}\right]} \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$- \frac{1-a}{2} e^{\left[j\omega\left(\frac{N-1}{2}\right) - j\left(\omega - \frac{2\pi}{N-1}\right)\frac{N}{2} + j\left(\omega - \frac{2\pi}{N-1}\right)\frac{1}{2}\right]} \frac{\sin\left(\left(\omega - \frac{2\pi}{N-1}\right)\frac{N}{2}\right)}{\sin\left(\left(\omega - \frac{2\pi}{N-1}\right)\frac{1}{2}\right)}$$

$$- \frac{1-a}{2} e^{\left[j\omega\left(\frac{N-1}{2}\right) - j\left(\omega + \frac{2\pi}{N-1}\right)\frac{N}{2} + j\left(\omega + \frac{2\pi}{N-1}\right)\frac{1}{2}\right]} \frac{\sin\left(\left(\omega + \frac{2\pi}{N-1}\right)\frac{N}{2}\right)}{\sin\left(\left(\omega + \frac{2\pi}{N-1}\right)\frac{1}{2}\right)}$$

$$= a e^{\left[\frac{j\omega N}{2} - \frac{j\omega}{2} - \frac{j\omega N}{2} + \frac{j\omega}{2}\right]} \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

$$- \frac{1-a}{2} e^{\left[\frac{j\omega N}{2} - \frac{j\omega}{2} - \frac{j\omega N}{2} + \frac{j\pi N}{N-1} + \frac{j\omega}{2} - \frac{j\pi}{N-1}\right]} \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)}$$

$$- \frac{1-a}{2} e^{\left[\frac{j\omega N}{2} - \frac{j\omega}{2} - \frac{j\omega N}{2} - \frac{j\pi N}{N-1} + \frac{j\omega}{2} + \frac{j\pi}{N-1}\right]} \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

$$= a \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} - \frac{1-a}{2} e^{\frac{j\pi}{\cancel{N-1}}\cancel{(N-1)}} \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)}$$

$$\boxed{e^{\pm j\pi} = -1}$$

$$- \frac{1-a}{2} e^{\frac{-j\pi}{\cancel{N-1}}\cancel{(N-1)}} \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

$$= a \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + \frac{1-a}{2} \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} + \frac{1-a}{2} \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

6.8.4 Hanning Window

The Hanning window is one type of raised cosine window. The equation for Hanning window sequence $w_C(n)$ is obtained by putting a = 0.5 in equations (6.62) and (6.63).

$$\begin{aligned} \text{Hanning window, } w_C(n) &= 0.5 + 0.5\cos\frac{2\pi n}{N-1} \quad ; \text{ for } -\frac{N-1}{2} \text{ to } +\frac{N-1}{2} \\ &= 0 \quad ; \text{ other } n \end{aligned} \qquad .....(6.65)$$

Alternatively,

$$\begin{aligned} \text{Hanning window, } w_C(n) &= 0.5 - 0.5\cos\frac{2\pi n}{N-1} \quad ; \text{ for } n = 0 \text{ to } N-1 \\ &= 0 \quad ; \text{ other } n \end{aligned} \qquad .....(6.66)$$

The Hanning window sequence defined by equation (6.65) can be used only for odd values of N, but the window sequence defined by equation (6.66) can be used for both odd and even values of N.

The frequency response or frequency spectrum of Hanning window $W_C(e^{jw})$ is obtained by taking Fourier transform of Hanning window sequence $w_C(n)$, which can also be obtained from equation (6.64) by putting a = 0.5.

$$\therefore\ W_C(e^{j\omega}) = \mathcal{F}\{w_C(n)\} = 0.5\ \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + 0.25\ \frac{\sin\left(\frac{\omega N}{2}-\frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2}-\frac{\pi}{N-1}\right)}$$

$$+\ 0.25\ \frac{\sin\left(\frac{\omega N}{2}+\frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2}+\frac{\pi}{N-1}\right)} \qquad .....(6.67)$$

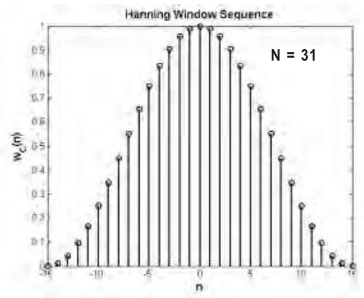


Fig a: Hanning window sequence.

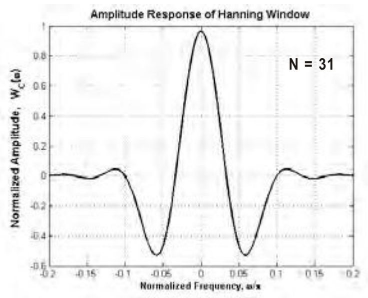


Fig b: Amplitude response of Hanning window.

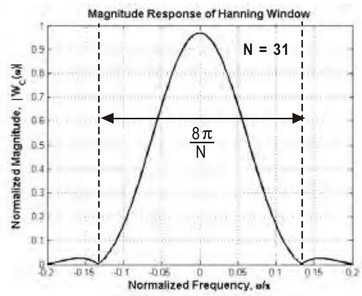


Fig c: Magnitude response of Hanning window.

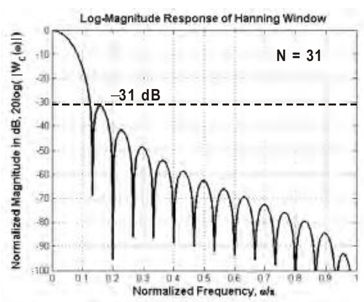


Fig d: Log-magnitude response of Hanning window,.

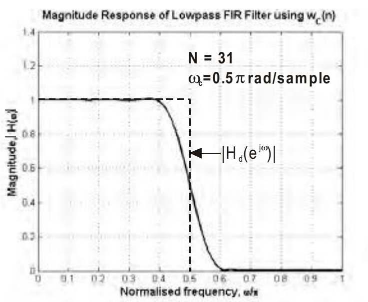


Fig e: Magnitude response of FIR lowpass filter designed using Hanning window.

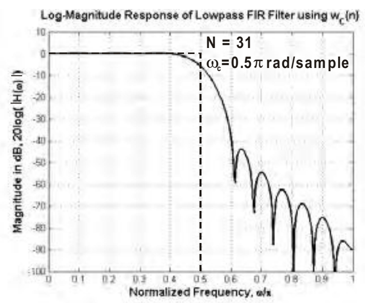


Fig f: Log-magnitude response of FIR lowpass filter designed using Hanning window.

Fig 6.24: *Hanning window sequence and its frequency response (when N = 31).*

The magnitude and log-magnitude response of Hanning window for N =31 are shown in fig 6.24(c) and (d). In the log-magnitude response of $W_C(e^{jw})$ the magnitude of the first side-lobe is –31 dB. An improvement of 6 dB over the triangular window. When compared to triangular window, the main-lobe width is same but the magnitude of side-lobe is reduced, hence the Hanning window is preferrable to triangular window.

The magnitude response $|H(e^{jw})|$ and log-magnitude response of the lowpass filter designed by using Hanning window are shown in fig 6.24(e) and (f). Most notable is the improved stopband attenuation characteristic. The largest peak is approximately 44 dB relative to the passband level. At higher frequencies the stopband attenuation is even greater.

6.8.5 Hamming Window

Hamming noted that a reduction in the first side-lobe level can be achieved by adding a small constant value to the raised cosine window. The equation for Hamming window sequence $w_H(n)$ is obtained by putting a = 0.54 in equations (6.62) and (6.63).

$$\therefore \text{ Hamming window, } w_H(n) = 0.54 + 0.46\cos\frac{2\pi n}{N-1} \; ; \; \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$$
$$= 0 \; ; \; \text{other } n \qquad .....(6.68)$$

Alternatively,

$$\text{Hamming window, } w_H(n) = 0.54 - 0.46\cos\frac{2\pi n}{N-1} \; ; \; \text{for } n = 0 \text{ to } N-1$$
$$= 0 \; ; \; \text{other } n \qquad .....(6.69)$$

The Hamming window sequence defined by equation (6.68) can be used only for odd values of N, but the window sequence defined by equation (6.69) can be used for both odd and even values of N.

The frequency response or frequency spectrum of Hamming window $W_H(e^{jw})$ is obtained by taking Fourier transform of Hamming window sequence $w_H(n)$, which can also be obtained from equation (6.64) by putting a = 0.54.

$$\therefore W_H(e^{j\omega}) = \mathcal{F}\{w_H(n)\} = 0.54\,\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + 0.23\,\frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} + 0.23\,\frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)} \qquad .....(6.70)$$

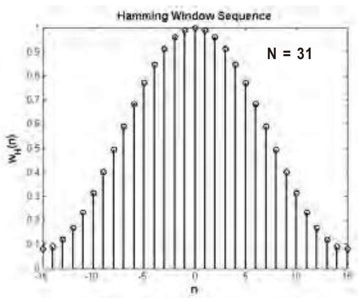


Fig a: *Hamming window sequence.*

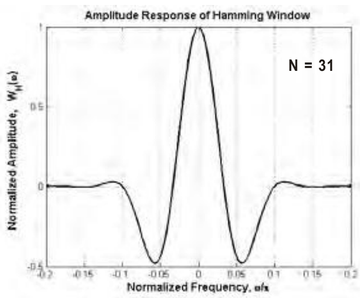


Fig b: *Amplitude response of Hamming window.*

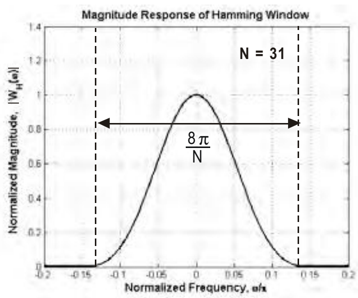


Fig c: *Magnitude response of Hamming window.*

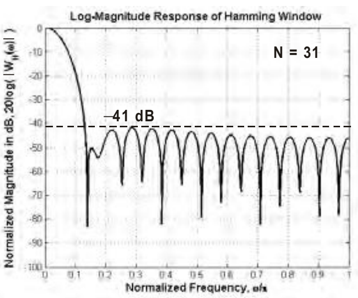


Fig d: *Log-magnitude response of Hamming window,.*

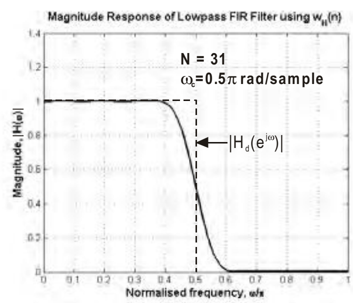


Fig e: *Magnitude response of FIR lowpass filter designed using Hamming window.*

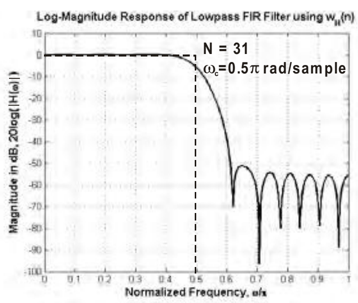


Fig f: *Log-magnitude response of FIR lowpass filter designed using Hamming window.*

Fig 6.25: *Hamming window sequence and its frequency response (when N = 31).*

The magnitude and log-magnitude response of hamming window for N = 31 are shown in fig 6.25(c) and (d). Hamming reduced the side-lobe magnitude while maintaining the main-lobe width, equal to 8p/N. The magnitude of the first side-lobe has been reduced to –41 dB, an improvement of 10 dB relative to the Hanning window. But this improvement is achieved at the expense of the side-lobe magnitudes at higher frequencies, which are almost constant with frequency. [With the Hanning window, the side-lobe amplitudes decrease with frequency].

The magnitude response $|H(e^{jw})|$ and log-magnitude response of lowpass filter designed using the Hamming window are shown in fig 6.25(e) and (f). It is noted that the first side-lobe peak is reduced to –51 dB, an improvement of 7 dB relative to the Hanning window filter. However, as the frequency increases, the stopband attenuation does not increase as much as with the filter produced by the Hanning window.

The stopband attenuation in the lowpass filter magnitude response is limited by the side-lobe level of the window function. Even though the Hamming window achieved an attenuation of 51 dB (or gain of –51 dB) in the stopband for our lowpass filter, it may not be sufficient for some applications.

6.8.6 Blackman Window

The Blackman window $w_B(n)$ is another type of cosine window defined by the equation,

$$\text{Blackman window, } w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{N-1} + 0.08\cos\dfrac{4\pi n}{N-1} & ;\ \text{for } -\dfrac{N-1}{2} \text{ to } +\dfrac{N-1}{2} \\ 0 & ;\ \text{other } n \end{cases} \quad .....(6.71)$$

Alternatively,

$$\text{Blackman window, } w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{N-1} + 0.08\cos\dfrac{4\pi n}{N-1} & ;\ \text{for } n = 0 \text{ to } N-1 \\ 0 & ;\ \text{other } n \end{cases} \quad .....(6.72)$$

The Blackman window sequence defined by equation (6.71) can be used only for odd values of N, but the window sequence defined by equation (6.72) can be used for both odd and even values of N.

The frequency response or frequency spectrum of Blackman window $W_B(e^{jw})$ is obtained by taking Fourier tranform of Blackman window sequence $w_B(n)$.

$$\therefore\ W_B(e^{j\omega}) = \mathcal{F}\{w_B(n)\} = 0.42\,\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + 0.25\,\frac{\sin\left(\frac{\omega N}{2}-\frac{N\pi}{N-1}\right)}{\sin\left(\frac{\omega}{2}-\frac{\pi}{N-1}\right)} + 0.25\,\frac{\sin\left(\frac{\omega N}{2}+\frac{N\pi}{N-1}\right)}{\sin\left(\frac{\omega}{2}+\frac{\pi}{N-1}\right)}$$
$$+\ 0.04\,\frac{\sin\left(\frac{\omega N}{2}-\frac{2N\pi}{N-1}\right)}{\sin\left(\frac{\omega}{2}-\frac{2\pi}{N-1}\right)} + 0.04\,\frac{\sin\left(\frac{\omega N}{2}+\frac{2N\pi}{N-1}\right)}{\sin\left(\frac{\omega}{2}+\frac{2\pi}{N-1}\right)} \quad .....(6.73)$$

The magnitude and log-magnitude response of Blackman window for N = 31 are shown in fig 6.26(c) and (d). In Blackman window the width of main-lobe is 12p/N, which is highest among windows. It can be observed that the magnitude of the first side-lobe is –58 dB and the side-lobe magnitude decreases with frequency. This desirable feature is achieved at the expense of increased main-lobe width. However, the main-lobe width can be reduced by increasing the value of N.

The magnitude response $|H(e^{jw})|$ and log-magnitude response of lowpass filter designed using blackman window are shown in fig6.26 (e) and (f). It is observed at the first side-lobe peak is –78 dB, an improvement of 27 dB relative to Hamming window filter. However, as the frequency increases, the stopband attenuation does not increase as much as with the filter produced by the Hanning window.

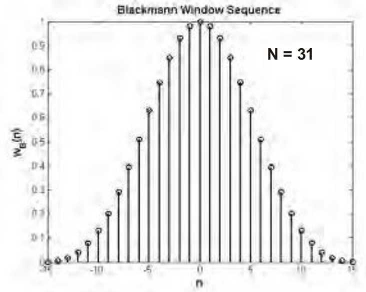


Fig a: Blackman window sequence.

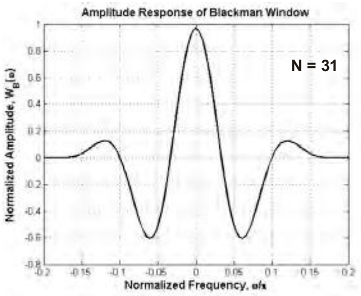


Fig b: Amplitude response of Blackman window.

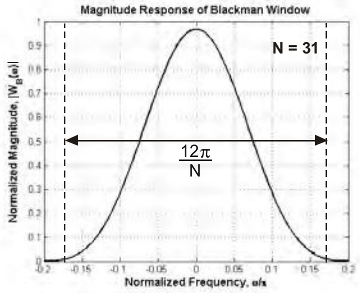


Fig c: Magnitude response of Blackman window.

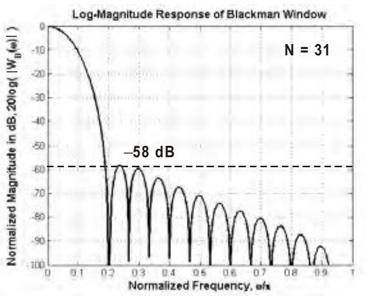


Fig d: Log-magnitude response of Blackman window,.

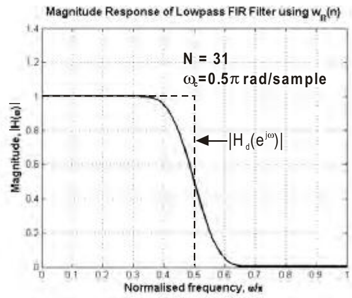


Fig e: Magnitude response of FIR lowpass filter designed using Blackman window.

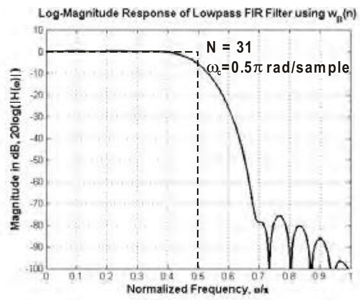


Fig f: Log-magnitude response of FIR lowpass filter designed using Blackman window.

***Fig 6.26:** Blackman window sequence and its frequency response (when N = 31).*

6.8.7 Kaiser Window

The design of window function is basically a mathematical problem of finding a time-limited function whose Fourier Transform best approximates a bandlimited function. The approximation should be such that the maximum energy is confined to mainlobe for a given peak side-lobe amplitude. The prolate spheroidal functions have this desirable property but these functions are difficult to compute. Kaiser has developed a simple approximation to these functions in terms of zero-order modified Bessel functions of the first kind, which is denoted by $I_0(x)$. The kaiser window function is in the form,

$$\text{Kaiser window function, } w_K(n) = \frac{I_0(\beta_1)}{I_0(a)} \quad ; \text{ for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$$
$$= 0 \quad ; \text{ other } n \qquad .....(6.74)$$

$$\text{where, } \beta_1 = a\left[1-\left(\frac{2n}{N-1}\right)^2\right]^{0.5}$$

Alternatively,

$$\text{Kaiser window function, } w_K(n) = \frac{I_0(\beta_2)}{I_0(a_2)} \quad ; \text{ for } n = 0 \text{ to } N-1$$
$$= 0 \quad ; \text{ other } n \qquad .....(6.75)$$

$$\text{where, } \beta_2 = a\left[\left(\frac{N-1}{2}\right)^2-\left(n-\frac{N-1}{2}\right)^2\right]^{0.5} \quad ; \quad a_2 = a\frac{N-1}{2}$$

The Kaiser window sequence defined by equation (6.74) can be used only for odd values of N, but the window sequenced defined by equation (6.75) can be used for both odd and even values of N.

The parameter "a" is an independent variable that can be varied to control the side-lobe levels with respect to the main-lobe peak. The modified Bessel function of the first kind $I_0(x)$ is given by,

$$I_0(x) = 1 + \sum_{k=1}^{\infty} \left[\frac{1}{k!} \left(\frac{x}{2} \right)^k \right]^2 = 1 + \sum_{k=1}^{\infty} \frac{\left[(0.5x)^2 \right]^k}{(k!)^2}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{\left(0.25x^2\right)^k}{(k!)^2} = 1 + \frac{0.25x^2}{(1!)^2} + \frac{\left(0.25x^2\right)^2}{(2!)^2} + \frac{\left(0.25x^2\right)^3}{(3!)^2} + ... \qquad .....(6.76)$$

The series of equation (6.76) can be used to compute $I_0(b_1)$, $I_0(a)$, $I_0(b_2)$, $I_0(a_2)$ and can be computed for any desired accuracy. Usually 25 terms of the series are sufficient for most practical purposes.

The frequency response or frequency spectrum of Kaiser window, $W_K(e^{jw})$ is obtained by taking Fourier transform of Kaiser window sequence $w_K(n)$.

$$\therefore\ W_K(e^{j\omega}) = \mathcal{F}\{w_K(n)\} = \frac{2}{I_0(a)} \; \frac{\sin\left(\frac{N-1}{2} \left(\omega^2 - \left(\frac{2a}{N-1} \right)^2 \right)^{0.5} \right)}{\left(\omega^2 - \left(\frac{2a}{N-1} \right)^2 \right)^{0.5}} \qquad .....(6.77)$$

Fig 6.27 to 6.29 shows the Kaiser window sequence and its frequency response for three different values of "a". With increase in value of "a" the magnitude of first side-lobe reduces, but the width of main-lobe increases. The width of the main-lobe can be reduced by increasing the length N of the window sequence. In the lowpass filter designed using Kaiser window the stopband attenuation increases with increase in the value of "a".

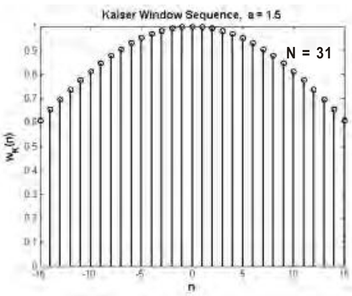


Fig a: *Kaiser window sequence.*

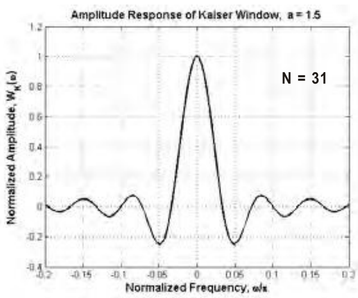


Fig b: *Amplitude response of Kaiser window.*

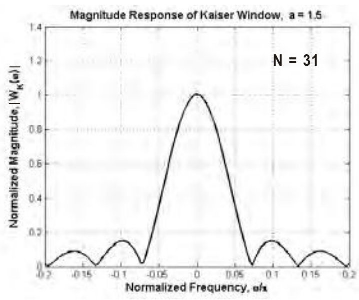


Fig c: *Magnitude response of Kaiser window.*

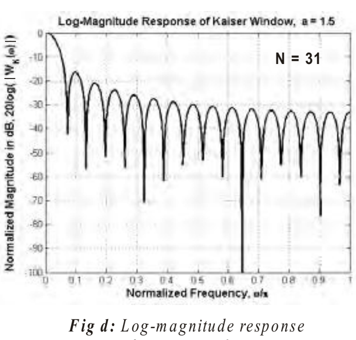


Fig d: *Log-magnitude response of Kaiser window.*

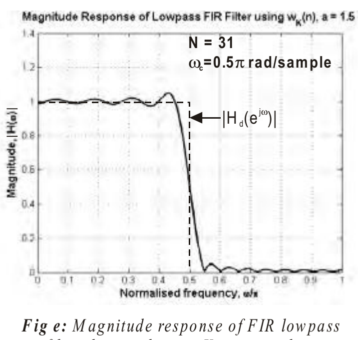


Fig e: *Magnitude response of FIR lowpass filter designed using Kaiser window.*

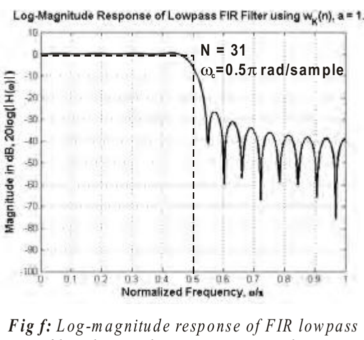


Fig f: *Log-magnitude response of FIR lowpass filter designed using Kaiser window.*

Fig 6.27: *Kaiser window sequence and its frequency response, for a = 1.5 and N = 31.*

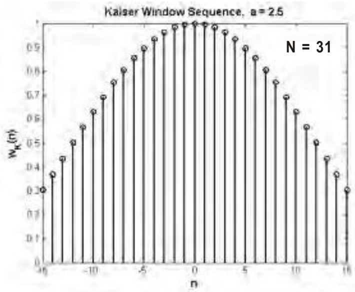


Fig a: *Kaiser window sequence.*

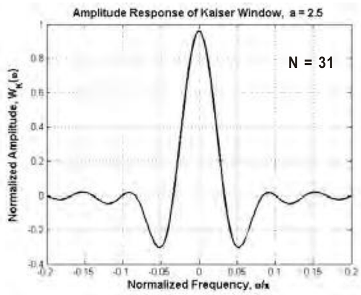


Fig b: *Amplitude response of Kaiser window.*

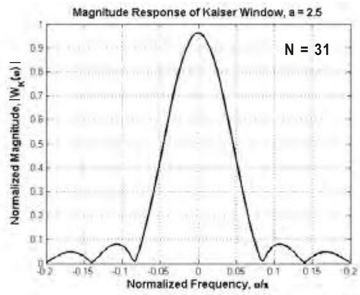


Fig c: *Magnitude response of Kaiser window.*

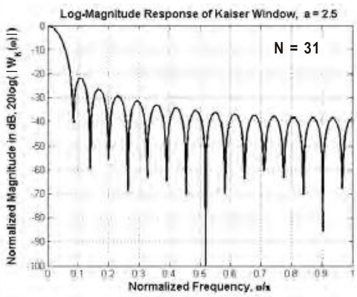


Fig d: *Log-magnitude response of Kaiser window.*

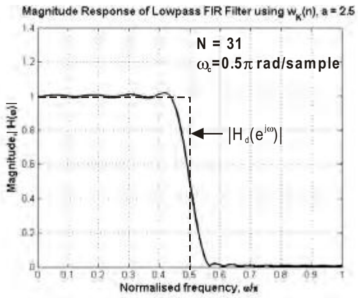


Fig e: *Magnitude response of FIR lowpass filter designed using Kaiser window.*

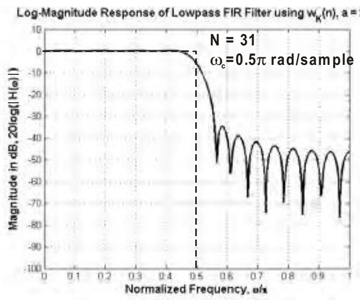


Fig f: *Log-magnitude response of FIR lowpass filter designed using Kaiser window.*

Fig 6.28: *Kaiser window sequence and its frequency response for a = 2.5 and N = 31..*

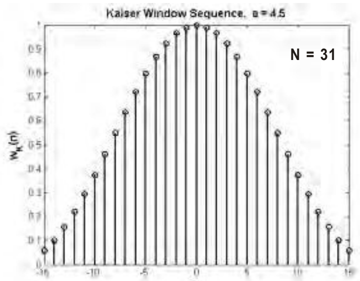


Fig a: *Kaiser window sequence.*

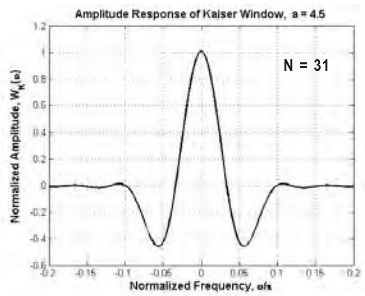


Fig b: *Amplitude response of Kaiser window.*

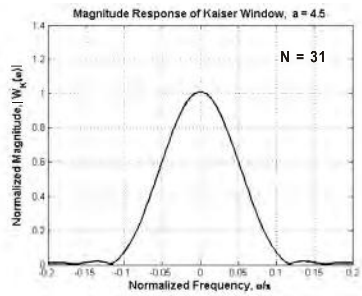


Fig c: *Magnitude response of Kaiser window.*

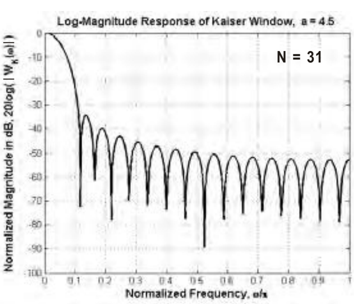


Fig d: *Log-magnitude response of Kaiser window,.*

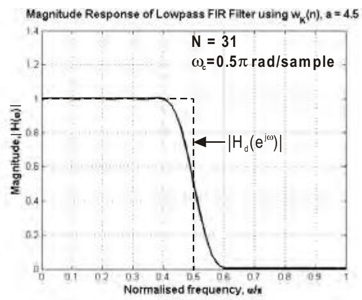


Fig e: *Magnitude response of FIR lowpass filter designed using Kaiser window.*

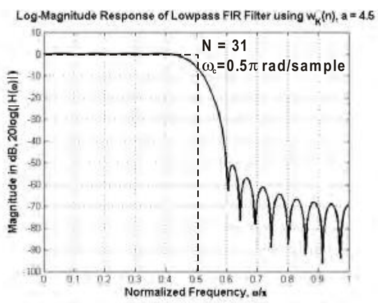


Fig f: *Log-magnitude response of FIR lowpass filter designed using Kaiser window.*

Fig 6.29: *Kaiser window sequence and its frequency response for a = 4.5 and N = 31.*

Table 6.4 : Window Sequences for FIR Filter Design

Name of window	Window sequence
Rectangular	$w_R(n) = 1 \quad ; \quad \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \quad ; \quad \text{other } n$
	$w_R(n) = 1 \quad ; \quad \text{for } n = 0 \text{ to } N-1$ $= 0 \quad ; \quad \text{other } n$
Triangular	$w_T(n) = 1 - \frac{2\lvert n\rvert}{N-1} \quad ; \quad \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \quad ; \quad \text{other } n$
	$w_T(n) = 1 - \frac{2\lvert n-(N-1)/2\rvert}{N-1} \quad ; \quad \text{for } n = 0 \text{ to } N-1$ $= 0 \quad ; \quad \text{other } n$
Hanning	$w_C(n) = 0.5 + 0.5 \cos \frac{2\pi n}{N-1} \quad ; \quad \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \quad ; \quad \text{other } n$
	$w_C(n) = 0.5 - 0.5 \cos \frac{2\pi n}{N-1} \quad ; \quad \text{for } n = 0 \text{ to } N-1$ $= 0 \quad ; \quad \text{other } n$
Hamming	$w_H(n) = 0.54 + 0.46 \cos \frac{2\pi n}{N-1} \quad ; \quad \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \quad ; \quad \text{other } n$
	$w_H(n) = 0.54 - 0.46 \cos \frac{2\pi n}{N-1} \quad ; \quad \text{for } n = 0 \text{ to } N-1$ $= 0 \quad ; \quad \text{other } n$
Blackman	$w_B(n) = 0.42 + 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1} \quad ; \quad \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \quad ; \quad \text{other } n$
	$w_B(n) = 0.42 - 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1} \quad ; \quad \text{for } n = 0 \text{ to } N-1$ $= 0 \quad ; \quad \text{other } n$
Kaiser	$w_k(n) = \frac{I_0(\beta_1)}{I_0(a)} \quad ; \quad \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \quad ; \quad \text{other } n$ where, $\beta_1 = a\left[1 - \left(\frac{2n}{N-1}\right)^2\right]^{0.5}$
	$w_K(n) = \frac{I_0(\beta_2)}{I_0(a_2)} \quad ; \quad \text{for } n = 0 \text{ to } N-1$ $= 0 \quad ; \quad \text{other } n$ where, $\beta_2 = a\left[\left(\frac{N-1}{2}\right)^2 - \left(n - \frac{N-1}{2}\right)^2\right]^{0.5}$ $a_2 = a\,\frac{N-1}{2}$

6.8.8 Summary of Various Features of Windows

The main advantage of windowing is that it is reasonably straightforward to obtain the filter impulse response with minimal computational effort. The major reasons for the relative success of windows is their simplicity and ease of use and the fact that closed form expressions are often available for the window coefficients. The main disadvantage of this technique is that the resulting FIR filters satisfy no known optimality criterion (such as specified attenuation at w_s and w_p) hence their performance have to be considerably improved in most cases.

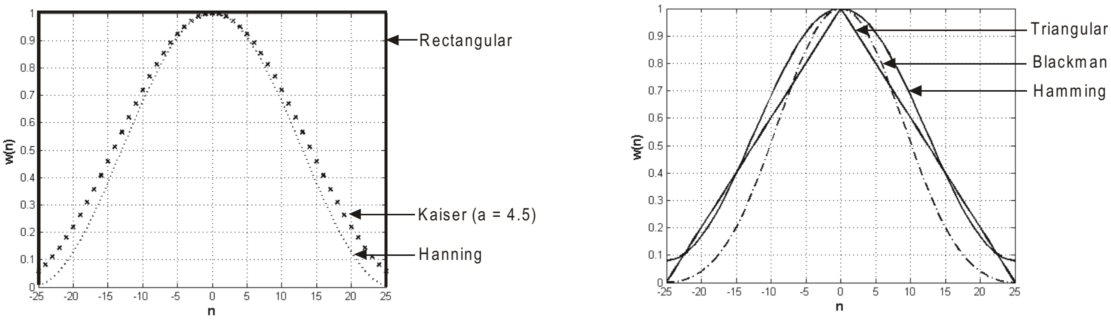


Fig 6.30 : Shapes of various window functions.

Table 6.5 : Frequency - Domain Characteristics of Some Window Functions

Type of window	Approximate width of main-lobe	Magnitude of first side-lobe
Rectangular	4p/N	–13 dB
Bartlett	8p/N	–25 dB
Hanning	8p/N	–31 dB
Hamming	8p/N	–41 dB
Blackman	12p/N	–58 dB

Note : In filter specifications gain and magnitude are same and will be in negative dB. The attenuation is inverse of gain and so it is negative of magnitude or gain in dB. Hence attenuation will be in positive dB.

6.9 FIR Filter Design Using Windows

Method - 1 : Symmetry condition h(N – 1 – n) = h(n)

1. The specifications of digital FIR filter are,

   i) The desired frequency response, $H_d(e^{jw}) = Ce^{-j\alpha w}$

   where, C = Constant (usually, C = 1 = Normalized magnitude)

   $$\alpha = \frac{N-1}{2}$$

   ii) The cutoff frequency w_c for lowpass and highpass, and w_{c1} and w_{c2} for bandpass and bandstop filters

   Note : If analog filter cutoff frequency F_c and sampling frequency F_s are specified, then calculate the cutoff frequency of digital filter w_c using the equation, $\omega_c = \frac{2\pi F_c}{F_s}$.

   iii) The number of samples of impulse response, N.

Table 6.6 : The Normalized Ideal (Desired) Frequency Response and Impulse Response for FIR Filter Design Using Windows

Type of filter	Ideal (desired) frequency response	Ideal (desired) impulse response
Lowpass filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \ -\omega_c \le \omega \le +\omega_c \\ 0 & ; \ -\pi \le \omega < -\omega_c \\ 0 & ; \ \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_c \text{ and } +\omega_c < \omega \le +\pi\right]$
Highpass filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \ -\pi \le \omega \le -\omega_c \\ e^{-j\omega\alpha} & ; \ \omega_c \le \omega \le \pi \\ 0 & ; \ -\omega_c < \omega < +\omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_c < \omega < +\omega_c\right]$
Bandpass filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \ -\omega_{c2} \le \omega \le -\omega_{c1} \\ e^{-j\omega\alpha} & ; \ \omega_{c1} \le \omega \le \omega_{c2} \\ 0 & ; \ -\pi \le \omega < -\omega_{c2} \\ 0 & ; \ -\omega_{c1} < \omega < +\omega_{c1} \\ 0 & ; \ \omega_{c2} < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_{c2}\ ;\ -\omega_{c1} < \omega < +\omega_{c1} \text{ and } +\omega_{c2} < \omega \le +\pi\right]$
Bandstop filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \ -\pi \le \omega \le -\omega_{c2} \\ e^{-j\omega\alpha} & ; \ -\omega_{c1} \le \omega \le +\omega_{c1} \\ e^{-j\omega\alpha} & ; \ \omega_{c2} \le \omega \le \pi \\ 0 & ; \ -\omega_{c2} < \omega < -\omega_{c1} \\ 0 & ; \ \omega_{c1} < \omega < \omega_{c2} \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega}) e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{+\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_{c2} < \omega < -\omega_{c1} \text{ and } +\omega_{c1} < \omega < +\omega_{c2}\right]$

2. Determine the desired impulse response, $h_d(n)$ by taking inverse Fourier transform of the desired frequency response, $H_d(e^{jw})$.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

(For limits of integration in the above equation refer table 6.6).

3. Choose the desired window sequence w(n) defined for n = 0 to N – 1 from table 6.4. Multiply $h_d(n)$ with w(n) to get the impulse response h(n) of the filter. calculate N-samples of the impulse response, for n = 0 to N – 1.

\ Impulse response, $h(n) = h_d(n)$ ´ $w(n)$; for $n = 0$ to $N - 1$

The impulse response is symmetric with centre of symmetry at (N – 1)/2 and so h(N – 1 – n) = h(n). Hence it is sufficient if we calculate h(n) for n = 0 to (N – 1)/2.

4. Take $\mathcal{Z}$-transform of the impulse response h(n) to get the transfer function H(z) of the filter.

$$\therefore \text{ Transfer function, } H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

5. Draw a suitable structure for realization of FIR filter.

Method - 2 : Symmetry condition h(–n) = h(n)

1. The specifications of digital FIR filter are,

i) The desired frequency response, $H_d(e^{jw}) = C$

where, C = Constant (usually, C = 1 = Normalized magnitude)

ii) The cutoff frequency w_c for lowpass and highpass, and w_{c1} and w_{c2} for bandpass and bandstop filters

> *Note : If analog filter cutoff frequency F_c and sampling frequency F_s are specified, then calculate the cutoff frequency of digital filter w_c using the equation,* $\omega_c = \dfrac{2\pi F_c}{F_s}$.

iii) The number of samples of impulse response, N.

2. Determine the desired impulse response, $h_d(n)$ by taking inverse Fourier transform of the desired frequency response, $H_d(e^{jw})$.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

(For limits of integration in the above equation refer table 6.3).

3. Choose the desired window sequence w(n) defined for $n = -\frac{N-1}{2}$ to $+\frac{N-1}{2}$ 1 from table 6.4. Multiply $h_d(n)$ with w(n) to get the impulse response h(n) of the filter. calculate N-samples of the impulse response, for $n = -\frac{N-1}{2}$ to $+\frac{N-1}{2}$.

\ Impulse response, $h(n) = h_d(n)$ ´ $w(n)$; for $n = -\frac{N-1}{2}$ to $+\frac{N-1}{2}$.

The impulse response is symmetric with centre of symmetry at n = 0, and so h(–n) = h(n). Hence it is sufficient if we calculate h(n) for n = 0 to (N – 1)/2.

4. Take $\mathcal{Z}$-transform of the impulse response h(n) to get the noncausal transfer function of FIR filter, $H_N(z)$.

$$\therefore\ H_N(z) = \mathcal{Z}\{h(n)\} = \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\, z^{-n}$$

5. Convert the noncausal transfer function, $H_N(z)$ to causal transfer function, H(z) by multiplying $H_N(z)$ by $z^{-(N-1)/2}$.

$$\therefore\ \text{Transfer function, } H(z) = z^{-\frac{N-1}{2}} \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\, z^{-n}$$

Applying symmetry condition, h(−n) = h(n) Refer equation (6.55).

$$\text{Alternatively, Transfer function, } H(z) = z^{-\frac{N-1}{2}} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^n + z^{-n}\right] \right]$$

6. Draw a suitable structure for realization of FIR filter.

Design verification

1. Determine the frequency response, $H(e^{j\omega})$.

Method - 1 : Choose a linear phase magnitude function $|H(e^{j\omega})|$ from table 6.2. Using h(n), obtain an equation for $|H(e^{j\omega})|$.

Method - 2 : The frequency response, $|H(e^{j\omega})|$ can be obtained by replacing z by $e^{j\omega}$ in the transfer function, H(z).

$$\therefore\ \text{Frequency response, } H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}}$$

2. Calculate frequency response for various values of ω in the range 0 to π.
3. Calculate the magnitude response, $|H(e^{j\omega})|$ and sketch the magnitude response to verify the design.

Example 6.5

Design a linear phase FIR lowpass filter using rectangular window by taking 7 samples of window sequence and with a cutoff frequency, $\omega_c = 0.2\pi$ rad/sample.

Solution

Let us choose symmetric impulse response with symmetry condition h(N – 1 – n) = h(n). Therefore, the desired ideal frequency response $H_d(e^{j\omega})$ for FIR lowpass filter is,

$$H_d(e^{j\omega}) = e^{-j\omega\alpha} \quad ; \ -\omega_c \le \omega \le +\omega_c$$
$$= 0 \quad ; \ \text{otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{j\omega})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{j\omega(n-\alpha)}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_c}^{+\omega_c} = \frac{1}{2\pi}\left[\frac{e^{j\omega_c(n-\alpha)}}{j(n-\alpha)} - \frac{e^{-j\omega_c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$\therefore\ h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}}{2j}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}\ ;\ \text{for all n, except } n = \alpha$$

When n = a, the $h_d(n)$ becomes 0/0, which is indeterminate.

$$\therefore\ \text{When } n = \alpha\ ;\ h_d(n) = \underset{(n-\alpha)\to 0}{\text{Lt}} \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}$$

$$= \frac{1}{\pi}\underset{(n-\alpha)\to 0}{\text{Lt}} \frac{\sin \omega_c(n-\alpha)}{(n-\alpha)} = \frac{1}{\pi}\times\omega_c = \frac{\omega_c}{\pi}$$

Using L' Hospital rule, $\underset{\theta\to 0}{\text{Lt}} \frac{\sin A\theta}{\theta} = A$

The impulse response h(n) of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

Rectangular window sequence, $w_R(n) = 1\ ;\ \text{for } n = 0 \text{ to } (N-1)$

$= 0\ ;\ \text{otherwise}$

$$\therefore\ \text{Impulse response, } h(n) = h_d(n)\times w_R(n)$$

$$= h_d(n)\ ;\ \text{for } n = 0 \text{ to } N-1$$

Here, $N = 7\ ;\ \omega_c = 0.2\pi$ rad / sample ; $\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3\ ;\ N-1 = 6$

Hence, calculate h(n) for n = 0 to 6.

Since, the impulse response h(n) satisfies the symmetry condition h(N – 1 – n) = h(n), calculate h(n) for n = 0 to 3.

When $n = 0\ ;\ h(0) = \frac{\sin(0.2\pi\times(0-3))}{\pi\times(0-3)} = 0.1009$

When $n = 1\ ;\ h(1) = \frac{\sin(0.2\pi\times(1-3))}{\pi\times(1-3)} = 0.1514$

When $n = 2\ ;\ h(2) = \frac{\sin(0.2\pi\times(2-3))}{\pi\times(2-3)} = 0.1871$

Note : Calculate sin q by keeping the calculator in radian mode.

When $n = 3\ ;\ h(3) = \frac{0.2\pi}{\pi} = 0.2$

When $n = 4\ ;\ h(4) = h(6-4) = h(2) = 0.1871$

When $n = 5\ ;\ h(5) = h(6-5) = h(1) = 0.1514$

When $n = 6\ ;\ h(6) = h(6-6) = h(0) = 0.1009$

Using symmetry condition, $h(N-1-n) = h(n) \Rightarrow h(6-n) = h(n)$.

The transfer function H(z) of FIR lowpass filter is given by,

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{6} h(n)\, z^{-n}$$

$$= h(0) + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3} + h(4)\,z^{-4} + h(5)\,z^{-5} + h(6)\,z^{-6}$$

$$= h(0) + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3} + h(2)\,z^{-4} + h(1)\,z^{-5} + h(0)\,z^{-6}$$

$$= h(0)\left[1+z^{-6}\right] + h(1)\left[z^{-1}+z^{-5}\right] + h(2)\left[z^{-2}+z^{-4}\right] + h(3)\,z^{-3}$$

$$= 0.1009\left[1+z^{-6}\right] + 0.1514\left[z^{-1}+z^{-5}\right] + 0.1871\left[z^{-2}+z^{-4}\right] + 0.2\,z^{-3}$$

Using symmetry condition, $h(N-1-n) = h(n)$.

Structure

Let, $H(z) = \frac{Y(z)}{X(z)} = 0.1009\left[1+z^{-6}\right] + 0.1514\left[z^{-1}+z^{-5}\right] + 0.1871\left[z^{-2}+z^{-4}\right] + 0.2z^{-3}$

$$\therefore\ Y(z) = 0.1009\left[X(z) + z^{-6}\,X(z)\right] + 0.1514\left[z^{-1}X(z) + z^{-5}\,X(z)\right] + 0.1871\left[z^{-2}X(z) + z^{-4}\,X(z)\right] + 0.2z^{-3}\,X(z)$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

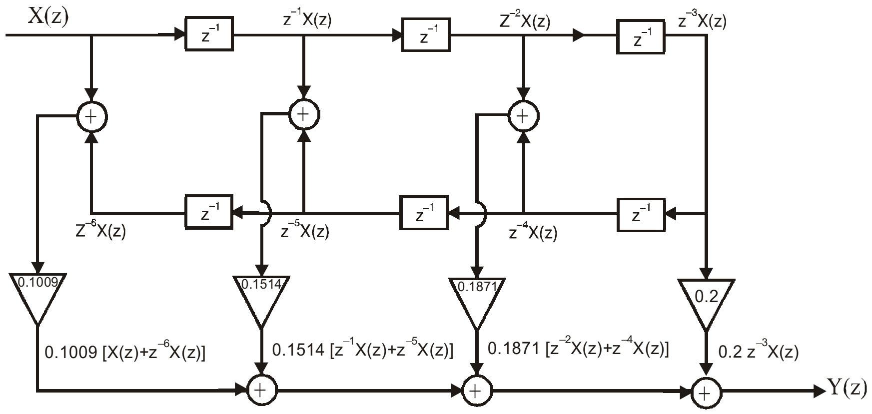


Fig 1 : Linear phase structure of FIR lowpass filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at (N – 1)/2, the magnitude response $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where,}\quad A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

Refer table 6.2 case (i)

$$\therefore\ A(\omega) = h(3) + \sum_{n=1}^{3} 2h(3-n)\cos\omega n$$

$$= h(3) + 2h(2)\cos\omega + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega$$

$$= 0.2 + 2\times 0.1871\cos\omega + 2\times 0.1514\cos 2\omega + 2\times 0.1009\cos 3\omega$$

$$= 0.2 + 0.3742\cos\omega + 0.3028\cos 2\omega + 0.2018\cos 3\omega$$

Using the above equation, the amplitude response A(w) and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

Table 1 : A(w) and $|H(e^{jw})|$ for various values of w.

w	A(w)	$\lvert H(e^{jw})\rvert=\lvert A(w)\rvert$
$\frac{0\times\pi}{16}$	1.0788	1.0788
$\frac{1\times\pi}{16}$	1.0145	1.0145
$\frac{2\times\pi}{16}$	0.8370	0.8370
$\frac{3\times\pi}{16}$	0.5876	0.5876
$\frac{4\times\pi}{16}$	0.3219	0.3219
$\frac{5\times\pi}{16}$	0.0940	0.0940
$\frac{6\times\pi}{16}$	–0.0573	0.0573
$\frac{7\times\pi}{16}$	–0.1188	0.1188
$\frac{8\times\pi}{16}$	–0.1028	0.1028

w	A(w)	$\lvert H(e^{jw})\rvert=\lvert A(w)\rvert$
$\frac{9\times\pi}{16}$	–0.0406	0.0406
$\frac{10\times\pi}{16}$	0.0291	0.0291
$\frac{11\times\pi}{16}$	0.0741	0.0741
$\frac{12\times\pi}{16}$	0.0780	0.0780
$\frac{13\times\pi}{16}$	0.0441	0.0441
$\frac{14\times\pi}{16}$	–0.0088	0.0088
$\frac{15\times\pi}{16}$	–0.0550	0.0550
$\frac{16\times\pi}{16}$	–0.0732	0.0732

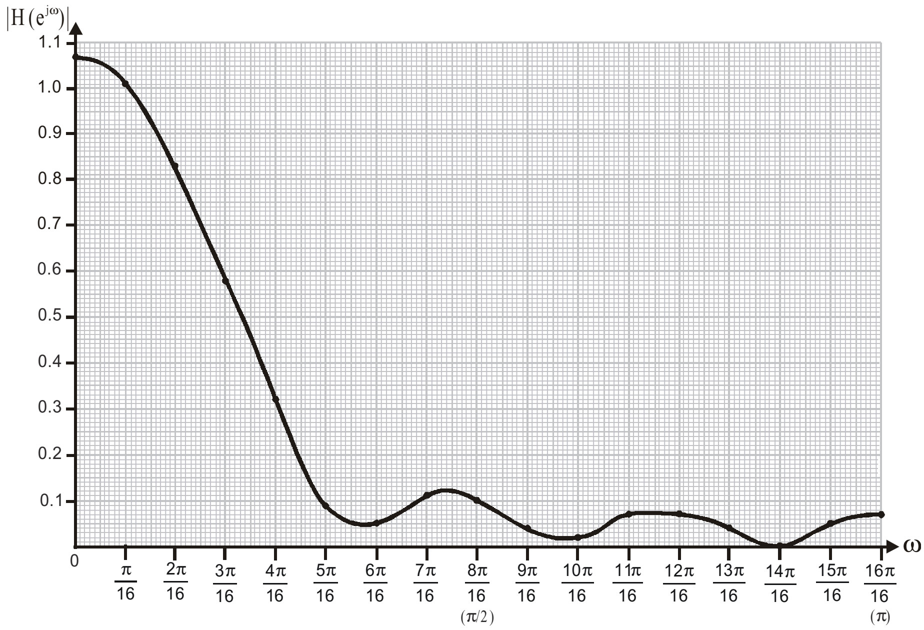


***Fig 2 :** Magnitude response of FIR lowpass filter.*

Alternate Method for Filter Design

Let the symmetry condition be h(–n) = h(n). Therefore, the desired ideal frequency response $H_d(e^{jw})$ for FIR lowpass filter is,

$$H_d(e^{j\omega}) = 1 \quad ; \; -\omega_c \le \omega \le +\omega_c$$
$$= 0 \quad ; \; \text{otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{jw})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} 1 \times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{+\omega_c} = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] = \frac{\sin \omega_c n}{\pi n} \quad ; \quad \text{for all n, except n = 0}$$

When n = 0, the $h_d(n)$ become 0/0, which is indeterminate.

$$\therefore \text{ When } n = 0 \; ; \; h_d(n) = \underset{n\to 0}{\text{Lt}} \frac{\sin \omega_c n}{\pi n}$$

$$= \frac{1}{\pi} \underset{n\to 0}{\text{Lt}} \frac{\sin \omega_c n}{n} = \frac{1}{\pi} \times \omega_c = \frac{\omega_c}{\pi}$$

Using L' Hospital rule,

$$\underset{\theta\to 0}{\text{Lt}} \frac{\sin A\theta}{\theta} = A$$

The impulse response h(n) of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

$$\text{Rectangular window sequence, } w_R(n) = 1 \quad ; \quad \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$$
$$= 0 \quad ; \quad \text{otherwise}$$

$$\therefore \text{ Impulse response, } h(n) = h_d(n) \times w_R(n)$$
$$= h_d(n) \quad ; \quad \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$$

Here, $N = 7;\ \omega_c = 0.2\pi$ rad / sample ; $\frac{N-1}{2} = \frac{7-1}{2} = 3$

Hence calculate h(n) for n = –3 to +3.

Since, h(n) satisfies the symmetry condition h(–n) = h(n), calculate h(n) for n = 0 to 3.

When $n = 0$; $h(0) = \frac{\omega_c}{\pi} = \frac{0.2\pi}{\pi} = 0.2$

When $n = 1$; $h(1) = \frac{\sin(0.2\pi\times 1)}{\pi\times 1} = 0.1871$

When $n = 2$; $h(2) = \frac{\sin(0.2\pi\times 2)}{\pi\times 2} = 0.1514$

When $n = 3$; $h(3) = \frac{\sin(0.2\pi\times 3)}{\pi\times 3} = 0.1009$

When $n = -1$; $h(-1) = h(1) = 0.1871$

When $n = -2$; $h(-2) = h(2) = 0.1514$

When $n = -3$; $h(-3) = h(3) = 0.1009$

Using symmetry condition h(– n) = h(n)

The transfer function H(z) of FIR lowpass filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\ \mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}} \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} h(n)\, z^{-n} = z^{-3} \sum_{n=-3}^{3} h(n)\, z^{-n}$$

$$= z^{-3}\left[h(-3)z^3 + h(-2)z^2 + h(-1)z + h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

$$= z^{-3}\left[h(3)z^3 + h(2)z^2 + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

Using symmetry condition, h(–n) = h(n).

$$= z^{-3}\left[h(3)\left[z^3 + z^{-3}\right] + h(2)\left[z^2 + z^{-2}\right] + h(1)\left[z + z^{-1}\right] + h(0)\right]$$

$$= h(3)\left[z^0 + z^{-6}\right] + h(2)\left[z^{-1} + z^{-5}\right] + h(1)\left[z^{-2} + z^{-4}\right] + h(0)\,z^{-3}$$

$$= 0.1009\left[1 + z^{-6}\right] + 0.1514\left[z^{-1} + z^{-5}\right] + 0.1871\left[z^{-2} + z^{-4}\right] + 0.2\,z^{-3}$$

It is observed that the transfer function obtained in both the methods are same.

Alternate method for Frequency Response

Frequency response, $H\left(e^{j\omega}\right) = H(z)\Big|_{z = e^{j\omega}}$

$$\therefore\ H\left(e^{j\omega}\right) = 0.1009\left[1 + z^{-6}\right] + 0.1514\left[z^{-1} + z^{-5}\right] + 0.1871\left[z^{-2} + z^{-4}\right] + 0.2z^{-3}\Big|_{z = e^{j\omega}}$$

$$= 0.1009\left[1 + e^{-j6\omega}\right] + 0.1514\left[e^{-j\omega} + e^{-j5\omega}\right] + 0.1871\left[e^{-j2\omega} + e^{-j4\omega}\right] + 0.2\,e^{-j3\omega}$$

$$= 0.1009 + 0.1009\left[\cos 6\omega - j\sin 6\omega\right] + 0.1514\left[\cos\omega - j\sin\omega + \cos 5\omega - j\sin 5\omega\right]$$

$$+ 0.1871\left[\cos 2\omega - j\sin 2\omega + \cos 4\omega - j\sin 4\omega\right] + 0.2\left[\cos 3\omega - j\sin 3\omega\right]$$

$$= [0.1009 + 0.1009\cos 6\omega + 0.1514\cos\omega + 0.1514\cos 5\omega + 0.1871\cos 2\omega + 0.1871\cos 4\omega + 0.2\cos 3\omega]$$

$$+ j[-0.1009\sin 6\omega - 0.1514\sin\omega - 0.1514\sin 5\omega - 0.1871\sin 2\omega - 0.1871\sin 4\omega - 0.2\sin 3\omega]$$

Using the above equation the frequency response $H(e^{jw})$ and magnitude function $|H(e^{jw})|$ of lowpass filter are calculated for various values of w and listed in table 2. It is observed that the magnitude response obtained by both the methods are same.

Table 2 : H(e^jw) and |H(e^jw)| for various values of w.

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	1.0788 + j0	1.0788
$\frac{1\times\pi}{16}$	0.8435 – j0.563	1.0141
$\frac{2\times\pi}{16}$	0.3203 – j0.7733	0.8370
$\frac{3\times\pi}{16}$	–0.1146 – j0.576	0.5872
$\frac{4\times\pi}{16}$	–0.2276 – j0.2276	0.3218
$\frac{5\times\pi}{16}$	–0.0922 – j0.0183	0.0940
$\frac{6\times\pi}{16}$	0.0529 – j0.0219	0.0572
$\frac{7\times\pi}{16}$	0.0660 – j0.0988	0.1188
$\frac{8\times\pi}{16}$	0 – j0.1028	0.1028

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{9\times\pi}{16}$	–0.0225 – j0.0337	0.0405
$\frac{10\times\pi}{16}$	0.0269 + j0.0114	0.0292
$\frac{11\times\pi}{16}$	0.0727 – j0.0144	0.0741
$\frac{12\times\pi}{16}$	0.0552 – j0.0552	0.0738
$\frac{13\times\pi}{16}$	0.0086 – j0.0432	0.0440
$\frac{14\times\pi}{16}$	0.0033 + j0.0081	0.0087
$\frac{15\times\pi}{16}$	0.0457 + j0.0305	0.0549
$\frac{16\times\pi}{16}$	0.0732 – j0	0.0732

Example 6.6

Design a linear phase FIR highpass filter using hamming window, with a cutoff frequency, $w_c = 0.8p$ rad/sample and N = 7.

Solution

Let us choose symmetric impulse response with symmetry condition h(N – 1 – n) = h(n). Therefore, the desired ideal frequency response $H_d(e^{jw})$ for FIR highpass filter is,

$$H_d(e^{j\omega}) = e^{-j\omega\alpha} \quad ; \; -\pi \le \omega \le -\omega_c \;\text{ and }\; +\omega_c \le \omega \le +\pi$$
$$= 0 \quad ; \text{ otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{jw})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega(n-\alpha)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_c(n-\alpha)}}{j(n-\alpha)} - \frac{e^{-j\pi(n-\alpha)}}{j(n-\alpha)}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi(n-\alpha)}}{j(n-\alpha)} - \frac{e^{j\omega_c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}}{2j} - \frac{e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}}{2j}\right]$$

$$= \frac{\sin \pi(n-\alpha) - \sin \omega_c(n-\alpha)}{\pi(n-\alpha)} \quad ; \text{ for all n, except } n = \alpha$$

When n = a, the $h_d(n)$ becomes 0/0, which is indeterminate.

When $n = \alpha$; $h_d(n) = \underset{(n-\alpha)\to 0}{Lt} \dfrac{\sin \pi(n-\alpha) - \sin \omega_c(n-\alpha)}{\pi(n-\alpha)}$

$$= \frac{1}{\pi}\left[\underset{(n-\alpha)\to 0}{Lt} \frac{\sin \pi(n-\alpha)}{(n-\alpha)} - \underset{(n-\alpha)\to 0}{Lt} \frac{\sin \omega_c(n-\alpha)}{(n-\alpha)}\right]$$

$$= \frac{1}{\pi}(\pi - \omega_c)$$

$$= 1 - \frac{\omega_c}{\pi}$$

Using L' Hospital rule, $\underset{\theta\to 0}{Lt} \dfrac{\sin A\theta}{\theta} = A$

The impulse response h(n) of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

The Hamming window sequence $w_H(n)$ is given by,

$$w_H(n) = 0.54 - 0.46 \cos\left(\tfrac{2\pi n}{N-1}\right) \quad ; \text{ for } n = 0 \text{ to } N-1$$

$$= 0 \quad ; \text{ otherwise}$$

$\therefore$ $h(n) = h_d(n)\ w_H(n)$

$$= \frac{\sin \pi(n-\alpha) - \sin \omega_c(n-\alpha)}{\pi(n-\alpha)}\left[0.54 - 0.46 \cos\left(\tfrac{2\pi n}{N-1}\right)\right] \quad ; \text{ for } n \neq \alpha$$

$$= \left(1 - \frac{\omega_c}{\pi}\right)\left[0.54 - 0.46 \cos\left(\tfrac{2\pi n}{N-1}\right)\right] \quad ; \text{ for } n = \alpha$$

Given that, $N = 7$; $\omega_c = 0.8\pi$ rad / sample

$$\therefore\ \alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3 \ ; \ N-1 = 6$$

Hence calculate h(n) for n = 0 to 6.

Since, h(n) satisfies the symmetry condition, h(N – 1 – n) = h(n), calculate h(n) for n = 0 to 3.

$$\therefore\ h(n) = \frac{-\sin\omega_c(n-3)}{\pi(n-3)}\left[0.54 - 0.46 \cos\frac{n\pi}{3}\right] \quad ; \text{ for } n \neq 3$$

$$= \left(1 - \frac{\omega_c}{\pi}\right)\left[0.54 - 0.46 \cos\frac{n\pi}{3}\right] \quad ; \text{ for } n = 3$$

Since n and a are integers, sin (n – a) p = 0.

When $n = 0$; $h(0) = \dfrac{-\sin(0.8\pi(0-3))\left[0.54 - 0.46 \cos \frac{0\times\pi}{3}\right]}{\pi\times(0-3)} = -0.0081$

When $n = 1$; $h(1) = \dfrac{-\sin(0.8\pi(1-3))\left[0.54 - 0.46 \cos \frac{1\times\pi}{3}\right]}{\pi\times(1-3)} = 0.0469$

When $n = 2$; $h(2) = \dfrac{-\sin(0.8\pi(2-3))\left[0.54 - 0.46 \cos \frac{2\times\pi}{3}\right]}{\pi\times(2-3)} = -0.1441$

When $n = 3$; $h(3) = \left(1 - \dfrac{0.8\pi}{\pi}\right)\left[0.54 - 0.46 \cos \dfrac{3\times\pi}{3}\right] = 0.2$

When $n = 4$; $h(4) = h(6-4) = h(2) = -0.1441$

When $n = 5$; $h(5) = h(6-5) = h(1) = 0.0469$

When $n = 6$; $h(6) = h(6-6) = h(0) = -0.0081$

Using symmetry condition
h(N – 1 – n) = h(n) Þ h(6 – n) = h(n).

The transfer function H(z) of FIR highpass filter is given by,

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{6} h(n)\, z^{-n}$$

$$= h(0) + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3} + h(4)\,z^{-4} + h(5)\,z^{-5} + h(6)\,z^{-6}$$

$$= h(0) + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3} + h(2)\,z^{-4} + h(1)\,z^{-5} + h(0)\,z^{-6}$$

Using symmetry condition $h(N-1-n) = h(n)$

$$= h(0)\left[1 + z^{-6}\right] + h(1)\left[z^{-1} + z^{-5}\right] + h(2)\left[z^{-2} + z^{-4}\right] + h(3)\,z^{-3}$$

$$= -0.0081\left[1 + z^{-6}\right] + 0.0469\left[z^{-1} + z^{-5}\right] - 0.1441\left[z^{-2} + z^{-4}\right] + 0.2\,z^{-3}$$

Structure

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)} = -0.0081\left[1 + z^{-6}\right] + 0.0469\left[z^{-1} + z^{-5}\right] - 0.1441\left[z^{-2} + z^{-4}\right] + 0.2\,z^{-3}$$

$$\therefore\quad Y(z) = -0.0081\left[X(z) + z^{-6}\,X(z)\right] + 0.0469\left[z^{-1}X(z) + z^{-5}\,X(z)\right]$$

$$- 0.1441\left[z^{-2}X(z) + z^{-4}\,X(z)\right] + 0.2\,z^{-3}X(z)$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

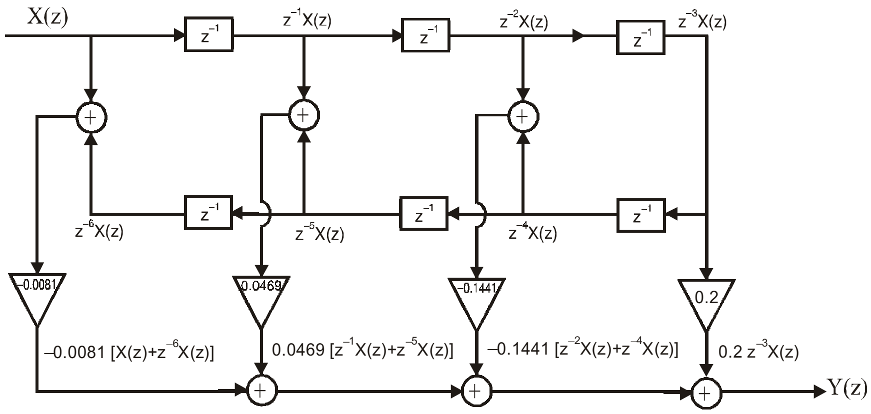


Fig 1 : Linear phase structure of FIR highpass filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at (N – 1)/2, the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$,

$$\text{where,}\quad A(\omega) = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2} - n\right)\cos\omega n$$

Refer table 6.2 case (i)

$$\therefore\quad A(\omega) = h(3) + \sum_{n=1}^{3} 2h(3-n)\cos\omega n$$

$$= h(3) + 2h(2)\cos\omega + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega$$

$$= 0.2 + 2\times(-0.1441)\cos\omega + 2\times 0.0469\,\cos 2\omega + 2\times(-0.0081)\,\cos 3\omega$$

$$= 0.2 - 0.2882\,\cos\omega + 0.0938\,\cos 2\omega - 0.0162\,\cos 3\omega$$

Using the above equation, the amplitude function $A(\omega)$ and magnitude function $|H(e^{j\omega})|$ are calculated for various values of ω and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

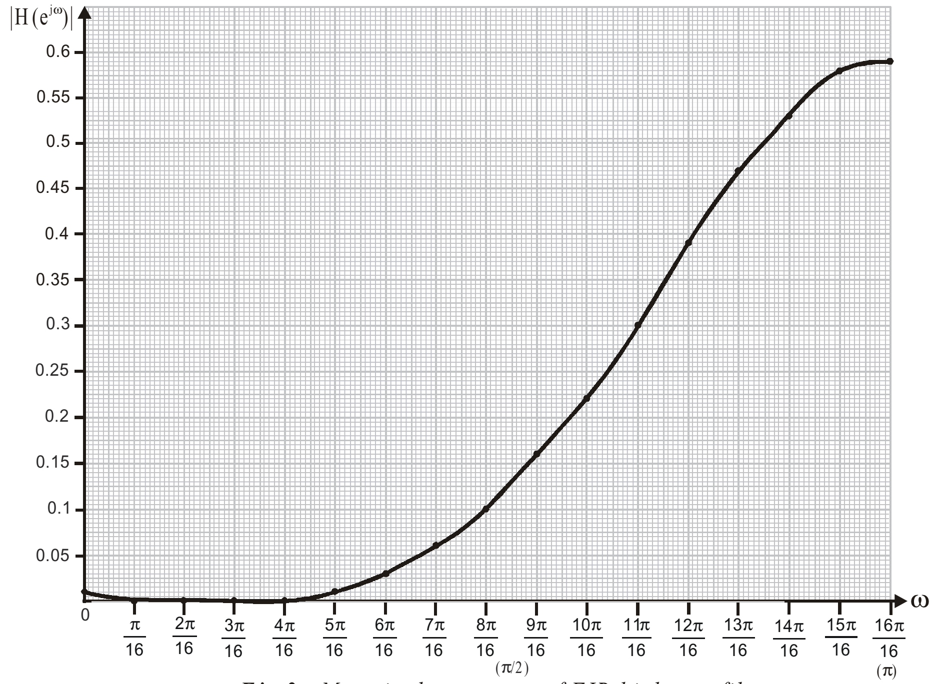


Fig 2 : *Magnitude response of FIR highpass filter.*

Table 1 : A(w) and $|H(e^{jw})|$ for various values of w.

w	A(w)	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{0\times\pi}{16}$	–0.0106	0.0106
$\frac{1\times\pi}{16}$	–0.0094	0.0094
$\frac{2\times\pi}{16}$	–0.0061	0.0061
$\frac{3\times\pi}{16}$	–0.0005	0.0005
$\frac{4\times\pi}{16}$	0.0076	0.0076
$\frac{5\times\pi}{16}$	0.0198	0.0198
$\frac{6\times\pi}{16}$	0.0383	0.0383
$\frac{7\times\pi}{16}$	0.0661	0.0661
$\frac{8\times\pi}{16}$	0.1062	0.1062

w	A(w)	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{9\times\pi}{16}$	0.1605	0.1605
$\frac{10\times\pi}{16}$	0.2289	0.2289
$\frac{11\times\pi}{16}$	0.3083	0.3083
$\frac{12\times\pi}{16}$	0.3923	0.3923
$\frac{13\times\pi}{16}$	0.4723	0.4723
$\frac{14\times\pi}{16}$	0.5387	0.5387
$\frac{15\times\pi}{16}$	0.5827	0.5827
$\frac{16\times\pi}{16}$	0.5982	0.5982

Alternate Method for Filter Design

Let the symmetry condition be h(– n) = h(n). Therefore, the desired ideal frequency response $H_d(e^{jw})$ for FIR highpass filter is,

$$H_d(e^{j\omega}) = 1 \quad ; \; -\pi \le \omega \le -\omega_c \;\text{ and }\; +\omega_c \le \omega \le +\pi$$

$$= 0 \quad ; \text{ otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{jw})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} 1\times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} 1\times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_c}^{\pi} = \frac{1}{2\pi}\left[\frac{e^{-j\omega_c n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_c n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} - \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right]$$

$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$$= \frac{\sin\pi n - \sin\omega_c n}{\pi n} \;;\; \text{for all n, except } n = 0.$$

When $n = 0$, the $h_d(n)$ becomes 0/0, which is indeterminate.

$$\text{When } n = 0 \;;\; h_d(0) = \underset{n\to 0}{\text{Lt}}\; \frac{\sin\pi n - \sin\omega_c n}{\pi n}$$

Using L' Hospital rule, $\underset{\theta\to 0}{\text{Lt}}\; \frac{\sin A\theta}{\theta} = A$

$$= \left[\frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\;\frac{\sin\pi n}{n} - \frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\;\frac{\sin\omega_c n}{n}\right] = 1 - \frac{\omega_c}{\pi}$$

The impulse response of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

Hamming window sequence, $w_H(n) = 0.54 + 0.46\cos\left(\frac{2\pi n}{N-1}\right)$; $n = -\frac{N-1}{2}$ to $+\frac{N-1}{2}$

$= 0$; otherwise

∴ Impulse response, $h(n) = h_d(n)\, w_H(n)$

Here, $N = 7$; $\frac{N-1}{2} = 3$; $\omega_c = 0.8$ rad / sample.

Hence, calculate h(n) for n = –3 to 3.

Since, the impulse response h(n) satisfies the symmetry condition, h(–n) = h(n), calculate h(n) for n = 0 to 3.

$$\therefore\; h(n) = -\frac{\sin\omega_c n}{\pi n}\left[0.54 + 0.46\cos\frac{\pi n}{3}\right] \;;\; \text{for } n \neq 0$$

For integer n $\sin \pi n = 0$.

$$= \left(1 - \frac{\omega_c}{\pi}\right)\left[0.54 + 0.46\cos\frac{\pi n}{3}\right] \;;\; \text{for } n = 0$$

$$\text{When } n = 0 \;;\; h(0) = \left(1 - \frac{0.8\pi}{\pi}\right)\left[0.54 + 0.46\cos\frac{\pi\times 0}{3}\right] = 0.2$$

$$\text{When } n = 1 \;;\; h(1) = -\frac{\sin(0.8\pi\times 1)}{\pi\times 1}\left[0.54 + 0.46\cos\frac{\pi\times 1}{3}\right] = -0.1441$$

$$\text{When } n = 2 \;;\; h(2) = -\frac{\sin(0.8\pi\times 2)}{\pi\times 2}\left[0.54 + 0.46\cos\frac{\pi\times 2}{3}\right] = 0.0469$$

$$\text{When } n = 3 \;;\; h(3) = -\frac{\sin(0.8\pi\times 3)}{\pi\times 3}\left[0.54 + 0.46\cos\frac{\pi\times 3}{3}\right] = -0.0081$$

When $n = -1$; $h(-1) = h(1) = -0.1441$

When $n = -2$; $h(-2) = h(2) = 0.0469$

When $n = -3$; $h(-3) = h(3) = -0.0081$

Using symmetry condition h(n) = h(–n)

The transfer function H(z) of the digital FIR highpass filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\, \mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}} \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} h(n)\, z^{-n} = z^{-3}\sum_{n=-3}^{3} h(n)\, z^{-n}$$

$$= z^{-3}\left[h(-3)z^3 + h(-2)z^2 + h(-1)z + h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

Using symmetry condition h(n) = h(–n)

$$= z^{-3}\left[h(3)z^3 + h(2)z^2 + h(1)z + h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

$$\therefore\ H(z) = z^{-3}\left[h(3)\left[z^3 + z^{-3}\right] + h(2)\left[z^2 + z^{-2}\right] + h(1)\left[z + z^{-1}\right] + h(0)\right]$$

$$= h(3)\left[z^0 + z^{-6}\right] + h(2)\left[z^{-1} + z^{-5}\right] + h(1)\left[z^{-2} + z^{-4}\right] + h(0)\,z^{-3}$$

$$= -0.0081\left[1 + z^{-6}\right] + 0.0469\left[z^{-1} + z^{-5}\right] - 0.1441\left[z^{-2} + z^{-4}\right] + 0.2\,z^{-3}$$

$$= -0.0081 - 0.0081z^{-6} + 0.0469z^{-1} + 0.0469z^{-5} - 0.1441z^{-2} - 0.1441z^{-4} + 0.2z^{-3}$$

It is observed that the transfer function obtained in both the methods are same.

Alternate Method for Frequency Response

Frequency response, $H(e^{j\omega}) = H(z)\Big|_{z\,=\,e^{j\omega}}$

$$\therefore\ H(e^{j\omega}) = -0.0081\left[1 + z^{-6}\right] + 0.0469\left[z^{-1} + z^{-5}\right] + -0.1441\left[z^{-2} + z^{-4}\right] + 0.2z^{-3}\Big|_{z\,=\,e^{j\omega}}$$

$$= -0.0081\left[1 + e^{-j6\omega}\right] + 0.0469\left[e^{-j\omega} + e^{-j5\omega}\right] - 0.1441\left[e^{-j2\omega} + e^{-j4\omega}\right] + 0.2\,e^{-j3\omega}$$

$$= -0.0081\left[1 + \cos 6\omega - j\sin 6\omega\right] + 0.0469\left[\cos\omega - j\sin\omega + \cos 5\omega - j\sin 5\omega\right]$$

$$- 0.1441\left[\cos 2\omega - j\sin 2\omega + \cos 4\omega - j\sin 4\omega\right] + 0.2\left[\cos 3\omega - j\sin 3\omega\right]$$

$$= [-0.0081 - 0.0081\cos 6\omega + 0.0469\cos\omega + 0.0469\cos 5\omega - 0.1441\cos 2\omega - 0.1441\cos 4\omega + 0.2\cos 3\omega]$$

$$+ j[0.0081\sin 6\omega - 0.0469\sin\omega - 0.0469\sin 5\omega + 0.1441\sin 2\omega + 0.1441\sin 4\omega - 0.2\sin 3\omega]$$

Using the above equation the frequency response $H(e^{jw})$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 2. It is observed that the magnitude response obtained by both the methods are same.

Table 2 : $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w.

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	–0.0106 – j0	0.0106
$\frac{1\times\pi}{16}$	–0.0078 + j0.0052	0.0093
$\frac{2\times\pi}{16}$	–0.0023 + j0.0056	0.0060
$\frac{3\times\pi}{16}$	0.0001 + j0.0005	0.0005
$\frac{4\times\pi}{16}$	–0.0054 – j0.0054	0.0076
$\frac{5\times\pi}{16}$	–0.0194 – j0.0038	0.0197
$\frac{6\times\pi}{16}$	–0.0354 + j0.0146	0.0382
$\frac{7\times\pi}{16}$	–0.0367 + j0.0549	0.0660
$\frac{8\times\pi}{16}$	0 + j0.1062	0.1062

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{9\times\pi}{16}$	0.0892 + j0.1335	0.1605
$\frac{10\times\pi}{16}$	0.2115 + j0.0876	0.2289
$\frac{11\times\pi}{16}$	0.3024 – j0.0601	0.3083
$\frac{12\times\pi}{16}$	0.2774 – j0.2774	0.3923
$\frac{13\times\pi}{16}$	0.0921 – j0.4632	0.4722
$\frac{14\times\pi}{16}$	–0.2061 – j0.4977	0.5386
$\frac{15\times\pi}{16}$	–0.4845 – j0.3237	0.5826
$\frac{16\times\pi}{16}$	–0.5982 – j0	0.5982

Example 6.7

Design a linear phase FIR bandpass filter to pass frequencies in the range 0.4p to 0.65p rad/sample by taking 7 samples of hanning window sequence.

Solution

Let us choose symmetric impulse response with symmetry condition $h(N-1-n) = h(n)$. Therefore, the desired ideal frequency response $H_d(e^{jw})$ for bandpass filter is,

$$H_d(e^{j\omega}) = e^{-j\omega\alpha} \quad ; \quad -\omega_{c2} \le \omega \le -\omega_{c1} \quad \& \quad +\omega_{c1} \le \omega \le +\omega_{c2}$$

$$= 0 \quad ; \quad \text{otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{jw})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega(n-\alpha)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_{c1}}^{\omega_{c2}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}(n-\alpha)}}{j(n-\alpha)} - \frac{e^{-j\omega_{c2}(n-\alpha)}}{j(n-\alpha)}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2}(n-\alpha)}}{j(n-\alpha)} - \frac{e^{j\omega_{c1}(n-\alpha)}}{j(n-\alpha)}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega_{c2}(n-\alpha)} - e^{-j\omega_{c2}(n-\alpha)}}{2j} - \frac{e^{j\omega_{c1}(n-\alpha)} - e^{-j\omega_{c1}(n-\alpha)}}{2j}\right]$$

$$= \frac{\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)}{\pi(n-\alpha)} \quad ; \quad \text{for all } n \text{ except } n = \alpha.$$

When n = a, the $h_d(n)$ becomes 0/0 which is indeterminate.

$$\text{When } n = \alpha\ ; \quad h_d(n) = \underset{(n-\alpha)\to 0}{\text{Lt}} \frac{\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)}{\pi(n-\alpha)}$$

$$= \frac{1}{\pi}\left[\underset{(n-\alpha)\to 0}{\text{Lt}} \frac{\sin\omega_{c2}(n-\alpha)}{(n-\alpha)} - \underset{(n-\alpha)\to 0}{\text{Lt}} \frac{\sin\omega_{c1}(n-\alpha)}{(n-\alpha)}\right]$$

$$= \frac{\omega_{c2} - \omega_{c1}}{\pi}$$

Using L′ Hospital rule

$$\underset{\theta\to 0}{\text{Lt}} \frac{\sin A\theta}{\theta} = A$$

The impulse response h(n) of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

The hanning window sequence $w_C(n)$ is given by,

$$w_C(n) = 0.5 - 0.5\cos\frac{2\pi n}{N-1} \quad ; \quad \text{for } n = 0 \text{ to } (N-1)$$

$$= 0 \quad ; \quad \text{otherwise}$$

$$\therefore\ h(n) = h_d(n)\, w_C(n) = \left[\frac{\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)}{\pi(n-\alpha)}\right]\left[0.5 - 0.5\cos\left(\tfrac{2\pi n}{N-1}\right)\right] ; \text{ for } n \ne \alpha$$

$$= \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)\left[0.5 - 0.5\cos\left(\tfrac{2\pi n}{N-1}\right)\right] ; \text{ for } n = \alpha$$

Given that N = 7 ; w_{c1} = 0.4p rad/sample and w_{c2} = 0.65p rad/sample

Here, $\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$; $N - 1 = 6$

Hence calculate h(n) for n = 0 to 6.

Since, h(n) satisfies the symmetry condition, h(N – 1 – n) = h(n), calculate h(n) for n = 0 to 3.

$$\therefore \quad h(n) = \frac{\left[\sin\omega_{c2}(n-3) - \sin\omega_{c1}(n-3)\right]\left[0.5 - 0.5\cos\frac{n\pi}{3}\right]}{\pi(n-3)} \quad ; \quad \text{for } n \neq 3$$

$$= \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)\left(0.5 - 0.5\cos\frac{n\pi}{3}\right) \quad ; \quad \text{for } n = 3$$

When $n = 0$; $h(0) = \frac{\left[\sin\left(0.65\pi(0-3)\right) - \sin\left(0.4\pi(0-3)\right)\right]\left[0.5 - 0.5\cos\frac{0\times\pi}{3}\right]}{\pi(0-3)} = 0$

When $n = 1$; $h(1) = \frac{\left[\sin\left(0.65\pi(1-3)\right) - \sin\left(0.4\pi(1-3)\right)\right]\left[0.5 - 0.5\cos\frac{1\times\pi}{3}\right]}{\pi(1-3)} = -0.0556$

When $n = 2$; $h(2) = \frac{\left[\sin\left(0.65\pi(2-3)\right) - \sin\left(0.4\pi(2-3)\right)\right]\left[0.5 - 0.5\cos\frac{2\times\pi}{3}\right]}{\pi(2-3)} = -0.0143$

When $n = 3$; $h(3) = \left(\frac{0.65\pi - 0.4\pi}{\pi}\right)\left(0.5 - 0.5\cos\frac{3\pi}{3}\right) = 0.25$

When $n = 4$; $h(4) = h(6-4) = h(2) = -0.0143$

When $n = 5$; $h(5) = h(6-5) = h(1) = -0.0556$

When $n = 6$; $h(6) = h(6-6) = h(0) = 0$

> Using symmetry condition
> $h(N-1-n) = h(n) \Rightarrow h(6-n) = h(n)$.

The transfer function H(z) of FIR bandpass filter is given by,

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{6} h(n)\, z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(2)z^{-4} + h(1)z^{-5} + h(0)z^{-6}$$

$$= h(0)\left[1 + z^{-6}\right] + h(1)\left[z^{-1} + z^{-5}\right] + h(2)\left[z^{-2} + z^{-4}\right] + h(3)z^{-3}$$

> Using symmetry condition,
> $h(N-1-n) = h(n)$.

$$= 0 \times \left[1 + z^{-6}\right] - 0.0556\left[z^{-1} + z^{-5}\right] - 0.0143\left[z^{-2} + z^{-4}\right] + 0.25z^{-3}$$

$$= -0.0556\left[z^{-1} + z^{-5}\right] - 0.0143\left[z^{-2} + z^{-4}\right] + 0.25z^{-3}$$

Structure

Let, $H(z) = \frac{Y(z)}{X(z)} = -0.0556\left[z^{-1} + z^{-5}\right] - 0.0143\left[z^{-2} + z^{-4}\right] + 0.25z^{-3}$

$$\therefore Y(z) = -0.0556\left[z^{-1}X(z) + z^{-5}X(z)\right] - 0.0143\left[z^{-2}X(z) + z^{-4}X(z)\right] + 0.25z^{-3}X(z)$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

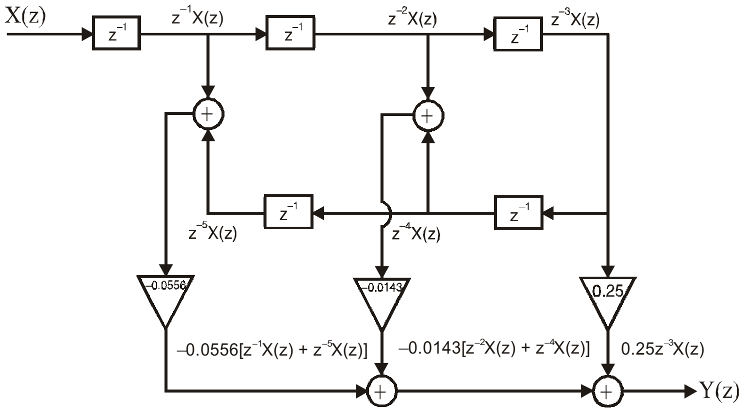


Fig 1 : Linear phase structure for FIR bandpass filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at (N – 1)/2 the magnitude response $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where,}\ \ A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

Refer table 6.2 case (i)

$$\therefore\ \ A(\omega) = h(3) + \sum_{n=1}^{3} 2h(3-n)\cos\omega n$$

$$= h(3) + 2h(2)\cos\omega + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega$$

$$= 0.25 + 2\times(-0.0143)\cos\omega + 2\times(-0.0556)\cos 2\omega + 2\times 0\cos 3\omega$$

$$= 0.25 - 0.0286\cos\omega - 0.1112\cos 2\omega$$

Using the above equation, the amplitude function A(w) and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

Table 1: A(w) and $|H(e^{jw})|$ for various values of w.

w	A(w)	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{0\times\pi}{16}$	0.1102	0.1102
$\frac{1\times\pi}{16}$	0.1192	0.1192
$\frac{2\times\pi}{16}$	0.1449	0.1449
$\frac{3\times\pi}{16}$	0.1836	0.1836
$\frac{4\times\pi}{16}$	0.2297	0.2297
$\frac{5\times\pi}{16}$	0.2766	0.2766
$\frac{6\times\pi}{16}$	0.3176	0.3176
$\frac{7\times\pi}{16}$	0.3471	0.3471
$\frac{8\times\pi}{16}$	0.3612	0.3612

w	A(w)	$\lvert H(e^{jw})\rvert = A(w)$
$\frac{9\times\pi}{16}$	0.3583	0.3583
$\frac{10\times\pi}{16}$	0.3395	0.3395
$\frac{11\times\pi}{16}$	0.3084	0.3084
$\frac{12\times\pi}{16}$	0.2702	0.2702
$\frac{13\times\pi}{16}$	0.2312	0.2312
$\frac{14\times\pi}{16}$	0.1977	0.1977
$\frac{15\times\pi}{16}$	0.1753	0.1753
$\frac{16\times\pi}{16}$	0.1674	0.1674

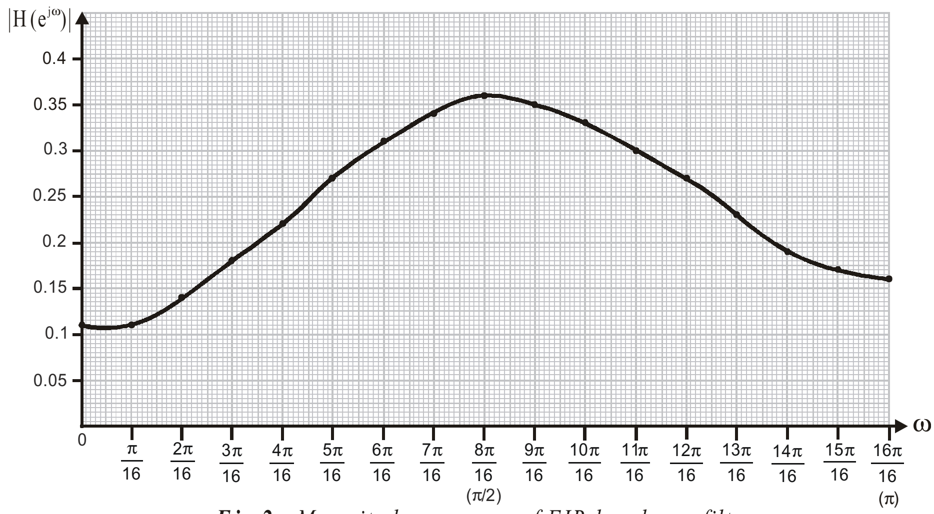


Fig 2 : Magnitude response of FIR bandpass filter.

Alternate Method for Filter Design

Let the symmetry condition be h(–n) = h(n). Therefore, the desired ideal frequency response $H_d(e^{jw})$ for FIR bandpass filter is,

$$H_d\left(e^{j\omega}\right) = 1 \;;\quad -\omega_{c2} \le \omega \le -\omega_{c1} \;\;\&\;\; +\omega_{c1} \le \omega \le \omega_{c2}$$

$$= 0 \;;\quad \text{otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{jw})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right)e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} 1\times e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{+\omega_{c1}}^{\omega_{c2}} 1\times e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c1}}^{\omega_{c2}} = \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}n}}{jn} - \frac{e^{-j\omega_{c2}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2}n}}{jn} - \frac{e^{j\omega_{c1}n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j} - \frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n} \;;\quad \text{for all n, except n} = 0$$

When n = 0, the $h_d(n)$ becomes 0/0 which is indeterminate.

$$\text{When } n = 0\;;\;\; h_d(0) = \underset{n\to 0}{\text{Lt}}\left[\frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n}\right] = \frac{1}{\pi}\left[\underset{n\to 0}{\text{Lt}}\frac{\sin\omega_{c2}n}{n} - \underset{n\to 0}{\text{Lt}}\frac{\sin\omega_{c1}n}{n}\right]$$

Using L' Hospital rule

$$\underset{\theta\to 0}{\text{Lt}}\frac{\sin A\theta}{\theta} = A$$

$$= \frac{\omega_{c2} - \omega_{c1}}{\pi}$$

The impulse response h(n) of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

$$\text{Hanning window sequence,}\;\; w_C(n) = 0.5 + 0.5\cos\tfrac{2\pi n}{N-1} \;;\quad \text{for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}$$

$$= 0 \;;\quad \text{otherwise}$$

\ Impulse response, $h(n) = h_d(n)$ ´ $w_C(n)$

$$= \left(\frac{\sin\omega_{c2} - \sin\omega_{c1}}{\pi n}\right)\left(0.5 + 0.5\cos\tfrac{2\pi n}{N-1}\right) \;;\quad \text{for } n \ne 0$$

$$= \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)\left(0.5 + 0.5\cos\tfrac{2\pi n}{N-1}\right) \;;\quad \text{for } n = 0$$

Given that, N = 7 ; w_{c1} = 0.4p rad/sample ; w_{c2} = 0.65p rad/sample

$$\therefore\ \alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3\ ;\ N-1=6$$

Hence calculate h(n) for n = 0 to 6.

Since, h(n) satisfies the symmetry condition, h(– n) = h(n), calculate h(n) for n = o to 3.

$$\therefore\ h(n) = \left(\frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n}\right)\left(0.5 + 0.5\cos\frac{\pi n}{3}\right)\ ;\ \text{for } n \neq 0$$

$$= \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)\left(0.5 + 0.5\cos\frac{\pi n}{3}\right)\ ;\ \text{for } n = 0$$

$$\text{When } n=0\ ;\ h(0) = \left[\frac{0.65\pi - 0.4\pi}{\pi}\right]\left[0.5 + 0.5\cos\frac{\pi\times 0}{3}\right] = 0.25$$

$$\text{When } n=1\ ;\ h(1) = \left[\frac{\sin(0.65\pi\times 1) - \sin(0.4\pi\times 1)}{\pi\times 1}\right]\left[0.5 + 0.5\cos\frac{\pi\times 1}{3}\right] = -0.0143$$

$$\text{When } n=2\ ;\ h(2) = \left[\frac{\sin(0.65\pi\times 2) - \sin(0.4\pi\times 2)}{\pi\times 2}\right]\left[0.5 + 0.5\cos\frac{\pi\times 2}{3}\right] = -0.0556$$

$$\text{When } n=3\ ;\ h(3) = \left[\frac{\sin(0.65\pi\times 3) - \sin(0.4\pi\times 3)}{\pi\times 3}\right]\left[0.5 + 0.5\cos\frac{\pi\times 3}{3}\right] = 0$$

When $n=-1$; $h(-1) = h(1) = -0.0143$

When $n=-2$; $h(-2) = h(2) = -0.0556$

When $n=-3$; $h(-3) = h(3) = 0$

Using symmetry condtion, h(–n) = h(n).

The transfer function H(z) of FIR bandpass filter is,

$$H(z) = z^{-\frac{N-1}{2}}\ \mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}} \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\,z^{-n} = z^{-3}\sum_{n=-3}^{3} h(n)\,z^{-n}$$

$$= z^{-3}\left[h(-3)z^{3} + h(-2)z^{2} + h(-1)z + h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

Using symmetry condtion, h(–n) = h(n).

$$= z^{-3}\left[h(3)z^{3} + h(2)z^{2} + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

$$= z^{-3}\left[h(3)\left[z^{3} + z^{-3}\right] + h(2)\left[z^{2} + z^{-2}\right] + h(1)\left[z + z^{-1}\right] + h(0)\right]$$

$$= h(3)\left[z^{0} + z^{-6}\right] + h(2)\left[z^{-1} + z^{-5}\right] + h(1)\left[z^{-2} + z^{-4}\right] + h(0)\,z^{-3}$$

$$= -0.0556\left[z^{-1} + z^{-5}\right] - 0.0143\left[z^{-2} + z^{-4}\right] + 0.25z^{-3}$$

h(3) = 0

It is observed that the transfer function obtained in both the methods are same.

Alternate Method for Frequency Response

Frequency response, $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

$$\therefore\ H(e^{j\omega}) = -0.0556\left[z^{-1} + z^{-5}\right] - 0.0143\left[z^{-2} + z^{-4}\right] + 0.25z^{-3}\Big|_{z=e^{j\omega}}$$

$$= -0.0556\left(e^{-j\omega} - e^{-j5\omega}\right) - 0.0143\left(e^{-j2\omega} - e^{-j4\omega}\right) + 0.25e^{-j3\omega}$$

$$= -0.0556\left(\cos\omega - j\sin\omega + \cos5\omega - j\sin5\omega\right)$$

$$- 0.0143\left(\cos2\omega - j\sin2\omega + \cos4\omega - j\sin4\omega\right) + 0.25\left(\cos3\omega - j\sin3\omega\right)$$

$$= \left[-0.0556\cos\omega - 0.0556\cos5\omega - 0.0143\cos2\omega - 0.0143\cos4\omega + 0.25\cos3\omega\right]$$

$$+ j\left[0.0556\sin\omega + 0.0556\sin5\omega + 0.0143\sin2\omega + 0.0143\sin4\omega - 0.25\cos3\omega\right]$$

Using the above equation the frequency response $H(e^{jw})$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 2. It is observed that the magnitude response obtained in both the methods are same.

Table 2: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	0.1102 – j0	0.1102
$\frac{1\times\pi}{16}$	0.0991 – j0.0662	0.1191
$\frac{2\times\pi}{16}$	0.0554 – j0.1339	0.1449
$\frac{3\times\pi}{16}$	–0.0358 – j0.1801	0.1836
$\frac{4\times\pi}{16}$	–0.1624 – j0.1624	0.2296
$\frac{5\times\pi}{16}$	–0.2713 – j0.0539	0.2766
$\frac{6\times\pi}{16}$	–0.2935 + j0.1215	0.3176
$\frac{7\times\pi}{16}$	–0.1928 + j0.2886	0.3470
$\frac{8\times\pi}{16}$	0 + j0.3612	0.3612

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{9\times\pi}{16}$	0.1990 + j0.2979	0.3582
$\frac{10\times\pi}{16}$	0.3137 + j0.1299	0.3395
$\frac{11\times\pi}{16}$	0.3025 – j0.0601	0.3084
$\frac{12\times\pi}{16}$	0.1910 – j0.1910	0.2701
$\frac{13\times\pi}{16}$	0.0451 – j0.2267	0.2311
$\frac{14\times\pi}{16}$	–0.0756 – j0.1827	0.1977
$\frac{15\times\pi}{16}$	–0.1457 – j0.097	0.1750
$\frac{16\times\pi}{16}$	–0.1674 – j0	0.1674

Example 6.8

Design a linear phase FIR bandstop filter to reject frequencies in the range 0.4p to 0.65p rad/sample using rectangular window, by taking 7 samples of window sequence.

Solution

Let us choose symmetric impulse response with symmetry condition, h(N – 1 – n) = h(n). Therefore, the desired ideal frequency response $H_d(e^{jw})$ for bandstop filter is,

$$H_d(e^{j\omega}) = e^{-j\omega\alpha} \quad ; -\pi \le \omega \le -\omega_{c2} \text{ and } -\omega_{c1} \le \omega \le +\omega_{c1} \text{ and } +\omega_{c2} \le \omega \le +\pi$$
$$= 0 \quad ; \text{ otherwise}$$

The desired impulse response $h_d(n)$ obtained by taking inverse Fourier transform of $H_d(e^{jw})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{j\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{j\omega(n-\alpha)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_{c1}}^{\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_{c2}}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c2}(n-\alpha)}}{j(n-\alpha)} - \frac{e^{-j\pi(n-\alpha)}}{j(n-\alpha)}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c1}(n-\alpha)}}{j(n-\alpha)} - \frac{e^{-j\omega_{c1}(n-\alpha)}}{j(n-\alpha)}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi(n-\alpha)}}{j(n-\alpha)} - \frac{e^{j\omega_{c2}(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega_{c1}(n-\alpha)} - e^{-j\omega_{c1}(n-\alpha)}}{2j} + \frac{e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}}{2j} - \frac{e^{j\omega_{c2}(n-\alpha)} - e^{-j\omega_{c2}(n-\alpha)}}{2j}\right]$$

$$= \frac{\sin\omega_{c1}(n-\alpha) + \sin\pi(n-\alpha) - \sin\omega_{c2}(n-\alpha)}{\pi(n-\alpha)} \quad ; \text{ for all n, except } n = \alpha \qquad \boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$\therefore$ When $n = \alpha$;

When $n = \alpha$, the $h_d(n)$ becomes 0/0 which is indeterminate.

$$h_d(n) = \underset{(n-\alpha)\to 0}{Lt} \frac{\sin \omega_{c1}(n-\alpha) + \sin \pi(n-\alpha) - \sin \omega_{c2}(n-\alpha)}{\pi(n - \alpha)}$$

$$= \frac{1}{\pi}\left[\underset{(n-\alpha)\to 0}{Lt} \frac{\sin \omega_{c1}(n-\alpha)}{(n-\alpha)} + \underset{(n-\alpha)\to 0}{Lt} \frac{\sin \pi(n-\alpha)}{(n-\alpha)} - \underset{(n-\alpha)\to 0}{Lt} \frac{\sin \omega_{c2}(n-\alpha)}{(n-\alpha)}\right]$$

$$= \frac{1}{\pi}\left[\omega_{c1} + \pi - \omega_{c2}\right]$$

$$= 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)$$

Using L' Hospital rule

$$\underset{\theta \to 0}{Lt} \frac{\sin A\theta}{\theta} = A$$

The impulse response h(n) of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

The window sequence for rectangular window $w_R(n)$ is given by,

$$w_R(n) = 1 \;\; ; \text{ for } n = 0 \text{ to } N - 1$$
$$= 0 \;\; ; \text{ otherwise}$$

$$\therefore\; h(n) = h_d(n) \times w_R(n) = h_d(n) = \frac{\sin \omega_{c1}(n-\alpha) + \sin \pi(n-\alpha) - \sin \omega_{c2}(n-\alpha)}{\pi(n-\alpha)} \;\; ; \text{ for } n \neq \alpha$$

$$= 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right) \;\; ; \text{ for } n = \alpha$$

Given that, $N = 7$; $\omega_{c1} = 0.4\pi$ rad / sample ; $\omega_{c2} = 0.65\pi$ rad / sample

$$\therefore\; \alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3 \;;\; N - 1 = 6$$

Hence calculate h(n) for n = 0 to 6.

Since, h(n) satisfies the symmetry condition, h(N – 1 – n) = h(n), calculate h(n) for n = 0 to 3.

$$\therefore\; h(n) = \frac{\sin\omega_{c1}(n-3) - \sin\omega_{c2}(n-3)}{\pi(n-3)} \;\; ; \text{ for } n \neq 3$$

$$= 1 - \frac{\omega_{c2} - \omega_{c1}}{\pi} \;\; ; \text{ for } n = 3$$

Since n and a are integers, $\sin(n - a)p = 0$.

When $n = 0$; $h(0) = \dfrac{\sin(0.4\pi(0-3)) - \sin(0.65\pi(0-3))}{\pi \times (0-3)} = -0.0458$

When $n = 1$; $h(1) = \dfrac{\sin(0.4\pi(1-3)) - \sin(0.65\pi(1-3))}{\pi \times (1-3)} = 0.2223$

When $n = 2$; $h(2) = \dfrac{\sin(0.4\pi(2-3)) - \sin(0.65\pi(2-3))}{\pi \times (2-3)} = 0.0191$

When $n = 3$; $h(3) = 1 - \left(\dfrac{0.65\pi - 0.4\pi}{\pi}\right) = 0.75$

When $n = 4$; $h(4) = h(6-4) = h(2) = 0.0191$

When $n = 5$; $h(5) = h(6-5) = h(1) = 0.2223$

When $n = 6$; $h(6) = h(6-6) = h(0) = -0.0458$

Using symmetry condition
$h(N-1-n) = h(n) \Rightarrow h(6-n) = h(n)$.

The transfer function H(z) of FIR bandstop filter is given by,

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{6} h(n)\, z^{-n}$$

$$\therefore\ H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(2)z^{-4} + h(1)z^{-5} + h(0)z^{-6}$$

$$= h(0)\left[1 + z^{-6}\right] + h(1)\left[z^{-1} + z^{-5}\right] + h(2)\left[z^{-2} + z^{-4}\right] + h(3)z^{-3}$$

$$= -0.0458\left[1 + z^{-6}\right] + 0.2223\left[z^{-1} + z^{-5}\right] + 0.0191\left[z^{-2} + z^{-4}\right] + 0.75z^{-3}$$

Using symmetry condition, $h(N - 1 - n) = h(n)$.

Structure

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = -0.0458\left[1 + z^{-6}\right] + 0.2223\left[z^{-1} + z^{-5}\right] + 0.0191\left[z^{-2} + z^{-4}\right] + 0.75z^{-3}$$

$$\therefore\ Y(z) = -0.0458\left[X(z) + z^{-6}X(z)\right] + 0.2223\left[z^{-1}X(z) + z^{-5}X(z)\right]$$

$$+ 0.0191\left[z^{-2}X(z) + z^{-4}X(z)\right] + 0.75z^{-3}X(z)$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

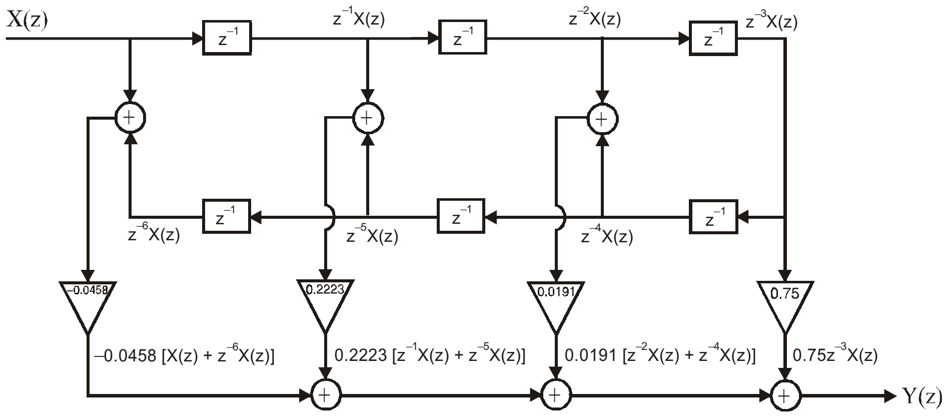


Fig 1 : Linear phase structure for FIR bandstop filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at $(N - 1)/2$ the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$,

$$\text{where, } A(\omega) = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2} - n\right)\cos\omega n$$

Refer table 6.2 case (i)

$$\therefore\ A(\omega) = h(3) + \sum_{n=1}^{3} 2h(3 - n)\cos\omega n$$

$$= h(3) + 2h(2)\cos\omega + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega$$

$$= 0.75 + 2 \times 0.0191\cos\omega + 2 \times 0.2223\cos 2\omega + 2 \times (-0.0458)\cos 3\omega$$

$$= 0.75 + 0.0382\cos\omega + 0.4446\cos 2\omega - 0.0914\cos 3\omega$$

Using the above equation the magnitude response, $A(\omega)$ and magnitude function $|H(e^{j\omega})|$ are calculated for various values of ω and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

Table 1: A(w) and |H(e^jw)| for various values of w.

w	A(w)	$\|H(e^{jw})\| = \|A(w)\|$
$\frac{0\times\pi}{16}$	1.1414	1.1414
$\frac{1\times\pi}{16}$	1.1223	1.1223
$\frac{2\times\pi}{16}$	1.0646	0.0646
$\frac{3\times\pi}{16}$	0.9697	0.9697
$\frac{4\times\pi}{16}$	0.8416	0.8416
$\frac{5\times\pi}{16}$	0.6907	0.6907
$\frac{6\times\pi}{16}$	0.5346	0.5346
$\frac{7\times\pi}{16}$	0.3974	0.3974
$\frac{8\times\pi}{16}$	0.3054	0.3054

w	A(w)	$\|H(e^{jw})\| = \|A(w)\|$
$\frac{9\times\pi}{16}$	0.2810	0.2810
$\frac{10\times\pi}{16}$	0.3365	0.3365
$\frac{11\times\pi}{16}$	0.4689	0.4689
$\frac{12\times\pi}{16}$	0.6583	0.6583
$\frac{13\times\pi}{16}$	0.8705	0.8705
$\frac{14\times\pi}{16}$	1.0640	1.0640
$\frac{15\times\pi}{16}$	1.1992	1.1992
$\frac{16\times\pi}{16}$	1.2478	1.2478

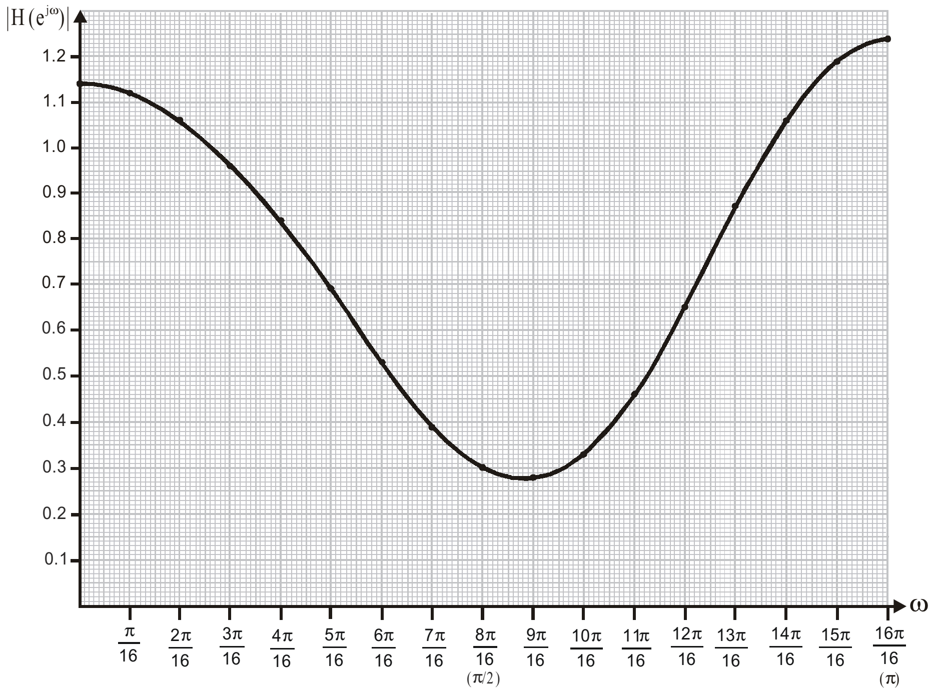


Fig 2 : Magnitude response of FIR bandstop filter.

Alternate Method for Filter Design

Let the symmetry condition be h(– n) = h(n). Therefore, the desired ideal frequency response $H_d(e^{jw})$ for FIR bandstop filter is,

$$H_d(e^{j\omega}) = 1 \quad ; -\pi \le \omega \le -\omega_{c2} \text{ and } -\omega_{c1} \le \omega \le +\omega_{c1} \text{ and } +\omega_{c2} \le \omega \le +\pi$$

$$= 0 \quad ; \text{otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{jw})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} 1\times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} 1\times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} 1\times e^{j\omega n}\, d\omega$$

$$\therefore\ h_d(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c1}}^{\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c2}}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c2}n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c1}n}}{jn} - \frac{e^{-j\omega_{c1}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_{c2}n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j}\right] + \left[\frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j}\right] - \left[\frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{\sin\pi n + \sin\omega_{c1}n - \sin\omega_{c2}n}{\pi n}\ ;\ \text{for all n, except } n = 0.$$

When n = 0, the $h_d(n)$ becomes 0/0 which is indeterminate.

$$\text{When } n = 0\ ;\ h_d(n) = \underset{n\to 0}{\text{Lt}}\ \frac{\sin\pi n + \sin\omega_{c1}n - \sin\omega_{c2}n}{\pi n}$$

$$= \frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\frac{\sin\pi n}{n} + \frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\frac{\sin\omega_{c1}n}{n} + \frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\frac{\sin\omega_{c2}n}{n}$$

$$= \frac{1}{\pi}\times\pi + \frac{1}{\pi}\times\omega_{c1} - \frac{1}{\pi}\times\omega_{c2} = 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)$$

Using L' Hospital rule

$$\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin A\theta}{\theta} = A$$

The impulse response h(n) of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

$$\text{Rectangular window sequence, } w_R(n) = 1\ ;\ n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}$$

$$= 0\ ;\ \text{otherwise}$$

$$\therefore \text{ Impulse response, } h(n) = h_d(n)\times w_R(n)$$

$$= h_d(n)\ ;\ \text{for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}$$

Here, N = 7, w_{c1} = 0.4p rad/sample ; w_{c2} = 0.65p rad/sample ; $\frac{N-1}{2} = \frac{7-1}{2} = 3$

Hence, calculate h(n) for n = –3 to 3.

Since, h(n) satisfies the symmetry condition, h(–n) = h(n), calculate h(n) for n = 0 to 3.

$$\therefore\ h(n) = \frac{\sin\omega_{c1} - \sin\omega_{c2}}{\pi n}\ ;\ \text{for } n \neq 0$$

$$= 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)\ ;\ \text{for } n = 0$$

For integer n, sin pn = 0

$$\text{When } n = 0\ ;\ h(0) = 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right) = 1 - \left(\frac{0.65\pi - 0.4\pi}{\pi}\right) = 0.75$$

$$\text{When } n = 1\ ;\ h(1) = \frac{\sin(0.4\pi\times 1) - \sin(0.65\pi\times 1)}{\pi\times 1} = 0.0191$$

$$\text{When } n = 2\ ;\ h(2) = \frac{\sin(0.4\pi\times 2) - \sin(0.65\pi\times 2)}{\pi\times 2} = 0.2223$$

$$\text{When } n = 3\ ;\ h(2) = \frac{\sin(0.4\pi\times 3) - \sin(0.65\pi\times 3)}{\pi\times 3} = -0.0457$$

When $n = -1$; $h(-1) = h(1) = 0.0191$

When $n = -2$; $h(-2) = h(2) = 0.2223$

When $n = -3$; $h(-3) = h(3) = -0.0457$

Using symmetry condtion, h(–n) = h(n).

The transfer function H(z) of the digital FIR bandstop filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\ \mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} h(n)\,z^{-n} = z^{-3}\sum_{n=-3}^{3} h(n)\,z^{-n}$$

$$= z^{-3}\left[h(-3)z^{3} + h(-2)z^{2} + h(-1)z + h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

Using symmetry condtion, h(–n) = h(n).

$$\therefore\ H(z) = z^{-3}\left[h(3)z^3 + h(2)z^2 + h(1)z + h(0)z + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

$$= z^{-3}\left[h(3)\left[z^3 + z^{-3}\right] + h(2)\left[z^2 + z^{-2}\right] + h(1)\left[z + z^{-1}\right] + h(0)\right]$$

$$= h(3)\left[z^0 + z^{-6}\right] + h(2)\left[z^{-1} + z^{-5}\right] + h(1)\left[z^{-2} + z^{-4}\right] + h(0)\,z^{-3}$$

$$= -0.0457\left[1 + z^{-6}\right] + 0.2223\left[z^{-1} + z^{-5}\right] + 0.0191\left[z^{-2} + z^{-4}\right] + 0.75z^{-3}$$

It is observed that the transfer function obtained in both the methods are same.

Alternate Method for Frequency Response

Frequency response, $H\left(e^{j\omega}\right) = H(z)\Big|_{z\,=\,e^{j\omega}}$

$$\therefore\ H\left(e^{j\omega}\right) = -0.0457\left[1 + z^{-6}\right] + 0.2223\left[z^{-1} + z^{-5}\right] + 0.0191\left[z^{-2} + z^{-4}\right] + 0.75z^{-3}\Big|_{z=e^{j\omega}}$$

$$= -\,0.0457\left[1 + e^{-j6\omega}\right] + 0.2223\left[e^{-j\omega} + e^{-j5\omega}\right] + 0.0191\left[e^{-j2\omega} + e^{-j4\omega}\right] + 0.75e^{-j3\omega}$$

$$= -0.0457 - 0.0457\left[\cos 6\omega - j\sin 6\omega\right] + 0.2223\left[\cos\omega - j\sin\omega + \cos 5\omega - j\sin 5\omega\right]$$

$$+\,0.0191\left[\cos 2\omega - j\sin 2\omega + \cos 4\omega - j\sin 4\omega\right] + 0.75\left[\cos 3\omega - j\sin 3\omega\right]$$

$$= [-0.0457 - 0.0457\ \cos 6\omega + 0.2223\cos\omega + 0.2223\ \cos 5\omega + 0.0191\ \cos 2\omega + 0.0191\ \cos 4\omega + 0.75\ \cos 3\omega]$$

$$+\ j[0.0457\ \sin 6\omega - 0.2223\sin\omega - 0.2223\sin 5\omega - 0.0191\ \sin 2\omega - 0.0191\sin 4\omega - 0.75\sin 3\omega]$$

Using the above equation the frequency response $H(e^{jw})$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 2. It is observed that the magnitude response obtained in both the methods are same.

Table 2: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	1.1414 – j0	1.1414
$\frac{1\times\pi}{16}$	0.9330 + j0.6234	1.1221
$\frac{2\times\pi}{16}$	0.4074 – j0.9836	1.0646
$\frac{3\times\pi}{16}$	–0.1899 – j0.9511	0.9697
$\frac{4\times\pi}{16}$	–0.5951 – j0.5951	0.8416
$\frac{5\times\pi}{16}$	–0.6774 – j0.1347	0.6907
$\frac{6\times\pi}{16}$	–0.4939 + j0.2046	0.5346
$\frac{7\times\pi}{16}$	–0.2208 + j0.3304	0.3974
$\frac{8\times\pi}{16}$	0 + j0.3054	0.3054

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{9\times\pi}{16}$	0.1561 + j0.2336	0.2809
$\frac{10\times\pi}{16}$	0.3109 + j0.1287	0.3364
$\frac{11\times\pi}{16}$	0.4599 – j0.0914	0.4689
$\frac{12\times\pi}{16}$	0.4655 – j0.4655	0.6583
$\frac{13\times\pi}{16}$	0.1698 – j0.8538	0.8705
$\frac{14\times\pi}{16}$	–0.4072 – j0.9830	1.0640
$\frac{15\times\pi}{16}$	–0.9971 – j0.6662	1.1991
$\frac{16\times\pi}{16}$	–1.2478 – j0	1.2478

6.10 Design of FIR Filters by Frequency Sampling Technique

In this method the ideal (desired) frequency response is sampled at sufficient number of points (i.e., N-points). These samples are the DFT coefficients of the impulse response of the filter. Hence the impulse response of the filter is determined by taking inverse DFT.

Let, $H_d(e^{jw})$ = Ideal desired frequency response

$H(k)$ = DFT sequence obtained by sampling $H_d(e^{jw})$

$h(n)$ = Impulse response of FIR filter.

The impulse response h(n) is obtained by taking inverse DFT of H(k). For practical realizability the samples of impulse response should be real. This can happen if all the complex terms appear in complex conjugate pairs. It can be observed that the complex DFT coefficients exists only as conjugate pairs. This suggest that the terms of H(k) can be matched by comparing the exponentials. The term H(k) $e^{+j2pnk/N}$ should be matched by the term that has the exponential $e^{-j2pnk/N}$ as a factor.

Procedure for Type-1 Design

1. Choose the ideal (desired) frequency response $H_d(e^{jw})$.
2. Sample $H_d(e^{jw})$ at N-points by taking $w = w_k$ = 2pk/N where k = 0, 1, 2, 3, ...(N - 1), to generate the sequence H(k) .

$$\therefore\ H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \quad ; \quad \text{for } k = 0, 1, \ldots (N-1)$$

3. Compute the N samples of impulse response h(n) using the following equation.

When N is odd,

$$\text{Impulse response, } h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[H(k)\, e^{j2\pi nk/N}\right]\right] \qquad .....(6.76)$$

When N is even,

$$\text{Impulse response, } h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} \mathrm{Re}\left[H(k)\, e^{j2\pi nk/N}\right]\right] \qquad \boxed{\text{Here, } H\left(\tfrac{N}{2}\right) = 0} \qquad .....(6.77)$$

where, "Re" stands for "real part of".

4. Take $\mathcal{Z}$-transform of the impulse response h(n) to get the filter transfer function, H(z).

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

Procedure for Type-2 Design

1. Choose the ideal (desired) frequency response $H_d(e^{jw})$.
2. Sample $H_d(e^{jw})$ at N-points by taking $w = w_k$ = 2p(2k + 1)/2N, where k = 0, 1, 2, 3, ... (N – 1), to generate the sequence H(k).

$$\therefore\ H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi(2k+1)}{2N}} \quad ; \quad \text{for } k = 0, 1, \ldots (N-1)$$

3. Compute the N samples of impulse response h(n) using the following equation.

When N is odd,

Here, $H\left(\frac{N-1}{2}\right) = 0$

$$\text{Impulse response, } h(n) = \frac{2}{N}\sum_{k=0}^{\frac{N-3}{2}} \text{Re}\left[H(k)\, e^{\frac{jn\pi(2K+1)}{N}}\right] \quad .....(6.78)$$

When N is even,

$$\text{Impulse response, } h(n) = \frac{2}{N} \times 2\sum_{k=0}^{\frac{N}{2}-1} \text{Re}\left[H(k)\, e^{\frac{jn\pi(2K+1)}{N}}\right] \quad .....(6.79)$$

where "Re" stands for "real part of".

4. Take $\mathcal{Z}$-transform of the impulse response h(n) to get the filter transfer function, H(z)

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

Example 6.9

Determine the coefficients of a linear-phase FIR filter of length N = 15 which has a symmetric unit sample response and a frequency response that satisfies the conditions

$$H\left(\frac{2\pi k}{15}\right) = 1 \quad ; \quad \text{for } k = 0,\ 1,\ 2,\ 3$$
$$= 0.4 \quad ; \quad \text{for } k = 4$$
$$= 0 \quad ; \quad \text{for } k = 5,\ 6,\ 7$$

Solution

For linear phase FIR filter the phase function, $\theta(\omega) = -\alpha\omega$ where $\alpha = \frac{N-1}{2}$.

Here, N = 15, $\therefore\ \alpha = \frac{15-1}{2} = 7$.

Also, here $\omega = \omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{15}$. Hence we can go for type-1 design.

In this problem the samples of the magnitude response of the ideal (desired) filter are directly given for various values of k.

$$\therefore\ H(k) = H_d(\omega)\big|_{\omega=\omega_k} = 1\, e^{-j\alpha\omega_k} = e^{-j7\times\frac{2\pi k}{15}} \quad ; \ k = 0,\ 1,\ 2,\ 3$$
$$= 0.4\, e^{-j\alpha\omega_k} = 0.4\, e^{-j7\times\frac{2\pi k}{15}} \quad ; \ k = 4$$
$$= 0 \quad ; \ k = 5,\ 6,\ 7$$

The samples of impulse response h(n) are given by,

Using equation(6.76).

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \text{Re}\left[H(k)\, e^{\frac{j2\pi nk}{N}}\right]\right]$$
$$= \frac{1}{15}\left[H(0) + 2\sum_{k=1}^{7} \text{Re}\left[H(k)\, e^{\frac{j2\pi nk}{15}}\right]\right]$$
$$= \frac{1}{15}\left[H(0) + 2\sum_{k=1}^{3} \text{Re}\left[H(k)\, e^{\frac{j2\pi nk}{15}}\right] + 2\ \text{Re}\left[H(4)\, e^{\frac{j2\pi n\times 4}{15}}\right]\right]$$

$$\therefore\ h(n) = \frac{1}{15}\left[1 + 2\sum_{k=1}^{3} \text{Re}\left[e^{-j7\times\frac{2\pi k}{15}} \times e^{\frac{j2\pi nk}{15}}\right] + 2\ \text{Re}\left[0.4\ e^{-j7\times\frac{2\pi\times 4}{15}} \times e^{\frac{j8\pi n}{15}}\right]\right] \qquad \boxed{H(0) = 1}$$

$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{3} \text{Re}\left[e^{\frac{j2\pi k}{15}(n-7)}\right] + 2\ \text{Re}\left[0.4\ e^{\frac{j8\pi}{15}(n-7)}\right]\right]$$

$$\boxed{\begin{aligned} & e^{j\theta} = \cos\theta + j\sin\theta \\ & \therefore\ \text{Re}[e^{j\theta}] = \cos\theta \end{aligned}}$$

$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{3} \cos\frac{2\pi k}{15}(n-7) + 0.8\cos\frac{8\pi}{15}(n-7)\right]$$

$$= \frac{1}{15}\left[1 + 2\cos\frac{2\pi(n-7)}{15} + 2\cos\frac{4\pi(n-7)}{15} + 2\cos\frac{6\pi(n-7)}{15} + 0.8\cos\frac{8\pi(n-7)}{15}\right]$$

Here N = 15, \ N – 1 = 14, $\frac{N-1}{2} = 7$.

Hence, calculate h(n) for n = 0 to 14

Since h(n) satisfies the symmetry condition h(N – 1 – n) = h(n) with centre of symmetry at (N – 1)/2, calculate h(n) for n = 0 to 7.

When n = 0 ; $h(0) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(0-7)}{15} + 2\cos\frac{4\pi(0-7)}{15} + 2\cos\frac{6\pi(0-7)}{15} + 0.8\cos\frac{8\pi(0-7)}{15}\right]$

$= -0.0141$

When n = 1 ; $h(1) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(1-7)}{15} + 2\cos\frac{4\pi(1-7)}{15} + 2\cos\frac{6\pi(1-7)}{15} + 0.8\cos\frac{8\pi(1-7)}{15}\right]$

$= -0.0019$

When n = 2 ; $h(2) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(2-7)}{15} + 2\cos\frac{4\pi(2-7)}{15} + 2\cos\frac{6\pi(2-7)}{15} + 0.8\cos\frac{8\pi(2-7)}{15}\right]$

$= 0.04$

When n = 3 ; $h(3) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(3-7)}{15} + 2\cos\frac{4\pi(3-7)}{15} + 2\cos\frac{6\pi(3-7)}{15} + 0.8\cos\frac{8\pi(3-7)}{15}\right]$

$= 0.0122$

When n = 4 ; $h(4) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(4-7)}{15} + 2\cos\frac{4\pi(4-7)}{15} + 2\cos\frac{6\pi(4-7)}{15} + 0.8\cos\frac{8\pi(4-7)}{15}\right]$

$= -0.0914$

When n = 5 ; $h(5) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(5-7)}{15} + 2\cos\frac{4\pi(5-7)}{15} + 2\cos\frac{6\pi(5-7)}{15} + 0.8\cos\frac{8\pi(5-7)}{15}\right]$

$= -0.0181$

When n = 6 ; $h(6) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(6-7)}{15} + 2\cos\frac{4\pi(6-7)}{15} + 2\cos\frac{6\pi(6-7)}{15} + 0.8\cos\frac{8\pi(6-7)}{15}\right]$

$= 0.3130$

When n = 7 ; $h(7) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(7-7)}{15} + 2\cos\frac{4\pi(7-7)}{15} + 2\cos\frac{6\pi(7-7)}{15} + 0.8\cos\frac{8\pi(7-7)}{15}\right]$

$= 0.52$

When n = 8, h(8) = h(15 – 1 – 8) = h(6) = 0.3130

When n = 9, h(9) = h(15 – 1 – 9) = h(5) = –0.0181

When n = 10, h(10) = h(15 – 1 – 10) = h(4) = –0.0914

$\boxed{\begin{aligned} & \text{Using symmetry condition} \\ & h(N-1-n) = h(n) \end{aligned}}$

When n = 11, $h(11) = h(15-1-11) = h(3) = 0.0122$

When n = 12, $h(12) = h(15-1-12) = h(2) = 0.04$

When n = 13, $h(13) = h(15-1-13) = h(1) = -0.0019$

When n = 14, $h(14) = h(15-1-14) = h(0) = -0.0141$

The transfer function H(z) of the filter is given by $\mathcal{Z}$-transform of h(n)

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{14} h(n)\, z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7}$$
$$+ h(8)z^{-8} + h(9)z^{-9} + h(10)z^{-10} + h(11)z^{-11} + h(12)z^{-12} + h(13)z^{-13} + h(14)z^{-14}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7}$$
$$+ h(6)z^{-8} + h(5)z^{-9} + h(4)z^{-10} + h(3)z^{-11} + h(2)z^{-12} + h(1)z^{-13} + h(0)z^{-14}$$

$$= h(0)\left[1 + z^{-14}\right] + h(1)\left[z^{-1} + z^{-13}\right] + h(2)\left[z^{-2} + z^{-12}\right] + +h(3)\left[z^{-3} + z^{-11}\right]$$
$$+ h(4)\left[z^{-4} + z^{-10}\right] + h(5)\left[z^{-5} + z^{-9}\right] + h(6)\left[z^{-6} + z^{-8}\right] + h(7)z^{-7}$$

Using symmetry condition $h(N-1-n) = h(n)$

$$= -0.0141\left[1 + z^{-14}\right] - 0.0019\left[z^{-1} + z^{-13}\right] + 0.04\left[z^{-2} + z^{-12}\right] + 0.0122\left[z^{-3} + z^{-11}\right]$$
$$- 0.0914\left[z^{-4} + z^{-10}\right] - 0.0181\left[z^{-5} + z^{-9}\right] + 0.3130\left[z^{-6} + z^{-8}\right] + 0.52\,z^{-7}$$

Structure

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)} = -0.0141\left[1 + z^{-14}\right] - 0.0019\left[z^{-1} + z^{-13}\right] + 0.04\left[z^{-2} + z^{-12}\right]$$
$$+ 0.0122\left[z^{-3} + z^{-11}\right] - 0.0914\left[z^{-4} + z^{-10}\right] - 0.0181\left[z^{-5} + z^{-9}\right]$$
$$+ 0.3130\left[z^{-6} + z^{-8}\right] + 0.52\,z^{-7}$$

$$\therefore\ Y(z) = -0.0141\left[X(z) + z^{-14}X(z)\right] - 0.0019\left[z^{-1}X(z) + z^{-13}X(z)\right] + 0.04\left[z^{-2}X(z) + z^{-12}X(z)\right]$$
$$+ 0.0122\left[z^{-3}X(z) + z^{-11}X(z)\right] - 0.0914\left[z^{-4}X(z) + z^{-10}X(z)\right] - 0.0181\left[z^{-5}X(z) + z^{-9}X(z)\right]$$
$$+ 0.3130\left[z^{-6}X(z) + z^{-8}X(z)\right] + 0.52\,z^{-7}X(z)$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

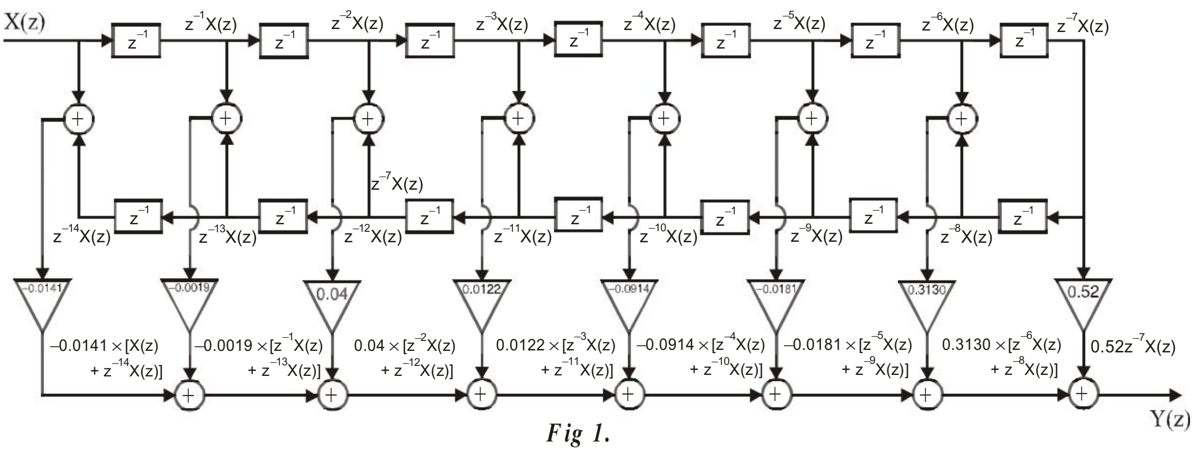


Fig 1.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at (N – 1)/2 the magnitude response $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where, } A(\omega) = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2} - n\right)\cos\omega n$$

Refer table 6.2 case (i)

$$\therefore\ A(\omega) = h(7) + \sum_{n=1}^{7} 2h(7-n)\cos\omega n$$

$$= h(7) + 2h(6)\cos\omega + 2h(5)\cos 2\omega + 2h(4)\cos 3\omega + 2h(3)\cos 4\omega$$
$$+ 2h(2)\cos 5\omega + 2h(1)\cos 6\omega + 2h(0)\cos 7\omega$$
$$= 0.52 + 2\times 0.3130\cos\omega + 2\times -0.0181\cos 2\omega + 2\times -0.0914\cos 3\omega$$
$$+ 2\times 0.0122\cos 4\omega + 2\times 0.04\cos 5\omega$$
$$+ 2\times -0.0019\cos 6\omega + 2\times -0.0141\cos 7\omega$$
$$= 0.52 + 0.626\cos\omega - 0.0362\cos 2\omega - 0.1828\cos 3\omega + 0.0244\cos 4\omega$$
$$+ 0.08\cos 5\omega - 0.0038\cos 6\omega - 0.0282\cos 7\omega$$

Using the above equation, the magnitude response A(w) and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

Table 1: A(w) and $|H(e^{jw})|$ for various values of w

w	A(w)	$\|H(e^{jw})\| = A(w)$
$\frac{0\times\pi}{16}$	0.9994	0.9994
$\frac{1\times\pi}{16}$	1.0032	1.0032
$\frac{2\times\pi}{16}$	1.0009	1.0009
$\frac{3\times\pi}{16}$	0.9856	0.9856
$\frac{4\times\pi}{16}$	0.9909	0.9909
$\frac{5\times\pi}{16}$	1.0323	1.0323
$\frac{6\times\pi}{16}$	1.0360	1.0360
$\frac{7\times\pi}{16}$	0.8900	0.8900
$\frac{8\times\pi}{16}$	0.5844	0.5844
$\frac{9\times\pi}{16}$	0.2542	0.2542
$\frac{10\times\pi}{16}$	0.0497	0.0497
$\frac{11\times\pi}{16}$	–0.0061	0.0061
$\frac{12\times\pi}{16}$	0.0020	0.0020
$\frac{13\times\pi}{16}$	–0.0009	0.0009
$\frac{14\times\pi}{16}$	–0.0067	0.0067
$\frac{15\times\pi}{16}$	–0.0014	0.0014
$\frac{16\times\pi}{16}$	0.0094	0.0094

w	A(w)	$\|H(e^{jw})\| = A(w)$
$\frac{17\times\pi}{16}$	0.0014	0.0014
$\frac{18\times\pi}{16}$	–0.0067	0.0067
$\frac{19\times\pi}{16}$	–0.0009	0.0009
$\frac{20\times\pi}{16}$	0.0020	0.0020
$\frac{21\times\pi}{16}$	–0.0061	0.0061
$\frac{22\times\pi}{16}$	0.0497	0.0497
$\frac{23\times\pi}{16}$	0.2542	0.2542
$\frac{24\times\pi}{16}$	0.5844	0.5844
$\frac{25\times\pi}{16}$	0.8900	0.8900
$\frac{26\times\pi}{16}$	1.0360	1.0360
$\frac{27\times\pi}{16}$	1.0323	1.0323
$\frac{28\times\pi}{16}$	0.9909	0.9909
$\frac{29\times\pi}{16}$	0.9857	0.9857
$\frac{30\times\pi}{16}$	1.0009	1.0009
$\frac{31\times\pi}{16}$	1.0032	1.0032
$\frac{32\times\pi}{16}$	0.9994	0.9994

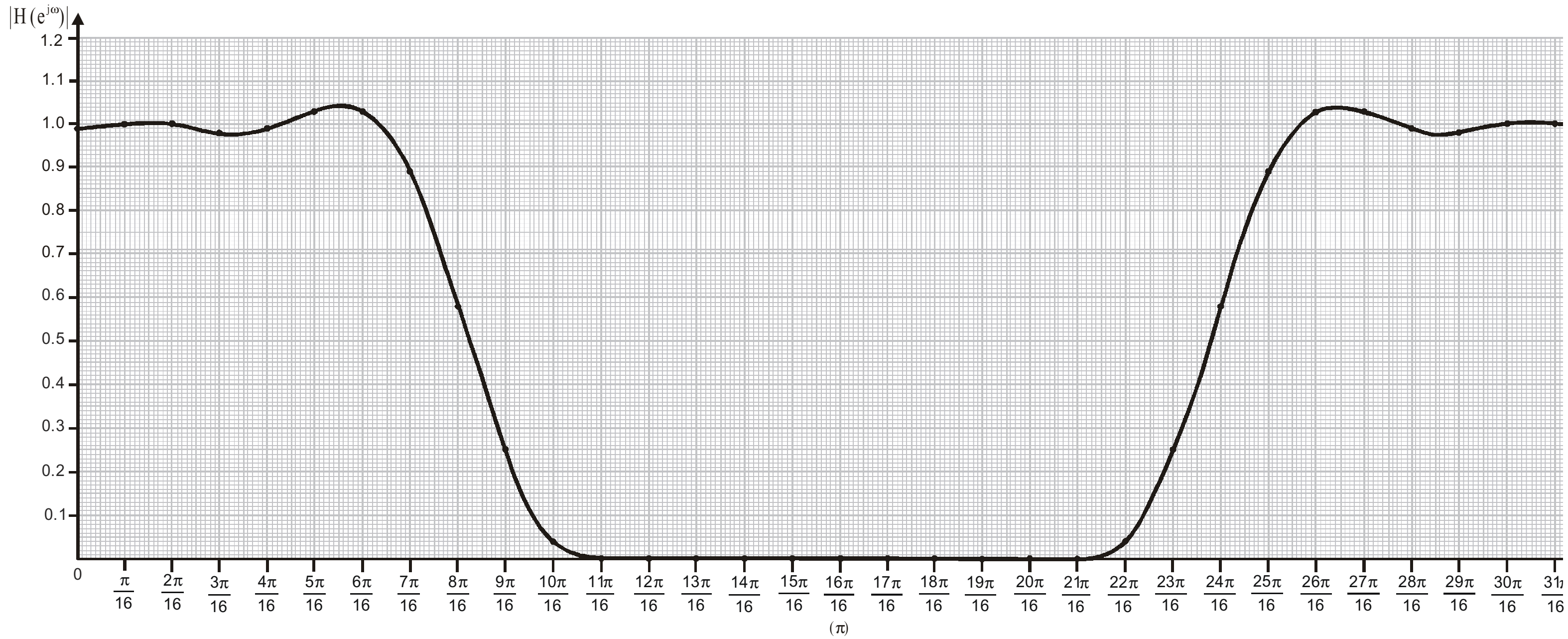


***Fig 2 :** Magnitude response of FIR.*

Alternate Method for Frequency Response

Frequency response, $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

$$\therefore\ H(e^{j\omega}) = -0.0141[1+z^{-14}] - 0.0019[z^{-1}+z^{-13}] + 0.04[z^{-2}+z^{-12}] + 0.0122[z^{-3}+z^{-11}]$$

$$-0.0914[z^{-4}+z^{-10}] - 0.0181[z^{-5}+z^{-9}] + 0.3130[z^{-6}+z^{-8}] + 0.52\,z^{-7}\Big|_{z=e^{j\omega}}$$

$$= -0.0141[1+e^{-j14\omega}] - 0.0019[e^{-j\omega}+e^{-j13\omega}] + 0.04[e^{-j2\omega}+e^{-j12\omega}] + 0.0122[e^{-j3\omega}+e^{-j11\omega}]$$

$$-0.0914[e^{-j4\omega}+e^{-j10\omega}] - 0.0181[e^{-j5\omega}+e^{-j9\omega}] + 0.3130[e^{-j6\omega}+e^{-j8\omega}] + 0.52\,e^{-j7\omega}$$

$$= -0.0141 - 0.0141[\cos 14\omega - j\sin 14\omega] - 0.0019[\cos\omega - j\sin\omega + \cos 13\omega - j\sin 13\omega] + 0.04[\cos 2\omega - j\sin 2\omega + \cos 12\omega - j\sin 12\omega]$$

$$+0.0122[\cos 3\omega - j\sin 3\omega + \cos 11\omega - j\sin 11\omega] - 0.0914[\cos 4\omega - j\sin 4\omega + \cos 10\omega - j\sin 10\omega]$$

$$-0.0181[\cos 5\omega - j\sin 5\omega + \cos 9\omega - j\sin 9\omega] + 0.3130[\cos 6\omega - j\sin 6\omega + \cos 8\omega - j\sin 8\omega] + 0.52[\cos 7\omega - j\sin 7\omega]$$

$$= [-0.0141 - 0.0141\cos 14\omega - 0.0019\cos\omega - 0.0019\cos 13\omega + 0.04\cos 2\omega + 0.04\cos 12\omega + 0.0122\cos 3\omega + 0.0122\cos 11\omega$$

$$-0.0914\cos 4\omega - 0.0914\cos 10\omega - 0.0181\cos 5\omega - 0.0181\cos 9\omega + 0.3130\cos 6\omega + 0.3130\cos 8\omega + 0.52\cos 7\omega]$$

$$+ j[0.0141\sin 14\omega + 0.0019\sin\omega + 0.0019\sin 13\omega - 0.04\sin 2\omega - 0.04\sin 12\omega - 0.0122\sin 3\omega - 0.0122\sin 11\omega$$

$$+0.0914\sin 4\omega + 0.0914\sin 10\omega + 0.0181\sin 5\omega + 0.0181\sin 9\omega - 0.3130\sin 6\omega - 0.3130\sin 8\omega - 0.52\sin 7\omega]$$

Using the above equation the frequency response $H(e^{jw})$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 2. It is observed that the magnitude response obtained in both the methods are same.

Table 2: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	0.9994 + j0	0.9994
$\frac{1\times\pi}{16}$	0.1956 – j0.9839	1.0031
$\frac{2\times\pi}{16}$	–0.9247 – j0.3828	1.0000
$\frac{3\times\pi}{16}$	–0.5476 – j0.8196	0.9857
$\frac{4\times\pi}{16}$	0.7006 + j0.7006	0.9907
$\frac{5\times\pi}{16}$	0.8582 – j0.5735	1.0321
$\frac{6\times\pi}{16}$	–0.3964 – j0.9571	1.0359
$\frac{7\times\pi}{16}$	–0.8728 + j0.1736	0.8898
$\frac{8\times\pi}{16}$	0 + j0.5844	0.5844
$\frac{9\times\pi}{16}$	0.2495 + j0.0497	0.2544
$\frac{10\times\pi}{16}$	0.0191 – j0.046	0.0498
$\frac{11\times\pi}{16}$	0.0051 + j0.0035	0.0061
$\frac{12\times\pi}{16}$	0 + j0	0
$\frac{13\times\pi}{16}$	–0.0006 – j0.0009	0.0010
$\frac{14\times\pi}{16}$	–0.0062 + j0.0027	0.0067
$\frac{15\times\pi}{16}$	–0.0002 – j0.0015	0.0015
$\frac{16\times\pi}{16}$	–0.0009 + j0	0.0009

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{17\times\pi}{16}$	–0.0002 + j0.0015	0.0015
$\frac{18\times\pi}{16}$	–0.0062 – j0.0027	0.0067
$\frac{19\times\pi}{16}$	0.0006 + j0.0009	0.0010
$\frac{20\times\pi}{16}$	0 + j0	0
$\frac{21\times\pi}{16}$	0.0051 – j0.0035	0.0061
$\frac{22\times\pi}{16}$	0.0191 + j0.046	0.0498
$\frac{23\times\pi}{16}$	0.2495 – j0.0497	0.2544
$\frac{24\times\pi}{16}$	0 – j0.5844	0.5844
$\frac{25\times\pi}{16}$	–0.8728 – j0.1736	0.8898
$\frac{26\times\pi}{16}$	–0.3964 + j0.9571	1.0359
$\frac{27\times\pi}{16}$	0.8582 + j0.5735	1.0321
$\frac{28\times\pi}{16}$	0.7006 – j0.7006	0.9907
$\frac{29\times\pi}{16}$	–0.5476 – j0.8196	0.9857
$\frac{30\times\pi}{16}$	–0.9247 + j0.3828	1.0000
$\frac{31\times\pi}{16}$	0.1956 + j0.9839	1.0031
$\frac{32\times\pi}{16}$	0.9994 + j0	0.9994

Example 6.10

Design a linear phase FIR lowpass filter with a cutoff frequency of 0.5p rad/sample by taking 11 samples of ideal frequency response.

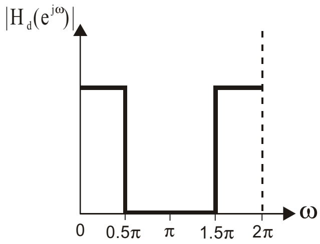


Fig 1 : Ideal magnitude response of FIR lowpass filter.

Solution

The magnitude response of ideal lowpass filter is shown in fig 1. The desired frequency response $H_d(e^{jw})$ of linear phase FIR lowpass filter with cutoff frequency of 0.5p rad/sample is given by,

$$H_d(e^{j\omega}) = e^{-j\alpha\omega} \quad ; \quad 0 \le \omega \le 0.5\pi \text{ and } 1.5\pi \le \omega \le 2\pi$$

$$= 0 \quad ; \quad 0.5\pi < \omega < 1.5\pi$$

$$\text{where, } \alpha = \frac{N-1}{2} = \frac{11-1}{2} = 5$$

The DFT sequence H(k) is obtained by sampling $H_d(e^{jw})$ at 11 equidistant frequency points in a period of 2p. The 11 frequencies for type-1 design are given by,

$$\omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{11} \quad ; \text{ for } k = 0 \text{ to } 10$$

When $k = 0$; $\omega_k = \frac{2\pi \times 0}{11} = 0$	When $k = 6$; $\omega_k = \frac{2\pi \times 6}{11} = 1.09\pi$
When $k = 1$; $\omega_k = \frac{2\pi \times 1}{11} = 0.18\pi$	When $k = 7$; $\omega_k = \frac{2\pi \times 7}{11} = 1.27\pi$
When $k = 2$; $\omega_k = \frac{2\pi \times 2}{11} = 0.36\pi$	When $k = 8$; $\omega_k = \frac{2\pi \times 8}{11} = 1.45\pi$
When $k = 3$; $\omega_k = \frac{2\pi \times 3}{11} = 0.55\pi$	When $k = 9$; $\omega_k = \frac{2\pi \times 9}{11} = 1.64\pi$
When $k = 4$; $\omega_k = \frac{2\pi \times 4}{11} = 0.73\pi$	When $k = 10$; $\omega_k = \frac{2\pi \times 10}{11} = 1.82\pi$
When $k = 5$; $\omega_k = \frac{2\pi \times 5}{11} = 0.91\pi$	

From the above calculations the following observations can be made.

For k = 0 to 2, the samples lie in the range 0 £ w £ 0.5 p

For k = 3 to 8, the samples lie in the range 0.5p < w < 1.5 p

For k = 9 to 10, the samples lie in the range 1.5p £ w < 2p

The sampling points on the ideal frequency response are shown in fig 2. The magnitude of samples of H(k) (Magnitude spectrum) are shown in fig 3.

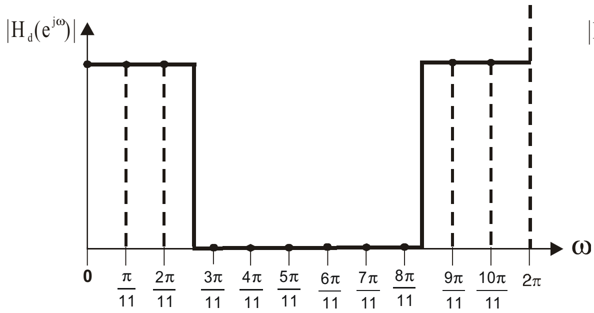


Fig 2 : Sampling points of $H_d(e^{j\omega})$.

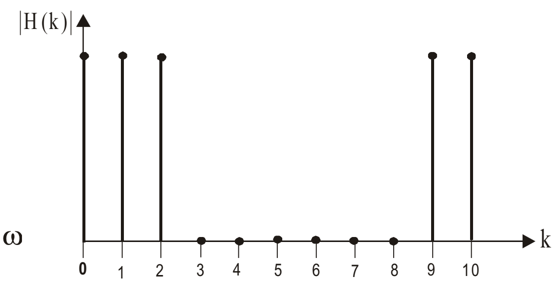


Fig 3 : Magnitude spectrum of H(k).

Based on the above discussions, the equation for DFT coefficients H(k) can be written as shown below.

$$H(k) = H_d(e^{j\omega})\Big|_{\omega=\omega_k} = e^{-j\alpha\omega_k} = e^{-j5\times\frac{2\pi k}{11}} \quad ; \text{ for } k = 0,\ 1,\ 2$$

$$= 0 \quad ; \text{ for } k = 3 \text{ to } 8$$

$$= e^{-j\alpha\omega_k} = e^{-j5\times\frac{2\pi k}{11}} \quad ; \text{ for } k = 9,\ 10$$

The samples of impulse response, h(n) are given by,

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \text{Re}\left[H(k)\, e^{\frac{j2\pi nk}{N}}\right]\right] \qquad \boxed{\text{Using equation(6.76).}}$$

$$= \frac{1}{11}\left[H(0) + 2\sum_{k=1}^{5} \text{Re}\left[H(k)\, e^{\frac{j2\pi nk}{11}}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{2} \text{Re}\left[e^{-j5\times\frac{2\pi k}{11}}\, e^{\frac{j2\pi nk}{11}}\right]\right] \qquad \boxed{H(0) = 1}$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{2} \text{Re}\left[e^{\frac{j2\pi k}{11}(n-5)}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\,\text{Re}\left[e^{\frac{j2\pi}{11}(n-5)}\right] + 2\,\text{Re}\left[e^{\frac{j4\pi}{11}(n-5)}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\cos\frac{2\pi(n-5)}{11} + 2\cos\frac{4\pi(n-5)}{11}\right] \qquad \boxed{\begin{array}{l} e^{j\theta} = \cos\theta + j\sin\theta \\ \therefore\ \text{Re}[e^{j\theta}] = \cos\theta \end{array}}$$

Here, N = 11, \ N – 1 = 10, $\frac{N-1}{2} = 5$.

Hence calculate h(n) for n = 0 to 10.

Since, h(n) satisfies the symmetry condition h(N – 1 – n) = h(n) with centre of symmetry at (N – 1)/2, calculate h(n) for n = 0 to to 5.

$$\text{When } n = 0\ ;\ h(0) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(0-5)}{11} + 2\cos\frac{4\pi(0-5)}{11}\right] = 0.0694$$

$$\text{When } n = 1\ ;\ h(1) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(1-5)}{11} + 2\cos\frac{4\pi(1-5)}{11}\right] = -0.054$$

$$\text{When } n = 2\ ;\ h(2) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(2-5)}{11} + 2\cos\frac{4\pi(2-5)}{11}\right] = -0.1094$$

$$\text{When } n = 3\ ;\ h(3) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(3-5)}{11} + 2\cos\frac{4\pi(3-5)}{11}\right] = 0.0474$$

$$\text{When } n = 4\ ;\ h(4) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(4-5)}{11} + 2\cos\frac{4\pi(4-5)}{11}\right] = 0.3194$$

$$\text{When } n = 5\ ;\ h(5) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(5-5)}{11} + 2\cos\frac{4\pi(5-5)}{11}\right] = 0.4545$$

When $n = 6$; $h(6) = h(11 - 1 - 6) = h(4) = 0.3194$

When $n = 7$; $h(7) = h(11 - 1 - 7) = h(3) = 0.0474$

When $n = 8$; $h(8) = h(11 - 1 - 8) = h(2) = -0.1094$

When $n = 9$; $h(9) = h(11 - 1 - 9) = h(1) = -0.054$

When $n = 10$; $h(10) = h(11 - 1 - 10) = h(0) = 0.0694$

Using symmetry condition $h(N - 1 - n) = h(n)$

The transfer function H(z) of the filter is given by $\mathcal{Z}$-transform of h(n).

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{10} h(n)\, z^{-n}$$

$$= h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4} + h(5)\, z^{-5}$$
$$+ h(6)\, z^{-6} + h(7)\, z^{-7} + h(8)\, z^{-8} + h(9)\, z^{-9} + h(10)\, z^{-10}$$

Using symmetry condition $h(N - 1 - n) = h(n)$

$$= h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4} + h(5)\, z^{-5}$$
$$+ h(4)\, z^{-6} + h(3)\, z^{-7} + h(2)\, z^{-8} + h(1)\, z^{-9} + h(0)\, z^{-10}$$

$$= h(0)\left[1 + z^{-10}\right] + h(1)\left[z^{-1} + z^{-9}\right] + h(2)\left[z^{-2} + z^{-8}\right] + h(3)\left[z^{-3} + z^{-7}\right]$$
$$+ h(4)\left[z^{-4} + z^{-6}\right] + h(5)\, z^{-5}$$

$$= 0.0694\left[1 + z^{-10}\right] - 0.054\left[z^{-1} + z^{-9}\right] - 0.1094\left[z^{-2} + z^{-8}\right] + 0.0474\left[z^{-3} + z^{-7}\right]$$
$$+ 0.3194\left[z^{-4} + z^{-6}\right] + 0.4545\, z^{-5}$$

Structure

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)} = 0.0694\left[1 + z^{-10}\right] - 0.054\left[z^{-1} + z^{-9}\right] - 0.1094\left[z^{-2} + z^{-8}\right]$$
$$+ 0.0474\left[z^{-3} + z^{-7}\right] + 0.3194\left[z^{-4} + z^{-6}\right] + 0.4545\, z^{-5}$$

$$\therefore\ Y(z) = 0.0694\left[X(z) + z^{-10}\, X(z)\right] - 0.054\left[z^{-1}X(z) + z^{-9}\, X(z)\right] - 0.1094\left[z^{-2}X(z) + z^{-8}\, X(z)\right]$$
$$+ 0.0474\left[z^{-3}\, X(z) + z^{-7}\, X(z)\right] + 0.3194\left[z^{-4}\, X(z) + z^{-6}\, X(z)\right] + 0.4545\, z^{-5}\, X(z)$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

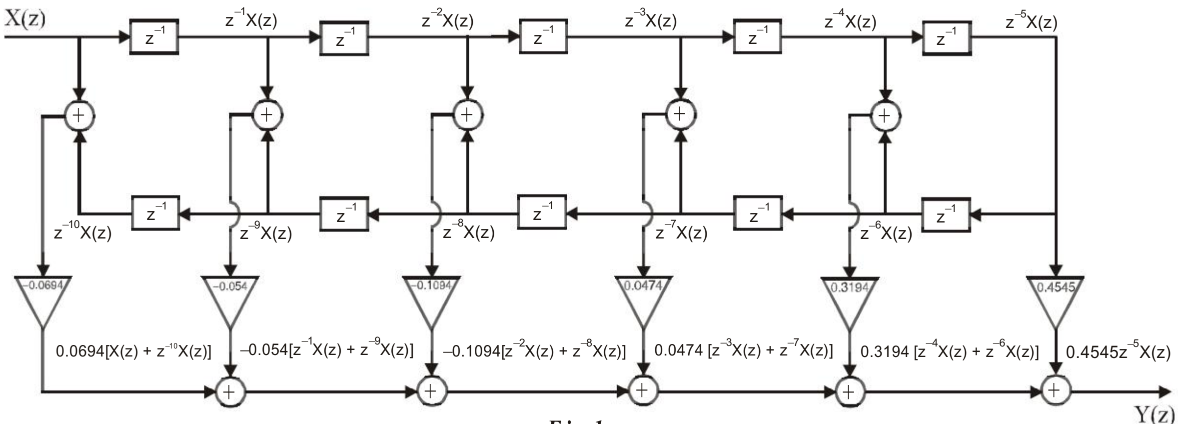


Fig 1.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at (N – 1)/2 the magnitude response $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where, } A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n \qquad \boxed{\text{Refer table 6.2 case (i)}}$$

$$\therefore\; A(\omega) = h(5) + \sum_{n=1}^{5} 2h(5-n)\cos\omega n$$

$$= h(5) + 2h(4)\cos\omega + 2h(3)\cos 2\omega + 2h(2)\cos 3\omega + 2h(1)\cos 4\omega + 2h(0)\cos 5\omega$$

$$= 0.4545 + 2 \times 0.3194\cos\omega + 2 \times 0.0474\cos 2\omega + 2 \times 0.1094\cos 3\omega + 2 \times -0.054\cos 4\omega + 2 \times 0.0694\cos 5\omega$$

$$= 0.4545 + 0.6388\cos\omega + 0.0948\cos 2\omega + 0.2188\cos 3\omega - 0.108\cos 4\omega + 0.1388\cos 5\omega$$

Using the above equation, the magnitude response A(w) and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

Table 1: A(w) and $|H(e^{jw})|$ for various values of w

w	A(w)	$\lvert H(e^{jw})\rvert = A(w)$
$\frac{0\times\pi}{16}$	1.0001	1.0001
$\frac{1\times\pi}{16}$	0.9874	0.9874
$\frac{2\times\pi}{16}$	0.9748	0.9748
$\frac{3\times\pi}{16}$	1.0048	1.0048
$\frac{4\times\pi}{16}$	1.0707	1.0707
$\frac{5\times\pi}{16}$	1.0911	1.0911
$\frac{6\times\pi}{16}$	0.9623	0.9623
$\frac{7\times\pi}{16}$	0.6521	0.6521
$\frac{8\times\pi}{16}$	0.2517	0.2517
$\frac{9\times\pi}{16}$	–0.0710	0.0710
$\frac{10\times\pi}{16}$	–0.1873	0.1873
$\frac{11\times\pi}{16}$	–0.1019	0.1019
$\frac{12\times\pi}{16}$	0.0542	0.0542
$\frac{13\times\pi}{16}$	0.1294	0.1294
$\frac{14\times\pi}{16}$	0.0682	0.0682
$\frac{15\times\pi}{16}$	–0.0559	0.0559
$\frac{16\times\pi}{16}$	–0.1175	0.1175

w	A(w)	$\lvert H(e^{jw})\rvert = A(w)$
$\frac{17\times\pi}{16}$	–0.0559	0.0559
$\frac{18\times\pi}{16}$	0.0682	0.0682
$\frac{19\times\pi}{16}$	0.1294	0.1294
$\frac{20\times\pi}{16}$	0.0542	0.0542
$\frac{21\times\pi}{16}$	–0.1019	0.1019
$\frac{22\times\pi}{16}$	–0.1873	0.1873
$\frac{23\times\pi}{16}$	–0.0710	0.0710
$\frac{24\times\pi}{16}$	0.2517	0.2517
$\frac{25\times\pi}{16}$	0.6521	0.6521
$\frac{26\times\pi}{16}$	0.9623	0.9623
$\frac{27\times\pi}{16}$	1.0911	1.0911
$\frac{28\times\pi}{16}$	1.0707	1.0707
$\frac{29\times\pi}{16}$	1.0048	1.0048
$\frac{30\times\pi}{16}$	0.9748	0.9748
$\frac{31\times\pi}{16}$	0.9874	0.9874
$\frac{32\times\pi}{16}$	1.0001	1.0001

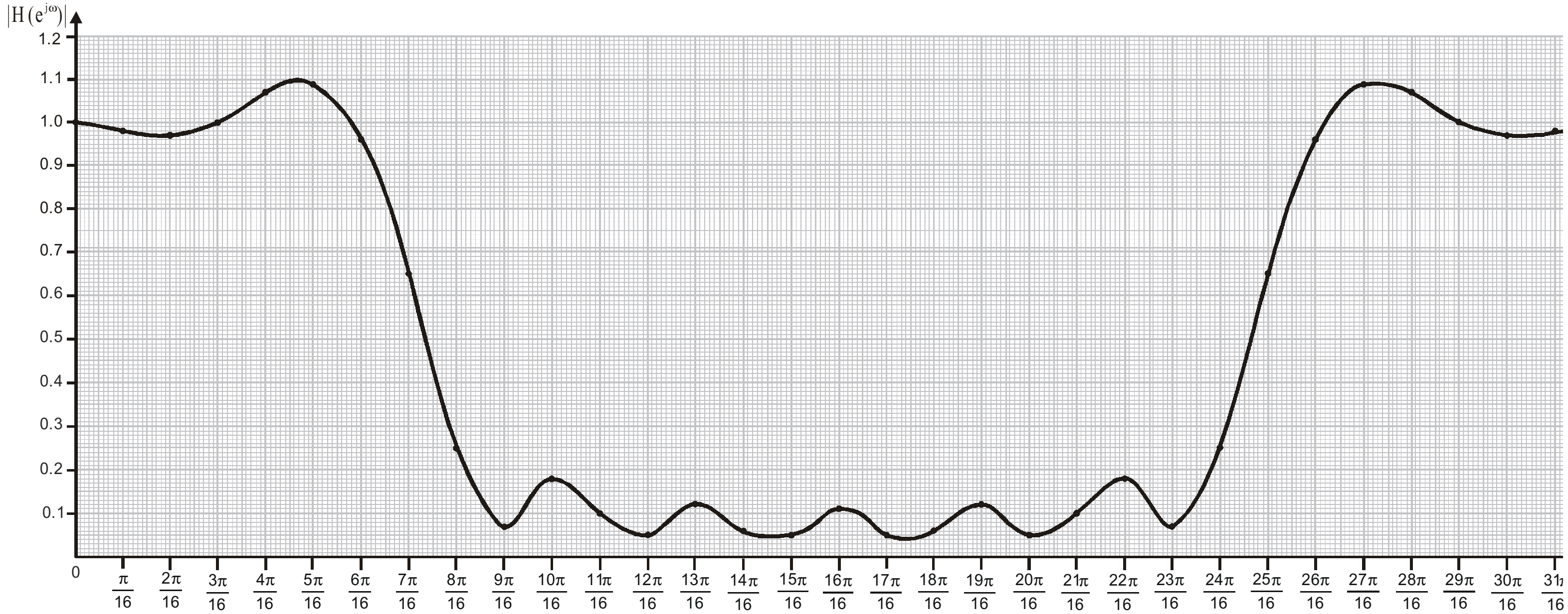


Fig 2 : *Magnitude response of FIR Lowpass filter.*

Alternate Method for Frequency Response

Frequency response, $H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}}$

$$\therefore\ H\left(e^{j\omega}\right) = 0.0694\left[1+z^{-10}\right] - 0.054\left[z^{-1}+z^{-9}\right] - 0.1094\left[z^{-2}+z^{-8}\right] + 0.0474\left[z^{-3}+z^{-7}\right] + 0.3194\left[z^{-4}+z^{-6}\right] + 0.4545z^{-5}\Big|_{z=e^{j\omega}}$$

$$= 0.0694\left[1+e^{-j10\omega}\right] - 0.054\left[e^{-j\omega}+e^{-j9\omega}\right] - 0.1094\left[e^{-j2\omega}+e^{-j8\omega}\right] + 0.0474\left[e^{-j3\omega}+e^{-j7\omega}\right] + 0.3194\left[e^{-j4\omega}+e^{-j6\omega}\right] + 0.4545\,e^{-j5\omega}$$

$$= 0.0694 + 0.0694\left[\cos 10\omega - j\sin 10\omega\right] - 0.054\left[\cos\omega - j\sin\omega + \cos 9\omega - j\sin 9\omega\right] - 0.1094\left[\cos 2\omega - j\sin 2\omega + \cos 8\omega - j\sin 8\omega\right] + 0.0474\left[\cos 3\omega - j\sin 3\omega + \cos 7\omega - j\sin 7\omega\right] + 0.3194\left[\cos 4\omega - j\sin 4\omega + \cos 6\omega - j\sin 6\omega\right] + 0.4545\left[\cos 5\omega - j\sin 5\omega\right]$$

$$= [\,0.0694 + 0.0694\cos 10\omega - 0.054\cos\omega - 0.054\cos 9\omega - 0.1094\cos 2\omega - 0.1094\cos 8\omega + 0.0494\cos 3\omega + 0.0474\cos 7\omega + 0.3194\cos 4\omega + 0.3194\cos 6\omega + 0.4545\cos 5\omega]$$
$$+ j[-0.0694\sin 10\omega + 0.054\sin\omega + 0.054\sin 9\omega + 0.1094\sin 2\omega + 0.1094\sin 8\omega - 0.0474\sin 3\omega - 0.0474\sin 7\omega - 0.3194\sin 4\omega - 0.3194\sin 6\omega - 0.4545\sin 5\omega]$$

Using the above equation the frequency response $H(e^{jw})$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 2. It is observed that the magnitude response obtained in both the methods are same.

Table 2: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	1.0001 + j0	1.0001
$\frac{1\times\pi}{16}$	0.5486 − j0.821	0.9874
$\frac{2\times\pi}{16}$	−0.373 − j0.9006	0.9747
$\frac{3\times\pi}{16}$	−0.9855 − j0.1959	1.0047
$\frac{4\times\pi}{16}$	−0.757 + j0.757	1.0705
$\frac{5\times\pi}{16}$	0.2128 + j1.0701	1.0910
$\frac{6\times\pi}{16}$	0.889 + j0.3681	0.9621
$\frac{7\times\pi}{16}$	0.5421 − j0.3622	0.6519
$\frac{8\times\pi}{16}$	0 − j0.2517	0.2517
$\frac{9\times\pi}{16}$	0.0591 + j0.0394	0.0710
$\frac{10\times\pi}{16}$	0.1731 − j0.0717	0.1873
$\frac{11\times\pi}{16}$	0.0199 − j0.1	0.1019
$\frac{12\times\pi}{16}$	0.0383 + j0.0382	0.0540
$\frac{13\times\pi}{16}$	0.1269 − j0.0253	0.1293
$\frac{14\times\pi}{16}$	0.0261 − j0.063	0.0681
$\frac{15\times\pi}{16}$	0.0311 + j0.0466	0.0560
$\frac{16\times\pi}{16}$	0.1175 + j0	0.1175

w	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{17\times\pi}{16}$	0.0311 − j0.0466	0.0560
$\frac{18\times\pi}{16}$	0.0261 + j0.063	0.0681
$\frac{19\times\pi}{16}$	0.1269 + j0.0253	0.1293
$\frac{20\times\pi}{16}$	0.0383 − j0.0382	0.0540
$\frac{21\times\pi}{16}$	0.0199 + j0.1	0.1019
$\frac{22\times\pi}{16}$	0.1731 + j0.0717	0.1873
$\frac{23\times\pi}{16}$	0.0591 − j0.0394	0.0710
$\frac{24\times\pi}{16}$	0 + j0.2517	0.2517
$\frac{25\times\pi}{16}$	0.5421 + j0.3622	0.6519
$\frac{26\times\pi}{16}$	0.889 − j0.3681	0.9621
$\frac{27\times\pi}{16}$	0.2128 − j1.0701	1.0910
$\frac{28\times\pi}{16}$	−0.757 − j0.757	1.0705
$\frac{29\times\pi}{16}$	−0.9855 + j0.1959	1.0047
$\frac{30\times\pi}{16}$	−0.373 + j0.9006	0.9747
$\frac{31\times\pi}{16}$	0.5486 + j0.821	0.9874
$\frac{32\times\pi}{16}$	1.0001 + j0	1.0001

6.11 Summary of Important Concepts

1. The filters are frequency selective devices.
2. The specification of a digital filter is the desired frequency response $H_d(e^{jw})$.
3. The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{jw})$.
4. The desired impulse response $h_d(n)$ is an infinite duration signal.
5. The filters designed by using finite samples of impulse response are called FIR (Finite Impulse Response) filters.
6. The transfer function H(z) of the filter is obtained by taking $\mathcal{Z}$-transform of impulse response.
7. An LTI system will behave as frequency selective device or filter.
8. The phase delay and group delay are defined to examine the linearity of phase characteristics of frequency response of FIR filter.
9. The phase delay, t_p is defined as, $t_p = -q(w)/w$, where $q(w) = \text{Ð}\, H(e^{jw})$.
10. The group delay, t_g is defined as, $t_g = -d\,q(w)/dw$, where $q(w) = \text{Ð}\, H(e^{jw})$.
11. The linear phase FIR filters has a constant delay within the desired frequency range.
12. The linear phase FIR filters with constant group and phase delay will have symmetric impulse response with symmetry condition $h(N - 1 - n) = h(n)$ and with centre of symmetry at a, where $a = (N - 1)/2$.
13. The linear phase FIR filters with only constant group delay will have antisymmetric impulse response with symmetry condition $h(N - 1 - n) = -h(n)$ and with centre of symmetry at a, where $a = (N - 1)/2$.
14. The frequency response of a digital filter is periodic with period equal to $2p$.
15. The samples of impulse response of digital filter are Fourier coefficients in the Fourier series representation of the frequency response of the filter.
16. The abrupt truncation of impulse response results in oscillations in the passband and stopband, and this effect is called Gibbs phenomenon.
17. The windows are finite duration sequences used to truncate and modify the impulse response of FIR filters.
18. The desirable features of a window are small width of main-lobe and side-lobes with very low magnitude (or large attenuation), in the frequency spectrum.
19. In the frequency spectrum, the width of main-lobe and peak side-lobe magnitude are characteristic constants of a particular window.
20. In windows, the width of the main-lobe can be reduced only by increasing the value of N.
21. In windows, except Kaiser window, there is no adjustable parameter to increase the side-lobe attenuation.
22. Kaiser has introduced a variable parameter "a" to modify the characteristics of window. With increase in value of "a", the width of main-lobe increases and side-lobe magnitude decreases. The width of main-lobe can be reduced by increasing the value of N.
23. In frequency sampling technique of FIR filter design, one period of ideal frequency response is sampled at N equal frequency intervals.
24. The samples obtained by sampling ideal frequency response are DFT coefficients.
25. The complex DFT coefficients obtained by sampling frequency response always exist as conjugate pairs.

6.12. Short Questions and Answers

Q6.1 ***How does an LTI system behave as a frequency selective filters?***

Let, x(n), h(n) and y(n) are input, impulse response and output of an LTI system.

Let, $\mathcal{F}\{x(n)\}=X(e^{jw})$, $\mathcal{F}\{h(n)\} = H(e^{jw})$ and $\mathcal{F}\{y(n)\}=Y(e^{jw})$.

By convolution property of Fourier transform, we can say that $Y(e^{jw}) = X(e^{jw})H(e^{jw})$, which implies that the input spectrum $X(e^{jw})$ is modified by the frequency response $H(e^{jw})$ to yield the output spectrum $Y(e^{jw})$. The $H(e^{jw})$ acts as a spectral shaping function to the different frequency components of the input signal. Hence the LTI systems can be considered as frequency selective filters.

Q6.2 ***How are phase distortion and delay distortion introduced?***

The phase distortion is introduced when the phase characteristics of a filter is not linear within the desired frequency band.

The delay distortion is introduced when the delay is not a constant within the desired frequency range.

Q6.3 ***What are FIR filters?***

The specifications of the desired filter will be given in terms of ideal frequency response $H_d(e^{jw})$. The impulse response $h_d(n)$ of desired filter can be obtained by inverse Fourier transform of $H_d(e^{jw})$, which consists of infinite samples. The filters designed by selecting finite number of samples of impulse response are called FIR filters.

Q6.4 ***Write the steps involved in FIR filter design.***

i. Choose the desired (ideal) frequency response $H_d(e^{jw})$.
ii. Take inverse fourier transform of $H_d(e^{jw})$ to get $h_d(n)$.
iii. Convert the infinite duration $h_d(n)$ to finite duration sequence h(n).
iv. Take $\mathcal{Z}$-transform of h(n) to get the transfer function H(z) of the FIR filter.

Q6.5 ***What are the advantages of FIR filters?***

i. Linear phase FIR filters can be easily designed.
ii. Efficient realizations of FIR filter exist as both recursive and nonrecursive structures.
iii. FIR filters realized nonrecursively are always stable.
iv. The roundoff noise can be made small in nonrecursive realization of FIR filters.

Q6.6 ***What are the disadvantages of FIR filters?***

i. The duration of impulse response should be large to realize sharp cutoff filters.
ii. The non-integral delay can lead to problems in some signal processing applications.

Q6.7 ***What is the necessary and sufficient condition for the linear phase characteristic of an FIR filter?***

The necessary and sufficient condition for the linear phase characteristic of a FIR filter is that the phase function should be a linear function of w, which in turn requires constant phase delay or constant phase and group delay.

Q6.8 ***Write the frequency response of linear phase LTI system with constant phase delay and constant group delay.***

$$\left.\begin{array}{l}\text{Frequency response of LTI}\\ \text{system with constant phase delay}\end{array}\right\} H(e^{j\omega}) = \pm|H(e^{j\omega})|e^{-j\alpha\omega}$$

where, a is a constant phase delay.

$$\left.\begin{array}{l}\text{Frequency response of LTI}\\ \text{system with constant group delay}\end{array}\right\} H(e^{j\omega}) = \pm|H(e^{j\omega})|e^{j(\beta-\alpha\omega)}$$

where, b is a constant group delay.

Q6.9 ***What are the conditions to be satisfied for constant phase delay in linear phase FIR filters ?***

The conditions for constant phase delay are,

Phase delay, $\alpha = \frac{N-1}{2}$ (i.e., phase delay is constant)

Impulse response, $h(n) = h(N-1-n)$ (i.e., impulse response is symmetric)

Q6.10 ***How is the constant group and phase delay achieved in linear phase FIR filters?***

$$\left.\begin{array}{l}\text{Frequency response of FIR filter with}\\ \text{constant group and phase delay}\end{array}\right\} H(e^{j\omega}) = \pm|H(e^{j\omega})|e^{j(\beta-\alpha\omega)}$$

The following conditions have to be satisfied to achieve constant group and phase delay.

Phase delay, $\alpha = \frac{N-1}{2}$ (i.e., phase delay is constant)

Group delay, $\beta = \pm\frac{\pi}{2}$ (i.e., group delay is constant)

Impulse response, $h(n) = -h(N-1-n)$ (i.e., impulse response is antisymmetric)

Q6.11. ***The frequency response of a digital filter is, $H(e^{jw}) = (0.7 + 0.6\cos w - 0.9\cos 2w)e^{-j7.5w}$***

Determine the phase delay and group delay.

<u>Solution</u>

Given that, $H(e^{jw}) = (0.7 + 0.6\cos w - 0.9\cos 2w)\, e^{-j7.5w}$ .....(1)

Let, $H(e^{j\omega}) = \left|H(e^{j\omega})\right| \angle H(e^{j\omega})$ .....(2)

On comparing equations (1) and (2) we get,

$$\angle H(e^{j\omega}) = -7.5\,\omega$$

Let, $\angle H(e^{j\omega}) = \theta(\omega) = -7.5\,\omega$

Phase delay, $\tau_p = -\dfrac{\theta(\omega)}{\omega} = -\dfrac{-7.5\,\omega}{\omega} = 7.5$

Group delay, $\tau_g = -\dfrac{d\theta(\omega)}{d\omega} = -\dfrac{d}{d\omega}(-7.5\omega) = 7.5$

Q6.12. ***The frequency response of a digital filter is, $H(e^{jw}) = (0.4 + 0.7\cos 2w - 0.5\cos 4w)e^{-j(0.3p + 4w)}$ Determine the phase delay and group delay.***

Solution

Given that, $H(e^{jw}) = (0.4 + 0.7\cos 2w - 0.5\cos 4w)\, e^{-j(0.3p + 4w)}$ .....(1)

Let, $H(e^{j\omega}) = \left|H(e^{j\omega})\right| \angle H(e^{j\omega})$ .....(2)

On comparing equations (1) and (2) we get,

$$\angle H(e^{j\omega}) = -(0.3\pi + 4\omega)$$

Let, $\angle H(e^{j\omega}) = \theta(\omega) = -(0.3\pi + 4\omega)$

Phase delay, $\tau_p = -\dfrac{\theta(\omega)}{\omega} = -\dfrac{-(0.3\pi + 4\omega)}{\omega} = \dfrac{0.3\pi}{\omega} + 4$

Group delay, $\tau_g = -\dfrac{d\theta(\omega)}{d\omega} = -\dfrac{d}{d\omega}\left(-(0.3\pi + 4\omega)\right) = 4$

Q6.13. ***What are the possible types of impulse response for linear phase FIR filters?***

There are six types of impulse response for linear phase FIR filters

(i). Symmetric impulse response and N is odd with centre of symmetry at (N – 1)/2.

(ii). Symmetric impulse response and N is even with centre of symmetry at (N – 1)/2.

(iii). Antisymmetric impulse response and N is odd with centre of antisymmetry at (N – 1)/2.

(iv). Antisymmetric impulse response and N is even with centre of antisymmetry at (N – 1)/2.

(v). Symmetric impulse response and N is odd with centre of symmetry at n = 0.

(vi). Antisymmetric impulse response and N is odd with centre of antisymmetry at n = 0.

Q6.14 ***Write the magnitude and phase function of FIR filter when impulse response is symmetric and N is odd.***

Magnitude function, $|H(e^{j\omega})| = \left| h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{N-1} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n \right|$

Phase function, $\angle H(e^{j\omega}) = \theta(\omega) = -\alpha\omega$

Q6.15 ***Write the magnitude and phase function of FIR filter when impulse response is symmetric and N is even.***

Magnitude function, $|H(e^{j\omega})| = \left| \sum_{n=1}^{N/2} 2h\left(\frac{N}{2} - n\right)\cos\left(\omega\left(n - \frac{1}{2}\right)\right) \right|$

Phase function, $\angle H(e^{j\omega}) = \theta(\omega) = -\alpha\omega$

Q6.16 ***Write the magnitude and phase function of FIR filter when impulse response is antisymmetric and N is odd.***

Magnitude function, $|H(e^{j\omega})| = \left| \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\sin\omega n \right|$

Phase function, $\angle H(e^{j\omega}) = \theta(\omega) = \beta - \alpha\omega$

Q6.17. Write the magnitude and phase function of FIR filter when impulse response is antisymmetric and N is even.

Magnitude function, $|H(e^{j\omega})| = \left|\sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right)\sin\left(\omega\left(n - \frac{1}{2}\right)\right)\right|$

Phase function, $\angle H(e^{j\omega}) = \theta(\omega) = \beta - \alpha\omega$

Q6.18 List the well known design techniques for linear phase FIR filter.

There are three well known method of design techniques for linear phase FIR filters. They are,

i. Fourier series method and window method.

ii. Frequency sampling method.

iii. Optimal filter design methods.

Q6.19 Write the two concepts that leads to the Fourier series or window method of designing FIR filters.

The following two concepts leads to the design of FIR filters by Fourier series method.

i. The frequency response of a digital filter is periodic with period equal to 2p.

ii. Any periodic function can be expressed as a linear combination of complex exponentials.

Q6.20. Write the procedure for designing FIR filter by Fourier series method.

i. Choose the desired frequency response $H_d(e^{jw})$ of the filter.

ii. Evaluate the Fourier series coefficients from frequency response, which gives the desired impulse response $h_d(n)$.

iii. Truncate the infinite sequence $h_d(n)$ to a finite N-point sequence h(n), for n = –(N – 1)/2 to +(N – 1)/2

iv. Take $\mathcal{Z}$-transform of h(n) to get a noncausal filter transfer function H(z).

v. Multiply H(z) by $z^{-(N-1)/2}$ to convert the noncausal transfer function to a realizable causal FIR filter transfer function.

Q6.21. How is causality brought-in in the Fourier series method of filter design?

The transfer function obtained in Fourier series method of filter design will represent an unrealizable noncausal system. If we multiply the noncausal transfer function by $z^{-(N-1)/2}$ then it will be converted to a transfer function of causal system.

Q6.22. What is Gibbs phenomenon (or Gibbs oscillation)?

In FIR filter design by Fourier series method, the infinite duration impulse response is truncated to finite duration impulse response. The abrupt truncation of impulse response introduces oscillations in the passband and stopband. This effect is known as Gibbs phenomenon (or Gibbs oscillations).

Q6.23 Write the procedure for designing FIR filter using windows.

i. Choose the desired frequency response of the filter $H_d(e^{jw})$.

ii. Take inverse fourier transform of $H_d(e^{jw})$ to obtain the desired impulse response $h_d(n)$.

iii. Choose a window sequence w(n) and multiply $h_d(n)$ by w(n) to convert the infinite duration impulse response to finite duration impulse response h(n).

iv. The transfer function H(z) of the filter is obtained by taking $\mathcal{Z}$-transform of h(n).

Q6.24 ***What are the desirable characteristics of the frequency response of window function ?***

The desirable characteristics of the frequency response of window function are,

i. The width of the mainlobe should be small and it should contain as much of the total energy as possible.

ii. The sidelobes should decrease in energy rapidly as w tends to p.

Q6.25 ***Write the procedure for FIR filter design by frequency sampling method.***

i. Choose the desired frequency response $H_d(e^{jw})$.

ii. Take N-samples of $H_d(e^{jw})$ to generate the sequence H(k).

iii. Take inverse DFT of H(k) to get the impulse response h(n).

iv. The transfer function H(z) of the filter is obtained by taking $\mathcal{Z}$-transform of impulse response.

Q6.26 ***What is the drawback in FIR filter design using windows and frequency sampling method? How is it overcome?***

The FIR filter design by window and frequency sampling method does not have precise control over the critical frequencies such as w_p and w_s.

This drawback can be overcome by designing FIR filter using Chebyshev approximation technique. In this technique an error function is used to approximate the ideal frequency response, in order to satisfy the desired specifications.

Q6.27 ***What is meant by optimum equiripple design criterion? Why it is followed?***

In FIR filter design by Chebyshev approximation technique, the weighted approximation error between the desired frequency response and the actual frequency response is spread evenly across the passband and stopband. The resulting filter will have ripples in both the passband and stopband. This concept of design is called optimum equiripple design criterion.

The optimum equiripple criterion is used to design FIR filter in order to satisfy the specifications of passband and stopband.

Q6.28 ***Write the characteristic features of rectangular window.***

i. The main-lobe width is equal to $4p/N$

ii. The maximum side-lobe magnitude is –13dB.

iii. The side-lobe magnitude does not decrease significantly with increasing w.

Q6.29 ***List the features of FIR filter design using rectangular window.***

i. The width of the transition region is related to the width of the main-lobe of window spectrum.

ii. Gibbs oscillations are noticed in the passband and stopband.

iii. The attenuation in the stopband is constant and cannot be varied.

Q6.30 ***How can the transition width of the FIR filter can be reduced in design using windows?***

In FIR filters designed using windows, the width of the transition region is related to the width of the main-lobe in window spectrum. If the main-lobe width is narrow then the transition region in FIR filter will be small. In general, the width of main-lobe is xp/N, where $x = 4$ or 8 or 12 and N is the length of the window sequence used for designing the filter. Hence the width of main-lobe can be reduced by increasing the value of N, which in turn reduces the width of the transition region in the FIR filter.

Q6.31 ***Why are Gibbs oscillations are developed in rectangular window and how can it be eliminated or reduced?***

The Gibbs oscillations in rectangular window are due to the sharp transitions from 1 to 0 at the edges of window sequence.

These oscillation can be eliminated or reduced by replacing the sharp transition by gradual transition. This is the motivation for development of triangular and cosine windows.

Q6.32 ***List the characteristics of FIR filters designed using windows.***

i. The width of the transition band depends on the type of window.

ii. The width of the transition band can be made narrow by increasing the value of N where N is the length of the window sequence.

iii. The attenuation in the stopband is fixed for a given window, except in case of Kaiser window where it is variable.

Q6.33 ***Write the characteristic features of triangular window.***

i. The main-lobe width is equal to 8p/N.

ii. The maximum side-lobe magnitude is –25dB.

iii. The sidelobe magnitude slightly decreases with increasing w.

Q6.34 ***Why is triangular window is not a good choice for designing FIR filters ?***

In FIR filters designed using triangular window the transition from passband to stopband is not sharp and the attenuation in stopband is less when compared to filters designed with rectangular window. For the above two reasons the triangular window is not a good choice.

Q6.35 ***Write the frequency response of Hanning window.***

$$\left.\begin{array}{l}\text{Frequency response}\\ \text{of Hanning window}\end{array}\right\} W_C(e^{j\omega}) = 0.5\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + 0.25\frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} + 0.25\frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

Q6.36 ***Write the frequency response of Hamming window.***

$$\left.\begin{array}{l}\text{Frequency response}\\ \text{of Hamming window}\end{array}\right\} W_H(e^{j\omega}) = 0.54\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + 0.23\frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} + 0.23\frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

Q6.37 ***Give the equation for Hanning window function.***

$$\begin{aligned}\text{Hanning window, } w_C(n) &= 0.5 + 0.5\cos\left(\frac{2\pi n}{N-1}\right); \quad \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}\\ &= 0 \qquad ; \quad \text{other } n\end{aligned}$$

Alternatively,

$$\begin{aligned}\text{Hanning window, } w_C(n) &= 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right); \quad \text{for } n = 0 \text{ to } N-1\\ &= 0 \qquad ; \quad \text{other } n\end{aligned}$$

Q6.38 ***List the features of Hanning window spectrum.***

i. The main-lobe width is equal to 8p/N.

ii. The maximum side-lobe magnitude is –31dB.

iii. The sidelobe magnitude decreases with increasing w.

Q6.39 *Compare the rectangular window and Hanning window.*

Rectangular window	Hanning window
i. The width of main-lobe in window spectrum is 4p/N. ii. The maximum side-lobe magnitude in window spectrum is –13dB. iii. In window spectrum the side-lobe magnitude slightly decreases with increasing w. iv. In FIR filter designed using rectangular window, the minimum stopband attenuation is 22 dB.	i. The width of main-lobe in window spectrum is 8p/N. ii. The maximum side-lobe magnitude in window spectrum is –31dB. iii. In window spectrum the side-lobe magnitude decreases with increasing w. iv. In FIR filter designed using Hanning window, the minimum stopband attenuation is 44 dB.

Q6.40 *Write the equation for Hamming window function.*

$$\text{Hamming window, } w_H(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right); \quad \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } +\frac{N-1}{2}$$
$$= 0 \quad ; \quad \text{other } n$$

Alternatively,

$$\text{Hamming window, } w_H(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right); \quad \text{for } n = 0 \text{ to } N-1$$
$$= 0 \quad ; \quad \text{other } n$$

Q6.41 *Compare the rectangular window and Hamming window.*

Rectangular window	Hamming window
i. The width of main-lobe in window spectrum is 4p/N. ii. The maximum side-lobe magnitude in window spectrum is –13dB. iii. In window spectrum the side-lobe magnitude slightly decreases with increasing w. iv. In FIR filter designed using rectangular window, the minimum stopband attenuation is 22 dB.	i. The width of main-lobe in window spectrum is 8p/N. ii. The maximum side-lobe magnitude in window spectrum is –41dB. iii. In window spectrum the side-lobe magnitude remains constant. iv. In FIR filter designed using Hamming window, the minimum stopband attenuation is 51dB.

Q6.42 *List the features of Hamming window spectrum.*

i. The main-lobe width is equal to 8p/N.

ii. The maximum side-lobe magnitude is –41dB.

iii. The side-lobe magnitude remains constant for increasing w.

Q6.43 Compare the Hanning and Hamming window.

Hanning window	Hamming window
i. The width of main-lobe in window spectrum is 8p/N.	i. The width of main-lobe in window spectrum is 8p/N.
ii. The maximum side-lobe magnitude in window spectrum is –31dB.	ii. The maximum side-lobe magnitude in window spectrum is –41dB.
iii. In window spectrum the side-lobe magnitude decreases with increasing w.	iii. In window spectrum the side-lobe magnitude remains constant. Here the increased side-lobe attenuation is achieved at the expense of constant attenuation at high frequencies.
iv. In FIR filter designed using Hanning window, the minimum stopband attenuation is 44 dB.	iv. In FIR filter designed using Hamming window, the minimum stopband attenuation is 51 dB.

Q6.44 Write the equation for Blackman window sequence.

$$\text{Blackman window, } w_B(n) = 0.42 + 0.5\ \cos\frac{2\pi n}{N-1} + 0.08\ \cos\frac{4\pi n}{N-1} \quad ; \text{ for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$$
$$= 0 \quad ; \text{ other } n$$

Alternatively,

$$\text{Blackman window, } w_B(n) = 0.42 - 0.5\ \cos\frac{2\pi n}{N-1} + 0.08\ \cos\frac{4\pi n}{N-1} \quad ; \text{ for } n = 0 \text{ to } N-1$$
$$= 0 \quad ; \text{ other } n$$

Q6.45 Compare the Hamming and Blackman window.

Hamming window	Blackman window
i. The width of main-lobe in window spectrum is 8p/N.	i. The width of main-lobe in window spectrum is 12p/N.
ii. The maximum side-lobe magnitude in window spectrum is –41dB.	ii. The maximum side-lobe magnitude in window spectrum is –58dB.
iii. The higher value of side-lobe attenuation is achieved at the expense of constant attenuation at high frequencies.	iii. The higher value of side-lobe attenuation is achieved at the expense of increased main-lobe width.
iv. In window spectrum the side-lobe magnitude remains constant with increasing w.	iv. In window spectrum the side-lobe magnitude decreases rapidly with increasing w.
v. In FIR filter designed using Hamming window, the minimum stopband attenuation is 51dB.	v. In FIR filter designed using Blackman window, the minimum stopband attenuation is 78 dB.

Q6.46 List the features of Blackman window spectrum.

i. The main-lobe width is 12p/N.

ii. The maximum side-lobe magnitude is –58 dB.

iii. The side-lobe magnitude decreases with increasing w.

iv. The side-lobe attenuation in Blackman window is the highest among windows, which is achieved at the expense of increased main-lobe width. However, the main-lobe width can be reduced by increasing the value of N.

Q6.47 What is the mathematical problem involved in the design of window function?

The mathematical problem involved in designing window function (or sequence) is that of finding a time limited function whose Fourier transform best approximates a bandlimited function. The approximation should be such that the maximum energy is confined to main-lobe for a given peak side-lobe amplitude.

Q6.48 Write the expression for Kaiser window function.

$$\text{Kaiser window function, } w_K(n) = \frac{I_0(\beta_1)}{I_0(a)} \quad ; \text{ for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$$

$$= 0 \qquad ; \text{ other } n$$

$$\text{where, } \beta_1 = a\left[1-\left(\frac{2n}{N-1}\right)^2\right]^{0.5}$$

Alternatively,

$$\text{Kaiser window function, } w_K(n) = \frac{I_0(\beta_2)}{I_0(a_2)} \quad ; \text{ for } n = 0 \text{ to } N-1$$

$$= 0 \qquad ; \text{ other } n$$

$$\text{where, } \beta_2 = a\left[\left(\frac{N-1}{2}\right)^2 - \left(n-\frac{N-1}{2}\right)^2\right]^{0.5} \quad ; \quad a_2 = a\frac{N-1}{2}$$

$$I_0(x) = 1 + \frac{0.25x^2}{(1!)^2} + \frac{\left(0.25x^2\right)^2}{(2!)^2} + \frac{\left(0.25x^2\right)^3}{(3!)^2} + \,.....$$

The series of $I_0(x)$ is used to compute $I_0(b_1)$, $I_0(a)$, $I_0(b_2)$, $I_0(a_2)$ and can be computed for any desired accuracy. Usually 25 terms of the series are sufficient for most practical purposes.

Q6.49 List the desirable features of Kaiser window spectrum.

i. The width of the main-lobe and the peak side-lobe are variable.

ii. The parameter "a" in the Kaiser window function, is an independent variable that can be varied to control the side-lobe levels with respect to main-lobe peak.

iii. The width of the main-lobe in the window spectrum (and so the transition region in the FIR filter) can be varied by varying the length N of the window sequence.

Q6.50 Compare the Hamming window and Kaiser windows.

Hamming window	Kaiser window
i. The width of main-lobe in window spectrum is 8p/N.	i. The width of main-lobe in window spectrum depends on the values of "a" and N.
ii. The maximum side-lobe magnitude in window spectrum is fixed at –41 dB.	ii. The maximum side-lobe magnitude with respect to peak of main-lobe is variable using the parameter "a".

6.14 Exercises

I. Fill in the blanks with appropriate words

1. The _______ is due to nonlinear phase characteristics of the filter.
2. The filters designed by using finite number of samples of impulse response are called _______.
3. In FIR filters _______ function is a linear function of w.
4. In FIR filters with constant phase delay the impulse response is _______.
5. In FIR filters with constant group and phase delay the impulse response is _______.
6. In linear phase filters when impulse response is antisymmetric and N is odd, the magnitude function is _______.
7. In linear phase filters when impulse response is antisymmetric and N is even, the magnitude function is _______.
8. The oscillations developed due to truncation of impulse response is called _______.
9. The linear phase FIR filter design by Chebyshev approximation technique is called _______.
10. In Fourier series method of FIR filter design the causality is brought by multiplying the transfer function with _______.
11. The width of the main-lobe in window spectrum can be reduced by increasing the length of _______.
12. The width of _______ region of FIR filter directly depends on the width of main-lobe in window spectrum.
13. The _______ can be eliminated by replacing the sharp transitions in window sequence by gradual transition.
14. In rectangular window the width of main-lobe is equal to _______.
15. In _______ window spectrum the width of main-lobe is double that of rectangular window for same value of N.
16. In _______ window spectrum the width of main-lobe is triple that of rectangular window for same value of N.
17. In _______ window spectrum the side-lobe magnitude is variable.
18. The _______ window spectrum has the highest attenuation for side-lobes.
19. In _______ window spectrum the increase in side-lobe attenuation is achieved at the expense of constant attenuation at high frequencies.
20. In _______ window spectrum the higher side-lobe attenuation is achieved at the expense of increased main-lobe width.

Answers

1.	Phase distortion	8.	Gibbs oscillation	15.	Hamming
2.	FIR filters	9.	optimum equiripple design	16.	Blackman
3.	phase	10.	$z^{-(N-1)/2}$	17.	Kaiser
4.	symmetric	11.	window sequence.	18.	Blackman
5.	antisymmetric	12.	transition	19.	Hamming
6.	antisymmetric	13.	Gibbs oscillation	20.	Blackman
7.	symmetric	14.	4p/N		

II. State whether the following statements are True/False

1. The filter output is a delayed and amplitude scaled version of the input signal.
2. The filter that causes phase distortion has a variable frequency delay and the filter with linear phase has a constant phase delay.
3. The ideal filters are noncausal.
4. FIR filters realized nonrecursively are always unstable.
5. In FIR filters the impulse response should have large number of samples to realize sharp cutoff filters.
6. In linear phase filters when impulse response is symmetric with odd number of samples, the magnitude function will be antisymmetric.
7. In linear phase filters when impulse response is symmetric with even number of samples, the magnitude function will be symmetric.
8. The frequency response of a digital filter is periodic with period equal to sampling frequency.
9. The truncation of impulse response result in oscillations in passband and stopband.
10. In a good window the width of main-lobe in its spectrum should be large in order to have maximum energy.
11. In a good window the side-lobes should increase in energy rapidly as ω tends to π.
12. The frequency response of digital filter is periodic with period equal to sampling frequency.
13. The transfer function obtained by taking $\mathcal{Z}$-transform of the truncated Fourier coefficients is causal.
14. The Gibbs oscillations can be reduced by multiplying the impulse response by an appropriate window function.
15. The FIR filters designed using windows and frequency sampling method will not have control over ω_p and ω_s.
16. The width of main-lobe in window spectrum increases with increase in length of window sequence.
17. The transition width of FIR filter can be varied only when it is designed with kaiser window.
18. In windows, generally the relative peak of side-lobe with respect to main-lobe is fixed.
19. In kaiser window the peak side-lobe is variable but the width of main-lobe is fixed.
20. In hamming window spectrum the magnitude of side-lobes remains constant with increasing ω.

Answers

1.	True	5.	True	9.	True	13.	False	17.	False
2.	True	6.	False	10.	False	14.	True	18.	True
3.	True	7.	False	11.	False	15.	True	19.	False
4.	False	8.	True	12.	True	16.	False	20.	True

III. Choose the right answer for the following questions

1. The frequency response of a digital filter is periodic in the range

a) $0 < \omega < 2\pi$ **b)** $-\pi < \omega < \pi$

c) $0 < \omega < \pi$ **d)** $0 \le \omega \le 2\pi$ or $-\pi \le \omega \le \pi$

2. The characteristics of ideal linear phase FIR filter are,

a) $|H(e^{j\omega})|$ = constant and $\angle H(e^{j\omega}) = 1/\omega$.

b) $|H(e^{j\omega})|$ = constant and $\angle H(e^{j\omega}) = -\alpha\omega$.

c) $|H(e^{j\omega})| = -\alpha\omega$ and $\angle H(e^{j\omega})$ = constant.

d) $|H(e^{j\omega})| = 1/\omega$ and $\angle H(e^{j\omega})$ = constant.

3. ***If q(w) is the phase function of FIR filter then group delay and phase delay of FIR filters are defined respectively as,***

a) $\dfrac{-d\theta(\omega)}{d\omega}$, $\dfrac{-\theta(\omega)}{\omega}$ **b)** $\dfrac{-d\theta(\omega)}{d\omega}$, $-\omega\,\theta(\omega)$

c) $\dfrac{\theta(\omega)}{\omega}$, $\dfrac{d\theta(\omega)}{d\omega}$ **d)** $-\omega\,\theta(\omega)$, $\dfrac{d\theta(\omega)}{d\omega}$

---

4. ***The frequency response of FIR filter with constant group delay will be in the form,***

a) $H(e^{jw}) = C\ e^{-jaw}$ **b)** $H(e^{jw}) = C\ e^{jaw}$

c) $H(e^{jw}) = C\ e^{-j(b-aw)}$ **d)** $H(e^{jw}) = C\ e^{j(b-aw)}$

---

5. ***In FIR filters the Gibbs oscillations are due to***

a) non-linear magnitude characteristics.

b) non-linear phase characteristics.

c) Sharp transition from pass-band to stop-band.

d) Gradual transition from pass-band to stop-band.

---

6. ***If w_C is the cutoff frequency of lowpass filter, then the response lies only in the range of,***

a) $-w_C$ £ w £ p **b)** $-w_C$ £ w £ w_C

c) –p £ w £ $-w_C$ **d)** $-w_C$ £ w £ p

---

7. ***If w_C is the cutoff frequency of highpass filter, then the response lies only in the range of,***

a) w_C £ w £ p and –p £ w £ 0

b) –p £ w £ $-w_C$ and w_C £ w £ p

c) $-w_C$ £ w £ –p and –w £ w £ w_C

d) $-w_C$ £ w £ 0 and 0 £ w £ w_C

---

8. ***If w_{C1} and w_{C2} are the cutoff frequencies of bandpass filter, then the response lies only in the range of,***

a) $-w_{C2}$ £ w £ 0 and $+w_{C2}$ £ w £ p

b) –p £ w £ $-w_{C2}$ and $-w_{C1}$ £ w £ 0

c) $-w_{C2}$ £ w £ $-w_{C1}$ and w_{C1} £ w £ w_{C2}

d) w_{C1} £ w £ w_{C2} and w_{C2} £ w £ p

---

9. ***If w_{C1} and w_{C2} are the cutoff frequencies of bandstop filter, then the response lies only in the range of,***

a) $-w_{C2}$ £ w £ $-w_{C1}$ and w_{C1} £ w £ w_{C2} and w_{C2} £ w £ p

b) –p £ w £ $-w_{C2}$ and $-w_{C1}$ £ w £ 0 and 0 £ w £ w_{C1}

c) $-w_{C2}$ £ w £ 0 and w_{C1} £ w £ w_{C2} and w_{C2} £ w £ p

d) –p £ w £ $-w_{C2}$ and $-w_{C1}$ £ w £ w_{C1} and w_{C2} £ w £ p

---

10. Symmetric impulse response having even number of samples can be used to design

a) lowpass and highpass filters.

b) lowpass and bandpass filters.

c) lowpass and bandstop filters.

d) only lowpass filters.

---

11. Raised cosine windows also called generalized

a) Hamming window. **b)** Hanning window.

c) Rectangular window. **d)** Blackman window.

---

12. The symmetric impulse response having odd number of samples has,

a) Symmetric magnitude function. **b)** Antisymmetric magnitude function.

c) Both a and b. **d)** None of these.

---

13. The symmetric impulse response having even number of samples cannot be used to design,

a) Lowpass filter. **b)** Bandstop filter.

c) Highpass filter. **d)** Bandpass filter.

---

14. The width of the main-lobe in rectangular window spectrum is,

a) $\frac{4\pi}{N}$ **b)** $\frac{16\pi}{N}$ **c)** $\frac{8\pi}{N}$ **d)** $\frac{2\pi}{N}$

---

15. In Hamming window spectrum the side-lobe magnitude remains constant with,

a) decreasing w **b)** constant w

c) increasing w **d)** None of these.

---

16. In which window sequence, the width of the main-lobe can be adjusted by varying the length N of the window?

a) Hamming **b)** Hanning **c)** Bartlett **d)** Kaiser

---

17. The condition for the impulse respone to be antisymmetric is,

a) $h(n) = -h(N-1-n)$ **b)** $h(n) = h(-n)$

c) $h(n) = h(N-1-n)$ **d)** All the above.

---

18. The width of the main-lobe should be ______ and it should contain as much of the total energy as possible.

a) Large **b)** Medium **c)** Very large **d)** Small

---

19. Symmetric impulse response having odd number of samples, N = 7 with centre of symmetry a is equal to,

a) 2 **b)** 5 **c)** 3.5 **d)** 3

---

20. Frequency response of LTI system, with constant phase delay

a) $H(w) = \pm |H(w)|\, e^{-jaw}$ **b)** $\pm |H(w)|\, e^{j(b - aw)}$

c) $H(w) = \pm |H(w)|\, e^{jaw}$ **d)** $\pm |H(w)|\, e^{-j(b - aw)}$

Answers

1. d	5. a	9. d	13. c	17. a
2. b	6. b	10. b	14. a	18. d
3. a	7. b	11. a	15. c	19. d
4. c	8. c	12. a	16. d	20. a

IV. Answer the following questions

1. Show that an LTI system can behave as a filter.
2. Prove that linear phase characteristics can be achieved if impulse response is symmetric with symmetry condition $h(N - 1 - n) = h(n)$ with centre of symmetry at $\alpha = (N - 1)/2$.
3. Derive the frequency response of linear phase FIR filter when impulse response is symmetric with centre of symmetry at $(N - 1)/2$ and N is odd.
4. Derive the frequency response of linear phase FIR filter when impulse response is symmetric with centre of symmetry at $(N - 1)/2$ and N is even.
5. Derive the frequency response of linear phase FIR filter when impulse response is antisymmetric with centre of antisymmetry at $(N - 1)/2$ and N is odd.
6. Derive the frequency response of linear phase FIR filter when impulse response is antisymmetric with centre of antisymmetry at $(N - 1)/2$ and N is even.
7. Discuss the FIR filter design by Fourier series method.
8. Discuss the FIR filter design by window method.
9. Explain the characteristics of rectangular window with typical sketches.
10. Explain the characteristics of Bartlett window with typical sketches.
11. Explain the characteristics of Hamming window with typical sketches.
12. Explain the characteristics of Hanning window with typical sketches.
13. Explain the characteristics of Blackman window with typical sketches.
14. Explain the characteristics of Kaiser window with typical sketches.
15. Discuss the frequency sampling method of FIR filter design.

V. Solve the following problems

E6.1 Design a FIR lowpass filter with cutoff frequency of 2 kHz and sampling frequency of 6 kHz with 9 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.

E6.2 Design a FIR highpass filter with cutoff frequency of 2.3 kHz and sampling frequency of 8 kHz with 9 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.

E6.3 Design a FIR bandpass filter to pass frequencies in the range 2.5 kHz to 3.8 kHz sampling frequency of 9KHz with 9 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.

E6.4 Design a FIR bandstop filter to reject frequencies in the range 2.5 kHz to 3.8 kHz and sampling frequency of 9kHz with 9 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.

E6.5 ***Design a linear phase FIR lowpass filter using hamming window by taking 5 samples of window sequence and with a cutoff frequency, w_c = 0.35p rad/sample.***

E6.6 ***Design a linear phase FIR highpass filter using rectangular window, with a cutoff frequency, w_c = 0.48p rad/sample and N = 5.***

E6.7 ***Design a linear phase FIR bandpass filter to pass frequencies in the range 0.35p to 0.48p rad/sample by taking 5 samples of rectangular window sequence.***

E6.8 ***Design a linear phase FIR bandstop filter to reject frequencies in the range 0.35p to 0.48p rad/sample using rectangular window, by taking 5 samples of window sequence.***

E6.9 ***Determine the coefficients of a linear phase FIR filter of length N = 11 which has a symmetric unit sample response and a frequency response that satisfies the conditions***

$$H\left(\frac{2\pi k}{11}\right) = 1 \quad ; \quad \text{for } k = 0, 1, 2, 3$$
$$= 0 \quad ; \quad \text{for } k = 4, 5$$

E6.10 ***Design a linear phase FIR lowpass filter for the desired frequency response as given below, by frequency sampling technique for N = 7.***

$$H_d(e^{j\omega}) = e^{-j3\omega} \; ; \; 0 \le \omega \le 0.6\pi \text{ and } 1.4\pi \le \omega \le 2\pi$$
$$= 0 \quad ; \; 0.6\pi < \omega < 1.4\pi$$

Answers

E6.1 $H(z) = 0.67z^{-4} + 0.2739\left[z^{-3}+z^{-5}\right] - 0.1394\left[z^{-2}+z^{-6}\right] + 0.0033\left[z^{-1}+z^{-7}\right] + 0.0671\left[1+z^{-8}\right]$

$\left|H\left(e^{j\omega}\right)\right| = |A(\omega)| = 0.67 + 0.5478\cos\omega - 0.2788\cos2\omega + 0.0066\cos3\omega + 0.1342\cos4\omega$

E6.2 $H(z) = 0.425z^{-4} - 0.3095\left[z^{-3}+z^{-5}\right] + 0.0722\left[z^{-2}+z^{-6}\right] + 0.0806\left[z^{-1}+z^{-7}\right] - 0.0643\left[1+z^{-8}\right]$

$\left|H\left(e^{j\omega}\right)\right| = |A(\omega)| = 0.425 - 0.619\cos\omega + 0.1444\cos2\omega + 0.1612\cos3\omega - 0.1286\cos4\omega$

E6.3 $H(z) = 0.289z^{-4} - 0.1644\left[z^{-3}+z^{-5}\right] - 0.0767\left[z^{-2}+z^{-6}\right] + 0.1971\left[z^{-1}+z^{-7}\right] - 0.1254\left[1+z^{-8}\right]$

$\left|H\left(e^{j\omega}\right)\right| = |A(\omega)| = 0.289 - 0.3288\cos\omega - 0.1534\cos2\omega + 0.3942\cos3\omega - 0.2508\cos4\omega$

E6.4 $H(z) = 0.711z^{-4} + 0.1644\left[z^{-3}+z^{-5}\right] + 0.0767\left[z^{-2}+z^{-6}\right] - 0.1971\left[z^{-1}+z^{-7}\right] + 0.1254\left[1+z^{-8}\right]$

$\left|H\left(e^{j\omega}\right)\right| = |A(\omega)| == 0.711 + +0.3288\cos\omega + 0.1534\cos2\omega - 0.3942\cos3\omega + 0.2508\cos4\omega$

E6.5 $H(z) = 0.35z^{-2} + 0.1531\left[z^{-1}+z^{-3}\right] + 0.0103\left[1+z^{-4}\right]$

$\left|H\left(e^{j\omega}\right)\right| = |A(\omega)| = 0.35 + 0.3062\cos\omega + 0.0206\cos2\omega$

E6.6 $H(z) = 0.52z^{-2} - 0.3176\left[z^{-1}+z^{-3}\right] - 0.0199\left[1+z^{-4}\right]$

$\left|H\left(e^{j\omega}\right)\right| = |A(\omega)| = 0.52 - 0.6352\cos\omega - 0.0398\cos2\omega$

E6.7	$H(z) = 0.13z^{-2} + 0.0340\left[z^{-1} + z^{-3}\right] - 0.1088\left[1 + z^{-4}\right]$ $\left\lvert H\left(e^{j\omega}\right)\right\rvert = \lvert A(\omega)\rvert = 0.13 + 0.068\cos\omega - 0.2176\cos2\omega$
E6.8	$H(z) = 0.87z^{-2} - 0.0340\left[z^{-1} + z^{-3}\right] + 0.1088\left[1 + z^{-4}\right]$ $\left\lvert H\left(e^{j\omega}\right)\right\rvert = \lvert A(\omega)\rvert = 0.87 - 0.068\cos\omega + 0.2176\cos2\omega$
E6.9	$H(z) = -0.0496\left[1 + z^{-10}\right] + 0.0989\left[z^{-1} + z^{-9}\right] - 0.0338\left[z^{-2} + z^{-8}\right] - 0.1270\left[z^{-3} + z^{-7}\right]$ $+ 0.2935\left[z^{-4} + z^{-6}\right] + 0.6363z^{-5}$ $\left\lvert H\left(e^{j\omega}\right)\right\rvert = \lvert A(\omega)\rvert = 0.6363 + 0.587\cos\omega - 0.254\cos2\omega - 0.0676\cos3\omega + 0.1978\cos4\omega - 0.0992\cos5\omega$
E6.10	$H(z) = 0.0635\left[1 + z^{-6}\right] - 0.1781\left[z^{-1} + z^{-5}\right] + 0.2574\left[z^{-2} + z^{-4}\right] + 0.7142z^{-3}$ $\left\lvert H\left(e^{j\omega}\right)\right\rvert = \lvert A(\omega)\rvert = 0.7142 + 0.5148\cos\omega - 0.3562\cos2\omega + 0.127\cos3\omega$

Hamming window	Kaiser window
iii. In window spectrum the side-lobe magnitude remains constant with increasing w. iv. In FIR filter designed using hamming window, the minimum stopband attenuation is fixed at 51 dB.	iii. In window spectrum the side-lobe magnitude decreases with increasing w. iv. In FIR filter designed using Kaiser window, the minimum stopband attenuation is variable and depends on the value of "a".

6.13. MATLAB Programs

Program 6.1

Write a MATLAB program to determine the impulse response of FIR lowpass Filter by Fourier series method and hence plot the frequency response.

```
%Program to plot frequency response of FIR lowpass filter
clear all
clc
wc=.5*pi;
N=11;
hd=zeros(1,N);


hd(1)=wc/pi;

k = 1 : 1 : ((N-1)/2)+1;
hd(k+1)=(sin(wc*k))./(pi*k);
hn(k)=hd(k)
a=(N-1)/2;
w= 0 : pi/16 : pi;

Hw1=hn(1)*exp(-j*w*a);
Hw2=0;

for m=1:1:a
     Hw3= hn(m+1)*((exp(j*w*(m-a)))+ (exp(-j*w*(m+a))));
     Hw2=Hw2+Hw3;
end
Hw=Hw2+Hw1

H_mag=abs(Hw)
plot(w/pi,H_mag,'k');
grid;
title('Magnitude Response','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

The magnitude response of FIR lowpass filter designed by Fourier series method is shown in fig p6.1.

```
hn =
     0.5000     0.3183     0.0000    -0.1061    -0.0000     0.0637
```

```
Hw =
      Columns 1 through 8
          1.0517                0.5659 - 0.8470i   -0.3667 - 0.8853i    -0.9277 - 0.1845i
          -0.7143 + 0.7143i    0.2120 + 1.0658i    0.9768 + 0.4046i     0.7051 - 0.4711i

      Columns 9 through 16
           0.0000 - 0.5000i    -0.1264 - 0.0845i   0.0529 - 0.0219i     0.0169 - 0.0850i
          -0.0072 - 0.0072i     0.0531 - 0.0106i   0.0160 - 0.0386i     0.0104 + 0.0155i

      Column 17
          0.0517 + 0.0000i

H_mag =
      Columns 1 through 16
          1.0517   1.0187    0.9582     0.9459    1.0102    1.0867    1.0573    0.8480
          0.5000   0.1520    0.0573     0.0867    0.0102    0.0541   0.0418    0.0187

      Column 17

          0.0517
```

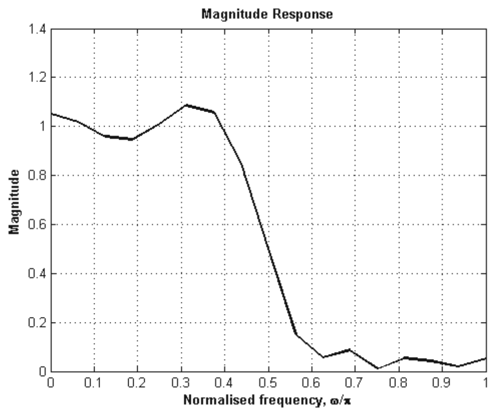


Fig P6.1 : Magnitude response.

Note : Verify the result with example 6.1.

Program 6.2

Write a MATLAB program to determine the impulse response of FIR highpass Filter by Fourier series method and hence plot the frequency response.

```
%Program to plot frequency response of highpass filter
clear all
clc

wc=.6*pi;
N=7;
hd=zeros(1,N);

hd(1)=1-(wc/pi);

k = 1 : 1 : ((N-1)/2)+1;
hd(k+1)=(-sin(wc*k))./(pi*k);
hn(k)=hd(k)
```

```
a=(N-1)/2;
w= 0 : pi/16 : pi;

Hw1=hn(1)*exp(-j*w*a);
Hw2=0;

for m=1:1:a
      Hw3= hn(m+1)*((exp(j*w*(m-a)))+ (exp(-j*w*(m+a))));
      Hw2=Hw2+Hw3;
end
Hw=Hw2+Hw1

H_mag=abs(Hw)
plot(w/pi,H_mag,'k');grid;
title('Magnitude Response','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

The magnitude response of FIR highpass filter designed by Fourier series method is shown in fig p6.2.

```
hn =
       0.4000    -0.3027     0.0935     0.0624

Hw =
     Columns 1 through 8
       0.1064             0.0688  - 0.0460i   0.0079 - 0.0191i    0.0110 + 0.0551i
       0.0823 + 0.0823i   0.1278  +0.0254i    0.0732 - 0.0303i   -0.0221 + 0.0330i

     Columns 9 through 16
      -0.0000 + 0.2129i    0.2303 + 0.3447i    0.5679 + 0.2352i    0.7720 - 0.1536i
       0.6479 - 0.6479i    0.1950 - 0.9802i   -0.3995 - 0.9645i   -0.8838 - 0.5906i

     Column 17
      -1.0678 - 0.0000i
H_mag =
     Columns 1 through 16
       0.1064    0.0827    0.0207    0.0562    0.1163    0.1303    0.0792    0.0397
       0.2129    0.4146    0.6146    0.7871    0.9163    0.9994    1.0439    1.0630

     Column 17
         1.0678
```

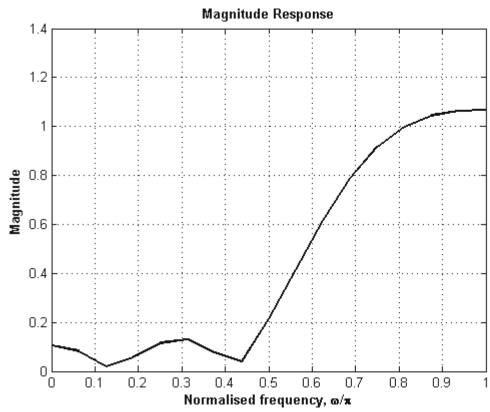


Fig P6.2 : Magnitude response.

Note : Verify the result with example 6.2.

Program 6.3

Write a MATLAB program to determine the impulse response of FIR bandpass Filter by Fourier series method and hence plot the frequency response.

```
%Program to plot frequency response of bandpass filter
clear all
clc

wc1=.375*pi;
wc2=.75*pi;
N=7;
hd=zeros(1,N);

hd(1)=(wc2-wc1)/pi;

k = 1 : 1 : ((N-1)/2)+1;
hd(k+1)=((sin(wc2*k))-(sin(wc1*k)))./(pi*k);
hn(k)=hd(k)

a=(N-1)/2;
w= 0 : pi/16 : pi;

Hw1=hn(1)*exp(-j*w*a);
Hw2=0;

for m=1:1:a
      Hw3= hn(m+1)*((exp(j*w*(m-a)))+ (exp(-j*w*(m+a))));
      Hw2=Hw2+Hw3;
end
Hw=Hw2+Hw1

H_mag=abs(Hw)
plot(w/pi,abs(H_mag),'k');grid;
title('Magnitude Response','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

The magnitude response of FIR bandpass filter designed by Fourier series method is shown in fig p6.3.

```
hn =
       0.3750    -0.0690    -0.2717     0.1156

Hw =
     Columns 1 through 8
       -0.0751             -0.0583 + 0.0389i   -0.0185 + 0.0446i   -0.0014 - 0.0071i
       -0.0805 - 0.0805i   -0.2741 - 0.0545i   -0.4553 + 0.1886i   -0.4009 + 0.6000i

      Columns 9 through 16
       -0.0000 + 0.9184i    0.5736 + 0.8584i    0.9476 + 0.3925i    0.8694 - 0.1729i
        0.4498 - 0.4498i    0.0638 - 0.3206i   -0.0114 - 0.0275i    0.1530 + 0.1022i

      Column 17
       0.2616 + 0.0000i
```

```
H_mag=
     Columns 1 through 16
        0.0751    0.0701    0.0482    0.0072    0.1139    0.2795    0.4928    0.7216
        0.9184    1.0324    1.0257    0.8864    0.6361    0.3269    0.0298    0.1840

     Column 17
        0.2616
```

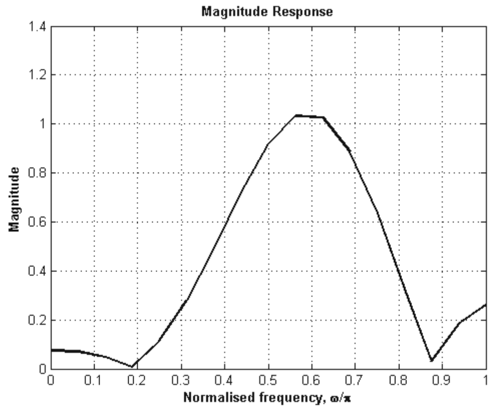


Fig P6.3 : Magnitude response.

Note : Verify the result with example 6.3.

Program 6.4

Write a MATLAB program to determine the impulse response of FIR bandstop Filter by Fourier series method and hence plot the frequency response.

```
%Program to plot frequency response of bandstop filter
clear all
clc

wc1=.375*pi;
wc2=.75*pi;
N=7;
hd=zeros(1,N);

hd(1)=1-((wc2-wc1)/pi);

k = 1 : 1 : ((N-1)/2)+1;
hd(k+1)=((sin(wc1*k))-(sin(wc2*k)))./(pi*k);
hn(k)=hd(k)

a=(N-1)/2;
w= 0 : pi/16 : pi;

Hw1=hn(1)*exp(-j*w*a);
Hw2=0;

for m=1:1:a
     Hw3= hn(m+1)*((exp(j*w*(m-a)))+ (exp(-j*w*(m+a))));
     Hw2=Hw2+Hw3;
end
Hw=Hw2+Hw1
```

```
H_mag=abs(Hw)
plot(w/pi,abs(H_mag),'k');grid;
title('Magnitude Response','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

The magnitude response of FIR bandstop filter designed by Fourier series method is shown in fig p6.4.

```
hn =
     0.6250     0.0690     0.2717    -0.1156
Hw =
   Columns 1 through 8
     1.0751                0.8897 - 0.5945i   0.4011 - 0.9684i   -0.1937 - 0.9737i
    -0.6266 - 0.6266i   -0.7067 - 0.1406i   -0.4686 + 0.1941i   -0.1547 + 0.2315i

Columns 9 through 16
     -0.0000 + 0.0816i   -0.0180 - 0.0270i   -0.0237 - 0.0098i    0.1114 - 0.0222i
      0.2573 - 0.2573i    0.1313 - 0.6602i   -0.3713 - 0.8964i   -0.9844 - 0.6578i

   Column 17
     -1.2616 - 0.0000i

H_mag =
   Columns 1 through 16
     1.0751     1.0701     1.0482   0.9928     0.8861     0.7205     0.5072     0.2784
     0.0816     0.0324     0.0257   0.1136     0.3639     0.6731     0.9702     1.1840

   Column 17
     1.2616
```

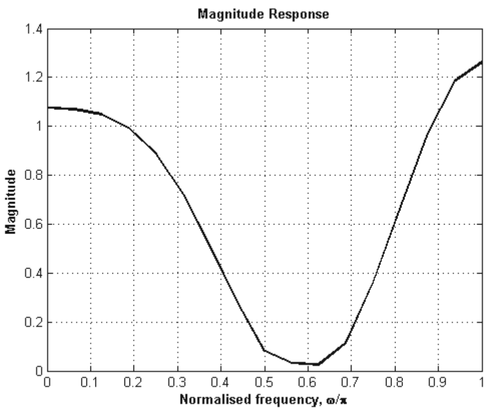


Fig P6.4 : *Magnitude response.*

Note : Verify the result with example 6.4.

Program 6.5

Write a MATLAB program to determine the impulse response of FIR lowpass Filter using rectangular window and hence plot the frequency response.

```
%Program to plot frequency response of lowpass filter using
rectangular window
```

```
clear all
clc

wc=.2*pi;

N=7;
hd=zeros(1,N);
a=(N-1)/2;
hna=wc/pi;

k = 1 : 1 : ((N-1)/2);
n=k-1-((N-1)/2);
hd(k)=(sin(wc*n))./(pi*n);
hn(k)=hd(k);
hn=[hn hna]

a=(N-1)/2;

w= 0 :pi/16 : pi;

Hw1=hna*exp(-j*w*a);
Hw2=0;

for m=1:1:a
      Hw3= hn(m)*((exp(j*w*(1-m)))+ (exp(-j*w*(1-m+2*a))));
      Hw2=Hw2+Hw3;
end
Hw=Hw2+Hw1

H_mag=abs(Hw)
plot(w/pi,H_mag,'k');grid;
title('Magnitude Response','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

The magnitude response of FIR lowpass filter designed using rectangular window is shown in fig p6.5.

```
hn =
      0.1009     0.1514     0.1871     0.2000

Hw =
  Columns 1 through 8
      1.0787              0.8435 - 0.5636i    0.3203 - 0.7733i   -0.1146 - 0.5763i
     -0.2276 - 0.2276i   -0.0923 - 0.0184i    0.0530 - 0.0219i    0.0660 - 0.0988i

  Columns 9 through 16
      0.0000 - 0.1027i   -0.0225 - 0.0337i    0.0270 + 0.0112i    0.0728 - 0.0145i
      0.0552 - 0.0552i    0.0086 - 0.0432i    0.0034 + 0.0082i    0.0458 + 0.0306i

  Column 17
      0.0733 + 0.0000i
H_mag =
  Columns 1 through 16
      1.0787     1.0145     0.8370     0.5876     0.3219     0.0941     0.0573     0.1188
      0.1027     0.0406     0.0292     0.0742     0.0781     0.0441     0.0089     0.0551

  Column 17
      0.0733
```

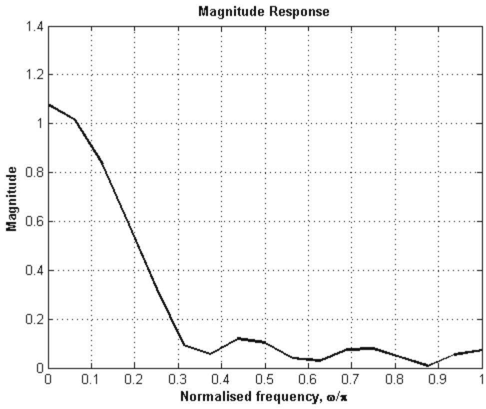


Fig P6.5 : Magnitude response.

Note : Verify the result with example 6.5.

Program 6.6

Write a MATLAB program to determine the impulse response of FIR highpass Filter using Hamming window and hence plot the frequency response.

```
%Program to plot frequency response of highpass filter using Hamming
window
clear all
clc

wc=.8*pi;

N=7;
hd=zeros(1,N);
a=(N-1)/2;
hna=1-(wc/pi);

k = 1 : 1 : ((N-1)/2);
n=k-1-((N-1)/2);
w_ham(k)=.54-.46*cos(2*pi*(k-1)/(N-1));
hd(k)=(-sin(wc*n))./(pi*n);

for s=1:length(k)
hn(s)=hd(s)*w_ham(s);
end

hn = [hn hna]

a=(N-1)/2;
w= 0 : pi/16 : pi;

Hw1=hna*exp(-j*w*a);
Hw2=0;

for m=1:1:a
      Hw3= hn(m)*((exp(j*w*(1-m)))+ (exp(-j*w*(1-m+2*a))));
      Hw2=Hw2+Hw3;
end
Hw=Hw2+Hw1
```

```
H_mag=abs(Hw)
plot(w/pi,H_mag,'k');grid;
title('Magnitude Response','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

The magnitude response of FIR highpass filter designed using rectangular window is shown in fig p6.6.

```
hn =
     -0.0081     0.0469    -0.1441     0.2000

Hw =
   Columns 1 through 8
     -0.0104              -0.0077 + 0.0052i  -0.0023 + 0.0056i   0.0001 + 0.0005i
     -0.0054 - 0.0054i    -0.0195 - 0.0039i  -0.0354 + 0.0147i  -0.0367 + 0.0549i

   Columns 9 through 16
     -0.0000 + 0.1062i     0.0892 + 0.1335i   0.2116 + 0.0876i   0.3024 - 0.0602i
      0.2774 - 0.2774i     0.0921 - 0.4633i  -0.2062 - 0.4977i  -0.4845 - 0.3237i

   Column 17
     -0.5981 - 0.0000i

H_mag =
   Columns 1 through 16
     0.0104    0.0093    0.0060    0.0005    0.0077    0.0198    0.0383    0.0661
     0.1062    0.1605    0.2290    0.3083    0.3923    0.4723    0.5387    0.5827

   Column 17
     0.5981
```

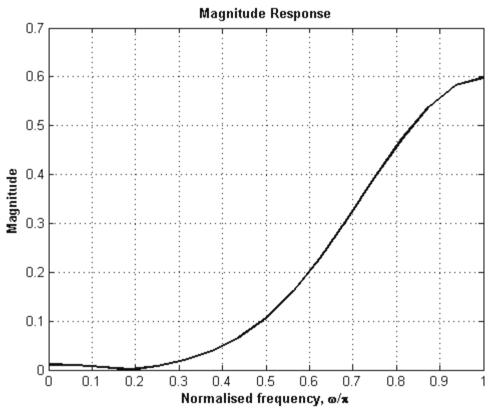


Fig P6.6 : Magnitude response.

Note : Verify the result with example 6.6.

Program 6.7

Write a MATLAB program to determine the impulse response of FIR bandpass Filter using Hanning window and hence plot the frequency response.

```
%Program to plot frequency response of bandpass filter using Hanning
window
clear all
clc

wc1=.4*pi;
wc2=.65*pi;

N=7;
hd=zeros(1,N);
a=(N-1)/2;
hna=(wc2-wc1)/pi;

k = 1 : 1 : ((N-1)/2);
n=k-1-((N-1)/2);
w_han(k)=.5-.5*cos(2*pi*(k-1)/(N-1));
hd(k)=(sin(wc2*n)-sin(wc1*n))./(pi*n);

for s=1:length(k)
hn(s)=hd(s)*w_han(s);
end

hn = [hn hna]
a=(N-1)/2;
w= 0 : pi/16 : pi;
Hw1=hna*exp(-j*w*a);
Hw2=0;

for m=1:1:a
      Hw3= hn(m)*((exp(j*w*(1-m)))+ (exp(-j*w*(1-m+2*a))));
      Hw2=Hw2+Hw3;
end
Hw=Hw2+Hw1

H_mag=abs(Hw)
plot(w/pi,H_mag,'k');grid;
title('Magnitude Response','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

The magnitude response of FIR bandpass filter designed using Hanning window is shown in fig p6.7.

```
hn =
       0    -0.0556    -0.0143     0.2500

Hw =
     Columns 1 through 8
        0.1102              0.0991 - 0.0662i    0.0555 - 0.1339i   -0.0358 - 0.1801i
       -0.1624 - 0.1624i  -0.2713 - 0.0540i   -0.2934 + 0.1216i   -0.1928 + 0.2886i

     Columns 9 through 16
       -0.0000 + 0.3612i    0.1991 + 0.2979i    0.3137 + 0.1299i    0.3025 - 0.0602i
        0.1911 - 0.1911i    0.0451 - 0.2269i   -0.0757 - 0.1828i   -0.1459 - 0.0975i
```

```
      Column 17
         -0.1675 - 0.0000i

H_mag =
      Columns 1 through 16
         0.1102    0.1192    0.1449    0.1836    0.2297    0.2766    0.3176    0.3471
         0.3612    0.3583    0.3396    0.3085    0.2703    0.2313    0.1979    0.1754

      Column 17
         0.1675
```

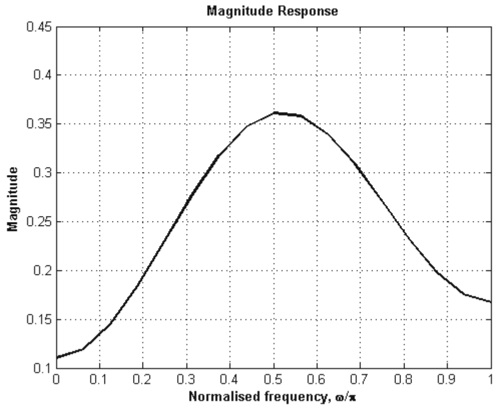


Fig P6.7 : *Magnitude response.*

Note : Verify the result with example 6.7.

Program 6.8

Write a MATLAB program to determine the impulse response of FIR bandstop filter using rectangular window and hence plot the frequency response.

```
%To plot frequency response of bandstop filter using rectangular window
clear all
clc

wc1=.4*pi;
wc2=.65*pi;

N=7;

hd=zeros(1,N);
a=(N-1)/2;
hna=1-((wc2-wc1)/pi);

k = 1 : 1 : ((N-1)/2);
n=k-1-((N-1)/2);
hd(k)=(sin(wc1*n)-sin(wc2*n))./(pi*n);
hn(k)=hd(k);
hn=[hn hna]

a=(N-1)/2;
w= 0 : pi/16 : pi;

Hw1=hna*exp(-j*w*a);
Hw2=0;
for m=1:1:a
```

```
        Hw3= hn(m)*((exp(j*w*(1-m)))+ (exp(-j*w*(1-m+2*a))));
        Hw2=Hw2+Hw3;
    end
    Hw=Hw2+Hw1

    H_mag=abs(Hw)
    plot(w/pi,H_mag,'k');grid;
    title('Magnitude Response','fontweight','b');
    xlabel('Normalised frequency, \omega/\pi','fontweight','b');
    ylabel('Magnitude','fontweight','b');
```

OUTPUT

The magnitude response of FIR bandstop filter designed using rectangular window is shown in fig p6.8.

```
hn =
      -0.0458     0.2223     0.0191     0.7500

Hw =
    Columns 1 through 8
      1.1413              0.9330 + 0.6234i    0.4074 - 0.9836i   -0.1892 - 0.9512i
      -0.5952 - 0.5952i  -0.6776 - 0.1348i  -0.4941 + 0.2047i   -0.2209 + 0.3305i

    Columns 9 through 16
      -0.0000 + 0.3054i    0.1561 + 0.2336i    0.3108 + 0.1287i    0.4598 - 0.0915i
       0.4654 - 0.4654i    0.1698 - 0.8538i   -0.4072 - 0.9831i   -0.9973 - 0.6663i

    Column 17
      -1.2479 - 0.0000i

H_mag =
    Columns 1 through 16
      1.1413    1.1222    1.0647    0.9698    0.8418    0.6909    0.5348    0.3975
      0.3054    0.2809    0.3364    0.4688    0.6582    0.8705    1.0641    1.1994

    Column 17
      1.2479
```

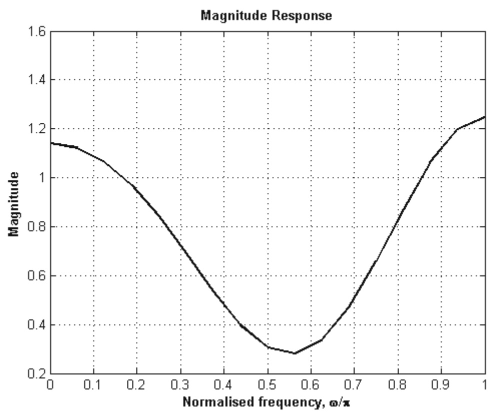


Fig P6.8 : Magnitude response.

Note : Verify the result with example 6.8.

Solution for Exercise Problems

E6.1. ***Design a FIR lowpass filter with cutoff frequency of 2 kHz and sampling frequency of 6 kHz with 9 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.***

Solution

Given that, $F_c = 2$ kHz ; $F_s = 6$ kHz

$$\therefore\ \omega_c = \Omega_c T = \frac{\Omega_c}{F_s} = \frac{2\pi F_c}{F_s} = \frac{2\pi \times 2 \times 10^3}{6 \times 10^3} = 0.67\,\pi \text{ rad / sample}$$

The desired frequency response $H_d(e^{j\omega})$ of lowpass filter is,

$$H_d(e^{j\omega}) = 1 \;;\; \text{for } -\omega_c \le \omega \le +\omega_c$$
$$= 0 \;;\; \text{otherwise}$$

The desired impulse response $h_d(n)$ of the lowpass filter is,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\,e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \times e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] = \frac{1}{\pi n}\sin\omega_c n \;;\; \text{for all n, except n} = 0$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

When n = 0, the factor $\frac{\sin\omega_c n}{\pi n}$ becomes 0 / 0, which is indeterminate.

$$\text{When } n = 0\;;\; h_d(n) = h_d(0) = \underset{n \to 0}{\text{Lt}}\frac{\sin\omega_c n}{\pi n}$$

$$= \frac{1}{\pi}\underset{n \to 0}{\text{Lt}}\frac{\sin\omega_c n}{n} = \frac{\omega_c}{\pi}$$

Using L' Hospital rule,

$$\underset{\theta \to 0}{\text{Lt}}\frac{\sin A\theta}{\theta} = A$$

The impulse response h(n) of FIR filter is obtained by truncating $h_d(n)$ to 9 samples.

$$\therefore\ h(n) = h_d(n) = \frac{\sin\omega_c n}{\pi n} \;;\; \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2},\ \text{except } n = 0$$

$$= \frac{\omega_c}{\pi} \;;\; \text{for } n = 0$$

$$\text{Here, } N = 9,\ \therefore\ \frac{N-1}{2} = \frac{9-1}{2} = 4$$

Hence, calculate h(n) for n = –4 to +4.

Since, the impulse response h(n) satisfies the symmetry condition, h(–n) = h(n), calculate h(n) for n = 0 to 4.

$$\text{When } n = 0\;;\; h(0) = \frac{\omega_c}{\pi} = \frac{0.67\pi}{\pi} = 0.67$$

$$\text{When } n = 1\;;\; h(1) = \frac{\sin(0.67\pi \times 1)}{\pi \times 1} = 0.2739$$

Note : Calculate $\sin\theta$ by keeping the calculator in radian mode.

$$\text{When } n = 2\;;\; h(2) = \frac{\sin(0.67\pi \times 2)}{\pi \times 2} = -0.1394$$

$$\text{When } n = 3\;;\; h(3) = \frac{\sin(0.67\pi \times 3)}{\pi \times 3} = 0.0033$$

$$\text{When } n = 4\;;\; h(4) = \frac{\sin(0.67\pi \times 4)}{\pi \times 4} = 0.0671$$

When n = –1 ; h(–1) = h(1) = 0.2739

When n = –2 ; h(–2) = h(2) = –0.1394

When n = –3 ; h(–3) = h(3) = 0.0033

When n = –4 ; h(–4) = h(4) = 0.0671

Using symmetry condition, h(–n) = h(n).

The transfer function H(z) of the digital lowpass filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\,\mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)z^{-n} = z^{-4}\sum_{n=-4}^{4} h(n)z^{-n}$$

$$\therefore\ H(z) = z^{-4}\left[h(-4)z^4 + h(-3)z^3 + h(-2)z^2 + h(1)z + h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}\right]$$

Using symmetry condition, $h(-n) = h(n)$.

$$= z^{-4}\left[h(4)z^4 + h(3)z^3 + h(2)z^2 + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}\right]$$

$$= z^{-4}\left[h(0) + h(1)\left[z + z^{-1}\right] + h(2)\left[z^2 + ^{-2}\right] + h(3)\left[z^3 + z^{-3}\right] + h(4)\left[z^4 + z^{-4}\right]\right]$$

$$= h(0)z^{-4} + h(1)\left[z^{-3} + z^{-5}\right] + h(2)\left[z^{-2} + z^{-6}\right] + h(3)\left[z^{-1} + z^{-7}\right] + h(4)\left[z^0 + z^{-8}\right]$$

$$= 0.67\,z^{-4} + 0.2739\left[z^{-3} + z^{-5}\right] - 0.1394\left[z^{-2} + z^{-6}\right] + 0.0033\left[z^{-1} + z^{-7}\right] + 0.0671\left[1 + z^{-8}\right]$$

Structure

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)} = 0.67\,z^{-4} + 0.2739\left[z^{-3} + z^{-5}\right] - 0.1394\left[z^{-2} + z^{-6}\right] + 0.0033\left[z^{-1} + z^{-7}\right] + 0.0671\left[1 + z^{-8}\right]$$

$$\therefore\ Y(z) = 0.67\,z^{-4}X(z) + 0.2739\left[z^{-3}X(z) + z^{-5}X(z)\right] - 0.1394\left[z^{-2}X(z) + z^{-6}X(z)\right] + 0.0033\left[z^{-1}X(z) + z^{-7}X(z)\right] + 0.0671\left[X(z) + z^{-8}X(z)\right]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

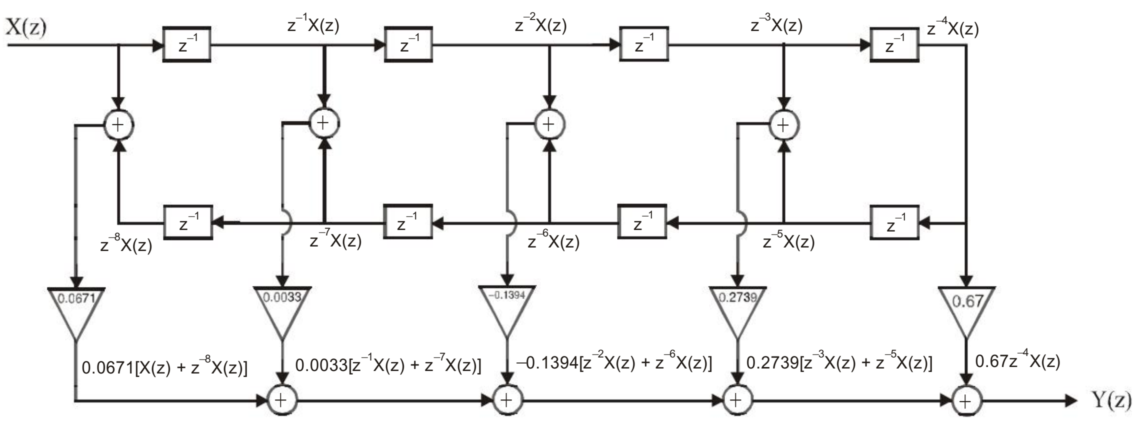


Fig 1 : Linear phase structure of FIR lowpass filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0, the magnitude response, $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where,}\ A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$$

Refer table 6.2 case (v)

$$= h(0) + \sum_{n=1}^{4} 2h(n)\cos\omega n$$

$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega + 2h(3)\cos 3\omega + 2h(4)\cos 4\omega$$

$$= 0.67 + 2\times 0.2739\cos\omega + 2\times -0.1394\cos 2\omega + 2\times 0.0033\cos 3\omega + 2\times 0.0671\cos 4\omega$$

$$= 0.67 + 0.5478\cos\omega - 0.2788\cos 2\omega + 0.0066\cos 3\omega + 0.1342\cos 4\omega$$

Using the above equation, the amplitude response $A(w)$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the tabulated values, the magnitude response is sketched as shown in fig 2.

TABLE 1: $A(w)$ and $|H(e^{jw})|$ for various values of w

w	A(w)	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{0\times\pi}{16}$	1.0798	1.0798
$\frac{1\times\pi}{16}$	1.0500	1.0500
$\frac{2\times\pi}{16}$	0.9814	0.9814
$\frac{3\times\pi}{16}$	0.9226	0.9226
$\frac{4\times\pi}{16}$	0.9184	0.9184
$\frac{5\times\pi}{16}$	0.9796	0.9796
$\frac{6\times\pi}{16}$	1.0706	1.0706
$\frac{7\times\pi}{16}$	1.1256	1.1256
$\frac{8\times\pi}{16}$	1.083	1.083

w	A(w)	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{9\times\pi}{16}$	0.9192	0.9192
$\frac{10\times\pi}{16}$	0.6636	0.6636
$\frac{11\times\pi}{16}$	0.3839	0.3839
$\frac{12\times\pi}{16}$	0.1531	0.1531
$\frac{13\times\pi}{16}$	0.0142	0.0142
$\frac{14\times\pi}{16}$	–0.0357	0.0357
$\frac{15\times\pi}{16}$	–0.0354	0.0354
$\frac{16\times\pi}{16}$	–0.029	0.029

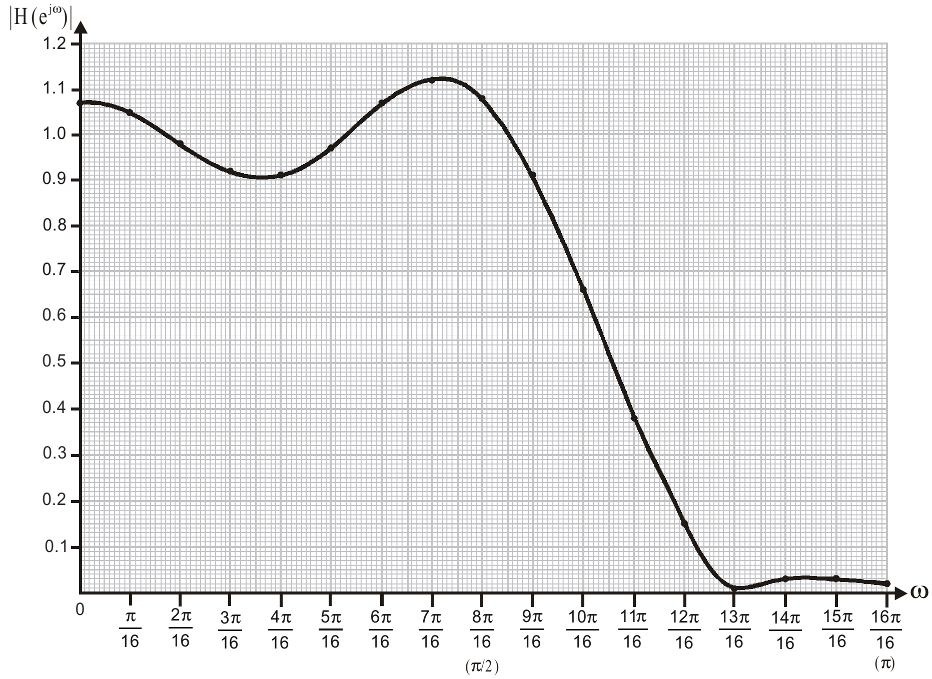


***Fig 2 :** Magnitude response of FIR lowpass filter.*

E6.2. *Design a FIR highpass filter with cutoff frequency of 2.3 kHz and sampling frequency of 8 kHz with 9 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.*

Solution

Given that, F_c = 2.3 KHz ; F_s = 8 KHz

$$\therefore \ \omega_c = \Omega_c T = \frac{\Omega_c}{F_s} = \frac{2\pi F_c}{F_s} = \frac{2\pi \times 2.3 \times 10^3}{8 \times 10^3} = 0.575\pi \ \text{ rad / sample}$$

The desired frequency response $H_d(e^{jw})$ of highpass filter is,

$$H_d(e^{j\omega}) = 1 \ ; \ \text{for } -\pi \le \omega \le -\omega_c \quad \text{and} \quad \omega_c \le \omega \le \pi$$

$$= 0 \ ; \ \text{otherwise}$$

The desired impulse response $h_d(n)$ of the highpass filter is,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} 1 \times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} 1 \times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_c}^{\pi} = \frac{1}{2\pi}\left[\frac{e^{-j\omega_c n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_c n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} - \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{\pi n}\left[\sin \pi n - \sin \omega_c n\right] \ ; \ \text{for all n, except n = 0.}$$

When n = 0, the $h_d(n)$ becomes 0/0, which is indeterminate.

$$\text{When } n = 0 \ ; \ h_d(n) = h_d(0) = \underset{n \to 0}{\text{Lt}}\left[\frac{\sin \pi n - \sin \omega_c n}{\pi n}\right]$$

Using L' Hospital rule, $\underset{\theta \to 0}{\text{Lt}} \dfrac{\sin A\theta}{\theta} = A$

$$= \underset{n \to 0}{\text{Lt}} \frac{\sin \pi n}{\pi n} - \underset{n \to 0}{\text{Lt}} \frac{\sin \omega_c n}{\pi n}$$

$$= \frac{1}{\pi} \underset{n \to 0}{\text{Lt}} \frac{\sin \pi n}{n} - \frac{1}{\pi} \underset{n \to 0}{\text{Lt}} \frac{\sin \omega_c n}{n}$$

$$= \frac{1}{\pi} \times \pi - \frac{1}{\pi} \times \omega_c = 1 - \frac{\omega_c}{\pi}$$

The impulse response h(n) of FIR filter is obtained by truncating $h_d(n)$ to 9 samples.

$$\therefore \ h(n) = h_d(n) = \frac{\sin \pi n - \sin \omega_c n}{\pi n} = -\frac{\sin \omega_c n}{\pi n} \ ; \ \text{for } n = -\tfrac{N-1}{2} \ \text{ to } \ +\tfrac{N-1}{2}\ , \ \text{except } n = 0$$

For any integer n, sin pn = 0

$$= 1 - \frac{\omega_c}{\pi} \ ; \ \text{for } n = 0$$

Here, $N = 9, \quad \therefore \quad \frac{N-1}{2} = \frac{9-1}{2} = 4$

Hence, calculate h(n) for n= –4 to 4.

Since, the impulse response h(n) satisfies the symmetry condition, h(–n) = h(n), calculate h(n) for n = 0 to 4.

When $n = 0\ ;\ h(0) = 1 - \frac{\omega_c}{\pi} = 1 - \frac{0.575\pi}{\pi} = 0.425$

When $n = 1\ ;\ h(1) = -\frac{\sin(0.575\pi \times 1)}{\pi \times 1} = -0.3095$

When $n = 2\ ;\ h(2) = -\frac{\sin(0.575\pi \times 2)}{\pi \times 2} = 0.0722$

When $n = 3\ ;\ h(3) = -\frac{\sin(0.575\pi \times 3)}{\pi \times 3} = 0.0806$

When $n = 4\ ;\ h(4) = -\frac{\sin(0.575\pi \times 4)}{\pi \times 4} = -0.0643$

When $n = -1\ ;\ h(-1) = h(1) = -0.3095$

When $n = -2\ ;\ h(-2) = h(2) = 0.0722$

When $n = -3\ ;\ h(-3) = h(3) = 0.0806$

When $n = -4\ ;\ h(-4) = h(4) = -0.0643$

Using symmetry condition, h(–n) = h(n).

The transfer function H(z) of the digital highpass filter is given by,

$$H(z) = z^{-\frac{(N-1)}{2}} \mathcal{Z}\{h(n)\} = z^{-\frac{(N-1)}{2}} \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)z^{-n} = z^{-4} \sum_{n=-4}^{+4} h(n)z^{-n}$$

$$= z^{-4}\left[h(-4)z^4 + h(-3)z^3 + h(-2)z^2 + h(-1)z + h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}\right]$$

$$= z^{-4}\left[h(4)z^4 + h(3)z^3 + h(2)z^2 + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}\right]$$

Using symmetry condition, h(–n) = h(n).

$$= z^{-4}\left[h(0) + h(1)\left[z + z^{-1}\right] + h(2)\left[z^2 + z^{-2}\right] + h(3)\left[z^3 + z^{-3}\right] + h(4)\left[z^4 + z^{-4}\right]\right]$$

$$= h(0)z^{-4} + h(1)\left[z^{-3} + z^{-5}\right] + h(2)\left[z^{-2} + z^{-6}\right] + h(3)\left[z^{-1} + z^{-7}\right] + h(4)\left[z^0 + z^{-8}\right]$$

$$= 0.425z^{-4} - 0.3095\left[z^{-3} + z^{-5}\right] + 0.0722\left[z^{-2} + z^{-6}\right] + 0.0806\left[z^{-1} + z^{-7}\right] - 0.0643\left[1 + z^{-8}\right]$$

Structure

Let, $H(z) = \frac{Y(z)}{X(z)} = 0.425z^{-4} - 0.3095\left[z^{-3} + z^{-5}\right] + 0.0722\left[z^{-2} + z^{-6}\right] + 0.0806\left[z^{-1} + z^{-7}\right] - 0.0643\left[1 + z^{-8}\right]$

$$\therefore\ Y(z) = 0.425z^{-4}X(z) - 0.3095\left[z^{-3}X(z) + z^{-5}X(z)\right] + 0.0722\left[z^{-2}X(z) + z^{-6}X(z)\right] + 0.0806\left[z^{-1}X(z) + z^{-7}X(z)\right] - 0.0643\left[X(z) + z^{-8}X(z)\right]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

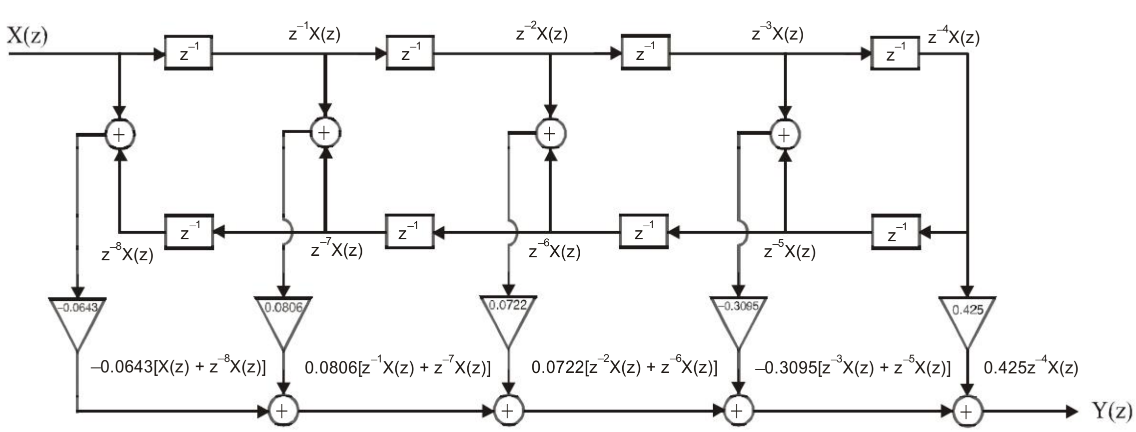


Fig 1 : Linear phase structure of FIR highpass filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0, the magnitude response $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where, } A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n \qquad \boxed{\text{Refer table 6.2 case (v)}}$$

$$= h(0) + \sum_{n=1}^{4} 2h(n)\cos\omega n$$

$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega + 2h(3)\cos 3\omega + 2h(4)\cos 4\omega$$

$$= 0.425 + 2\times -0.3095\cos\omega + 2\times 0.0722\cos 2\omega + 2\times 0.0806\cos 3\omega + 2\times -0.0643\cos 4\omega$$

$$= 0.425 - 0.619\cos\omega + 0.1444\cos 2\omega + 0.1612\cos 3\omega - 0.1286\cos 4\omega$$

Using the above equation, the amplitude response $A(w)$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the tabulated values, the magnitude response is sketched as shown in fig 2.

TABLE 1: $A(w)$ and $|H(e^{jw})|$ for various values of w

w	$A(w)$	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{0\times\pi}{16}$	–0.017	0.017
$\frac{1\times\pi}{16}$	–0.0055	0.0055
$\frac{2\times\pi}{16}$	0.0169	0.0169
$\frac{3\times\pi}{16}$	0.0250	0.0250
$\frac{4\times\pi}{16}$	0.0019	0.0019
$\frac{5\times\pi}{16}$	–0.0413	0.0413
$\frac{6\times\pi}{16}$	–0.0629	0.0629
$\frac{7\times\pi}{16}$	–0.0096	0.0096
$\frac{8\times\pi}{16}$	0.152	0.152

w	$A(w)$	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{9\times\pi}{16}$	0.4109	0.4109
$\frac{10\times\pi}{16}$	0.7087	0.7087
$\frac{11\times\pi}{16}$	0.9626	0.9626
$\frac{12\times\pi}{16}$	1.1052	1.1052
$\frac{13\times\pi}{16}$	1.1173	1.1173
$\frac{14\times\pi}{16}$	1.0372	1.0372
$\frac{15\times\pi}{16}$	0.9405	0.9405
$\frac{16\times\pi}{16}$	0.8986	0.8986

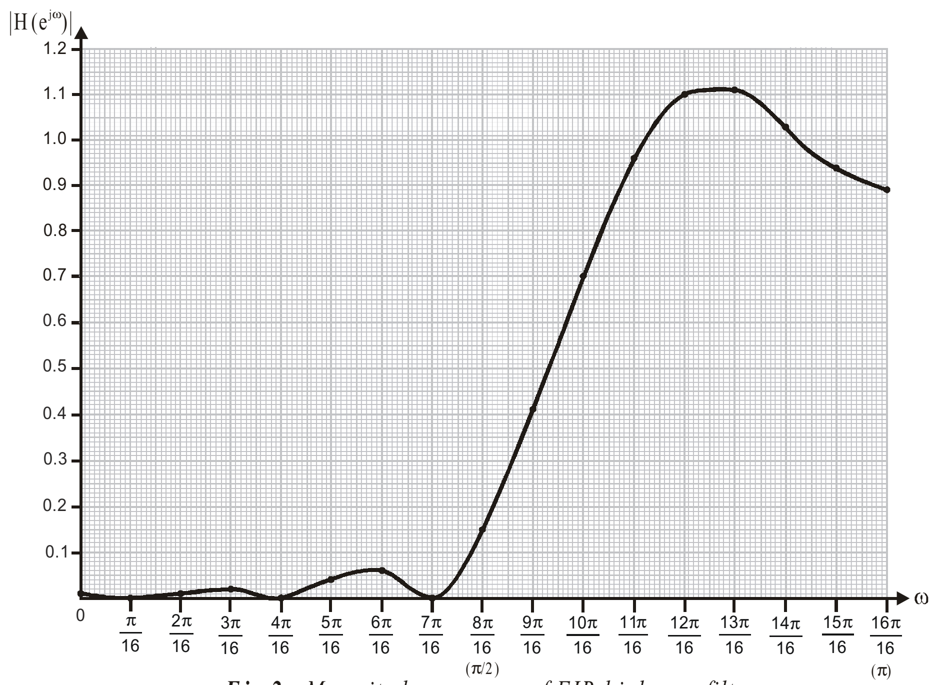


Fig 2 : Magnitude response of FIR highpass filter.

E6.3. ***Design a FIR bandpass filter to pass frequencies in the range 2.5 kHz to 3.8 kHz sampling frequency of 9 kHz with 9 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.***

Solution

Given that, $F_{c1} = 2.5$ kHz ; $F_{c2} = 3.8$ kHz ; $F_s = 9$ kHz

$$\therefore \quad \omega_{c1} = \Omega_{c1}T = \frac{\Omega_{c1}}{F_s} = \frac{2\pi F_{c1}}{F_s} = \frac{2\pi\times 2.5\times 10^3}{9\times 10^3} = 0.556\pi \text{ rad / sample}$$

$$\omega_{c2} = \Omega_{c2}T = \frac{\Omega_{c2}}{F_s} = \frac{2\pi F_{c2}}{F_s} = \frac{2\pi\times 3.8\times 10^3}{9\times 10^3} = 0.845\pi \text{ rad / sample}$$

The desired frequency response $H_d(e^{j\omega})$ of bandpass filter is,

$$H_d\left(e^{j\omega}\right) = 1 \;;\; \text{for } -\omega_{c2} \le \omega \le -\omega_{c1} \;\text{ and }\; \omega_{c1} \le \omega \le \omega_{c2}$$

$$= 0 \;;\; \text{otherwise}$$

The desired impulse response $h_d(n)$ of the bandpass filter is,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} 1\times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} 1\times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c1}}^{\omega_{c2}} = \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}n}}{jn} - \frac{e^{-j\omega_{c2}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2}n}}{jn} - \frac{e^{j\omega_{c1}n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j} - \frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j}\right]$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n} \;;\; \text{for all n, except n} = 0.$$

When n = 0, the $h_d(n)$ becomes 0/0, which is indeterminate.

When, $n = 0 \;;\; h_d(n) = h_d(0) = \underset{n\to 0}{\text{Lt}}\left[\frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n}\right]$

$$= \frac{1}{\pi}\left[\underset{n\to 0}{\text{Lt}}\frac{\sin\omega_{c2}n}{n} - \underset{n\to 0}{\text{Lt}}\frac{\sin\omega_{c1}n}{n}\right]$$

$$= \frac{1}{\pi}\left(\omega_{c2} - \omega_{c1}\right)$$

Using L' Hospital rule,

$$\underset{\theta\to 0}{\text{Lt}}\frac{\sin A\theta}{\theta} = A$$

The impulse response h(n) of FIR filter is obtained by truncating $h_d(n)$ to 9 samples.

$$\therefore\; h(n) = h_d(n) = \frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n} \;;\; \text{for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2},\ \text{except } n = 0$$

$$= \frac{\omega_{c2} - \omega_{c1}}{\pi} \qquad ;\; \text{for } n = 0$$

Here, $N = 9$, $\therefore\; \frac{N-1}{2} = \frac{9-1}{2} = 4$

Hence, calculate h(n) for n= –4 to 4.

Since, the impulse response h(n) satisfies the symmetry condition, h(–n) = h(n), calculate h(n) for n = 0 to 4.

When $n = 0 \;;\; h(0) = \frac{\omega_{c2} - \omega_{c1}}{\pi} = \frac{0.845\pi - 0.556\pi}{\pi} = 0.289$

When $n = 1 \;;\; h(1) = \frac{\sin(0.845\pi\times 1) - \sin(0.556\pi\times 1)}{\pi\times 1} = -0.1644$

When $n = 2 \;;\; h(2) = \frac{\sin(0.845\pi\times 2) - \sin(0.556\pi\times 2)}{\pi\times 2} = -0.0767$

When $n = 3 \;;\; h(3) = \frac{\sin(0.845\pi\times 3) - \sin(0.556\pi\times 3)}{\pi\times 3} = 0.1971$

When $n = 4 \;;\; h(4) = \frac{\sin(0.845\pi\times 4) - \sin(0.556\pi\times 4)}{\pi\times 4} = -0.1254$

When $n = -1 \;;\; h(-1) = h(1) = -0.1644$

When $n = -2 \;;\; h(-2) = h(2) = -0.0767$

When $n = -3 \;;\; h(-3) = h(3) = 0.1971$

When $n = -4 \;;\; h(-4) = h(4) = -0.1254$

The transfer function H(z) of the digital bandpass filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)z^{-n} = z^{-4}\sum_{n=-4}^{+4} h(n)z^{-n}$$

$$= z^{-4}\left[h(-4)z^4 + h(-3)z^3 + h(-2)z^2 + h(-1)z + h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}\right]$$

$$= z^{-4}\left[h(4)z^4 + h(3)z^3 + h(2)z^2 + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}\right]$$

Using symmetry condition, h(–n) = h(n).

$$= z^{-4}\left[h(0) + h(1)\left[z + z^{-1}\right] + h(2)\left[z^2 + z^{-2}\right] + h(3)\left[z^3 + z^{-3}\right] + h(4)\left[z^4 + z^{-4}\right]\right]$$

$$= h(0)z^{-4} + h(1)\left[z^{-3} + z^{-5}\right] + h(2)\left[z^{-2} + z^{-6}\right] + h(3)\left[z^{-1} + z^{-7}\right] + h(4)\left[1 + z^{-8}\right]$$

$$= 0.289z^{-4} - 0.1644\left[z^{-3} + z^{-5}\right] - 0.0767\left[z^{-2} + z^{-6}\right] + 0.1971\left[z^{-1} + z^{-7}\right] - 0.1254\left[1 + z^{-8}\right]$$

Structure

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=0.289z^{-4}-0.1644\left[z^{-3}+z^{-5}\right]-0.0767\left[z^{-2}+z^{-6}\right]+0.1971\left[z^{-1}+z^{-7}\right]-0.1254\left[1+z^{-8}\right]$$

$$Y(z)=0.289z^{-4}X(z)-0.1644\left[z^{-3}X(z)+z^{-5}X(z)\right]-0.0767\left[z^{-2}X(z)+z^{-6}X(z)\right]+0.1971\left[z^{-1}X(z)+z^{-7}X(z)\right]$$
$$-0.1254\left[X(z)+z^{-8}X(z)\right]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

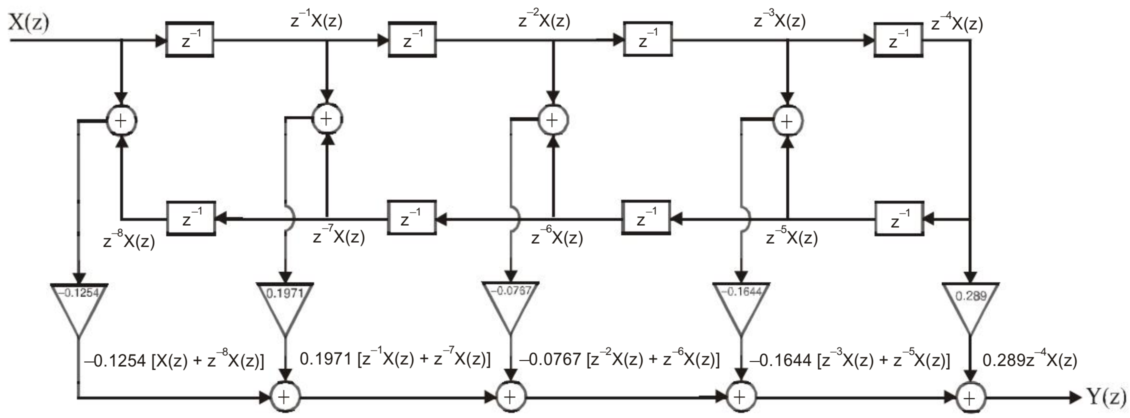


***Fig 1 :** Linear phase structure of FIR bandpass filter.*

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0, the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$,

$$\text{where, } A(\omega)=h(0)+\sum_{n=1}^{\frac{N-1}{2}}2h(n)\cos\omega n$$

Refer table 6.2 case (v)

$$=h(0)+\sum_{n=1}^{4}2h(n)\cos\omega n$$
$$=h(0)+2h(1)\cos\omega n+2h(2)\cos 2\omega+2h(3)\cos 3\omega+2h(4)\cos 4\omega$$
$$=0.289+2\times-0.1644\cos\omega+2\times-0.0767\cos 2\omega+2\times 0.1971\cos 3\omega+2\times-0.1254\cos 4\omega$$
$$=0.289-0.3288\cos\omega-0.1534\cos 2\omega+0.3942\cos 3\omega-0.2508\cos 4\omega$$

Using the above equation, the amplitude response $A(\omega)$ and magnitude function $|H(e^{j\omega})|$ are calculated for various values of ω and listed in table 1. Using the tabulated values, the magnitude response is sketched as shown in fig 2.

TABLE 1: $A(\omega)$ and $|H(e^{j\omega})|$ for various values of ω

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$\frac{0\times\pi}{16}$	–0.0498	0.0498
$\frac{1\times\pi}{16}$	–0.0247	0.0247
$\frac{2\times\pi}{16}$	0.0276	0.0276
$\frac{3\times\pi}{16}$	0.0573	0.0573
$\frac{4\times\pi}{16}$	0.0285	0.0285
$\frac{5\times\pi}{16}$	–0.0442	0.0442
$\frac{6\times\pi}{16}$	–0.0925	0.0925
$\frac{7\times\pi}{16}$	–0.0297	0.0297
$\frac{8\times\pi}{16}$	0.1916	0.1916

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$\frac{9\times\pi}{16}$	0.5365	0.5365
$\frac{10\times\pi}{16}$	0.8874	0.8874
$\frac{11\times\pi}{16}$	1.0943	1.0943
$\frac{12\times\pi}{16}$	1.0510	1.0510
$\frac{13\times\pi}{16}$	0.7579	0.7579
$\frac{14\times\pi}{16}$	0.3334	0.3334
$\frac{15\times\pi}{16}$	–0.0353	0.0353
$\frac{16\times\pi}{16}$	–0.1806	0.1806

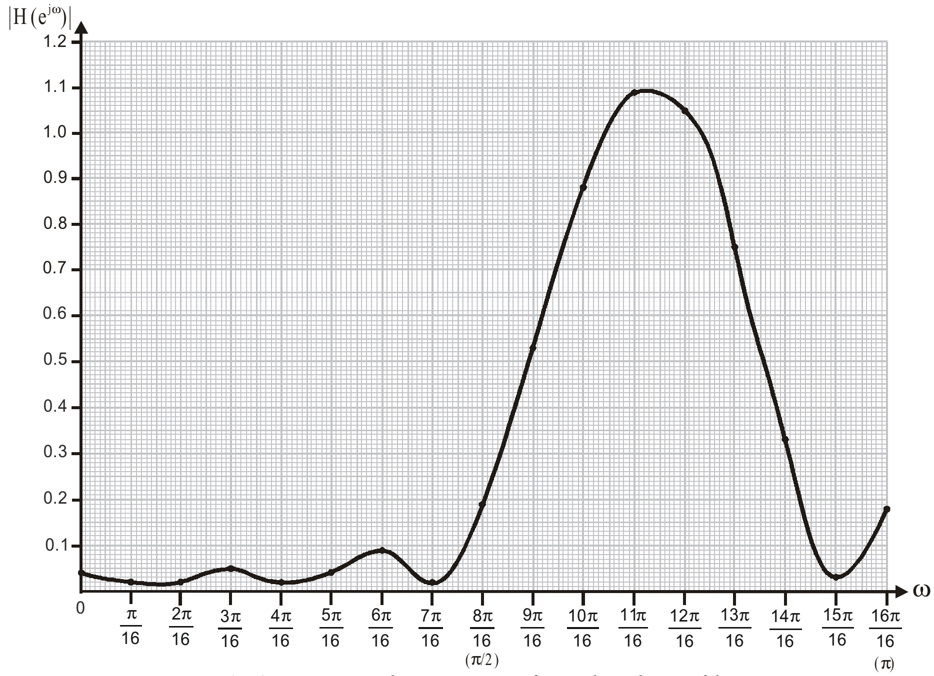


Fig 2 : Magnitude response of FIR bandpass filter.

E6.4. *Design a FIR bandstop filter to reject frequencies in the range 2.5 kHz to 3.8 kHz and sampling frequency of 9kHz with 9 samples using Fourier series method. Determine the frequency response and verify the design by sketching the magnitude response.*

Solution

Given that, $F_{c1} = 2.5$ kHz ; $F_{c2} = 3.8$ kHz ; $F_s = 9$ kHz

$$\therefore \quad \omega_{c1} = \Omega_{c1}T = \frac{\Omega_{c1}}{F_s} = \frac{2\pi F_{c1}}{F_s} = \frac{2\pi \times 2.5 \times 10^3}{9 \times 10^3} = 0.556\pi \text{ rad / sample}$$

$$\omega_{c2} = \Omega_{c2}T = \frac{\Omega_{c2}}{F_s} = \frac{2\pi F_{c2}}{F_s} = \frac{2\pi \times 3.8 \times 10^3}{9 \times 10^3} = 0.845\pi \text{ rad / sample}$$

The desired frequency response $H_d(e^{jw})$ of bandstop filter is,

$$H_d(e^{j\omega}) = 1 \; ; \; -\pi \le \omega \le -\omega_{c2} \;\&\; -\omega_{c1} \le \omega \le \omega_{c1} \;\&\; +\omega_{c2} \le \omega \le \pi$$
$$= 0 \; ; \; \text{otherwise}$$

The desired impulse response $h_d(n)$ of the bandstop filter is,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} 1 \times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} 1 \times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{+\omega_{c2}}^{\pi} 1 \times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c1}}^{\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c2}}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c2}n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c1}n}}{jn} - \frac{e^{-j\omega_{c1}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_{c2}n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} + \frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j} - \frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{\sin\pi n + \sin\omega_{c1}n - \sin\omega_{c2}n}{\pi n} \quad ; \quad \text{for all n, except n = 0}$$

When n = 0, the $h_d(n)$ becomes 0/0, which is indeterminate.

$$\text{When } n = 0 \; ; \quad h_d(n) = h_d(0) = \underset{n \to 0}{\text{Lt}}\left[\frac{\sin\pi n}{\pi n} + \frac{\sin\omega_{c1}n}{\pi n} - \frac{\sin\omega_{c2}n}{\pi n}\right]$$

$$= \frac{1}{\pi}\underset{n \to 0}{\text{Lt}}\frac{\sin\pi n}{n} + \frac{1}{\pi}\underset{n \to 0}{\text{Lt}}\frac{\sin\omega_{c1}n}{n} - \frac{1}{\pi}\underset{n \to 0}{\text{Lt}}\frac{\sin\omega_{c2}n}{n}$$

$$= \frac{1}{\pi} \times \pi + \frac{1}{\pi} \times \omega_{c1} - \frac{1}{\pi} \times \omega_{c2} = 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)$$

Using L' Hospital rule,
$$\underset{\theta \to 0}{\text{Lt}}\frac{\sin A\theta}{\theta} = A$$

The impulse response h(n) of FIR filter is obtained by truncating $h_d(n)$ to 9 samples.

$$\therefore\ h(n) = h_d(n) = \frac{\sin\omega_{c1}n - \sin\omega_{c2}n}{\pi n} \quad ; \text{ for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}, \text{ except } n = 0$$

$$= 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right) \quad ; \text{ for } n = 0$$

For any integer n, $\sin \pi n = 0$

Here, $N = 9$, $\therefore\ \dfrac{N-1}{2} = \dfrac{9-1}{2} = 4$

Hence, calculate h(n) for n = –4 to +4.

Since, the impulse response h(n) satisfies the symmetry condition, h(n) = h(–n), calculate h(n) for n = 0 to 4.

When $n = 0$; $h(0) = 1 - \dfrac{(\omega_{c2} - \omega_{c1})}{\pi} = 0.711$

When $n = 1$; $h(1) = \dfrac{\sin(0.556\pi \times 1) - \sin(0.845\pi \times 1)}{\pi \times 1} = 0.1644$

When $n = 2$; $h(2) = \dfrac{\sin(0.556\pi \times 2) - \sin(0.845\pi \times 2)}{\pi \times 2} = 0.0767$

When $n = 3$; $h(3) = \dfrac{\sin(0.556\pi \times 3) - \sin(0.845\pi \times 3)}{\pi \times 3} = -0.1971$

When $n = 4$; $h(4) = \dfrac{\sin(0.556\pi \times 4) - \sin(0.845\pi \times 4)}{\pi \times 4} = 0.1254$

When $n = -1$; $h(-1) = h(1) = 0.1644$

When $n = -2$; $h(-2) = h(2) = 0.0767$

When $n = -3$; $h(-3) = h(3) = -0.1971$

When $n = -4$; $h(-4) = h(4) = 0.1254$

Using symmetry condition, h(–n) = h(n).

The transfer function H(z) of the digital bandstop filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\ \mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}} \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)z^{-n} = z^{-4} + \sum_{n=-4}^{4} h(n)z^{-n}$$

$$= z^{-4}\left[h(-4)z^4 + h(-3)z^3 + h(-2)z^2 + h(-1)z + h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}\right]$$

$$= z^{-4}\left[h(4)z^4 + h(3)z^3 + h(2)z^2 + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}\right]$$

Using symmetry condition, h(–n) = h(n).

$$= z^{-4}\left[h(0) + h(1)\left[z + z^{-1}\right] + h(2)\left[z^2 + z^{-2}\right] + h(3)\left[z^3 + z^{-3}\right] + h(4)\left[z^4 + z^{-4}\right]\right]$$

$$= h(0)z^{-4} + h(1)\left[z^{-3} + z^{-5}\right] + h(2)\left[z^{-2} + z^{-6}\right] + h(3)\left[z^{-1} + z^{-7}\right] + h(4)\left[z^0 + z^{-8}\right]$$

$$= 0.711z^{-4} + 0.1644\left[z^{-3} + z^{-5}\right] + 0.0767\left[z^{-2} + z^{-6}\right] - 0.1971\left[z^{-1} + z^{-7}\right] + 0.1254\left[1 + z^{-8}\right]$$

Structure

Let, $H(z) = \dfrac{Y(z)}{X(z)} = 0.711z^{-4} + 0.1644\left[z^{-3} + z^{-5}\right] + 0.0767\left[z^{-2} + z^{-6}\right] - 0.1971\left[z^{-1} + z^{-7}\right] + 0.1254\left[1 + z^{-8}\right]$

$$\therefore\ Y(z) = 0.711z^{-4}X(z) + 0.1644\left[z^{-3}X(z) + z^{-5}X(z)\right] + 0.0767\left[z^{-2}X(z) + z^{-6}X(z)\right] - 0.1971\left[z^{-1}X(z) + z^{-7}X(z)\right]$$
$$+ 0.1254\left[X(z) + z^{-8}X(z)\right]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

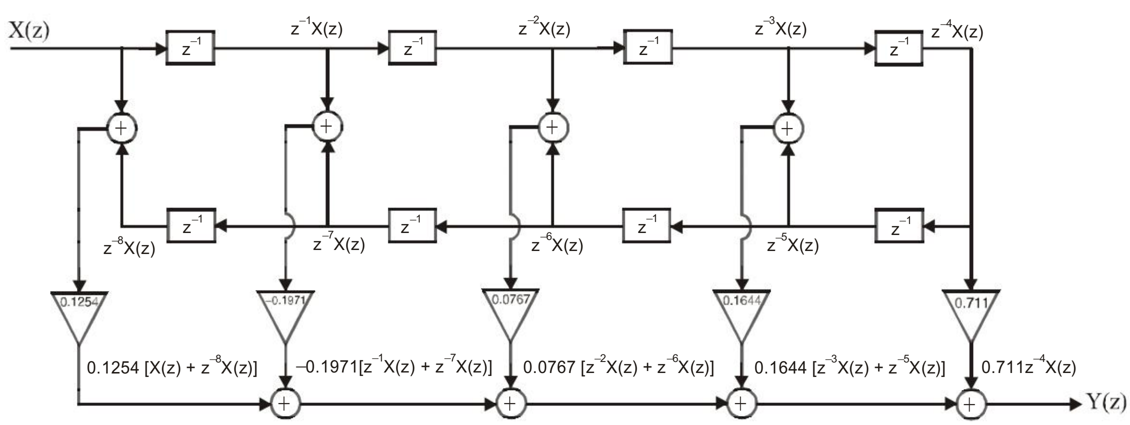


Fig 1 : Linear phase structure of FIR bandstop filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0, the magnitude function $|H(e^{j\omega})|$ is given by $|A(\omega)|$,

$$\text{where, } A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n \qquad \boxed{\text{Refer table 6.2 case (v)}}$$

$$= h(0) + \sum_{n=1}^{4} 2h(n)\cos\omega n$$

$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega + 2h(3)\cos 3\omega + 2h(4)\cos 4\omega$$

$$= 0.711 + 2\times 0.1644\cos\omega + 2\times 0.0767\cos 2\omega + 2\times -0.1971\cos 3\omega + 2\times 0.1254\cos 4\omega$$

$$= 0.711 + +0.3288\cos\omega + 0.1534\cos 2\omega - 0.3942\cos 3\omega + 0.2508\cos 4\omega$$

Using the above equation, the amplitude response $A(\omega)$ and magnitude function $|H(e^{j\omega})|$ are calculated for various values of ω and listed in table 1. Using the tabulated values, the magnitude response is sketched as shown in fig 2.

TABLE 1: $A(\omega)$ and $|H(e^{j\omega})|$ for various values of ω

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$\frac{0\times\pi}{16}$	1.0498	1.0498
$\frac{1\times\pi}{16}$	1.0247	1.0247
$\frac{2\times\pi}{16}$	0.9723	0.9723
$\frac{3\times\pi}{16}$	0.9426	0.9426
$\frac{4\times\pi}{16}$	0.9714	0.9714
$\frac{5\times\pi}{16}$	1.0442	1.0442
$\frac{6\times\pi}{16}$	1.0925	1.0925
$\frac{7\times\pi}{16}$	1.0297	1.0297
$\frac{8\times\pi}{16}$	0.8084	0.8084

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$\frac{9\times\pi}{16}$	0.4634	0.4634
$\frac{10\times\pi}{16}$	0.1125	0.1125
$\frac{11\times\pi}{16}$	–0.0943	0.0943
$\frac{12\times\pi}{16}$	–0.0510	0.0510
$\frac{13\times\pi}{16}$	0.2420	0.2420
$\frac{14\times\pi}{16}$	0.6665	0.6665
$\frac{15\times\pi}{16}$	1.0353	1.0353
$\frac{16\times\pi}{16}$	1.1806	1.1806

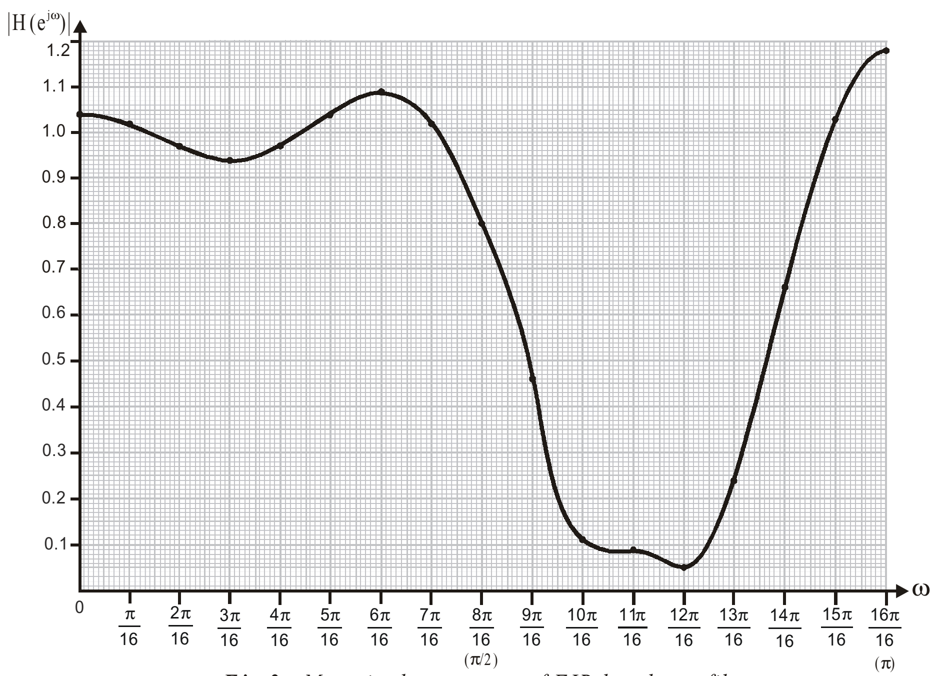


Fig 2 : Magnitude response of FIR bandstop filter.

E6.5. *Design a linear phase FIR lowpass filter using hamming window by taking 5 samples of window sequence and with a cutoff frequency, ω_c = 0.35π rad/sample.*

Solution

Given, $\omega_c = 0.35\pi$

Let the symmetry condition be h(– n) = h(n). Therefore, the desired ideal frequency response for FIR lowpass filter is,

$$H_d(e^{j\omega}) = 1 \quad ; \quad -\omega_c \le \omega \le +\omega_c$$
$$= 0 \quad ; \quad \text{otherwise}$$

The $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{j\omega})$.

By definition of inverse Fourier transform,

$$\therefore\ h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} 1\times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}}$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_c n}-e^{-j\omega_c n}}{2j}\right] = \frac{1}{\pi n}\sin\omega n \quad ; \quad \text{for all n except n = 0}$$

When n = 0, the $h_d(n)$ becomes 0/0, which is indeterminate.

$$\therefore\ \text{When, } n = 0\ ;\ h_d(n) = h_d(0) = \underset{n\to 0}{\text{Lt}}\ \frac{\sin\omega_c n}{\pi n}$$

Using L' Hospital rule, $\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin A\theta}{\theta} = A$

$$= \frac{1}{\pi}\ \underset{n\to 0}{\text{Lt}}\ \frac{\sin\omega_c n}{n} = \frac{1}{\pi}\omega_c = \frac{\omega_c}{\pi}$$

The impulse response of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

$$\text{Hamming window sequence, } w_H(n) = 0.54 + 0.46\cos\left(\tfrac{2\pi n}{N-1}\right) \quad ;\ \text{for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}$$

$$= 0 \qquad ;\ \text{otherwise}$$

$$\therefore\ \text{Impulse response, } h(n) = h_d(n)\times w_H(n)$$

Here, $N = 5,\ N-1 = 4\ ;\ \omega_c = 0.35\pi$ rad / sample ; $h(-n) = h(n)$

$$\text{When } n = 0\ ;\ h(0) = \left[\frac{0.35\pi}{\pi}\right]\left[0.54 + 0.46\frac{\cos 2\pi\times 0}{4}\right] = 0.35$$

$$\text{When } n = 1\ ;\ h(1) = \frac{(\sin 0.35\pi\times 1)}{\pi\times 1}\left[0.54 + 0.46\frac{\cos 2\pi\times 1}{4}\right] = 0.1531$$

Note : Calculate sinq by keeping the calculator in radian mode.

$$\text{When } n = 2\ ;\ h(2) = \frac{(\sin 0.35\pi\times 2)}{\pi\times 2}\left[0.54 + 0.46\frac{\cos 2\pi\times 2}{4}\right] = 0.0103$$

$$\text{When } n = -1\ ;\ h(-1) = h(1) = 0.1531$$

$$\text{When } n = -2\ ;\ h(-2) = h(2) = 0.0103$$

Using symmetry condition, $h(-n) = h(n)$.

The transfer function H(z) of FIR lowpass filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\, \mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n)\, z^{-n} = z^{-2}\sum_{n=-2}^{2} h(n)\, z^{-n}$$

$$= z^{-2}\left[h(-2)z^2 + h(-1)z + h(0) + h(1)z^{-1} + h(2)z^{-2}\right]$$

$$= z^{-2}\left[h(2)z^2 + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2}\right]$$

Using symmetry condition, $h(-n) = h(n)$.

$$= z^{-2}\left[h(2)\left[z^2 + z^{-2}\right] + h(1)\left[z + z^{-1}\right] + h(0)\right]$$

$$= z^{-2}h(0) + h(1)\left[z^{-1} + z^{-3}\right] + h(2)\left[z^0 + z^{-4}\right]$$

$$= 0.35z^{-2} + 0.1531\left[z^{-1} + z^{-3}\right] + 0.0103\left[1 + z^{-4}\right]$$

Structure

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = 0.35z^{-2} + 0.1531\left[z^{-1} + z^{-3}\right] + 0.0103\left[1 + z^{-4}\right]$$

$$\therefore\ Y(z) = 0.35z^{-2}X(z) + 0.1531\left[z^{-1}X(z) + z^{-3}X(z)\right] + 0.0103\left[X(z) + z^{-4}X(z)\right]$$

The above equation, the linear phase FIR lowpass filter structure is drawn as shown fig 1.

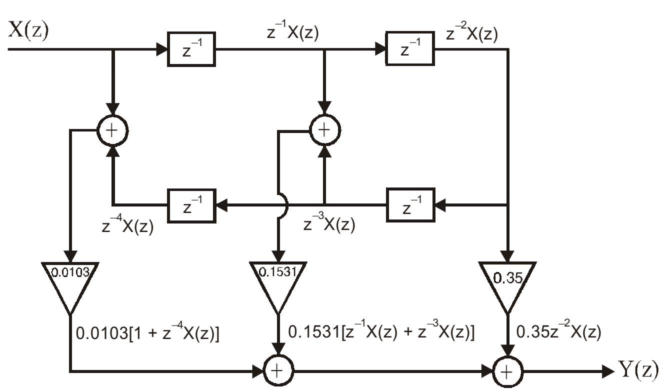


Fig 1 : *Linear phase structure of FIR lowpass filter.*

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0, the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$,

$$\text{where, } A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$$

Refer table 6.2 case (v)

$$\therefore\ A(\omega) = h(0) + \sum_{n=1}^{2} 2h(n)\cos\omega n$$

$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega$$

$$= 0.35 + 2 \times 0.1531\cos\omega + 2 \times 0.0103\cos 2\omega$$

$$= 0.35 + 0.3062\cos\omega + 0.0206\cos 2\omega$$

Using the above equation, the amplitude response, $A(\omega)$ and magnitude function $|H(e^{j\omega})|$ are calculated for various values of ω and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

TABLE 1: $A(\omega)$ and $|H(e^{j\omega})|$ for various values of ω

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$\frac{0\times\pi}{16}$	0.6768	0.6768
$\frac{1\times\pi}{16}$	0.6693	0.6693
$\frac{2\times\pi}{16}$	0.6474	0.6474
$\frac{3\times\pi}{16}$	0.6124	0.6124
$\frac{4\times\pi}{16}$	0.5665	0.5665
$\frac{5\times\pi}{16}$	0.5122	0.5122
$\frac{6\times\pi}{16}$	0.4526	0.4526
$\frac{7\times\pi}{16}$	0.3907	0.3907
$\frac{8\times\pi}{16}$	0.3294	0.3294

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$\frac{9\times\pi}{16}$	0.2712	0.2712
$\frac{10\times\pi}{16}$	0.2182	0.2182
$\frac{11\times\pi}{16}$	0.1720	0.1720
$\frac{12\times\pi}{16}$	0.1334	0.1334
$\frac{13\times\pi}{16}$	0.1032	0.1032
$\frac{14\times\pi}{16}$	0.0816	0.0816
$\frac{15\times\pi}{16}$	0.0687	0.0687
$\frac{16\times\pi}{16}$	0.0644	0.0644

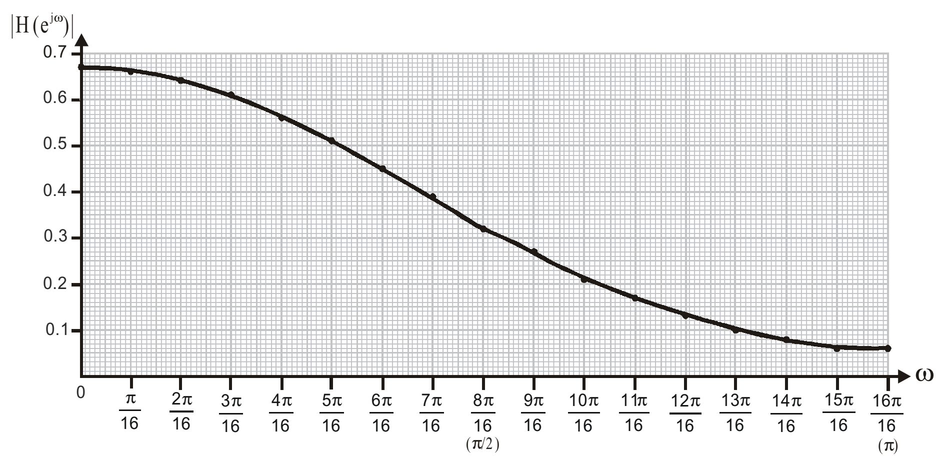


Fig 2 : Magnitude response of FIR lowpass filter.

E6.6 ***Design a linear phase FIR highpass filter using rectangular window, with a cutoff frequency, ω_c = 0.48π rad/sample and N = 5.***

Solution

Given, $\omega_c = 0.48\pi$

Let the symmetry condtion be h(–n) = h(n).Therefore, the desired ideal frequency response for FIR highpass filter is,

$$H_d(e^{j\omega}) = 1 \quad ; \quad -\pi \le \omega \le -\omega_c \ \text{ and } \ +\omega_c \le \omega \le +\pi$$

$$= 0 \quad ; \quad \text{otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{j\omega})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} 1\times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} 1\times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_c}^{\pi}$$

$$\therefore\ h_d(n) = \frac{1}{2\pi}\left[\frac{e^{-j\omega_c n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_c n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} - \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right]$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{\sin \pi n - \sin \omega_c n}{\pi n}\ ;\ \text{for all n, except n = 0}$$

When n = 0, the $h_d(n)$ becomes 0/0, which is indeterminate.

$$\text{When n = 0 ; } h_d(n) = h_d(0) = \underset{n \to 0}{\text{Lt}}\ \frac{\sin \pi n - \sin \omega_c n}{\pi n}$$

$$= \frac{1}{\pi}\ \underset{n \to 0}{\text{Lt}}\ \frac{\sin \pi n}{n} - \underset{n \to 0}{\text{Lt}}\ \frac{\sin \omega_c n}{n}$$

Using L' Hospital rule,

$$\underset{\theta \to 0}{\text{Lt}}\ \frac{\sin A\theta}{\theta} = A$$

$$= \frac{1}{\pi} \times \pi - \frac{1}{\pi}\omega_c = 1 - \frac{\omega_c}{\pi}$$

The impulse response h(n) of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

\ Impulse response, $h(n) = h_d(n)\, w_R(n)$

$$\text{Rectangular window sequence, } w_R(n) = 1\ ;\ n = -\tfrac{N-1}{2}\ \text{to}\ +\tfrac{N-1}{2}$$
$$= 0\ ;\ \text{otherwise}$$

Here N = 5, N – 1 = 4 ; w_c = 0.48p rad/sample ; h(–n) = h(n)

$$\text{When n = 0 ; } h(0) = \left(1 - \frac{0.48\pi}{\pi}\right) = 0.52$$

$$\text{When n = 1 ; } h(1) = \frac{-\sin(0.48 \times 1)}{\pi \times 1} = -0.3176$$

For any integer n, sin pn = 0

$$\text{When n = 2 ; } h(2) = \frac{-\sin(0.48 \times 2)}{\pi \times 2} = -0.0199$$

When n = –1 ; h(–1) = h(1) = –0.3176

When n = –2 ; h(–2) = h(2) = –0.0199

Using symmetry condition, h(–n) = h(n).

The transfer function H(z) of FIR highpass filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\, \mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}} \sum_{n=-\frac{N-1}{2}}^{+\frac{N-1}{2}} h(n) z^{-n} = z^{-2} \sum_{n=-2}^{2} h(n)\, z^{-n}$$

$$= z^{-2}\left[h(-2)z^{2} + h(-1)z + h(0) + h(1)z^{-1} + h(2)z^{-2}\right]$$

$$= z^{-2}\left[h(2)z^{2} + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2}\right]$$

$$= z^{-2}\left[h(0) + h(1)\left[z + z^{-1}\right] + h(2)\left[z^{2} + z^{-2}\right]\right]$$

Using symmetry condition, h(–n) = h(n).

$$= h(0)z^{-2} + h(1)\left[z^{-1} + z^{-3}\right] + h(2)\left[1 + z^{-4}\right]$$

$$= 0.52z^{-2} - 0.3176\left[z^{-1} + z^{-3}\right] - 0.0199\left[1 + z^{-4}\right]$$

Structure

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = 0.52z^{-2} - 0.3176\left[z^{-1} + z^{-3}\right] - 0.0199\left[1 + z^{-4}\right]$$

$$\therefore\ Y(z) = 0.52z^{-2}X(z) - 0.3176\left[z^{-1}X(z) + z^{-3}X(z)\right] - 0.0199\left[X(z) + z^{-4}X(z)\right]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

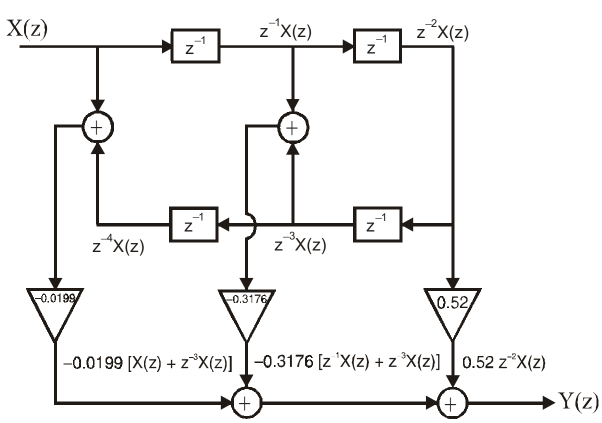


***Fig 1 :** Linear phase structure of FIR highpass filter.*

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0, the magnitude response $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where,} \quad A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n \qquad \boxed{\text{Refer table 6.2 case (v)}}$$

$$\therefore \; A(\omega) = h(0) + \sum_{n=1}^{2} 2h(n)\cos\omega n$$

$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega$$

$$= 0.52 + 2\times(-0.3176)\cos\omega + 2\times(-0.0199)\cos 2\omega$$

$$= 0.52 - 0.6352\cos\omega - 0.0398\cos 2\omega$$

Using the above equation, the amplitude response $A(w)$ and magnitude function $(e^{jw})|$ are calculated for various values of w and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

TABLE 1: $A(w)$ and $|H(e^{jw})|$ for various values of w

w	$A(w)$	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{0\times\pi}{16}$	–0.155	0.155
$\frac{1\times\pi}{16}$	–0.1397	0.1397
$\frac{2\times\pi}{16}$	–0.0949	0.0949
$\frac{3\times\pi}{16}$	–0.0233	0.0233
$\frac{4\times\pi}{16}$	0.0708	0.0708
$\frac{5\times\pi}{16}$	0.1823	0.1823
$\frac{6\times\pi}{16}$	0.3050	0.3050
$\frac{7\times\pi}{16}$	0.4328	0.4328
$\frac{8\times\pi}{16}$	0.5598	0.5598

w	$A(w)$	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{9\times\pi}{16}$	0.6806	0.6806
$\frac{10\times\pi}{16}$	0.7912	0.7912
$\frac{11\times\pi}{16}$	0.8881	0.8881
$\frac{12\times\pi}{16}$	0.9691	0.9691
$\frac{13\times\pi}{16}$	1.0329	1.0329
$\frac{14\times\pi}{16}$	1.0787	1.0787
$\frac{15\times\pi}{16}$	1.1062	1.1062
$\frac{16\times\pi}{16}$	1.1154	1.1154

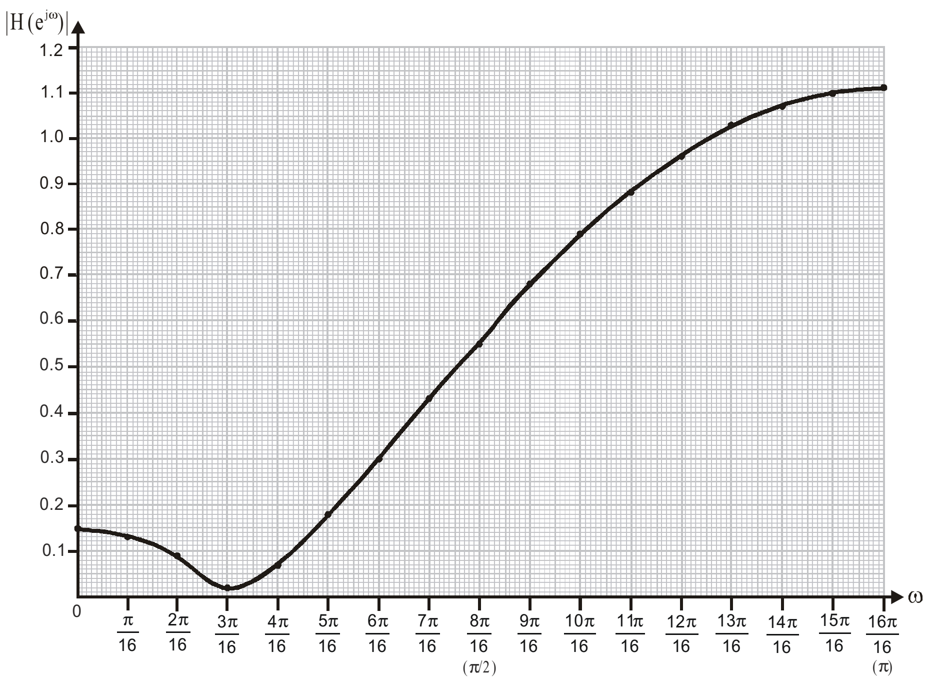


Fig 2 : Magnitude response of FIR highpass filter.

E6.7. Design a linear phase FIR bandpass filter to pass frequencies in the range 0.35p to 0.48p rad/sample by taking 5 samples of rectangular window sequence.

Solution

Given that, $w_{c1} = 0.35p$; $w_{c2} = 0.48p$

Let the symmetry condition $h(-n) = h(n)$. Therefore, the desired ideal frequency response $H_d(e^{jw})$ for FIR bandpass filter is,

$$H_d(e^{j\omega}) = 1 \;;\; -\omega_{c2} \le \omega \le -\omega_{c1} \;\;\&\;\; +\omega_{c1} \le \omega \le +\omega_{c2}$$

$$= 0 \;;\; \text{otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{j\omega})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} 1\times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} 1\times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c1}}^{\omega_{c2}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}n}}{jn} - \frac{e^{-j\omega_{c2}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2}n}}{jn} - \frac{e^{j\omega_{c1}n}}{jn}\right]$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j} - \frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j}\right]$$

$$= \frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n} \quad ; \text{ for all n, except n = 0}$$

When n = 0, the $h_d(n)$ becomes 0/0, which is indeterminate.

$$\text{When } n = 0\ ;\ h_d(n) = h_d(0) = \underset{n\to 0}{\text{Lt}}\ \frac{\sin \omega_{c2}n - \sin \omega_{c1}n}{\pi n}$$

Using L' Hospital rule, $\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin A\theta}{\theta} = A$

$$= \frac{1}{\pi}\ \underset{n\to 0}{\text{Lt}}\ \frac{\sin\omega_{c2}n}{n} - \underset{n\to 0}{\text{Lt}}\ \frac{\sin\omega_{c1}n}{n}$$

$$= \frac{1}{\pi}(\omega_{c2} - \omega_{c1})$$

The impulse response h(n) of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

$$\text{Rectangular window sequence, } w_R(n) = 1 \quad ; \text{ for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}$$
$$= 0 \quad ; \text{ otherwise}$$

$$\therefore \text{ Impulse response, } h(n) = h_d(n) \times w_R(n)$$
$$= h_d(n) \quad ; \text{ for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}$$

Here, N = 5 ; $w_{c1} = 0.35p$ rad/sample ; $w_{c2} = 0.48p$ rad/sample ; h(–n) = h(n) ; N – 1 = 4

Hence calculate h(n) for n = –2 to 2.

Since, h(n) satisfies the symmetry condition, h(– n) = h(n), calculate h(n) for n = 0 to 2.

$$\text{When } n = 0\ ;\ h(0) = \left[\frac{0.48\pi - 0.35\pi}{\pi}\right] = 0.13$$

$$\text{When } n = 1\ ;\ h(1) = \left[\frac{\sin(0.48\pi\times 1) - \sin(0.35\pi\times 1)}{\pi\times 1}\right] = 0.0340$$

$$\text{When } n = 2\ ;\ h(2) = \left[\frac{\sin(0.48\pi\times 2) - \sin(0.35\pi\times 2)}{\pi\times 2}\right] = -0.1088$$

When n = –1 ; h(–1) = h(1) = 0.0340

When n = –2 ; h(–2) = h(2) = –0.1088

Using symmetry condition, h(–n) = h(n).

The transfer function H(z) of FIR bandpass filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\, \mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}} \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} h(n)z^{-n} = z^{-2}\sum_{n=-2}^{2} h(n)z^{-n}$$

$$= z^{-2}\left[h(-2)z^2 + h(-1)z + h(0) + h(1)z^{-1} + h(2)z^{-2}\right]$$

Using symmetry condition, h(–n) = h(n).

$$= z^{-2}\left[h(2)\left[z^2 + z^{-2}\right] + h(1)\left[z + z^{-1}\right] + h(0)\right]$$

$$= z^{-2}h(0) + h(1)\left[z^{-1} + z^{-3}\right] + h(2)\left[z^0 + z^{-4}\right]$$

$$= 0.13z^{-2} + 0.0340\left[z^{-1} + z^{-3}\right] - 0.1088\left[1 + z^{-4}\right]$$

Structure

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = 0.13z^{-2} + 0.0340\left[z^{-1} + z^{-3}\right] - 0.1088\left[1 + z^{-4}\right]$$

$$\therefore\ Y(z) = 0.13z^{-2}X(z) + 0.0340\left[z^{-1}X(z) + z^{-3}X(z)\right] - 0.1088\left[X(z) + z^{-4}X(z)\right]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

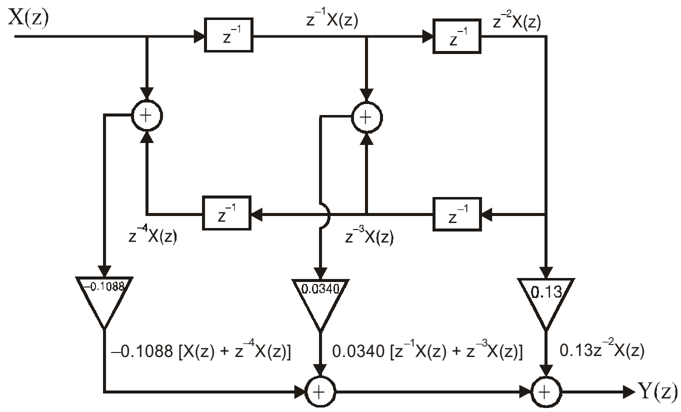


Fig 1 : Linear phase structure for FIR bandpass filter.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0 the magnitude response $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where, } A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$$

Refer table 6.2 case (v)

$$= h(0) + \sum_{n=1}^{2} 2h(n)\cos\omega n$$

$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega$$

$$= 0.13 + 2 \times 0.0340\cos\omega + 2 \times (-0.1080)\cos 2\omega$$

$$= 0.13 + 0.068\cos\omega - 0.2176\cos 2\omega$$

Using the above equation, the amplitude response, $A(w)$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

TABLE 1: $A(w)$ and $|H(e^{jw})|$ for various values of w

w	$A(w)$	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{0\times\pi}{16}$	–0.0196	0.0196
$\frac{1\times\pi}{16}$	–0.0043	0.0043
$\frac{2\times\pi}{16}$	0.0389	0.0389
$\frac{3\times\pi}{16}$	0.1032	0.1032
$\frac{4\times\pi}{16}$	0.1780	0.1780
$\frac{5\times\pi}{16}$	0.2510	0.2510
$\frac{6\times\pi}{16}$	0.3098	0.3098
$\frac{7\times\pi}{16}$	0.3443	0.3443
$\frac{8\times\pi}{16}$	0.3476	0.3476

w	$A(w)$	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{9\times\pi}{16}$	0.3177	0.3177
$\frac{10\times\pi}{16}$	0.2578	0.2578
$\frac{11\times\pi}{16}$	0.1754	0.1754
$\frac{12\times\pi}{16}$	0.0819	0.0819
$\frac{13\times\pi}{16}$	–0.0098	0.0098
$\frac{14\times\pi}{16}$	–0.0866	0.0866
$\frac{15\times\pi}{16}$	–0.1377	0.1377
$\frac{16\times\pi}{16}$	–0.1556	0.1556

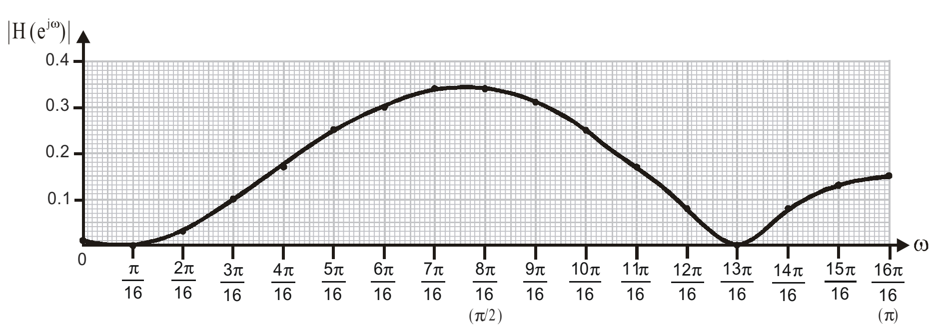


Fig 2 : Linear phase structure of FIR bandpass filter.

E6.8 Design a linear phase FIR bandstop filter to reject frequencies in the range 0.35p to 0.48p rad/sample using rectangular window, by taking 5 samples of window sequence.

Solution

Given that, $\omega_{c1} = 0.35\pi$; $\omega_{c2} = 0.48\pi$

Let the symmetry condition be $h(-n) = h(n)$. Therefore, the desired ideal frequency response $H_d(e^{j\omega})$ for FIR bandstop filter is,

$$H_d(e^{j\omega}) = 1 \quad ; -\pi \le \omega \le -\omega_{c2} \text{ and } -\omega_{c1} \le \omega \le +\omega_{c1} \text{ and } \omega_{c2} \le \omega \le +\pi$$
$$= 0 \quad ; \text{otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse Fourier transform of $H_d(e^{j\omega})$.

By definition of inverse Fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} 1 \times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} 1 \times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} 1 \times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c1}}^{\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c2}}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c2}n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c1}n}}{jn} - \frac{e^{-j\omega_{c1}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_{c2}n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} + \frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j} - \frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j}\right]$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{\sin\pi n + \sin\omega_{c1}n - \sin\omega_{c2}n}{\pi n} \quad ; \text{ for all } n, \text{ except } n = 0.$$

When $n = 0$, the $h_d(n)$ becomes 0/0, which is indeterminate.

$$\text{When } n = 0 \; ; \; h_d(n) = h_d(0) = \underset{n\to 0}{\text{Lt}} \frac{\sin\pi n + \sin\omega_{c1}n - \sin\omega_{c2}n}{\pi n}$$

$$= \frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\frac{\sin\pi n}{n} + \frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\frac{\sin\omega_{c1}n}{n} + \frac{1}{\pi}\underset{n\to 0}{\text{Lt}}\frac{\sin\omega_{c2}n}{n}$$

Using L' Hospital rule, $\underset{\theta\to 0}{\text{Lt}} \frac{\sin A\theta}{\theta} = A$

$$= \frac{1}{\pi}\times\pi + \frac{1}{\pi}\times\omega_{c1} - \frac{1}{\pi}\times\omega_{c2} = 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)$$

The impulse response $h(n)$ of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

$$\text{Rectangular window sequence, } w_R(n) = 1 \; ; \; n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}$$
$$= 0 \; ; \; \text{otherwise}$$

$$\therefore \text{ Impulse response, } h(n) = h_d(n) \times w_R(n)$$
$$= h_d(n) \; ; \; \text{for } n = -\tfrac{N-1}{2} \text{ to } +\tfrac{N-1}{2}$$

Here, $N = 5$, $N - 1 = 4$; $\omega_{c1} = 0.4\pi$ rad/sample ; $\omega_{c2} = 0.65\pi$ rad/sample

Hence, calculate $h(n)$ for $n = -2$ to 2.

Since, $h(n)$ satisfies the symmetry condition, $h(-n) = h(n)$, calculate $h(n)$ for $n = 0$ to 2.

$$\text{When } n = 0 \; ; \; h(0) = 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right) = 0.87$$

$$\text{When } n = 1 \; ; \; h(1) = \frac{\sin(0.35\pi\times 1) - \sin(0.48\pi \times 1)}{\pi \times 1} = -0.0340$$

$$\text{When } n = 2 \; ; \; h(2) = \frac{\sin(0.35\pi\times 2) - \sin(0.48\pi \times 2)}{\pi \times 2} = 0.1088$$

When $n = -1$; $h(-1) = h(1) = -0.0340$

When $n = -2$; $h(-2) = h(2) = 0.1088$

Using symmetry condition, $h(-n) = h(n)$.

The transfer function $H(z)$ of the digital FIR bandstop filter is given by,

$$H(z) = z^{-\frac{N-1}{2}}\, \mathcal{Z}\{h(n)\} = z^{-\frac{N-1}{2}} \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} h(n)\, z^{-n} = z^{-2}\sum_{n=-2}^{2} h(n)\, z^{-n}$$

$$= z^{-2}\left[h(-2)z^{2} + h(-1)z + h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2}\right]$$

Using symmetry condition, $h(-n) = h(n)$.

$$= z^{-2}\left[h(1)\left[z + z^{-1}\right] + h(2)\left[z^{2} + z^{-2}\right] + h(0)\right]$$

$$= h(0)z^{-2} + h(1)\left[z^{-1} + z^{-3}\right] + h(2)\left[1 + z^{-4}\right]$$

$$= 0.87\,z^{-2} - 0.0340\left[z^{-1} + z^{-3}\right] + 0.1088\left[1 + z^{-4}\right]$$

Structure

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)} = 0.87\,z^{-2} - 0.0340\left[z^{-1} + z^{-3}\right] + 0.1088\left[1 + z^{-4}\right]$$

$$\therefore\quad Y(z) = 0.87\,z^{-2}\,X(z) - 0.0340\left[z^{-1}X(z) + z^{-3}\,X(z)\right] + 0.1088\left[X(z) + z^{-4}\,X(z)\right]$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

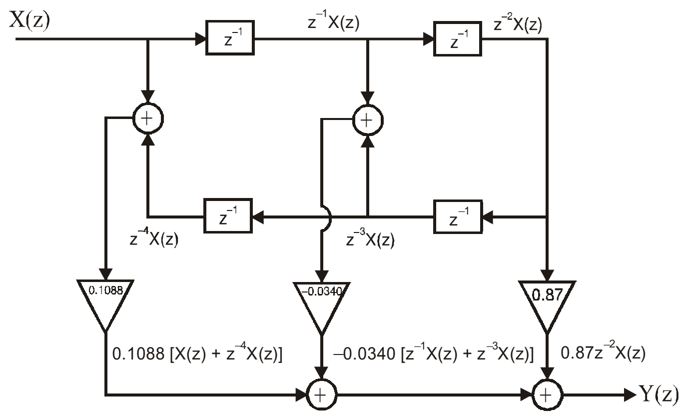


***Fig 1 :** Linear phase structure for FIR bandstop filter.*

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at n = 0 the magnitude response $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where,}\quad A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$$

Refer table 6.2 case (v)

$$= h(0) + \sum_{n=1}^{3} 2h(n)\cos\omega n$$

$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega$$

$$= 0.87 + 2\times(-0.0340)\cos\omega + 2\times 0.1080\cos 2\omega$$

$$= 0.87 - 0.068\cos\omega + 0.2176\cos 2\omega$$

Using the above equation, the amplitude response, $A(w)$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

TABLE 1: $A(w)$ and $|H(e^{jw})|$ for various values of w

w	$A(w)$	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$	w	$A(w)$	$\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
$\frac{0\times\pi}{16}$	1.0196	1.0196	$\frac{9\times\pi}{16}$	0.6822	0.6822
$\frac{1\times\pi}{16}$	1.0043	1.0043	$\frac{10\times\pi}{16}$	0.7421	0.7421
$\frac{2\times\pi}{16}$	0.9610	0.9610	$\frac{11\times\pi}{16}$	0.8245	0.8245
$\frac{3\times\pi}{16}$	0.8967	0.8967	$\frac{12\times\pi}{16}$	0.9180	0.9180
$\frac{4\times\pi}{16}$	0.8219	0.8219	$\frac{13\times\pi}{16}$	1.0098	1.0098
$\frac{5\times\pi}{16}$	0.7489	0.7489	$\frac{14\times\pi}{16}$	1.0866	1.0866
$\frac{6\times\pi}{16}$	0.6901	0.6901	$\frac{15\times\pi}{16}$	1.1377	1.1377
$\frac{7\times\pi}{16}$	0.6556	0.6556	$\frac{16\times\pi}{16}$	1.1556	1.1556
$\frac{8\times\pi}{16}$	0.6524	0.6524			

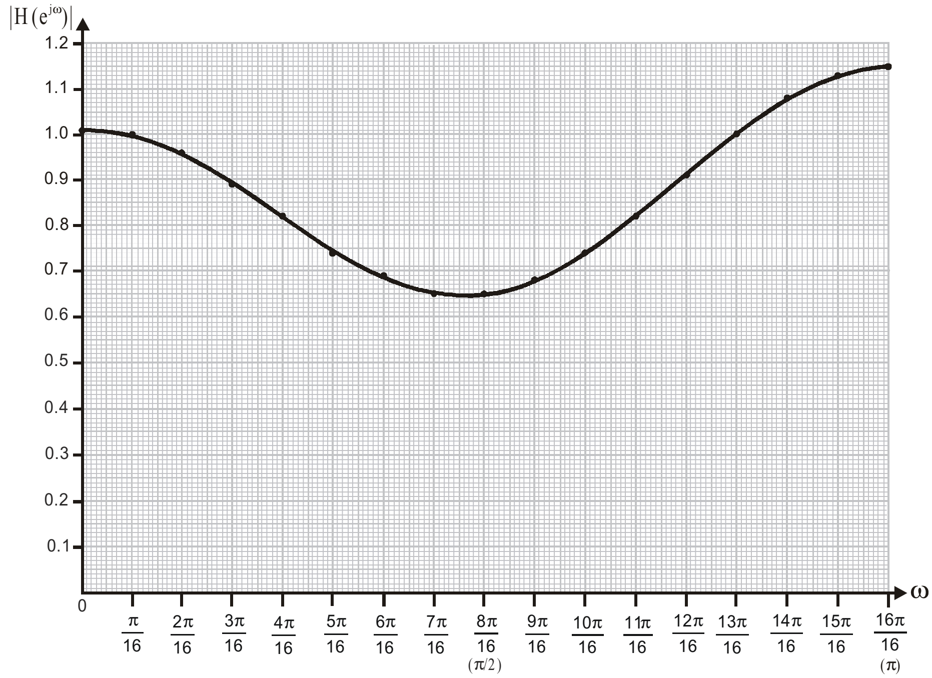


Fig 2 : *Magnitude response of FIR bandstop filter.*

E6.9 ***Determine the coefficients of a linear phase FIR filter of length N = 11 which has a symmetric unit sample response and a frequency response that satisfies the conditions***

$$H\left(\frac{2\pi k}{11}\right) = 1 \quad ; \quad \text{for } k = 0, 1, 2, 3$$
$$= 0 \quad ; \quad \text{for } k = 4, 5$$

Solution

For linear phase FIR filter the phase function, $\theta(\omega) = -\alpha\omega$ where $\alpha = \frac{N-1}{2}$.

Here, N = 11, $\therefore \ \alpha = \dfrac{11-1}{2} = 5$.

Also, here $\omega = \omega_k = \dfrac{2\pi k}{N} = \dfrac{2\pi k}{11}$. Hence we can go for type-1 design.

In this problem the samples of the magnitude response of the ideal (desired) filter are directly given for various values of k.

$$\therefore \ H(k) = H_d(e^{j\omega})\Big|_{\omega = \omega_k} = 1\, e^{-j\alpha\omega_k} = e^{-j5\times\frac{2\pi k}{11}} \quad ; \ k = 0, 1, 2, 3$$
$$= 0 \quad ; \ k = 4, 5, 6, 7$$
$$= 1\, e^{-j\alpha\omega_k} = e^{-j5\times\frac{2\pi k}{11}} \quad ; \ k = 8, 9, 10$$

The samples of impulse response h(n) are given by,

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi nk}{N}}\right]\right]$$

Using equation(6.76).

$$= \frac{1}{11}\left[H(0) + 2\sum_{k=1}^{5} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi nk}{11}}\right]\right]$$

$$= \frac{1}{11}\left[H(0) + 2\sum_{k=1}^{3} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi nk}{11}}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{3} \mathrm{Re}\left[e^{-j5\times\frac{2\pi k}{11}} \times e^{\frac{j2\pi nk}{11}}\right]\right]$$

H(0) = 1

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{3} \mathrm{Re}\left[e^{\frac{j2\pi k}{11}(n-5)}\right]\right]$$

$e^{j\theta} = \cos\theta + j\sin\theta$

$\therefore \ \mathrm{Re}[e^{j\theta}] = \cos\theta$

$$\therefore\ h(n) = \frac{1}{11}\left[1 + 2\sum_{k=1}^{3}\cos\frac{2\pi k}{11}(n-5)\right]$$

$$= \frac{1}{11}\left[1 + 2\cos\frac{2\pi(n-5)}{11} + 2\cos\frac{4\pi(n-5)}{11} + 2\cos\frac{6\pi(n-5)}{11}\right]$$

Here N = 11, \ N – 1 = 10 ; $\frac{N-1}{2} = 5$

Hence, calculate h(n) for n = 0 to 10

Since h(n) satisfies the symmetry condition h(N – 1 – n) = h(n) with centre of symmetry at (N – 1)/2, calculate h(n) for n = 0 to 5.

When n = 0 ; $h(0) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(0-5)}{11} + 2\cos\frac{4\pi(0-5)}{11} + 2\cos\frac{6\pi(0-5)}{11}\right] = -0.0496$

When n = 1 ; $h(1) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(1-5)}{11} + 2\cos\frac{4\pi(1-5)}{11} + 2\cos\frac{6\pi(1-5)}{11}\right] = 0.0989$

When n = 2 ; $h(2) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(2-5)}{11} + 2\cos\frac{4\pi(2-5)}{11} + 2\cos\frac{6\pi(2-5)}{11}\right] = -0.0338$

When n = 3 ; $h(3) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(3-5)}{11} + 2\cos\frac{4\pi(3-5)}{11} + 2\cos\frac{6\pi(3-5)}{11}\right] = -0.1270$

When n = 4 ; $h(4) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(4-5)}{11} + 2\cos\frac{4\pi(4-5)}{11} + 2\cos\frac{6\pi(4-5)}{11}\right] = 0.2935$

When n = 5 ; $h(5) = \frac{1}{11}\left[1 + 2\cos\frac{2\pi(5-5)}{11} + 2\cos\frac{4\pi(5-5)}{11} + 2\cos\frac{6\pi(5-5)}{11}\right] = 0.6363$

When n = 6 ; h(6) = h(11 – 1 – 6) = h(4) = 0.2935

When n = 7 ; h(7) = h(11 – 1 – 7) = h(3) = –0.1270

When n = 8 ; h(8) = h(11 – 1 – 8) = h(2) = –0.0338

When n = 9 ; h(9) = h(11 – 1 – 9) = h(1) = 0.0989

When n = 10 ; h(10) = h(11 – 1 – 10) = h(0) = –0.0496

Using symmetry condition h(N – 1 – n) = h(n)

The transfer function H(z) of the filter is given by Ƶ-transform of h(n).

$$\therefore\ H(z) = Ƶ\{h(n)\} = \sum_{n=0}^{N-1} h(n)\,z^{-n} = \sum_{n=0}^{10} h(n)\,z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8} + h(9)z^{-9} + h(10)z^{-10}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(4)z^{-6} + h(3)z^{-7} + h(2)z^{-8} + h(1)z^{-9} + h(0)z^{-10}$$

Using symmetry condition, h(N – 1 – n) = h(n)

$$= h(0)\left[1 + z^{-10}\right] + h(1)\left[z^{-1} + z^{-9}\right] + h(2)\left[z^{-2} + z^{-8}\right] + h(3)\left[z^{-3} + z^{-7}\right] + h(4)\left[z^{-4} + z^{-6}\right] + h(5)z^{-5}$$

$$= -0.0496\left[1 + z^{-10}\right] + 0.0989\left[z^{-1} + z^{-9}\right] - 0.0338\left[z^{-2} + z^{-8}\right] - 0.1270\left[z^{-3} + z^{-7}\right] + 0.2935\left[z^{-4} + z^{-6}\right] + 0.6363z^{-5}$$

Structure

Let, $H(z) = \frac{Y(z)}{X(z)} = -0.0496\left[1 + z^{-10}\right] + 0.0989\left[z^{-1} + z^{-9}\right] - 0.0338\left[z^{-2} + z^{-8}\right] - 0.1270\left[z^{-3} + z^{-7}\right] + 0.2935\left[z^{-4} + z^{-6}\right] + 0.6363\,z^{-5}$

$$\therefore\ Y(z) = -0.0496\left[X(z) + z^{-10}X(z)\right] + 0.0989\left[z^{-1}X(z) + z^{-9}X(z)\right] - 0.0338\left[z^{-2}X(z) + z^{-8}X(z)\right] - 0.1270\left[z^{-3}X(z) + z^{-7}X(z)\right] + 0.2935\left[z^{-4}X(z) + z^{-6}X(z)\right] + 0.6363\,z^{-5}X(z)$$

The above equation can be used to draw the FIR filter structure as shown in fig 1.

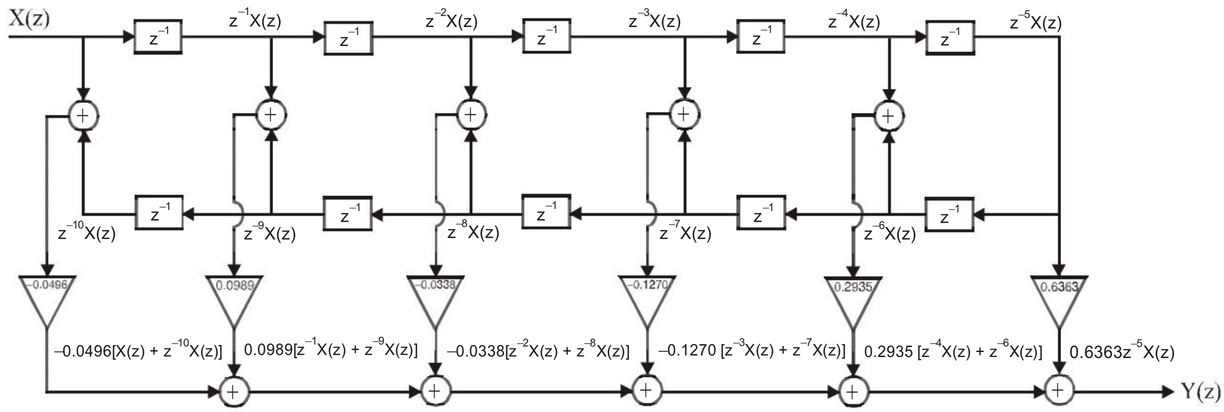


Fig 1.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at (N –1)/2 the magnitude response $|H(e^{jw})|$ is given by $|A(w)|$,

$$\text{where, } A(\omega) = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2} - n\right)\cos\omega n$$

Refer table 6.2 case (i)

$$= h(5) + \sum_{n=1}^{5} 2h(5-n)\cos\omega n$$

$$= h(5) + 2h(4)\cos\omega + 2h(3)\cos 2\omega + 2h(2)\cos 3\omega + 2h(1)\cos 4\omega + 2h(0)\cos 5\omega$$

$$= 0.6363 + 2 \times 0.2935\cos\omega + 2\times(-0.1270)\cos 2\omega + 2\times(-0.0338)\cos 3\omega$$
$$+ 2\times 0.0989\cos 4\omega + 2\times(-0.0496)\cos 5\omega$$

$$= 0.6363 + 0.587\cos\omega - 0.254\cos 2\omega - 0.0676\cos 3\omega + 0.1978\cos 4\omega - 0.0992\cos 5\omega$$

Using the above equation, the amplitude response, $A(w)$ and magnitude function $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using these values the magnitude response is plotted as shown in fig 2.

TABLE 1: $A(w)$ and $|H(e^{jw})|$ for various values of w

w	A(w)	$\|H(e^{jw})\| = \|A(w)\|$
$\frac{0\times\pi}{16}$	1.0003	1.0003
$\frac{1\times\pi}{16}$	1.0059	1.0059
$\frac{2\times\pi}{16}$	1.0111	1.0111
$\frac{3\times\pi}{16}$	0.9978	0.9978
$\frac{4\times\pi}{16}$	0.9715	0.9715
$\frac{5\times\pi}{16}$	0.9667	0.9667
$\frac{6\times\pi}{16}$	1.0113	1.0113
$\frac{7\times\pi}{16}$	1.0804	1.0804
$\frac{8\times\pi}{16}$	1.0881	1.0881
$\frac{9\times\pi}{16}$	0.9412	0.9412
$\frac{10\times\pi}{16}$	0.6204	0.6204
$\frac{11\times\pi}{16}$	0.2205	0.2205
$\frac{12\times\pi}{16}$	–0.0945	0.0945
$\frac{13\times\pi}{16}$	–0.1993	0.1993
$\frac{14\times\pi}{16}$	–0.0977	0.0977
$\frac{15\times\pi}{16}$	0.0770	0.0770
$\frac{16\times\pi}{16}$	0.1599	0.1599

w	A(w)	$\|H(e^{jw})\| = \|A(w)\|$
$\frac{17\times\pi}{16}$	0.0770	0.0770
$\frac{18\times\pi}{16}$	–0.0977	0.0977
$\frac{19\times\pi}{16}$	–0.1993	0.1993
$\frac{20\times\pi}{16}$	–0.0945	0.0945
$\frac{21\times\pi}{16}$	0.2205	0.2205
$\frac{22\times\pi}{16}$	0.6204	0.6204
$\frac{23\times\pi}{16}$	0.9412	0.9412
$\frac{24\times\pi}{16}$	1.0881	1.0881
$\frac{25\times\pi}{16}$	1.0804	1.0804
$\frac{26\times\pi}{16}$	1.0113	1.0113
$\frac{27\times\pi}{16}$	0.9667	0.9667
$\frac{28\times\pi}{16}$	0.9715	0.9715
$\frac{29\times\pi}{16}$	0.9978	0.9978
$\frac{30\times\pi}{16}$	1.0111	1.0111
$\frac{31\times\pi}{16}$	1.0059	1.0059
$\frac{32\times\pi}{16}$	1.0003	1.0003

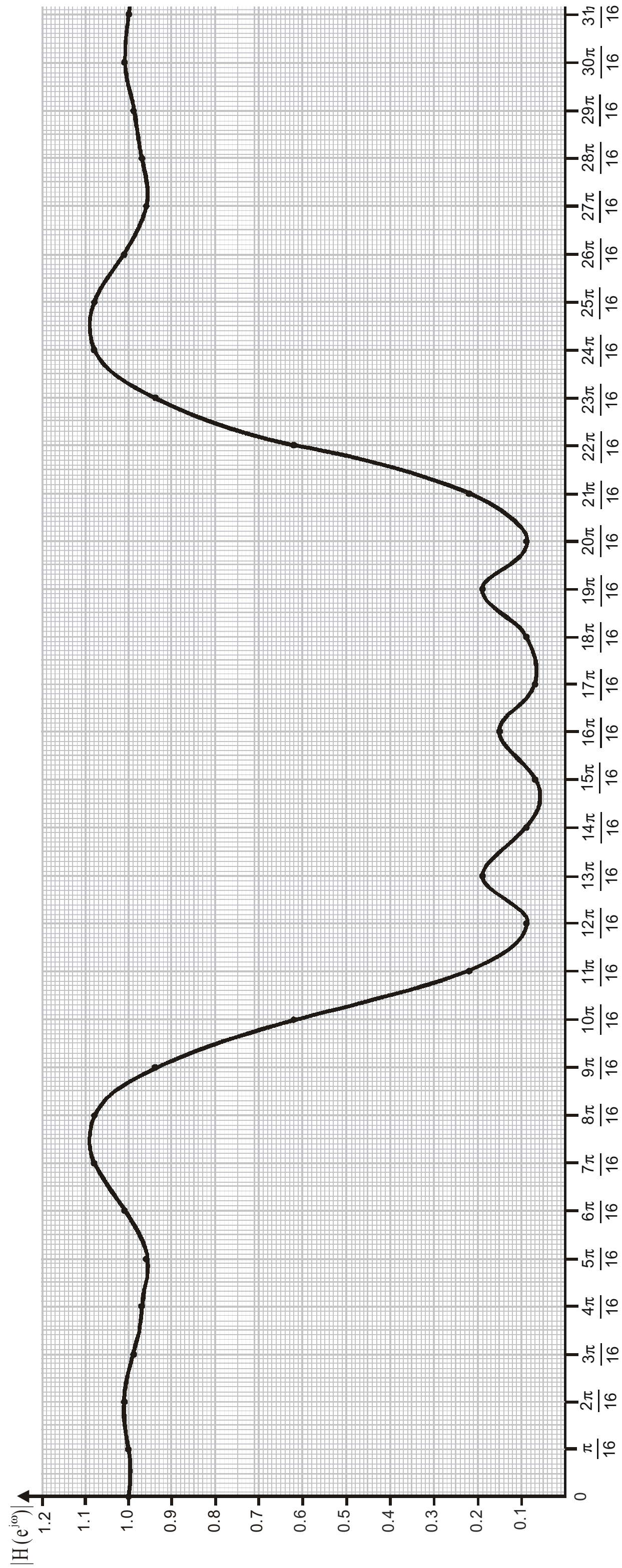


Fig 2 : Magnitude response of FIR Linear phase filter.

E6.10 Design a linear phase FIR lowpass filter for the desired frequency response as given below, by frequency samplingtechnique for N = 7.

$$H_d(e^{j\omega}) = e^{-j3\omega} \quad ; \quad 0 \le \omega \le 0.6\pi \text{ and } 1.4\pi \le \omega \le 2\pi$$
$$= 0 \quad ; \quad 0.6\pi < \omega < 1.4\pi$$

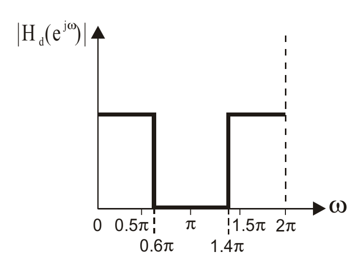


Fig 1 : Ideal magnitude response of FIR lowpass filter.

Solution

The magnitude response for ideal lowpass filter is shown in fig 1.

The desired frequency response $H_d(e^{j\omega})$ of linear phase FIR lowpass filter with cutoff frequency 0.6π rad/sample is given by,

$$H_d(e^{j\omega}) = e^{-j3\omega} \quad ; \quad 0 \le \omega \le 0.6\pi \text{ and } 1.4\pi \le \omega \le 2\pi$$
$$= 0 \quad ; \quad 0.6\pi < \omega < 1.4\pi$$

where, $N = 7$; $\alpha = 3$

The DFT sequence H(k) is obtained by sampling $H_d(e^{j\omega})$ at 7 equidistant frequency points in a period of 2π. The 7 frequencies for type-1 design are given by,

$$\omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{7} \quad ; \text{ for k = 0 to 6.}$$

$$\therefore \; H(k) = H_d(\omega)\Big|_{\omega = \frac{2\pi k}{N} = \frac{2\pi k}{7}}$$

When $k = 0$; $\omega_k = \frac{2\pi \times 0}{7} = 0$

When $k = 1$; $\omega_k = \frac{2\pi \times 1}{7} = 0.28\pi$

When $k = 2$; $\omega_k = \frac{2\pi \times 2}{7} = 0.57\pi$

When $k = 3$; $\omega_k = \frac{2\pi \times 3}{7} = 0.86\pi$

When $k = 4$; $\omega_k = \frac{2\pi \times 4}{7} = 1.14\pi$

When $k = 5$; $\omega_k = \frac{2\pi \times 5}{7} = 1.43\pi$

When $k = 6$; $\omega_k = \frac{2\pi \times 6}{7} = 1.71\pi$

From the above calculations the following observations can be made.

For k = 0 to 2, the samples lie in the range $0 \le \omega \le 0.6\pi$

For k = 3 to 4, the samples lie in the range $0.6\pi < \omega < 1.4\pi$

For k = 5 to 6, the samples lie in the range $1.4\pi \le \omega \le 2\pi$

The sampling points of the ideal frequency response are shown in fig 2. The magnitude samples of H(k) (Magnitude spectrum) are shown in fig 3.

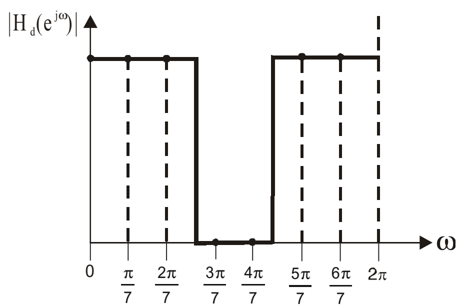


Fig 2 : Sampling points of $H_d(e^{j\omega})$.

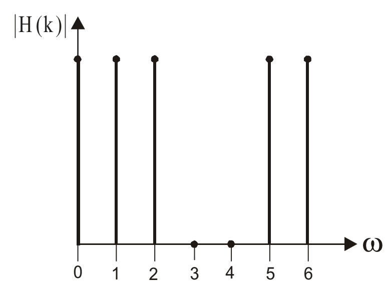


Fig 3 : Magnitude spectrum of H(k).

Based on the above discussions, the equation for DFT coefficients H(k) can be written as shown below.

$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \omega_k} = e^{-j3\times\frac{2\pi k}{7}} \quad ; \text{ for } k = 0, 1, 2$$
$$= 0 \quad ; \text{ for } k = 3, 4$$
$$= e^{-j3\times\frac{2\pi k}{7}} \quad ; \text{ for } k = 5, 6$$

The samples of impulse response, h(n) are given by,

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi nk}{N}}\right]\right]$$

Using equation(6.76).

$$= \frac{1}{7}\left[H(0) + 2\sum_{k=1}^{2} \mathrm{Re}\left[H(k)\, e^{\frac{j2\pi nk}{7}}\right]\right]$$

$$= \frac{1}{7}\left[1 + 2\sum_{k=1}^{2} \mathrm{Re}\left[e^{-j3\times\frac{2\pi k}{7}}\, e^{\frac{j2\pi nk}{7}}\right]\right]$$

H(0) = 1

$$= \frac{1}{7}\left[1 + 2\sum_{k=1}^{2} \mathrm{Re}\left[e^{\frac{j2\pi k(n-3)}{7}}\right]\right]$$

$$= \frac{1}{7}\left[1 + 2\cos\frac{2\pi(n-3)}{7} + 2\cos\frac{4\pi(n-3)}{7}\right]$$

$e^{j\theta} = \cos\theta + j\sin\theta$

$\therefore \; \mathrm{Re}[e^{j\theta}] = \cos\theta$

Here $N = 7$, \ $N - 1 = 6$; $\dfrac{N-1}{2} = 3$

Hence, calculate h(n) for n = 0 to 6

Since h(n) satisfies the symmetry condition $h(N-1-n) = h(n)$ with centre of symmetry at $(N-1)/2$, calculate h(n) for n = 0 to 3.

$$\text{When } n=0\ ;\ h(0) = \frac{1}{7}\left[1 + 2\cos\frac{2\pi(0-3)}{7} + 2\cos\frac{4\pi(0-3)}{7}\right] = 0.0635$$

$$\text{When } n=1\ ;\ h(1) = \frac{1}{7}\left[1 + 2\cos\frac{2\pi(1-3)}{7} + 2\cos\frac{4\pi(1-3)}{7}\right] = -0.1781$$

$$\text{When } n=2\ ;\ h(2) = \frac{1}{7}\left[1 + 2\cos\frac{2\pi(2-3)}{7} + 2\cos\frac{4\pi(2-3)}{7}\right] = 0.2574$$

$$\text{When } n=3\ ;\ h(3) = \frac{1}{7}\left[1 + 2\cos\frac{2\pi(3-3)}{7} + 2\cos\frac{4\pi(3-3)}{7}\right] = 0.7142$$

When $n = 4$; $h(4) = h(7-1-4) = h(2) = 0.2574$

When $n = 5$; $h(5) = h(7-1-5) = h(1) = -0.1781$

When $n = 6$; $h(6) = h(7-1-6) = h(0) = 0.0635$

Using symmetry condition $h(N-1-n) = h(n)$

The transfer function H(z) of the filter is given by $\mathcal{Z}$-transform of h(n).

$$\therefore\ H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{6} h(n)\, z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(2)z^{-4} + h(1)z^{-5} + h(0)z^{-6}$$

Using symmetry condition $h(N-1-n) = h(n)$

$$= h(0)\left[1 + z^{-6}\right] + h(1)\left[z^{-1} + z^{-5}\right] + h(2)\left[z^{-2} + z^{-4}\right] + +h(3)z^{-3}$$

$$= 0.0635\left[1 + z^{-6}\right] - 0.1781\left[z^{-1} + z^{-5}\right] + 0.2574\left[z^{-2} + z^{-4}\right] + 0.7142z^{-3}$$

Structure

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = 0.0635\left[1 + z^{-6}\right] - 0.1781\left[z^{-1} + z^{-5}\right] + 0.2574\left[z^{-2} + z^{-4}\right] + 0.7142z^{-3}$$

$$\therefore\ Y(z) = 0.0635\left[X(z) + z^{-6}X(z)\right] - 0.1781\left[z^{-1}X(z) + z^{-5}X(z)\right] + 0.2574\left[z^{-2}X(z) + z^{-4}X(z)\right] + 0.7142z^{-3}X(z)$$

The above equation can be used to draw the FIR filter structure as shown in fig 4.

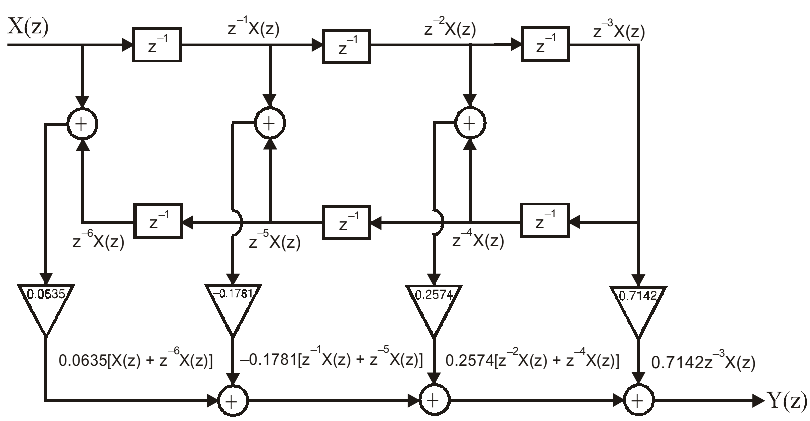


Fig 4.

Frequency Response

When impulse response is symmetric and N is odd with centre of symmetry at $(N-1)/2$ the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$,

$$\text{where, } A(\omega) = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2} - n\right)\cos\omega n$$

Refer table 6.2 case (i)

$$= h(3) + \sum_{n=1}^{3} 2h(3-n)\cos\omega n$$

$$= h(3) + 2h(2)\cos\omega + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega$$

$$= 0.7142 + 2 \times 0.2574\cos\omega + 2 \times (-0.1781)\cos 2\omega + 2 \times 0.0635\cos 3\omega$$

$$= 0.7142 + 0.5148\cos\omega - 0.3562\cos 2\omega + 0.127\cos 3\omega$$

Using the above equation, the magnitude response, A(ω) and magnitude function $|H(e^{j\omega})|$ are calculated for various values of ω and listed in table 1. Using these values the magnitude response is plotted as shown in fig 5.

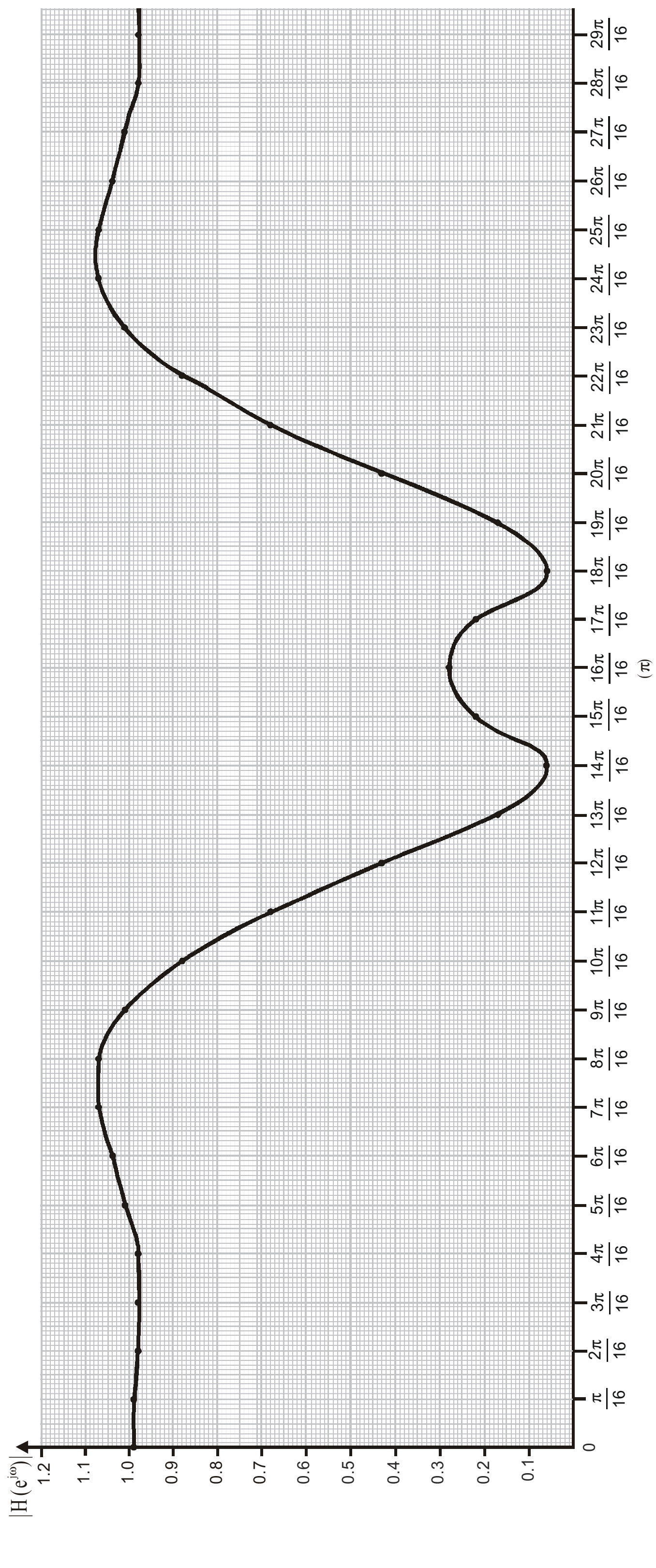


Fig 5 : Magnitude response of FIR Linear phase filter.

TABLE 1: A(w) and |H(e^jw)| for various values of w

w	A(w)	|H(e^jw)| = |A(w)|
$\frac{0\times\pi}{16}$	0.9998	0.9998
$\frac{1\times\pi}{16}$	0.9956	0.9956
$\frac{2\times\pi}{16}$	0.9865	0.9865
$\frac{3\times\pi}{16}$	0.9811	0.9811
$\frac{4\times\pi}{16}$	0.9884	0.9884
$\frac{5\times\pi}{16}$	1.0119	1.0119
$\frac{6\times\pi}{16}$	1.0457	1.0457
$\frac{7\times\pi}{16}$	1.0731	1.0731
$\frac{8\times\pi}{16}$	1.0704	1.0704
$\frac{9\times\pi}{16}$	1.0134	1.0134
$\frac{10\times\pi}{16}$	0.8863	0.8863
$\frac{11\times\pi}{16}$	0.6890	0.6890
$\frac{12\times\pi}{16}$	0.4399	0.4399
$\frac{13\times\pi}{16}$	0.1746	0.1746
$\frac{14\times\pi}{16}$	–0.0618	0.0618
$\frac{15\times\pi}{16}$	–0.2253	0.2253
$\frac{16\times\pi}{16}$	–0.2838	0.2838
$\frac{17\times\pi}{16}$	–0.2253	0.2253
$\frac{18\times\pi}{16}$	–0.0618	0.0618
$\frac{19\times\pi}{16}$	0.1746	0.1746
$\frac{20\times\pi}{16}$	0.4399	0.4399
$\frac{21\times\pi}{16}$	0.6890	0.6890
$\frac{22\times\pi}{16}$	0.8863	0.8863
$\frac{23\times\pi}{16}$	1.0134	1.0134
$\frac{24\times\pi}{16}$	1.0704	1.0704
$\frac{25\times\pi}{16}$	1.0731	1.0731
$\frac{26\times\pi}{16}$	1.0457	1.0457
$\frac{27\times\pi}{16}$	1.0119	1.0119
$\frac{28\times\pi}{16}$	0.9884	0.9884
$\frac{29\times\pi}{16}$	0.9811	0.9811
$\frac{30\times\pi}{16}$	0.9865	0.9865
$\frac{31\times\pi}{16}$	0.9956	0.9956
$\frac{32\times\pi}{16}$	0.9998	0.9998

Chapter 7

IIR Filters

7.1 Introduction

The specification of a digital filter will be desired frequency response, $H_d(e^{j\omega})$. The desired impulse response, $h_d(n)$ of the digital filter can be obtained by taking inverse Fourier transform of $H_d(e^{j\omega})$. Now, the $h_d(n)$ will be an infinite duration discrete time signal defined for all values of n in the range $-\infty$ to $+\infty$. The filters designed by considering all the infinite samples of impulse response are called IIR (***Infinite Impulse Response***) filters.

In digital domain, the processing of infinite samples of impulse response is practically not possible. Hence direct design of IIR filter is not possible. Therefore, the IIR filters are designed via analog filters.

In design of IIR filter, the specification of an IIR filter is transformed to specification of an analog filter and an analog filter with transfer function, H(s) is designed to satisfy the specification. Then the analog filter is transformed to digital filter with transfer function, H(z).

We know that the analog filter with transfer function H(s) is stable if all its poles lie in the left half of the s-plane. Consequently, if the conversion technique is to be effective, it should possess the following desirable properties.

1. The imaginary axis in the s-plane should map into the unit circle in the z-plane. Thus there will be a direct relationship between the two frequency variables in the two domains.
2. The left-half of the s-plane should map into the interior of the unit circle in the z-plane. Thus a stable analog filter will be converted to a stable digital filter.

The analog filter is designed by approximating the ideal frequency response using an error function. A number of solutions to the approximation problem of analog filter design are well developed. The popular among them are Butterworth and Chebyshev approximation. The popular transformation techniques used for transforming analog filter transfer function H(s) to digital filter transfer function H(z) are bilinear and impulse invariant transformation. The digital transfer function H(z) can be realized in a software that runs on a digital hardware (or it can be implemented in firmware).

The frequency response $H(e^{j\omega})$ of the digital filter can be obtained by letting $z = e^{j\omega}$ in the transfer function H(z) of the filter.

The designed transfer function of the filter should represent a stable and causal system. For stability and causality of analog filter, the analog transfer function should satisfy the following requirements.

1. The H(s) should be a rational function of "s" and the coefficients of "s" should be real.
2. The poles should lie on the left half of s-plane.
3. The number of zeros should be less than or equal to number of poles.

For stability and causality of digital filter, the digital transfer function should satisfy the following requirements.

1. The H(z) should be a rational function of "z" and the coefficients of "z" should be real.
2. The poles should lie inside the unit circle in z-plane.
3. The number of zeros should be less than or equal to number of poles.

Advantages of Digital Filters

1. The values of resistors,capacitors and inductors used in the analog filters changes with temperature. Since digital filters do not have these components, they have high thermal stability.
2. In digital filters the precision of the filter depends on the length (or size) of the registers used to store the filter coefficients. Hence by increasing the register bit-length (in hardware) the performance characteristics of the filter like accuracy, dynamic range, stability and frequency response tolerance, can be enhanced.
3. The digital filters are programmable. Hence the filter coefficients can be changed at any time to implement adaptive features.
4. A single filter can be used to process multiple signals by using the techniques of multiplexing.

Disadvantages of Digital Filters

1. The bandwidth of the discrete signal is limited by the sampling frequency. The bandwidth of real discrete signal is half the sampling frequency.
2. The performance of the digital filter depends on the hardware (i.e., depends on the bit length of the registers in the hardware) used to implement the filter.

Important Features of IIR Filters

1. The physically realizable IIR filters do not have linear phase.
2. The IIR filter specifications include the desired characteristics for the magnitude response only.

Table 7.1 : Comparison of Digital and Analog Filters

Digital Filter	Analog Filter
1. Operates on digital samples (or sampled version) of the signal.	1. Operates on analog signals (or actual signals).
2. It is governed (or defined) by linear difference equation.	2. It is governed (or defined) by linear differential equation.

Table 7.1 : continued...

Digital Filter	Analog Filter
3. It consists of adders, multipliers and delays implemented in digital logic (either in hardware or software or both)	3. It consists of electrical components like resistors, capacitors and inductors.
4. In digital filters the filter coefficients are designed to satisfy the desired frequency response.	4. In analog filters the approximation problem is solved to satisfy the desired frequency response.

7.2 Frequency Response of Analog and Digital IIR Filters

The filters are frequency selective devices and so they are designed to pass the spectral content of the input signal in a specified band of frequencies. Hence, based on frequency response the filters are classified into four basic types. They are lowpass, highpass, bandpass and bandstop filters.

The ideal magnitude response, $|H_d(jW)|$ of the four basic types of analog filters are shown in fig 7.1 (a), (b), (c) and (d). The ideal magnitude response has sudden transition from passband to stopband which is practically not realizable. Hence the ideal response is approximated using a filter approximation function.

The approximation problem is solved to meet a specified tolerance in the passband and stopband. The shaded areas in the fig 7.1 shows the tolerance regions of the ideal frequency response. In the passband the magnitude is approximated to unity within an error of d_p. In the stopband the magnitude is approximated to zero within an error of d_s. Here the d_p and d_s are the limits of the tolerance in the passband and stopband. The d_p and d_s are also called ripples.

The magnitude response of practical or approximated analog filters, $|H(jW)|$ are shown in fig 7.1 (e), (f), (g) and (h). The frequency repsonse of practical analog filter shows edges for passband and stopband so that the tolerances are within specified limits. Now, the specification of practical analog filter will be the following.

W_p = Passband edge frequency in rad/second.

W_s = Stopband edge frequency in rad/second.

A_p = Gain at passband edge frequency

A_s = Gain at stopband edge frequency

The ideal magnitude response, $|H_d(e^{jw})|$ of the four basic types of digital IIR filters are shown in fig 7.2 (a), (b), (c) and (d). The ideal magnitude response has sudden transition from passband to stopband which is practically not possible. The transformation of analog to digital filter will preserve the magnitude response, and so the magnitude response of digital filter will be similar to analog filter, (but the frequency response of digital filter is periodic with period $2p$). Therefore the practical frequency response of digital IIR filters will be similar to analog filter as shown in fig 7.2 (e), (f), (g) and (h). The magnitude response of practical digital IIR filter shows edges for passband and stopband so that the tolerances are within specified limits. Now, the specifications of practical digital IIR filter will be the following.

w_p = Passband edge frequency in rad/sample

w_s = Stopband edge frequency in rad/sample

A_p = Gain at passband edge frequency

A_s = Gain at stopband edge frequency

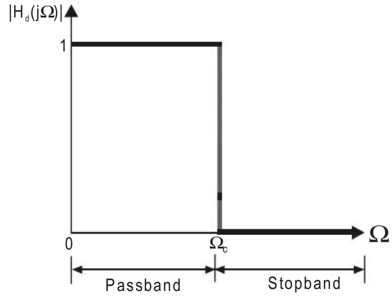


Fig a : Normalized magnitude response of ideal analog lowpass filter.

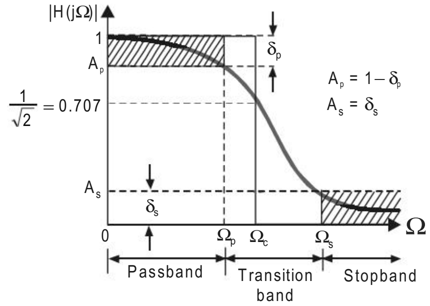


Fig e : Normalized magnitude response of practical analog lowpass filter.

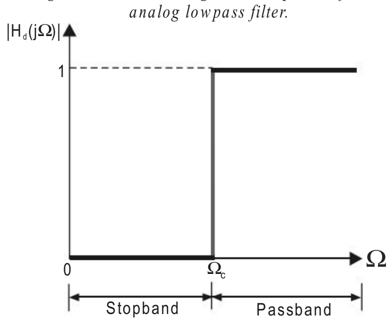


Fig b : Normalized magnitude response of ideal analog highpass filter.

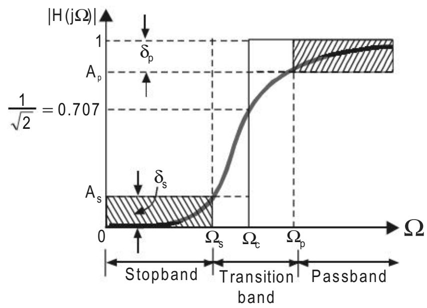


Fig f : Normalized magnitude response of practical analog highpass filter.

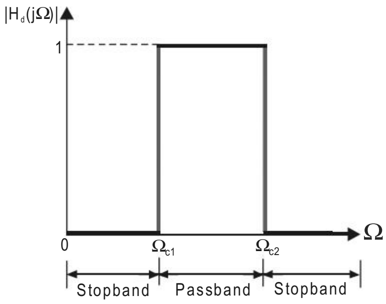


Fig c : Normalized magnitude response of ideal analog bandpass filter.

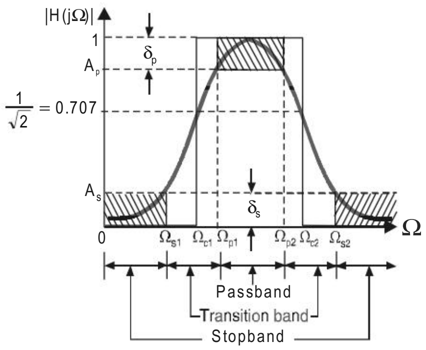


Fig g : Normalized magnitude response of practical analog bandpass filter.

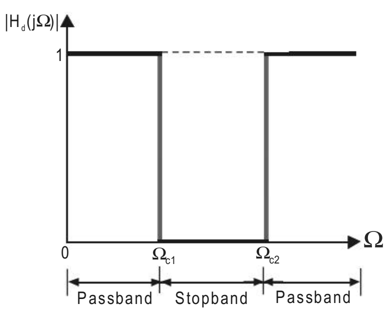


Fig d : Normalized magnitude response of ideal analog bandstop filter.

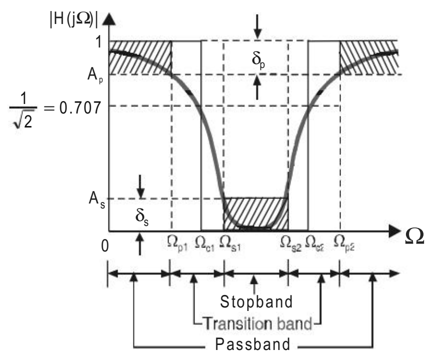


Fig h : Normalized magnitude response of practical analog bandstop filter.

Fig 7.1 : Normalized frequency response of ideal and practical analog filters.

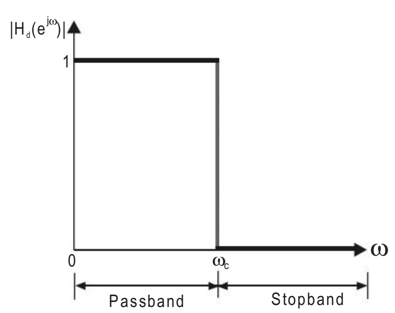


Fig a : *Normalized magnitude response of ideal digital IIR lowpass filter.*

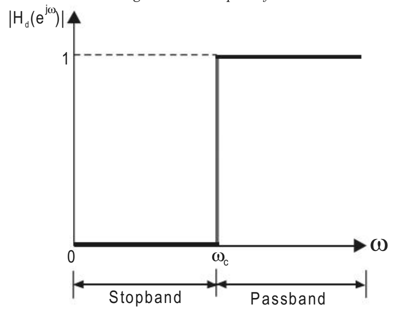


Fig b : *Normalized magnitude response of ideal digital IIR highpass filter.*

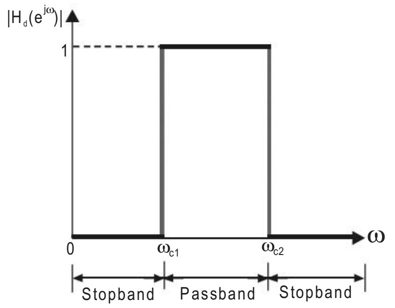


Fig c : *Normalized magnitude response of ideal digital IIR bandpass filter.*

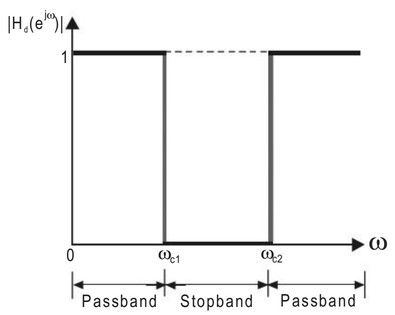


Fig d : *Normalized magnitude response of ideal digital IIR bandstop filter.*

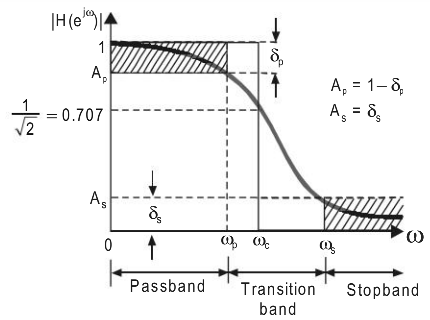


Fig e : *Normalized magnitude response of practical digital IIR lowpass filter.*

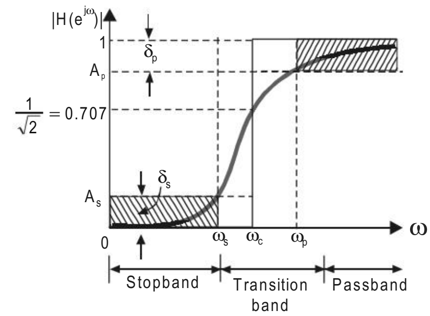


Fig f : *Normalized magnitude response of practical digital IIR highpass filter.*

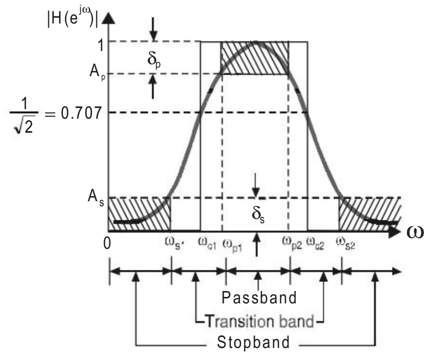


Fig g : *Normalized magnitude response of practical digital IIR bandpass filter.*

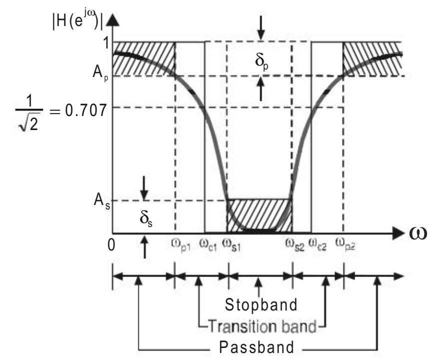


Fig h : *Normalized magnitude response of practical digital IIR bandstop filter.*

Fig 7.2 : *Normalized frequency response of ideal and practical digital IIR filters.*

7.3 Impulse Invariant Transformation

The objective of ***impulse invariant transformation*** is to develop an IIR filter transfer function whose impulse response is the sampled version of the impulse response of the analog filter. The main idea behind this technique is to preserve the frequency response characteristics of the analog filter. It can be stated that the frequency response of digital filter will be identical with the frequency response of the corresponding analog filter if the sampling time period T is selected sufficiently small (or the sampling frequency should be high) to minimize (or avoid completely) the effects of aliasing.

Let, h(t) = Impulse response of analog filter

The Laplace transform of the analog impulse response h(t) gives the transfer function of analog filter.

$\therefore$ Transfer function of analog filter, $H(s) = \mathcal{L}\{h(t)\}$.

When H(s) has N number of distinct poles, it can be expressed as shown in equation (7.1) by partial fraction expansion.

$$H(s) = \sum_{i=1}^{N} \frac{A_i}{s+p_i} = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + ... + \frac{A_N}{s+p_N} \qquad .....(7.1)$$

On taking inverse Laplace transform of equation (7.1) we get,

$$\boxed{\mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}}$$

$$h(t) = \sum_{i=1}^{N} A_i\, e^{-p_i t}\, u(t) = A_1\, e^{-p_1 t}\, u(t) + A_2\, e^{-p_2 t}\, u(t) + ... + A_N\, e^{-p_N t}\, u(t) \qquad ....(7.2)$$

where, u(t) = Continuous time unit step function.

Let, T = Sampling period.

h(n) = Impulse response of digital filter.

The impulse response of the digital filter is obtained by uniformly sampling the impulse response of the analog filter.

$$\therefore\ h(n) = h(t)\big|_{t = nT} = h(nT)$$

Therefore the impulse response h(n) can be obtained from equation (7.2) by replacing t by nT.

$$\begin{aligned}\therefore\ h(n) &= h(t)\big|_{t = nT} = h(nT) = \sum_{i=1}^{N} A_i\, e^{-p_i nT}\, u(nT)\\ &= A_1\, e^{-p_1 nT}\, u(nT) + A_2\, e^{-p_2 nT}\, u(nT) + ... + A_N\, e^{-p_N nT}\, u(nT)\end{aligned} \qquad .....(7.3)$$

On taking Z-transform of equation (7.3) we get,

$$\boxed{\mathcal{Z}\left\{e^{-anT}\,u(nT)\right\} = \frac{1}{1 - e^{-aT} z^{-1}}}$$

$$\begin{aligned}H(z) = \mathcal{Z}\{h(n)\} &= A_1 \frac{1}{1 - e^{-p_1 T} z^{-1}} + A_2 \frac{1}{1 - e^{-p_2 T} z^{-1}} + ...\\ &\quad + A_N \frac{1}{1 - e^{-p_N T} z^{-1}} = \sum_{i=1}^{N} A_i \frac{1}{1 - e^{-p_i T} z^{-1}}\end{aligned} \qquad .....(7.4)$$

Comparing the expression of H(s) and H(z) [i.e., equations (7.1) and (7.4)] we can say that,

$$\boxed{\frac{1}{s+p_i} \xrightarrow[\text{(is transformed to)}]{} \frac{1}{1-e^{-p_iT}\,z^{-1}}} \qquad .....(7.5)$$

by impulse invariant transformation, where T is the sampling time period.

When a discrete time signal is obtained by sampling analog signal, the frequency spectrum of discrete signal will be scaled by a factor 1/T (Refer section 4.7 of Chapter 4). Due to this fact, the transfer function obtained by impulse invariant method is amplified by the factor 1/T for small values of T. If this amplification is undesirable then the transfer function obtained by impulse invariant tranformation can be multiplied by T to obtain magnitude normalized transfer function $H_N(z)$.

\ $H_N(z) = T$ ´ $H(z)$ .....(7.6)

7.3.1 Relation Between Analog and Digital Filter Poles in Impulse Invariant Transformation

The analog poles are given by the roots of the term $(s + p_i)$, for i = 1, 2, 3, ..., N. The digital poles are given by the roots of the term $(1 - e^{-p_iT}\,z^{-1})$, for i = 1, 2, 3, ..., N. From equation (7.5) we can say that the analog pole at $s = -p_i$ is transformed into a digital pole at $z = e^{-p_iT}$

Consider the digital pole, $z_i = e^{-p_iT}$ ..... (7.7)

Put, $-p_i = s_i$ in equation (7.7).

$$\therefore\ z_i = e^{-p_iT} = e^{s_iT} \qquad ..... (7.8)$$

We know that, "s_i" is a point on s-plane. Let the coordinates of s_i be s_i and jW_i as shown in fig 7.3.

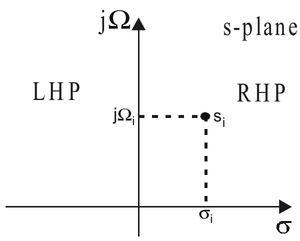


Fig 7.3 : s-plane.

$$\therefore\ s_i = \sigma_i + j\Omega_i \qquad ..... (7.9)$$

Using equation (7.9), the equation (7.8) can be written as,

$$z_i = e^{(\sigma_i + j\Omega_i)T} = e^{\sigma_iT}\,e^{j\Omega_iT}$$

We know that "z_i" is a complex number. Hence "z_i" can be expressed in polar coordinates as, $z_i = |z_i|$ Ð z_i.

$$\therefore\ |z_i|\,\angle z_i = e^{\sigma_iT}\,e^{j\Omega_iT} \qquad ..... (7.10)$$

On separating the magnitude and phase of equation (7.10) we get,

$$|z_i| = e^{\sigma_iT} \text{ and } \angle z_i = \Omega_iT \qquad ..... (7.11)$$

From equation (7.11) the following observations can be made.

1. If $s_i < 0$ (i.e., s_i is negative), then the analog pole "s_i" lie on Left Half (LHP) of s-plane. In this case, $|z_i| < 1$, hence the corresponding digital pole "z_i" will lie inside the unit circle in z-plane.
2. If $s_i = 0$ (i.e., real part is zero), then the analog pole "s_i" lie on imaginary axis of s-plane. In this case, $|z_i| = 1$, hence the corresponding digital pole "z_i" will lie on the unit circle in z-plane.
3. If $s_i > 0$ (i.e., s_i is positive), then the analog pole "s_i" lie on the Right Half (RHP) of s-plane. In this case $|z_i| > 1$, hence the corresponding digital pole will lie outside the unit circle in z-plane.

The above discussions are applicable for mapping any point on s-plane to z-plane. In general the impulse invariant transformation maps all points in the s-plane given by,

$$s_i = \sigma_i + j\Omega_i + j\frac{2\pi k}{T}, \quad \text{for } k = 0, \pm 1, \pm 2 \ldots.. \qquad \ldots..(7.12)$$

into a single point in the z-plane as

For integer k, $e^{j2pk} = 1$

$$z_i = e^{\left(\sigma_i + j\Omega_i + \frac{j2\pi k}{T}\right)T} = e^{\sigma_i T}\, e^{j\Omega_i T}\, e^{j2\pi k} = e^{\sigma_i T}\, e^{j\Omega_i T} \qquad \ldots..(7.13)$$

From equations (7.12) and (7.13) we can say that the strip of width 2p/T in the s-plane for values of s in the range –p/T £ W £ +p/T is mapped into the entire z-plane. Similarly the strip of width 2p/T in the s-plane for values of s in the range p/T £ W £ 3p/T is also mapped into the entire z-plane. Likewise the strip of width 2p/T in the s-plane for values of s in the range -3p/T £ W £ -p/T is also mapped into the entire z-plane.

In general any strip of width 2p/T in the s-plane for values of s in the range, (2k–1)p/T £ W £ (2k+1)p/T (where k is an integer), is mapped into the entire z-plane. The left half portion of each strip in s-plane maps into the interior of the unit circle in z-plane, right half portion of each strip in s-plane maps into the exterior of the unit circle in z-plane and the imaginary axis of each strip in s-plane maps into the unit circle in z-plane as shown in fig 7.4. Therefore we can say that the impulse invariant mapping is many-to-one mapping (and does not provide one-to-one mapping).

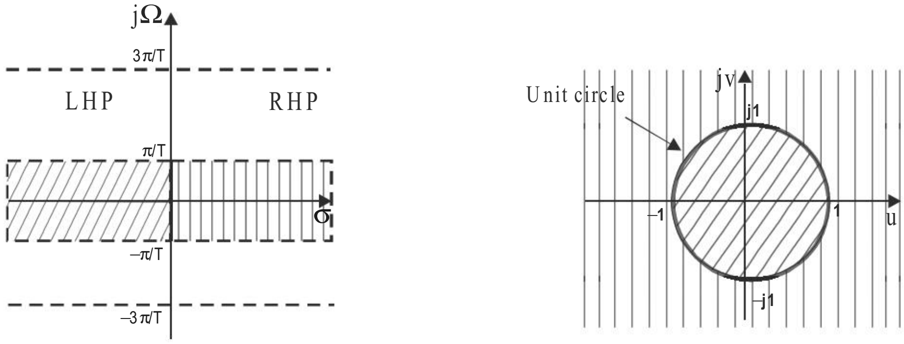


Fig 7.4 a : s-plane. *Fig 7.4 b : z-plane.*

Fig 7.4 : Mapping of s-plane into z-plane in impulse invariant transformation.

The stability of a filter (or system) is related to the location of the poles. For a stable analog filter the poles should lie on the left half of the s-plane. Since the left half of s-plane maps inside the unit circle in z-plane we can say that, for a stable digital filter the poles should lie inside the unit circle in z-plane.

7.3.2 Relation Between Analog and Digital Frequency in Impulse Invariant Transformation

Let, W = Analog frequency in rad/second.

w = Digital frequency in rad/sample.

Let, $z = re^{jw}$ be a point on z-plane,

and s = s + jW be the corresponding point in s-plane.

Then by impulse invariant transformation,

$$z = e^{sT} \qquad \ldots..(7.14)$$

Put, $z = r\,e^{j\omega}$ and $s = \sigma + j\Omega$ in equation (7.14).

$$\therefore\; r\,e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$r\,e^{j\omega} = e^{\sigma T}\, e^{j\Omega T} \qquad .....(7.15)$$

On equating the phase on either side of equation (7.15) we get,

$$\boxed{\text{Digital frequency, } \omega = \Omega T} \quad \text{or} \quad \boxed{\text{Analog frequency, } \Omega = \frac{\omega}{T}} \qquad .....(7.16)$$

When impulse invariant transformation is employed the equation (7.16) can be used to compute the digital frequency for a given analog frequency and vice versa.

The mapping of analog to digital frequency is not one-to-one. Since ω is unique over the range ($-\pi$ to $+\pi$), the mapping $\omega = \Omega T$ implies that the interval $-\pi/T \le \Omega \le +\pi/T$ maps into the corresponding values of $-\pi \le \omega \le +\pi$. In general the interval $(2k-1)\,\pi/T \le \Omega \le (2k+1)\,\pi/T$ (where k is an integer) maps into the corresponding values of $-\pi \le \omega \le +\pi$. Thus the mapping from the analog frequency Ω to the digital frequency ω is many-to-one. This reflects the effects of aliasing due to sampling.

7.3.3 Useful Impulse Invariant Transformation

The following transformations are given without proof. The equation (7.17) can be used when the analog real poles has a multiplicity of m. The equations (7.18) and (7.19) can be used when the analog poles are complex conjugate.

$$\boxed{\frac{1}{(s + p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!}\,\frac{d^{m-1}}{dp_i^{m-1}}\,\frac{1}{1 - e^{-p_i T} z^{-1}}} \qquad .....(7.17)$$

$$\boxed{\frac{(s + a)}{(s + a)^2 + b^2} \longrightarrow \frac{1 - e^{-aT}(\cos bT)\, z^{-1}}{1 - 2e^{-aT}(\cos bT)\, z^{-1} + e^{-2aT} z^{-2}}} \qquad ....(7.18)$$

$$\boxed{\frac{b}{(s + a)^2 + b^2} \longrightarrow \frac{e^{-aT}(\sin bT)\, z^{-1}}{1 - 2e^{-aT}(\cos bT)\, z^{-1} + e^{-2aT} z^{-2}}} \qquad ....(7.19)$$

Example 7.1

For the analog transfer function, $H(s) = \dfrac{2}{s^2 + 3s + 2}$, determine H(z) using impulse invariant transformation if **(a)** T = 1 second and **(b)** T = 0.1 second.

Solution

Given that, $H(s) = \dfrac{2}{s^2 + 3s + 2} = \dfrac{2}{(s+1)\,(s+2)}$

By partial fraction expansion technique we can write,

$$H(s) = \frac{2}{(s+1)\,(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \frac{2}{\cancel{(s+1)}\,(s+2)} \times \cancel{(s+1)}\bigg|_{s=-1} = \frac{2}{-1+2} = 2$$

$$B = \frac{2}{(s+1)\,\cancel{(s+2)}} \times \cancel{(s+2)}\bigg|_{s=-2} = \frac{2}{-2+1} = -2$$

The roots of quadratic,

$s^2 + 3s + 2 = 0$ are,

$$s = \frac{-3 \pm \sqrt{3^2 - 4 \times 2}}{2}$$

$$= \frac{-3 \pm 1}{2} = -1,\; -2$$

$$\therefore\ H(s) = \frac{2}{s+1} + \frac{-2}{s+2}$$

By impulse invariant transformation we know that,

$$\boxed{\frac{A_i}{s+p_i} \xrightarrow{\text{(is transformed to)}} \frac{A_i}{1-e^{-p_iT}\,z^{-1}}}$$

$$\therefore\ H(z) = \frac{2}{1-e^{-p_1T}\,z^{-1}} + \frac{-2}{1-e^{-p_2T}\,z^{-1}} \quad \text{where } p_1 = 1 \quad \text{and} \quad p_2 = 2$$

$$H(z) = \frac{2}{1-e^{-T}\,z^{-1}} + \frac{-2}{1-e^{-2T}\,z^{-1}}$$

(a) When T = 1 second

$$H(z) = \frac{2}{1-e^{-1}\,z^{-1}} + \frac{-2}{1-e^{-2}\,z^{-1}}$$

$$H(z) = \frac{2}{1-0.3679z^{-1}} + \frac{-2}{1-0.1353z^{-1}} = \frac{2(1-0.1353z^{-1}) - 2(1-0.3679z^{-1})}{(1-0.3679z^{-1})\,(1-0.1353z^{-1})}$$

$$= \frac{2-0.2706z^{-1} - 2 + 0.7358z^{-1}}{1-0.1353\,z^{-1} - 0.3679z^{-1} + 0.0498z^{-2}} = \frac{0.4652\,z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.4652\,z^{-1}}{1-0.5032\,z^{-1}+0.0498\,z^{-2}} = \frac{0.4652\,z^{-1}}{z^{-2}(z^2-0.5032\,z+0.0498)}$$

$$= \frac{0.4652\,z}{z^2-0.5032\,z+0.0498}$$

(b) When T = 0.1 second

$$H(z) = \frac{2}{1-e^{-0.1}\,z^{-1}} + \frac{-2}{1-e^{-0.2}\,z^{-1}}$$

$$= \frac{2}{1-0.9048z^{-1}} + \frac{-2}{1-0.8187z^{-1}} = \frac{2(1-0.8187z^{-1}) - 2(1-0.9048z^{-1})}{(1-0.9048z^{-1})\,(1-0.8187z^{-1})}$$

$$= \frac{2-1.6374z^{-1} - 2 + 1.8096\,z^{-1}}{1-0.8187\,z^{-1} - 0.9048z^{-1} + 0.7408\,z^{-2}} = \frac{0.1722\,z^{-1}}{1-1.7235\,z^{-1}+0.7408\,z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.1722\,z^{-1}}{1-1.7235\,z^{-1}+0.7408\,z^{-2}} = \frac{0.1722\,z^{-1}}{z^{-2}(z^2-1.7235\,z+0.7408)}$$

$$= \frac{0.1722\,z}{z^2-1.7235\,z+0.7408}$$

Since, T < 1, we can compute magnitude normalized transfer function, $H_N(z)$.

$$H_N(z) = T \times H(z) = 0.1 \times \frac{0.1722\,z^{-1}}{1-1.7235\,z^{-1}+0.7408\,z^{-2}} = \frac{0.0172\,z^{-1}}{1-1.7235\,z^{-1}+0.7408z^{-2}}$$

Alternatively,

$$H_N(z) = T \times H(z) = 0.1 \times \frac{0.1722\,z}{z^2-1.7235\,z+0.7408} = \frac{0.0172\,z}{z^2-1.7235\,z+0.7408}$$

Example 7.2

Convert the analog filter with system transfer function,

$$H(s) = \frac{(s+0.1)}{(s+0.1)^2+9}$$

into a digital IIR filter by means of the impulse invariant method.

Solution

Method - I

Given that, $H(s) = \frac{s+0.1}{(s+0.1)^2+9} = \frac{s+0.1}{(s+0.1)^2+3^2}$

Using transformation of equation (7.18) we can write,

$$H(z) = \frac{1-e^{-0.1T}\left(\cos 3T\right)z^{-1}}{1-2e^{-0.1T}\left(\cos 3T\right)z^{-1}+e^{-2\times 0.1T}\,z^{-2}} = \frac{1-e^{-0.1}\left(\cos 3\right)z^{-1}}{1-2e^{-0.1}\left(\cos 3\right)z^{-1}+e^{-0.2}\,z^{-2}} \qquad \boxed{\text{Put, } T=1}$$

$$= \frac{1+0.8958\,z^{-1}}{1+1.7916\,z^{-1}+0.8187\,z^{-2}}$$

Alternatively,

$$H(z) = \frac{1+0.8958\,z^{-1}}{1+1.7916\,z^{-1}+0.8187\,z^{-2}} = \frac{1+0.8958\,z^{-1}}{z^{-2}(z^2+1.7916\,z+0.8187)} = \frac{z^2+0.8958\,z}{z^2+1.7916\,z+0.8187}$$

Method - II

Given that, $H(s) = \frac{(s+0.1)}{(s+0.1)^2+9} = \frac{s+0.1}{s^2+2\times 0.1\times s+0.1^2+9}$

$$= \frac{s+0.1}{s^2+0.2s+9.01} = \frac{s+0.1}{(s+0.1-j3)\,(s+0.1+j3)}$$

The roots of the quadratic $s^2 + 0.2s + 9.01 = 0$ are

$$s = \frac{-0.2 \pm \sqrt{0.2^2-4\times 9.01}}{2}$$

$$= \frac{-0.2}{2} \pm \frac{1}{2}\sqrt{-36} = -0.1 \pm j3$$

$$\therefore\ (s^2+0.2s+9.01)$$

$$= (s-(-0.1+j3))(s-(-0.1-j3))$$

$$= (s+0.1-j3)(s+0.1+j3)$$

By partial fraction expansion H(s) can be expressed as,

$$H(s) = \frac{s+0.1}{(s+0.1-j3)\,(s+0.1+j3)} = \frac{A}{s+0.1-j3} + \frac{A^*}{s+0.1+j3}$$

$$A = \frac{s+0.1}{\cancel{(s+0.1-j3)}\,(s+0.1+j3)} \times \cancel{(s+0.1-j3)}\Bigg|_{s=-0.1+j3} = \frac{-0.1+j3+0.1}{-0.1+j3+0.1+j3} = \frac{j3}{j6} = 0.5$$

$$A^* = (0.5)^* = 0.5$$

$$\therefore\ H(s) = \frac{0.5}{s+0.1-j3} + \frac{0.5}{s+0.1+j3}$$

By impulse invariant transformation we know that,

$$\boxed{\frac{A_i}{s+p_i} \xrightarrow[\text{(is transformed to}]{} \frac{A_i}{1-e^{-p_iT}\,z^{-1}} \quad \text{and let,} \quad T=1}$$

$$\therefore\ H(z) = \frac{0.5}{1-e^{-(0.1-j3)T}\,z^{-1}} + \frac{0.5}{1-e^{-(0.1+j3)T}\,z^{-1}} = \frac{0.5}{1-e^{-0.1}\,e^{j3}\,z^{-1}} + \frac{0.5}{1-e^{-0.1}\,e^{-j3}\,z^{-1}}$$

$$= \frac{0.5(1-e^{-0.1}\,e^{-j3}\,z^{-1}) + 0.5(1-e^{-0.1}\,e^{j3}\,z^{-1})}{(1-e^{-0.1}\,e^{j3}\,z^{-1})\,(1-e^{-0.1}\,e^{-j3}\,z^{-1})}$$

$$\therefore\ H(z) = \frac{0.5\ -\ 0.5\,e^{-0.1}\,e^{-j3}\,z^{-1}\ +\ 0.5\ -\ 0.5\,e^{-0.1}\,e^{j3}\,z^{-1}}{1\ -\ e^{-0.1}\,e^{-j3}\,z^{-1}\ -\ e^{-0.1}\,e^{j3}\,z^{-1}\ +\ e^{-0.1}\,e^{j3}\,e^{-0.1}\,e^{-j3}\,z^{-2}}$$

$$= \frac{1\ -\ 0.5\,e^{-0.1}\,z^{-1}(e^{j3}\ +\ e^{-j3})}{1\ -\ e^{-0.1}\,z^{-1}\,(e^{j3}\ +\ e^{-j3})\ +\ e^{-0.2}\,z^{-2}}$$

$$= \frac{1\ -\ 0.5\ \times\ (2\cos 3)\,e^{-0.1}\,z^{-1}}{1\ -\ e^{-0.1}\,z^{-1}\,(2\ \cos 3)\ +\ e^{-0.2}\,z^{-2}}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{1\ -\ (\cos 3)\,e^{-0.1}\,z^{-1}}{1\ -\ 2(\cos\ 3)\,e^{-0.1}\,z^{-1}\ +\ e^{-0.2}\,z^{-2}}$$

Note : Evaluate cos q by keeping calculator in radian mode.

$$= \frac{1 + 0.8958\,z^{-1}}{1 + 1.7916\,z^{-1} + 0.8187\,z^{-2}}$$

Alternatively,

$$H(z) = \frac{1 + 0.8958\,z^{-1}}{1 + 1.7916\,z^{-1} + 0.8187\,z^{-2}} = \frac{1 + 0.8958\,z^{-1}}{z^{-2}(z^2 + 1.7916\,z + 0.8187)}$$

$$= \frac{z^2 + 0.8958\,z}{z^2 + 1.7916\,z + 0.8187}$$

Example 7.3

Using impulse invariant transformation convert the following analog filter transfer function to digital filter transfer function by taking sampling time, T = 0.5 second.

$$H(s) = \frac{2.8s^2 + 4.8s + 2.9}{(s + 3)\ (s^2 + s + 0.85)}$$

Solution

Method - I

Given that, $H(s) = \dfrac{2.8s^2 + 4.8s + 2.9}{(s + 3)\ (s^2 + s + 0.85)}$

By partial fraction expansion technique H(s) can be expressed as,

$$H(s) = \frac{2.8s^2 + 4.8s + 2.9}{(s + 3)\ (s^2 + s + 0.85)} = \frac{A}{s + 3} + \frac{Bs + C}{s^2 + s + 0.85}$$

On cross multiplying the above equation we get,

$$2.8s^2 + 4.8s + 2.9 = A(s^2 + s + 0.85) + (Bs + C)\ (s + 3)$$

$$2.8s^2 + 4.8s + 2.9 = As^2 + As + 0.85A + Bs^2 + 3Bs + Cs + 3C$$

On equating coefficients of s^2 we get,

$$A + B = 2.8$$

$$\backslash\ \ B = 2.8 - A$$

On equating coefficients of s we get,

$$A + 3B + C = 4.8$$

Put, $B = 2.8 - A$

$$\backslash\ \ A + 3\ (2.8 - A) + C = 4.8$$

$$A + 8.4 - 3A + C = 4.8$$

$$C = 4.8 - 8.4 + 2A$$

$$\backslash\ C = 2A - 3.6$$

On equating constants we get,

$$0.85\ A + 3C = 2.9$$

Put, $C = 2A - 3.6$

$$\backslash\ \ 0.85A + 3\ (2A - 3.6) = 2.9$$

$$0.85A + 6A - 10.8 = 2.9$$

$$6.85A = 2.9 + 10.8$$

$$A = \frac{2.9 + 10.8}{6.85} = 2$$

Here, A = 2

$$\therefore \quad B = 2.8 - A = 2.8 - 2 = 0.8$$

$$C = 2A - 3.6 = 2 \times 2 - 3.6 = 0.4$$

$(a + b)^2 = a^2 + 2ab + b^2$

$$\therefore \quad H(s) = \frac{2}{s+3} + \frac{0.8s + 0.4}{s^2 + s + 0.85} = \frac{2}{s+3} + \frac{0.8\left(s + \frac{0.4}{0.8}\right)}{\left[s^2 + (2 \times 0.5)\,s + 0.5^2\right] - 0.5^2 + 0.85}$$

$$= \frac{2}{s+3} + \frac{0.8\,(s + 0.5)}{(s+0.5)^2 + 0.6} = \frac{2}{s+3} + \frac{0.8(s+0.5)}{(s+0.5)^2 + \left(\sqrt{0.6}\right)^2}$$

$$= 2 \times \frac{1}{s+3} + 0.8 \times \frac{(s+0.5)}{(s+0.5)^2 + 0.7746^2}$$

Now, using impulse invariant transformation,

Using equations (7.17) and (7.18).

$$H(z) = 2 \times \frac{1}{1 - e^{-3T} z^{-1}} + 0.8 \times \frac{1 - e^{-0.5T}(\cos 0.7746T)\, z^{-1}}{1 - 2e^{-0.5T}(\cos 0.7746T)\, z^{-1} + e^{-2 \times 0.5T}\, z^{-2}}$$

Put, T = 0.5

$$= \frac{2}{1 - e^{-3 \times 0.5} z^{-1}} + \frac{0.8 - 0.8\, e^{-0.5 \times 0.5}\,(\cos 0.7746 \times 0.5)\, z^{-1}}{1 - 2e^{-0.5 \times 0.5}(\cos 0.7746 \times 0.5)\, z^{-1} + e^{-2 \times 0.5 \times 0.5}\, z^{-2}}$$

$$= \frac{2}{1 - 0.2231 z^{-1}} + \frac{0.8 - 0.5769\, z^{-1}}{1 - 1.4422 z^{-1} + 0.6065\, z^{-2}}$$

$$= \frac{2(1 - 1.4422\, z^{-1} + 0.6065\, z^{-2}) + (0.8 - 0.5769\, z^{-1})\,(1 - 0.2231 z^{-1})}{(1 - 0.2231 z^{-1})\,(1 - 1.4422\, z^{-1} + 0.6065\, z^{-2})}$$

$$= \frac{2 - 2.8844\, z^{-1} + 1.213\, z^{-2} + 0.8 - 0.1785\, z^{-1} - 0.5769\, z^{-1} + 0.1287 z^{-2}}{1 - 1.4422 z^{-1} + 0.6065\, z^{-2} - 0.2231 z^{-1} + 0.3218\, z^{-2} - 0.1353\, z^{-3}}$$

$$= \frac{2.8 - 3.6398\, z^{-1} + 1.3417\, z^{-2}}{1 - 1.6653\, z^{-1} + 0.9283\, z^{-2} - 0.1353\, z^{-3}}$$

Alternatively,

$$H(z) = \frac{2.8 - 3.6398\, z^{-1} + 1.3417\, z^{-2}}{1 - 1.6653\, z^{-1} + 0.9283\, z^{-2} - 0.1353\, z^{-3}} = \frac{z^{-3}(2.8\, z^3 - 3.6398\, z^2 + 1.3417\, z)}{z^{-3}(z^3 - 1.6653\, z^2 + 0.9283\, z - 0.1353)}$$

$$= \frac{2.8\, z^3 - 3.6398\, z^2 + 1.3417\, z}{z^3 - 1.6653\, z^2 + 0.9283\, z - 0.1353}$$

Since, T < 1, we can compute magnitude normalized transfer function, $H_N(z)$.

$$H_N(z) = T \times H(z) = 0.5 \times \frac{2.8 - 3.6398\, z^{-1} + 1.3417\, z^{-2}}{1 - 1.6653\, z^{-1} + 0.9283\, z^{-2} - 0.1353\, z^{-3}} = \frac{1.4 - 1.8199\, z^{-1} + 0.6709\, z^{-2}}{1 - 1.6653\, z^{-1} + 0.9283\, z^{-2} - 0.1353\, z^{-3}}$$

Alternatively,

$$H_N(z) = T \times H(z) = 0.5 \times \frac{2.8z^3 - 3.6398\, z^2 + 1.3417\, z}{z^3 - 1.6653\, z^2 + 0.9283\, z - 0.1353} = \frac{1.4z^3 - 1.8199\, z^2 + 0.6709\, z}{z^3 - 1.6653\, z^2 + 0.9283\, z - 0.1353}$$

Method - II

Given that, $H(s) = \dfrac{2.8s^2 + 4.8s + 2.9}{(s+3)\,(s^2 + s + 0.85)}$

$$= \frac{2.8s^2 + 4.8s + 2.9}{(s+3)\,(s + 0.5 - j0.7746)\,(s + 0.5 + j0.7746)}$$

The roots of the quadratic $s^2 + s + 0.85 = 0$ are,

$$s = \frac{-1 \pm \sqrt{1^2 - 4 \times 0.85}}{2}$$

$$= \frac{-1 \pm j1.5492}{2}$$

$$= -0.5 \pm j0.7746$$

By partial fraction expansion technique H(s) can be expressed as,

$$H(s) = \frac{2.8s^2 + 4.8s + 2.9}{(s+3)\,(s+0.5-j0.7746)\,(s+0.5+j0.7746)} = \frac{A}{s+3} + \frac{B}{s+0.5-j0.7746} + \frac{B^*}{s+0.5+j0.7746}$$

$$A = \left.\frac{2.8s^2 + 4.8s + 2.9}{\cancel{(s+3)}\,(s+0.5-j0.7746)\,(s+0.5+j0.7746)} \times \cancel{(s+3)}\right|_{s=-3} = \left.\frac{2.8s^2 + 4.8s + 2.9}{s^2 + s + 0.85}\right|_{s=-3}$$

$$= \frac{2.8 \times (-3)^2 + 4.8(-3) + 2.9}{(-3)^2 + (-3) + 0.85} = 2$$

$$B = \left.\frac{2.8s^2 + 4.8s + 2.9}{(s+3)\,\cancel{(s+0.5-j0.7746)}\,(s+0.5+j0.7746)} \times \cancel{(s+0.5-j0.7746)}\right|_{s=-0.5+j0.7746}$$

$$= \frac{2.8(-0.5+j0.7746)^2 + 4.8(-0.5+j0.7746) + 2.9}{(-0.5+j0.7746+3)\,(-0.5+j0.7746+0.5+j0.7746)}$$

$$= \frac{2.8(-0.5+j0.7746)^2 + 0.5 + j3.7181}{(2.5+j0.7746)\,(j1.5492)} = 0.4$$

$$B^* = (0.4)^* = 0.4$$

$$\therefore\ H(s) = \frac{2}{s+3} + \frac{0.4}{s+0.5-j0.7746} + \frac{0.4}{s+0.5+j0.7746}$$

$$= 2 \times \frac{1}{s+3} + 0.4 \times \frac{1}{s+(0.5-j0.7746)} + 0.4 \times \frac{1}{s+(0.5+j0.7746)}$$

Using impulse invariant transformation, H(s) is transformed to H(z) as shown below.

$$H(z) = 2 \times \frac{1}{1-e^{-3T}\,z^{-1}} + 0.4 \times \frac{1}{1-e^{-(0.5\,-j0.7746)T}\,z^{-1}} + 0.4 \times \frac{1}{1-e^{-(0.5+j0.7746)T}z^{-1}}$$

Using equation (7.17).

Put, T = 0.5

$$= \frac{2}{1-e^{-3\times0.5}\,z^{-1}} + \frac{0.4}{1-e^{-(0.5\,-j0.7746)\times0.5}\,z^{-1}} + \frac{0.4}{1-e^{-(0.5+j0.7746)\times0.5}z^{-1}}$$

$$= \frac{2}{1-e^{-1.5}\,z^{-1}} + \frac{0.4}{1-e^{-0.25}\,e^{j0.3873}\,z^{-1}} + \frac{0.4}{1-e^{-0.25}\,e^{-j0.3873}z^{-1}}$$

$$= \frac{2}{1-0.2231\,z^{-1}} + \frac{0.4\left(1-e^{-0.25}\,e^{-j0.3873}z^{-1}\right) + 0.4\left(1-e^{-0.25}\,e^{j0.3873}z^{-1}\right)}{\left(1-e^{-0.25}\,e^{j0.3873}\,z^{-1}\right)\left(1-e^{-0.25}\,e^{-j0.3873}\,z^{-1}\right)}$$

$$= \frac{2}{1-0.2231\,z^{-1}} + \frac{0.4 - 0.4e^{-0.25}\,e^{-j0.3873}z^{-1} + 0.4 - 0.4e^{-0.25}\,e^{j0.3873}z^{-1}}{1-e^{-0.25}\,e^{-j0.3873}\,z^{-1} - e^{-0.25}\,e^{j0.3873}\,z^{-1} + e^{-0.5}z^{-2}}$$

$$= \frac{2}{1-0.2231z^{-1}} + \frac{0.8 - 0.4e^{-0.25}\left(e^{j0.3873} + e^{-j0.3873}\right)z^{-1}}{1-e^{-0.25}\left(e^{j0.3873} + e^{-j0.3873}\right)z^{-1} + e^{-0.5}\,z^{-2}}$$

$$= \frac{2}{1-0.2231z^{-1}} + \frac{0.8 - 0.4e^{-0.25}\left(2\cos 0.3873\right)z^{-1}}{1-e^{-0.25}\left(2\cos 0.3873\right)z^{-1} + e^{-0.5}\,z^{-2}}$$

$$= \frac{2}{1-0.2231z^{-1}} + \frac{0.8 - 0.5769\,z^{-1}}{1-1.4422\,z^{-1} + 0.6065\,z^{-2}}$$

$$= \frac{2\left(1-1.4422\,z^{-1} + 0.6065\,z^{-2}\right) + \left(0.8-0.5769\,z^{-1}\right)\left(1-0.2231z^{-1}\right)}{\left(1-0.2231z^{-1}\right)\left(1-1.4422\,z^{-1} + 0.6065\,z^{-2}\right)}$$

$$\therefore \quad H(z) = \frac{2 - 2.8844\,z^{-1} + 1.213\,z^{-2} + 0.8 - 0.1785\,z^{-1} - 0.5769\,z^{-1} + 0.1287\,z^{-2}}{1 - 1.4422\,z^{-1} + 0.6065\,z^{-2} - 0.2231z^{-1} + 0.3218\,z^{-2} - 0.1353\,z^{-3}}$$

$$= \frac{2.8 - 3.6398\,z^{-1} + 1.3417\,z^{-2}}{1 - 1.6653\,z^{-1} + 0.9283\,z^{-2} - 0.1353\,z^{-3}}$$

Alternatively,

$$H(z) = \frac{2.8 - 3.6398\,z^{-1} + 1.3417\,z^{-2}}{1 - 1.6653\,z^{-1} + 0.9283\,z^{-2} - 0.1353\,z^{-3}}$$

$$= \frac{z^{-3}\left(2.8\,z^{3} - 3.6398\,z^{2} + 1.3417\,z\right)}{z^{-3}\left(z^{3} - 1.6653\,z^{2} + 0.9283\,z - 0.1353\right)}$$

$$= \frac{2.8\,z^{3} - 3.6398\,z^{2} + 1.3417\,z}{z^{3} - 1.6653\,z^{2} + 0.9283\,z - 0.1353}$$

7.4 Bilinear Transformation

The ***bilinear transformation*** is a conformal mapping that transforms the imaginary axis of s-plane into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. In this mapping all points in the left half of s-plane are mapped inside the unit circle in the z-plane and all points in the right half of s-plane are mapped outside the unit circle in the z-plane.

The bilinear transformation can be linked to the trapezoidal formula for numerical integration. Any analog system is governed by a differential equation in time domain. Consider the first order differential equation of an analog system as shown in equation (7.20).

$$\text{Let,} \quad \frac{dy(t)}{dt} = x(t) \qquad .....(7.20)$$

On integrating both sides of equation (7.20) we get,

$$\int_{(n-1)T}^{nT} \frac{dy(t)}{dt}\,dt = \int_{(n-1)T}^{nT} x(t)\,dt$$

$$\big[y(t)\big]_{(n-1)T}^{nT} = \int_{(n-1)T}^{nT} x(t)\,dt$$

$$y(nT) - y\big((n-1)T\big) = \int_{(n-1)T}^{nT} x(t)\,dt \qquad .....(7.21)$$

The trapezoidal rule when integration is approximated by two trapezoids is,

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\big[f(a) + f(b)\big]$$

The integral on the right side of equation (7.21) can be approximated by the trapezoidal rule, so that,

$$y(nT) - y[(n-1)T] = \left(\tfrac{T}{2}\right)\big[x(nT) + x\big((n-1)T\big)\big] \qquad .....(7.22)$$

For discrete time system, the equation (7.22) can be written as,

$$y(n) - y(n-1) = \tfrac{T}{2}\,\big[x(n) + x(n-1)\big] \qquad .....(7.23)$$

On taking $\mathcal{Z}$-transform of equation (7.23) we get,

$$Y(z) - z^{-1}\,Y(z) = \frac{T}{2}\left[X(z) + z^{-1}\,X(z)\right]$$

$$\left[1 - z^{-1}\right]Y(z) = \frac{T}{2}\left[1 + z^{-1}\right]X(z)$$

$$\frac{2(1 - z^{-1})}{T(1 + z^{-1})}\,Y(z) = X(z) \qquad .....(7.24)$$

On taking Laplace transform of equation (7.20) we get,

$$s\,Y(s) = X(s) \qquad .....(7.25)$$

On comparing equations (7.24) and (7.25) we can say that,

$$\boxed{s\,Y(s) \xrightarrow[\text{(is transformed to)}]{} \frac{2}{T}\,\frac{1 - z^{-1}}{1 + z^{-1}}\,Y(z)} \qquad .....(7.26)$$

by bilinear transformation, where T is the sampling time period.

Hence in the s-domain transfer function, if "s" is substituted by the term $\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ the resulting transfer function will be z-domain transfer function.

7.4.1 Relation Between Analog and Digital Filter Poles in Bilinear Transformation

The mapping of s-domain function to z-domain function by bilinear transformation is a one to one mapping, that is, for every point in z-plane, there is exactly one corresponding point in s-plane and vice versa. The transformation is accomplished when,

$$\boxed{s = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}} \qquad .....(7.27)$$

The equation (7.27) can be rearranged as shown below to express "z" in terms of "s".

$$s = \frac{2}{T}\,\frac{1 - z^{-1}}{1 + z^{-1}} \quad \Rightarrow \quad \frac{T}{2}s = \frac{1 - z^{-1}}{1 + z^{-1}} \quad \Rightarrow \quad \frac{T}{2}s = \frac{z^{-1}(z-1)}{z^{-1}(z+1)}$$

$$\therefore \quad \frac{T}{2}s = \frac{z-1}{z+1} \qquad .....(7.28)$$

On cross multiplying equation (7.28) we get,

$$\frac{T}{2}s(z+1) = z-1 \quad \Rightarrow \quad \frac{T}{2}s\,z + \frac{T}{2}s = z-1 \quad \Rightarrow \quad \frac{T}{2}s\,z - z = -1 - \frac{T}{2}s$$

$$\therefore \quad -z\left(1 - \frac{T}{2}s\right) = -\left(1 + \frac{T}{2}s\right)$$

$$\therefore \quad z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s} \qquad .....(7.29)$$

In equation (7.29), the variable "s" represent a point on s-plane and "z" is the corresponding point in z-plane.

Let, $s_i = \sigma_i + j\Omega_i$.

On substituting, $s_i = \sigma_i + j\Omega_i$ in equation (7.29) we get,

$$z_i = \frac{1+\frac{T}{2}(\sigma_i + j\Omega_i)}{1-\frac{T}{2}(\sigma_i + j\Omega_i)} = \frac{1+\frac{T}{2}\sigma_i + j\frac{T}{2}\Omega_i}{1-\frac{T}{2}\sigma_i - j\frac{T}{2}\Omega_i} \qquad .....(7.30)$$

The magnitude of equation (7.30) is given by,

$$|z_i| = \left[\frac{\left(1+\frac{T}{2}\sigma_i\right)^2 + \left(\frac{T}{2}\Omega_i\right)^2}{\left(1-\frac{T}{2}\sigma_i\right)^2 + \left(-\frac{T}{2}\Omega_i\right)^2}\right]^{\frac{1}{2}} \qquad .....(7.31)$$

From equation (7.31) the following observations can be made,

1. If $s_i < 0$ (i.e., s_i is negative), then the point $s_i = s_i + jW_i$, lie on the left half of s-plane. In this case, $|z_i| < 1$, hence the corresponding point in z-plane will lie inside the unit circle in z-plane.
2. If $s_i = 0$ (i.e., real part is zero), then the point $s_i = s_i + jW_i$ lie on the imaginary axis in the s-plane. In this case, $|z_i| = 1$, hence the corresponding point in z-plane will lie on the unit circle in z-plane.
3. If $s_i > 0$ (i.e., s_i is positive), then the point $s_i = s_i + jW_i$ lie on the right half of s-plane. In this case, $|z_i| > 1$, hence the corresponding point in z-plane will lie outside the unit circle in z-plane.

The above discussions are applicable for mapping poles and zeros from s-plane to z-plane. The stability of the filter is associated with location of poles. We know that for a stable analog filter the poles should lie on the left half of s-plane. In bilinear transformation, the points on left half of s-plane are mapped as points inside unit circle in z-plane. Hence for stability of digital filter the digital poles should lie inside the unit circle in z-plane.

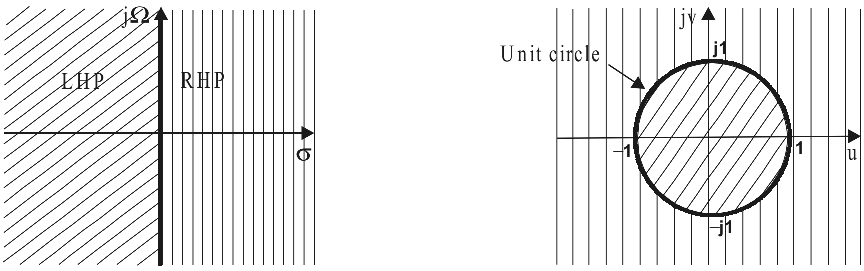


Fig 7.5a : s-plane. *Fig 7.5b : z-plane.*

Fig 7.5 : Mapping of s-plane into z-plane in bilinear transformation.

7.4.2 Relation Between Analog and Digital Frequency in Bilinear Transformation

Let, $s = jW$ be points on imaginary axis and the corresponding points on the z-plane on unit circle are given by $z = e^{jw}$. For bilinear transformation,

$$s = \frac{2}{T}\,\frac{1 - z^{-1}}{1 + z^{-1}} \qquad .....(7.32)$$

Put, $s = jW$ and $z = e^{jw}$ in equation (7.32)

$$\therefore\ j\Omega = \frac{2}{T}\,\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{2}{T}\,\frac{\left(e^{\frac{j\omega}{2}}\,e^{\frac{-j\omega}{2}} - e^{-j\omega}\right)}{\left(e^{\frac{j\omega}{2}}\,e^{\frac{-j\omega}{2}} + e^{-j\omega}\right)} \qquad \boxed{e^{j\theta}\,e^{-j\theta} = 1}$$

$$j\Omega = \frac{2}{T}\,\frac{e^{\frac{-j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)}{e^{\frac{-j\omega}{2}}\left(e^{\frac{j\omega}{2}} + e^{\frac{-j\omega}{2}}\right)} = \frac{2}{T}\,\frac{2j\sin\frac{\omega}{2}}{2\cos\frac{\omega}{2}}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\therefore\ \Omega = \frac{2}{T}\,\frac{\sin\frac{\omega}{2}}{\cos\frac{\omega}{2}} = \frac{2}{T}\tan\frac{\omega}{2} \qquad .....(7.33)$$

$$\boxed{\therefore\ \text{Analog frequency, } \Omega = \frac{2}{T}\tan\frac{\omega}{2}}$$

The equation (7.33) relates the analog frequency, Ω and digital frequency, ω.

From equation (7.33) we get,

$$\frac{\Omega T}{2} = \tan\frac{\omega}{2} \quad \Rightarrow \quad \frac{\omega}{2} = \tan^{-1}\frac{\Omega T}{2}$$

$$\boxed{\therefore\ \text{Digital frequency, } \omega = 2\tan^{-1}\frac{\Omega T}{2}} \qquad .....(7.34)$$

The equation (7.34) can be used to estimate the digital frequency ω for a given analog frequency, Ω. The equation (7.33) is used to calculate the analog frequency for a given digital frequency. From the above analysis it is evident that the analog frequency Ω and digital frequency ω has a nonlinear relationship, because the entire negative imaginary axis in the s-plane (from $\Omega = -\infty$ to 0) is mapped into the lower half of unit circle in z-plane (from $\omega = -\pi$ to 0) and the entire positive imaginary axis in the s-plane (from $\Omega = 0$ to $+\infty$) is mapped into the upper half of unit circle in z-plane (from $\omega = 0$ to $+\pi$). This nonlinear mapping introduces a distortion in the frequency axis, which is called frequency warping.

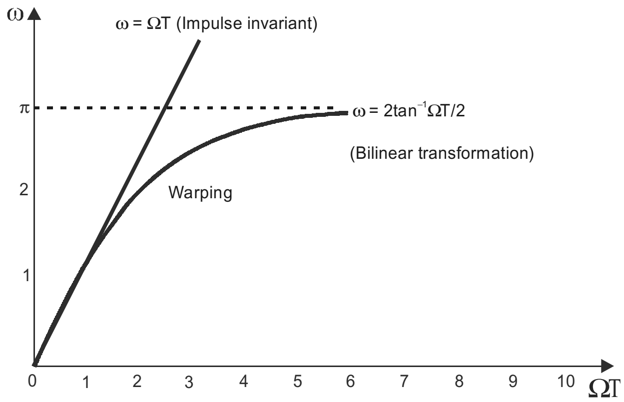


***Fig 7.6 :** Correspondence between analog and digital frequencies resulting from the bilinear transformation.*

The effect of warping on the magnitude response can be explained by considering an analog filter with a number of passbands as shown in fig 7.7. The corresponding digital filter will have same number of passbands, but with disproportionate bandwidth, as shown in fig 7.7.

In designing digital filter using bilinear transformation the effect of warping on amplitude response can be eliminated by prewarping the analog filter. In this method, the specified digital frequencies are converted to analog equivalent using equation (7.33). This analog frequencies are called prewarp frequencies. Using the prewarp frequencies, the analog filter transfer function is designed and then it is transformed to digital filter transfer function.

The effect of warping on the phase response can be explained by considering an analog filter with linear phase response as shown in fig 7.8. The phase response of corresponding digital filter will be nonlinear.

From the above discussions it can be stated that the bilinear transformation preserves the magnitude response of an analog filter only if the specification requires piecewise constant magnitude, but the phase response of the analog filter is not preserved. Therefore the bilinear transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant values. A linear phase analog filter cannot be transformed to a linear phase digital filter using bilinear transformation.

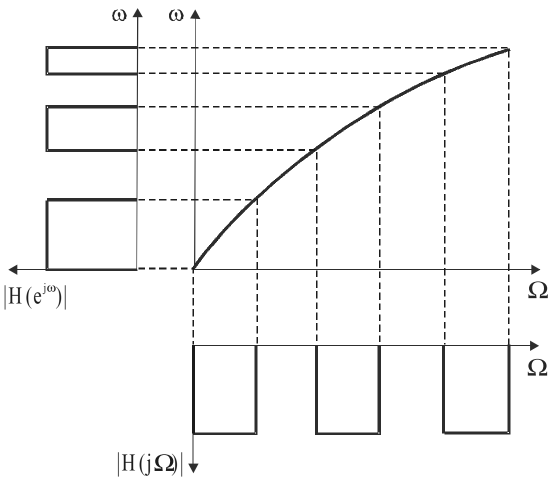


Fig 7.7 : The warping effect on magnitude response.

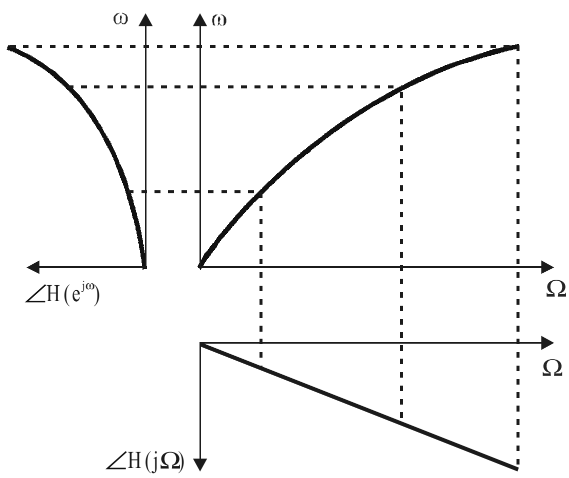


Fig 7.8 : The warping effect on phase response.

Example 7.4

For the analog transfer function, $H(s) = \dfrac{2}{s^2 + 3s + 2}$, determine H(z) using bilinear transformation if **(a)** T = 1 second and **(b)** T = 0.1 second.

Solution

Given that, $H(s) = \dfrac{2}{s^2 + 3s + 2}$

Put, $s = \dfrac{2}{T}\,\dfrac{1 - z^{-1}}{1 + z^{-1}}$ in H(s) to get H(z).

$$\therefore\ H(z) = \frac{2}{\left(\dfrac{2}{T}\,\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2 + 3\left(\dfrac{2}{T}\,\dfrac{1-z^{-1}}{1+z^{-1}}\right) + 2}$$

$$= \frac{2}{\dfrac{4}{T^2}\,\dfrac{\left(1-z^{-1}\right)^2}{\left(1+z^{-1}\right)^2} + \dfrac{6}{T}\,\dfrac{\left(1-z^{-1}\right)}{\left(1+z^{-1}\right)} + 2}$$

$$\therefore\ H(z)=\cfrac{2}{\cfrac{4\left(1-z^{-1}\right)^2+6T\left(1-z^{-1}\right)\left(1+z^{-1}\right)+2T^2\left(1+z^{-1}\right)^2}{T^2\left(1+z^{-1}\right)^2}}$$

$$=\frac{2T^2\left(1+z^{-1}\right)^2}{4\left(1-z^{-1}\right)^2+6T\left(1-z^{-2}\right)+2T^2\left(1+z^{-1}\right)^2}$$

$\boxed{(a+b)\,(a-b)=a^2-b^2}$

(a) T = 1 second

$$\therefore\ H(z)=\frac{2\left(1+z^{-1}\right)^2}{4\left(1-z^{-1}\right)^2+6\left(1-z^{-2}\right)+2\left(1+z^{-1}\right)^2}$$

$$=\frac{2\left(1+2z^{-1}+z^{-2}\right)}{4\left(1-2z^{-1}+z^{-2}\right)+6\left(1-z^{-2}\right)+2\left(1+2z^{-1}+z^{-2}\right)}$$

$\boxed{\begin{array}{l}(a+b)^2=a^2+2ab+b^2\\ (a-b)^2=a^2-2ab+b^2\end{array}}$

$$=\frac{2+4z^{-1}+2z^{-2}}{12-4z^{-1}}=\frac{2+4z^{-1}+2z^{-2}}{12\left(1-\frac{4}{12}z^{-1}\right)}$$

$$=\frac{\frac{2}{12}+\frac{4}{12}z^{-1}+\frac{2}{12}z^{-2}}{1-\frac{4}{12}z^{-1}}=\frac{0.1667+0.3333\,z^{-1}+0.1667\,z^{-2}}{1-0.3333\,z^{-1}}$$

Alternatively,

$$H(z)=\frac{0.1667+0.3333\,z^{-1}+0.1667\,z^{-2}}{1-0.3333\,z^{-1}}=\frac{z^{-2}(0.1667\,z^2+0.3333\,z+0.1667)}{1-0.3333\,z^{-1}}$$

$$=\frac{0.1667\,z^2+0.3333\,z+0.1667}{z^2-0.3333\,z}$$

(b) T = 0.1 second

$$H(z)=\frac{2\times0.1^2\left(1+z^{-1}\right)^2}{4\left(1-z^{-1}\right)^2+6\times0.1\left(1-z^{-2}\right)+2\times0.1^2\left(1+z^{-1}\right)^2}$$

$$=\frac{0.02\left(1+2z^{-1}+z^{-2}\right)}{4\left(1-2z^{-1}+z^{-2}\right)+0.6\left(1-z^{-2}\right)+0.02\left(1+2z^{-1}+z^{-2}\right)}$$

$$=\frac{0.02+0.04z^{-1}+0.02z^{-2}}{4.62-7.96\,z^{-1}+3.42\,z^{-2}}$$

$$=\frac{\frac{0.02}{4.62}+\frac{0.04}{4.62}z^{-1}+\frac{0.02}{4.62}z^{-2}}{1-\frac{7.96}{4.62}z^{-1}+\frac{3.42}{4.62}z^{-2}}=\frac{0.0043+0.0087\,z^{-1}+0.0043\,z^{-2}}{1-1.7229\,z^{-1}+0.7403\,z^{-2}}$$

Alternatively,

$$H(z)=\frac{0.0043+0.0087\,z^{-1}+0.0043\,z^{-2}}{1-1.7229\,z^{-1}+0.7403\,z^{-2}}=\frac{z^{-2}(0.0043\,z^2+0.0087\,z+0.0043)}{z^{-2}(z^2-1.7229\,z+0.7403)}$$

$$=\frac{0.0043\,z^2+0.0087\,z+0.0043}{z^2-1.7229\,z+0.7403}$$

Example 7.5

Obtain H(z) from H(s) when T = 1 second and $H(s) = \dfrac{2s}{s^2 + 0.2s + 1}$

Solution

Given that, $H(s) = \dfrac{2s}{s^2 + 0.2s + 1}$

Put , $s = \dfrac{2}{T}\ \dfrac{1 - z^{-1}}{1 + z^{-1}}$ in H(s) to get H(z).

$$\therefore\ H(z) = \frac{2\left(\dfrac{2}{T}\ \dfrac{1 - z^{-1}}{1 + z^{-1}}\right)}{\left(\dfrac{2}{T}\ \dfrac{1 - z^{-1}}{1 + z^{-1}}\right)^2 + 0.2\left(\dfrac{2}{T}\ \dfrac{1 - z^{-1}}{1 + z^{-1}}\right) + 1} = \frac{\dfrac{4(1 - z^{-1})}{1 + z^{-1}}}{\dfrac{4(1 - z^{-1})^2}{(1 + z^{-1})^2} + \dfrac{0.4(1 - z^{-1})}{1 + z^{-1}} + 1} \qquad \boxed{\text{Put, T = 1}}$$

$$= \frac{\dfrac{4(1 - z^{-1})}{(1 + z^{-1})}}{\dfrac{4(1 - z^{-1})^2 + 0.4(1 - z^{-1})\ (1 + z^{-1}) + (1 + z^{-1})^2}{(1 + z^{-1})^2}} = \frac{4(1 - z^{-1})(1 + z^{-1})}{4(1 - z^{-1})^2 + 0.4(1 - z^{-1})\ (1 + z^{-1}) + (1 + z^{-1})^2}$$

$$= \frac{4(1 - z^{-2})}{4(1 - 2z^{-1} + z^{-2}) + 0.4(1 - z^{-2}) + (1 + 2z^{-1} + z^{-2})}$$

$(a + b)\ (a - b) = a^2 - b^2$
$(a + b)^2 = a^2 + 2ab + b^2$
$(a - b)^2 = a^2 - 2ab + b^2$

$$= \frac{4 - 4z^{-2}}{5.4 - 6z^{-1} + 4.6z^{-2}} = \frac{\dfrac{4}{5.4} - \dfrac{4}{5.4}z^{-1}}{1 - \dfrac{6}{5.4}z^{-1} + \dfrac{4.6}{5.4}z^{-2}}$$

$$= \frac{0.7407 - 0.7407\,z^{-1}}{1 - 1.111z^{-1} + 0.8519\,z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.7407 - 0.7407\,z^{-1}}{1 - 1.111z^{-1} + 0.8519\,z^{-2}} = \frac{0.7407 - 0.7407\,z^{-1}}{z^{-2}(z^2 - 1.111z + 0.8519)}$$

$$= \frac{0.7407\,z^2 - 0.7407\,z}{z^2 - 1.111z + 0.8519}$$

Example 7.6

Obtain H(z) from H(s) when T =1 second, and $H(s) = \dfrac{s^3}{(s + 1)\ (s^2 + s + 1)}$

Solution

Given that, $H(s) = \dfrac{s^3}{(s + 1)\ (s^2 + s + 1)}$

Put, $s = \dfrac{2}{T}\ \dfrac{1 - z^{-1}}{1 + z^{-1}}$ in H(s) to get H(z).

$$\therefore\ H(z) = \frac{\left(\frac{2}{T}\ \frac{1-z^{-1}}{1+z^{-1}}\right)^3}{\left(\frac{2}{T}\ \frac{1-z^{-1}}{1+z^{-1}}+1\right)\left[\left(\frac{2}{T}\ \frac{1-z^{-1}}{1+z^{-1}}\right)^2+\frac{2}{T}\ \frac{1-z^{-1}}{1+z^{-1}}+1\right]}$$

$$= \frac{\frac{8(1-z^{-1})^3}{(1+z^{-1})^3}}{\left[\frac{2(1-z^{-1})}{1+z^{-1}}+1\right]\left[\frac{4(1-z^{-1})^2}{(1+z^{-1})^2}+\frac{2(1-z^{-1})}{1+z^{-1}}+1\right]}$$ $\boxed{\text{Put, T = 1}}$

$$= \frac{\frac{8(1-z^{-1})^3}{(1+z^{-1})^3}}{\left[\frac{2(1-z^{-1})+(1+z^{-1})}{(1+z^{-1})}\right]\left[\frac{4(1-z^{-1})^2+2(1-z^{-1})\ (1+z^{-1})+(1+z^{-1})^2}{(1+z^{-1})^2}\right]}$$

$$= \frac{8(1-z^{-1})^3}{\left[2(1-z^{-1})+(1+z^{-1})\right]\left[4(1-z^{-1})^2+2(1-z^{-1})\ (1+z^{-1})+(1+z^{-1})^2\right]}$$

$$= \frac{8(1-z^{-1})(1-2z^{-1}+z^{-2})}{\left[2-2z^{-1}+1+z^{-1}\right]\left[4(1-2z^{-1}+z^{-2})+2(1-z^{-2})+(1+2z^{-1}+z^{-2})\right]}$$

$$= \frac{8(1-2z^{-1}+z^{-2}-z^{-1}+2z^{-2}-z^{-3})}{\left[3-z^{-1}\right]\left[7-6z^{-1}+3z^{-2}\right]}$$

$$= \frac{8\left[1-3z^{-1}+3z^{-2}-z^{-3}\right]}{21-18z^{-1}+9z^{-2}-7z^{-1}+6z^{-2}-3z^{-3}}$$

$$= \frac{8-24z^{-1}+24z^{-2}-8z^{-3}}{21-25z^{-1}+15z^{-2}-3z^{-3}}$$

$$= \frac{\frac{8}{21}-\frac{24}{21}z^{-1}+\frac{24}{21}z^{-2}-\frac{8}{21}z^{-3}}{1-\frac{25}{21}z^{-1}+\frac{15}{21}z^{-2}-\frac{3}{21}z^{-3}}$$

$$= \frac{0.381-1.1429z^{-1}+1.1429\,z^{-2}-0.381z^{-3}}{1-1.1905\,z^{-1}+0.7143\,z^{-2}-0.1429\,z^{-3}}$$

Alternatively,

$$H(z) = \frac{0.381-1.1429z^{-1}+1.1429\,z^{-2}-0.381z^{-3}}{1-1.1905\,z^{-1}+0.7143\,z^{-2}-0.1429\,z^{-3}}$$

$$= \frac{z^{-3}\left[0.381z^3-1.1429z^2+1.1429\,z-0.381\right]}{z^{-3}\left[z^3-1.1905\,z^2+0.7143\,z-0.1429\right]}$$

$$= \frac{0.381z^3-1.1429z^2+1.1429\,z-0.381}{z^3-1.1905\,z^2+0.7143\,z-0.1429}$$

Example 7.7

Convert the analog filter with system function H(s) into digital filter using bilinear transformation.

$$H(s) = \frac{s+0.3}{(s+0.3)^2+16} \quad ; \quad \text{Take } T = 0.5$$

Solution

Given that, $H(s) = \dfrac{s+0.3}{(s+0.3)^2+16} = \dfrac{s+0.3}{s^2+0.6s+0.09+16} = \dfrac{s+0.3}{s^2+0.6s+16.09}$

Put, $s = \dfrac{2}{T}\,\dfrac{1-z^{-1}}{1+z^{-1}}$ in H(s) to get H(z).

$$\therefore\ H(z) = \frac{\dfrac{2}{T}\,\dfrac{1-z^{-1}}{1+z^{-1}}+0.3}{\left(\dfrac{2}{T}\,\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2+0.6\left(\dfrac{2}{T}\,\dfrac{1-z^{-1}}{1+z^{-1}}\right)+16.09} = \frac{\dfrac{2(1-z^{-1})}{T(1+z^{-1})}+0.3}{\dfrac{4(1-z^{-1})^2}{T^2(1+z^{-1})^2}+\dfrac{1.2(1-z^{-1})}{T(1+z^{-1})}+16.09}$$

$$= \frac{\dfrac{2(1-z^{-1})+0.3T(1+z^{-1})}{T(1+z^{-1})}}{\dfrac{4(1-z^{-1})^2+1.2T(1-z^{-1})(1+z^{-1})+16.09\,T^2(1+z^{-1})^2}{T^2(1+z^{-1})^2}}$$

$$= \frac{\left[2(1-z^{-1})+0.3T(1+z^{-1})\right]T(1+z^{-1})}{4(1-z^{-1})^2+1.2T(1-z^{-2})+16.09\,T^2(1+z^{-1})^2}$$

$$= \frac{\left[2(1-z^{-1})+0.3\times0.5(1+z^{-1})\right]0.5(1+z^{-1})}{4(1-z^{-1})^2+1.2\times0.5(1-z^{-2})+16.09\times0.5^2(1+z^{-1})^2} \qquad \boxed{\text{Put, } T = 0.5}$$

$$= \frac{(1-z^{-1})(1+z^{-1})+0.075(1+z^{-1})^2}{4(1-z^{-1})^2+0.6(1-z^{-2})+4.0225(1+z^{-1})^2}$$

$$= \frac{(1-z^{-2})+0.075(1+2z^{-1}+z^{-2})}{4(1-2z^{-1}+z^{-2})+0.6(1-z^{-2})+4.0225(1+2z^{-1}+z^{-2})}$$

$$= \frac{1.075+0.15z^{-1}-0.925z^{-2}}{8.6225+0.045z^{-1}+7.4225z^{-2}}$$

$$= \frac{\dfrac{1.075}{8.6225}+\dfrac{0.15}{8.6225}z^{-1}-\dfrac{0.925}{8.6225}z^{-2}}{1+\dfrac{0.045}{8.6225}z^{-1}+\dfrac{7.4225}{8.6225}z^{-2}} = \frac{0.1247+0.0174z^{-1}-0.1073z^{-2}}{1+0.0052z^{-1}+0.8608z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.1247+0.0174z^{-1}-0.1073z^{-2}}{1+0.0052z^{-1}+0.8608z^{-2}}$$

$$= \frac{z^{-2}(0.1247z^2+0.0174z-0.1073)}{z^{-2}(z^2+0.0052z+0.8608)}$$

$$= \frac{0.1247z^2+0.0174z-0.1073}{z^2+0.0052z+0.8608}$$

7.5 Specifications of Digital IIR Lowpass Filter

Let, $H(e^{jw})$ = Frequency response of IIR filter.

$|H(e^{jw})|$ = Magnitude response of IIR filter.

The magnitude response, $|H(e^{jw})|$ of IIR filter will have a passband, transition band and stop band. The specification of the IIR filter can be expressed in any one of the following three different ways.

Case i : Gain at passband and stopband edge frequency

Case ii : Attenuation at passband and stopband edge frequency

Case iii : Ripple at passband and stopband edge frequency

Case i : Gain at passband and stopband edge frequency

The gain can be expressed either in normal values or in decibels (dB).

The maximum value of normalized gain is unity and so the gain at band edge frequencies will be less than 1. Therefore, the dB-gain will be negative.

Let, w_p = Passband edge digital frequency in rad/sample.

w_s = Stopband edge digital frequency in rad/sample.

$A_p = |H(e^{jw})|_{w = w_p}$ = Gain (or magnitude) at passband edge frequency.

$A_s = |H(e^{jw})|_{w = w_s}$ = Gain (or magnitude) at stopband edge frequency.

$A_{p,dB} = 20 \log [|H(e^{jw})|_{w = w_p}]$ = dB-Gain (or dB-magnitude) at passband edge frequency.

$A_{s,dB} = 20 \log [|H(e^{jw})|_{w = w_s}]$ = dB-Gain (or dB-magnitude) at stopband edge frequency.

The gain in normal values can be converted to dB-gain or vice versa as shown below.

$$A_{p,dB} = 20 \log A_p \quad \Rightarrow \quad A_p = 10^{(A_{p,dB}/20)}$$

$$A_{s,dB} = 20 \log A_s \quad \Rightarrow \quad A_s = 10^{(A_{s,dB}/20)}$$

Example

Let, $A_p = 0.8$

$A_s = 0.2$

The gain in normal values can be converted to dB-gain as shown below.

$\therefore \quad A_{p,dB} = 20 \log A_p = 20 \log 0.8 = -1.9382 \text{ dB} \approx -2\text{dB}$

$A_{s,dB} = 20 \log A_s = 20 \log 0.2 = -13.9794 \text{ dB} \approx -14\text{dB}$

The dB-gain can be converted back to gain in normal values as shown below.

$A_p = 10^{(A_{p,dB}/20)} = 10^{(-1.9382/20)} = 0.8$

$A_s = 10^{(A_{s,dB}/20)} = 10^{(-13.9794/20)} = 0.2$

The magnitude response and log-magnitude response of a digital IIR lowpass filter are shown in fig 7.9.

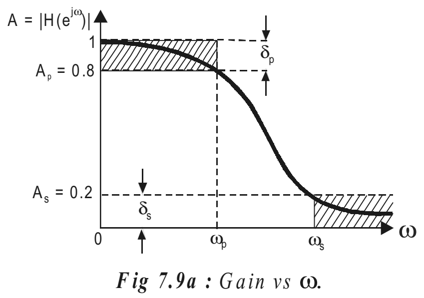


Fig 7.9a : Gain vs ω.

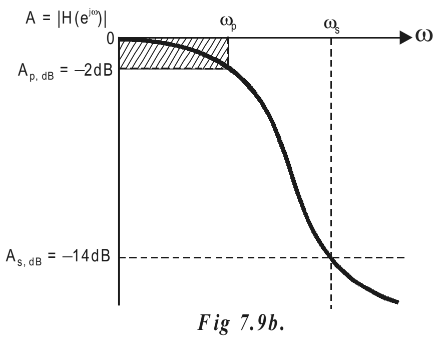


Fig 7.9b.

Fig 7.9 : Magnitude response of digital IIR lowpass filter.

Case ii : Attenuation at passband and stopband edge frequency

Alternatively, the specification of the filter can be attenuation at passband and stopband edge frequencies. The attenuation in normal value is inverse of the gain in normal value. The attenuation is usually expressed in decibels (dB). Since the gain at edge frequencies are less than 1, the attenuation in normal values will be greater than1, and the dB-attenuation is positive.

$$\text{Let, } \alpha_p = \frac{1}{A_p} = \frac{1}{\left|H(e^{j\omega})\right|_{\omega=\omega_p}} = \text{Attenuation at passband edge frequency}$$

$$\alpha_s = \frac{1}{A_s} = \frac{1}{\left|H(e^{j\omega})\right|_{\omega=\omega_s}} = \text{Attenuation at stopband edge frequency}$$

$$\alpha_{p,dB} = 20\log\left[\frac{1}{A_p}\right] = 20\log\left[\frac{1}{\left|H(e^{j\omega})\right|_{\omega=\omega_p}}\right] = \text{dB-Attenuation at passband edge frequency}$$

$$\alpha_{s,dB} = 20\log\left[\frac{1}{A_s}\right] = 20\log\left[\frac{1}{\left|H(e^{j\omega})\right|_{\omega=\omega_s}}\right] = \text{dB-Attenuation at stopband edge frequency}$$

The attenuation in normal values can be conveted to dB-attenuation or vice-versa as shown below.

$$\alpha_{p,dB} = 20\log\alpha_p \quad \Rightarrow \quad \alpha_p = 10^{(\alpha_{p,dB}/20)}$$
$$\alpha_{s,dB} = 20\log\alpha_s \quad \Rightarrow \quad \alpha_s = 10^{(\alpha_{s,dB}/20)}$$

The attenuation can be converted to gain or vice versa using the following equations.

$A_p = \frac{1}{\alpha_p}$	$\alpha_p = \frac{1}{A_p}$
$A_s = \frac{1}{\alpha_s}$	$\alpha_s = \frac{1}{A_s}$
$A_{p,dB} = -\alpha_{p,dB}$	$\alpha_{p,dB} = -A_{p,dB}$
$A_{s,dB} = -\alpha_{s,dB}$	$\alpha_{s,dB} = -A_{s,dB}$

Example

Let, $\alpha_{p,dB} = +1.9382\text{dB} \approx +2\text{dB}$

$\alpha_{s,dB} = +13.9794\text{dB} \approx +14\text{dB}$

The dB-attenuation can be converted to normal values as shown below.

$$\alpha_p = 10^{(\alpha_{p,dB}/20)} = 10^{(1.9382/20)} = 1.25$$
$$\alpha_s = 10^{(\alpha_{s,dB}/20)} = 10^{(13.9794/20)} = 5$$

The attenuation can be converted to gain as shown below.

$$A_p = \frac{1}{\alpha_p} = \frac{1}{1.25} = 0.8$$

$$A_s = \frac{1}{\alpha_s} = \frac{1}{5} = 0.2$$

The gain in normal values can be converted to dB-gain as shown below.

$$A_{p,dB} = 20\log A_p = 20\log 0.8 = -1.9382\ \text{dB}$$
$$A_{s,dB} = 20\log A_s = 20\log 0.2 = -13.9794\ \text{dB}$$

Note : The dB-gain and dB-attenuation are numerically same, but dB-gain is negative and dB-attenuation is positive.

The attenuation response and log-attenuation response of a digital IIR lowpass filter are shown in fig 7.10.

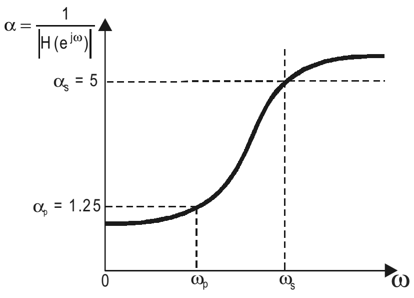


Fig 7.10a : *Attenuation vs ω*

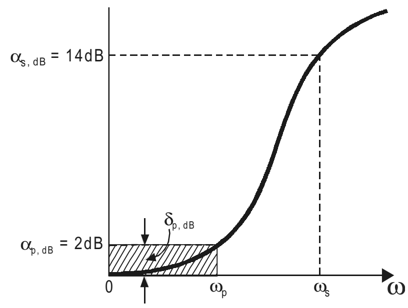


Fig 7.10b : *dB-attenuation vs ω*

Fig 7.10 : *Attenuation response of digital IIR lowpass filter.*

Case iii : Ripple at passband and stopband edge frequency

Sometimes, the specifications are given in terms of ripple or tolerance in the passband and stopband. The ripple can be in normal values or in decibels (dB).

Let, d_p = Passband ripple.

d_s = Stopband ripple.

$d_{p,\,dB} = 20 \log d_p$ = Passband ripple in dB.

$d_{s,dB} = 20 \log d_s$ = Stopband ripple in dB.

The dB-ripples can be converted to normal values as shown below.

$$d_p = 1 - 10^{(-d_{p,dB}/20)}$$

$$d_s = 10^{(-d_{s,dB}/20)}$$

The ripples in normal values can be converted to gain or attenuation as shown below.

$$A_p = 1 - \delta_p \qquad \alpha_p = \frac{1}{A_p} = \frac{1}{1-\delta_p}$$

$$A_s = \delta_s \qquad \alpha_s = \frac{1}{A_s} = \frac{1}{\delta_s}$$

The ripples in dB can be converted to dB-gain or dB-attenuation as shown below. Usually, the ripples are specified as positive dB.

$$A_{p,dB} = -\delta_{p,dB} \qquad \alpha_{p,dB} = \delta_{p,dB}$$

$$A_{s,dB} = -\delta_{s,dB} \qquad \alpha_{s,dB} = \delta_{s,dB}$$

The ripples in dB can be converted to gain or attenuation in normal values as shown below.

$$A_p = 10^{(-\delta_{p,dB}/20)} \qquad \alpha_p = 10^{(\delta_{p,dB}/20)}$$

$$A_s = 10^{(-\delta_{s,dB}/20)} \qquad \alpha_s = 10^{(\delta_{s,dB}/20)}$$

Example

Let, $\delta_{p,dB} = +1.9382 \text{ dB} \approx +2 \text{ dB}$

$\delta_{s,dB} = +13.9794 \text{ dB} \approx +14 \text{ dB}$

The dB-ripples can be converted to ripples in normal values as shown below.

$$\delta_p = 1 - 10^{(-\delta_{p,dB}/20)} = 1 - 10^{(-1.9382/20)} = 0.2$$

$$\delta_s = 10^{(-\delta_{s,dB}/20)} = 10^{(-13.9794/20)} = 0.2$$

The dB ripples can be converted to dB-gain and dB-attenuation as shown below.

$$A_{p,dB} = -\delta_{p,dB} = -\,1.9382 \text{ dB} \approx -2 \text{ dB}$$

$$A_{s,dB} = -\delta_{s,dB} = -\,13.9794 \text{ dB} \approx -14 \text{ dB}$$

$$\alpha_{p,dB} = \delta_{p,dB} = +\,1.9382 \text{ dB} \approx +2 \text{ dB}$$

$$\alpha_{s,dB} = \delta_{s,dB} = +\,13.9794 \text{ dB} \approx +14 \text{ dB}$$

The dB-ripples can be converted to gain and attenuation in normal values.

$$A_p = 10^{(-\delta_{p,dB}/20)} = 10^{(-1.9382/20)} = 0.8$$

$$A_s = 10^{(-\delta_{s,dB}/20)} = 10^{(-13.9794/20)} = 0.2$$

$$\alpha_p = 10^{(\delta_{p,dB}/20)} = 10^{(1.9382/20)} = 1.25$$

$$\alpha_s = 10^{(\delta_{s,dB}/20)} = 10^{(13.9794/20)} = 5$$

7.6 Design of Lowpass Digital Butterworth Filter

The popular methods of designing IIR digital filter involves the design of equivalent analog filter and then converting the analog filter to digital filter. Hence to design a Butterworth IIR digital filter, first an analog butterworth filter transfer function is determined using the given specifications. Then the analog filter transfer function is converted to a digital filter transfer function by using either impulse invariant transformation or bilinear transformation.

7.6.1 Analog Butterworth Filter

The analog butterworth filter is designed by approximating the ideal analog filter frequency response, $H(j\Omega)$ using an error function. The error function is selected such that the magnitude is maximally flat in the passband and monotonically decreasing in the stopband. (Strictly saying, the magnitude is maximally flat at the origin i.e., at $\Omega = 0$, and monotonically decreasing with increasing Ω).

The magnitude response of lowpass filter obtained by this approximation is given by,

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} \qquad \text{..... (7.35)}$$

Since, $|H(j\Omega)|^2 = H(j\Omega)\, H^*(j\Omega) = H(j\Omega)\, H(-j\Omega)$, the equation (7.35) can be written as shown below.

$$H(j\Omega)\, H(-j\Omega) = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} \qquad \text{..... (7.36)}$$

We know that the frequency response $H(j\Omega)$ of an analog filter is obtained by letting $s = j\Omega$ in the analog transfer function $H(j\Omega)$. Hence substituting Ω by s/j in equation (7.36) gives the system transfer function.

$$\therefore\ H(s)\,H(-s) = \frac{1}{1+\left(\frac{s/j}{\Omega_c}\right)^{2N}} = \frac{1}{1+\left(\frac{s^2}{j^2\,\Omega_c^2}\right)^{N}} \qquad .....\ (7.37)$$

In equation (7.37), when s/W_c is replaced by s_n (i.e., letting W_c = 1 rad/sec) the transfer function is called normalized transfer function.

$$\therefore\ H(s_n)\,H(-s_n) = \frac{1}{1\ +\ (-s_n^2)^N} \qquad .....\ (7.38)$$

The transfer function of equation (7.38) will have 2N poles which are given by the roots of the denominator polynomial. It can be shown that the poles of the transfer function symmetrically lies on a unit circle in s-plane with angular spacing of p/N. (Refer example 7.8 to example 7.13).

Properties of Butterworth Filters

1. The Butterworth filters are all pole designs. (i.e., the zeros of the filters exist at infinity).
2. At the cutoff frequency W_c the magnitude of normalized Butterworth filter is $1/\sqrt{2}$ (i.e., $|H(jW)| = 1/\sqrt{2} = 0.707$). Hence the dB magnitude at the cutoff frequency will be 3 dB less than the maximum value.
3. The filter order N completely specifies the filter.
4. The magnitude is maximally flat at the origin.
5. The magnitude is a monotonically decreasing function of W.
6. The magnitude response approaches the ideal response as the value of N increases.

7.6.2 Poles of Butterworth Lowpass Filter

Let us equate the denominator polynomial of equation (7.38) to zero and solve the 2N poles of Butterworth lowpass filter.

$$\therefore\ 1+(-s_n^2)^N = 0 \quad \Rightarrow \quad 1+(-1)^N\ s_n^{2N} = 0 \qquad .....(7.39)$$

case i : When N is odd

When N is odd, $(-1)^N = -1$

Hence the equation (7.39) can be written as,

$$1-s_n^{2N} = 0 \quad \Rightarrow \quad s_n^{2N} = 1 \quad \Rightarrow \quad s_n = 1^{\frac{1}{2N}}$$

Now, s_n will have 2N values which are given by 2N roots of unity. These 2N roots can be evaluated by taking 1 as e^{j2pk}, where k is an iteger.

For integer k,
$e^{j2pk} = \cos 2pk + j\sin 2pk$
$= 1 + j0 = 1$

$$\therefore\ s_n = 1^{\frac{1}{2N}} = \left(e^{j2\pi k}\right)^{\frac{1}{2N}} = e^{j\frac{\pi k}{N}}$$

Therefore, when N is odd, the 2N poles of Butterworth filter are given by the equation,

$$s_n = e^{j\frac{\pi k}{N}} \quad ; \quad \text{for } k = 1,\ 2,\ 3,\ .....\ 2N \qquad ......(7.40)$$

case ii : When N is even

When N is even, $(-1)^N = 1$

Hence the equation (7.39) can be written as,

$$1 + s_n^{2N} = 0 \quad \Rightarrow \quad s_n^{2N} = -1 \quad \Rightarrow \quad s_n = (-1)^{\frac{1}{2N}}$$

Now, s_n will have 2N values which are given by 2N roots of –1. These 2N roots can be evaluated by taking –1 as $e^{j(2k-1)\pi}$, where k is an integer.

$$\therefore \; s_n = (-1)^{\frac{1}{2N}} = \left(e^{j(2k-1)\pi}\right)^{\frac{1}{2N}} = e^{j\frac{(2k-1)\pi}{2N}}$$

For integer k,

$$e^{j(2k-1)\pi} = \cos(2k-1)\pi + j\sin(2k-1)\pi = -1 + j0 = -1$$

Therefore, when N is even the 2N poles of Butterworth filter are given by,

$$\therefore \; s_n = e^{j\frac{(2k-1)\pi}{2N}} \quad ; \quad \text{for } k = 1, 2, 3, \ldots\ldots 2N \qquad \ldots\ldots(7.41)$$

Example 7.8

Determine the poles of lowpass Butterworth filter for N = 1. Sketch the location of poles on s-plane and hence determine the normalized transfer function of lowpass filter.

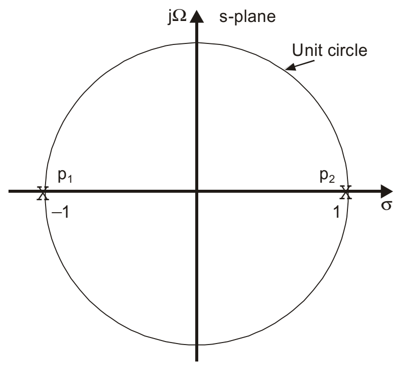


Fig 1 : Location of poles on s-plane, when N = 1.

Solution

When N = 1, from equation (7.40), $s_n = e^{j\pi k}$; for k = 1, 2

When k = 1 ; $s_n = e^{j\pi\times 1} = 1\angle\pi = \cos\pi + j\sin\pi = -1 + j0 = p_1$

When k = 2 ; $s_n = e^{j\pi\times 2} = 1\angle 2\pi = \cos 2\pi + j\sin 2\pi = 1 + j0 = p_2$

The transfer function is formed using the poles lying on left half of s-plane. The pole lying on left half of s-plane is p_1.

$$\therefore \; s_n = p_1 = -1 \;\Rightarrow\; s_n - p_1 = 0 \;\Rightarrow\; s_n + 1 = 0$$

$$\therefore \; \text{Normalized transfer function, } H(s_n) = \frac{1}{s_n - p_1} = \frac{1}{s+1}$$

Example 7.9

Determine the poles of lowpass Butterworth filter for N = 2. Sketch the location of poles on s-plane and hence determine the normalized transfer function of lowpass filter.

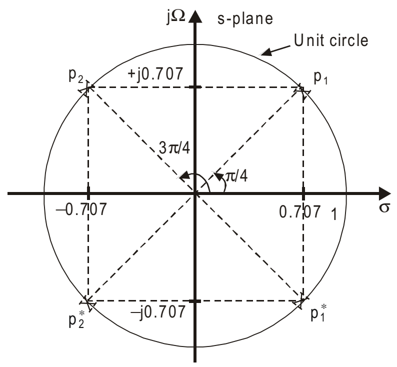


Fig 1 : Location of poles on s-plane, when N = 2.

Solution

When N = 2, from equation (7.41),

$$s_n = e^{j\frac{(2k-1)\pi}{4}} \quad ; \text{ for } k = 1, 2, 3, 4$$

When k = 1 ; $s_n = e^{j\frac{(2-1)\pi}{4}} = e^{j\frac{\pi}{4}} = 1\angle\pi/4 = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} = 0.707 + j0.707 = p_1$

When k = 2 ; $s_n = e^{j\frac{(4-1)\pi}{4}} = e^{j\frac{3\pi}{4}} = 1\angle 3\pi/4 = \cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4} = -0.707 + j0.707 = p_2$

When k = 3 ; $s_n = e^{j\frac{(6-1)\pi}{4}} = e^{j\frac{5\pi}{4}} = 1\angle 5\pi/4 = \cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4} = -0.707 - j0.707 = p_2^*$

When k = 4 ; $s_n = e^{j\frac{(8-1)\pi}{4}} = e^{j\frac{7\pi}{4}} = 1\angle 7\pi/4 = \cos\frac{7\pi}{4} + j\sin\frac{7\pi}{4} = 0.707 - j0.707 = p_1^*$

The transfer function is formed using the poles lying on left half of s-plane. The poles lying on left half of s-plane are p_2 and p_2^*.

\ $s_n = p_2 = -0.707 + j0.707$ Þ $s_n - p_2 = 0$ Þ $(s_n + 0.707 - j0.707) = 0$

$s_n = p_2^* = -0.707 - j0.707$ Þ $s_n - p_2^* = 0$ Þ $(s_n + 0.707 + j0.707) = 0$

$\therefore$ Normalized transfer function, $H(s_n) = \dfrac{1}{(s_n - p_2)\,(s_n - p_2^*)}$

$$\therefore\ H(s_n) = \frac{1}{(s_n + 0.707 - j0.707)\,(s_n + 0.707 + j0.707)}$$

$$= \frac{1}{(s_n + 0.707)^2 - (j0.707)^2}$$

$\boxed{(a + b)\,(a - b) = a^2 - b^2}$

$$= \frac{1}{(s_n + 0.707)^2 + 0.707^2}$$

$$= \frac{1}{s_n^2 + 2 \times 0.707\,s_n + 0.707^2 + 0.707^2} = \frac{1}{s_n^2 + 1.414\,s_n + 1}$$

Example 7.10

Determine the poles of lowpass Butterworth filter for N = 3. Sketch the location of poles on s-plane and hence determine the normalized transfer function of lowpass filter.

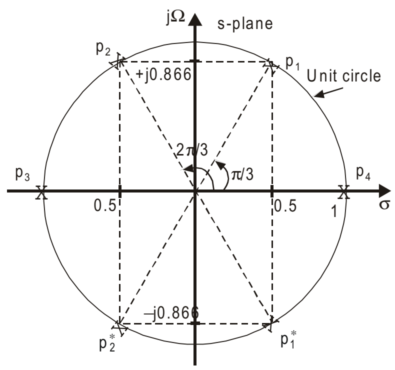


Fig 1 : Location of poles on s-plane, when N = 3.

Solution

When N = 3, from equation (7.40),

$$s_n = e^{j\frac{\pi k}{3}}\ ;\ \text{for } k = 1,\ 2,\ 3,\ 4,\ 5,\ 6$$

When k = 1 ; $s_n = e^{j\frac{\pi\times 1}{3}} = 1\angle\pi/3 = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = 0.5 + j0.866 = p_1$

When k = 2 ; $s_n = e^{j\frac{\pi\times 2}{3}} = 1\angle 2\pi/3 = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = -0.5 + j0.866 = p_2$

When k = 3 ; $s_n = e^{j\frac{\pi\times 3}{3}} = 1\angle 3\pi/3 = \cos\frac{3\pi}{3} + j\sin\frac{3\pi}{3} = -1 + j0 = p_3$

When k = 4 ; $s_n = e^{j\frac{\pi\times 4}{3}} = 1\angle 4\pi/3 = \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} = -0.5 - j0.866 = p_2^*$

When k = 5 ; $s_n = e^{j\frac{\pi\times 5}{3}} = 1\angle 5\pi/3 = \cos\frac{5\pi}{3} + j\sin\frac{5\pi}{3} = 0.5 - j0.866 = p_1^*$

When k = 6 ; $s_n = e^{j\frac{\pi\times 6}{3}} = 1\angle 6\pi/3 = \cos\frac{6\pi}{3} + j\sin\frac{6\pi}{3} = 1 + j0 = p_4$

The transfer function is formed using the poles lying on left half of s-plane. The poles lying on left half of s-plane are p_2, p_2^* and p_3.

$\therefore\ s_n = p_2 = -0.5 + j0.866 \Rightarrow s_n - p_2 = 0 \Rightarrow (s_n + 0.5 - j0.866) = 0$

$s_n = p_2^* = -0.5 - j0.866 \Rightarrow s_n - p_2^* = 0 \Rightarrow (s_n + 0.5 + j0.866) = 0$

$s_n = p_3 = -1 \Rightarrow s_n - p_3 = 0 \Rightarrow (s_n + 1) = 0$

$\therefore$ Normalized transfer function, $H(s_n) = \dfrac{1}{(s_n - p_3)\,(s_n - p_2)\,(s_n - p_2^*)}$

$$\therefore\ H(s_n) = \frac{1}{(s_n + 1)\,(s_n + 0.5 - j0.866)\,(s_n + 0.5 + j0.866)}$$

$$= \frac{1}{(s_n + 1)\,((s_n + 0.5)^2 - (j0.866)^2)}$$ $\boxed{(a + b)\,(a - b) = a^2 - b^2}$

$$= \frac{1}{(s_n + 1)\,((s_n + 0.5)^2 + 0.866^2)} = \frac{1}{(s_n + 1)\,(s_n^2 + 2\times 0.5\,s_n + 0.5^2 + 0.866^2)}$$

$$= \frac{1}{(s_n + 1)\,(s_n^2 + s + 1)}$$

Example 7.11

Determine the poles of lowpass Butterworth filter for N = 4. Sketch the location of poles on s-plane and hence determine the normalized transfer function of lowpass filter.

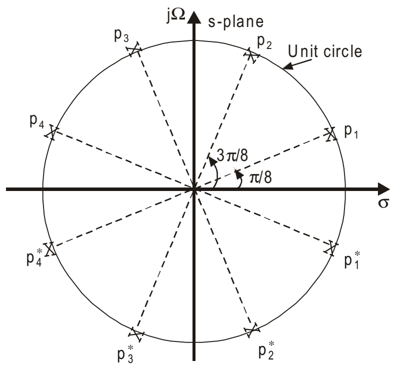


Fig 1 : Location of poles on s-plane, when N = 4.

Solution

When N = 4, from equation (7.41),

$$s_n = e^{j\frac{(2k-1)\pi}{8}}\ ;\ \text{for } k = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8$$

When k = 1 ; $s_n = e^{j\frac{(2-1)\pi}{8}} = e^{j\frac{\pi}{8}} = 1\angle\pi/8 = \cos\frac{\pi}{8} + j\sin\frac{\pi}{8} = 0.924 + j0.383 = p_1$

When k = 2 ; $s_n = e^{j\frac{(4-1)\pi}{8}} = e^{j\frac{3\pi}{8}} = 1\angle 3\pi/8 = \cos\frac{3\pi}{8} + j\sin\frac{3\pi}{8} = 0.383 + j0.924 = p_2$

When k = 3 ; $s_n = e^{j\frac{(6-1)\pi}{8}} = e^{j\frac{5\pi}{8}} = 1\angle 5\pi/8 = \cos\frac{5\pi}{8} + j\sin\frac{5\pi}{8} = -0.383 + j0.924 = p_3$

$$\text{When } k = 4\ ;\quad s_n = e^{j\frac{(8-1)\pi}{8}} = e^{j\frac{7\pi}{8}} = 1\angle 7\pi / 8 = \cos\frac{7\pi}{8} + j\sin\frac{7\pi}{8} = -0.924 + j0.383 = p_4$$

$$\text{When } k = 5\ ;\quad s_n = e^{j\frac{(10-1)\pi}{8}} = e^{j\frac{9\pi}{8}} = 1\angle 9\pi / 8 = \cos\frac{9\pi}{8} + j\sin\frac{9\pi}{8} = -0.924 - j0.383 = p_4^*$$

$$\text{When } k = 6\ ;\quad s_n = e^{j\frac{(12-1)\pi}{8}} = e^{j\frac{11\pi}{8}} = 1\angle 11\pi / 8 = \cos\frac{11\pi}{8} + j\sin\frac{11\pi}{8} = -0.383 - j0.924 = p_3^*$$

$$\text{When } k = 7\ ;\quad s_n = e^{j\frac{(14-1)\pi}{8}} = e^{j\frac{13\pi}{8}} = 1\angle 13\pi / 8 = \cos\frac{13\pi}{8} + j\sin\frac{13\pi}{8} = 0.383 - j0.924 = p_2^*$$

$$\text{When } k = 8\ ;\quad s_n = e^{j\frac{(16-1)\pi}{8}} = e^{j\frac{15\pi}{8}} = 1\angle 15\pi / 8 = \cos\frac{15\pi}{8} + j\sin\frac{15\pi}{8} = 0.924 - j0.383 = p_1^*$$

The transfer function is formed using the poles lying on left half of s-plane. The poles lying on left half of s-plane are p_3, p_3^*, p_4 and p_4^*.

$$\therefore\ s_n = p_3 = -0.383 + j0.924 \;\Rightarrow\; s_n - p_3 = 0 \;\Rightarrow\; (s_n + 0.383 - j0.924) = 0$$

$$s_n = p_3^* = -0.383 - j0.924 \;\Rightarrow\; s_n - p_3^* = 0 \;\Rightarrow\; (s_n + 0.383 + j0.924) = 0$$

$$s_n = p_4 = -0.924 + j0.383 \;\Rightarrow\; s_n - p_4 = 0 \;\Rightarrow\; (s_n + 0.924 - j0.383) = 0$$

$$s_n = p_4^* = -0.924 - j0.383 \;\Rightarrow\; s_n - p_4^* = 0 \;\Rightarrow\; (s_n + 0.924 + j0.383) = 0$$

$$\therefore\ \text{Normalized transfer function, } H(s_n) = \frac{1}{(s_n - p_3)\,(s_n - p_3^*)\,(s_n - p_4)\,(s_n - p_4^*)}$$

$$\therefore\ H(s_n) = \frac{1}{(s_n + 0.383 - j0.924)\,(s_n + 0.383 + j0.924)\,(s_n + 0.924 + j0.383)\,(s_n + 0.924 + j0.383)}$$

$$= \frac{1}{\left((s_n + 0.383)^2 - (j0.924)^2\right)\left((s_n + 0.924)^2 - (j0.383)^2\right)} \qquad \boxed{(a+b)\,(a-b) = a^2 - b^2}$$

$$= \frac{1}{\left((s_n + 0.383)^2 + 0.924^2\right)\left((s_n + 0.924)^2 + 0.383^2\right)}$$

$$= \frac{1}{(s_n^2 + 2 \times 0.383\, s_n + 0.383^2 + 0.924^2)\,(s_n^2 + 2 \times 0.924\, s_n + 0.924^2 + 0.383^2)}$$

$$= \frac{1}{(s_n^2 + 0.766\, s_n + 1)\,(s_n^2 + 1.848\, s_n + 1)}$$

Example 7.12

Determine the poles of lowpass Butterworth filter for N = 5. Sketch the location of poles on s-plane and hence determine the normalized transfer function of lowpass filter.

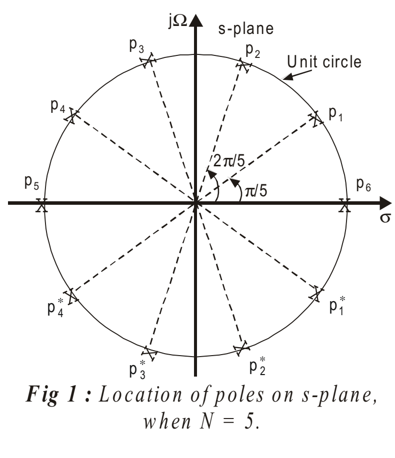


***Fig 1 :** Location of poles on s-plane, when N = 5.*

Solution

When N = 5, from equation (7.40), $s_n = e^{j\frac{\pi k}{5}}$; for k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

When k = 1 ; $s_n = e^{j\frac{\pi\times 1}{5}} = 1\angle\pi / 5 = \cos\frac{\pi}{5} + j\sin\frac{\pi}{5} = 0.809 + j0.588 = p_1$

When k = 2 ; $s_n = e^{j\frac{\pi\times 2}{5}} = 1\angle 2\pi / 5 = \cos\frac{2\pi}{5} + j\sin\frac{2\pi}{5} = 0.309 + j0.951 = p_2$

When k = 3 ; $s_n = e^{j\frac{\pi\times 3}{5}} = 1\angle 3\pi / 5 = \cos\frac{3\pi}{5} + j\sin\frac{3\pi}{5} = -0.309 + j0.951 = p_3$

When k = 4 ; $s_n = e^{j\frac{\pi\times 4}{5}} = 1\angle 4\pi / 5 = \cos\frac{4\pi}{5} + j\sin\frac{4\pi}{5} = -0.809 + j0.587 = p_4$

When k = 5 ; $s_n = e^{j\frac{\pi\times 5}{5}} = 1\angle 5\pi / 5 = \cos\frac{5\pi}{5} + j\sin\frac{5\pi}{5} = -1 + j0 = p_5$

When k = 6 ; $s_n = e^{j\frac{\pi\times 6}{5}} = 1\angle 6\pi / 5 = \cos\frac{6\pi}{5} + j\sin\frac{6\pi}{5} = -0.809 - j0.587 = p_4^*$

When k = 7 ; $s_n = e^{j\frac{\pi\times 7}{5}} = 1\angle 7\pi / 5 = \cos\frac{7\pi}{5} + j\sin\frac{7\pi}{5} = -0.309 - j0.951 = p_3^*$

When k = 8 ; $s_n = e^{j\frac{\pi\times 8}{5}} = 1\angle 8\pi / 5 = \cos\frac{8\pi}{5} + j\sin\frac{8\pi}{5} = 0.309 - j0.951 = p_2^*$

When k = 9 ; $s_n = e^{j\frac{\pi\times 9}{5}} = 1\angle 9\pi / 5 = \cos\frac{9\pi}{5} + j\sin\frac{9\pi}{5} = 0.809 - j0.588 = p_1^*$

When k = 10 ; $s_n = e^{j\frac{\pi\times 10}{5}} = 1\angle 10\pi / 5 = \cos\frac{10\pi}{5} + j\sin\frac{10\pi}{5} = 1 + j0 = p_6$

The transfer function is formed using the poles lying on left half of s-plane. The poles lying on left half of s-plane are p_3, p_3^*, p_4, p_4^* and p_5.

$\therefore$ $s_n = p_3 = -0.309 + j0.951 \Rightarrow s_n - p_3 = 0 \Rightarrow (s_n + 0.309 - j0.951) = 0$

$s_n = p_3^* = -0.309 - j0.951 \Rightarrow s_n - p_3^* = 0 \Rightarrow (s_n + 0.309 + j0.951) = 0$

$s_n = p_4 = -0.809 + j0.587 \Rightarrow s_n - p_4 = 0 \Rightarrow (s_n + 0.809 - j0.587) = 0$

$s_n = p_4^* = -0.809 - j0.587 \Rightarrow s_n - p_4^* = 0 \Rightarrow (s_n + 0.809 + j0.587) = 0$

$s_n = p_5 = -1 \Rightarrow s_n - p_5 = 0 \Rightarrow (s_n + 1) = 0$

$\therefore$ Normalized transfer function, $H(s_n) = \dfrac{1}{(s_n - p_5)\,(s_n - p_3)\,(s_n - p_3^*)\,(s_n - p_4)\,(s_n - p_4^*)}$

$$\therefore\ H(s_n) = \frac{1}{(s_n+1)\,(s+0.309-j0.951)\,(s_n+0.309+j0.951)\,(s_n+0.809-j0.587)\,(s_n+0.809+j0.587)}$$

$$= \frac{1}{(s_n+1)\,((s_n+0.309)^2-(j0.951)^2)\,((s_n+0.809)^2-(j0.587)^2)} \qquad \boxed{(a+b)\,(a-b) = a^2 - b^2}$$

$$= \frac{1}{(s_n+1)\,((s_n+0.309)^2+0.951^2)\,((s_n+0.809)^2+0.587^2)}$$

$$= \frac{1}{(s_n+1)\,(s_n^2+2\times 0.309\,s_n+0.309^2+0.951^2)\,(s_n^2+2\times 0.809\,s_n+0.809^2+0.587^2)}$$

$$= \frac{1}{(s_n+1)\,(s_n^2+0.618\,s_n+1)\,(s_n^2+1.618\,s_n+1)}$$

Example 7.13

Determine the poles of lowpass Butterworth filter for N = 6. Sketch the location of poles on s-plane and hence determine the normalized transfer function of lowpass filter.

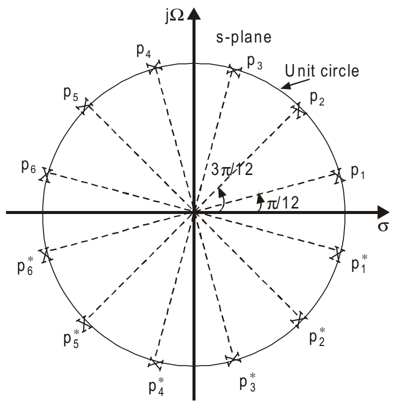


Fig 1 : Location of poles on s-plane, when N = 6.

Solution

When N = 6, from equation (7.41),

$$s_n = e^{j\frac{(2k-1)\pi}{12}} \quad ; \text{ for } k = 1,\ 2,\ 3,\ \ldots\ldots\ 12$$

$$\text{When } k = 1\ ; \quad s_n = e^{j\frac{(2-1)\pi}{12}} = e^{j\frac{\pi}{12}} = 1\angle\pi/12 = \cos\frac{\pi}{12} + j\sin\frac{\pi}{12} = 0.966 + j0.259 = p_1$$

$$\text{When } k = 2\ ; \quad s_n = e^{j\frac{(4-1)\pi}{12}} = e^{j\frac{3\pi}{12}} = 1\angle 3\pi/12 = \cos\frac{3\pi}{12} + j\sin\frac{3\pi}{12} = 0.707 + j0.707 = p_2$$

$$\text{When } k = 3\ ; \quad s_n = e^{j\frac{(6-1)\pi}{12}} = e^{j\frac{5\pi}{12}} = 1\angle 5\pi/12 = \cos\frac{5\pi}{12} + j\sin\frac{5\pi}{12} = 0.259 + j0.966 = p_3$$

$$\text{When } k = 4\ ; \quad s_n = e^{j\frac{(8-1)\pi}{12}} = e^{j\frac{7\pi}{12}} = 1\angle 7\pi/12 = \cos\frac{7\pi}{12} + j\sin\frac{7\pi}{12} = -0.259 + j0.966 = p_4$$

$$\text{When } k = 5\ ; \quad s_n = e^{j\frac{(10-1)\pi}{12}} = e^{j\frac{9\pi}{12}} = 1\angle 9\pi/12 = \cos\frac{9\pi}{12} + j\sin\frac{9\pi}{12} = -0.707 + j0.707 = p_5$$

$$\text{When } k = 6\ ; \quad s_n = e^{j\frac{(12-1)\pi}{12}} = e^{j\frac{11\pi}{12}} = 1\angle 11\pi/12 = \cos\frac{11\pi}{12} + j\sin\frac{11\pi}{12} = -0.966 + j0.259 = p_6$$

$$\text{When } k = 7\ ; \quad s_n = e^{j\frac{(14-1)\pi}{12}} = e^{j\frac{13\pi}{12}} = 1\angle 13\pi/12 = \cos\frac{13\pi}{12} + j\sin\frac{13\pi}{12} = -0.966 - j0.259 = p_6^*$$

$$\text{When } k = 8\ ; \quad s_n = e^{j\frac{(16-1)\pi}{12}} = e^{j\frac{15\pi}{12}} = 1\angle 15\pi/12 = \cos\frac{15\pi}{12} + j\sin\frac{15\pi}{12} = -0.707 - j0.707 = p_5^*$$

$$\text{When } k = 9\ ; \quad s_n = e^{j\frac{(18-1)\pi}{12}} = e^{j\frac{17\pi}{12}} = 1\angle 17\pi/12 = \cos\frac{17\pi}{12} + j\sin\frac{17\pi}{12} = -0.259 - j0.966 = p_4^*$$

$$\text{When } k = 10\ ; \quad s_n = e^{j\frac{(20-1)\pi}{12}} = e^{j\frac{19\pi}{12}} = 1\angle 19\pi/12 = \cos\frac{19\pi}{12} + j\sin\frac{19\pi}{12} = 0.259 - j0.966 = p_3^*$$

$$\text{When } k = 11\ ; \quad s_n = e^{j\frac{(22-1)\pi}{12}} = e^{j\frac{21\pi}{12}} = 1\angle 21\pi/12 = \cos\frac{21\pi}{12} + j\sin\frac{21\pi}{12} = 0.707 - j0.707 = p_2^*$$

$$\text{When } k = 12\ ; \quad s_n = e^{j\frac{(24-1)\pi}{12}} = e^{j\frac{23\pi}{12}} = 1\angle 23\pi/12 = \cos\frac{23\pi}{12} + j\sin\frac{23\pi}{12} = 0.966 - j0.259 = p_1^*$$

The transfer function is formed using the poles lying on left half of s-plane. The poles lying on left half of s-plane are p_4, p_4^*, p_5, p_5^*, p_6 and p_6^*.

$$\therefore\ s_n = p_4 = -0.259 + j0.966 \ \Rightarrow\ s_n - p_4 = 0 \ \Rightarrow\ (s_n + 0.259 - j0.966) = 0$$

$$s_n = p_4^* = -0.259 - j0.966 \ \Rightarrow\ s_n - p_4^* = 0 \ \Rightarrow\ (s_n + 0.259 + j0.966) = 0$$

$$s_n = p_5 = -0.707 + j0.707 \ \Rightarrow\ s_n - p_5 = 0 \ \Rightarrow\ (s_n + 0.707 - j0.707) = 0$$

$$s_n = p_5^* = -0.707 - j0.707 \;\Rightarrow\; s_n - p_5^* = 0 \;\Rightarrow\; (s_n + 0.707 + j0.707) = 0$$

$$s_n = p_6 = -0.966 + j0.259 \;\Rightarrow\; s_n - p_6 = 0 \;\Rightarrow\; (s_n + 0.966 - j0.259) = 0$$

$$s_n = p_6^* = -0.966 - j0.259 \;\Rightarrow\; s_n - p_6^* = 0 \;\Rightarrow\; (s_n + 0.966 + j0.259) = 0$$

$$\therefore \text{ Normalized transfer function, } H(s_n) = \frac{1}{(s_n - p_4)\,(s_n - p_4^*)\,(s_n - p_5)\,(s_n - p_5^*)\,(s_n - p_6)\,(s_n - p_6^*)}$$

$$\therefore\; H(s_n) = \frac{1}{(s_n + 0.259 - j0.966)\,(s_n + 0.259 + j0.966)\,(s_n + 0.707 - j0.707)\,(s_n + 0.707 + j0.707)\,(s_n + 0.966 - j0.259)\,(s_n + 0.966 + j0.259)}$$

$$= \frac{1}{\left((s_n + 0.259)^2 - (j0.966)^2\right)\left((s_n + 0.707)^2 - (j0.707)^2\right)\left((s_n + 0.966)^2 - (j0.259)^2\right)}$$

$$= \frac{1}{\left((s_n + 0.259)^2 + 0.966^2\right) + \left((s_n + 0.707)^2 + 0.707^2\right)\left((s_n + 0.966)^2 + 0.259^2\right)}$$

$$\therefore\; H(s_n) = \frac{1}{(s_n^2 + 2 \times 0.259\, s_n + 0.259^2 + 0.966^2)\,(s_n^2 + 2 \times 0.707\, s_n + 0.707^2 + 0.707^2)\,(s_n^2 + 2 \times 0.966\, s_n + 0.966^2 + 0.259^2)}$$

$$= \frac{1}{(s_n^2 + 0.518\, s_n + 1)\,(s_n^2 + 1.414\, s_n + 1)\,(s_n^2 + 1.932\, s_n + 1)}$$

7.6.3 Transfer function of Analog Butterworth Lowpass Filter

For a stable and causal filter the poles should lie on the left half of s-plane. Hence the desired filter transfer function is formed by choosing the N-number of left half poles. When N is even, all the poles are complex and exist as conjugate pair. When N is odd, one of the poles is real and all other poles are complex and exist as conjugate pair. Therefore the transfer function of Butterworth filters will be a product of second order factors. (Refer example 7.8 to example 7.13). The analog filter transfer function of normalized and unnormalized butterworth lowpass filters are given below.

Normalized Butterworth Lowpass Filter Transfer Function

Let, N be the order of the filter.

Let, $H(s_n)$ be the normalized Butterworth lowpass filter transfer function.

When N is even,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k\, s_n + 1} \qquad .....(7.42)$$

When N is odd,

$$H(s_n) = \frac{1}{s_n + 1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k\, s_n + 1} \qquad .....(7.43)$$

$$\text{where, } b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right] \qquad .....(7.44)$$

Table 7.2 : Summary of Butterworth Lowpass Filter Normalized Transfer Function

Order, N	Normalized tansfer function, $H(s_n)$
1	$\dfrac{1}{s_n + 1}$
2	$\dfrac{1}{s_n^2 + 1.414\,s_n + 1}$
3	$\dfrac{1}{(s_n + 1)\,(s_n^2 + s_n + 1)}$
4	$\dfrac{1}{(s_n^2 + 0.765\,s_n + 1)\,(s_n^2 + 1.848\,s_n + 1)}$
5	$\dfrac{1}{(s_n + 1)\,(s_n^2 + 0.618\,s_n + 1)\,(s_n^2 + 1.618\,s_n + 1)}$
6	$\dfrac{1}{(s_n^2 + 1.932\,s_n + 1)\,(s_n^2 + 1.414\,s_n + 1)\,(s_n^2 + 0.518\,s_n + 1)}$

Unnormalized Butterworth Lowpass Filter Transfer Function

The unnormalized transfer function is obtained by replacing s_n by s/W_c, in the normalized transfer function, where W_c is the 3-dB cutoff frequency of the lowpass filter.

Let, N be the order of the filter.

Let, H(s) be the unnormalized Butterworth lowpass filter transfer function.

When N is even, H(s) is obtained by letting s_n ® s/W_c in equation (7.42).

$$\therefore\ H(s) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k\,s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \prod_{k=1}^{\frac{N}{2}} \frac{\Omega_c^2}{s^2 + b_k\,\Omega_c s + \Omega_c^2} \qquad .....(7.45)$$

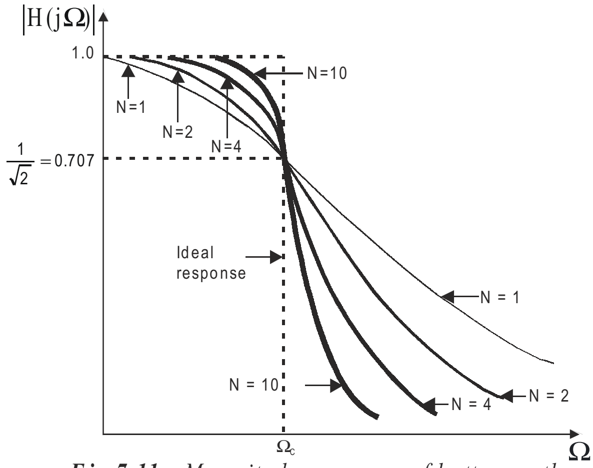
When N is odd, H(s) is obtained by letting s_n ® s/W_c in equation (7.43).

$$\therefore\ H(s) = \frac{1}{s_n + 1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k\,s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k\,\Omega_c + \Omega_c^2} \qquad .....(7.46)$$

7.6.4 Frequency Response of Analog Lowpass Butterworth Filter

The frequency response of Butterworth filter depends on the order N. The magnitude response (frequency response) for different values of N are shown in fig 7.11. From fig 7.11 it can be observed that the approximated magnitude response approaches the ideal response as the value of N increases.



3 *Fig 7.11 : Magnitude response of butterworth low pass filter for various values of N.*

7.6.5 Order of the Lowpass Butterworth Filter

In Butterworth filters the frequency response of the filter depends on the order, N. Hence the order N has to be estimated to satisfy the given specifications.

Usually the specifications of the filter are given in terms of gain at a passband and stopband frequency.

Let, A_p = Gain or Magnitude at a passband frequency Ω_p.

A_s = Gain or Magnitude at a stopband frequency Ω_s.

Calculate a parameter N_1 using equation (7.47) and correct it to nearest integer. Choose N such that $N \geq N_1$.

$$N_1 = \frac{1}{2} \frac{\log\left[\dfrac{\left(1/A_s^2\right) - 1}{\left(1/A_p^2\right) - 1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} \qquad \text{..... (7.47)}$$

Sometimes, the specifications of the filter are given in terms of dB-attenuation at a passband and stopband frequency.

Let, $\alpha_{p,\,dB}$ = dB-attenuation at a passband frequency Ω_p.

$\alpha_{s,\,dB}$ = dB-attenuation at a stopband frequency Ω_s.

Calculate a parameter N_1 using equation (7.48) and correct it to nearest integer. Choose N such that $N \geq N_1$.

$$N_1 = \frac{\log\left[\left(\dfrac{10^{0.1\alpha_{s,dB}} - 1}{10^{0.1\alpha_{p,dB}} - 1}\right)^{\frac{1}{2}}\right]}{\log\dfrac{\Omega_s}{\Omega_p}} \qquad \text{..... (7.48)}$$

7.6.6 Cutoff Frequency of Lowpass Butterworth Filter

The IIR filters are designed to satisfy a prescribed gain or attenuation at a passband and stopband frequency. But practically the 3-dB cutoff frequency, Ω_c is used to decide the useful frequency range of the filter. Therefore, in Butterworth filter design the passband and stopband specifications are used to estimate the order, N of the filter and N^{th} order normalized Butterworth lowpass filter is designed. Then the normalized lowpass filter is unnormalized using the cutoff frequency.

The cutoff frequency of lowpass Butterworth filter can be calculated using the following equations.

Case i : When the specifications are A_p, A_s, ω_p, ω_s

$$\text{Cutoff frequency, } \Omega_c = \frac{\Omega_s}{\left[\left(1/A_s^2\right)-1\right]^{\frac{1}{2N}}} \qquad \text{..... (7.49)}$$

Alternatively,

$$\text{Cutoff frequency, } \Omega_c = \frac{\Omega_p}{\left[\left(1/A_p^2\right)-1\right]^{\frac{1}{2N}}} \qquad \text{..... (7.50)}$$

The equation (7.49) is preferable to equation (7.50), because the cutoff frequency Ω_c calculated using equation (7.49) ensures smallest amplitude distortion (or ripple) in the passband.

For bilinear transformation,

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} \quad ; \quad \Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$$

For impulse invariant transformation,

$$\Omega_p = \frac{\omega_p}{T} \quad ; \quad \Omega_s = \frac{\omega_s}{T}$$

where T is the sampling time.

Case ii : When the specifications are $\alpha_{p,dB}$, $\alpha_{s,dB}$, ω_p, ω_s

$$\text{Cutoff frequency, } \Omega_c = \frac{\Omega_s}{\left(10^{0.1\alpha_{s,dB}}-1\right)^{\frac{1}{2N}}} \qquad \text{.....(7.51)}$$

Alternatively,

$$\text{Cutoff frequency, } \Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_{p,dB}}-1\right)^{\frac{1}{2N}}} \qquad \text{.....(7.52)}$$

The calculation of Ω_p and Ω_s are same as that of case (i).

The equation(7.51) is preferable to equation (7.52), because, the cutoff frequency, Ω_c calculated by using equation (7.51) ensures smallest magnitude distortion (or ripple) in the passband.

7.6.7 Design Procedure for Lowpass Digital Butterworth IIR Filter

Let, ω_p = Passband edge digital frequency in rad/sample.

ω_s = Stopband edge digital frequency in rad/sample.

$T = \frac{1}{F_s} =$ Sampling time in sec.

where, F_s = sampling frequency in Hz.

A_p = Gain at a passband frequency ω_p.

A_s = Gain at a stopband frequency ω_s.

Note-1: If passband dB-attenuation, $\alpha_{p,dB}$ and stopband dB-attenuation, $\alpha_{s,dB}$ are specified, then convert them to A_p and A_s as shown below.

$$A_p = 10^{(-\alpha_{p,dB}/20)}$$

$$A_s = 10^{(-\alpha_{s,dB}/20)}$$

$\alpha_{p,dB}$ and $\alpha_{s,dB}$ are positive dB

Bd dnabssap teshni sdenitfieicaeps s t t a-Bd dnabssap fo dae , noitaune $\alpha_{p,dB}$. Remember that $\alpha_{p,dB}$ equal to $\delta_{p,dB}$ (refer section 7.5).

Note-3: If T is not specified then take T = 1 second.

1. Choose either bilinear or impulse invariant transformation, and determine the specifications of equivalent analog filter. The gain or attenuation of analog filter is same as digital filter. The band edge frequencies are calculated using the following equations.

Let, Ω_p = Passband edge analog frequency corresponding to ω_p.

Ω_s = Stopband edge analog frequency corresponding to ω_s.

For bilinear transformation,

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} \qquad \text{.....(7.53)}$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} \qquad \text{.....(7.54)}$$

Note : If either T or F_s is not specified then take T = 1 second.

If F_s is specified, then $T = \frac{1}{F_s}$

For impulse invariant transformation,

$$\Omega_p = \frac{\omega_p}{T} \qquad \text{.....(7.55)}$$

$$\Omega_s = \frac{\omega_s}{T} \qquad \text{.....(7.56)}$$

2. Decide the order N of the filter. In order to estimate the order N, calculate a parameter N_1 using the following equation.

$$N_1 = \frac{1}{2}\,\frac{\log\left[\dfrac{(1/A_s^2)-1}{(1/A_p^2)-1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} \qquad \text{.....(7.57)}$$

Choose N such that, $N \geq N_1$. Usually N is chosen as nearest integer just greater than N_1.

3. Determine the normalized transfer function, $H(s_n)$ of the analog lowpass filter.

When N is even,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1} \qquad \text{.....(7.58)}$$

When N is odd,

$$H(s_n) = \frac{1}{s_n + 1}\prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k s_n + 1} \qquad \text{.....(7.59)}$$

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right] \qquad \text{.....(7.60)}$$

4. Calculate the analog cutoff frequency, W_c.

$$\text{Cutoff frequency, } \Omega_c = \frac{\Omega_s}{\left[\left(1/A_s^2\right) - 1\right]^{\frac{1}{2N}}} \qquad \text{.....(7.61)}$$

5. Determine the unnormalized analog transfer function H(s) of the lowpass filter.

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

When the order N is even, H(s) is obtained by letting s_n ® s/W_c in equation (7.58).

$$\therefore\ H(s) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}} = \prod_{k=1}^{\frac{N}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \qquad \text{.....(7.62)}$$

When the order N is odd, H(s) is obtained by letting s_n ® s/W_c in equation (7.59).

$$\therefore\ H(s) = \frac{1}{s_n + 1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}} = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2} \qquad \text{.....(7.63)}$$

6. Determine the transfer function of digital filter, H(z). Using the chosen transformation in step-1, transform H(s) to H(z). When impulse invariant transformation is employed, if T < 1, then multiply H(z) by T to normalize the magnitude.

7. Realize the digital filter transfer function H(z) by a suitable structure.

8. Verify the design by sketching the frequency response $H(e^{jw})$.

$$H(e^{j\omega}) = H(z)\Big|_{z = e^{j\omega}}$$

Note : The basic filter design is lowpass filter design. The highpass, bandpass or bandstop filters are obtained from lowpass filter design by frequency transformation.

7.7 Design of Lowpass Digital Chebyshev Filter

For designing a Chebyshev IIR digital filter, first an analog filter is designed using the given specifications. Then the analog filter transfer function is transformed to digital filter transfer function by using either impulse invariant transformation or bilinear transformation.

Analog Chebyshev Filter

The analog Chebyshev filter is designed by approximating the ideal frequency response using an error function. The approximation function is selected such that the error is minimized over a prescribed band of frequencies. There are two types of Chebyshev approximation. In type-1 approximation, the error function is selected such that, the magnitude response is equiripple in the passband and monotonic in the stopband. In type-2 approximation the error function is selected such that, the magnitude response is monotonic in passband and equiripple in stopband. The type-2 magnitude response is also called inverse Chebyshev response. The type-1 design is presented in this book.

The magnitude response of Type-1 lowpass filter is given by,

$$|H(j\Omega)|^2 = \frac{1}{1 + \in^2 C_N^2\left(\dfrac{\Omega}{\Omega_c}\right)} \qquad \text{..... (7.64)}$$

where Î is attenuation constant and $C_N(W/W_c)$ is the Chebyshev polynomial of the first kind of degree N.

$$\text{The attenuation constant, } \in = \left[\frac{1}{A_p^2} - 1\right]^{\frac{1}{2}} \qquad \text{..... (7.65)}$$

where, A_p is the gain or magnitude at passband edge frequency W_p

For small values of N the Chebyshev polynomial is given by,

$$C_N(x) = \begin{cases} \cos(N \cos^{-1} x) & ; \text{ for } |x| \le 1 \\ \cosh(N \cosh^{-1} x) & ; \text{ for } |x| > 1 \end{cases} \qquad \text{..... (7.66)}$$

For large values of N the Chebyshev polynomial is given by the recurrence relation,

$$C_N(x) = 2xC_{N-1}(x) - C_{N-2}(x) \qquad \text{..... (7.67)}$$

with initial values $C_0(x) = 1$ and $C_1(x) = x$

The transfer function of the analog system can be obtained from equation (7.64) by substituting W by s/j.

$$\therefore \; H(s)\,H(-s) = \frac{1}{1 + \in^2 C_N^2\left(\dfrac{s/j}{\Omega_c}\right)} \qquad \text{..... (7.68)}$$

For the normalized transfer function, let us replace s/W_c by s_n.

$$\therefore \; H(s_n)\,H(-s_n) = \frac{1}{1 + \in^2 C_N^2(-js_n)} \qquad \text{..... (7.69)}$$

For the transfer function of equation (7.69) we can determine 2N poles which are given by the roots of the denominator polynomial. It can be shown that the poles of the transfer function symmetrically lies on an ellipse in s-plane.

Properties of Chebyshev Filters (Type-1)

1. The magnitude $|H(jW)|$ oscillates between 1 and $1/\sqrt{1+\in^2}$ within the passband and so the filter is called equiripple in the passband.
2. The normalized magnitude response has a value of $1/\sqrt{1+\in^2}$ at cutoff frequency W_c.
3. The magnitude is monotonic outside the passband.
4. The Chebyshev Type-1 filters are all pole designs.
5. With large values of N, the transition from passband to stopband becomes more sharp and approaches ideal characteristics.

7.7.1 Transfer Function of Analog Chebyshev Lowpass Filter

For a stable and causal filter the poles should lie on the left half of s-plane. Hence the desired filter transfer function is obtained by selecting N number of left half poles. When N is even all the poles are complex and exist as conjugate pair. When N is odd, one of the pole is real and all other poles are complex and exist as conjugate pair. Therefore the transfer function of Chebyshev filters will be a product of second-order factors.

Normalized Chebyshev Lowpass Filter Transfer Function

Let, N be the order of the filter.

Let, $H(s_n)$ be the normalized Chebyshev lowpass filter transfer function.

When N is even,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{B_k}{s_n^2 + b_k\, s_n + c_k} \qquad ..... (7.70)$$

When N is odd,

$$H(s_n) = \frac{B_0}{s + c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s_n^2 + b_k\, s_n + c_k} \qquad ......(7.71)$$

$$\text{where, } b_k = 2\, y_N\, \sin\left(\frac{(2k-1)\pi}{2N}\right) \qquad ..... (7.72)$$

$$c_k = y_N^2 + \cos^2\left(\frac{(2k-1)\pi}{2N}\right) \qquad ..... (7.73)$$

$$c_0 = y_N \qquad ..... (7.74)$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right\} \qquad ..... (7.75)$$

For even values of N the parameter B_k are evaluated using the equation (7.76)

$$H(s_n)\Big|_{s_n = 0} = \frac{1}{(1 + \epsilon^2)^{\frac{1}{2}}} \qquad ..... (7.76)$$

For odd values of N the parameter B_k are evaluated using the equation (7.77)

$$H(s_n)\Big|_{s_n = 0} = 1 \qquad ..... (7.77)$$

While evaluating B_k using equation (7.76) or (7.77), it is normal practice to take, $B_0 = B_1 = B_2 = ... = B_k$.

Unnormalized Chebyshev Lowpass Filter Transfer Function

The unnormalized transfer function is obtained by replacing s_n by s/Ω_c in the normalized transfer function, where Ω_c is the cutoff frequency of the lowpass filter.

Let, N be the order of the filter.

Let, H(s) be the normalized Chebyshev lowpass filter transfer function.

When N is even, H(s) is obtained by letting $s_n \rightarrow s/\Omega_c$ in equation (7.70).

$$\therefore\ H(s) = \prod_{k=1}^{\frac{N}{2}} \frac{B_k}{s_n^2 + b_k\, s_n + c_k}\Bigg|_{s_n = \frac{s}{\Omega_c}} = \prod_{k=1}^{\frac{N}{2}} \frac{B_k\, \Omega_c^2}{s^2 + b_k\, \Omega_c\, s + c_k\, \Omega_c^2} \qquad .....(7.78)$$

When N is odd, H(s) is obtained by letting $s_n \rightarrow s/\Omega_c$ in equation (7.71).

$$\therefore\ H(s) = \frac{B_0}{s_n + c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s_n^2 + b_k\, s_n + c_k}\Bigg|_{s_n = \frac{s}{\Omega_c}} = \frac{B_0\, \Omega_c}{s + c_0\, \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k\, \Omega_c^2}{s^2 + b_k\, \Omega_c\, s + c_k\, \Omega_c^2} \qquad .....(7.79)$$

7.7.2 Order of Analog Lowpass Chebyshev Filter

In Chebyshev filters the frequency response of the filter depends on the order, N.

Hence the order has to be estimated to satisfy the given specifications.

Usually the specifications of the filter are given in terms of gain at a passband and stopband frequency.

Let, A_p = Gain or Magnitude at a passband frequency, Ω_p.

A_s = Gain or Magnitude at a stopband frequency, Ω_s.

Calculate a parameter N_1, using equation (7.80) and correct it to nearest integer. Then choose N such that $N \geq N_1$.

$$N_1 = \frac{\cosh^{-1}\left[\left(\frac{\left(1/A_s^2\right) - 1}{\left(1/A_p^2\right) - 1}\right)^{\frac{1}{2}}\right]}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} \qquad .....(7.80)$$

Sometimes, the specifications of the filter are given in terms of dB-attenuation at a passband and stopband frequency.

Let, $\alpha_{p,dB}$ = dB-attenuation at a passband frequency, Ω_p.

Let, $\alpha_{s,dB}$ = dB-attenuation at a stopband frequency, Ω_s.

Calculate a parameter N_1 using equation (7.81) and correct it to nearest integer. Choose N such that $N \geq N_1$.

$$N_1 = \frac{\cosh^{-1}\left[\left(\frac{10^{0.1\alpha_{s,dB}} - 1}{10^{0.1\alpha_{p,dB}} - 1}\right)^{\frac{1}{2}}\right]}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} \qquad .....(7.81)$$

7.7.3 Cutoff Frequency of Analog Lowpass Chebyshev Filter

The IIR filters are designed to satisfy a prescribed gain or attenuation at a passband and stopband frequency. But practically the cutoff frequency, Ω_c is used to decide the useful frequency range of the filter. Therefore, in Chebyshev filter design the passband and stopband specifications are used to estimate the order, N of the filter and N^{th} order normalized Chebyshev lowpass filter is designed. Then the normalized lowpass filter is unnormalized using the cutoff frequency.

In Chebyshev filters the passband edge frequency, Ω_p is considered as cutoff frequency, Ω_c and this cutoff is not equal to 3 dB cutoff frequency, Ω_{3dB}.

The 3 dB cutoff frequency of Chebyshev filter is given by, $\Omega_{3dB} = \Omega_c \cosh\left(\frac{1}{N} \cosh^{-1} \frac{1}{\epsilon}\right)$ ..... (7.82)

7.7.4 Frequency Response of Analog Chebyshev Lowpass Filter

The frequency response of Chebyshev filter depends on the order N as shown in fig 7.12. It can be observed that the approximated magnitude response approaches the ideal response as the value of N increases. The magnitude response of Type-1 and Type-2 Chebyshev filters are shown in fig 7.13.

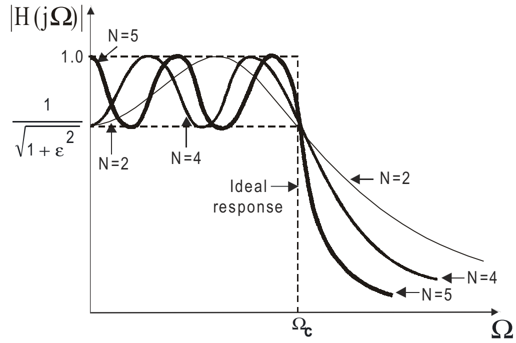


Fig 7.12 : Magnitude response of Chebyshev type-1 lowpass filter for various value of N.

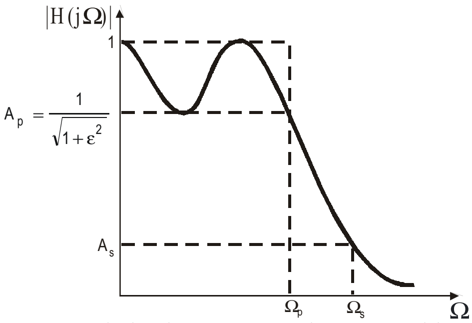


Fig a : Chebyshev type-1, when N is odd.

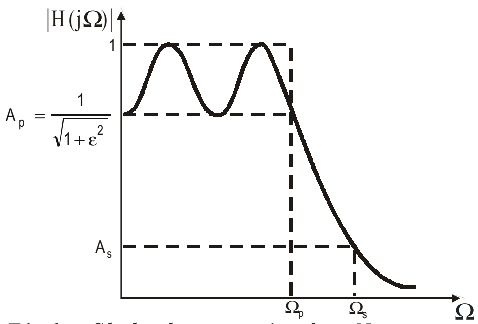


Fig b : Chebyshev type-1, when N is even.

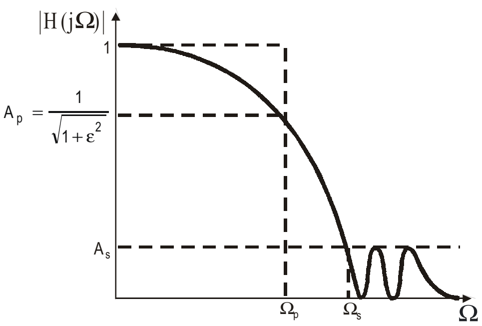


Fig c : Chebyshev type-2, when N is odd.

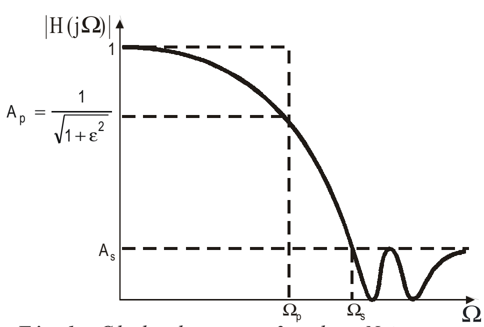


Fig d : Chebyshev type-2, when N is even.

Fig 7.13 : Magnitude response of analog Chebyshev filters.

7.7.5 Design Procedure for Lowpass Digital Chebyshev IIR Filter

Let, ω_p = Passband edge digital frequency in rad/sample.

ω_s = Stopband edge digital frequency in rad/sample.

$T = \dfrac{1}{F_s} =$ Sampling time in seconds.

where, F_s = sampling frequency in Hz.

A_p = Gain at a passband frequency ω_p.

A_s = Gain at a stopband frequency ω_s.

Note-1: If passband dB-attenuation, $\alpha_{p,dB}$ and stopband dB-attenuation, $\alpha_{s,dB}$ are specified, then convert them to A_p and A_s as shown below.

$A_p = 10^{(-\alpha_{p,dB}/20)}$

$A_s = 10^{(-\alpha_{s,dB}/20)}$

$\alpha_{p,dB}$ and $\alpha_{s,dB}$ are positive dB

Bd dnabssap teshnti sdemï fi ienaSp2s s t t a-Bd dnabssap fo dae , noi taune

$\alpha_{p,dB}$. Remember that $\alpha_{p,dB}$ equal to $\delta_{p,dB}$. (refer section 7.5).

3: If T is not specified then take T = 1 second.

1. Choose either bilinear or impulse invariant transformation, and determine the specifications of equivalent analog filter. The gain or attenuation of analog filter is same as digital filter. The band edge frequencies are calculated using the following equations.

Let, Ω_p = Passband edge analog frequency corresponding to ω_p.

Ω_s = Stopband edge analog frequency corresponding to ω_s.

For bilinear transformation,

$$\Omega_p = \frac{2}{T}\,\tan\frac{\omega_p}{2} \qquad .....(7.83)$$

$$\Omega_s = \frac{2}{T}\,\tan\frac{\omega_s}{2} \qquad .....(7.84)$$

Note : If either T or F_s is not specified then take T= 1 sec.

If F_s is specified, then $T = \dfrac{1}{F_s}$

For impulse invariant transformation,

$$\Omega_p = \frac{\omega_p}{T} \qquad .....(7.85)$$

$$\Omega_s = \frac{\omega_s}{T} \qquad .....(7.86)$$

2. Decide the order N of the filter. In order to estimate the order N, calculate a parameter N_1 using the following equation. Choose N such that $N \ge N_1$. Usually N is chosen as nearest integer just greater than N_1.

$$N_1 = \frac{\cosh^{-1}\left[\left(\dfrac{(1/A_s^2) - 1}{(1/A_p^2) - 1}\right)^{\frac{1}{2}}\right]}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)} \qquad .....(7.87)$$

3. Determine the normalized transfer function $H(s_n)$, of the filter.

When the order N is even,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{B_k}{s_n^2 + b_k s_n + c_k} \qquad .....(7.88)$$

When the order N is odd,

$$H(s) = \frac{B_0}{s + c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s^2 + b_k s + c_k} \qquad .....(7.89)$$

where, $$b_k = 2\, y_N \sin\left(\frac{(2k-1)\pi}{2N}\right) \qquad .....(7.90)$$

$$c_k = y_N^2 + \cos^2\left(\frac{(2k-1)\pi}{2N}\right) \qquad .....(7.91)$$

$$c_0 = y_N \qquad .....(7.92)$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right\} \qquad .....(7.93)$$

$$\epsilon = \left[\left(1/A_p^2\right) - 1\right]^{\frac{1}{2}} \qquad .....(7.94)$$

For even values of N, find B_k such that,

$$H(0) = \frac{1}{(1 + \epsilon^2)^{\frac{1}{2}}} \qquad .....(7.95)$$

For odd values of N, find B_k such that,

$$H(0) = 1 \qquad .....(7.96)$$

(It is normal practice to take $B_0 = B_1 = B_2 ..... = B_k$).

4. Determine the unnormalized analog transfer function H(s) of the lowpass filter.

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

Here, $W_c = W_p$ = Passband edge frequency.

When the order N is even, H(s) is obtained by letting s_n ® s/W_c in equation (7.88).

$$\therefore\ H(s) = \prod_{k=1}^{\frac{N}{2}} \frac{B_k}{s_n^2 + b_k s_n + c_k}\Bigg|_{s_n = \frac{s}{\Omega_c}} = \prod_{k=1}^{\frac{N}{2}} \frac{B_k\, \Omega_c^2}{s^2 + b_k\, \Omega_c\, s + c_k\, \Omega_c^2} \qquad .....(7.97)$$

When the order N is odd, H(s) is obtained by letting $s_n \rightarrow s/\Omega_c$ in equation (7.89).

$$\therefore\ H(s) = \frac{B_0}{s\ +\ c_0} \prod_{k\,=\,1}^{\frac{N-1}{2}} \frac{B_k}{s_n^2 + b_k\ s_n + c_k}\Bigg|_{s_n = \frac{s}{\Omega_c}} = \frac{B_0\ \Omega_c}{s + c_0\,\Omega_c}\ \frac{B_k\ \Omega_c^2}{s^2 + b_k\ \Omega_c s +\ c_k\,\Omega_c^2} \qquad .....(7.98)$$

5. Determine the transfer function of digital filter, H(z). Using the chosen transformation, in step-1transform H(s) to H(z). When impulse invariant transformation is employed, if T < 1, then multiply H(z) by T to normalize the magnitude.

6. Realize the digital filter transfer function H(z) by a suitable structure.

7. Verify the design by sketching the frequency response $H(e^{j\omega})$.

$$H(e^{j\omega}) = H(z)\big|_{z = e^{j\omega}}$$

> *Note : The highpass, bandpass and bandstop filters are obtained from lowpass filter design by frequency transformation.*

7.8 Frequency Transformation

The four basic types of filters are lowpass, highpass, bandpass and bandstop filters.

The highpass or bandpass or bandstop filters are designed by designing a lowpass filter and then using frequency transformation, the transfer function of the desired filter is obtained. The frequency transformation can be carried in s-domain (analog) or in z-domain (digital).

7.8.1 Analog Frequency Transformation

Using analog frequency transformation the following filters can be designed from the normalized lowpass filter. For normalized lowpass the cutoff frequency, $\Omega_c = 1$ rad/second.

1. Lowpass filter with cutoff frequency, Ω_c.

2. Highpass filter with cutoff frequency, Ω_c.

3. Bandpass filter with center frequency, Ω_0 and quality factor, Q.

4. Bandstop filter with center frequency, Ω_0 and quality factor, Q.

$$\text{where, } \Omega_0 = \sqrt{\Omega_p\,\Omega_s} \quad \text{and} \quad Q = \frac{\Omega_0}{\Omega_s - \Omega_p}$$

To design a filter, first design a normalized lowpass filter from the given specifications, and determine the analog normalized transfer function (either Butterworth or Chebyshev transfer function) of the lowpass filter. Then choose the transformation from the table 7.2 and determine the analog transfer function of the desired filter.

Table 7.2 : Summary of Transformation for Analog Filter

Filter Type	Transformation
Lowpass	$s_n \rightarrow \dfrac{s}{\Omega_c}$
Highpass	$s_n \rightarrow \dfrac{\Omega_c}{s}$
Bandpass	$s_n \rightarrow \dfrac{Q\,(s^2 + \Omega_0^2)}{\Omega_0 s}$
Bandstop	$s_n \rightarrow \dfrac{\Omega_0 s}{Q\,(s^2 + \Omega_0^2)}$

From the analog transfer function H(s) the digital transfer function H(z) is obtained by either bilinear transformation or impulse invariant transformation.

7.8.2 Digital Frequency Transformation

Table 7.3 : Summary of Transformation for Digital Filter

Filter Type	Transformation	Design Parameters
Lowpass	$z^{-1} \rightarrow \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\dfrac{\omega_c' + \omega_c}{2}\right)}{\sin\left(\dfrac{\omega_c' - \omega_c}{2}\right)}$
Highpass	$z^{-1} \rightarrow \dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = \dfrac{\cos\left(\dfrac{\omega_c' + \omega_c}{2}\right)}{\cos\left(\dfrac{\omega_c' - \omega_c}{2}\right)}$
Bandpass	$z^{-1} \rightarrow -\dfrac{z^{-2} - \dfrac{2\alpha k}{k+1} z^{-1} + \dfrac{k-1}{k+1}}{\dfrac{k-1}{k+1} z^{-2} - \dfrac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\dfrac{\omega_s + \omega_p}{2}\right)}{\cos\left(\dfrac{\omega_s - \omega_p}{2}\right)} = \cos\omega_0$ $k = \cos\left(\dfrac{\omega_s - \omega_p}{2}\right) \tan\dfrac{\omega_c}{2}$
Bandstop	$z^{-1} \rightarrow \dfrac{z^{-2} - \dfrac{2\alpha k}{k+1} z^{-1} + \dfrac{k-1}{k+1}}{\dfrac{k-1}{k+1} z^{-2} - \dfrac{2\alpha k}{k+1} z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\dfrac{\omega_s + \omega_p}{2}\right)}{\cos\left(\dfrac{\omega_s - \omega_p}{2}\right)} = \cos\omega_0$ $k = \cos\left(\dfrac{\omega_s - \omega_p}{2}\right) \tan\dfrac{\omega_c'}{2}$

Using digital frequency transformation the following filters can be designed from the lowpass digital filter with cutoff frequency, w_c'.

1. Lowpass filter with cutoff frequency, w_c.
2. Highpass filter with cutoff frequency w_c.
3. Bandpass filter with center frequency w_0 and lower and upper cutoff frequency w_1 and w_2.
4. Bandstop filter with center frequency w_0 and lower and upper cutoff frequency w_1 and w_2.

To design a filter, first design a lowpass digital filter from the given specifications, (either Butterworth or Chebyshev) and determine H(z). Then choose the transformation from table 7.3 and determine the digital transfer function of the desired filter.

Example 7.14

The normalized transfer function of an analog filter is given by,

$$H(s_n) = \frac{1}{s_n^2 + 1.4142 s_n + 1}$$

Convert the analog filter to a digital filter with a cutoff frequency of 0.4p, using bilinear transformation.

Solution

To preserve the magnitude response the prewarping of analog filter has to be performed. For this the analog cutoff frequency is determined using bilinear transformation and the analog transfer function is unnormalized using this analog cutoff frequency. Then the analog transfer function is converted to digital filter transfer function using bilinear transformation.

Given that, digital cutoff frequency, w_c = 0.4p rad/sample. Let T = 1 second.

In Bilinear transformation,

Analog cutoff frequency, $\Omega_c = \frac{2}{T}\tan\frac{0.4\pi}{2} = 2\tan\frac{0.4\pi}{2} = 1.4531 \text{ rad / second}$

Normalized analog transfer function,

$$H(s_n) = \frac{1}{s_n^2 + 1.4142 s_n + 1}$$

The analog transfer function is unnormalized by replacing s_n by s/W_c

$$\therefore \left.\begin{array}{r}\text{Unnormalized}\\ \text{analog filter}\\ \text{transfer function}\end{array}\right\} H(s) = \frac{1}{\left(\frac{s}{\Omega_c}\right)^2 + 1.4142\left(\frac{s}{\Omega_c}\right) + 1}$$

$$= \frac{\Omega_c^2}{s^2 + 1.4142\,\Omega_c s + \Omega_c^2}$$

$$= \frac{1.4531^2}{s^2 + (1.4142 \times 1.4531)s + 1.4531^2}$$

$$= \frac{2.1115}{s^2 + 2.055s + 2.1115}$$

The H(z) is obtained by substituting, $s = \frac{2}{T}\,\frac{1 - z^{-1}}{1 + z^{-1}}$ in H(s)

$$\left.\begin{array}{l}\therefore\ \text{Digital filter}\\ \text{transfer function}\end{array}\right\} H(z)=\frac{2.1115}{\left[\dfrac{2(1-z^{-1})}{1+z^{-1})}\right]^2+2.055\left[\dfrac{2(1-z^{-1})}{1+z^{-1}}\right]+2.1115}$$

$$\therefore\ H(z)=\frac{2.1115}{\dfrac{4(1-z^{-1})^2+4.11(1-z^{-1})\,(1+z^{-1})+2.1115(1+z^{-1})^2}{(1+z^{-1})^2}}$$

$$=\frac{2.1115(1+z^{-1})^2}{4(1-2z^{-1}+z^{-2})+4.11(\ 1-z^{-2})+2.1115(1+2z^{-1}+z^{-2})}$$

$$=\frac{2.1115(1+2z^{-1}+z^{-2})}{4(1-2z^{-1}+z^{-2})+4.11(\ 1-z^{-2})+2.1115(1+2z^{-1}+z^{-2})}$$

$$=\frac{2.1115+4.223\,z^{-1}+2.1115\,z^{-2}}{10.2215-3.777z^{-1}+2.0015z^{-2}}$$

$$=\frac{\dfrac{2.1115}{10.2215}+\dfrac{4.223}{10.2215}z^{-1}+\dfrac{2.1115}{10.2215}z^{-2}}{1-\dfrac{3.777}{10.2215}z^{-1}+\dfrac{2.0015}{10.2215}z^{-2}}$$

$$=\frac{0.2066\ +\ 0.4131z^{-1}\ +\ 0.2066z^{-2}}{1-0.3695z^{-1}\ +\ 0.1958\,z^{-2}}$$

Alternatively,

$$H(z)=\frac{0.2066+0.4131z^{-1}\ +\ 0.2066z^{-2}}{1-0.3695z^{-1}\ +\ 0.1958\,z^{-2}}$$

$$=\frac{z^{-2}(0.2066\,z^2\ +\ 0.4131z\ +\ 0.2066)}{z^{-2}(z^2-0.3695z\ +\ 0.1958)}=\frac{0.2066\,z^2\ +\ 0.4131z\ +\ 0.2066}{z^2-0.3695z\ +\ 0.1958}$$

Example 7.15

Design a Butterworth digital IIR lowpass filter using bilinear transformation by taking T = 0.1second, to satisfy the following specifications.

$$0.6 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.35\pi$$

$$|H(e^{j\omega})| \le 0.1 \quad ; \quad \text{for } 0.7\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternatively,

Passband ripple $\le$ 4.436 dB

Stopband attenuation $\ge$ 20 dB

Passband edge frequency = 0.35π rad/sample

Stopband edge frequency = 0.7π rad/sample

The above specifications can be converted to A_p and A_s as shown below.

$$A_p=10^{\left(-\delta_{p,dB}/20\right)}=10^{(-4.436/20)}=0.6$$

$$A_s=10^{\left(-\alpha_{s,dB}/20\right)}=10^{(-20/20)}=0.1$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.35\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.7\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.6$

Gain in normal value at stopband edge, $A_s = 0.1$

Sampling time, T = 0.1second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.6$

Gain in normal value at stopband edge, $A_s = 0.1$

Gain is same in analog and digital filter.

For bilinear transformation,

Passband edge analog frequency, $\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$

Using equation (7.53).

$$= \frac{2}{0.1}\tan\frac{0.35\pi}{2} = 12.256 \text{ rad / second}$$

Stopband edge analog frequency, $\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$

Using equation (7.54).

$$= \frac{2}{0.1}\tan\frac{0.7\pi}{2}$$

$$= 39.2522 \text{ rad / second}$$

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\frac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]}{\log\frac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\frac{\left(1/0.1^2\right)-1}{\left(1/0.6^2\right)-1}\right]}{\log\frac{39.2522}{12.256}}$$

Using equation (7.57).

$$= \frac{1}{2}\frac{\log\left[\frac{99}{1.7778}\right]}{\log\frac{39.2522}{12.256}} = 1.7267$$

Choose order N, such that $N \geq N_1$ and N is an integer.

Let, order, N = 2.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For even N,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.58).

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Using equation (7.60).

Here, N = 2, $\therefore\ k = \frac{N}{2} = \frac{2}{2} = 1$

$$\therefore\ H(s_n) = \frac{1}{s_n^2 + b_1 s_n + 1}$$

When k = 1 ; $b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times 2}\right] = 1.4142$

Calculate sin θ using calculator in radian mode.

$$\therefore\ H(s_n) = \frac{1}{s_n^2 + 1.4142 s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2) - 1\right]^{\frac{1}{2N}}} = \frac{39.2522}{\left[(1/0.1^2) - 1\right]^{\frac{1}{4}}} = 12.4439 \text{ rad / sec}$$

Using equation (7.61).

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{s_n^2 + 1.4142 s_n + 1}\Big|_{s_n = \frac{s}{\Omega_c}}$$

$$\therefore\ H(s) = \frac{1}{\frac{s^2}{\Omega_c^2} + 1.4142\frac{s}{\Omega_c} + 1} = \frac{1}{\frac{s^2 + 1.4142\Omega_c s + \Omega_c^2}{\Omega_c^2}}$$

$$= \frac{\Omega_c^2}{s^2 + 1.4142\Omega_c s + \Omega_c^2} = \frac{12.4439^2}{s^2 + 1.4142 \times 12.4439 s + 12.4439^2}$$

$$= \frac{154.8506}{s^2 + 17.5982 s + 154.8506}$$

Digital IIR lowpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{154.8506}{s^2 + 17.5982 s + 154.8506}\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{154.8506}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 17.5982\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + 154.8506}$$

$$= \frac{154.8506}{\frac{4(1-z^{-1})^2}{T^2(1+z^{-1})^2} + \frac{35.1964(1-z^{-1})}{T(1+z^{-1})} + 154.8506}$$

$$= \frac{154.8506}{\frac{4(1-z^{-1})^2 + 35.1964\,T(1-z^{-1})(1+z^{-1}) + 154.8506\,T^2(1+z^{-1})^2}{T^2(1+z^{-1})^2}}$$

Put, T = 0.1

$$\therefore\ H(z) = \frac{154.8506\,T^2(1+z^{-1})^2}{4\left(1-z^{-1}\right)^2 + 35.1964\,T\left(1-z^{-1}\right)\left(1+z^{-1}\right) + 154.8506\,T^2\left(1+z^{-1}\right)^2}$$

$$= \frac{154.8506 \times 0.1^2(1+2z^{-1}+z^{-2})}{4\left(1-2z^{-1}+z^{-2}\right) + 35.1964 \times 0.1\left(1-z^{-2}\right) + 154.8506 \times 0.1^2\left(1+2z^{-1}+z^{-2}\right)}$$

$$= \frac{1.5485(1+2z^{-1}+z^{-2})}{4\left(1-2z^{-1}+z^{-2}\right) + 3.5196\left(1-z^{-2}\right) + 1.5485\left(1+2z^{-1}+z^{-2}\right)}$$

$(a+b)(a-b) = a^2 - b^2$
$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab - b^2$

$$= \frac{1.5485 + 3.097\,z^{-1} + 1.5485\,z^{-2}}{9.0681 - 4.903\,z^{-1} + 2.0289\,z^{-2}}$$

$$= \frac{\frac{1.5485}{9.0681} + \frac{3.097}{9.0681}z^{-1} + \frac{1.5485}{9.0681}z^{-2}}{1 - \frac{4.903}{9.0681}z^{-1} + \frac{2.0289}{9.0681}z^{-2}} = \frac{0.1708 + 0.3415z^{-1} + 0.1708\,z^{-2}}{1 - 0.5407\,z^{-1} + 0.2237\,z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.1708 + 0.3415\,z^{-1} + 0.1708\,z^{-2}}{1 - 0.5407\,z^{-1} + 0.2237\,z^{-2}}$$

$$= \frac{z^{-2}\left(0.1708\,z^2 + 0.3415\,z + 0.1708\right)}{z^{-2}\left(z^2 - 0.5407\,z + 0.2237\right)} = \frac{0.1708\,z^2 + 0.3415\,z + 0.1708}{z^2 - 0.5407\,z + 0.2237}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let,}\ H(z) = \frac{Y(z)}{X(z)} = \frac{0.1708 + 0.3415\,z^{-1} + 0.1708\,z^{-2}}{1 - 0.5407\,z^{-1} + 0.2237\,z^{-2}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.5407z^{-1}Y(z) + 0.2237z^{-2}Y(z) = 0.1708\,X(z) + 0.3415z^{-1}X(z) + 0.1708z^{-2}X(z)$$

$$\backslash\ \ Y(z) = 0.1708X(z) + 0.3415z^{-1}X(z) + 0.1708z^{-2}X(z) + 0.5407z^{-1}Y(z) - 0.2237z^{-2}Y(z) \qquad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

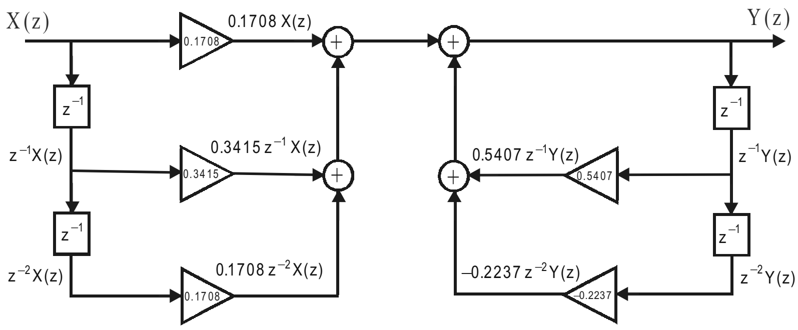


Fig 1 : *Direct form-I structure of 2nd order digital IIR lowpass filter.*

Direct form-II structure of digital IIR lowpass filter

$$\text{Let,}\ H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.1708 + 0.3415\,z^{-1} + 0.1708\,z^{-2}}{1 - 0.5407\,z^{-1} + 0.2237\,z^{-2}}$$

$$\text{where,}\quad \frac{W(z)}{X(z)} = \frac{1}{1 - 0.5407\,z^{-1} + 0.2237\,z^{-2}} \qquad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.1708 + 0.3415\,z^{-1} + 0.1708\,z^{-2} \qquad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) - 0.5407z^{-1}W(z) + 0.2237z^{-2}W(z) = X(z)$$

$$\backslash \;\; W(z) = X(z) + 0.5407z^{-1}W(z) - 0.2237z^{-2}W(z) \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$Y(z) = 0.1708W(z) + 0.3415z^{-1}W(z) + 0.1708z^{-2}W(z) \qquad .....(5)$$

Using equations (4) and (5), the direct form-II structure is drawn as shown in fig 2.

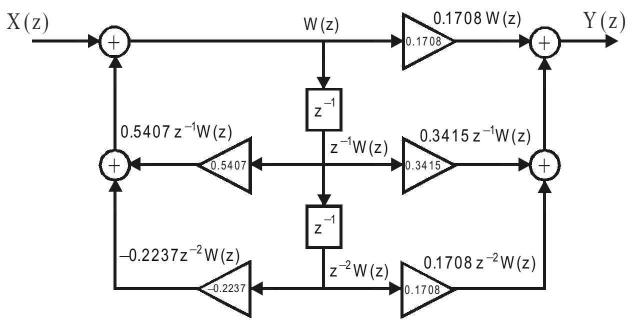


Fig 2 : Direct form-II structure of 2nd order digital IIR lowpass filter.

Frequency Response, $H(e^{jw})$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \left.\frac{0.1708 + 0.3415\,z^{-1} + 0.1708\,z^{-2}}{1 - 0.5407\,z^{-1} + 0.2237\,z^{-2}}\right|_{z=e^{j\omega}}$$

$$= \frac{0.1708 + 0.3415\,e^{-j\omega} + 0.1708\,e^{-j2\omega}}{1 - 0.5407\,e^{-j\omega} + 0.2237\,e^{-j2\omega}}$$

$$= \frac{0.1708 + 0.3415(\cos\omega - j\sin\omega) + 0.1708(\cos 2\omega - j\sin 2\omega)}{1 - 0.5407(\cos\omega - j\sin\omega) + 0.2237(\cos 2\omega - j\sin 2\omega)} \qquad \boxed{e^{-j\theta} = \cos\theta - j\sin\theta}$$

$$= \frac{(0.1708 + 0.3415\cos\omega + 0.1708\cos 2\omega) + j(-0.3415\sin\omega - 0.1708\sin 2\omega)}{(1 - 0.5407\cos\omega + 0.2237\cos 2\omega) + j(0.5407\sin\omega - 0.2237\sin 2\omega)}$$

$$\text{Let,}\;\; H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} = \frac{(0.1708 + 0.3415\cos\omega + 0.1708\cos 2\omega) + j(-0.3415\sin\omega - 0.1708\sin 2\omega)}{(1 - 0.5407\cos\omega + 0.2237\cos 2\omega) + j(0.5407\sin\omega - 0.2237\sin 2\omega)}$$

where, $H_N(e^{jw}) = (0.1708 + 0.3415\cos w + 0.1708\cos 2w) + j(-0.3415\sin w - 0.1708\sin 2w)$

$H_D(e^{jw}) = (1 - 0.5407\cos w + 0.2237\cos 2w) + j(0.5407\sin w - 0.2237\sin 2w)$

The frequency response $H(e^{jw})$ and hence the magnitude response $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the values listed in table 1, the magnitude response of lowpass filter is sketched as shown in fig 3.

Note : Verify the result with MATLAB program 7.1.

Table 1: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w.

w	$H_N(e^{jw})$	$H_D(e^{jw})$	$H(e^{jw})$	$\|H(e^{jw})\|$
$\frac{0\times\pi}{16}$	0.6831 + j0	0.683 + j0	1.0000 + j0	1.0000
$\frac{1\times\pi}{16}$	0.6635 – j0.1320	0.6764 + j0.0199	0.9743 – j0.2238	0.9997
$\frac{2\times\pi}{16}$	0.6071 – j0.2515	0.6586 + j0.0487	0.8887 – j0.4476	0.9951
$\frac{3\times\pi}{16}$	0.5201 – j0.3475	0.6360 + j0.0937	0.7216 – j0.6527	0.9730
$\frac{4\times\pi}{16}$	0.4123 – j0.4123	0.6177 + j0.1586	0.4654 – j0.7870	0.9143
$\frac{5\times\pi}{16}$	0.2952 – j0.4417	0.6140 + j0.2429	0.1696 – j0.7865	0.8046
$\frac{6\times\pi}{16}$	0.1807 – j0.4363	0.6349 + j0.3414	–0.0659 – j0.6518	0.6551
$\frac{7\times\pi}{16}$	0.0796 – j0.4003	0.6878 + j0.4447	–0.1838 – j0.4632	0.4983
$\frac{8\times\pi}{16}$	0 – j0.3415	0.7763 + j0.5407	–0.2063 – j0.2962	0.3610
$\frac{9\times\pi}{16}$	–0.0536 – j0.2696	0.8988 + j0.6159	–0.1804 – j0.1763	0.2523
$\frac{10\times\pi}{16}$	–0.0807 – j0.1947	1.0487 + j0.6577	–0.1388 – j0.0986	0.1703
$\frac{11\times\pi}{16}$	–0.0843 – j0.1261	1.2148 + j0.6562	–0.0971 – j0.0531	0.1099
$\frac{12\times\pi}{16}$	–0.0707 – j0.0707	1.3823 + j0.6060	–0.0617 – j0.0241	0.0662
$\frac{13\times\pi}{16}$	–0.0478 – j0.0319	1.5352 + j0.5071	–0.0343 – j0.0095	0.0355
$\frac{14\times\pi}{16}$	–0.0239 – j0.0099	1.6577 + j0.3651	–0.0150 – j0.0027	0.0152
$\frac{15\times\pi}{16}$	–0.0063 – j0.0013	1.7370 + j0.1911	–0.0037 – j0.0003	0.0037
$\frac{16\times\pi}{16}$	0 + j0	1.7644 + j0	0 + j0	0

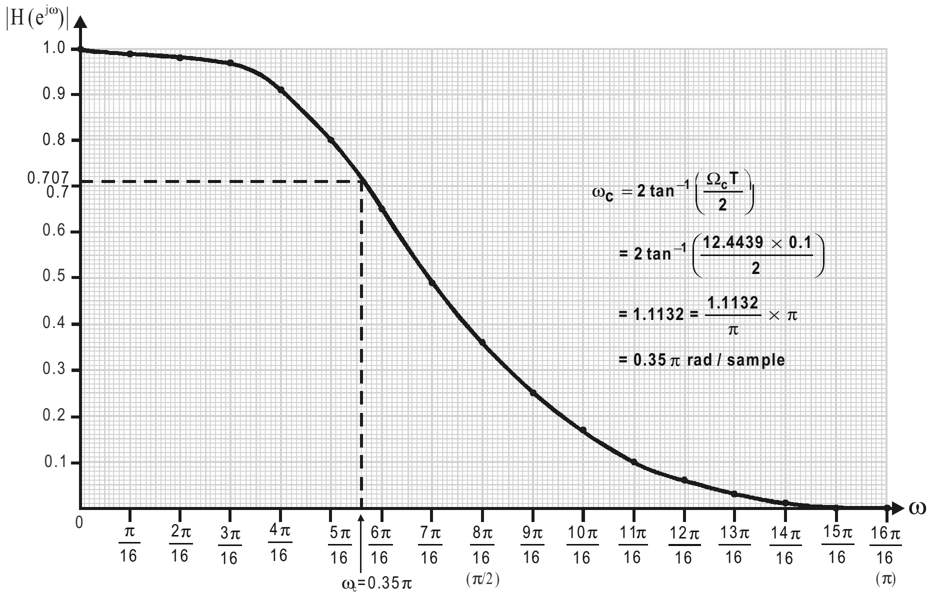


***Fig 3 :** Frequency response of 2^{nd} order digital Butterworth IIR lowpass filter.*

Example 7.16

Design a Butterworth digital IIR highpass filter using bilinear transformation by taking T = 0.1second, to satisfy the following specifications.

$$0.6 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0.7\pi \le \omega \le \pi$$

$$|H(e^{j\omega})| \le 0.1 \quad ; \quad 0 \le \omega \le 0.35\pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternatively,

Passband ripple $\le$ 4.436 dB

Stopband attenuation $\ge$ 20 dB

Passband edge frequency = 0.7π rad/sample

Stopband edge frequency = 0.35π rad/sample

The above specifications can be converted to A_p and A_s as shown below.

$$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{\left(-4.436/20\right)} = 0.6$$

$$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{\left(-20/20\right)} = 0.1$$

Solution

Specifications of digital IIR highpass filter

Passband edge digital frequency, ω_p = 0.7π rad/sample

Stopband edge digital frequency, ω_s = 0.35π rad/sample

Gain in normal value at passband edge, A_p = 0.6

Gain in normal value at stopband edge, A_s = 0.1

Sampling time, T = 0.1second

The highpass filter is designed via lowpass filter using frequency transformation technique. Hence the given specifications of IIR highpass filter are converted to corresponding specification of IIR lowpass filter.

Specifications of digital IIR lowpass filter

The specification of lowpass filter is obtained by taking passband edge of highpass as stopband edge of lowpass and stopband edge of highpass as passband edge of lowpass. The gain of passband and stopband remain same.

$\therefore$ Passband edge digital frequency, ω_p = 0.35π rad/sample

$\therefore$ Stopband edge digital frequency, ω_s = 0.7π rad/sample

Gain in normal value at passband edge, A_p = 0.6

Gain in normal value at stopband edge, A_s = 0.1

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, A_p = 0.6

Gain in normal value at stopband edge, A_s = 0.1

Gain is same in analog and digital filter.

For bilinear transformation,

Passband edge analog frequency, $\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$ | Using equation (7.53).

$$= \frac{2}{0.1}\tan\frac{0.35\pi}{2} = 12.256 \text{ rad / second}$$

Stopband edge analog frequency, $\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$ | Using equation (7.54).

$$= \frac{2}{0.1}\tan\frac{0.7\pi}{2}$$

$$= 39.2522 \text{ rad / second}$$

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\frac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]}{\log\frac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\frac{\left(1/0.1^2\right)-1}{\left(1/0.6^2\right)-1}\right]}{\log\frac{39.2522}{12.256}}$$

Using equation (7.57).

$$= \frac{1}{2}\frac{\log\left[\frac{99}{1.7778}\right]}{\log\frac{39.2522}{12.256}} = 1.7267$$

Choose order N, such that $N \geq N_1$ and N is an integer.

Let, order, N = 2.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For even N,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.58).

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Using equation (7.60).

Here, N = 2, $\therefore\ k = \frac{N}{2} = \frac{2}{2} = 1$

$$\therefore\ H(s_n) = \frac{1}{s_n^2 + b_1 s_n + 1}$$

$$\text{When } k = 1\ ;\ b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times 2}\right] = 1.4142$$

Calculate $\sin\theta$ using calculator in radian mode.

$$\therefore\ H(s_n) = \frac{1}{s_n^2 + 1.4142 s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth highpass filter

The highpass filter with cutoff frequency, Ω_c can be obtained from normalized lowpass filter using the transformation, $s_n \rightarrow \Omega_c/s$.

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{\Omega_c}{s}}$$

where, W_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2) - 1\right]^{\frac{1}{2N}}} = \frac{39.2522}{\left[(1/0.1^2)-1\right]^{\frac{1}{4}}} = 12.4439 \text{ rad / second}$$

Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{\Omega_c}{s}} = \frac{1}{s_n^2 + 1.4142 s_n + 1}\Big|_{s_n = \frac{\Omega_c}{s}}$$

$$= \frac{1}{\frac{\Omega_c^2}{s^2} + 1.4142\frac{\Omega_c}{s} + 1} = \frac{1}{\frac{\Omega_c^2 + 1.4142\Omega_c s + s^2}{s^2}} = \frac{s^2}{s^2 + 1.4142\Omega_c s + \Omega_c^2}$$

$$= \frac{s^2}{s^2 + 1.4142 \times 12.4439 s + 12.4439^2} = \frac{s^2}{s^2 + 17.5982 s + 154.8506}$$

Digital IIR highpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{s^2}{s^2 + 17.5982 s + 154.8506}\Bigg|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 17.5982\frac{2}{T}\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + 154.8506}$$

$$= \frac{\frac{4(1-z^{-1})^2}{T^2(1+z^{-1})^2}}{\frac{4(1-z^{-1})^2}{T^2(1+z^{-1})^2} + \frac{35.1964(1-z^{-1})}{T(1+z^{-1})} + 154.8506}$$

$$= \frac{\frac{4(1-z^{-1})^2}{\cancel{T^2(1+z^{-1})^2}}}{\frac{4(1-z^{-1})^2 + 35.1964T(1-z^{-1})(1+z^{-1}) + 154.8506T^2(1+z^{-1})^2}{\cancel{T^2(1+z^{-1})^2}}}$$

$$= \frac{4(1-z^{-1})^2}{4(1-z^{-1})^2 + 35.1964T(1-z^{-1})(1+z^{-1}) + 154.8506T^2(1+z^{-1})^2}$$

$(a+b)(a-b) = a^2 - b^2$
$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab - b^2$

$$= \frac{4(1-2z^{-1}+z^{-2})}{4(1-2z^{-1}+z^{-2}) + 35.1964 \times 0.1(1-z^{-2}) + 154.8506 \times 0.1^2(1+2z^{-1}+z^{-2})}$$

Put, T = 0.1

$$= \frac{4\left(1-2z^{-1}+z^{-2}\right)}{4\left(1-2z^{-1}+z^{-2}\right)+3.5196\left(1-z^{-2}\right)+1.5485\left(1+2z^{-1}+z^{-2}\right)}$$

$$= \frac{4-8z^{-1}+4z^{-2}}{9.0681-4.903\,z^{-1}+2.0289\,z^{-2}}$$

$$= \frac{\dfrac{4}{9.0681}-\dfrac{8}{9.0681}z^{-1}+\dfrac{4}{9.0681}z^{-2}}{1-\dfrac{4.903}{9.0681}z^{-1}+\dfrac{2.0289}{9.0681}z^{-2}}$$

$$= \frac{0.4411-0.8822\,z^{-1}+0.4411z^{-2}}{1-0.5407\,z^{-1}+0.2237\,z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.4411-0.8822\,z^{-1}+0.4411z^{-2}}{1-0.5407\,z^{-1}+0.2237\,z^{-2}}$$

$$= \frac{z^{-2}\left(0.4411z^{2}-0.8822\,z+0.4411\right)}{z^{-2}\left(z^{2}-0.5407\,z+0.2237\right)}$$

$$= \frac{0.4411z^{2}-0.8822\,z+0.4411}{z^{2}-0.5407\,z+0.2237}$$

Direct form-I structure of digital IIR highpass filter

Let, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{0.4411-0.8822\,z^{-1}+0.4411z^{-2}}{1-0.5407\,z^{-1}+0.2237\,z^{-2}}$

On cross multiplying the above equation we get,

$$Y(z)-0.5407z^{-1}Y(z)+0.2237z^{-2}Y(z) = 0.4411X(z)-0.8822z^{-1}X(z)+0.4411z^{-2}X(z)$$

\ $Y(z) = 0.4411X(z)-0.8822z^{-1}X(z)+0.4411z^{-2}X(z)+0.5407z^{-1}X(z)-0.2237z^{-2}X(z)$ .....(1)

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

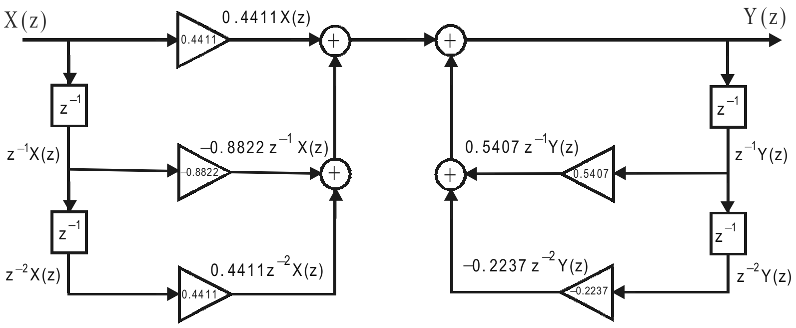


Fig 1 : Direct form-I structure of 2nd order digital IIR highpass filter.

Direct form-II structure of digital IIR highpass filter

Let, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{W(z)}{X(z)} \times \dfrac{Y(z)}{W(z)} = \dfrac{0.4411-0.8822\,z^{-1}+0.4411z^{-2}}{1-0.5407\,z^{-1}+0.2237\,z^{-2}}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 - 0.5407\,z^{-1} + 0.2237\,z^{-2}} \qquad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.4411 - 0.8822z^{-1} + 0.4411z^{-2} \qquad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) - 0.5407z^{-1}W(z) + 0.2237z^{-2}W(z) = X(z)$$

$$\backslash\ W(z) = X(z) + 0.5407z^{-1}W(z) - 0.2237z^{-2}W(z) \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$Y(z) = 0.4411W(z) - 0.8822z^{-1}W(z) + 0.4411z^{-2}W(z) \qquad .....(5)$$

Using equations (4) and (5), the direct form-II structure is drawn as shown in fig 2.

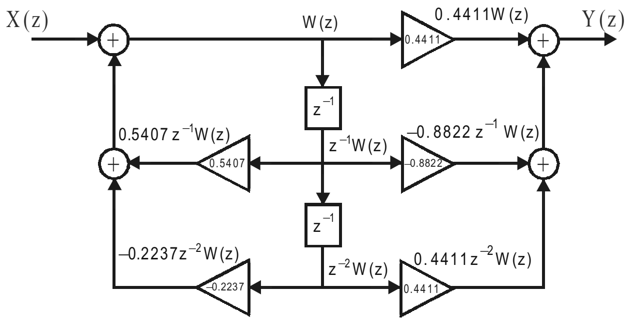


***Fig 2 :** Direct form-II structure of 2^{nd} order digital IIR highpass filter.*

Frequency Response, $H(e^{jw})$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{0.4411 - 0.8822z^{-1} + 0.4411z^{-2}}{1 - 0.5407\,z^{-1} + 0.2237\,z^{-2}}\Bigg|_{z=e^{j\omega}}$$

$$\boxed{e^{-j\theta} = \cos\theta - j\sin\theta}$$

$$= \frac{0.4411 - 0.8822\,e^{-j\omega} + 0.4411e^{-j2\omega}}{1 - 0.5407\,e^{-j\omega} + 0.2237\,e^{-j2\omega}}$$

$$= \frac{0.4411 - 0.8822(\cos\omega - j\sin\omega) + 0.4411(\cos 2\omega - j\sin 2\omega)}{1 - 0.5407(\cos\omega - j\sin\omega) + 0.2237(\cos 2\omega - j\sin 2\omega)}$$

$$= \frac{(0.4411 - 0.8822\cos\omega + 0.4411\cos 2\omega) + j(0.8822\sin\omega - 0.4411\sin 2\omega)}{(1 - 0.5407\cos\omega + 0.2237\cos 2\omega) + j(0.5407\sin\omega - 0.2237\sin 2\omega)}$$

Let, $$H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} = \frac{(0.4411 - 0.8822\cos\omega + 0.4411\cos 2\omega) + j(0.8822\sin\omega - 0.4411\sin 2\omega)}{(1 - 0.5407\cos\omega + 0.2237\cos 2\omega) + j(0.5407\sin\omega - 0.2237\sin 2\omega)}$$

where, $H_N(e^{jw}) = (0.4411 - 0.8822\cos w + 0.4411\cos 2w) + j(0.8822\sin w - 0.4411\sin 2w)$

$H_D(e^{jw}) = (1 - 0.5407\cos w + 0.2237\cos 2w) + j(0.5407\sin w - 0.2237\sin 2w)$

The frequency response $H(e^{jw})$ and hence the magnitude response $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the values listed in table 1, the magnitude response of highpass filter is sketched as shown in fig 3.

Note : Verify the result with MATLAB program 7.2.

Table 1: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w

w	$H_N(e^{jw})$	$H_D(e^{jw})$	$H(e^{jw})$	$\|H(e^{jw})\|$
$\frac{0\times\pi}{16}$	0 + j0	0.683 + j0	0 + j0	0
$\frac{1\times\pi}{16}$	–0.0166 + j0.0033	0.6764 + j0.0199	–0.0244 + j0.0056	0.0250
$\frac{2\times\pi}{16}$	–0.0620 + j0.0257	0.6586 + j0.0487	–0.0908 + j0.0451	0.1016
$\frac{3\times\pi}{16}$	–0.1236 + j0.0826	0.636 + j0.0937	–0.1715 + j0.1551	0.2312
$\frac{4\times\pi}{16}$	–0.1827 + j0.1827	0.6177 + j0.1586	–0.2062 + j0.3487	0.4051
$\frac{5\times\pi}{16}$	–0.2178 + j0.326	0.614 + j0.2429	–0.1251 + j0.5804	0.5938
$\frac{6\times\pi}{16}$	–0.2084 + j0.5031	0.6349 + j0.3414	0.0759 + j0.7516	0.7554
$\frac{7\times\pi}{16}$	–0.1385 + j0.6964	0.6878 + j0.4447	0.3196 + j0.8058	0.8669
$\frac{8\times\pi}{16}$	0 + j0.8822	0.7763 + j0.5407	0.533 + j0.7652	0.9325
$\frac{9\times\pi}{16}$	0.2057 + j1.0341	0.8988 + j0.6159	0.6922 + j0.6762	0.9677
$\frac{10\times\pi}{16}$	0.4668 + j1.1270	1.0487 + j0.6577	0.8032 + j0.5709	0.9854
$\frac{11\times\pi}{16}$	0.7624 + j1.1470	1.2148 + j0.6562	0.8786 + j0.4647	0.9939
$\frac{12\times\pi}{16}$	1.0649 + j1.0649	1.3823 + j0.6060	0.9295 + j0.3629	0.9978
$\frac{13\times\pi}{16}$	1.3434 + j0.8976	1.5352 + j0.5071	0.9631 + j0.2665	0.9993
$\frac{14\times\pi}{16}$	1.5681 + j0.6495	1.6577 + j0.3651	0.9845 + j0.175	0.9999
$\frac{15\times\pi}{16}$	1.7139 + j0.3409	1.737 + j0.1911	0.9962 + j0.0867	1.0000
$\frac{16\times\pi}{16}$	1.7644 + j0	1.7644 + j0	1.0000 + j0	1.0000

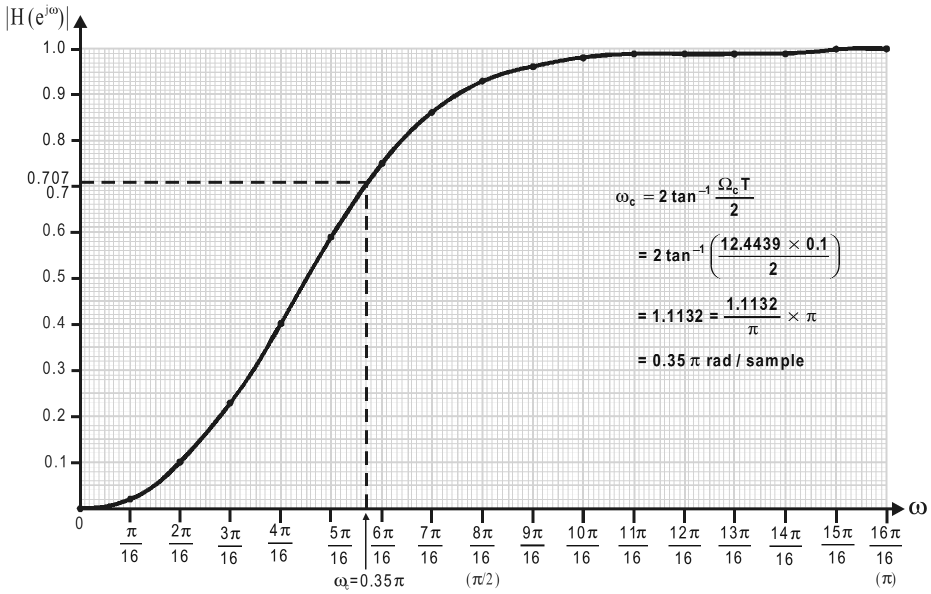


Fig 3 : Frequency response of 2nd order digital Butterworth IIR highpass filter.

Example 7.17

Design a Butterworth digital IIR lowpass filter using bilinear transformation by taking T = 0.5second, to satisfy the following specifications.

$$0.707 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.45\pi$$

$$|H(e^{j\omega})| \le 0.2 \quad ; \quad \text{for } 0.65\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternatively,

Passband ripple $\le$ 3.01 dB

Stopband attenuation $\ge$ 13.97 dB

Passband edge frequency = 0.45π rad/sample

Stopband edge frequency = 0.65π rad/sample

The above specifications can be converted to A_p and A_s as shown below.

$$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{(-3.01/20)} = 0.707$$

$$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{(-13.97/20)} = 0.2$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.45\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.65\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.707$

Gain in normal value at stopband edge, $A_s = 0.2$

Sampling time, T = 0.5second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.707$

Gain in normal value at stopband edge, $A_s = 0.2$

Gain is same in analog and digital filter.

For bilinear transformation,

Passband edge analog frequency, $\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$

$$= \frac{2}{0.5}\tan\frac{0.45\pi}{2}$$

$$= 3.4163 \text{ rad / second}$$

Using equation (7.53).

Stopband edge analog frequency, $\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$

$$= \frac{2}{0.5}\tan\frac{0.65\pi}{2}$$

$$= 6.5274 \text{ rad / second}$$

Using equation (7.54).

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\dfrac{\left(1/A_s^2\right)-1}{\left(1/A_s^2\right)-1}\right]}{\log\dfrac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\dfrac{\left(1/0.2^2\right)-1}{\left(1/0.707^2\right)-1}\right]}{\log\dfrac{6.5274}{3.4163}}$$

Using equation (7.57).

$$= \frac{1}{2}\frac{\log\left[\dfrac{24}{1.0006}\right]}{\log\dfrac{6.5274}{3.4163}} = 2.4538$$

Choose order N, such that N ³ N_1 and N is an integer.

Let, order, N = 3.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For odd N,

$$H(s_n) = \frac{1}{s_n+1}\prod_{k=1}^{\frac{N-1}{2}}\frac{1}{s_n^2+b_k\,s_n+1}$$

Using equation (7.59).

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Using equation (7.60).

$$\text{Here, } N=3, \qquad \therefore\ k = \frac{N-1}{2} = \frac{3-1}{2} = 1$$

$$\therefore\ H(s_n) = \frac{1}{(s_n+1)\ (s_n^2+b_1\,s_n+1)}$$

$$\text{When } k=1\ ;\ b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times 3}\right] = 1$$

Calculate sin q using calculator in radian mode.

$$\therefore\ H(s_n) = \frac{1}{(s_n+1)\ (s_n^2+s_n+1)}$$

$$= \frac{1}{s_n^3+s_n^2+s_n+s_n^2+s_n+1} = \frac{1}{s_n^3+2s_n^2+2s_n+1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n=\frac{s}{\Omega_c}}$$

where, W $_c$ = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[\left(1/A_s^2\right)-1\right]^{\frac{1}{2N}}} = \frac{6.5274}{\left[\left(1/0.2^2\right)-1\right]^{\frac{1}{2\times 3}}} = 3.8433 \text{ rad / second}$$

Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n=\frac{s}{\Omega_c}} = \frac{1}{s_n^3+2s_n^2+2s_n+1}\Big|_{s_n=\frac{s}{\Omega_c}}$$

$$\therefore\ H(s) = \frac{1}{\left(\frac{s}{\Omega_c}\right)^3 + 2\left(\frac{s}{\Omega_c}\right)^2 + 2\frac{s}{\Omega_c} + 1} = \frac{1}{\frac{s^3 + 2\Omega_c s^2 + 2\Omega_c^2 s + \Omega_c^3}{\Omega_c^3}}$$

$$= \frac{\Omega_c^3}{s^3 + 2\Omega_c s^2 + 2\Omega_c^2 s + \Omega_c^3}$$

$$= \frac{3.8433^3}{s^3 + 2 \times 3.8433 s^2 + 2 \times 3.8433^2 s + 3.8433^3}$$

$$= \frac{56.7692}{s^3 + 7.6866 s^2 + 29.5419 s + 56.7692}$$

Digital IIR lowpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{56.7692}{s^3 + 7.6866 s^2 + 29.5419 s + 56.7692}\Bigg|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{56.7692}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + 7.6866\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 29.5419\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + 56.7692}$$

$$= \frac{56.7692}{\frac{8(1-z^{-1})^3 + 7.6866 \times 4T(1-z^{-1})^2(1+z^{-1}) + 29.5419 \times 2T^2(1-z^{-1})(1+z^{-1})^2 + 56.7692 \times T^3(1+z^{-1})^3}{T^3(1+z^{-1})^3}}$$

Put, T = 0.5

$$= \frac{56.7692 \times 0.5^3(1+z^{-1})^3}{8(1-z^{-1})^3 + 7.6866 \times 4 \times 0.5(1-z^{-1})^2(1+z^{-1}) + 29.5419 \times 2 \times 0.5^2(1-z^{-1})(1+z^{-1})^2 + 56.7692 \times 0.5^3(1+z^{-1})^3}$$

$$= \frac{7.0962(1+z^{-1})^3}{8(1-z^{-1})^3 + 15.3732(1-z^{-1})^2(1+z^{-1}) + 14.771(1-z^{-1})(1+z^{-1})^2 + 7.0962(1+z^{-1})^3}$$

$(a+b)(a-b) = a^2 - b^2$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$$= \frac{7.0962(1+3z^{-1}+3z^{-2}+z^{-3})}{8(1-3z^{-1}+3z^{-2}-z^{-3}) + 15.3732(1-z^{-1})(1-z^{-2}) + 14.771(1-z^{-2})(1+z^{-1}) + 7.0962(1+3z^{-1}+3z^{-2}+z^{-3})}$$

$$= \frac{7.0962 + 21.2886 z^{-1} + 21.2886 z^{-2} + 7.0962 z^{-3}}{8(1-3z^{-1}+3z^{-2}-z^{-3}) + 15.3732(1-z^{-1}-z^{-2}+z^{-3}) + 14.771(1+z^{-1}-z^{-2}-z^{-3}) + 7.0962(1+3z^{-1}+3z^{-2}+z^{-3})}$$

$$= \frac{7.0962 + 21.2886 z^{-1} + 21.2886 z^{-2} + 7.0962 z^{-3}}{45.2404 - 3.3136 z^{-1} + 15.1444 z^{-2} - 0.3016 z^{-3}}$$

$$= \frac{\frac{7.0962}{45.2404} + \frac{21.2886}{45.2404}z^{-1} + \frac{21.2886}{45.2404}z^{-2} + \frac{7.0962}{45.2404}z^{-3}}{1 - \frac{3.3136}{45.2404}z^{-1} + \frac{15.1444}{45.2404}z^{-2} - \frac{0.3016}{45.2404}z^{-3}}$$

$$= \frac{0.1569 + 0.4706 z^{-1} + 0.4706 z^{-2} + 0.1569 z^{-3}}{1 - 0.0732 z^{-1} + 0.3348 z^{-2} - 0.0067 z^{-3}}$$

Alternatively,

$$H(z) = \frac{0.1569 + 0.4706\,z^{-1} + 0.4706\,z^{-2} + 0.1569\,z^{-3}}{1 - 0.0732\,z^{-1} + 0.3348\,z^{-2} - 0.0067\,z^{-3}}$$

$$= \frac{z^{-3}\left(0.1569\,z^{3} + 0.4706\,z^{2} + 0.4706\,z + 0.1569\right)}{z^{-3}\left(z^{3} - 0.0732\,z^{2} + 0.3348\,z - 0.0067\right)}$$

$$= \frac{0.1569\,z^{3} + 0.4706\,z^{2} + 0.4706\,z + 0.1569}{z^{3} - 0.0732\,z^{2} + 0.3348\,z - 0.0067}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{0.1569 + 0.4706\,z^{-1} + 0.4706\,z^{-2} + 0.1569\,z^{-3}}{1 - 0.0732\,z^{-1} + 0.3348\,z^{-2} - 0.0067\,z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.0732z^{-1}Y(z) + 0.3348z^{-2}Y(z) - 0.00677\,z^{-3}Y(z) = 0.1569X(z) + 0.4706z^{-1}X(z) + 0.4706z^{-2}X(z) + 0.1569z^{-3}X(z)$$

$$\backslash\;\; Y(z) = 0.1569X(z) + 0.4706z^{-1}X(z) + 0.4706z^{-2}X(z) + 0.1569z^{-3}X(z) + 0.0732z^{-1}Y(z) - 0.3348z^{-2}Y(z) + 0.0067z^{-3}Y(z) \quad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

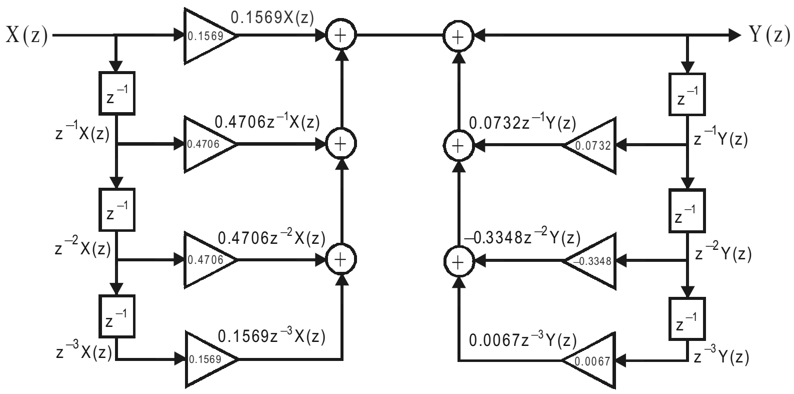


Fig 1 : Direct form-I structure of 3rd order digital IIR lowpass filter..

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.1569 + 0.4706\,z^{-1} + 0.4706\,z^{-2} + 0.1569\,z^{-3}}{1 - 0.0732\,z^{-1} + 0.3348\,z^{-2} - 0.0067\,z^{-3}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1 - 0.0732\,z^{-1} + 0.3348\,z^{-2} - 0.0067\,z^{-3}} \quad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.1569 + 0.4706\,z^{-1} + 0.4706\,z^{-2} + 0.1569\,z^{-3} \quad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) - 0.0732z^{-1}W(z) + 0.3348z^{-2}W(z) - 0.0067z^{-3}W(z) = X(z)$$

$$\backslash\;\; W(z) = X(z) + 0.0732z^{-1}W(z) - 0.3348z^{-2}W(z) + 0.0067z^{-3}W(z) \quad .....(4)$$

On cross multiplying equation (3) we get,

$$Y(z) = 0.1569W(z) + 0.4706z^{-1}W(z) + 0.4706z^{-2}W(z) + 0.1569z^{-3}W(z) \quad .....(5)$$

Using equations (4) and (5), the direct form-II structure is drawn as shown in fig 2.

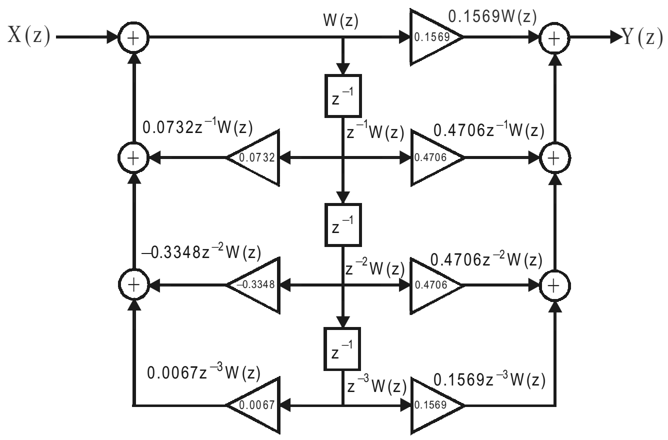


Fig 2 : Direct form-II structure of 3rd order digital IIR lowpass filter.

Frequency Response, $H(e^{jw})$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{0.1569 + 0.4706\,z^{-1} + 0.4706\,z^{-2} + 0.1569\,z^{-3}}{1 - 0.0732\,z^{-1} + 0.3348\,z^{-2} - 0.0067\,z^{-3}}\Bigg|_{z=e^{j\omega}}$$

$$= \frac{0.1569 + 0.4706\,e^{-j\omega} + 0.4706\,e^{-j2\omega} + 0.1569\,e^{-j3\omega}}{1 - 0.0732\,e^{-j\omega} + 0.3348\,e^{-j2\omega} - 0.0067\,e^{-j3\omega}}$$

$$= \frac{0.1569 + 0.4706(\cos\omega - j\sin\omega) + 0.4706(\cos 2\omega - j\sin 2\omega) + 0.1569(\cos 3\omega - j\sin 3\omega)}{1 - 0.0732(\cos\omega - j\sin\omega) + 0.3348(\cos 2\omega - j\sin 2\omega) - 0.0067(\cos 3\omega - j\sin 3\omega)}$$

$$= \frac{\begin{array}{l}(0.1569 + 0.4706\cos\omega + 0.4706\cos 2\omega + 0.1569\cos 3\omega)\\ \qquad + j(-0.4706\sin\omega - 0.4706\sin 2\omega - 0.1569\sin 3\omega)\end{array}}{\begin{array}{l}(1 - 0.0732\cos\omega + 0.3348\cos 2\omega - 0.0067\cos 3\omega)\\ \qquad + j(0.0732\sin\omega - 0.3348\sin 2\omega + 0.0067\sin 3\omega)\end{array}}$$

Let, $$H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} = \frac{\begin{array}{l}(0.1569 + 0.4706\cos\omega + 0.4706\cos 2\omega + 0.1569\cos 3\omega)\\ \qquad + j(-0.4706\sin\omega - 0.4706\sin 2\omega - 0.1569\sin 3\omega)\end{array}}{\begin{array}{l}(1 - 0.0732\cos\omega + 0.3348\cos 2\omega - 0.0067\cos 3\omega)\\ \qquad + j(0.0732\sin\omega - 0.3348\sin 2\omega + 0.0067\sin 3\omega)\end{array}}$$

where, $H_N(e^{jw}) = (0.1569 + 0.4706\cos w + 0.4706\cos 2w + 0.1569\cos 3w)$

$\qquad + j(-0.4706\sin w - 0.4706\sin 2w - 0.1569\sin 3w)$

$H_D(e^{jw}) = (1 - 0.0732\cos w + 0.3348\cos 2w - 0.0067\cos 3w)$

$\qquad + j(0.0732\sin w - 0.3348\sin 2w + 0.0067\sin 3w)$

The frequency response $H(e^{jw})$ and hence the magnitude response $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the values listed in table 1, the magnitude response of lowpass filter is sketched as shown in fig 3.

Note : Verify the result with MATLAB program 7.3.

Table 1 : $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w

w	$H_N(e^{jw})$	$H_D(e^{jw})$	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	1.255 + j0	1.2549 + j0	1 + j0	1.0000
$\frac{1\times\pi}{16}$	1.1837 – j0.3591	1.232 – j0.1101	0.979 – j0.204	1.0000
$\frac{2\times\pi}{16}$	0.9845 – j0.6578	1.1665 – j0.2025	0.9143 – j0.4052	1.0000
$\frac{3\times\pi}{16}$	0.6977 – j0.8501	1.0686 – j0.2621	0.7999 – j0.5993	0.9995
$\frac{4\times\pi}{16}$	0.3787 – j0.9143	0.953 – j0.2783	0.6243 – j0.7771	0.9968
$\frac{5\times\pi}{16}$	0.0844 – j0.8567	0.8378 – j0.2471	0.3701 – j0.9134	0.9855
$\frac{6\times\pi}{16}$	–0.1407 – j0.7075	0.7414 – j0.1717	0.0296 – j0.9474	0.9479
$\frac{7\times\pi}{16}$	–0.2732 – j0.5112	0.6801 – j0.0619	–0.3306 – j0.7817	0.8488
$\frac{8\times\pi}{16}$	–0.3137 – j0.3137	0.6652 + j0.0665	–0.5136 – j0.4202	0.6636
$\frac{9\times\pi}{16}$	– 0.2825 – j0.1510	0.7012 + j0.1943	–0.4296 – j0.0963	0.4402
$\frac{10\times\pi}{16}$	– 0.211 – j0.042	0.7851 + j0.3018	–0.2521 + j0.0434	0.2558
$\frac{11\times\pi}{16}$	– 0.1308 + j0.0129	0.906 + j0.3715	–0.1186 + j0.0629	0.1342
$\frac{12\times\pi}{16}$	– 0.0649 + j0.0269	1.047 + j0.3913	–0.046 + j0.0429	0.0629
$\frac{13\times\pi}{16}$	– 0.0237 + j0.0194	1.1877 + j0.3566	–0.0138 + j0.0205	0.0247
$\frac{14\times\pi}{16}$	– 0.0052 – j0.0077	1.3069 + j0.2709	–0.0026 + j0.0064	0.0070
$\frac{15\times\pi}{16}$	– 0.0003 + j0.0011	1.3867 + j0.1461	–0.0001 + j0.0008	0.0007
$\frac{16\times\pi}{16}$	0	1.4147 + j0	0 + j0	0

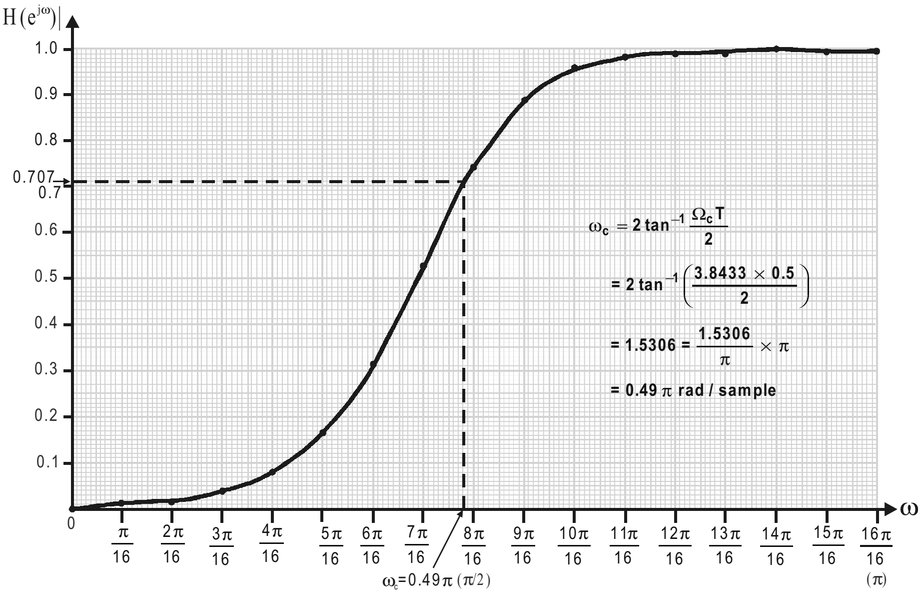


Fig 3 : *Frequency response of 3rd order digital Butterworth IIR highpass filter.*

Example 7.18

Design a Butterworth digital IIR highpass filter using bilinear transformation by taking T = 0.5second, to satisfy the following specifications.

$$0.707 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0.65\pi \le \omega \le \pi$$

$$|H(e^{j\omega})| \le 0.2 \quad ; \quad \text{for } 0 \le \omega \le 0.45\pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

> Alternatively,
>
> Passband ripple $\le$ 3.01dB
>
> Stopband attenuation $\ge$ 13.97dB
>
> Passband edge frequency = 0.65π rad/sample
>
> Stopband edge frequency = 0.45π rad/sample
>
> The above specifications can be converted to A_p and A_s as shown below.
>
> $$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{(-3.01/20)} = 0.707$$
>
> $$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{(-13.97/20)} = 0.2$$

Solution

Specifications of digital IIR highpass filter

Passband edge digital frequency, $\omega_p = 0.65\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.45\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.707$

Gain in normal value at stopband edge, $A_s = 0.2$

Sampling time, T = 0.5 second.

The highpass filter is designed via lowpass filter using frequency transformation technique. Hence the given specifications of IIR highpass filter are converted to corresponding specification of IIR lowpass filter.

Specifications of digital IIR lowpass filter

The specification of lowpass filter is obtained by taking passband edge of highpass as stopband edge of lowpass and stopband edge of highpass as passband edge of lowpass. The gain of passband and stopband remain same.

$\therefore$ Passband edge digital frequency, $\omega_p = 0.45\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.65\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.707$

Gain in normal value at stopband edge, $A_s = 0.2$

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.707$

Gain in normal value at stopband edge, $A_s = 0.2$

> Gain is same in analog and digital filter.

For bilinear transformation,

> Using equation (7.53).

$$\text{Passband edge analog frequency, } \Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$$

$$= \frac{2}{0.5}\tan\frac{0.45\pi}{2} = 3.4163 \text{ rad / second}$$

Stopband edge analog frequency, $\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$ | Using equation (7.54).

$$= \frac{2}{0.5}\tan\frac{0.65\pi}{2} = 6.5274 \text{ rad / second}$$

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\frac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]}{\log\frac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\frac{\left(1/0.2^2\right)-1}{\left(1/0.707^2\right)-1}\right]}{\log\frac{6.5274}{3.4163}}$$ Using equation (7.57).

$$= \frac{1}{2}\frac{\log\left[\frac{24}{1.0006}\right]}{\log\frac{6.5274}{3.4163}} = 2.4538$$

Choose order N, such that $N \geq N_1$ and N is an integer.

Let, order, N = 3.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For odd N,

$$H(s_n) = \frac{1}{s_n+1}\prod_{k=1}^{\frac{N-1}{2}}\frac{1}{s_n^2+b_k\,s_n+1}$$ Using equation (7.59).

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$ Using equation (7.60).

Here, N = 3, $\therefore\ k = \frac{N-1}{2} = \frac{3-1}{2} = 1$

$$\therefore\ H(s_n) = \frac{1}{(s_n+1)\left(s_n^2+b_1\,s_n+1\right)}$$

When $k = 1$; $b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times3}\right] = 1$ | Calculate sin θ using calculator in radian mode.

$$\therefore\ H(s_n) = \frac{1}{(s_n+1)\left(s_n^2+s_n+1\right)} = \frac{1}{s_n^3+s_n^2+s_n+s_n^2+s_n+1} = \frac{1}{s_n^3+2s_n^2+2s_n+1}$$

Unnormalized transfer function, H(s) of Butterworth highpass filter

The highpass filter with cutoff frequency, Ω_c can be obtained from normalized lowpass filter using the transformation, $s_n \rightarrow \Omega_c/s$.

$$\therefore\ H(s) = H(s_n)\Big|_{s_n=\frac{\Omega_c}{s}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[\left(1/A_s^2\right)-1\right]^{\frac{1}{2N}}} = \frac{6.5274}{\left[\left(1/0.2^2\right)-1\right]^{\frac{1}{2\times3}}} = 3.8433 \text{ rad / second}$$ Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{\Omega_c}{s}} = \frac{1}{s_n^3 + 2s_n^2 + 2s_n + 1}\Bigg|_{s_n = \frac{\Omega_c}{s}}$$

$$= \frac{1}{\left(\frac{\Omega_c}{s}\right)^3 + 2\left(\frac{\Omega_c}{s}\right)^2 + 2\frac{\Omega_c}{s} + 1} = \frac{1}{\frac{\Omega_c^3 + 2\Omega_c^2 s + 2\Omega_c s^2 + s^3}{s^3}}$$

$$= \frac{s^3}{s^3 + 2\Omega_c s^2 + 2\Omega_c^2 s + \Omega_c^3} = \frac{s^3}{s^3 + 2\times 3.8433\ s^2 + 2\times 3.8433^2 s + 3.8433^3}$$

$$= \frac{s^3}{s^3 + 7.6866s^2 + 29.5419s + 56.7692}$$

Digital IIR highpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{s^3}{s^3 + 7.6866s^2 + 29.5419s + 56.7692}\Bigg|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + 7.6866\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 29.5419\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + 56.7692} \qquad \boxed{\text{Put, } T = 0.5}$$

$$= \frac{\frac{8(1-z^{-1})^3}{\cancel{T^3(1+z^{-1})^3}}}{\frac{8(1-z^{-1})^3 + 7.6866\times 4T(1-z^{-1})^2(1+z^{-1}) + 29.5419\times 2T^2(1-z^{-1})(1+z^{-1})^2 + 56.7692\times T^3(1+z^{-1})^3}{\cancel{T^3(1+z^{-1})^3}}}$$

$(a + b)(a - b) = a^2 - b^2$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$$= \frac{8(1-z^{-1})^3}{8(1-z^{-1})^3 + 7.6866\times 4\times 0.5(1-z^{-1})^2(1+z^{-1}) + 29.5419\times 2\times 0.5^2(1-z^{-1})(1+z^{-1})^2 + 56.7692\times 0.5^3(1+z^{-1})^3}$$

$$= \frac{8(1-z^{-1})^3}{8(1-z^{-1})^3 + 15.3732(1-z^{-1})^2(1+z^{-1}) + 14.771(1-z^{-1})(1+z^{-1})^2 + 7.0962(1+z^{-1})^3}$$

$$= \frac{8(1-3z^{-1}+3z^{-2}-z^{-3})}{8(1-3z^{-1}+3z^{-2}-z^{-3}) + 15.3732(1-z^{-1})(1-z^{-2}) + 14.771(1-z^{-2})(1+z^{-1}) + 7.0962(1+3z^{-1}+3z^{-2}+z^{-3})}$$

$$= \frac{8 - 24z^{-1} + 24z^{-2} - 8z^{-3}}{8\left(1-3z^{-1}+3z^{-2}-z^{-3}\right) + 15.3732(1-z^{-1}-z^{-2}+z^{-3}) + 14.771(1+z^{-1}-z^{-2}-z^{-3}) + 7.0962(1+3z^{-1}+3z^{-2}+z^{-3})}$$

$$= \frac{8 - 24z^{-1} + 24z^{-2} - 8z^{-3}}{45.2404 - 3.3136z^{-1} + 15.1444z^{-2} - 0.3016z^{-3}}$$

$$= \frac{\frac{8}{45.2404} - \frac{24}{45.2404}z^{-1} + \frac{24}{45.2404}z^{-2} - \frac{8}{45.2404}z^{-3}}{1 - \frac{3.3136}{45.2404}z^{-1} + \frac{15.1444}{45.2404}z^{-2} - \frac{0.3016}{45.2404}z^{-3}}$$

$$= \frac{0.1768 - 0.5305z^{-1} + 0.5305z^{-2} - 0.1768z^{-3}}{1 - 0.0732z^{-1} + 0.3348z^{-2} - 0.0067z^{-3}}$$

Alternatively,

$$H(z)=\frac{0.1768-0.5305\,z^{-1}+0.5305\,z^{-2}-0.1768\,z^{-3}}{1-0.0732\,z^{-1}+0.3348\,z^{-2}-0.0067\,z^{-3}}$$

$$=\frac{z^{-3}\left(0.1768\,z^{3}-0.5305\,z^{2}+0.5305\,z-0.1768\right)}{z^{-3}\left(z^{3}-0.0732\,z^{2}+0.3348\,z-0.0067\right)}$$

$$=\frac{0.1768\,z^{3}-0.5305\,z^{2}+0.5305\,z-0.1768}{z^{3}-0.0732\,z^{2}+0.3348\,z-0.0067}$$

Direct form-I structure of digital IIR highpass filter

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=\frac{0.1768-0.5305\,z^{-1}+0.5305\,z^{-2}-0.1768\,z^{-3}}{1-0.0732\,z^{-1}+0.3348\,z^{-2}-0.0067\,z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z)-0.0732z^{-1}Y(z)+0.3348z^{-2}Y(z)-0.0067z^{-3}Y(z)=0.1768X(z)-0.5305z^{-1}X(z)$$
$$+\,0.5305z^{-2}X(z)-0.1768z^{-3}X(z)$$

\ $Y(z)=0.1768X(z)-0.5305z^{-1}X(z)+0.5305z^{-2}X(z)-0.1768z^{-3}X(z)+0.0732z^{-1}Y(z)$
$-\,0.3348z^{-2}Y(z)+0.0067z^{-3}Y(z)$ .....(1)

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

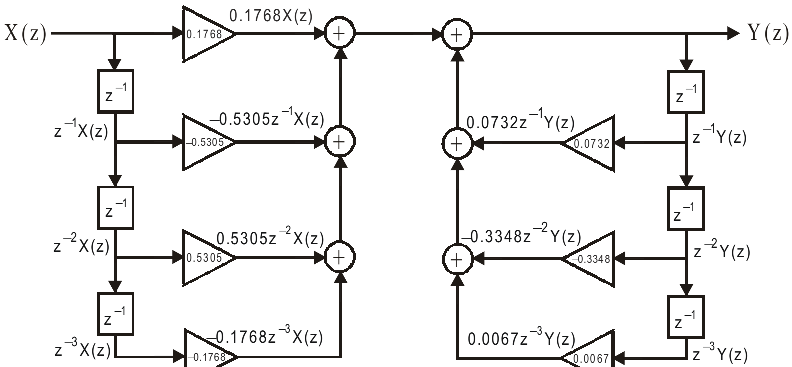


Fig 1 : Direct form-I structure of 3rd order digital IIR highpass filter..

Direct form-II structure of digital IIR highpass filter

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=\frac{W(z)}{X(z)}\times\frac{Y(z)}{W(z)}=\frac{0.1768-0.5305\,z^{-1}+0.5305\,z^{-2}-0.1768\,z^{-3}}{1-0.0732\,z^{-1}+0.3348\,z^{-2}-0.0067\,z^{-3}}$$

$$\text{where, } \frac{W(z)}{X(z)}=\frac{1}{1-0.0732\,z^{-1}+0.3348\,z^{-2}-0.0067\,z^{-3}} \quad .....(2)$$

$$\frac{Y(z)}{W(z)}=0.1768-0.5305\,z^{-1}+0.5305\,z^{-2}-0.1768\,z^{-3} \quad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z)-0.0732z^{-1}W(z)+0.3348z^{-2}W(z)-0.0067z^{-3}W(z)=X(z)$$

\ $W(z)=X(z)+0.0732z^{-1}W(z)-0.3348z^{-2}W(z)+0.0067z^{-3}W(z)$ .....(4)

On cross multiplying equation (3) we get,

$$Y(z) = 0.1768W(z) - 0.5305z^{-1}W(z) + 0.5305z^{-2}W(z) - 0.1768z^{-3}W(z) \qquad .....(5)$$

Using equations (4) and (5), the direct form-II structure is drawn as shown in fig 2.

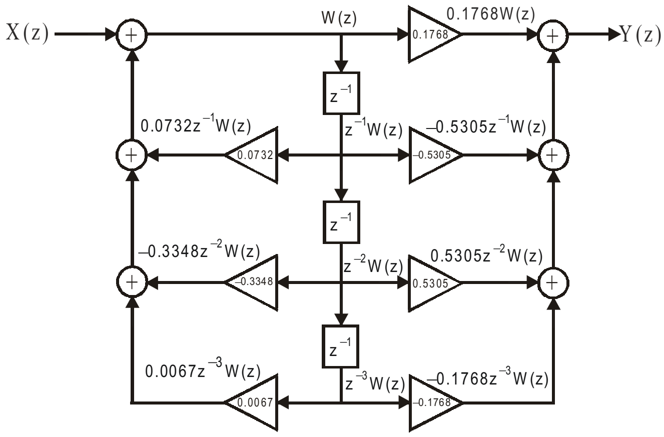


Fig 2 : Direct form-II structure of 3rd order digital IIR highpass filter.

Frequency Response, $H(e^{j\omega})$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \left.\frac{0.1768 - 0.5305z^{-1} + 0.5305z^{-2} - 0.1768z^{-3}}{1 - 0.0732z^{-1} + 0.3348z^{-2} - 0.0067z^{-3}}\right|_{z=e^{j\omega}}$$

$$= \frac{0.1768 - 0.5305e^{-j\omega} + 0.5305e^{-j2\omega} - 0.1768e^{-j3\omega}}{1 - 0.0732e^{-j\omega} + 0.3348e^{-j2\omega} - 0.0067e^{-j3\omega}}$$

$$= \frac{0.1768 - 0.5305(\cos\omega - j\sin\omega) + 0.5305(\cos 2\omega - j\sin 2\omega) - 0.1768(\cos 3\omega - j\sin 3\omega)}{1 - 0.0732(\cos\omega - j\sin\omega) + 0.3348(\cos 2\omega - j\sin 2\omega) - 0.0067(\cos 3\omega - j\sin 3\omega)}$$

$$= \frac{\begin{matrix}(0.1768 - 0.5305\cos\omega + 0.5305\cos 2\omega - 0.1768\cos 3\omega)\\ + j(0.5305\sin\omega - 0.5305\sin 2\omega + 0.1768\sin 3\omega)\end{matrix}}{\begin{matrix}(1 - 0.0732\cos\omega + 0.3348\cos 2\omega - 0.0067\cos 3\omega)\\ + j(0.0732\sin\omega - 0.3348\sin 2\omega + 0.0067\sin 3\omega)\end{matrix}}$$

$$\text{Let, } H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} == \frac{\begin{matrix}(0.1768 - 0.5305\cos\omega + 0.5305\cos 2\omega - 0.1768\cos 3\omega)\\ + j(0.5305\sin\omega - 0.5305\sin 2\omega + 0.1768\sin 3\omega)\end{matrix}}{\begin{matrix}(1 - 0.0732\cos\omega + 0.3348\cos 2\omega - 0.0067\cos 3\omega)\\ + j(0.0732\sin\omega - 0.3348\sin 2\omega + 0.0067\sin 3\omega)\end{matrix}}$$

where, $H_N(e^{jw}) = (0.1768 - 0.5305\cos w + 0.5305\cos 2w - 0.1768\cos 3w)$

$+ j(0.5305\sin w - 0.5305\sin 2w + 0.1768\sin 3w)$

$H_D(e^{jw}) = (1 - 0.0732\cos w + 0.3348\cos 2w - 0.0067\cos 3w)$

$+ j(0.0732\sin w - 0.3348\sin 2w + 0.0067\sin 3w)$

The frequency response $H(e^{jw})$ and hence the magnitude response $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the values listed in table 1, the magnitude response of highpass filter is sketched as shown in fig 3.

Note : Verify the result with MATLAB program 7.4.

Table 1: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w

w	$H_N(e^{jw})$	$H_D(e^{jw})$	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	0 + j0	1.2549 + j0	0 + j0	0
$\frac{1\times\pi}{16}$	–0.0004 – j0.0013	1.232 – j0.1101	–0.0002 – j0.0011	0.0011
$\frac{2\times\pi}{16}$	–0.0059 – j0.0088	1.1665 – j0.2025	–0.0036 – j0.0082	0.0089
$\frac{3\times\pi}{16}$	–0.0268 – j0.022	1.0686 – j0.2621	–0.0189 – j0.0252	0.0315
$\frac{4\times\pi}{16}$	–0.0733 – j0.0304	0.953 – j0.2783	–0.0623 – j0.0501	0.0799
$\frac{5\times\pi}{16}$	–0.1475 – j0.0145	0.8378 – j0.2471	–0.1573 – j0.0637	0.1697
$\frac{6\times\pi}{16}$	–0.238 + j0.0473	0.7414 + j0.1717	–0.3187 + j0.100	0.3189
$\frac{7\times\pi}{16}$	–0.3186 + j0.1703	0.6801 + j0.0619	–0.4872 + j0.2061	0.5290
$\frac{8\times\pi}{16}$	–0.3537 + j0.3537	0.6652 + j0.0665	–0.4738 + j0.5791	0.7482
$\frac{9\times\pi}{16}$	–0.3080 + j0.5763	0.7012 + j0.1943	–0.1964 + j0.8763	0.8981
$\frac{10\times\pi}{16}$	–0.1586 + j0.7976	0.7851 + j0.3018	0.1642 + j0.9528	0.9668
$\frac{11\times\pi}{16}$	0.0951 + j0.9657	0.906 + j0.3715	0.4640 + j0.8756	0.9910
$\frac{12\times\pi}{16}$	0.4269 + j1.0306	1.047 + j0.3913	0.6806 + j0.7300	0.9980
$\frac{13\times\pi}{16}$	0.7864 + j1.9583	1.1877 + j0.3566	0.8296 + j0.5578	0.9997
$\frac{14\times\pi}{16}$	1.1097 + j0.7415	1.3069 + j0.2709	0.9269 + j0.3752	1.0000
$\frac{15\times\pi}{16}$	1.3342 + j0.4047	1.3867 + j0.1461	0.9820 + j0.1884	0.9999
$\frac{16\times\pi}{16}$	1.4146 + j0	1.4147 + j0	1.9999 + j0	0.9999

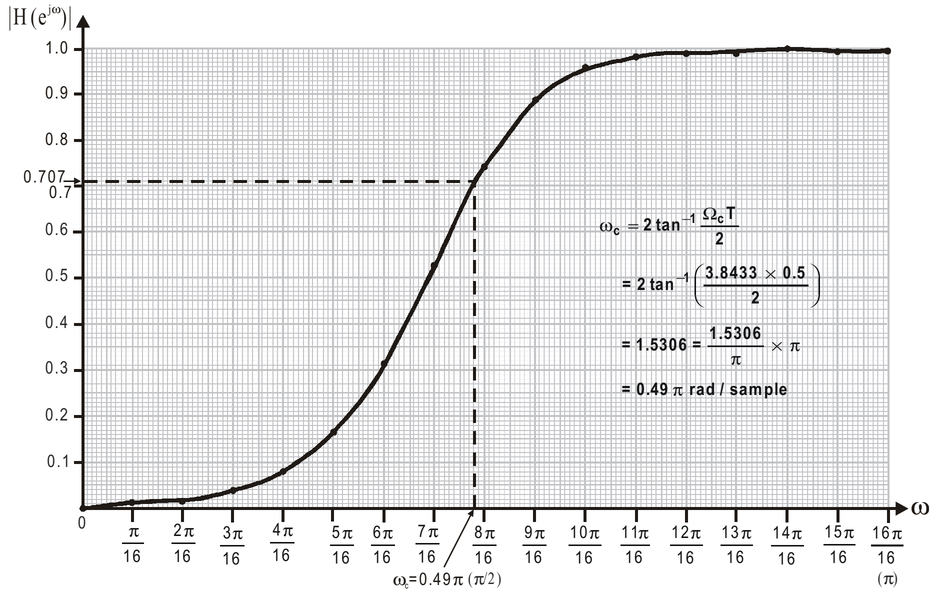


Fig 3 : *Frequency response of 3[rd] order digital Butterworth IIR highpass filter.*

Example 7.19

Design a Butterworth digital IIR lowpass filter using impulse invariant transformation by taking T = 1second, to satisfy the following specifications.

$$0.707 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.3\pi$$

$$|H(e^{j\omega})| \le 0.2 \quad ; \quad \text{for } 0.75\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

> Alternatively,
>
> Passband ripple $\le$ 3.01dB
>
> Stopband attenuation $\ge$ 13.97dB
>
> Passband edge frequency = 0.3π rad/sample
>
> Stopband edge frequency = 0.75π rad/sample
>
> The above specifications can be converted to A_p and A_s as shown below.
>
> $$A_p = 10^{\left(-\delta_{p_1,\text{dB}}/20\right)} = 10^{(-3.01/20)} = 0.707$$
>
> $$A_s = 10^{\left(-\alpha_{s,\text{dB}}/20\right)} = 10^{(-13.97/20)} = 0.2$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.3\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.75\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.707$

Gain in normal value at stopband edge, $A_s = 0.2$

Sampling time, T = 1second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.707$ — Gain is same in analog and digital filter.

Gain in normal value at stopband edge, $A_s = 0.2$

For impulse invariant transformation,

Passband edge analog frequency, $\Omega_p = \dfrac{\omega_p}{T} = \dfrac{0.3\pi}{1} = 0.9425\pi$ rad / second — Using equation (7.55).

Stopband edge analog frequency, $\Omega_s = \dfrac{\omega_s}{T} = \dfrac{0.75\pi}{1} = 2.3562$ rad / second — Using equation (7.56).

Order of the filter

$$N_1 = \frac{1}{2}\,\frac{\log\left[\dfrac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]}{\log\dfrac{\Omega_s}{\Omega_p}} = \frac{1}{2}\,\frac{\log\left[\dfrac{\left(1/0.2^2\right)-1}{\left(1/0.707^2\right)-1}\right]}{\log\dfrac{0.9425}{2.3562}} = \frac{1}{2}\,\frac{\log\left[\dfrac{24}{1.0006}\right]}{\log\dfrac{0.9425}{2.3562}} = 1.7339$$

Using equation (7.57).

Choose order N, such that $N \ge N_1$ and N is an integer.

Let, order, N = 2.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For even N,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.58).

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Using equation (7.60).

Here, $N = 2, \quad \therefore\ k = \frac{N}{2} = \frac{2}{2} = 1$

$$\therefore\ H(s_n) = \frac{1}{s_n^2 + b_1 s_n + 1}$$

Calculate sin q using calculator in radian mode.

When $k = 1\ ;\ b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2 \times 2}\right] = 1.4142$

$$\therefore\ H(s_n) = \frac{1}{s_n^2 + 1.4142 s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2) - 1\right]^{\frac{1}{2N}}} = \frac{2.3562}{\left[(1/0.2^2) - 1\right]^{\frac{1}{4}}} = 1.0645 \text{ rad / second}$$

Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{s_n^2 + 1.4142 s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{1}{\frac{s^2}{\Omega_c^2} + 1.4142\frac{s}{\Omega_c} + 1} = \frac{1}{\frac{s^2 + 1.4142\Omega_c s + \Omega_c^2}{\Omega_c^2}} = \frac{\Omega_c^2}{s^2 + 1.4142\Omega_c s + \Omega_c^2}$$

$$= \frac{1.0645^2}{s^2 + 1.4142 \times 1.0645 s + 1.0645^2} = \frac{1.1332}{s^2 + 1.5054 s + 1.1332}$$

To convert the analog transfer function to digital transfer function, the above equation can be modified as follows.

$$\therefore H(s) = \frac{1.1332}{s^2 + 0.7527 \times 2s + 0.7527^2 - 0.7527^2 + 1.1332}$$

$$= \frac{1.1332}{(s + 0.7527)^2 + 0.5666}$$

$$= \frac{1.1332}{0.7527} \times \frac{0.7527}{(s + 0.7527)^2 + 0.7527^2}$$

$$= 1.5055 \times \frac{0.7527}{(s + 0.7527)^2 + 0.7527^2}$$

$(s + a)^2 = s^2 + 2as + a^2$

$2a = 1.5054 \ \Rightarrow\ a = \frac{1.5054}{2} = 0.7527$

Digital IIR lowpass filter transfer function, H(z)

In impulse invariant transformation,

$$\frac{b}{(s+a)^2+b^2} \xrightarrow{\text{is transformed to}} \frac{e^{-aT}\left(\sin bT\right)z^{-1}}{1-2e^{-aT}\left(\cos bT\right)z^{-1}+e^{-2aT}\,z^{-2}}$$ Using equation (7.19).

Using the above transformation, the H(s) can be transformed to H(z) as shown below.

$$\therefore\ H(z) = 1.5055 \times \frac{e^{-0.7527\times 1}\left(\sin 0.7527\times 1\right)z^{-1}}{1-2e^{-0.7527\times 1}(\cos 0.7527\times 1)z^{-1}+e^{-2\times 0.7527\times 1}z^{-2}}$$ Put, T = 1

$$= 1.5055 \times \frac{0.3220\,z^{-1}}{1-0.6877\,z^{-1}+0.2219\,z^{-2}} = \frac{0.4848\,z^{-1}}{1-0.6877\,z^{-1}+0.2219\,z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.4848\,z^{-1}}{1-0.6877\,z^{-1}+0.2219\,z^{-2}} = \frac{0.4848\,z^{-1}}{z^{-2}(z^2-0.6877\,z+0.2219)}$$

$$= \frac{0.4848\,z}{z^2-0.6877\,z+0.2219}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{0.4848\,z^{-1}}{1-0.6877\,z^{-1}+0.2219\,z^{-2}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.6877z^{-1}Y(z) + 0.2219z^{-2}Y(z) = 0.4848z^{-1}X(z)$$

$$\therefore\ Y(z) = 0.4848z^{-1}X(z) + 0.6877z^{-1}Y(z) - 0.2219z^{-2}Y(z) \qquad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

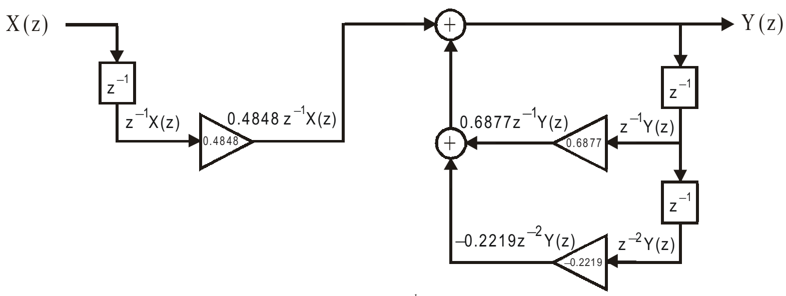


Fig 1 : Direct form-I structure of 2nd order digital IIR lowpass filter.

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.4848\,z^{-1}}{1-0.6877\,z^{-1}+0.2219\,z^{-2}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1-0.6877\,z^{-1}+0.2219\,z^{-2}} \qquad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.4848\,z^{-1} \qquad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) - 0.6877z^{-1} + 0.2219z^{-2}W(z) = X(z)$$

\ $W(z) = X(z) + 0.6877z^{-1}W(z) - 0.2219z^{-2}W(z)$ .....(4)

On cross multiplying equation (3) we get,

$Y(z) = 0.4848z^{-1}W(z)$ .....(5)

Using equation (4) and (5), the direct form-II structure is drawn as shown in fig 2.

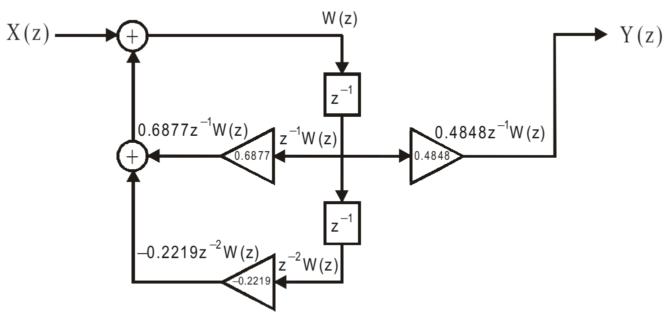


Fig 2 : *Direct form-II structure of 2^nd order digital IIR lowpass filter.*

Frequency Response, H(e^jw)

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{0.4848\,z^{-1}}{1 - 0.6877\,z^{-1} + 0.2219\,z^{-2}}\Bigg|_{z=e^{j\omega}}$$

$$= \frac{0.4848\,e^{-j\omega}}{1 - 0.6877\,e^{-j\omega} + 0.2219\,e^{-j2\omega}}$$

$$= \frac{0.4848(\cos\omega - j\sin\omega)}{1 - 0.6877(\cos\omega - j\sin\omega) + 0.2219(\cos 2\omega - j\sin 2\omega)}$$

$$= \frac{0.4848\cos\omega - j0.4848\sin\omega}{(1 - 0.6877\cos\omega + 0.2219\cos 2\omega) + j(0.6877\sin\omega - 0.2219\sin 2\omega)}$$

Let, $$H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} = \frac{0.4848\cos\omega - j0.4848\sin\omega}{(1 - 0.6877\cos\omega + 0.2219\cos 2\omega) + j(0.6877\sin\omega - 0.2219\sin 2\omega)}$$

where, $H_N(e^{jw}) = 0.4848\cos w - j0.4848\sin w$

$H_D(e^{jw}) = (1 - 0.6877\cos w + 0.2219\cos 2w) + j(0.6877\sin w - 0.2219\sin 2w)$

The frequency response $H(e^{jw})$ and hence the magnitude response $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the values listed in table 1, the magnitude response of lowpass filter is sketched as shown in fig 3.

Note : Verify the result with MATLAB program 7.5.

Table 1: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of **w**

w	$H_N(e^{jw})$	$H_D(e^{jw})$	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	0.4848 + j0	0.5342 + j0	0.9075 + j0	0.9075
$\frac{1\times\pi}{16}$	0.4755 – j0.0946	0.5305 + j0.0492	0.8723 – j0.2592	0.9099
$\frac{2\times\pi}{16}$	0.4479 – j0.1855	0.5215 + j0.1063	0.7550 – j0.5096	0.9101
$\frac{3\times\pi}{16}$	0.4031 – j0.2693	0.5131 + j0.1770	0.5403 – j0.7112	0.8932
$\frac{4\times\pi}{16}$	0.3428 – j0.3428	0.5137 + j0.2644	0.2560 – j0.7990	0.8390
$\frac{5\times\pi}{16}$	0.2693 – j0.4031	0.5330 + j0.3668	–0.0103 – j0.7492	0.7493
$\frac{6\times\pi}{16}$	0.1855 – j0.4479	0.5799 + j0.4784	–0.1888 – j0.6166	0.6449
$\frac{7\times\pi}{16}$	0.0946 – j0.4755	0.6608 + j0.5896	–0.2778 – j0.4718	0.5475
$\frac{8\times\pi}{16}$	0 – j0.4848	0.7781 + j0.6877	–0.3092 – j0.3498	0.4669
$\frac{9\times\pi}{16}$	–0.0946 – j0.4755	0.9292 + j0.7594	–0.3118 – j0.2569	0.4040
$\frac{10\times\pi}{16}$	–0.1855 – j0.4479	1.1062 + j0.7923	–0.3025 – j0.1882	0.3563
$\frac{11\times\pi}{16}$	–0.2693 – j0.4031	1.2971 + j0.7768	–0.2898 – j0.1372	0.3206
$\frac{12\times\pi}{16}$	–0.3428 – j0.3428	1.4863 + j0.7082	–0.2775 – j0.0984	0.2944
$\frac{13\times\pi}{16}$	–0.4031 – j0.2693	1.6567 + j0.5871	–0.2673 – j0.0678	0.2758
$\frac{14\times\pi}{16}$	–0.4479 – j0.1855	1.7923 + j0.4201	–0.2599 – j0.0426	0.2634
$\frac{15\times\pi}{16}$	–0.4755 – j0.0946	1.8795 + j0.2191	–0.2554 – j0.0205	0.2562
$\frac{16\times\pi}{16}$	–0.4848 + j0	1.9096 + j0	–0.2539 + j0	0.2539

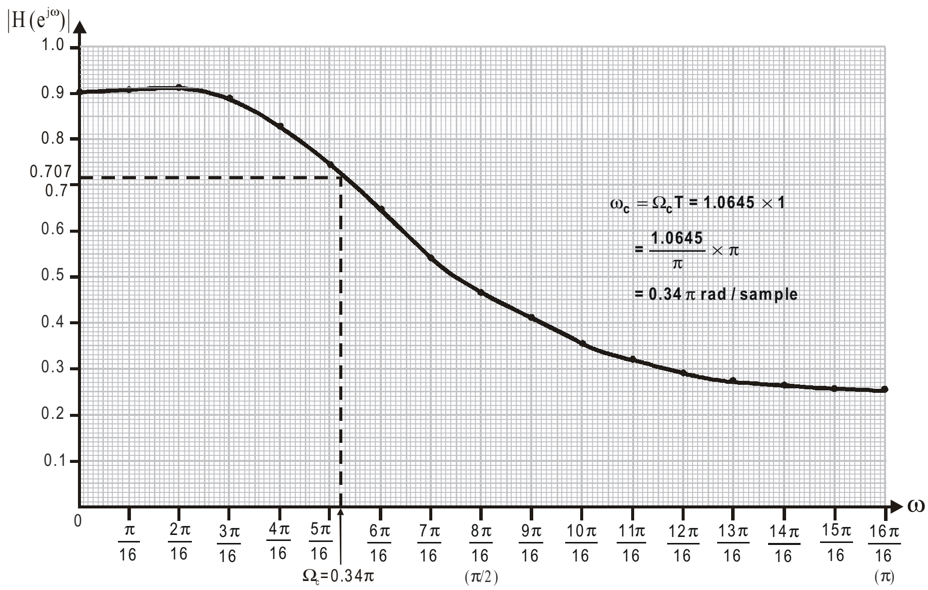


***Fig 3 :** Frequency response of 2^{nd} order digital Butterworth IIR lowpass filter.*

Example 7.20

Design a Butterworth digital IIR lowpass filter using impulse invariant transformation by taking T = 1 second, to satisfy the following specifications.

$$0.9 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad 0 \le \omega \le 0.35\pi$$

$$|H(e^{j\omega})| \le 0.275 \quad ; \quad 0.7\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

> Alternatively,
>
> Passband ripple ≤ 0.9151 dB
>
> Stopband attenuation ≥ 11.2133 dB
>
> Passband edge frequency = 0.35π rad/sample
>
> Stopband edge frequency = 0.7π rad/sample
>
> The above specifications can be converted to A_p and A_s as shown below.
>
> $$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{(-0.9151/20)} = 0.9$$
>
> $$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{(-11.2133/20)} = 0.275$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.35\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.7\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.9$

Gain in normal value at stopband edge, $A_s = 0.275$

Sampling time, T = 1 second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.9$

Gain in normal value at stopband edge, $A_s = 0.275$

Gain is same in analog and digital filter.

For impulse invariant transformation,

Passband edge analog frequency, $\Omega_p = \dfrac{\omega_p}{T} = \dfrac{0.35\pi}{1} = 1.0996$ rad / second

Using equation (7.53).

Stopband edge analog frequency, $\Omega_s = \dfrac{\omega_s}{T} = \dfrac{0.7\pi}{1} = 2.1991$ rad / second

Using equation (7.54).

Order of the filter

$$N = \frac{1}{2}\,\frac{\log\left[\dfrac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]}{\log\dfrac{\Omega_s}{\Omega_p}} = \frac{1}{2}\,\frac{\log\left[\dfrac{\left(1/0.275^2\right)-1}{\left(1/0.9^2\right)-1}\right]}{\log\dfrac{2.1991}{1.0996}} = \frac{1}{2}\,\frac{\log\left[\dfrac{12.2231}{0.2346}\right]}{\log\dfrac{2.1991}{1.0996}} = 2.8518$$

Using equation (7.57).

Choose order N, such that $N \ge N_1$ and N is an integer.

Let, order, N = 3.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For odd N,

$$H(s_n) = \frac{1}{s_n + 1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.59).

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Using equation (7.60).

Here, $N = 3$, $\quad \therefore\ k = \frac{N-1}{2} = \frac{3-1}{2} = 1$

$$\therefore\ H(s_n) = \frac{1}{s_n + 1}\ \frac{1}{s_n^2 + b_1 s_n + 1}$$

When $k = 1;\ b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2 \times 3}\right] = 1$

Calculate sin q using calculator in radian mode.

$$\therefore\ H(s_n) = \frac{1}{(s_n + 1)\ (s_n^2 + s_n + 1)} = \frac{1}{s_n^3 + 2s_n^2 + 2s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2) - 1\right]^{\frac{1}{2N}}} = \frac{2.1991}{\left(\frac{1}{0.275^2} - 1\right)^{\frac{1}{6}}} = 1.4489 \text{ rad / sec}$$

Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{(s_n + 1)\ (s_n^2 + s_n + 1)}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{1}{\left(\frac{s}{\Omega_c} + 1\right)\left(\frac{s^2}{\Omega_c^2} + \frac{s}{\Omega_c} + 1\right)} = \frac{1}{\left(\frac{s + \Omega_c}{\Omega_c}\right)\left(\frac{s^2 + s\,\Omega_c + \Omega_c^2}{\Omega_c^2}\right)}$$

$$= \frac{\Omega_c^3}{(s + \Omega_c)\ (s^2 + s\,\Omega_c + \Omega_c^2)} = \frac{1.4489^3}{(s + 1.4489)\ (s^2 + s \times 1.4489 + 1.4489^2)}$$

$$= \frac{3.0417}{(s + 1.4489)\ (s^2 + 1.4489\,s + 2.0993)} \qquad .....(1)$$

$$= \frac{3.0417}{s^3 + 2.8978s^2 + 4.1986s + 3.0417}$$

To convert the analog transfer function to digital transfer function using impulse invariant transformation, the equation (1) can be simplified as follows.

$$H(s) = \frac{3.0417}{(s + 1.4489)\ (s^2 + 1.4489\,s + 2.0993)}$$

By partial fraction expansion H(s) can be expressed as

$$= \frac{3.0417}{(s + 1.4489)\ (s^2 + 1.4489\,s + 2.0993)} = \frac{A}{s + 1.4489} + \frac{Bs + C}{s^2 + 1.4489\,s + 2.0993} \qquad .....(2)$$

On cross multiplying the equation (2) we get

$$3.0417 = A(s^2 + 1.4489s + 2.0993) + (Bs + C)(s + 1.4489)$$

$$3.0417 = As^2 + 1.4489As + 2.0993A + Bs^2 + 1.4489Bs + Cs + 1.4489C \qquad .....(3)$$

On equating the coefficients of s^2 in equation (3) we get,

$$A + B = 0$$

$$\therefore \; B = -A$$

On equating the coefficients of s in equation (3) we get,

$$1.4489\,A + 1.4489\,B + C = 0$$

Put, $B = -A$

$$\therefore \; 1.4489\,A + 1.4489(-A) + C = 0$$

$$\therefore \; C = 0$$

On equating constants we get,

$$2.0993\,A + 1.4489\,C = 3.0417$$

Put, $C = 0$,

$$\therefore \; 2.0993\,A = 3.0417$$

$$\therefore \; A = \frac{3.0417}{2.0993} = 1.4489$$

$$\therefore \; B = -A = -1.4489$$

$$\therefore \; H(s) = \frac{A}{s + 1.4489} + \frac{Bs + C}{s^2 + 1.4489s + 2.0993}$$

$$(s + a)^2 = s^2 + 2as + a^2$$

$$2a = 1.4489 \Rightarrow a = \frac{1.4489}{2} = 0.7245$$

$$= \frac{1.4489}{s + 1.4489} \; \frac{-1.4489s}{s^2 + 1.4489s + 2.0993}$$

$$= \frac{1.4489}{s + 1.4489} - \frac{1.4489s}{(s^2 + 2 \times 0.7245s + 0.7245^2) + \left(\sqrt{2.0993 - 0.7245^2}\right)^2}$$

$$= \frac{1.4489}{s + 1.4489} - 1.4489\,\frac{s + 0.7245 - 0.7245}{(s + 0.7245)^2 + 1.2548^2}$$

$$= \frac{1.4489}{s + 1.4489} - 1.4489\,\frac{s + 0.7245}{(s + 0.7245)^2 + 1.2548^2} + 1.4489\,\frac{0.7245}{(s + 0.7245)^2 + 1.2548^2}$$

$$= \frac{1.4489}{s + 1.4489} - 1.4489\,\frac{s + 0.7245}{(s + 0.7245)^2 + 1.2548^2} + \frac{1.4489 \times 0.7245}{1.2548}\,\frac{1.2548}{(s + 0.7245)^2 + 1.2548^2}$$

$$= \frac{1.4489}{s + 1.4489} - 1.4489\,\frac{s + 0.7245}{(s + 0.7245)^2 + 1.2548^2} + 0.8366\,\frac{1.2548}{(s + 0.7245)^2 + 1.2548^2} \qquad ......(4)$$

Digital IIR lowpass filter transfer function, H(z)

In impulse invariant transformation,

Using equations (7.17), (7.18) and (7.19).

$$\frac{1}{s + p_i} \xrightarrow{\text{is transformed to}} \frac{A_i}{1 - e^{-p_i T} z^{-1}}$$

$$\frac{(s + a)}{(s + a)^2 + b^2} \xrightarrow{\text{is transformed to}} \frac{1 - e^{-aT}(\cos bT) z^{-1}}{1 - 2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(s + a)^2 + b^2} \xrightarrow{\text{is transformed to}} \frac{e^{-aT}(\sin bT) z^{-1}}{1 - 2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

Using the above transformation, the H(s) of equation (4) can be transformed to H(z) as shown below.

Put, T = 1.

$$\therefore \; H(z) = \frac{1.4489}{1 - e^{-1.4489} z^{-1}} - 1.4489\,\frac{1 - e^{-0.7245}(\cos 1.2548) z^{-1}}{1 - 2e^{-0.7245}(\cos 1.2548) z^{-1} + e^{-2 \times 0.7245} z^{-2}}$$

$$+ 0.8366\,\frac{e^{-0.7245}(\sin 1.2548) z^{-1}}{1 - 2e^{-0.7245}(\cos 1.2548) z^{-1} + e^{-2 \times 0.7245} z^{-2}}$$

$$\therefore\ H(z) = \frac{1.4489}{1 - 0.2348\,z^{-1}} + \frac{-1.4489 + 0.2182\,z^{-1}}{1 - 0.3012\,z^{-1} + 0.2348\,z^{-2}} + \frac{0.3853\,z^{-1}}{1 - 0.3012\,z^{-1} + 0.2348\,z^{-2}}$$

$$= \frac{1.4489}{1 - 0.2348\,z^{-1}} + \frac{-1.4489 + 0.6035\,z^{-1}}{1 - 0.3012\,z^{-1} + 0.2348\,z^{-2}}$$

$$= \frac{1.4489\,(1 - 0.3012\,z^{-1} + 0.2348\,z^{-2}) + (-1.4489 + 0.6035\,z^{-1})\,(1 - 0.2348\,z^{-1})}{(1 - 0.2348\,z^{-1})\,(1 - 0.3012\,z^{-1} + 0.2348\,z^{-2})}$$

$$= \frac{1.4489\ \ - 0.4364\,z^{-1} + 0.3402\,z^{-2} - 1.4489 + 0.3402\,z^{-1} + 0.6035\,z^{-1} - 0.1417\,z^{-2}}{1 - 0.3012\,z^{-1} + 0.2348\,z^{-2} - 0.2348\,z^{-1} + 0.0707\,z^{-2} - 0.0551\,z^{-3}}$$

$$= \frac{0.5073\,z^{-1} + 0.1985\,z^{-2}}{1 - 0.536\,z^{-1} + 0.3055\,z^{-2} - 0.0551\,z^{-3}}$$

Alternatively,

$$H(z) = \frac{0.5073\,z^{-1} + 0.1985\,z^{-2}}{1 - 0.536\,z^{-1} + 0.3055\,z^{-2} - 0.0551\,z^{-3}} = \frac{z^{-3}(0.5073\,z^{-2} + 0.1985\,z)}{z^{-3}(1 - 0.536\,z^{2} + 0.3055\,z - 0.0551)}$$

$$= \frac{0.5073\,z^{-2} + 0.1985\,z}{1 - 0.536\,z^{2} + 0.3055\,z - 0.0551}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let,}\ H(z) = \frac{Y(z)}{X(z)} = \frac{0.5073\,z^{-1} + 0.1985\,z^{-2}}{1 - 0.536\,z^{-1} + 0.3055\,z^{-2} - 0.0551\,z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.536z^{-1}Y(z) + 0.3055z^{-2}Y(z) - 0.0551z^{-3}Y(z) = 0.5073z^{-1}X(z) + 0.1985z^{-2}X(z)$$

\ $Y(z) = 0.5073z^{-1}X(z) + 0.1985z^{-2}X(z) + 0.536z^{-1}X(z) - 0.3055z^{-2}Y(z) + 0.0551z^{-3}Y(z)$ .....(5)

Using equation (5), the direct form-I structure is drawn as shown in fig 1.

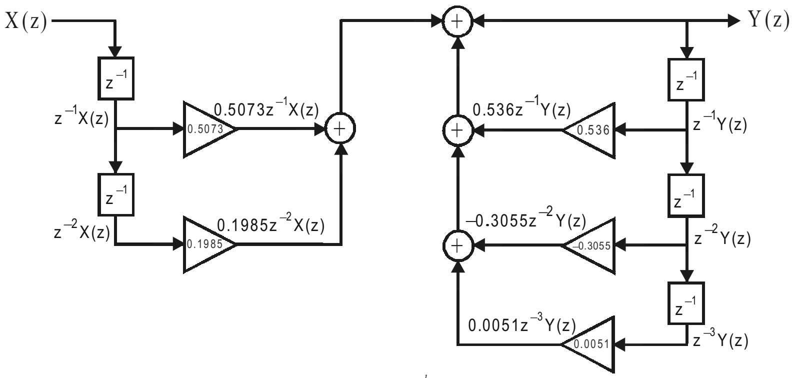


Fig 1 : *Direct form-I structure of 3rd order digital IIR lowpass filter.*

Direct form-II structure of digital IIR lowpass filter

$$\text{Let,}\ H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.5073\,z^{-1} + 0.1985z^{-2}}{1 - 0.536\,z^{-1} + 0.3055\,z^{-2} - 0.0551z^{-3}}$$

$$\text{where,}\ \frac{W(z)}{X(z)} = \frac{1}{1 - 0.536\,z^{-1} + 0.3055\,z^{-2} - 0.0551z^{-3}} \qquad .....(6)$$

$$\frac{Y(z)}{W(z)} = 0.5073\,z^{-1} + 0.1985\,z^{-2} \qquad .....(7)$$

On cross multiplying equation (6) we get,

$$W(z) - 0.536z^{-1}W(z) + 0.3055z^{-2}W(z) - 0.0551\,z^{-3}W(z) = X(z)$$

$$W(z) = X(z) + 0.536z^{-1}W(z) - 0.3055z^{-2}W(z) + 0.0551z^{-3}W(z) \qquad .....(8)$$

On cross multiplying equation (7) we get,

$$Y(z) = 0.5073z^{-1}W(z) + 0.1985z^{-2}W(z) \qquad .....(9)$$

Using equations (8) and (9), the direct form-II structure is drawn as shown in fig 2.

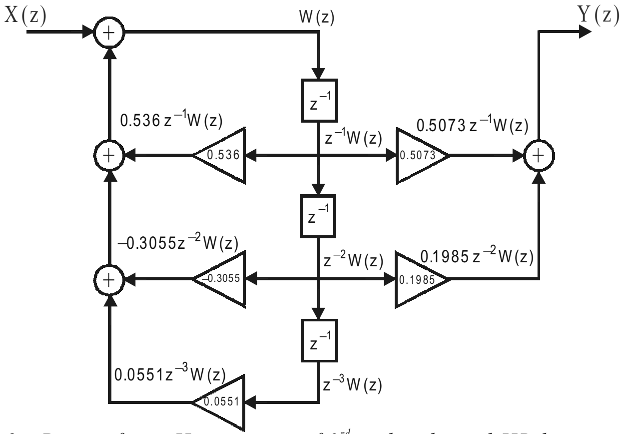


Fig 2 : Direct form-II structure of 3rd order digital IIR lowpass filter.

Frequency Response, $H(e^{jw})$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{0.5073z^{-1} + 0.1985z^{-2}}{1 - 0.536z^{-1} + 0.3055z^{-2} - 0.0551z^{-3}}\Bigg|_{z=e^{j\omega}}$$

$$= \frac{0.5073e^{-j\omega} + 0.1985e^{-j2\omega}}{1 - 0.536e^{-j\omega} + 0.3055e^{-j2\omega} - 0.0551e^{-j3\omega}}$$

$$\boxed{e^{-j\theta} = \cos\theta - j\sin\theta}$$

$$= \frac{0.5073(\cos\omega - j\sin\omega) + 0.1985(\cos 2\omega - j\sin 2\omega)}{1 - 0.536(\cos\omega - j\sin\omega) + 0.3055(\cos 2\omega - j\sin 2\omega) - 0.0551(\cos 3\omega - j\sin 3\omega)}$$

$$= \frac{(0.5073\cos\omega + 0.1985\cos\omega) + j(-0.5073\sin\omega - 0.1985\sin 2\omega)}{(1 - 0.536\cos\omega + 0.3055\cos 2\omega - 0.0551\cos 3\omega) + j(0.536\sin\omega - 0.3055\sin 2\omega + 0.0551\sin 3\omega)}$$

$$\text{Let, } H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} = \frac{(0.5073\cos + 0.1985\cos\omega) + j(-0.5073\sin\omega - 0.1985\sin 2\omega)}{\begin{gathered}(1 - 0.536\cos\omega + 0.3055\cos 2\omega - 0.0551\cos 3\omega)\\ + j(0.536\sin\omega - 0.3055\sin 2\omega + 0.0551\sin 3\omega)\end{gathered}}$$

where, $H_N(e^{jw}) = (0.5073\cos w + 0.1985\cos 2w) + j(-0.5073\sin w - 0.1985\sin 2w)$

$H_D(e^{jw}) = (1 - 0.536\cos w + 0.30554\cos 2w - 0.0551\cos 3w)$

$+ j(0.536\sin w - 0.3055\sin 2w + 0.0551\sin 3w)$

The frequency response $H(e^{jw})$ and hence the magnitude response $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the values listed in table 1, the magnitude response of lowpass filter is sketched as shown in fig 3.

Note : Verify the result with MATLAB program 7.6.

Table 1: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w

w	$H_N(e^{jw})$	$H_D(e^{jw})$	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	0.7058 + j0	0.7144 + j0	0.9880 + j0	0.9880
$\frac{1\times\pi}{16}$	0.6809 – j0.1749	0.7107 + j0.0183	0.9511 – j0.2706	0.9888
$\frac{2\times\pi}{16}$	0.6090 – j0.3345	0.6997 + j0.0400	0.8403 – j0.5261	0.9914
$\frac{3\times\pi}{16}$	0.4978 – j0.4652	0.6820 + j0.0696	0.6535 – j0.7488	0.9939
$\frac{4\times\pi}{16}$	0.3587 – j0.5572	0.6600 + j0.1125	0.3883 – j0.9104	0.9898
$\frac{5\times\pi}{16}$	0.2059 – j0.6052	0.6393 + j0.1742	0.0597 – j0.9629	0.9648
$\frac{6\times\pi}{16}$	0.0538 – j0.6090	0.6298 + j0.2581	–0.2662– j0.8579	0.8982
$\frac{7\times\pi}{16}$	–0.0844 – j0.5735	0.6438 + j0.3630	–0.4806 – j0.6198	0.7843
$\frac{8\times\pi}{16}$	–0.1985 – j0.5073	0.6945 + j0.4809	–0.5351 – j0.3500	0.6449
$\frac{9\times\pi}{16}$	– 0.2824 – j0.4216	0.7917 + j0.5968	–0.4834 – j0.1681	0.5118
$\frac{10\times\pi}{16}$	– 0.3345 – j0.3283	0.9382 + j0.6901	–0.3984 – j0.0569	0.4024
$\frac{11\times\pi}{16}$	– 0.3578 – j0.2384	1.1268 + j0.7387	–0.3191 – j0.0023	0.3191
$\frac{12\times\pi}{16}$	– 0.3587 – j0.1602	1.3400 + j0.7235	–0.2572 – j0.0193	0.2580
$\frac{13\times\pi}{16}$	– 0.3458 – j0.0985	1.5519 + j0.6341	–0.2132 – j0.0236	0.2145
$\frac{14\times\pi}{16}$	– 0.3283 – j0.0538	1.7323 + j0.4720	–0.1843 + j0.0192	0.1853
$\frac{15\times\pi}{16}$	– 0.3142 – j0.0230	1.8538 + j0.2521	–0.1681 + j0.0104	0.1684
$\frac{16\times\pi}{16}$	– 0.3088 + j0	1.8966 + j0	–0.1628 + j0	0.1628

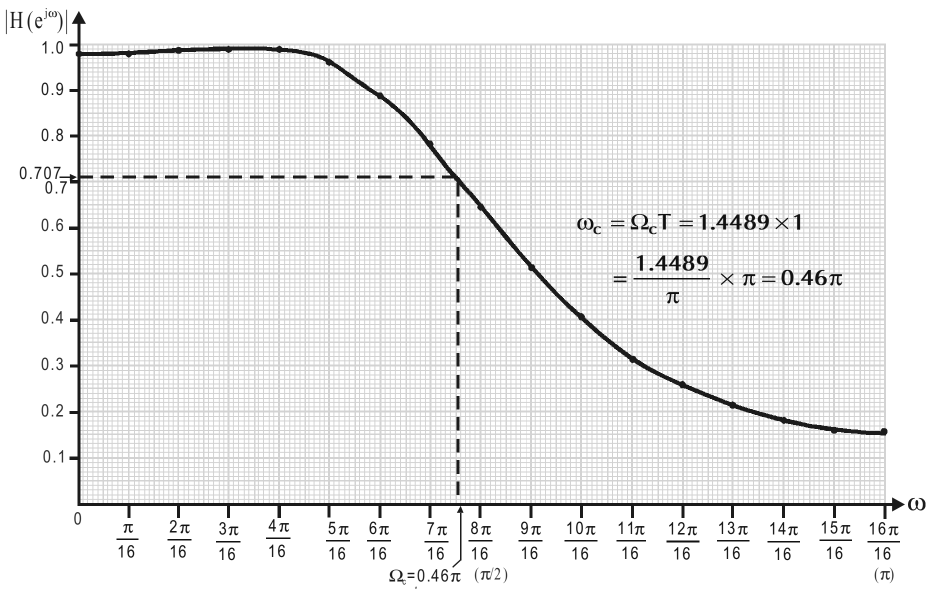


***Fig 3 :** Frequency response of 3^{rd} order digital Butterworth IIR lowpass filter.*

Example 7.21

Design a Butterworth digital IIR lowpass filter using impulse invariant transformation by taking T = 1second, to satisfy the following specifications.

$$0.8 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2 \quad ; \quad \text{for } 0.32\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter.

> Alternatively,
>
> Passband ripple ≤ 1.9 dB
>
> Stopband attenuation ≥ 13.97dB
>
> Passband edge frequency = 0.2π rad/sample
>
> Stopband edge frequency = 0.32π rad/sample
>
> The above specifications can be converted to A_p and A_s as shown below.
>
> $$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{(-1.9/20)} = 0.8$$
>
> $$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{(-13.97/20)} = 0.2$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.2\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.32\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.8$

Gain in normal value at stopband edge, $A_s = 0.2$

Sampling time, T = 1second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.8$

Gain in normal value at stopband edge, $A_s = 0.2$

Gain is same in analog and digital filter.

For impulse invariant transformation,

$$\text{Passband edge analog frequency, } \Omega_p = \frac{\omega_p}{T} = \frac{0.2\pi}{1} = 0.6283 \text{rad / second}$$

Using equation (7.55).

$$\text{Stopband edge analog frequency, } \Omega_s = \frac{\omega_s}{T} = \frac{0.32\pi}{1} = 1.0053 \text{ rad / second}$$

Using equation (7.56).

Order of the filter

$$N = \frac{1}{2}\,\frac{\log\left[\dfrac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]}{\log\dfrac{\Omega_s}{\Omega_p}} = \frac{1}{2}\,\frac{\log\left[\dfrac{\left(1/0.2^2\right)-1}{\left(1/0.8^2\right)-1}\right]}{\log\dfrac{1.0053}{0.6283}} = \frac{1}{2}\,\frac{\log\left[\dfrac{24}{0.5625}\right]}{\log\dfrac{1.0053}{0.6283}} = 3.9928$$

Using equation (7.57).

Choose order N, such that $N \ge N_1$ and N is an integer.

Let, order, N = 4.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For even N,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k\, s_n + 1}$$ Using equation (7.58).

where, $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$ Using equation (7.60).

Here, $N = 4$, $\therefore\ k = 1,\ 2$

$$\therefore\ H(s_n) = \frac{1}{s_n^2 + b_1\, s_n + 1} \times \frac{1}{s_n^2 + b_2\, s_n + 1}$$

When $k = 1$; $b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times 4}\right] = 0.7654$ Calculate sin q using calculator in radian mode.

When $k = 2$; $b_k = b_2 = 2\sin\left[\frac{(2\times 2-1)\pi}{2\times 4}\right] = 1.8478$

$$H(s_n) = \frac{1}{(s_n^2 + 0.7654\, s_n + 1)\ (s_n^2 + 1.8478\, s_n + 1)}$$

$$= \frac{1}{s_n^4 + 2.6132\, s_n^2 + 3.4143\, s_n^2 + 2.6132\, s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, W c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2) - 1\right]^{\frac{1}{2N}}} = \frac{1.0053}{\left[(1/0.2^2) - 1\right]^{\frac{1}{8}}} = 0.6757 \text{ rad / second}$$

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{(s_n^2 + 0.7654\, s_n + 1)(s_n^2 + 1.8478\, s_n + 1)}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$ Using equation (7.61).

$$= \frac{1}{\left(\frac{s^2}{\Omega_c^2} + 0.7654\frac{s}{\Omega_c} + 1\right)\left(\frac{s^2}{\Omega_c^2} + 1.8478\frac{s}{\Omega_c} + 1\right)} = \frac{1}{\left(\frac{s^2 + 0.7654\,\Omega_c\, s + \Omega_c^2}{\Omega_c^2}\right)\left(\frac{s^2 + 1.8478\,\Omega_c\, s + \Omega_c^2}{\Omega_c^2}\right)}$$

$$= \frac{\Omega_c^4}{(s^2 + 0.7654\,\Omega_c s + \Omega_c^2)(s^2 + 1.8478\,\Omega_c\, s + \Omega_c^2)} = \frac{0.6757^4}{(s^2 + 0.7654 \times 0.6757\, s + 0.6757^2)\ (s^2 + 1.8478 \times 0.6757\, s + 0.6757^2)}$$

$$= \frac{0.2085}{(s^2 + 0.5172 s + 0.4566)(s^2 + 1.2486\, s + 0.4566)} \qquad ....(1)$$

$$= \frac{0.2085}{s^4 + 1.2486\, s^3 + 0.4566\, s^2 + 0.5172 s^3 + 0.6458\, s^2 + 0.2362\, s + 0.4566\, s^2 + 0.5701 s + 0.2085}$$

$$= \frac{0.2085}{s^4 + 1.7658\, s^3 + 1.559\, s^2 + 0.8063\, s + 0.2085}$$

To convert the analog transfer function to digital transfer function using impulse invariant transformation, the equation (1) is simplified as follows.

The roots of the quadratic $s^2 + 0.5172s + 0.4566 = 0$ are,	The roots of the quadratic $s^2 + 1.2486s + 0.4566 = 0$ are,
$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= \frac{-0.5172 \pm \sqrt{0.5172^2 - 4 \times 1 \times 0.4566}}{2}$	$= \frac{-1.2486 \pm \sqrt{1.2486^2 - 4 \times 1 \times 0.4566}}{2}$
$= \frac{-0.5172 \pm j1.2486}{2} = -0.2586 \pm j0.6243$	$= \frac{-1.2486 \pm j0.5170}{2} = -0.6243 \pm j0.2586$
$= (s - (-0.2586 + j0.6243))(s - (-0.2586 - j0.6243))$	$= (s - (-0.6243 + j0.2586))(s - (-0.6243 - j0.2586))$
$= (s + 0.2586 - j0.6243)(s + 0.2586 + j0.6243)$	$= (s + 0.6243 - j0.2586)(s + 0.6243 + j0.2586)$

$$H(s) = \frac{0.2085}{(s^2 + 0.5172s + 0.4566)(s^2 + 1.2486s + 0.4566)}$$

$$= \frac{0.2085}{(s + 0.2586 - j0.6243)\ (s + 0.2586 + j0.6243)\ (s + 0.6243 - j0.2586)\ (s + 0.6243 + j0.2586)}$$

By partial fraction expansion H(s) can be expressed as,

$$H(s) = \frac{A_1}{(s + 0.2586 - j0.6243)} + \frac{A_1^*}{(s + 0.2586 + j0.6243)} + \frac{A_2}{(s + 0.6243 - j0.2586)} + \frac{A_2^*}{(s + 0.6243 + j0.2586)}$$

where, A_1, A_1^*, A_2, A_2^* are residues

$$A_1 = \left.\frac{0.2085 \times \cancel{(s + 0.2586 - j0.6243)}}{\cancel{(s + 0.2586 - j0.6243)}(s + 0.2586 + j0.6243)\ (s + 0.6243 - j0.2586)(s + 0.6243 + j0.2586)}\right|_{s = -0.2586 + j0.6243}$$

$$= \frac{0.2085}{(-0.2586 + j0.6243 + 0.2586 + j0.6243)\ (-0.2586 + j0.6243 + 0.6243 - j0.2586)\ (-0.2586 + j0.6243 + 0.6243 + j0.2586)}$$

$$= \frac{0.2085}{j1.2486(0.3657 + j0.3657)\ (0.3657 + j0.8829)} = -0.3121 + j0.1293$$

A_1^* = conjugate of $A_1 = -0.3121 - j0.1293$

$$A_2 = \left.\frac{0.2085 \times \cancel{(s + 0.6243 - j0.2586)}}{(s + 0.2586 - j0.6243)\ (s + 0.2586 + j0.6243)\cancel{(s + 0.6243 - j0.2586)}(s + 0.6243 + j0.2586))}\right|_{s = -0.6243 + j0.2586}$$

$$= \frac{0.2085}{(-0.6243 + j0.2586 + 0.2586 - j0.6243)\ (-0.6243 + j0.2586 + 0.2586 + j0.6243)\ (-0.6243 + j0.2586 + 0.6243 + j0.2586)}$$

$$= \frac{0.2085}{(-0.3657 - j0.3657)\ (-0.3657 + j0.8829)\ j0.5172} = 0.3121 - j0.7536$$

A_2^* = conjugate of $A_2 = 0.3121 + j0.7536$

$$\therefore\ H(s)=\frac{-0.3121+j0.1293}{s+0.2586-j0.6243}+\frac{-0.3121-j0.1293}{s+0.2586+j0.6243}$$
$$+\frac{0.3121-j0.7536}{s+0.6243-j0.2586}+\frac{0.3121+j0.7536}{s+0.6243+j0.2586}$$

Digital IIR lowpass filter transfer function, H(z)

For impulse invariant transformation,

$$\frac{A_i}{s+p_i}\longrightarrow\frac{A_i}{1-e^{-p_iT}\,z^{-1}}$$

Using equation (7.17).

Using the above transformation, the H(s) can be transformed to H(z) as shown below,

$$H(z)=\frac{-0.3121+j0.1293}{1-e^{-(0.2586-j0.6243)}\,z^{-1}}+\frac{-0.3121-j0.1293}{1-e^{-(0.2586+j0.6243)}\,z^{-1}}$$
$$+\frac{0.3121-j0.7536}{1-e^{-(0.6243-j0.2586)}z^{-1}}+\frac{0.3121+j0.7536}{1-e^{-(0.6243+j0.2586)}z^{-1}}$$

Put, T = 1

$$=\frac{(-0.3121+j0.1293)(1-e^{-0.2586-j0.6243}\,z^{-1})+(-0.3121-j0.1293)(1-e^{-0.2586+j0.6243}\,z^{-1})}{(1-e^{-0.2586+j0.6243}\,z^{-1})\,(1-e^{-0.2586-j0.6243}\,z^{-1})}$$
$$+\frac{(0.3121-j0.7536)(1-e^{-0.6243-j0.2586}z^{-1})+(0.3121+j0.7536)(1-e^{-0.6243+j0.2586}z^{-1})}{(1-e^{-0.6243+j0.2586}z^{-1})\,(1-e^{-0.6243-j0.2586}z^{-1})}$$

$$=\frac{\begin{array}{l}-0.3121+0.3121e^{-0.2586-j0.6243}\,z^{-1}+\cancel{j0.1293}-j0.1293\,e^{-0.2586-j0.6243}\,z^{-1}\\ \quad-0.3121+0.3121e^{-0.2586+j0.6243}\,z^{-1}-\cancel{j0.1293}+j0.1293\,e^{-0.2586+j0.6243}\,z^{-1}\end{array}}{1-e^{-0.2586-j0.6243}\,z^{-1}-e^{-0.2586+j0.6243}\,z^{-1}+e^{-2\times0.2586}\,z^{-2}}$$

$$+\frac{\begin{array}{l}0.3121-0.3121e^{-0.6243-j0.2586}\,z^{-1}-\cancel{j0.7536}+j0.7536\,e^{-0.6243-j0.2586}\,z^{-1}\\ \quad0.3121-0.3121e^{-0.6243+j0.2586}\,z^{-1}+\cancel{j0.7536}-j0.7536\,e^{-0.6243+j0.2586}\,z^{-1}\end{array}}{1-e^{-0.6243-j0.2586}\,z^{-1}-e^{-0.6243+j0.2586}\,z^{-1}+e^{-2\times0.6243}\,z^{-2}}$$

$$=\frac{-0.6242+0.3121e^{-0.2586}\left(e^{j0.6243}+e^{-j0.6243}\right)z^{-1}+j0.1293e^{-0.2586}\left(e^{j0.6243}-e^{-j0.6243}\right)z^{-1}}{1-e^{-0.2586}\,(e^{j0.6243}+e^{-j0.6243})z^{-1}+e^{-0.5172}\,z^{-2}}$$
$$+\frac{0.6242-0.3121e^{-0.6243}\left(e^{j0.2586}+e^{-j0.2586}\right)z^{-1}-j0.7536\,e^{-0.6243}\left(e^{j0.2586}-e^{-j0.2586}\right)z^{-1}}{1-e^{-0.6243}\,(e^{j0.2586}+e^{-j0.2586})z^{-1}+e^{-1.2486}\,z^{-2}}$$

$$=\frac{-0.6242+0.3121e^{-0.2586}\,(2\cos\ 0.6243)\,z^{-1}+j0.1293e^{-0.2586}\,(2\,j\sin0.6243)\,z^{-1}}{1-e^{-0.2586}\,(2\cos\ 0.6243)z^{-1}+e^{-0.5172}\,z^{-2}}$$
$$+\frac{0.6242-0.3121e^{-0.6243}\,(2\cos\ 0.2586)\,z^{-1}-j0.7536\,e^{-0.6243}\,(2\,j\sin0.2586)\,z^{-1}}{1-e^{-0.6243}\,(2\cos\ 0.2586)z^{-1}+e^{-1.2486}\,z^{-2}}$$

$$=\frac{-0.6242+0.3911z^{-1}-0.1167\,z^{-1}}{1-1.2530\,z^{-1}+0.5962\,z^{-2}}+\frac{0.6242-0.3232\,z^{-1}+0.2065\,z^{-1}}{1-1.0357\,z^{-1}+0.2869\,z^{-2}}$$

$$=\frac{-0.6242+0.2744\,z^{-1}}{1-1.2530\,z^{-1}+0.5962\,z^{-2}}+\frac{0.6242-0.1167\,z^{-1}}{1-1.0357\,z^{-1}+0.2869\,z^{-2}}$$

$$\therefore\ H(z) = \frac{-0.6242 + 0.2744\,z^{-1}\left(1 - 1.0357\,z^{-1} + 0.2869\,z^{-2}\right) + \left(0.6242 - 0.1167\,z^{-1}\right)\left(1 - 1.2530\,z^{-1} + 0.5962\,z^{-2}\right)}{\left(1 - 1.2530\,z^{-1} + 0.5962\,z^{-2}\right)\left(1 - 1.0357\,z^{-1} + 0.2869\,z^{-2}\right)}$$

$$= \frac{-0.6242 + 0.6465\,z^{-1} - 0.1791z^{-2} + 0.2744\,z^{-1} - 0.2842\,z^{-2} + 0.0787\,z^{-3} + 0.6242 - 0.7821z^{-1} + 0.3721z^{-2} - 0.1167\,z^{-1} + 0.1462\,z^{-2} - 0.0696\,z^{-3}}{1 - 1.0357\,z^{-1} + 0.2869\,z^{-2} - 1.2530\,z^{-1} + 1.2977\,z^{-2} - 0.3595\,z^{-3} + 0.5962\,z^{-2} - 0.6175\,z^{-3} + 0.171z^{-4}}$$

$$= \frac{0.0221z^{-1} + 0.055\,z^{-2} + 0.0091z^{-3}}{1 - 2.2887\,z^{-1} + 2.1808\,z^{-2} - 0.977\,z^{-3} + 0.171z^{-4}}$$

Alternatively,

$$H(z) = \frac{0.0221z^{-1} + 0.055\,z^{-2} + 0.0091z^{-3}}{1 - 2.2887\,z^{-1} + 2.1808\,z^{-2} - 0.977\,z^{-3} + 0.171z^{-4}}$$

$$= \frac{z^{-4}(0.0221z^{3} + 0.055\,z^{2} + 0.0091z)}{z^{-4}(z^{4} - 2.2887\,z^{3} + 2.1808\,z^{2} - 0.977z + 0.171)}$$

$$= \frac{0.0221z^{3} + 0.055\,z^{2} + 0.0091}{z^{4} - 2.2887\,z^{3} + 2.1808\,z^{2} - 0.977z + 0.171}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{0.0221z^{-1} + 0.055\,z^{-2} + 0.0091z^{-3}}{1 - 2.2887\,z^{-1} + 2.1808\,z^{-2} - 0.977\,z^{-3} + 0.171z^{-4}}$$

On cross multiplying the above equation we get,

$$Y(z) - 2.2887z^{-1}Y(z) + 2.1808z^{-2}Y(z) - 0.977z^{-3}Y(z) + 0.171z^{-4}Y(z)$$
$$= 0.0221z^{-1}X(z) + 0.055z^{-2}X(z) + 0.0091z^{-3}X(z)$$

$$\therefore\ Y(z) = 0.0221z^{-1}X(z) + 0.055z^{-2}X(z) + 0.0091z^{-3}X(z)$$
$$+ 2.2887z^{-1}Y(z) - 2.1808z^{-2}Y(z) + 0.977z^{-3}Y(z) - 0.171z^{-4}Y(z) \qquad .....(2)$$

Using equation (2), the direct form-I structure is drawn as shown in fig 1.

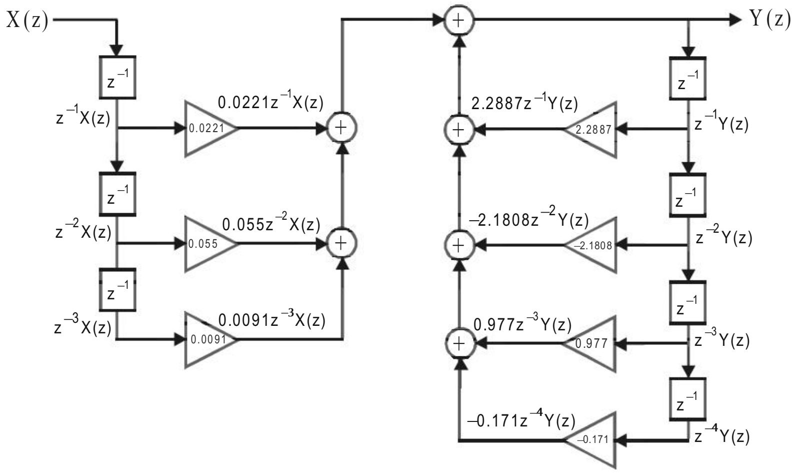


Fig 1 : Direct form-I structure of 4th order digital IIR lowpass filter.

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.0221z^{-1} + 0.055\,z^{-2} + 0.0091z^{-3}}{1 - 2.2887\,z^{-1} + 2.1808\,z^{-2} - 0.977\,z^{-3} + 0.171z^{-4}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1 - 2.2887\,z^{-1} + 2.1808\,z^{-2} - 0.977\,z^{-3} + 0.171z^{-4}} \qquad .....(3)$$

$$\frac{Y(z)}{W(z)} = 0.0221z^{-1} + 0.055\,z^{-2} + 0.0091z^{-3} \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$W(z) - 2.2887z^{-1}W(z) + 2.1808z^{-2}W(z) - 0.977z^{-3}W(z) + 0.171z^{-4}W(z) = X(z)$$

$$\therefore\; W(z) = X(z) + 2.2887z^{-1}W(z) - 2.1808z^{-2}W(z) + 0.977z^{-3}W(z) - 0.171z^{-4}W(z) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = 0.0221z^{-1}W(z) + 0.055z^{-2}W(z) + 0.0091z^{-3}W(z) \qquad .....(6)$$

Using equations (5) and (6), the direct form-II structure is drawn as shown in fig 2.

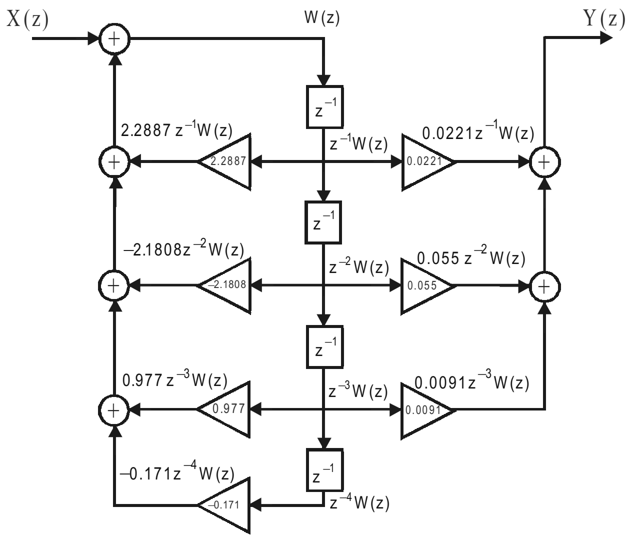


Fig 2 : Direct form-II structure of 4rd order digital IIR lowpass filter.

Note : Verify the result with MATLAB program 7.7.

Alternate method for transforming H(s) to H(z)

By partial fraction expansion H(s) can be expressed as,

$$H(s) = \frac{0.2085}{(s^2 + 0.5172s + 0.4566)\,(s^2 + 1.2486\,s + 0.4566)}$$

$$= \frac{As + B}{(s^2 + 0.5172s + 0.4566)} + \frac{Cs + D}{(s^2 + 1.2486\,s + 0.4566)} \qquad .....(7)$$

On cross multiplying equation (7) we get,

$$0.2085 = (As + B)\,(s^2 + 1.2486s + 0.4566) + (Cs + D)\,(s^2 + 0.5172s + 0.4566)$$

$$= As^3 + 1.2486\,As^2 + 0.4566\,As + Bs^2 + 1.2486\,Bs + 0.4566B$$

$$+ Cs^3 + 0.5172\,Cs^2 + 0.4566\,Cs + Ds^2 + 0.5172\,Ds + 0.4566\,D \qquad .....(8)$$

On equating the coefficients of s^3 in equation (8) we get,

$$A + C = 0$$

$$\therefore C = -A$$

On equating the coefficients of s^2 in equation (8) we get,

$$1.2486\ A + B + 0.5172\ C + D = 0$$

Put, $C = -A$

$$\therefore\ 1.2486A + B - 0.5172A + D = 0$$

$$\therefore\ 0.7314\ A + B + D = 0$$

$$\therefore\ D = -0.7314\ A - B$$

On equating coefficients of s in equation (8) we get,

$$0.4566\ A + 1.2486B + 0.4566C + 0.5172\ D = 0$$

Put, $C = -A$

and, $D = -0.7314\ A - B$

$$0.4566\ A + 1.2486\ B - 0.4566\ A + 0.5172\ (-0.7314\ A - B) = 0$$

$$-0.3783\ A + 0.7314\ B = 0$$

$$\therefore\ B = \frac{0.3783}{0.7314} A = 0.5172A$$

On equating constants of equation (8) we get,

$$0.4566\ B + 0.4566\ D = 0.2085$$

Put, $D = -0.7314\ A - B$

$$\therefore\ 0.4566\ B + 0.4566\ (-0.7314\ A - B) = 0.2085 \quad \Rightarrow \quad -0.334A = 0.085$$

$$\therefore\ A = -\frac{0.2085}{0.334} = -0.6243$$

Here, $A = -0.6243$,

$$\therefore\ B = 0.5172A = 0.5172 \times (-0.6243) = -0.3229$$

$$C = -A = 0.6243$$

$$D = -0.7314\ A - B = -0.7314\ (-0.6243) - (-0.3229) = 0.7795$$

Now, H(s) can be written as,

$$H(s) = \frac{-0.6243s - 0.3229}{(s^2 + 0.5172s + 0.4566)} + \frac{0.6243s + 0.7795}{(s^2 + 1.2486\,s + 0.4566)}$$

$$= \frac{-0.6243\left(s + \dfrac{0.3229}{0.6243}\right)}{(s^2 + 2 \times 0.2586s + 0.2586^2) + \left(\sqrt{0.4566 - 0.2586^2}\right)^2}$$

$$\boxed{(s + a_1)^2 = s^2 + 2a_1 s + a_1^2}$$

$$\boxed{2a_1 = 0.5172 \Rightarrow a_1 = \frac{0.5172}{2} = 0.2586}$$

$$+ \frac{0.6243\left(s + \dfrac{0.7795}{0.6243}\right)}{(s^2 + 2 \times 0.6243s + 0.6243^2) + \left(\sqrt{0.4566 - 0.6243^2}\right)^2}$$

$$\boxed{(s + a_2)^2 = s^2 + 2a_2 s + a_2^2}$$

$$\boxed{2a_2 = 1.2486 \Rightarrow a_2 = \frac{1.2486}{2} = 0.6243}$$

$$= \frac{-0.6243(s + 0.5172)}{(s + 0.2586)^2 + 0.6243^2} + \frac{0.6243(s + 1.2486)}{(s + 0.6243)^2 + 0.2586^2}$$

$$= \frac{-0.6243(s + 0.2586 + 0.2586)}{(s + 0.2586)^2 + 0.6243^2} + \frac{0.6243(s + 0.6243 + 0.6243)}{(s + 0.6243)^2 + 0.2586^2}$$

$$= -0.6243\ \frac{(s + 0.2586)}{(s + 0.2586)^2 + 0.6243^2} - 0.2586\ \frac{0.6243}{(s + 0.2586)^2 + 0.6243^2}$$

$$+ 0.6243\ \frac{(s + 0.6243)}{(s + 0.6243)^2 + 0.2586^2} + \frac{0.6243 \times 0.6243}{0.2586}\ \frac{0.2586}{(s + 0.6243)^2 + 0.2586^2}$$

$$\therefore \; H(s) = -0.6243 \, \frac{(s + 0.2586)}{(s+0.2586)^2 + 0.6243^2} - 0.2586 \, \frac{0.6243}{(s + 0.2586)^2 + 0.6243^2}$$

$$+ 0.6243 \, \frac{(s+0.6243)}{(s+0.6243)^2 + 0.2586^2} + 1.5072 \, \frac{0.2586}{(s+0.6243)^2 + 0.2586^2}$$

Digital IIR lowpass filter transfer function, H(z)

In impulse invariant transformation,

$$\frac{(s+a)}{(s+a)^2 + b^2} \xrightarrow{\text{is transformed to}} \frac{1 - e^{-aT}(\cos bT)\, z^{-1}}{1 - 2\, e^{-aT}(\cos bT)\, z^{-1} + e^{-2aT}\, z^{-2}}$$

$$\frac{b}{(s+a)^2 + b^2} \xrightarrow{\text{is transformed to}} \frac{e^{-aT}(\sin bT)\, z^{-1}}{1 - 2\, e^{-aT}(\cos bT)\, z^{-1} + e^{-2aT}\, z^{-2}}$$

Using equations (7.18) and (7.19).

Using the above transformation, the H(s) can be transformed to H(z) as shown below.

Put, T = 1

$$\therefore \; H(z) = -0.6243 \, \frac{1 - e^{-0.2586}(\cos 0.6243)\, z^{-1}}{1 - 2\, e^{-0.2586}(\cos 0.6243)\, z^{-1} + e^{-2 \times 0.2586}\, z^{-2}}$$

$$- 0.2586 \, \frac{e^{-0.2586}(\sin 0.6243)\, z^{-1}}{1 - 2\, e^{-0.2586}(\cos 0.6243)\, z^{-1} + e^{-2 \times 0.2586}\, z^{-2}}$$

$$+ 0.6243 \, \frac{1 - e^{-0.6243}(\cos 0.2586)\, z^{-1}}{1 - 2\, e^{-0.6243}(\cos 0.2586)\, z^{-1} + e^{-2 \times 0.6243}\, z^{-2}}$$

$$+ 1.5072 \, \frac{e^{-0.6243}(\sin 0.2586)\, z^{-1}}{1 - 2e^{-0.6243}(\cos 0.2586)\, z^{-1} + e^{-2 \times 0.6243}\, z^{-2}}$$

$$= \frac{-0.6243 + 0.3911z^{-1}}{1 - 1.2530z^{-1} + 0.5962z^{-2}} + \frac{-0.1167z^{-1}}{1 - 1.2530z^{-1} + 0.5962z^{-2}}$$

$$+ \frac{0.6243 - 0.3233z^{-1}}{1 - 1.0357z^{-1} + 0.2869z^{-2}} + \frac{0.2065z^{-1}}{1 - 1.0357z^{-1} + 0.2869z^{-2}}$$

$$= \frac{-0.6243 + 0.2744z^{-1}}{1 - 1.2530z^{-1} + 0.5962z^{-2}} + \frac{0.6243 - 0.1168z^{-1}}{1 - 1.0357z^{-1} + 0.2869z^{-2}}$$

$$= \frac{(-0.6243 + 0.2744\,z^{-1})(1 - 1.0357z^{-1} + 0.2869z^{-2}) + \left(0.6243 - 0.1168z^{-1}\right)\left(1 - 1.2530z^{-1} + 0.5962z^{-2}\right)}{\left(1 - 1.2530z^{-1} + 0.5962z^{-2}\right)\left(1 - 1.0357z^{-1} + 0.2869z^{-2}\right)}$$

$$= \frac{\begin{matrix}-0.6243 + 0.6466z^{-1} - 0.1791z^{-2} + 0.2744z^{-1} - 0.2842z^{-2} + 0.0787z^{-3} \\ + 0.6243 - 0.7822z^{-1} + 0.3722z^{-2} - 0.1168z^{-1} + 0.1464z^{-2} - 0.0696z^{-3}\end{matrix}}{\begin{matrix}1 - 1.0357z^{-1} + 0.2869z^{-2} - 1.2530z^{-1} + 1.2977z^{-2} - 0.3595z^{-3} \\ + 0.5962z^{-2} - 0.6175z^{-3} + 0.171z^{-4}\end{matrix}}$$

$$H(z) = \frac{0.022z^{-1} + 0.0554z^{2} + 0.0091z^{-3}}{1 - 2.2887z^{-1} + 2.1808z^{-2} - 0.977z^{-3} + 0.171z^{-4}}$$

Note : The H(z) obtained by both the methods are same. The small difference in the coefficients are due to the corrections (or rounding) made in calculations.

Example 7.22

Design a Butterworth digital IIR lowpass filter using bilinear transformation by taking T = 1second, to satisfy the following specifications.

$$0.707 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.08 \quad ; \quad 0.4\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter.

> Alternatively,
>
> Passband ripple $\le$ 3.0116dB
>
> Stopband attenuation $\ge$ 21.9382dB
>
> Passband edge frequency = 0.2π rad/sample
>
> Stopband edge frequency = 0.4π rad/sample
>
> The above specifications can be converted to A_p and A_s as shown below.
>
> $$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{(-3.0116/20)} = 0.707$$
>
> $$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{(-21.9382/20)} = 0.08$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.2\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.4\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.707$

Gain in normal value at stopband edge, $A_s = 0.08$

Sampling time, T = 1second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.707$

Gain in normal value at stopband edge, $A_s = 0.08$

Gain is same in analog and digital filter.

For bilinear transformation,

Passband edge analog frequency, $$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = \frac{2}{1}\tan\frac{0.2\pi}{2}$$
$$= 0.6498 \text{ rad / second}$$

Using equation (7.53).

Stopband edge analog frequency, $$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = \frac{2}{1}\tan\frac{0.4\pi}{2}$$
$$= 1.4531 \text{ rad / second}$$

Using equation (7.54).

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\frac{\left(1/A_s^2\right)-1}{\left(1/A_s^2\right)-1}\right]}{\log\frac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\frac{\left(1/0.08^2\right)-1}{\left(1/0.707^2\right)-1}\right]}{\log\frac{1.4531}{0.6498}}$$

Using equation (7.57).

$$= \frac{1}{2}\frac{\log\left[\frac{155.25}{1.0006}\right]}{\log\frac{1.4531}{0.6498}} = 3.1341$$

Choose order N_1 such that $N \geq N_1$ and N is an integer.

Let, order, N = 4.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For even N,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k\, s_n + 1}$$

Using equation (7.58).

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Using equation (7.60).

Here, N = 2, $\therefore$ k = 1, 2

$$\text{When } k = 1,\ b_k = b_1 = 2\sin\left[\frac{(2\times 1-1)\pi}{2\times 4}\right] = 0.7654$$

Calculate sin θ using calculator in radian mode.

$$\text{When } k = 2,\ b_k = b_2 = 2\sin\left[\frac{(2\times 2-1)\pi}{2\times 4}\right] = 1.8478$$

$$\therefore\ H(s_n) = \frac{1}{(s_n^2 + 0.7654\, s_n + 1)\ (s_n^2 + 1.8478\, s_n + 1)}$$

$$= \frac{1}{s_n^4 + 2.6132\, s_n^3 + 3.4143\, s_n^2 + 2.6132\, s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[\left(1/A_s^2\right)-1\right]^{\frac{1}{2N}}} = \frac{1.4531}{\left(\frac{1}{0.08^2}-1\right)^{\frac{1}{2\times 4}}} = 0.7734 \text{ rad / second}$$

Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{s_n^4 + 2.6132\, s_n^3 + 3.4143\, s_n^2 + 2.6132\, s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$\therefore\ H(s) = \frac{1}{\left(\frac{s}{\Omega_c}\right)^4 + 2.6132\left(\frac{s}{\Omega_c}\right)^3 + 3.4143\left(\frac{s}{\Omega_c}\right)^2 + 2.6132\left(\frac{s}{\Omega_c}\right) + 1}$$

$$= \frac{1}{\frac{s^4}{\Omega_c^4} + 2.6132\frac{s^3}{\Omega_c^3} + 3.4143\frac{s^2}{\Omega_c^2} + 2.6132\frac{s}{\Omega_c} + 1}$$

$$= \frac{\Omega_c^4}{s^4 + 2.6132\Omega_c\, s^3 + 3.4143\Omega_c^2\, s^2 + 2.6132\Omega_c^3\, s + \Omega_c^4}$$

$$= \frac{0.7734^4}{s^4 + 2.6132 \times 0.7734 s^3 + 3.4143 \times 0.7734^2 s^2 + 2.6132 \times 0.7734^3 s + 0.7734^4}$$

$$= \frac{0.3578}{s^4 + 2.021 s^3 + 2.0423 s^2 + 1.2089 s + 0.3578}$$

Digital IIR lowpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{0.3578}{s^4 + 2.021 s^3 + 2.0423 s^2 + 1.2089 s + 0.3578}\Bigg|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{0.3578}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^4 + 2.021\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + 2.0423\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.2089\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.3578}$$

$$= \frac{0.3578}{\dfrac{16(1-z^{-1})^4 + 2.021 \times 8T(1-z^{-1})^3(1+z^{-1}) + 2.0423 \times 4T^2(1-z^{-1})^2(1+z^{-1})^2 + 1.2089 \times 2T^3(1-z^{-1})(1+z^{-1})^3 + 0.3578 \times T^4(1+z^{-1})^4}{T^4(1+z^{-1})^4}}$$

$\boxed{\text{Put, } T = 1}$

$$= \frac{0.3578(1+z^{-1})^4}{16(1-z^{-1})^2(1-z^{-1})^2 + 16.168(1-3z^{-1}+3z^{-2}-z^{-3})(1+z^{-1}) + 8.1692(1-z^{-1})^2(1+z^{-1})^2 + 2.4178(1-z^{-1})(1+3z^{-1}+3z^{-2}+z^{-3}) + 0.3578(1+z^{-1})^2(1+z^{-1})^2}$$

$$= \frac{0.3578(1+2z^{-1}+z^{-2})(1+2z^{-1}+z^{-2})}{\begin{array}{l}16(1-2z^{-1}+z^{-2})(1-2z^{-1}+z^{-2}) + 16.168(1-3z^{-1}+3z^{-2}-z^{-3}+z^{-1}-3z^{-2}+3z^{-3}-z^{-4}) \\ +8.1692(1-2z^{-1}+z^{-2})(1+2z^{-1}+z^{-2}) + 2.4178(1+3z^{-1}+3z^{-2}+z^{-3}-z^{-1}-3z^{-2}-3z^{-3}-z^{-4}) \\ +0.3578(1+2z^{-1}+z^{-2})(1+2z^{-1}+z^{-2})\end{array}}$$

$$= \frac{0.3578(1+4z^{-1}+6z^{-2}+4z^{-3}+z^{-4})}{\begin{array}{l}16(1-4z^{-1}+6z^{-2}-4z^{-3}+z^{-4}) + 16.168(1-2z^{-1}+2z^{-3}-z^{-4}) + 8.1692(1-2z^{-2}+z^{-4}) \\ +2.4178(1+2z^{-1}-2z^{-3}-z^{-4}) + 0.3578(1+4z^{-1}+6z^{-2}+4z^{-3}+z^{-4})\end{array}}$$

$$= \frac{0.3578 + 1.4312z^{-1} + 2.1468z^{-2} + 1.4312z^{-3} + 0.3578z^{-4}}{43.1128 - 90.0388z^{-1} + 81.8084z^{-2} - 35.0684z^{-3} + 5.7412z^{-4}}$$

$$\therefore\ H(z)=\frac{\dfrac{0.3578}{43.1128}+\dfrac{1.4312}{43.1128}z^{-1}+\dfrac{2.1468}{43.1128}z^{-2}+\dfrac{1.4312}{43.1128}z^{-3}+\dfrac{0.3578}{43.1128}z^{-4}}{1-\dfrac{90.0388}{43.1128}z^{-1}+\dfrac{81.8084}{43.1128}z^{-2}-\dfrac{35.0684}{43.1128}z^{-3}+\dfrac{5.7412}{43.1128}z^{-4}}$$

$$=\frac{0.0083+0.0332\,z^{-1}+0.0498\,z^{-2}+0.0332\,z^{-3}+0.0083\,z^{-4}}{1-2.0892\,z^{-1}+1.8975\,z^{-2}-0.8133\,z^{-3}+0.1378\,z^{-4}}$$

Alternatively,

$$H(z)=\frac{0.0083+0.0332\,z^{-1}+0.0498\,z^{-2}+0.0332\,z^{-3}+0.0083\,z^{-4}}{1-2.0892\,z^{-1}+1.8975\,z^{-2}-0.8133\,z^{-3}+0.1378\,z^{-4}}$$

$$=\frac{z^{-4}(0.0083z^{4}+0.0332\,z^{3}+0.0498\,z^{2}+0.0332\,z+0.0083)}{z^{-4}(z^{4}-2.0892\,z^{3}+1.8975\,z^{2}-0.8133\,z+0.1378)}$$

$$=\frac{0.0083z^{4}+0.0332\,z^{3}+0.0498\,z^{2}+0.0332\,z+0.0083}{z^{4}-2.0892\,z^{3}+1.8975\,z^{2}-0.8133\,z+0.1378}$$

Note: Verify the result with MATLAB program 7.8.

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=\frac{0.0083+0.0332\,z^{-1}+0.0498\,z^{-2}+0.0332\,z^{-3}+0.0083\,z^{-4}}{1-2.0892\,z^{-1}+1.8975\,z^{-2}-0.8133\,z^{-3}+0.1378\,z^{-4}}$$

On cross multiplying the above equation we get,

$$Y(z)-2.0892z^{-1}Y(z)+1.8975z^{-2}Y(z)-0.8133\,z^{-3}Y(z)+0.1378\,z^{-4}Y(z)$$

$$=\ 0.0083X(z)+0.0332z^{-1}X(z)+0.0498z^{-2}X(z)+0.0332z^{-3}X(z)+0.0083z^{-4}X(z)$$

$$\therefore\ Y(z)=0.0083X(z)+0.0332z^{-1}X(z)+0.0498z^{-2}X(z)+0.0332z^{-3}X(z)+0.0083z^{-4}X(z)$$

$$+\ 2.0892z^{-1}Y(z)-1.8975z^{-2}Y(z)+0.8133\,z^{-3}Y(z)-0.1378\,z^{-4}Y(z) \qquad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

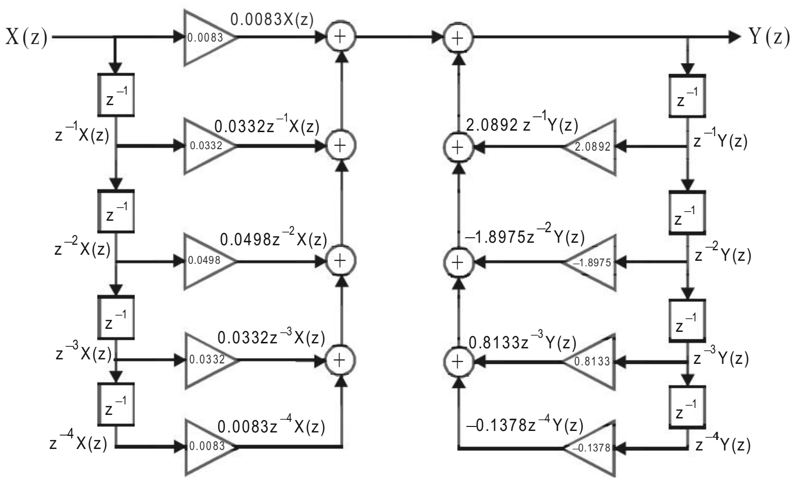


Fig 1 : *Direct form-I structure of 4^{th} order digital IIR lowpass filter..*

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.0083 + 0.0332\,z^{-1} + 0.0498\,z^{-2} + 0.0332\,z^{-3} + 0.0083\,z^{-4}}{1 - 2.0892\,z^{-1} + 1.8975\,z^{-2} - 0.8133\,z^{-3} + 0.1378\,z^{-4}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1 - 2.0892\,z^{-1} + 1.8975\,z^{-2} - 0.8133\,z^{-3} + 0.1378\,z^{-4}} \qquad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.0083 + 0.0332\,z^{-1} + 0.0498\,z^{-2} + 0.0332\,z^{-3} + 0.0083\,z^{-4} \qquad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) - 2.0892z^{-1}W(z) + 1.8975z^{-2}W(z) - 0.8133\,z^{-3}W(z) + 0.1378\,z^{-4}W(z) = X(z)$$

$$\therefore\; W(z) = X(z) + 2.0892z^{-1}W(z) - 1.8975z^{-2}W(z) + 0.8133\,z^{-3}W(z) - 0.1378\,z^{-4}W(z) \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$Y(z) = 0.0083W(z) + 0.0332z^{-1}W(z) + 0.0498z^{-2}W(z) + 0.0332z^{-3}W(z) + 0.0083z^{-4}W(z) \quad .....(5)$$

Using equations (4) and (5), the direct form-II structure is drawn as shown in fig 2.

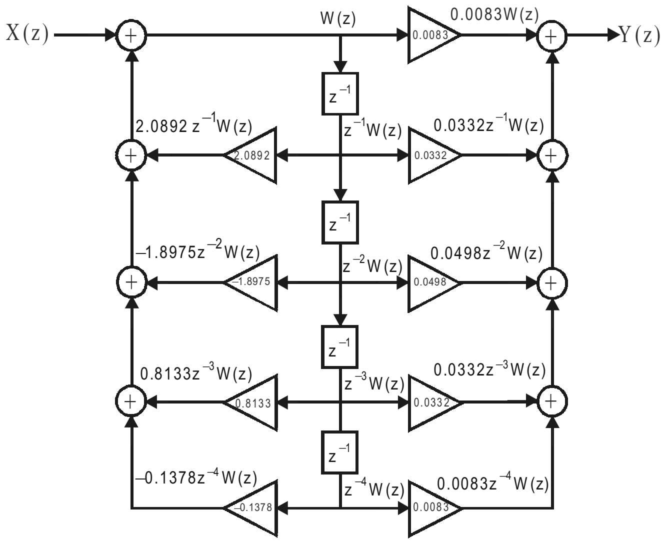


Fig 2 : *Direct form-II structure of 4^th order digital IIR lowpass filter.*

Example 7.23

Design a Chebyshev digital IIR lowpass filter using impulse invariant transformation by taking T = 1second, to satisfy the following specifications.

$$0.9 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad 0 \le \omega \le 0.25\pi$$

$$|H(e^{j\omega})| \le 0.24 \quad ; \quad 0.5\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter.

Alternatively,

Passband ripple $\le 0.9151\,\text{dB}$

Stopband attenuation $\ge 12.3958\,\text{dB}$

Passband edge frequency = 0.25π rad/sample

Stopband edge frequency = 0.5π rad/sample

The above specifications can be converted to A_p and A_s as shown below.

$$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{\left(-0.9151/20\right)} = 0.9$$

$$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{\left(-12.3958/20\right)} = 0.24$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.25\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.5\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.9$

Gain in normal value at stopband edge, $A_s = 0.24$

Sampling time, $T = 1$ second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.9$

Gain in normal value at stopband edge, $A_s = 0.24$

Gain is same in analog and digital filter.

For impulse invariant transformation,

Passband edge analog frequency, $\Omega_p = \dfrac{\omega_p}{T} = \dfrac{0.25\pi}{1} = 0.7854$ rad / second

Using equation (7.85).

Stopband edge analog frequency, $\Omega_s = \dfrac{\omega_s}{T} = \dfrac{0.5\pi}{1} = 1.5708$ rad / second

Using equation (7.86).

Order of the filter

$$N_1 = \frac{\cosh^{-1}\left[\dfrac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]^{\frac{1}{2}}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1}\left[\dfrac{\left(1/0.24^2\right)-1}{\left(1/0.9^2\right)-1}\right]^{\frac{1}{2}}}{\cosh^{-1}\dfrac{1.5708}{0.7854}}$$

Using equation (7.87).

$$= \frac{\cosh^{-1}\left[\dfrac{16.3611}{0.2346}\right]^{\frac{1}{2}}}{\cosh^{-1}\dfrac{1.5708}{0.7854}} = 2.1077$$

Choose order N_1 such that $N \ge N_1$ and N is an integer.

Let, order, N = 3.

Normalized transfer function, $H(s_n)$ of Chebyshev lowpass filter

For odd N,

$$H(s_n) = \frac{B_0}{s_n + c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s_n^2 + b_k s_n + c_k}$$ Using equation (7.89).

Here, $N = 3$, $\therefore\ k = \frac{N-1}{2} = \frac{3-1}{2} = 1$

$$\therefore\ H(s_n) = \frac{B_0}{s_n + c_0} \times \frac{B_1}{s_n^2 + b_1 s_n + c_1}$$

$$\in = \left[\left(1/A_p^2\right) - 1\right]^{\frac{1}{2}}$$ Using equation (7.94).

$$= \left[\left(1/0.9^2\right) - 1\right]^{\frac{1}{2}} = 0.4843$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\in^2}+1\right)^{\frac{1}{2}} + \frac{1}{\in}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\in^2}+1\right)^{\frac{1}{2}} + \frac{1}{\in}\right]^{-\frac{1}{N}}\right\}$$ Using equation (7.93).

$$= \frac{1}{2}\left[\left(\frac{1}{0.4843^2}+1\right)^{\frac{1}{2}} + \frac{1}{0.4843^2}\right]^{\frac{1}{3}} - \left[\left(\frac{1}{0.4843^2}+1\right)^{\frac{1}{2}} + \frac{1}{0.4843^2}\right]^{-\frac{1}{3}}$$

$$= \frac{1}{2}\ [1.6335 - 0.6122] = 0.5107$$

$$c_0 = y_N = 0.5107$$ Using equation (7.94).

$$b_k = 2\ y_N \sin\left[\frac{(2k-1)\pi}{2N}\right]$$ Using equation (7.90).

When $k = 1,\ b_k = b_1 = 2 \times 0.5107 \sin\left[\frac{(2\times 1 - 1)\pi}{2\times 3}\right] = 0.5107$

$$c_k = y_N^2 + \cos^2\left[\frac{(2k-1)\pi}{2N}\right]$$ Using equation (7.91).

When $k = 1,\ c_k = c_1 = 0.5107^2 + \cos^2\left[\frac{(2\times 1 - 1)\pi}{2\times 3}\right]$

$$= 0.5107^2 + \cos^2\left(\frac{\pi}{6}\right) = 0.5107^2 + \left[\frac{1+\cos\left(\frac{2\pi}{6}\right)}{2}\right]$$ $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= 0.2608 + 0.75 = 1.0108$$

$$\therefore\ H(s_n) = \frac{B_0}{s_n + c_0} \times \frac{B_1}{s_n^2 + b_1 s_n + c_1} = \frac{B_0}{s_n + 0.5107} \times \frac{B_1}{s_n^2 + 0.5107\ s_n + 1.0108}$$

To evaluate B_0 and B_1 , let, $H(s_n)\Big|_{s_n=0} = 1.$

When $s_n = 0$, $H(s_n) = \dfrac{B_0\,B_1}{(0.5107)\,(1.0108)} = 1.9372 B_0\,B_1$

$$\therefore\ 1.9372\ B_0\,B_1 = 1 \quad \Rightarrow \quad B_0\,B_1 = \frac{1}{1.9372} = 0.5162$$

Let, $B_0 = B_1$; $\therefore\ B_0^2 = 0.5162 \quad \Rightarrow \quad B_0 = \sqrt{0.5162} = 0.7185$

$$\therefore\ B_1 = B_0 = 0.7185$$

$$\therefore\ H(s_n) = \frac{B_0}{s_n + 0.5107} \times \frac{B_1}{(s_n^2 + 0.5107\,s_n + 1.0108)} = \frac{0.7185}{(s_n + 0.5107)} \times \frac{0.7185}{(s_n^2 + 0.5107\,s_n + 1.0108)}$$

$$= \frac{0.5162}{(s_n + 0.5107)(s_n^2 + 0.5107\,s_n + 1.0108)}$$

$$= \frac{0.5162}{s_n^3 + 1.0214\,s_n^2 + 1.2716\,s_n + 0.5162}$$

Unnormalized transfer function, H(s) of Chebyshev lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

Here, $\Omega_c = \Omega_p = 0.7854$ rad/sec.

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{0.5162}{(s_n + 0.5107)(s_n^2 + 0.5107\,s_n + 1.0108)}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{0.5162}{\left(\dfrac{s}{\Omega_c} + 0.5107\right)\left(\dfrac{s^2}{\Omega_c^2} + 0.5107\,\dfrac{s}{\Omega_c} + 1.0108\right)}$$

$$= \frac{0.5162}{\left(\dfrac{s + 0.5107\,\Omega_c}{\Omega_c}\right)\left(\dfrac{s^2 + 0.5107\,\Omega_c + 1.0108\,\Omega_c^2}{\Omega_c^2}\right)}$$

$$= \frac{0.5162\,\Omega_c^3}{(s + 0.5107\,\Omega_c)(s^2 + 0.5107\,\Omega_c\,s + 1.0108\,\Omega_c^2)}$$

$$= \frac{0.5162 \times 0.7854^3}{(s + 0.5107 \times 0.7854)(s^2 + 0.5107 \times 0.7854 s + 1.0108 \times 0.7854^2)}$$

$$= \frac{0.2501}{(s + 0.4011)\,(s^2 + 0.4011 s + 0.6235)} \qquad .....(1)$$

$$= \frac{0.2501}{s^3 + 0.8022\,s^2 + 0.7844\,s + 0.2501)}$$

To convert the analog transfer function to digital transfer function using impulse invariant transformation, the equation (1) is simplified as shown below.

By partial fraction expansion, the H(s) can be expressed as,

$$H(s) = \frac{0.2501}{(s+0.4011)(s^2+0.4011s+0.6235)} = \frac{A}{s+0.4011} + \frac{Bs+C}{s^2+0.4011s+0.6235} \quad .....(2)$$

On cross multiplying the equation (1) we get

$$0.2501 = A(s^2 + 0.4011s + 0.6235) + (Bs + C)(s + 0.4011)$$

$$0.2501 = As^2 + 0.4011\ As + 0.6235\ A + Bs^2 + 0.4011\ Bs + Cs + 0.4011\ C \quad .....(3)$$

On equating the coefficients of s^2 in equation (3) we get,

$$A + B = 0$$

$$\therefore B = -A$$

On equating the coefficients of s in equation (3) we get,

$$0.4011\ A + 0.4011\ B + C = 0$$

Put, $B = -A$

$$\therefore 0.4011\ A - 0.4011\ A + C = 0$$

$$\therefore C = 0$$

On equating constants of equation (3) we get,

$$0.6235\ A + 0.4011\ C = 0.2501$$

Put, $C = 0$.

$$\therefore 0.6235\ A = 0.2501$$

$$\therefore A = \frac{0.2501}{0.6235} = 0.4011$$

$$B = -A = -0.4011$$

$$\therefore H(s) = \frac{A}{(s+0.4011)} + \frac{Bs+C}{(s^2+0.4011s+0.6235)}$$

$$= \frac{0.4011}{(s+0.4011)} - \frac{0.4011s}{(s^2+0.4011s+0.6235)}$$

$$\boxed{(s+a)^2 = s^2 + 2as + a^2}$$

$$\boxed{2a = 0.4011 \Rightarrow a = \frac{0.4011}{2} = 0.2006}$$

$$= \frac{0.4011}{(s+0.4011)} - \frac{0.4011s}{(s^2 + 2\times 0.2006s + 0.2006^2) + \left(\sqrt{0.6235 - 0.2006^2}\right)^2}$$

$$= \frac{0.4011}{(s+0.4011)} - 0.4011\frac{s + 0.2006 - 0.2006}{(s+0.2006)^2 + 0.7637^2}$$

$$= \frac{0.4011}{(s+0.4011)} - 0.4011\frac{s+0.2006}{(s+0.2006)^2+0.7637^2} + 0.4011\frac{0.2006}{(s+0.2006)^2+0.7637^2}$$

$$= \frac{0.4011}{(s+0.4011)} - 0.4011\frac{s+0.2006}{(s+0.2006)^2+0.7637^2}$$

$$+ \frac{0.4011\times 0.2006}{0.7637}\ \frac{0.7637}{(s+0.2006)^2+0.7637^2}$$

$$= \frac{0.4011}{(s+0.4011)} - 0.4011\frac{s+0.2006}{(s+0.2006)^2+0.7637^2} + 0.1054\ \frac{07637}{(s+0.2006)^2+0.7637^2}$$

Digital IIR lowpass filter transfer function, H(z)

In impulse invariant transformation,

Using equations (7.17), (7.18) and (7.19).

$$\frac{A_i}{s+p_i} \xrightarrow{\text{is transformed to}} \frac{A_i}{1-e^{-p_iT}z^{-1}}$$

$$\frac{(s+a)}{(s+a)^2+b^2} \xrightarrow{\text{is transformed to}} \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \xrightarrow{\text{is transformed to}} \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

Using the above transformation, the H(s) can be transformed to H(z) as shown below.

Put, T = 1

$$\therefore\ H(z) = \frac{0.4011}{1-e^{-0.4011}z^{-1}} - 0.4011\,\frac{1-e^{-0.2006}(\cos\ 0.7637)z^{-1}}{1-2\,e^{-0.2006}(\cos\ 0.7637)\,z^{-1}+e^{-2\times0.2006}\,z^{-2}}$$

$$+0.1054\,\frac{e^{-0.2006}(\sin 0.7637)z^{-1}}{1-2\,e^{-0.2006}(\cos\ 0.7637)z^{-1}+e^{-2\,\times\,0.2006}z^{-2}}$$

$$= \frac{0.4011}{1-0.6696\,z^{-1}} + \frac{-0.4011+0.2371z^{-1}}{1-1.1820z^{-1}+0.6696\,z^{-2}} + \frac{0.0596z^{-1}}{1-1.1820z^{-1}+0.6696\,z^{-2}}$$

$$= \frac{0.4011}{1-0.6696\,z^{-1}} + \frac{-0.4011+0.2371z^{-1}+0.0596z^{-1}}{1-1.1820z^{-1}+0.6696\,z^{-2}}$$

$$= \frac{0.4011}{1-0.6696\,z^{-1}} + \frac{-0.4011\ +\ 0.2967z^{-1}}{1-1.1820z^{-1}+0.6696\,z^{-2}}$$

$$= \frac{0.4011\,(1-1.1820z^{-1}+0.6696\,z^{-2})+(1-0.6696\,z^{-1})(-0.4011\ +\ 0.2967z^{-1})}{(1-0.6696\,z^{-1})\,(1-1.1820z^{-1}+0.6696\,z^{-2})}$$

$$= \frac{0.4011-0.4741z^{-1}+0.2686z^{-2}-0.4011+0.2961z^{-1}+0.2686z^{-2}-0.1988z^{-2})}{1-1.1820z^{-1}+0.6696\,z^{-2}-0.6696\,z^{-1}+0.7915z^{-2}-0.4484z^{-3}}$$

$$= \frac{0.0906z^{-1}+0.0698z^{-2}}{1-1.8516z^{-1}+1.4611z^{-2}-0.4484z^{-3}}$$

Alternatively,

$$H(z) = \frac{0.0906z^{-1}+0.0698z^{-2}}{1-1.8516z^{-1}+1.4611z^{-2}-0.4484z^{-2}} = \frac{z^{-3}(0.0906z^{2}+0.0698z)}{z^{-3}(z^{3}-1.8516z^{2}+1.4611z-0.4484)}$$

$$= \frac{0.0906z^{2}+0.0698z}{z^{3}-1.8516z^{2}+1.4611z-0.4484}$$

Note: Verify the result with MATLAB program 7.9.

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{0.0906z^{-1}+0.0698z^{-2}}{1-1.8516z^{-1}+1.4611z^{-2}-0.4484z^{-2}}$$

On cross multiplying the above equation we get,

$$Y(z) - 1.8516z^{-1}Y(z) + 1.4611z^{-2}Y(z) - 0.4484z^{-3}\,Y(z) = 0.0906z^{-1}\,X(z) + 0.0698z^{-1}X(z)$$

$$\backslash\ \ Y(z) = 0.0906z^{-1}X(z) + 0.0698z^{-2}X(z) + 1.8516z^{-1}Y(z)\ - 1.4611z^{-2}Y(z)$$

$$+\ 0.4484z^{-3}Y(z) \qquad .....(4)$$

Using equation (4), the direct form -I structure is drawn as shown in fig 1.

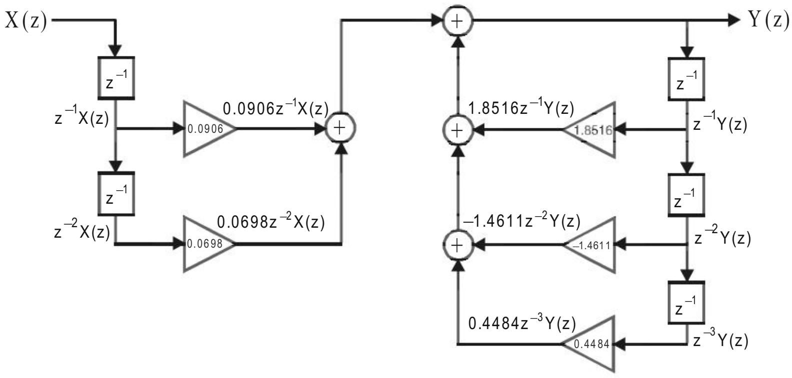


Fig 1 : Direct form -I structure of 3rd order digital IIR lowpass filter.

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=\frac{W(z)}{X(z)}\times\frac{Y(z)}{W(z)}=\frac{0.0906z^{-1}+0.0698z^{-2}}{1-1.8516z^{-1}+1.4611z^{-2}-0.4484z^{-2}}$$

$$\text{where, } \frac{W(z)}{X(z)}=\frac{1}{1-1.8516z^{-1}+1.4611z^{-2}-0.4484z^{-2}} \qquad .....(5)$$

$$\frac{Y(z)}{W(z)}=0.0906z^{-1}+0.0698z^{-2} \qquad .....(6)$$

On cross multiplying equation (5) we get,

$$W(z)-1.8516z^{-1}W(z)+1.4611z^{-2}W(z)-0.4484z^{-3}W(z)=X(z)$$

$$\backslash \quad W(z)=X(z)+1.8516z^{-1}W(z)-1.4611z^{-2}W(z)+0.4484z^{-3}W(z) \qquad .....(7)$$

On cross multiplying equation (6) we get,

$$Y(z)=0.0906z^{-1}X(z)+0.0698z^{-2}W(z) \qquad .....(8)$$

Using equations (7) and (8), the direct form-II structure is drawn as shown in fig 2.

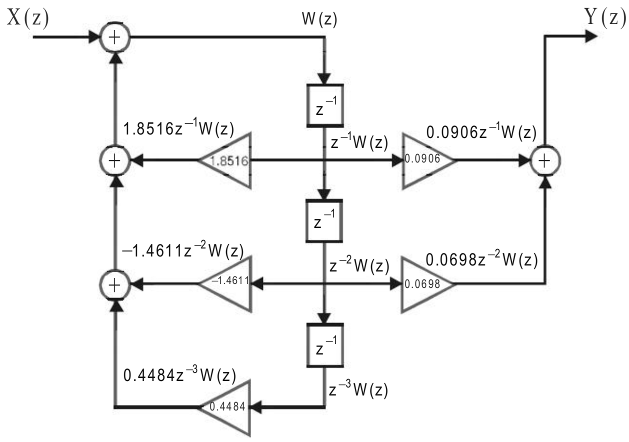


Fig 2 : Direct form -II structure of 3rd order digital IIR lowpass filter.

Example 7.24

Design a Chebyshev digital IIR lowpass filter using bilinear transformation by taking T = 1second, to satisfy the following specifications.

$$0.8 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2 \quad ; \quad \text{for } 0.32\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter.

> Alternatively,
>
> Passband ripple $\le$ 1.9dB
>
> Stopband attenuation $\ge$ 13.97dB
>
> Passband edge frequency = 0.2π rad/sample
>
> Stopband edge frequency = 0.32π rad/sample
>
> The above specifications can be converted to A_p and A_s as shown below.
>
> $$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{(-1.9/20)} = 0.8$$
>
> $$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{(-13.97/20)} = 0.2$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.2\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.32\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.8$

Gain in normal value at stopband edge, $A_s = 0.2$

sampling time, T = 1second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.8$

Gain in normal value at stopband edge, $A_s = 0.2$

Gain is same in analog and digital filter.

For bilinear transformation,

Passband edge analog frequency, $\Omega_p = \dfrac{2}{T}\tan\dfrac{\omega_p}{2}$

Using equation (7.83).

$$= \frac{2}{1}\tan\frac{0.2\pi}{2} = 0.6498 \text{ rad / second}$$

Stopband edge analog frequency, $\Omega_s = \dfrac{2}{T}\tan\dfrac{\omega_s}{2}$

Using equation (7.84).

$$= \frac{2}{1}\tan\frac{0.32\pi}{2} = 1.0995 \text{ rad / second}$$

Order of the filter

Using equation (7.87).

$$N_1 = \frac{\cosh^{-1}\left[\dfrac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]^{\frac{1}{2}}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1}\left[\dfrac{\left(1/0.2^2\right)-1}{\left(1/0.8^2\right)-1}\right]^{\frac{1}{2}}}{\cosh^{-1}\dfrac{1.0995}{0.6498}} = \frac{\cosh^{-1}6.5319}{\cosh^{-1}1.6921} = 2.2944$$

Choose order N_1 such that $N \ge N_1$ and N is an integer.

Let, order, N = 3.

Normalized transfer function, $H(s_n)$ of Chebyshev lowpass filter

For odd N,

$$H(s_n) = \frac{B_0}{s_n + c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s_n^2 + b_k s_n + c_k}$$ Using equation (7.89).

Here, $N = 3$, $\therefore\ k = \frac{N-1}{2} = \frac{3-1}{2} = 1$

$$\therefore\ H(s_n) = \frac{B_0}{s_n + c_0} \times \frac{B_1}{s_n^2 + b_1 s_n + c_1}$$

$$\in = \left[\left(1/A_p^2\right) - 1\right]^{\frac{1}{2}}$$ Using equation (7.94).

$$= \left[\left(1/0.8^2\right) - 1\right]^{\frac{1}{2}} = 0.75$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\in^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\in}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\in^2} + 1\right)^{\frac{1}{2}} + \frac{1}{\in}\right]^{-\frac{1}{N}}\right\}$$ Using equation (7.93).

$$= \frac{1}{2}\left[\left(\frac{1}{0.75^2} + 1\right)^{\frac{1}{2}} + \frac{1}{0.75}\right]^{\frac{1}{3}} - \left[\left(\frac{1}{0.75^2} + 1\right)^{\frac{1}{2}} + \frac{1}{0.75}\right]^{-\frac{1}{3}}$$

$$= \frac{1}{2}\ [1.4422 - 0.6934] = 0.3744$$

$$c_0 = y_N = 0.3744$$ Using equation (7.94).

$$b_k = 2\, y_N \sin\left[\frac{(2k-1)\pi}{2N}\right]$$ Using equation (7.90).

When $k = 1,\ b_k = b_1 = 2 \times 0.3744 \sin\left[\frac{(2 \times 1 - 1)\pi}{2 \times 3}\right] = 0.3744$

$$c_k = y_N^2 + \cos^2\left[\frac{(2k-1)\pi}{2N}\right]$$ Using equation (7.91).

When $k = 1,\ c_k = c_1 = y_N^2 + \cos^2\left[\frac{(2 \times 1 - 1)\pi}{2 \times 3}\right]$

$$= 0.3744^2 + \cos^2\frac{\pi}{6} = 0.3744^2 + \left[\frac{1 + \cos\left(\frac{2\pi}{6}\right)}{2}\right]$$ $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

$$= 0.1402 + 0.75 = 0.8902$$

$$\therefore\ H(s_n) = \frac{B_0}{s_n + c_0} \times \frac{B_1}{s_n^2 + b_1 s_n + c_1} = \frac{B_0}{s_n + 0.3744} \times \frac{B_1}{s_n^2 + 0.3744\, s_n + 0.8902}$$

To evaluate B_0 and B_1, let, $H(s_n)\Big|_{s_n=0} = 1$

When $s_n = 0$, $H(s_n) = \dfrac{B_0\,B_1}{(0.3744)\,(0.8902)} = 3B_0\,B_1$

$$\therefore\; 3\;B_0\;B_1 = 1 \quad\Rightarrow\quad B_0\;B_1 = \frac{1}{3} = 0.3333$$

Let, $B_0 = B_1$; $\therefore\; B_0^2 = 0.3333 \quad\Rightarrow\quad B_0 = \sqrt{0.3333} = 0.5774$

$\therefore\; B_1 = B_0 = 0.5774$

$$H(s_n) = \frac{B_0}{s_n + 0.3744} \times \frac{B_1}{s_n^2 + 0.3744\,s_n + 0.8902} = \frac{0.5774}{s_n + 0.3744} \times \frac{0.5774}{s_n^2 + 0.3744\,s_n + 0.8902}$$

$$= \frac{0.3333}{(s_n + 0.3744)\,(s_n^2 + 0.3744\,s_n + 0.8902)} = \frac{0.3333}{s_n^3 + 0.7488\,s_n^2 + 1.0304\,s_n + 0.3333}$$

Unnormalized transfer function, H(s) of Chebyshev lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

Here, $\Omega_c = \Omega_p = 0.6498$ rad/sec.

$$\therefore\; H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{0.3333}{(s_n^3 + 0.7488\,s_n^2 + 1.0304\,s_n + 0.3333)}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{0.3333}{\left(\dfrac{s^3}{\Omega_c^3} + 0.7488\,\dfrac{s^2}{\Omega_c^2} + 1.0304\,\dfrac{s}{\Omega_c} + 0.3333\right)}$$

$$= \frac{0.3333 \times \Omega_c^3}{s^3 + 0.7488\,\Omega_c\,s^2 + 1.0304\,\Omega_c^2\,s + 0.3333\,\Omega_c^3}$$

$$= \frac{0.3333 \times 0.6498^3}{s^3 + 0.7488 \times 0.6498\,s^2 + 1.0304 \times 0.6498^2 s + 0.3333 \times 0.6498^3}$$

$$= \frac{0.0914}{s^3 + 0.4865\,s^2 + 0.4351\,s + 0.0914}$$

Digital IIR lowpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{0.0914}{s^3 + 0.4865\,s^2 + 0.4351\,s + 0.0914}\Bigg|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{0.0914}{\left(\dfrac{2}{T}\,\dfrac{1-z^{-1}}{1+z^{-1}}\right)^3 + 0.4865\left(\dfrac{2}{T}\,\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.4351\left(\dfrac{2}{T}\,\dfrac{1-z^{-1}}{1+z^{-1}}\right) + 0.0914}$$

$$= \frac{0.0914}{\dfrac{8(1-z^{-1})^3}{T^3(1+z^{-1})^3} + \dfrac{1.946(1-z^{-1})^2}{T^2(1+z^{-1})^2} + \dfrac{0.8702(1-z^{-1})}{T\,(1+z^{-1})} + 0.0914}$$

$$\therefore H(z) = \frac{0.0914}{\dfrac{8(1-z^{-1})^3 + 1.946\,T(1-z^{-1})^2(1+z^{-1}) + 0.8702\,T^2(1-z^{-1})(1+z^{-1})^2 + 0.0914\,T^3(1+z^{-1})^3}{T^3(1+z^{-1})}}$$

$$= \frac{0.0914\,T^3(1+z^{-1})^3}{8(1-z^{-1})^3 + 1.946\,T(1-z^{-1})^2(1+z^{-1}) + 0.8702\,T^2(1-z^{-1})(1+z^{-1})^2 + 0.0914\,T^3(1+z^{-1})^3}$$

$\boxed{\text{Put, } T = 1}$

$$= \frac{0.0914(1+3z^{-1}+3z^{-2}+z^{-3})}{8(1-3z^{-1}+3z^{-2}-z^{-3}) + 1.946(1-2z^{-1}+z^{-2})(1+z^{-1}) + 0.8702(1-z^{-1})(1+2z^{-1}+z^{-2}) + 0.0914(1+3z^{-1}+3z^{-2}+z^{-3})}$$

$$= \frac{0.0914(1+3z^{-1}+3z^{-2}+z^{-3})}{8(1-3z^{-1}+3z^{-2}-z^{-3}) + 1.946(1-z^{-1}-z^{-2}+z^{-3}) + 0.8702(1+z^{-1}-z^{-2}-z^{-3}) + 0.0914(1+3z^{-1}+3z^{-2}+z^{-3})}$$

$$= \frac{0.0914 + 0.2742z^{-1} + 0.2742z^{-2} + 0.0914z^{-3}}{10.9076 - 24.893z^{-1} + 21.3666z^{-2} - 6.8328z^{-3}}$$

$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$
$\boxed{(a-b)^2 = a^2 - 2ab + b^2}$
$\boxed{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$
$\boxed{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$

$$= \frac{\dfrac{0.0914}{10.9076} + \dfrac{0.2742\,z^{-1}}{10.9076} + \dfrac{0.2742\,z^{-2}}{10.9076} + \dfrac{0.0914}{10.9076}z^{-3}}{1 - \dfrac{24.893z^{-1}}{10.9076} + \dfrac{21.3666z^{-2}}{10.9076} - \dfrac{6.8328z^{-3}}{10.9076}}$$

$$= \frac{0.0083 + 0.0251z^{-1} + 0.0251z^{-2} + 0.0083z^{-3}}{1 - 2.2821z^{-1} + 1.9589z^{-2} - 0.6264z^{-3}}$$

Alternatively,

$$H(z) = \frac{0.0083 + 0.0251z^{-1} + 0.0251z^{-2} + 0.0083z^{-3}}{1 - 2.2821z^{-1} + 1.9589z^{-2} - 0.6264z^{-3}}$$

$$= \frac{z^{-3}(0.0083z^3 + 0.0251z^2 + 0.0251z + 0.0083}{z^{-3}(z^3 - 2.2821z^2 + 1.9589z - 0.6264)}$$

$$= \frac{0.0083z^3 + 0.0251z^2 + 0.0251z + 0.0083}{z^3 - 2.2821z^2 + 1.9589z - 0.6264}$$

Note: Verify the result with MATLAB program 7.10.

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{0.0083 + 0.0251z^{-1} + 0.0251z^{-2} + 0.0083z^{-3}}{1 - 2.2821z^{-1} + 1.9589z^{-2} - 0.6264z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z) - 2.2821z^{-1}Y(z) + 1.9589z^{-2}Y(z) - 0.6264z^{-3}Y(z) = 0.0083\,X(z) + 0.0251z^{-1}X(z) + 0.0251z^{-2}X(z) + 0.0083z^{-3}X(z)$$

$$\therefore Y(z) = 0.0083X(z) + 0.0251z^{-1}X(z) + 0.0251z^{-2}X(z) + 0.0083z^{-3}X(z) + 2.2821z^{-1}Y(z) - 1.9589z^{-2}Y(z) + 0.6264z^{-3}Y(z) \quad .....(1)$$

Using equation (1), the direct form -I structure is drawn as shown in fig 1.

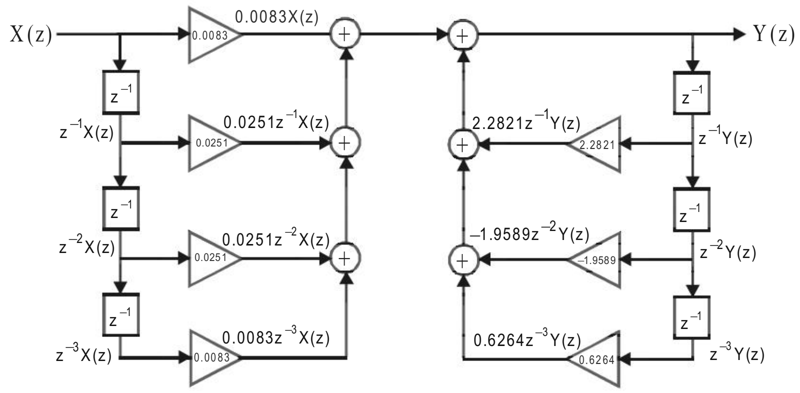


***Fig 1 :** Direct form-I structure of 3rd order digital IIR lowpass filter..*

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.0083 + 0.0251z^{-1} + 0.0251z^{-2} + 0.0083z^{-3}}{1 - 2.2821z^{-1} + 1.9589z^{-2} - 0.6264z^{-3}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1 - 2.2821z^{-1} + 1.9589z^{-2} - 0.6264z^{-3}} \quad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.0083 + 0.0251z^{-1} + 0.0251z^{-2} + 0.0083z^{-3} \quad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) - 2.2821z^{-1}W(z) + 1.9589z^{-2}W(z) - 0.6264\, z^{-3}W(z) = X(z)$$

$$W(z) = X(z) + 2.2821z^{-1}W(z) - 1.9589z^{-2}W(z) + 0.6264z^{-3}W(z) \quad .....(4)$$

On cross multiplying equation (3) we get,

$$Y(z) = 0.0083W(z) + 0.0168z^{-1}W(z) + 0.0168z^{-2}W(z) + 0.0083z^{-3}W(z) \quad .....(5)$$

Using equations (4) and (5), the direct form-II structure is drawn as shown in fig 2.

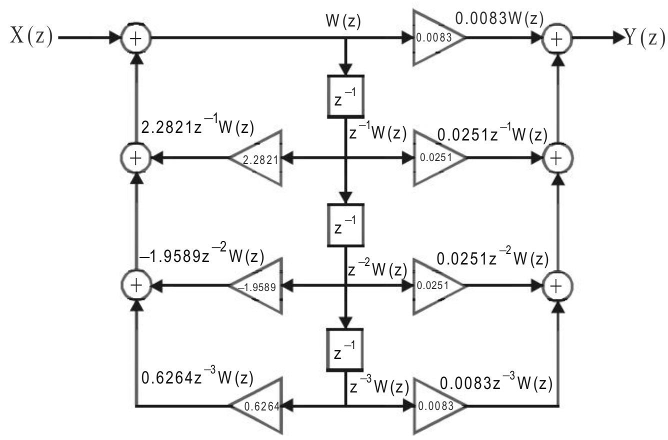


***Fig 2 :** Direct form-II structure of 3rd order digital IIR lowpass filter.*

7.9 Summary of Important Concepts

1. The filters designed by considering all the infinite samples of impulse response are called IIR filters.
2. Since IIR filter design involves processing of infinite samples, the direct design of IIR filters is not possible.
3. The IIR filters are designed via analog filters.
4. The analog filter is designed by approximating the ideal frequency response using an error function.
5. In analog filter design the approximation problem is solved to meet a specified tolerance in the passband and stopband.
6. The popular approximation method used for analog filter design are Butterworth and Chebyshev approximation.
7. The popular transformation method used for transforming analog filter to digital filter are impulse invariant transformation and bilinear transformation.
8. For stability of analog filter, the poles should lie on the left half of s-plane.
9. For stability of digital IIR filter, the poles should lie inside the unit circle in z-plane.
10. For casuality of analog and digital IIR filter the number of zeros should be less than or equal to number of poles.
11. For realizability the transfer function of analog filter should be a rational function of "s" and the coefficients of "s" should be real.
12. For realizability the transfer function of digital IIR filter should be a rational function of "z" and the coefficients of "z" should be real.
13. The frequency response of a practical filter will have a passband, transition band and stopband.
14. The specifications of analog filter are gain or attenuation at a passband edge and stopband edge frequencies.
15. The specifications of digital IIR filter are gain or attenuation at a passband edge and stopband edge frequencies.
16. The specifications of a filter are also specified in terms of ripple or tolerance in passband or stopband.
17. In impulse invariant transformation, the impulse response of digital filter is obtained by sampling the impulse response of the analog filter.
18. In impulse invariant transformation the analog pole at $s = -p_i$ is mapped into a digital pole at $z = e^{-p_iT}$.
19. In impulse invariant transformation any strip of width 2p/T in the s-plane for values of s in the range (2k – 1) p/T £ W £ (2k + 1) p/T (where k is an integer), is mapped into the entire z-plane.
20. In impulse invariant transformation an analog frequency, W is transformed to a digital frequency, w = WT.
21. The bilinear transformation is a conformal mapping that transforms the imaginary axis of s-plane into the unit circle in z-plane.
22. In bilinear transformation the mapping from s-plane to z-plane is accomplished when "s" is replaced by $\frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}$.
23. The bilinear transformation is one-to-one mapping whereas the impulse invariant transformation is many-to-one mapping.

24. In bilinear transformation an analog frequency, W is transformed to a digital frequency, $\omega = 2\tan^{-1}\frac{\Omega T}{2}$.
25. The distortion in the frequency axis due to nonlinear mapping of analog frequency to digital frequency in bilinear transformation is called frequency warping.
26. The prewarping is conversion of specified digital frequency to analog frequency using the relation $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$.
27. When expressed in dB, the gain and attenuation are numerically same but opposite in sign. The gain will be negative dB, whereas the attenuation will be positive dB.
28. When expressed in dB, the attenuation and ripple are same.
29. In Butterworth filter design, the error function is selected such that the magnitude is maximally flat in the passband and monotonically decreasing in the stopband.
30. The 2N poles of Butterworth normalized transfer function symmetrically lies on a unit circle in s-plane with angular spacing of p/N (= 2p/2N).
31. The transfer function of Butterworth filter is obtained by considering the N-poles lying on left half of s-plane.
32. When N is even, all poles of Butterworth filter are complex and exist as conjugate pair.
33. When N is odd, one pole of Butterworth filter is real and all other poles are complex and exist as conjugate pair.
34. For normalized transfer function of the filter, cutoff frequency, W_c = 1 rad/second.
35. In Butterworth approximation, the approximated magnitude ressponse approaches the ideal response as the order N increases.
36. In type-1 Chebyshev approximation, the error function is selected such that, the magnitude response is equiripple in the passband and monotonic in the stopband.
37. In type-2 Chebyshev approximation, the error function is selected such that, the magnitude response is monotonic in passband and equiripple in stopband.
38. The type-2 Chebyshev magnitude response is also called inverse Chebyshev resposne.
39. The 2N poles of Chebyshev transfer function symmetrically lies on an ellipse in s-plane.
40. The transfer function of Chebyshev filter is formed by considering the N-poles lying on left half of s-plane.
41. When N is even, all the poles of Chebyshev filter are complex and exist as conjugate pair.
42. When N is odd, one of the pole of Chebyshev filter is real and all other poles are complex and exist as conjugate pair.
43. The normalized transfer function of lowpass filter is transformed to lowpass filter with cutoff frequency, W_c, by the transformation $s_n \to \frac{s}{\Omega_c}$.
44. The normalized transfer function of lowpass filter is transformed to highpass filter with cutoff frequency, W_c, by the transformation $s_n \to \frac{\Omega_c}{s}$.
45. The frequency response of digital IIR filter obtained by impulse invariant tansformation will be amplified by the factor 1/T.

7.10. Short Questions and Answers

Q7.1 Define an IIR filter.

The filter designed by considering all the infinite samples of impulse response are called IIR filters. The impulse response is obtained by taking inverse fourier transform of ideal frequency response.

Q7.2 Distinguish between IIR and FIR filters.

The filter design starts from ideal frequency response. By taking inverse fourier transform of ideal frequency response, the desired impulse response is obtained, which consists of infinite number of samples.

The digital filters designed by selecting only N samples of the impulse response are called FIR filters. The digital filters designed by considering all the infinite samples of impulse response are called IIR filters.

Q7.3 Compare IIR and FIR filers.

IIR Filter	FIR filter
i. All the infinite samples of impulse response are considered.	i. Only N samples of impulse response are considered.
ii. The impulse response cannot be directly converted to digital filter transfer function.	ii. The impulse response can be directly converted to digital filter transfer function.
iii. The design involves design of analog filter and then transforming analog filter to digital filter.	iii. The digital filter can be directly designed to achieve the desired specifications.
iv. The specifications include the desired characteristics for magnitude response only.	iv. The specifications include the desired characteristics for both magnitude and phase response.
v. Linear phase characteristics cannot be achieved.	v. Linear phase filters can be easily designed.

Q7.4 Classify the filters based on frequency response.

Based on frequency response, the filters can be classified into lowpass, highpass, bandpass and bandstop filters.

Q7.5 What are the properties that are maintained same in the transformation of analog to digital filter? (or Mention the two properties that an analog filter should have for effective transformation).

The analog filters should be stable and causal for effective transformation to digital filters. While transforming the analog filter to digital filters these two properties (i.e., stability and causality) are maintained same, which means that the transformed digital filter should also be stable and causal.

Q7.6 What are the requirements for an analog filter to be stable and causal?

i. The analog filter transfer function H(s) should be a rational function of s and the coefficients of s should be real.
ii. The poles should lie on the left half of s-plane.
iii. The number of zeros should be less than or equal to number of poles.

Q7.7 What are the requirements for a digital filter to be stable and causal?

i. The digital filter transfer function H(z) should be a rational function of z and the coefficients of z should be real.
ii. The poles should lie inside the unit circle in z-plane.
iii. The number of zeros should be less than or equal to number of poles.

Q7.8 Sketch the various tolerance limits to approximate an ideal lowpass and highpass filter.

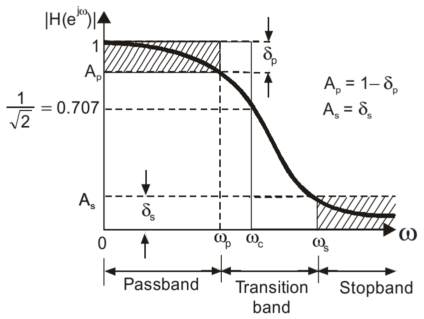


Fig Q7.8a : *Lowpass filter.*

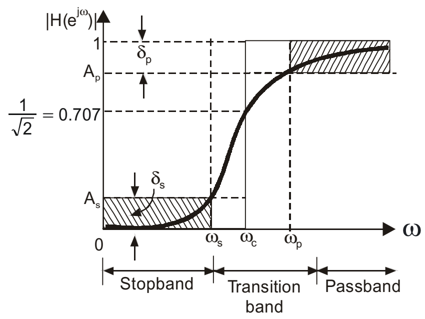


Fig Q7.8b : *Highpass filter.*

Q7.9 Define ripples in a filter.

The limits of the tolerance in the magnitude of passband and stopband are called ripples. The tolerance in passband is denoted as δ_p and that in stopband is denoted as δ_s.

Q7.10 Write a brief note on the design of IIR filter. (or How a digital IIR filter is designed?)

For designing a digital IIR filter, first an equivalent analog filter is designed using any one of the approximation technique and the given specifications. The result of the analog filter design will be an analog filter transfer function H(s). The analog filter transfer function is transformed to digital filter transfer function H(z) using either Bilinear or Impulse invariant transformation.

Q7.11. Mention any two techniques for digitizing the transfer function of an analog filter.

The bilinear transformation and the impulse invariant transformation are the two techniques available for digitizing the analog filter transfer function.

Q7.12 Compare the digital and analog filter.

Digital Filter	Analog Filter
1. Operates on digital samples (or sampled version) of the signal.	1. Operates on analog signals (or actual signals).
2. It is governed (or defined) by linear difference equation.	2. It is governed (or defined) by linear differential equation.
3. It consists of adders, multipliers and delays implemented in digital logic (either in hardware or software or both).	3. It consists of electrical components like resistors, capacitors and inductors.
4. In digital filters the filter coefficients are designed to satisfy the desired frequency response.	4. In analog filters the approximation problem is solved to satisfy the desired frequency response.

Q7.13 What are the advantages and disadvantages of digital filters?

Advantages of digital filters

i. High thermal stability due to absence of resistors, inductors and capacitors.

ii. The performance characteristics like accuracy, dynamic range, stability and tolerance can be enhanced by increasing the length of the registers.

iii. The digital filters are programmable.

iv. Multiplexing and adaptive filtering are possible.

Disadvantages of digital filters

i. The bandwidth of the discrete signal is limited by the sampling frequency.

ii. The performance of the digital filter depends on the hardware used to implement the filter.

Q7.14 Mention the important features of IIR filters.

i. The physically realizable IIR filters does not have linear phase.

ii. The IIR filter specifications includes the desired characteristics for the magnitude response only.

Q7.15. What is impulse invariant transformation?

The transformation of analog filter to digital filter without modifying the impulse response of the filter is called impulse invariant transformation (ie., in this transformation the impulse response of the digital filter will be the sampled version of the impulse response of the analog filter).

Q7.16 What is the main objective of impulse invariant transformation?

The objective of this method is to develop an IIR filter transfer function whose impulse response is the sampled version of the impulse response of the analog filter. Therefore the frequency response characteristics of the analog filter is preserved.

Q7.17 How are analog poles mapped to digital poles in impulse invariant transformation (or in bilinear transformation)?

In impulse invariant transformation (or In bilinear transformation) the mapping of analog to digital poles are as follows,

i. The analog poles on the left half of s-plane are mapped into the interior of unit circle in z-plane.

ii. The analog poles on the imaginary axis of s-plane are mapped into the unit circle in the z-plane.

iii. The analog poles on the right half of s-plane are mapped into the exterior of unit circle in z-plane.

Q7.18. What is the importance of poles in filter design?

The stability of a filter is related to the location of the poles. For a stable analog filter the poles should lie on the left half of s-plane. For a stable digital filter the poles should lie inside the unit circle in the z-plane.

Q7.19. Why an impulse invariant transformation is not considered to be one-to-one?

In impulse invariant transformation any strip of width 2p/T in the s-plane for values of s in the range (2k–1)/T £ W £ (2k+1)) p/T (where k is an integer) is mapped into the entire z-plane. The left half portion of each strip in s-plane maps into the interior of the unit circle in z-plane, right half portion of each strip in s-plane maps into the exterior of the unit circle in z-plane and the imaginary axis of each strip in s-plane maps into the unit circle in z-plane as shown in the figure below. Hence the impulse invariant transformation is many-to-one.

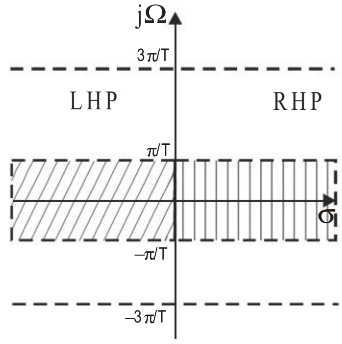


Fig Q7.19a : *s-plane.*

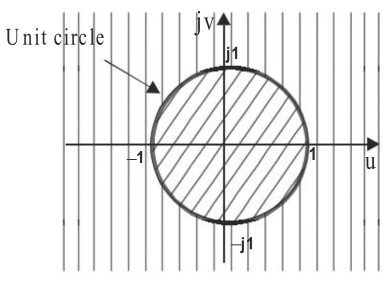


Fig Q7.19b : *z-plane.*

Fig Q7.19 : *Mapping of s-plane into z-plane in impulse invariant transformation.*

Q7.20. ***Write the impulse invariant transformation used to transform real poles with and without multiplicity.***

The impulse invariant transformation used to transform real pole (at s = $-p_i$) without multiplicity is

$$\frac{1}{s+p_i} \longrightarrow \frac{1}{1-e^{-p_i T} z^{-1}}$$

The impulse invariant transformation used to transform multiple real pole (at s = $-p_i$) is

$$\frac{1}{(s+p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{dp_i^{m-1}} \frac{1}{1-e^{-p_i T} z^{-1}}$$

where, m is the multiplicity.

Q7.21 ***Write the impulse invariant transform used to transform complex conjugate poles.***

$$\frac{(s+a)}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}\,(\cos bT)\,z^{-1}}{1-2e^{-aT}\,(\cos bT)\,z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT}\,(\sin bT)\,z^{-1}}{1-2e^{-aT}\,(\cos bT)\,z^{-1}+e^{-2aT}z^{-2}}$$

Q7.22 ***What is the relation between digital and analog frequency in impulse invariant transformation?***

The relation between analog and digital frequency in impulse invariant transformation is given by,

Digital frequency, $\omega = \Omega T$

where, Ω = Analog frequency, and T = Sampling time period.

Q7.23 ***What is aliasing?***

The phenomena of high frequency sinusoidal components acquiring the identity of low frequency sinusoidal components after sampling is called aliasing. The aliasing problem will arise if the sampling rate does not satisfy the Nyquist sampling criteria.

Q7.24 ***What is aliasing problem in impulse invariance method of designing digital filters? Why is it absent in bilinear transformation?***

In impulse invariant mapping, the analog frequencies in the interval $(2k-1)\pi/T \le \Omega \le (2k+1)\pi/T$ (where k is an integer) maps into corresponding values of digital frequencies in the interval $-\pi \le \omega \le \pi$. Hence the mapping of Ω to ω is many-to-one.

This will result in high frequency components acquiring the identity of the low frequency components if the analog filter is not band limited. This effect is called aliasing. The aliasing can be avoided in bandlimited filters by choosing very small values of sampling time (or very high sampling frequency).

The bilinear mapping is one-to-one mapping and so there is no effect of aliasing.

Q7.25 ***Obtain the impulse response of digital filter corresponding to an analog filter with impulse response $h(t) = 0.5e^{-2t}$ and with a sampling rate of 1.0 kHz using impulse invariant method.***

<u>Solution</u>

Given that, $h(t) = 0.5e^{-2t}$

Sampling frequency, $F = 1\text{ kHz} = 1 \times 10^3\text{ Hz}$

$$\therefore \text{ Sampling time, } T = \frac{1}{F} = \frac{1}{1\times10^3} = 10^{-3}\text{second.}$$

$$\left.\begin{array}{l}\text{Impulse response of}\\ \text{digital filter}\end{array}\right\} h(n) = h(t)\big|_{t=nT} = 0.5e^{-2t}\big|_{t=nT} = 0.5e^{-2nT}$$

$$= 0.5\left(e^{-2T}\right)^n = 0.5\left(e^{-2 \times 10^{-3}}\right)^n$$

$$= 0.5(0.998)^n \quad ; \quad \text{for } n \geq 0.$$

Q7.26 ***Given that, H(s) = 1/(s+1). By impulse invariant method, obtain the digital filter transfer function and the difference equation of digital fitler.***

Solution

Given that, $H(s) = \frac{1}{s+1}$

In impulse invariant transformation,

$$\frac{1}{s+p_i} \longrightarrow \frac{1}{1-e^{-p_iT}z^{-1}}$$

Let T = 1 second,

$$\therefore \left.\begin{array}{l}\text{Transfer function of}\\ \text{digital filter}\end{array}\right\} H(z) = \frac{1}{1-e^{-T}z^{-1}} = \frac{1}{1-e^{-1}z^{-1}} = \frac{1}{1-0.368z^{-1}}$$

We know that, $H(z) = \frac{Y(z)}{X(z)}$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1}{1-0.368z^{-1}}$$

On cross multiplyting we get,

$$Y(z) - 0.368z^{-1}\,Y(z) = X(z)$$

$$\therefore Y(z) = X(z) + 0.368z^{-1}\,Y(z)$$

On taking inverse Z-transform we get,

$$y(n) = x(n) + 0.368y(n-1)$$

Q7.27 ***What is bilinear transformation ?***

The bilinear transformation is a conformal mapping that transforms the s-plane to z-plane. In this mapping the imaginary axis of s-plane is mapped into the unit circle in z-plane, the left half of s-plane is mapped into interior of unit circle in z-plane and the right half of s-plane is mapped into exterior of unit circle in z-plane. The bilinear mapping is a one-to-one mapping and it is accomplished when

$$s = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}$$

Q7.28 ***Sketch the mapping of s-plane to z-plane in bilinear transformation.***

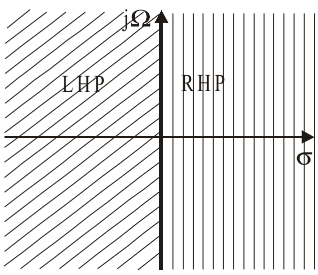


Fig Q7.28a : *s-plane.*

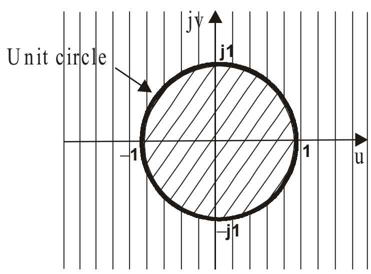


Fig Q7.28b : *z-plane.*

Q7.29 What is the relation between digital and analog frequency in bilinear transformation?

In bilinear transformation the digital frequency is given by,

$$\text{Digital frequency, } \omega = 2\tan^{-1}\frac{\Omega T}{2}$$

where, Ω = Analog frequency, and T = sampling time period.

Q7.30 How is bilinear transformation performed?

The bilinear transformation is performed by letting $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ in the analog filter transfer function.

$$\text{i.e., } H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

Q7.31 How is the analog frequency mapped to digital frequency in bilinear transformation?

In bilinear transformation, the digital frequency and analog frequency are related by the equation,

$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \quad \text{or} \quad \Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

From the above equations we can infer that the relation between analog and digital frequency is nonlinear. Here the entire negative imaginary axis in the s-plane (from $\Omega = -\infty$ to 0) is mapped into the lower half of unit circle in z-plane (from $\omega = -\pi$ to 0) and the entire positive imaginary axis in the s-plane (from $\Omega = 0$ to $+\infty$) is mapped into the upper half of unit circle in z-plane (from $\omega = 0$ to $+\pi$).

Q7.32 What is frequency warping?

In bilinear transformation the relation between analog and digital frequencies is nonlinear. When the s-plane is mapped into z-plane using bilinear transformation, this nonlinear relationship introduces distortion is frequency axis, which is called frequency warping.

Q7.33 What are the advantages and disadvantages of bilinear transformation?

Advantages of bilinear transformation

i. The bilinear transformation is one-to-one mapping.

ii. There is no aliasing and so the analog filter need not have a band limited frequency response.

iii. The effect of warping on amplitude response can be eliminated by prewarping the analog filter.

iv. Bilinear transformation can be used to design digital filters with prescribed magnitude response with piecewise constant values.

Disadvantages of bilinear transformation

i. The nonlinear relationship between analog and digital frequencies introduces frequency distortion which is called frequency warping.

ii. Using bilinear transformation, a linear phase analog filter cannot be transformed to a linear phase digital filter.

Q7.34 What is prewarping? Why is it employed?

In IIR filter design using bilinear transformation, the conversion of the specified digital frequencies to analog frequencies is called prewarping.

The prewarping is necessary to eliminate the effect of warping on amplitude response.

Q7.35 ***Explain the technique of prewarping.***

In IIR filter design using bilinear transformation the specified digital frequencies are converted to analog equivalent frequencies, which are called prewarp frequencies. Using the prewarp frequencies, the analog filter transfer function is designed and then it is transformed to digital filter transfer function.

Q7.36 ***Compare the impulse invariant and bilinear transformations.***

Impulse invariant transformation	Bilinear transformation
i. It is many-to-one mapping.	i. It is one-to-one mapping.
ii. The relation between analog and digital frequency is linear.	ii. The relation between analog and digital frequency is nonlinear.
iii. To prevent the problem of aliasing the analog filters should be bandlimited.	iii. There is no problem of aliasing and so the analog filter need not be bandlimited.
iv. The magnitude and phase response of analog filter can be preserved by choosing low sampling time or high sampling frequency.	iv. Due to the effect of warping, the phase response of analog filter cannot be preserved. But the magnitude response can be preserved by prewarping.

Q7.37 ***What is Butterworth approximation?***

In Butterworth approximation, the error function is selected such that the magnitude is maximally flat in the origin (i.e., at $W = 0$) and monotonically decreasing with increasing W.

Q7.38 ***How are the poles of Butterworth transfer function are located in s-plane?***

The poles of the normalized Butterworth transfer function symmetrically lies on an unit circle in s-plane with angular spacing of p/N.

Q7.39 ***Write the magnitude function of lowpass Butterworth filter.***

The magnitude function of lowpass Butterworth filter is given by,

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\left(\dfrac{\Omega}{\Omega_c}\right)^{2N}}}$$

where, W_c is the cutoff frequency and N is the order of the filter.

Q7.40 ***How the order of the filter affects the frequency response of butterworth filter.***

The magnitude response of butterworth filter is shown in fig Q7.40.,from which it can be observed that the magnitude response approaches the ideal response as the order of the filter is increased.

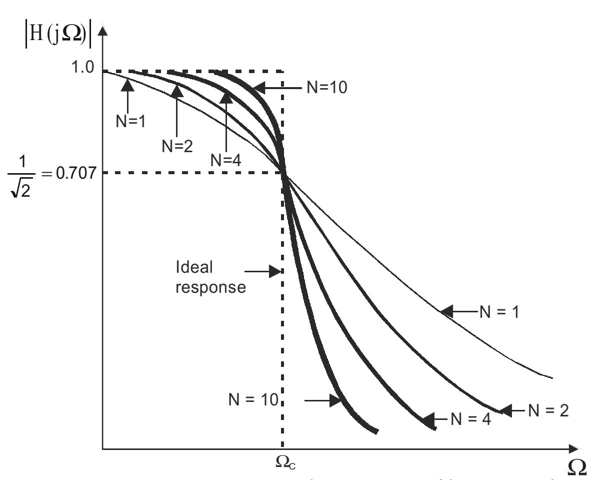


Fig Q7.40 : *Magnitude response of butterworth low pass filter for various values of N.*

Q7.41 Write the transfer function of unnormalized Butterworth lowpass filter.

When N is even,

$$\left.\begin{matrix}\text{Transfer function of analog} \\ \text{lowpass Butterworth filter}\end{matrix}\right\} H(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k \Omega_c\, s + \Omega_c^2}$$

When N is odd,

$$\left.\begin{matrix}\text{Transfer function of analog} \\ \text{lowpass Butterworth filter}\end{matrix}\right\} H(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c\, s + \Omega_c^2}$$

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

N = Order of the filter

Ω_c = Analog cutoff frequency

Q7.42 How will you choose the order N for a Butterworth filter?

Calculate a parameter N_1 using the following equation and correct it to nearest integer.

$$N_1 = \frac{1}{2} \frac{\log\left[\dfrac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

Choose the order N of the filter such that $N \geq N_1$.

Q7.43 Write the properties of Butterworth filter.

i. The Butterworth filters are all pole design.

ii. At the cutoff frequency Ω_c, the magnitude of normalized Butterworth filter is $1/\sqrt{2}$.

iii. The filter order N, completely specifies the filter and as the value of N increases the magnitude response approaches the ideal response.

iv. The magnitude is maximally flat at the origin and monotonically decreasing with increasing Ω.

Q7.44 What is Chebyshev approximation?

In Chebyshev approximation, the approximation function is selected such that the error is minimized over a prescribed band of frequencies.

Q7.45 What is type-1 Chebyshev approximation?

In type-1 Chebyshev approximation, the error function is selected such that, the magnitude response is equiripple in the passband and monotonic in the stopband.

Q7.46 What is type-2 Chebyshev approximation?

In type-2 Chebyshev approximation, the error function is selected such that, the magnitude response is monotonic is passband and equiripple in stopband. The type-2 magnitude response is called inverse Chebyshev response.

Q7.47 Write the magnitude function of Chebyshev lowpass filter.

The magnitude response of type-1 lowpass Chebyshev filter is given by,

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\epsilon^2\, C_N^2\left(\dfrac{\Omega}{\Omega_c}\right)}}$$

where Î is attenuation constant and C_N(W/W$_c$) is the Chebyshev polynomial of the first kind of degree N.

Q7.48 How does the order of the filter affect the frequency response of Chebyshev filter?

From the magnitude response of type-1 Chebyshev filter it can be observed that the magnitude response approaches the ideal response as the order of the filter is increased.

Q7.49 Sketch the magnitude response of type-1 Chebyshev filters.

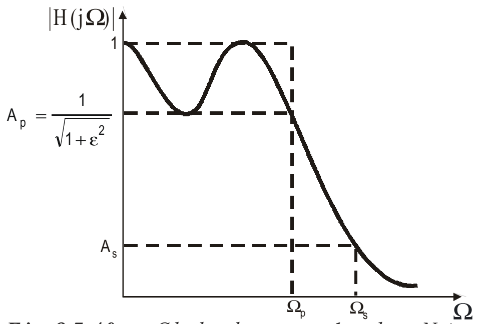


Fig Q7.49a : *Chebyshev type-1, when N is odd.*

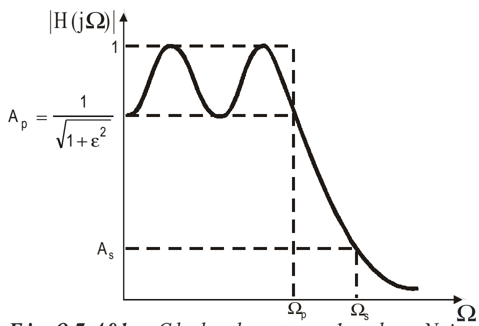


Fig Q7.49b : *Chebyshev type-1, when N is even.*

Q7.50 Sketch the magnitude response of type-2 Chebyshev filters.

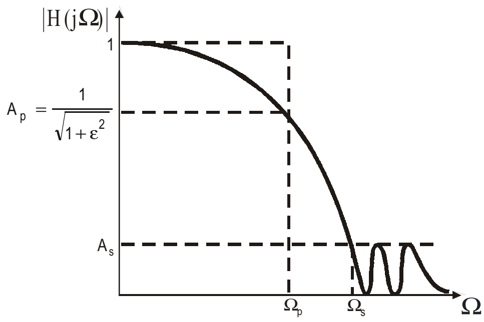


Fig Q7.50a : *Chebyshev type-2, when N is odd.*

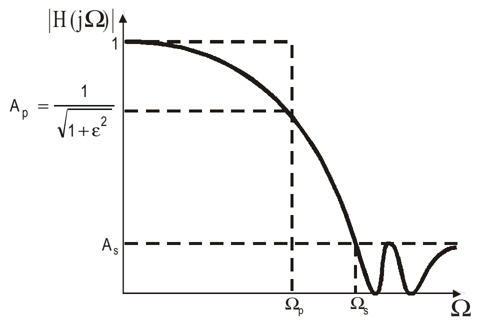


Fig Q7.50b : *Chebyshev type-2, when N is even.*

Q7.51 Write the transfer function of unnormalized Chebyshev lowpass filter.

When N is even,

$$H(s) = \prod_{k=1}^{\frac{N}{2}} \frac{B_k\Omega_c^2}{s^2 + b_k\Omega_c s + c_k\Omega_c^2}$$

When N is odd,

$$H(s) = \frac{B_0\Omega_c}{s+c_0\Omega_c}\prod_{k=1}^{\frac{N-1}{2}}\frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+c_k\Omega_c^2}$$

$$\text{where, } b_k = 2y_N \sin\left(\frac{(2k-1)\pi}{2N}\right) \quad ; \quad c_k = y_N^2 + \cos^2\frac{(2k-1)\pi}{2N} \quad ; \quad c_0 = y_N$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\in^2}+1\right)^{\frac{1}{2}}+\frac{1}{\in}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\in^2}+1\right)^{\frac{1}{2}}+\frac{1}{\in}\right]^{-\frac{1}{N}}\right\}$$

Q7.52 How will you determine the order N of Chebyshev filter?

Calculate a parameter N_1, using the following equation and correct it to nearest integer.

$$N_1 = \frac{\cosh^{-1}\left[\left(\frac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right)^{\frac{1}{2}}\right]}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$. Choose N such that $N \geq N_1$.

Q7.53 How are the poles of Chebyshev transfer function are located in s-plane?

The poles of the Chebyshev transfer function symmetrically lies on an ellipse in s-plane.

Q7.54 Write the properties of Chebyshev type-1 filters.

i. The magnitude response is equiripple in the passband and monotonic in the stopband.

ii. The Chebyshev type-1 filters are all pole design.

iii. The normalized magnitude function has a value of $1/\sqrt{1+\in^2}$ at the cutoff frequency Ω_c.

iv. The magnitude response approaches the ideal response as the value of N increases.

Q7.55 Compare the Butterworth and Chebyshev Type-1 filters.

Butterworth	**Chebyshev Type - 1**
i. All pole design.	i. All pole design.
ii. The poles lie on a circle in s- plane.	ii. The poles lie on an ellipse in s- plane.
iii. The magnitude response is maximally flat at the origin and monotonically decreasing function of Ω.	iii. The magnitude response is equiripple in passband and monotonically decreasing in the stopband.
iv. The normalized magnitude response has a value of $1/\sqrt{2}$ at the cutoff frequency Ω_c.	iv. The normalized magnitude response has a value of $1/\sqrt{1+\in^2}$ at the cutoff frequency Ω_c.
v. Only few parameters has to be calculated to determine the transfer function.	v. A large number of parameters has to be calculated to determine the transfer function.

7.11. MATLAB Programs

Program 7.1

Write a MATLAB program to design an IIR filter to satisfy the specifications of example 7.15.

```
%To design a Butterworth 2nd order lowpass filter using bilinear
transformation

clear all
clc

AP=0.6;                          %Gain at passband edge frequency
AS=0.1;                          %Gain at stopband edge frequency
PEF_D=0.35*pi;                   %Passband edge digital frequency
SEF_D=0.7*pi;                    %Stopband edge digital frequency
T=.1;                            %Sampling time
alpha_P=-20*log10(AP)            %Passband attenuation in dB
alpha_S=-20*log10(AS)            %stopband attenuation in dB

PEF_A=(2/T)*tan((PEF_D)/2)
SEF_A=(2/T)*tan((SEF_D)/2)
[N,CF]=buttord(PEF_A,SEF_A,alpha_P,alpha_S,'s')
                                 %Order and cutoff frequency

[Bn,An]=butter(N,1,'s');         %Normalised transfer function
display('Normalised Transfer Function is,')
Hsn=tf(Bn,An)

[B,A]=butter(N,CF,'s');          %Unnormalised transfer function
display('Unnormalised Transfer Function is,')
Hs=tf(B,A)

[num,den]=bilinear(B,A,1/T);     %Digital transfer function
display('Digital Transfer Function is,')
Hz=tf(num,den,T)

w=0:pi/16:pi;
display('Frequency Response is,')
Hw=freqz(num,den,w)              %Frequency response
display('Magnitude Response is,')
Hw_mag=abs(Hw)                   %Magnitude response
plot(w/pi,Hw_mag,'k');grid;
title('Magnitude Response of Butterworth 2nd Order Lowpass Filter','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

```
alpha_P  =  4.4370
alpha_S  =  20
PEF_A    =  12.2560
SEF_A    =  39.2522

N  = 2
CF = 12.4439
```

```
Normalised Transfer Function is,
   Transfer function:
               1
        -----------------
        s^2 + 1.414 s + 1

Unnormalised Transfer Function is,
   Transfer function:
               154.8
        ---------------------
        s^2 + 17.6 s + 154.8

Digital Transfer Function is,
   Transfer function:
        0.1708 z^2 + 0.3415 z + 0.1708
        ------------------------------
            z^2 - 0.5407 z + 0.2237

Sampling time: 0.1

Frequency Response is,
Hw =
 Columns 1 through 8
   1.0000                0.9743 - 0.2237i   0.8885 - 0.4474i   0.7215 - 0.6526i
   0.4654 - 0.7869i   0.1696 - 0.7865i  -0.0658 - 0.6518i  -0.1837 - 0.4632i

 Columns 9 through 16
  -0.2063 - 0.2962i  -0.1805 - 0.1763i  -0.1388 - 0.0987i  -0.0972 - 0.0514i
  -0.0617 - 0.0241i  -0.0343 - 0.0095i  -0.0151 - 0.0027i  -0.0037 - 0.0003i

 Column 17
   -0.0000 - 0.0000i


Magnitude Response is,
Hw_mag =
 Columns 1 through 16
    1.0000    0.9997    0.9948    0.9729    0.9142    0.8046    0.6551    0.4983
    0.3610    0.2523    0.1703    0.1099    0.0663    0.0356    0.0153    0.0038

  Column 17
    0.0000
```

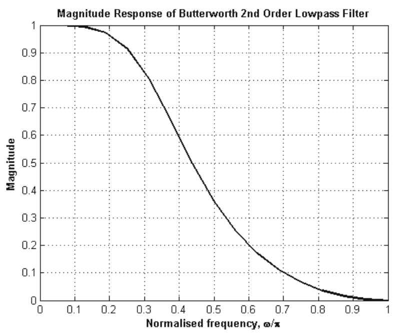


***Fig P7.1 :** Magnitude response.*

Program 7.2

Write a MATLAB program to design an IIR filter to satisfy the specifications of example 7.16.

```
%To design a Butterworth 2nd order highpass filter using bilinear
transformation

clear all
clc

AP=0.6;                               %Gain at passband edge frequency
AS=0.1;                               %Gain at stopband edge frequency
PEF_D=0.35*pi;                        %Passband edge digital frequency
SEF_D=0.7*pi;                         %Stopband edge digital frequency
T=.1;                                 %Sampling time
alpha_P=-20*log10(AP)                 %Passband attenuation in dB
alpha_S=-20*log10(AS)                 %stopband attenuation in dB

PEF_A=(2/T)*tan((PEF_D)/2)
SEF_A=(2/T)*tan((SEF_D)/2)

[N,CF]=buttord(PEF_A,SEF_A,alpha_P,alpha_S,'s')
                                      %Order and cutoff frequency

[Bn,An]=butter(N,1,'s');              %Normalised transfer function
display('Normalised Transfer Function is,')
Hsn=tf(Bn,An)

[B,A]=butter(N,CF,'high','s');        %Unnormalised transfer function
display('Unnormalised Transfer Function is,')
Hs=tf(B,A)

[num,den]=bilinear(B,A,1/T);          %Digital transfer function
display('Digital Transfer Function is,')
Hz=tf(num,den,T)

w=0:pi/16:pi;
display('Frequency Response is,')
Hw=freqz(num,den,w)                   %Frequency response
display('Magnitude Response is,')
Hw_mag=abs(Hw)                        %Magnitude response
plot(w/pi,Hw_mag,'k');grid;
title('Magnitude Response of Butterworth 2nd Order Highpass Filter','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

```
alpha_P   =  4.4370
alpha_S   =  20
PEF_A     =  12.2560
SEF_A     =  39.2522

N  =   2
CF =   12.4439
```

```
Normalised Transfer Function is,
   Transfer function:
                          1
                  -----------------
                  s^2 + 1.414 s + 1

Unnormalised Transfer Function is,
   Transfer function:
                         s^2
                  --------------------
                  s^2 + 17.6 s + 154.8

Digital Transfer Function is,
   Transfer function:
                  0.4411 z^2 - 0.8822 z + 0.4411
                  ------------------------------
                      z^2 - 0.5407 z + 0.2237

Sampling time: 0.1

Frequency Response is,
Hw =

  Columns 1 through 8
    0.0000              -0.0244 + 0.0056i   -0.0908 + 0.0457i   -0.1715 + 0.1551i
   -0.2063 + 0.3488i   -0.1252 + 0.5805i    0.0759 + 0.7517i    0.3196 + 0.8059i

  Columns 9 through 16
    0.5330 + 0.7652i    0.6922 + 0.6762i    0.8032 + 0.5709i    0.8786 + 0.4646i
    0.9295 + 0.3629i    0.9632 + 0.2666i    0.9845 + 0.1750i    0.9962 + 0.0867i

  Column 17
    1.0000 + 0.0000i

Magnitude Response is,
Hw_mag =

  Columns 1 through 16
    0.0000    0.0251    0.1017    0.2313    0.4052    0.5938    0.7555    0.8670
    0.9326    0.9676    0.9854    0.9939    0.9978    0.9994    0.9999    1.0000

  Column 17
     1.0000
```

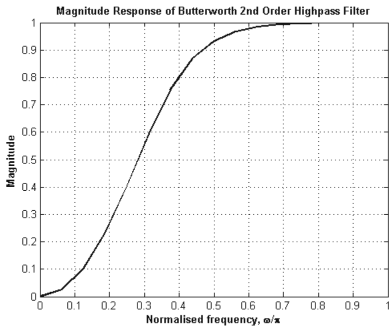


Fig P7.2 : Magnitude response.

Program 7.3

Write a MATLAB program to design an IIR filter to satisfy the specifications of example 7.17.

```
%To design a Butterworth 3rd order lowpass filter using bilinear transformation

clear all
clc

AP=0.707;                            %Gain at passband edge frequency
AS=0.2;                              %Gain at stopband edge frequency
PEF_D=0.45*pi;                       %Passband edge digital frequency
SEF_D=0.65*pi;                       %Stopband edge digital frequency
T=.5;                                %Sampling time
alpha_P=-20*log10(AP)                %Passband attenuation in dB
alpha_S=-20*log10(AS)                %stopband attenuation in dB

PEF_A=(2/T)*tan((PEF_D)/2)
SEF_A=(2/T)*tan((SEF_D)/2)

[N,CF]=buttord(PEF_A,SEF_A,alpha_P,alpha_S,'s')
                                     %Order and cutoff frequency

[Bn,An]=butter(N,1,'s');             %Normalised transfer function
display('Normalised Transfer Function is,')
Hsn=tf(Bn,An)

[B,A]=butter(N,CF,'s');              %Unnormalised transfer function
display('Unnormalised Transfer Function is,')
Hs=tf(B,A)

[num,den]=bilinear(B,A,1/T);         %Digital transfer function
display('Digital Transfer Function is,')
Hz=tf(num,den,T)

w=0:pi/16:pi;
display('Frequency Response is,')
Hw=freqz(num,den,w)                  %Frequency response
display('Magnitude Response is,')
Hw_mag=abs(Hw)                       %Magnitude response
plot(w/pi,Hw_mag,'k');grid;
title('Magnitude Response of Butterworth 3rd Order Lowpass Filter','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

```
alpha_P  =   3.0116
alpha_S  =  13.9794
PEF_A    =   3.4163
SEF_A    =   6.5274

N  =  3
CF =   3.8433
```

```
Normalised Transfer Function is,
   Transfer function:
                        1
                 ---------------------
                 s^3 + 2 s^2 + 2 s + 1

Unnormalised Transfer Function is,
   Transfer function:
                            56.77
                 ---------------------------------
                 s^3 + 7.687 s^2 + 29.54 s + 56.77

Digital Transfer Function is,
   Transfer function:
                 0.1569 z^3 + 0.4706 z^2 + 0.4706 z + 0.1569
                 -------------------------------------------
                   z^3 - 0.07324 z^2 + 0.3348 z - 0.006666

Sampling time: 0.5

Frequency Response is,
Hw =

  Columns 1 through 8
    1.0000              0.9790 - 0.2039i   0.9142 - 0.4051i   0.7999 - 0.5994i
    0.6243 - 0.7771i    0.3701 - 0.9134i   0.0295 - 0.9474i  -0.3307 - 0.7816i

  Columns 9 through 16
   -0.5136 - 0.4202i  -0.4296 - 0.0963i  -0.2521 + 0.0434i  -0.1186 + 0.0628i
   -0.0460 + 0.0429i  -0.0138 + 0.0205i  -0.0026 + 0.0065i  -0.0002 + 0.0008i

  Column 17
    0.0000 + 0.0000i

Magnitude Response is,
Hw_mag =

  Columns 1 through 16
    1.0000    1.0000    1.0000    0.9995    0.9968    0.9855    0.9478    0.8487
    0.6636    0.4402    0.2558    0.1342    0.0629    0.0248    0.0070    0.0008

  Column 17
    0.0000
```

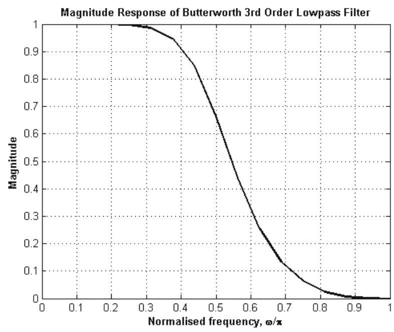


Fig P7.3 : *Magnitude response.*

Program 7.4

Write a MATLAB program to design an IIR filter to satisfy the specifications of example 7.18.

```
%To design a Butterworth 3rd order highpass filter using bilinear
transformation

clear all
clc

AP=0.707;                           %Gain at passband edge frequency
AS=0.2;                             %Gain at stopband edge frequency
PEF_D=0.45*pi;                      %Passband edge digital frequency
SEF_D=0.65*pi;                      %Stopband edge digital frequency
T=.5;                               %Sampling time
alpha_P=-20*log10(AP)               %Passband attenuation in dB
alpha_S=-20*log10(AS)               %stopband attenuation in dB

PEF_A=(2/T)*tan((PEF_D)/2)
SEF_A=(2/T)*tan((SEF_D)/2)

[N,CF]=buttord(PEF_A,SEF_A,alpha_P,alpha_S,'s')
                                    %Order and cutoff frequency

[Bn,An]=butter(N,1,'s');            %Normalised transfer function
display('Normalised Transfer Function is,')
Hsn=tf(Bn,An)

[B,A]=butter(N,CF,'high','s');      %Unnormalised transfer function
display('Unnormalised Transfer Function is,')
Hs=tf(B,A)

[num,den]=bilinear(B,A,1/T);        %Digital transfer function
display('Digital Transfer Function is,')
Hz=tf(num,den,T)

w=0:pi/16:pi;
display('Frequency Response is,')
Hw=freqz(num,den,w)                 %Frequency response
display('Magnitude Response is,')
Hw_mag=abs(Hw)                      %Magnitude response
plot(w/pi,Hw_mag,'k');grid;
title('Magnitude Response of Butterworth 3rd Order Highpass Filter','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

```
alpha_P   =   3.0116
alpha_S   =  13.9794
PEF_A     =   3.4163
SEF_A     =   6.5274

N  =  3
CF =  3.8433
```

```
Normalised Transfer Function is,
   Transfer function:
                  1
         ---------------------
         s^3 + 2 s^2 + 2 s + 1

Unnormalised Transfer Function is,
   Transfer function:
                         s^3
         --------------------------------
         s^3 + 7.687 s^2 + 29.54 s + 56.77

Digital Transfer Function is,
   Transfer function:
         0.1768 z^3 - 0.5305 z^2 + 0.5305 z - 0.1768
         -------------------------------------------
            z^3 - 0.07324 z^2 + 0.3348 z - 0.006666

Sampling time: 0.5

Frequency Response is,
Hw =

  Columns 1 through 8
    0.0000             -0.0002 - 0.0011i  -0.0036 - 0.0081i  -0.0189 - 0.0252i
   -0.0623 - 0.0500i   -0.1572 - 0.0637i  -0.3186 - 0.0099i  -0.4871 + 0.2061i

  Columns 9 through 16
   -0.4737 + 0.5790i   -0.1964 + 0.8762i   0.1642 + 0.9527i   0.4640 + 0.8756i
    0.6805 + 0.7300i    0.8296 + 0.5578i   0.9269 + 0.3752i   0.9821 + 0.1884i

  Column 17
    1.0000 + 0.0000i

Magnitude Response is,
Hw_mag =

  Columns 1 through 16
    0.0000    0.0011    0.0089    0.0315    0.0799    0.1697    0.3188    0.5289
    0.7481    0.8979    0.9667    0.9909    0.9980    0.9997    1.0000    1.0000

  Column 17
    1.0000
```

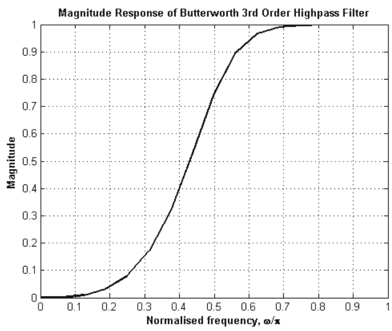


Fig P7.4 : *Magnitude response.*

Program 7.5

Write a MATLAB program to design an IIR filter to satisfy the specifications of example 7.19.

```
%To design a Butterworth 2nd order lowpass filter using impulse
invariant transformation

clear all
clc

AP=0.707;                               %Gain at passband edge frequency
AS=0.2;                                 %Gain at stopband edge frequency
PEF_D=0.3*pi;                           %Passband edge digital frequency
SEF_D=0.75*pi;                          %Stopband edge digital frequency
T=1;                                    %Sampling time
alpha_P=-20*log10(AP)                   %Passband attenuation in dB
alpha_S=-20*log10(AS)                   %stopband attenuation in dB

PEF_A=PEF_D/T
SEF_A=SEF_D/T

[N,CF]=buttord(PEF_A,SEF_A,alpha_P,alpha_S,'s')
                                        %Order and cutoff frequency

[Bn,An]=butter(N,1,'s');                %Normalised transfer function
display('Normalised Transfer Function is,')
Hsn=tf(Bn,An)

[B,A]=butter(N,CF,'s');                 %Unnormalised transfer function
display('Unnormalised Transfer Function is,')
Hs=tf(B,A)

[num,den]=impinvar(B,A,1/T); %Digital transfer function
display('Digital Transfer Function is,')
Hz=tf(num,den,T)

w=0:pi/16:pi;
display('Frequency Response is,')
Hw=freqz(num,den,w)           %Frequency response
display('Magnitude Response is,')
Hw_mag=abs(Hw)                          %Magnitude response
plot(w/pi,Hw_mag,'k');grid;
title('Magnitude Response of Butterworth 2nd Order Lowpass Filter','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

```
alpha_P  =    3.0116
alpha_S  =   13.9794
PEF_A    =    0.9425
SEF_A    =    2.3562

N  =  2
CF =   1.0645
```

```
Normalised Transfer Function is,
   Transfer function:
                    1
              -----------------
              s^2 + 1.414 s + 1

Unnormalised Transfer Function is,
   Transfer function:
                    1.133
              --------------------
              s^2 + 1.505 s + 1.133

Digital Transfer Function is,
   Transfer function:
                  0.4848 z
              ----------------------
              z^2 - 0.6876 z + 0.2219

Sampling time: 1

Frequency Response is,
Hw =

 Columns 1 through 8
   0.9074              0.8721 - 0.2591i   0.7549 - 0.5094i   0.5402 - 0.7112i
   0.2562 - 0.7990i  -0.0101 - 0.7493i  -0.1887 - 0.6167i  -0.2777 - 0.4718i

 Columns 9 through 16
  -0.3092 - 0.3499i  -0.3118 - 0.2570i  -0.3025 - 0.1883i  -0.2898 - 0.1372i
  -0.2776 - 0.0984i  -0.2674 - 0.0678i  -0.2599 - 0.0426i  -0.2554 - 0.0206i

 Column 17
  -0.2539 - 0.0000i

Magnitude Response is,
Hw_mag =

 Columns 1 through 16
    0.9074    0.9098    0.9107    0.8931    0.8391    0.7493    0.6449    0.5475
    0.4669    0.4041    0.3563    0.3207    0.2945    0.2759    0.2634    0.2562

 Column 17
    0.2539
```

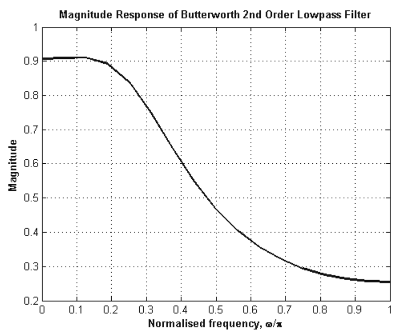


Fig P7.5 : *Magnitude response.*

Program 7.6

Write a MATLAB program to design an IIR filter to satisfy the specifications of example 7.20.

```
%To design a Butterworth 3rd order lowpass filter using impulse
invariant transformation

clear all
clc

AP=0.9;                                %Gain at passband edge frequency
AS=0.275;                              %Gain at stopband edge frequency
PEF_D=0.35*pi;                         %Passband edge digital frequency
SEF_D=0.7*pi;                          %Stopband edge digital frequency
T=1;                                   %Sampling time
alpha_P=-20*log10(AP)                  %Passband attenuation in dB
alpha_S=-20*log10(AS)                  %stopband attenuation in dB

PEF_A=PEF_D/T
SEF_A=SEF_D/T

[N,CF]=buttord(PEF_A,SEF_A,alpha_P,alpha_S,'s')
                                       %Order and cutoff frequency

[Bn,An]=butter(N,1,'s');               %Normalised transfer function
display('Normalised Transfer Function is,')
Hsn=tf(Bn,An)

[B,A]=butter(N,CF,'s');                %Unnormalised transfer function
display('Unnormalised Transfer Function is,')
Hs=tf(B,A)

[num,den]=impinvar(B,A,1/T); %Digital transfer function
display('Digital Transfer Function is,')
Hz=tf(num,den,T)

w=0:pi/16:pi;
display('Frequency Response is,')
Hw=freqz(num,den,w)          %Frequency response
display('Magnitude Response is,')
Hw_mag=abs(Hw)                         %Magnitude response
plot(w/pi,Hw_mag,'k');grid;
title('Magnitude Response of Butterworth 3rd Order Lowpass Filter','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

```
alpha_P =    0.9151
alpha_S =   11.2133
PEF_A   =    1.0996
SEF_A   =    2.1991

N  =  3
CF =  1.4489
```

```
Normalised Transfer Function is,
  Transfer function:
                     1
          ---------------------
          s^3 + 2 s^2 + 2 s + 1

Unnormalised Transfer Function is,
  Transfer function:
                       3.042
          ---------------------------------
          s^3 + 2.898 s^2 + 4.199 s + 3.042

Digital Transfer Function is,
  Transfer function:
          -4.441e-016 z^3 + 0.5074 z^2 + 0.1985 z
          ---------------------------------------
           z^3 - 0.536 z^2 + 0.3055 z - 0.05514

Sampling time: 1

Frequency Response is,
Hw =

 Columns 1 through 8
   0.9881             0.9512 - 0.2706i   0.8404 - 0.5261i   0.6535 - 0.7489i
   0.3884 - 0.9106i   0.0598 - 0.9629i  -0.2662 - 0.8581i  -0.4807 - 0.6200i

 Columns 9 through 16
  -0.5352 - 0.3600i  -0.4835 - 0.1682i  -0.3985 - 0.0569i  -0.3192 - 0.0024i
  -0.2573 + 0.0193i  -0.2132 + 0.0237i  -0.1843 + 0.0192i  -0.1681 + 0.0104i

 Column 17
  -0.1628 + 0.0000i


Magnitude Response is,
Hw_mag =

 Columns 1 through 16
    0.9881    0.9890    0.9915    0.9939    0.9900    0.9648    0.8985    0.7845
    0.6450    0.5119    0.4025    0.3192    0.2580    0.2145    0.1853    0.1684

 Column 17
    0.1628
```

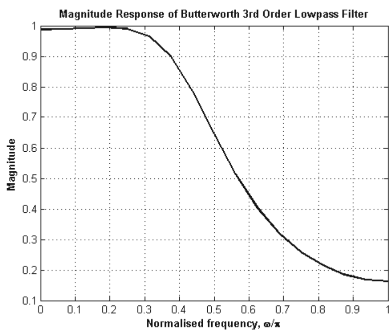


Fig P7.6 : Magnitude response.

Program 7.7

Write a MATLAB program to design an IIR filter to satisfy the specifications of example 7.21.

```
%To design a Butterworth 4th order lowpass filter using impulse invariant
transformation

clear all
clc

AP=0.8;                              %Gain at passband edge frequency
AS=0.2;                              %Gain at stopband edge frequency
PEF_D=0.2*pi;                        %Passband edge digital frequency
SEF_D=0.32*pi;                       %Stopband edge digital frequency
T=1;                                 %Sampling time
alpha_P=-20*log10(AP)                %Passband attenuation in dB
alpha_S=-20*log10(AS)                %stopband attenuation in dB

PEF_A=PEF_D/T
SEF_A=SEF_D/T

[N,CF]=buttord(PEF_A,SEF_A,alpha_P,alpha_S,'s')
                                     %Order and cutoff frequency

[Bn,An]=butter(N,1,'s');             %Normalised transfer function
display('Normalised Transfer Function is,')
Hsn=tf(Bn,An)

[B,A]=butter(N,CF,'s');              %Unnormalised transfer function
display('Unnormalised Transfer Function is,')
Hs=tf(B,A)

[num,den]=impinvar(B,A,1/T); %Digital transfer function
display('Digital Transfer Function is,')
Hz=tf(num,den,T)

w=0:pi/16:pi;
display('Frequency Response is,')
Hw=freqz(num,den,w)          %Frequency response
display('Magnitude Response is,')
Hw_mag=abs(Hw)                       %Magnitude response
plot(w/pi,Hw_mag,'k');grid;
title('Magnitude Response of Butterworth 4th Order Lowpass Filter','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

```
alpha_P   =    1.9382
alpha_S   =   13.9794
PEF_A     =    0.6283
SEF_A     =    1.0053

N  =    4
CF =    0.6757
```

```
Normalised Transfer Function is,
   Transfer function:
                                 1
                  -------------------------------------------
                  s^4 + 2.613 s^3 + 3.414 s^2 + 2.613 s + 1

Unnormalised Transfer Function is,
   Transfer function:
                                  0.2085
                  -------------------------------------------------
                  s^4 + 1.766 s^3 + 1.559 s^2 + 0.8063 s + 0.2085

Digital Transfer Function is,
   Transfer function:
                  4.441e-016 z^4 + 0.02189 z^3 + 0.05529 z^2 + 0.009073 z
                  ---------------------------------------------------------
                      z^4 - 2.289 z^3 + 2.181 z^2 - 0.977 z + 0.1711

Sampling time: 1

Frequency Response is,
Hw =

  Columns 1 through 8
    1.0003              0.7191 - 0.6952i   -0.0382 - 0.9928i   -0.7624 - 0.4102i
   -0.4124 + 0.2461i   -0.0836 + 0.2022i   -0.0012 + 0.1075i    0.0140 + 0.0566i

  Columns 9 through 16
    0.0142 + 0.0312i    0.0117 + 0.0180i    0.0092 + 0.0109i    0.0072 + 0.0068i
    0.0058 + 0.0043i    0.0048 + 0.0027i    0.0041 + 0.0016i    0.0038 + 0.0007i

  Column 17
    0.0037 + 0.0000i


Magnitude Response is,
Hw_mag =

  Columns 1 through 16
    1.0003    1.0002    0.9936    0.8657    0.4802    0.2188    0.1075    0.0583
    0.0343    0.0215    0.0142    0.0099    0.0072    0.0055    0.0044    0.0039

  Column 17
     0.0037
```

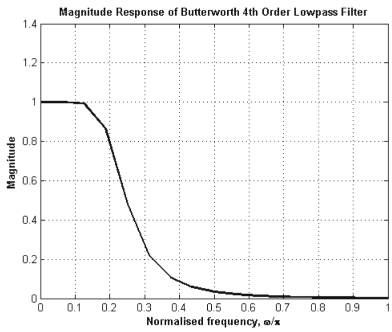


Fig P7.7 : Magnitude response.

Program 7.8

Write a MATLAB program to design an IIR filter to satisfy the specifications of example 7.22.

```
%To design a Butterworth 4th order lowpass filter using bilinear
transformation

clear all
clc

AP=0.707;                           %Gain at passband edge frequency
AS=0.08;                            %Gain at stopband edge frequency
PEF_D=0.2*pi;                       %Passband edge digital frequency
SEF_D=0.4*pi;                       %Stopband edge digital frequency
T=1;                                %Sampling time
alpha_P=-20*log10(AP)               %Passband attenuation in dB
alpha_S=-20*log10(AS)               %stopband attenuation in dB

PEF_A=(2/T)*tan((PEF_D)/2)
SEF_A=(2/T)*tan((SEF_D)/2)

[N,CF]=buttord(PEF_A,SEF_A,alpha_P,alpha_S,'s')
                                    %Order and cutoff frequency

[Bn,An]=butter(N,1,'s');            %Normalised transfer function
display('Normalised Transfer Function is,')
Hsn=tf(Bn,An)

[B,A]=butter(N,CF,'s');             %Unnormalised transfer function
display('Unnormalised Transfer Function is,')
Hs=tf(B,A)

[num,den]=bilinear(B,A,1/T); %Digital transfer function
display('Digital Transfer Function is,')
Hz=tf(num,den,T)

w=0:pi/16:pi;
display('Frequency Response is,')
Hw=freqz(num,den,w)          %Frequency response
display('Magnitude Response is,')
Hw_mag=abs(Hw)                      %Magnitude response
plot(w/pi,Hw_mag,'k');grid;
title('Magnitude Response of Butterworth 4th Order Lowpass Filter','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

```
alpha_P   =    3.0116
alpha_S   =   21.9382
PEF_A     =    0.6498
SEF_A     =    1.4531

N  =  4
CF =   0.7734
```

```
Normalised Transfer Function is,
   Transfer function:
                                  1
                  ------------------------------------------
                  s^4 + 2.613 s^3 + 3.414 s^2 + 2.613 s + 1

Unnormalised Transfer Function is,
   Transfer function:
                                0.3578
                  -----------------------------------------------
                  s^4 + 2.021 s^3 + 2.042 s^2 + 1.209 s + 0.3578

Digital Transfer Function is,
   Transfer function:
                  0.008299 z^4 + 0.0332 z^3 + 0.0498 z^2 + 0.0332 z + 0.008299
                  --------------------------------------------------------------
                          z^4 - 2.089 z^3 + 1.898 z^2 - 0.8134 z + 0.1378

Sampling time: 1

Frequency Response is,
Hw =

  Columns 1 through 8
    1.0000              0.7827 - 0.6224i   0.1659 - 0.9837i   -0.6317 - 0.6896i
   -0.5858 + 0.1509i   -0.1308 + 0.2296i  -0.0035 + 0.1114i    0.0143 + 0.0471i

  Columns 9 through 16
    0.0114 + 0.0192i    0.0068 + 0.0076i   0.0035 + 0.0028i    0.0016 + 0.0009i
    0.0006 + 0.0003i    0.0002 + 0.0001i   0.0000 + 0.0000i    0.0000 + 0.0000i

  Column 17
   -0.0000 - 0.0000i

Magnitude Response is,
Hw_mag =

  Columns 1 through 16
    1.0000    1.0000    0.9976    0.9352    0.6049    0.2642    0.1115    0.0492
    0.0224    0.0101    0.0045    0.0018    0.0007    0.0002    0.0000    0.0000

  Column 17
    0.0000
```

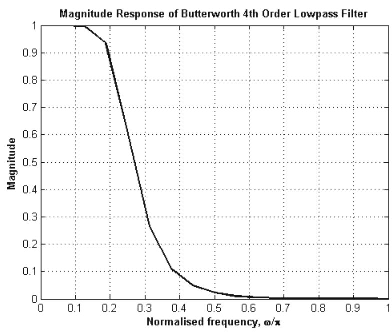


Fig P7.8 : Magnitude response.

Program 7.9

Write a MATLAB program to design an IIR filter to satisfy the specifications of example 7.23.

```
%To design a Chebyshev 3rd order lowpass filter using impulse
invariant transformation

clear all
clc

AP=0.9;                                 %Gain at passband edge frequency
AS=0.24;                                %Gain at stopband edge frequency
PEF_D=0.25*pi;                          %Passband edge digital frequency
SEF_D=0.5*pi;                           %Stopband edge digital frequency
T=1;                                    %Sampling time
alpha_P=-20*log10(AP)                   %Passband attenuation in dB
alpha_S=-20*log10(AS)                   %stopband attenuation in dB

PEF_A=PEF_D/T
SEF_A=SEF_D/T

[N,CF]=cheb1ord(PEF_A,SEF_A,alpha_P,alpha_S,'s')
                                        %Order and cutoff frequency

[Bn,An]=cheby1(N,alpha_P,1,'s');        %Normalised transfer function
display('Normalised Transfer Function is,')
Hsn=tf(Bn,An)

[B,A]=cheby1(N,alpha_P,CF,'s');         %Unnormalised transfer function
display('Unnormalised Transfer Function is,')
Hs=tf(B,A)

[num,den]=impinvar(B,A,1/T); %Digital transfer function
display('Digital Transfer Function is,')
Hz=tf(num,den,T)

w=0:pi/16:pi;
display('Frequency Response is,')
Hw=freqz(num,den,w)          %Frequency response
display('Magnitude Response is,')
Hw_mag=abs(Hw)                          %Magnitude response
plot(w/pi,Hw_mag,'k');grid;
title('Magnitude Response of Chebyshev 3rd Order Lowpass Filter','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

```
alpha_P  =   0.9151
alpha_S  =  12.3958
PEF_A    =   0.7854
SEF_A    =   1.5708

N  =  3
CF =  0.7854
```

```
Normalised Transfer Function is,
   Transfer function:
                           0.5162
              ----------------------------------
              s^3 + 1.021 s^2 + 1.272 s + 0.5162

Unnormalised Transfer Function is,
   Transfer function:
                            0.2501
              ------------------------------------
              s^3 + 0.8022 s^2 + 0.7844 s + 0.2501

Digital Transfer Function is,
   Transfer function:
              5.551e-017 z^3 + 0.09112 z^2 + 0.06993 z
              ----------------------------------------
                 z^3 - 1.852 z^2 + 1.461 z - 0.4484

Sampling time: 1

Frequency Response is,
Hw =

 Columns 1 through 8
   0.9997              0.7886 - 0.5273i   0.4090 - 0.8019i  -0.1055 - 0.9597i
  -0.7930 - 0.4270i  -0.4297 + 0.1435i  -0.1730 + 0.1408i  -0.0814 + 0.0955i

 Columns 9 through 16
  -0.0438 + 0.0642i  -0.0260 + 0.0442i  -0.0167 + 0.0311i  -0.0114 + 0.0221i
  -0.0083 + 0.0155i  -0.0064 + 0.0106i  -0.0052 + 0.0066i  -0.0046 + 0.0032i

 Column 17
  -0.0045 + 0.0000i


Magnitude Response is,
Hw_mag =

 Columns 1 through 16
    0.9997    0.9487    0.9002    0.9655    0.9007    0.4531    0.2230    0.1255
    0.0778    0.0513    0.0353    0.0248    0.0176    0.0124    0.0084    0.0056

 Column 171
    0.0045
```

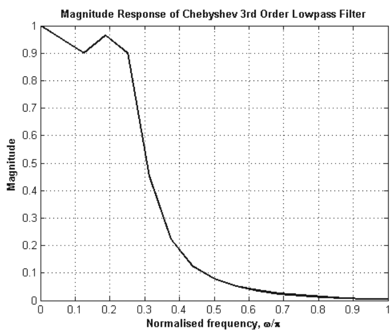


Fig P7.9 : *Magnitude response.*

Program 7.10

Write a MATLAB program to design an IIR filter to satisfy the specifications of example 7.24.

```
%To design a Chebyshev 3rd order lowpass filter using bilinear
transformation

clear all
clc

AP=0.8;                                   %Gain at passband edge frequency
AS=0.2;                                   %Gain at stopband edge frequency
PEF_D=0.2*pi;                             %Passband edge digital frequency
SEF_D=0.32*pi;                            %Stopband edge digital frequency
T=1;                                      %Sampling time
alpha_P=-20*log10(AP)                     %Passband attenuation in dB
alpha_S=-20*log10(AS)                     %stopband attenuation in dB

PEF_A=(2/T)*tan(PEF_D/2)
SEF_A=(2/T)*tan(SEF_D/2)

[N,CF]=cheb1ord(PEF_A,SEF_A,alpha_P,alpha_S,'s')
                                          %Order and cutoff frequency

[Bn,An]=cheby1(N,alpha_P,1,'s');          %Normalised transfer function
display('Normalised Transfer Function is,')
Hsn=tf(Bn,An)

[B,A]=cheby1(N,alpha_P,CF,'s');           %Unnormalised transfer function
display('Unnormalised Transfer Function is,')
Hs=tf(B,A)

[num,den]=bilinear(B,A,1/T); %Digital transfer function
display('Digital Transfer Function is,')
Hz=tf(num,den,T)

w=0:pi/16:pi;
display('Frequency Response is,')
Hw=freqz(num,den,w)             %Frequency response
display('Magnitude Response is,')
Hw_mag=abs(Hw)                            %Magnitude response
plot(w/pi,Hw_mag,'k');grid;
title('Magnitude Response of Chebyshev 3rd Order Lowpass Filter','fontweight','b');
xlabel('Normalised frequency, \omega/\pi','fontweight','b');
ylabel('Magnitude','fontweight','b');
```

OUTPUT

```
alpha_P   =    1.9382
alpha_S   =   13.9794
PEF_A     =    0.6498
SEF_A     =    1.0995

N  =  3
CF =   0.6498
```

```
Normalised Transfer Function is,
   Transfer function:
                              0.3333
            ----------------------------------
            s^3 + 0.7489 s^2 + 1.03 s + 0.3333

Unnormalised Transfer Function is,
   Transfer function:
                             0.09147
            -------------------------------------
            s^3 + 0.4867 s^2 + 0.4351 s + 0.09147

Digital Transfer Function is,
   Transfer function:
            0.008386 z^3 + 0.02516 z^2 + 0.02516 z + 0.008386
            ------------------------------------------------
                    z^3 - 2.274 z^2 + 1.967 z - 0.6263

Sampling time: 1

Frequency Response is,
Hw =

  Columns 1 through 8
    1.0000              0.5843 - 0.6284i   0.1071 - 0.8164i  -0.8586 - 0.3985i
   -0.2173 + 0.1864i  -0.0539 + 0.0878i  -0.0184 + 0.0427i  -0.0073 + 0.0223i

  Columns 9 through 16
   -0.0031 + 0.0120i  -0.0014 + 0.0065i  -0.0006 + 0.0035i  -0.0002 + 0.0018i
   -0.0001 + 0.0008i  -0.0000 + 0.0003i  -0.0000 + 0.0001i  -0.0000 + 0.0000i

  Column 17
    0.0000 + 0.0000i


Magnitude Response is,
Hw_mag =

  Columns 1 through 16
    1.0000    0.8581    0.8234    0.9466    0.2863    0.1030    0.0465    0.0234
    0.0124    0.0067    0.0035    0.0018    0.0008    0.0003    0.0001    0.0000

  Column 17
    0.0000
```

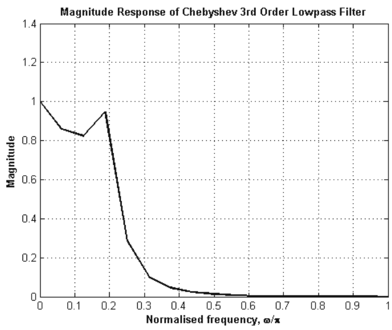


Fig P7.10 : Magnitude response.

7.12. Exercises

I. Fill in the blanks with appropriate words

1. The two popular techniques used to approximate the ideal frequency response are ——— and ——— approximation.
2. The two techniques used to transform analog filter to digital filter are ——— and ——— transformations.
3. The two properties which are preserved in analog to digital transformation are ——— and ———.
4. The tolerance in the passband and stopband are called ———.
5. In ——— transformation the impulse response of digital filter is the sampled version of the impulse response of analog filter.
6. In impulse invariant (or bilinear) mapping the ——— poles of s-plane are mapped into ——— of unit circle in z-plane.
7. In impulse invariant (or bilinear) mapping the right half poles of s-plane are mapped into ________ of unit circle in z-plane.
8. In impulse invariant (or bilinear) mapping the poles on the imaginary axis in s-plane are mapped into the ——— in z-plane.
9. In ——— transformation any strip of width 2p/T in s-plane is mapped into the entire z-plane.
10. The phenomena of high frequency components acquiring the identity of low frequency components is called ———.
11. The impulse invariant mapping is ——— mapping whereas bilinear mapping is ———.
12. The distortion in frequency axis due to nonlinear relationship between analog and digital frequency is called ———.
13. In bilinear transformation the effect of warping on ——— can be eliminated by ——— the analog filter.
14. In ——— approximation the magntidue response is maximally flat at the ———.
15. In Butterworth approximation the ——— is ——— decreasing function of frequency.
16. At the cutoff frequency the magnitude of the Butterworth filter is ——— times the maximum value.
17. In type-1 Chebyshev approximation the magnitude response is ——— in the passband and ——— in the stopband.
18. In type-2 Chebyshev approximation the magnitude response is monotonic in the ——— and equiripple in the ———.
19. The type-2 magnitude response is also called ——— response.
20. In Chebyshev approximation, the normalized magnitude response has a value of ——— at the cutoff frequency.

Answers

1.	Butterworth, Chebyshev	8.	unit circle	15.	magnitude function, monotonically
2.	impulse invariant, bilinear	9.	impulse invariant	16.	$1/\sqrt{2}$ or 0.707
3.	stability, causality	10.	aliasing	17.	equiripple,monotonic
4.	ripples	11.	many-to-one, one-to-one	18.	passband, stopband
5.	impulse invariant	12.	frequency warping	19.	inverse Chebyshev
6.	left half, interior	13.	amplitude response, prewarping	20.	$1/\sqrt{1+\epsilon^2}$
7.	exterior	14.	Buttrerworth, origin		

II. State whether the following statements are True/False

1. In IIR filters all the samples of impulse response are considered.
2. For direct relationship between analog and digital frequency, the imaginary axis in s-plane should map into unit circle in z-plane.
3. In analog to digital transformation the stability is preserved by mapping left half of s-plane into the interior of unit circle in the z-plane.
4. The bandwidth of the discrete signal is not affected by sampling frequency.
5. For a stable analog filter the poles should lie on the right half of s-plane.
6. For a stable digital filter the poles should lie on the unit circle.
7. The IIR filters will not have linear phase characteristics.
8. In impulse invariant transformation the frequency response characteristics of the analog filter is preserved.
9. In impulse invariant transformation the aliasing can be minimized by increasing the sampling time.
10. In impulse invariant transformation aliasing problem will arise if the sampling rate does not satisfy the Nyquist criteria.
11. Using impulse invariant transformation, only band limited analog filter can be transformed to digital filter without aliasing.
12. In impulse invariant transformation the problem of aliasing is due to many-to-one mapping.
13. In impulse invariant transformation the relation between analog and digital frequency is nonlinear.
14. In bilinear transformation the relation between analog and digital frequency is linear.
15. A linear phase analog filter can be transformed to linear phase digital filter using bilinear transformation.
16. In bilinear transformation the magnitude response of analog filter can be preserved by prewarping.
17. The poles of the Butterworth transfer function symmetrically lies on an unit circle in s-plane with angular spacing of 2p/2N.
18. In Butterworth (or Chebyshev) approximation the magnitude response approaches the ideal response as the order is increased.
19. In Chebyshev approximation the approximation function is selected such that the error is minimized over a prescribed band of frequencies.
20. The poles of Chebyshev transfer function symmetrically lies on an ellipse in s-plane.

Answers

1. True	5. False	9. False	13. False	17. True
2. True	6. False	10. True	14. False	18. True
3. True	7. True	11. True	15. False	19. True
4. False	8. True	12. True	16. True	20. True

III. Choose the right answer for the following questions

1. IIR filters are designed by considering all the

a) Infinite samples of frequency response

b) Finite samples of impulse response.

c) Infinite samples of impulse response.

d) None of the above.

2. ***For the analog and digital IIR filters to be casual, the number of zeros should be***

   a) ³ Number of poles. **b)** £ Numbers of poles.

   c) = Number of poles. **d)** Zero.

---

3. ***An analog filter has poles at s = 0, s = –2, s = –1. If impulse invariant transformation is employed then the corresponding poles of digital filters are respectively,***

   a) $0,\ e^{\frac{-T}{2}},\ e^{T}$ **b)** $1,\ e^{-2T},\ e^{T}$ **c)** $1,\ e^{2T},\ e^{-T}$ **d)** $0,\ e^{-2T},\ e^{-T}$

---

4. ***An analog filter transfer function is given by,*** $H(s) = \frac{3}{s+1}$. ***When the filter is transformed to digital filter using impulse invariant transformation, what are the poles and zeros of the filter?***

   a) Zero at z = 0, Pole at z = 0.368 **b)** Zero at z = 1, Pole at z = 0

   c) Zero at z = 0.368, Pole at z = 0 **d)** Zero at z = 0, Pole at z = 1

---

5. ***The digital lowpass Chebyshev filter with following specification is realized using impulse invariant transformation. What should be the attenuation constant and order N of the filter?***

   0.75 £ |H(w)| £ 1.0 ; 0 £ w £ 0.4p

   |H(w)| £ 0.05 ; 0.5p £ w £ p

   a) 0.9, $N \geq 10$ **b)** 0.1, $N \leq 20$ **c)** 0.882, $N \geq 6$ **d)** 0.7, $N \leq 5$

---

6. ***In Impulse invariant transformation the digital frequency 'w' for a given analog frequency, W is given by,***

   a) $\omega = \Omega T$ **b)** $\omega = \frac{\Omega}{T}$ **c)** $\omega = \frac{T}{\Omega}$ **d)** $\omega = \tan \Omega T$

---

7. ***In Impulse invariant transformation the analog system with transfer function,*** $H(s) = \frac{0.3}{s+0.7}$ ***is transformed to a digital system with transfer function,***

   a) $H(s) = \frac{-0.3}{1-e^{-0.7T}z^{-1}}$ **b)** $H(s) = \frac{0.3}{1-e^{-0.7T}z^{-1}}$

   c) $H(s) = \frac{0.7}{1-e^{-0.3T}z^{-1}}$ **d)** $H(s) = \frac{0.7}{1-e^{0.3T}z^{-1}}$

---

8. ***In bilinear transformation the analog system with transfer function,*** $H(s) = \frac{0.2}{s+0.9}$ ***is transformed to a digital system with transfer function,***

   a) $H(s) = \frac{0.2}{\frac{2}{T}\frac{1+z^{-1}}{1-z^{-1}}+0.9}$ **b)** $H(s) = \frac{0.2}{\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}+0.9}$

   c) $H(s) = \frac{0.2}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+0.9}$ **d)** $H(s) = \frac{0.2}{\frac{T}{2}\frac{1-z^{-1}}{1+z^{-1}}+0.9}$

---

9. ***The transfer function of a normalized lowpass filter can be transformed to a highpass filter with cutoff frequency,*** W_c ***by the transformation,***

   a) $s \rightarrow \frac{1}{s}$ **b)** $s \rightarrow \frac{\Omega_c}{s}$ **c)** $s \rightarrow \frac{s}{\Omega_c}$ **d)** $s \rightarrow \Omega_c$

---

10. The zeros of the Butterworth filters exist at

a) left half of s-plane. **b)** Origin

c) Infinity **d)** Right half of s-plane

11. The poles of Butterworth transfer function lie,

a) Symmetrically on a circle in s-plane

b) Symmetrically on an ellipse in s-plane

c) Antisymmetrically on a circle in s-plane

d) Antisymmetrically on an ellipse in s-plane

12. The poles of Buttterworth transfer function symmetrically lies on a circle in s-plane with angular spacing,

a) $\frac{\pi}{N}$ **b)** $\frac{\pi}{2N}$ **c)** $\frac{2\pi}{N}$ **d)** $\frac{\pi}{N^2}$

13. In Butterworth and Chebyshev transfer function, when N is even, the nature of poles are,

a) Complex and exist as conjugate pair

b) Complex but not conjugate pairs

c) One pole is complex and other poles are real

d) One pole is real and other poles are complex

14. The Butterworth and Chebyshev transfer function, when N is odd, the nature of poles are,

a) Complex and exist as conjugate pair

b) Complex but not conjugate pairs

c) One pole is complex and other poles are real

d) One pole is real and other poles are complex

15. Consider the digital lowpass butterworth filter with following specification.

0.9 £ |H(w)| £ 1.0 ; 0 £ w £ 0.2p

|H(w)| £ 0.1 ; 0.4p £ w £ p

What should be the order of the filter to realize the above specifications using bilinear transformation?

a) $N \geq 3$ **b)** $N \geq 20$ **c)** $N \geq 4$ **d)** $N \geq 5$

16. The relation between analog and digital frequency is nonlinear in case of

a) Impulse invariant transformation. **b)** Bilinear transformation.

c) Frequency sampling. **d)** All the above.

17. The normalized transfer function of 3rd order lowpass Butterworth filter is

a) $\frac{1}{s^3 + 1.414 s_n^2 + s_n + 1}$ **b)** $\frac{1}{(s_n + 1)(s_n^2 + s_n + 1)}$

c) $\frac{1}{s^2 (s_n + 1)}$ **d)** $\frac{1}{s_n^3 + s_n^2 + s_n + 1}$

18. The unnormalized transfer function of lowpass Butterworth filter is obtained from normalized transfer function by replacing s_n by,

a) $\frac{s_n}{\Omega_c}$ **b)** $s_n\,\Omega_c$ **c)** $\frac{s}{\Omega_c}$ **d)** $s\Omega_c$

19. Which of the following is true for a Chebyshev analog filter?

a) In type-1, the magnitude response is monotonic in passband and equiripple in stopband.

b) In type-1 the manitude response is monotonic in passband and stopband.

c) In type-2 the magnitude response is equiripple in passband and stopband.

d) In type-2 the magnitude response is monotonic in passband and equiripple in stopband.

20. The poles of Chebyshev transfer function lie,

a) Symmetrically on a circle in s-plane

b) Symmetrically on an ellipse in s-plane

c) Antisymmetrically on a circle in s-plane

d) Antisymmetrically on an ellipse in s-plane

Answers

1. c	5. c	9. b	13. a	17. b
2. b	6. a	10. c	14. d	18. c
3. d	7. b	11. a	15. a	19. d
4. a	8. c	12. a	16. b	20. b

IV. Answer the following questions

1. Discuss the advantages and disadvantages of digital filters.
2. Sketch the ideal and practical frequency response of four basic types of analog filters and mark the important filter specifications.
3. Sketch the ideal and practical frequency response of four basic types of digital IIR filters and mark the important filter specifications.
4. Derive the impulse invariant transformation to transform an analog system to digital system.
5. Explain the mapping of s-plane to z-plane in impulse invariant transformation.
6. Derive the relation between analog and digital frequency in impulse invariant transformation.
7. Derive the bilinear transformation to transform an analog system to digital system.
8. Explain the mapping of s-plane to z-plane in bilinear transformation.
9. Derive the relation between analog and digital frequency in bilinear transformation.
10. Discuss the Butterworth approximation.
11. Derive the expression to determine the poles of Butterworth filter.
12. Write the procedure for design of lowpass digital Butterworth IIR filter.
13. Discuss the Chebyshev approximation.
14. Write the procedure for design of lowpass digital Chebyshev IIR filter.
15. Discuss the frequency transformation IIR filters.

V. Solve the following problems

E7.1. ***For the analog transfer function,*** $H(s) = \dfrac{(s+1)}{(s+2)(s+4)}$ ***, determine H(z) using impulse invariant transformation if (a) T = 1 second and (b) T = 0.5 second.***

E7.2. ***Convert the analog filter with system transfer function,*** $H(s) = \dfrac{s+0.7}{s^2+1.4s+4.49}$ ***into a digital IIR filter by means of the impulse invariant method.***

E7.3. ***Using impulse invariant transformation convert the following analog filter transfer function to digital filter transfer function by taking sampling time, T = 0.5 second,*** $H(s) = \dfrac{1}{s^2+2s+10}$.

E7.4. ***For the analog transfer function,*** $H(s) = \dfrac{0.8}{s^2+1.6s+9.64}$***, determine H(z) using bilinear transformation if (a) T = 1 second and (b) T = 0.6 second.***

E7.5. ***Obtain H(z) from H(s) when T = 1 second and*** $H(s) = \dfrac{4s}{(s+0.5)(s+4)}$.

E7.6. ***Obtain H(z) from H(s) when T = 0.1 second, and*** $H(s) = \dfrac{0.6s^3}{s^3+4s^2+0.9s+1}$.

E7.7. ***Convert the analog filter with system function H(s) into digital filter using bilinear transformation.***

$$H(s) = \frac{(s+0.1)}{(s+0.1)^2+5} \quad ; \quad \text{Take, } T = 0.2$$

E7.8. ***Design a Butterworth digital IIR lowpass filter using bilinear transformation by taking T = 0.2 second, to satisfy the following specifications.***

$$0.8 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.4\pi$$

$$|H(e^{j\omega})| \le 0.3 \quad ; \quad \text{for } 0.7\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternate specification,
- ***Passband ripple*** $\le$ ***1.9382 dB***
- ***Stopband attenuation*** $\ge$ ***10.4576 dB***
- ***Passband edge frequency = 0.4π rad/sample***
- ***Stopband edge frequency = 0.7π rad/sample***

E7.9. ***Design a Butterworth digital IIR highpass filter using bilinear transformation by taking T = 0.2 second, to satisfy the following specifications.***

$$0.8 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0.7\pi \le \omega \le \pi$$

$$|H(e^{j\omega})| \le 0.3 \quad ; \quad \text{for } 0 \le \omega \le 0.4\pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternate specification,
- ***Passband ripple*** $\le$ ***1.9382 dB***
- ***Stopband attenuation*** $\ge$ ***10.4576 dB***
- ***Passband edge frequency = 0.7π rad/sample***
- ***Stopband edge frequency = 0.4π rad/sample***

E7.10. ***Design a Butterworth digital IIR lowpass filter using bilinear transformation by taking T = 0.3 second, to satisfy the following specifications.***

$$0.45 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.675\pi$$

$$|H(e^{j\omega})| \le 0.15 \quad ; \quad \text{for } 0.8\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternate specification,
- ***Passband ripple*** $\le$ ***6.9357 dB***
- ***Stopband attenuation*** $\ge$ ***16.4781 dB***
- ***Passband edge frequency = 0.675π rad/sample***
- ***Stopband edge frequency = 0.8π rad/sample***

E7.11. ***Design a Butterworth digital IIR highpass filter using bilinear transformation by taking $T = 0.3$second, to satisfy the following specifications.***

$$0.45 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0.8\pi \le \omega \le \pi$$

$$|H(e^{j\omega})| \le 0.15 \quad ; \quad \text{for } 0 \le \omega \le 0.675\pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternate specification,

Passband ripple ≤ 6.9357 dB

Stopband attenuation ≥ 16.4781 dB

Passband edge frequency = 0.8π rad/sample

Stopband edge frequency = 0.675π rad/sample

E7.12. ***Design a Butterworth digital IIR lowpass filter using impulse invariant transformation by taking $T = 0.8$second, to satisfy the following specifications.***

$$0.8 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.3\pi$$

$$|H(e^{j\omega})| \le 0.3 \quad ; \quad \text{for } 0.7\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternate specification,

Passband ripple ≤ 1.9382 dB

Stopband attenuation ≥ 10.4576 dB

Passband edge frequency = 0.3π rad/sample

Stopband edge frequency = 0.7π rad/sample

E7.13. ***Design a Butterworth digital IIR lowpass filter using impulse invariant transformation by taking $T = 1$second, to satisfy the following specifications.***

$$0.45 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad 0 \le \omega \le 0.5\pi$$

$$|H(e^{j\omega})| \le 0.15 \quad ; \quad 0.8\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternate specification,

Passband ripple ≤ 6.9357 dB

Stopband attenuation ≥ 16.4781 dB

Passband edge frequency = 0.5π rad/sample

Stopband edge frequency = 0.8π rad/sample

E7.14. ***Design a Butterworth digital IIR lowpass filter using impulse invariant transformation by taking $T = 1$second, to satisfy the following specifications.***

$$0.9 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0.3981\pi \le \omega \le \pi$$

$$|H(e^{j\omega})| \le 0.35 \quad ; \quad \text{for } 0.3981\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternate specification,

Passband ripple ≤ 0.9151 dB

Stopband attenuation ≥ 9.1186 dB

Passband edge frequency = 0.25π rad/sample

Stopband edge frequency = 0.3981π rad/sample

E7.15. ***Design a Butterworth digital IIR lowpass filter using bilinear transformation by taking $T = 0.6$second, to satisfy the following specifications.***

$$0.6 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad 0 \le \omega \le 0.3\pi$$

$$|H(e^{j\omega})| \le 0.02 \quad ; \quad 0.575\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter.

Alternate specification,

Passband ripple ≤ 4.4370 dB

Stopband attenuation ≥ 33.9794 dB

Passband edge frequency = 0.3π rad/sample

Stopband edge frequency = 0.575π rad/sample

E7.16. ***Design a Chebyshev digital IIR lowpass filter using impulse invariant transformation by taking $T = 1$second, to satisfy the following specifications.***

$$0.87 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.25\pi$$

$$|H(e^{j\omega})| \le 0.35 \quad ; \quad \text{for } 0.375\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter.

Alternate specification,

Passband ripple ≤ 1.2096 dB

Stopband attenuation ≥ 9.1136 dB

Passband edge frequency = 0.25π rad/sample

Stopband edge frequency = 0.375π rad/sample

E7.17. ***Design a Chebyshev digital IIR lowpass filter using bilinear transformation by taking $T = 0.5$second, to satisfy the following specifications.***

$$0.9 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.25\pi$$

$$|H(e^{j\omega})| \le 0.35 \quad ; \quad \text{for } 0.375\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter.

Alternate specification,

Passband ripple ≤ 0.9151 dB

Stopband attenuation ≥ 9.1186 dB

Passband edge frequency = 0.25π rad/sample

Stopband edge frequency = 0.375π rad/sample

Answers

E7.1 **a)** $H(z) = \dfrac{1-0.1938z^{-1}}{1-0.1536z^{-1}+0.0025z^{-2}}$

b) $H(z) = \dfrac{1-0.4842z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$; $H_N(z) = \dfrac{0.5-0.2421z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$

E7.2 $H(z) = \dfrac{1+0.2067z^{-1}}{1+0.4133z^{-1}+0.2466z^{-2}}$

E7.3 $H(z) = \dfrac{0.2017z^{-1}}{1-0.0858z^{-1}+0.3678z^{-2}}$; $H_N(z) = \dfrac{0.1009z^{-1}}{1-0.0858z^{-1}+0.3678z^{-2}}$

E7.4 **a)** $H(z) = \dfrac{0.0475+0.0950z^{-1}+0.0475z^{-2}}{1+0.6698z^{-1}+0.6199z^{-2}}$; **b)** $H(z) = \dfrac{0.0307+0.0613z^{-1}+0.0307z^{-2}}{1-0.1128z^{-1}+0.5911z^{-2}}$

E7.5 $H(z) = \dfrac{0.5333-0.5333z^{-2}}{1-0.2667z^{-1}-0.2z^{-2}}$

E7.6 $H(z) = \dfrac{0.4990-1.4970z^{-1}+1.4970z^{-2}-0.4990z^{-3}}{1-2.6592z^{-1}+2.3272z^{-2}-0.6671z^{-3}}$

E7.7 $H(z) = \dfrac{0.0943+0.0018z^{-1}-0.0925z^{-2}}{1-1.7735z^{-1}+0.9626z^{-2}}$

E7.8 $H(z) = \dfrac{0.3215+0.643z^{-1}+0.3215z^{-2}}{1+0.1122z^{-1}+0.1738z^{-2}}$

E7.9 $H(z) = \dfrac{0.2654-0.5308z^{-1}+0.2654z^{-2}}{1+0.1122z^{-1}+0.1738z^{-2}}$

E7.10 $H(z) = \dfrac{0.3138+0.9414z^{-1}+0.9414z^{-2}+0.3138z^{-3}}{1+0.8779z^{-1}+0.5389z^{-2}+0.0934z^{-3}}$

E7.11 $H(z) = \dfrac{0.0709-0.2128z^{-1}+0.2128z^{-2}-0.0709z^{-3}}{1+0.8779z^{-1}+0.5389z^{-2}+0.0934z^{-3}}$

E7.12 $H(z) = \dfrac{0.6979z^{-1}}{1-0.5379z^{-1}+0.1748z^{-2}}$; $H_N(z) = \dfrac{0.5583z^{-1}}{1-0.5379z^{-1}+0.1748z^{-2}}$

E7.13 $H(z) = \dfrac{0.4405z^{-1}+0.1843z^{-2}}{1-0.6695z^{-1}+0.3684z^{-2}-0.0685z^{-3}}$

E7.14 $H(z) = \dfrac{0.0767z^{-1}+0.1567z^{-2}+0.0215z^{-3}}{1-1.6058z^{-1}+1.2796z^{-2}-0.4967z^{-3}+0.0777z^{-4}}$

E7.15 $H(z) = \dfrac{0.0154+0.0617z^{-1}+0.0925z^{-2}+0.0617z^{-3}+0.0154z^{-4}}{1-1.7024z^{-1}+1.4160z^{-2}-0.5558z^{-3}+0.0889z^{-4}}$

E7.16 $H(z) = \dfrac{0.0874z^{-1}+0.0687z^{-2}}{1-1.8793z^{-1}+1.5215z^{-2}-0.4862z^{-3}}$

E7.17 $H(z) = \dfrac{0.0219+0.0656z^{-1}+0.0656z^{-2}+0.0219z^{-3}}{1-1.8444z^{-1}+1.4713z^{-2}-0.4519z^{-3}}$

Solution for Exercise Problems

E7.1 For the analog transfer function, $H(s) = \dfrac{(s+1)}{(s+2)(s+4)}$ ***, determine H(z) using impulse invariant transformation if (a) T = 1 second and (b) T = 0.5 second.***

Solution

Given that, $H(s) = \dfrac{(s+1)}{(s+2)\,(s+4)}$

By partial fraction expansion technique we can write,

$$H(s) = \frac{(s+1)}{(s+2)\,(s+4)} = \frac{A}{s+2} + \frac{B}{s+4}$$

$$A = \frac{(s+1)}{\cancel{(s+2)}\,(s+4)} \times \cancel{(s+2)}\bigg|_{s=-2} = \frac{-2+1}{-2+4} = \frac{-1}{2} = -0.5$$

$$B = \frac{(s+1)}{(s+2)\,\cancel{(s+4)}} \times \cancel{(s+4)}\bigg|_{s=-4} = \frac{-4+1}{-4+2} = \frac{-3}{-2} = \frac{3}{2} = 1.5$$

$$\therefore\; H(s) = \frac{-0.5}{s+2} + \frac{1.5}{s+4}$$

By impulse invariant transformation we know that,

$$\frac{A_i}{s+p_i} \xrightarrow{\text{(is transformed to)}} \frac{A_i}{1-e^{-p_iT}\,z^{-1}}$$

$$\therefore\; H(z) = \frac{-0.5}{1-e^{-p_1T}\,z^{-1}} + \frac{1.5}{1-e^{-p_2T}\,z^{-1}} \quad ; \text{ where, } p_1 = 2 \text{ and } p_2 = 4$$

$$H(z) = \frac{-0.5}{1-e^{-2T}\,z^{-1}} + \frac{1.5}{1-e^{-4T}\,z^{-1}}$$

(a) When T = 1 second

$$H(z) = \frac{-0.5}{1-e^{-2}\,z^{-1}} + \frac{1.5}{1-e^{-4}\,z^{-1}}$$

$$H(z) = \frac{-0.5}{1-0.1353z^{-1}} + \frac{1.5}{1-0.0183z^{-1}} = \frac{-0.5(1-0.0183z^{-1}) + 1.5(1-0.1353z^{-1})}{(1-0.1353z^{-1})\,(1-0.0183z^{-1})}$$

$$= \frac{-0.5+0.0092z^{-1}+1.5-0.203z^{-1}}{1-0.0183z^{-1}-0.1353z^{-1}+0.0025z^{-2}} = \frac{1-0.1938z^{-1}}{1-0.1536z^{-1}+0.0025z^{-2}}$$

Alternatively,

$$H(z) = \frac{1-0.1938z^{-1}}{1-0.1536z^{-1}+0.0025z^{-2}} = \frac{1-0.1938z^{-1}}{z^{-2}\left(z^2-0.1536z+0.0025\right)}$$

$$= \frac{z^2-0.1938z}{z^2-0.1536z+0.0025}$$

(b) When T = 0.5 second

$$H(z) = \frac{-0.5}{1-e^{-1}\,z^{-1}} + \frac{1.5}{1-e^{-2}\,z^{-1}}$$

$$= \frac{-0.5}{1-0.3679z^{-1}} + \frac{1.5}{1-0.1353z^{-1}} = \frac{-0.5(1-0.1353z^{-1}) + 1.5(1-0.3679z^{-1})}{(1-0.3679z^{-1})\,(1-0.1353z^{-1})}$$

$$= \frac{-0.5+0.0677z^{-1}+1.5-0.5519z^{-1}}{1-0.1353z^{-1}-0.3679z^{-1}+0.0498z^{-2}} = \frac{1-0.4842z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$$

Alternatively,

$$H(z) = \frac{1-0.4842z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}} = \frac{1-0.4842z^{-1}}{z^{-2}\left(z^2-0.5032z+0.0498\right)}$$

$$= \frac{z^2-0.4842z}{z^2-0.5032z+0.0498}$$

Since T < 1, we can compute magnitude normalized transfer function, $H_N(z)$

$$H_N(z) = T \times H(z) = \frac{0.5 \times (1 - 0.4842z^{-1})}{1 - 0.5032z^{-1} + 0.0498z^{-2}} = \frac{0.5 - 0.2421z^{-1}}{1 - 0.5032z^{-1} + 0.0498z^{-2}}$$

Alternatively,

$$H_N(z) = T \times H(z) = \frac{0.5 \times (z^2 - 0.4842z)}{z^2 - 0.5032z + 0.0498} = \frac{0.5z^2 - 0.2421z}{z^2 - 0.5032z + 0.0498}$$

E7.2. Convert the analog filter with system transfer function,

$$H(s) = \frac{(s + 0.7)}{s^2 + 1.4s + 4.49}$$

into a digital IIR filter by means of the impulse invariant method.

Solution

Given that, $H(s) = \dfrac{s + 0.7}{s^2 + 1.4s + 4.49}$

By partial fraction expansion H(s) can be expressed as,

$$H(s) = \frac{s + 0.7}{(s + 0.7 - j2)\,(s + 0.7 + j2)} = \frac{A}{(s + 0.7 - j2)} + \frac{A^*}{(s + 0.7 + j2)}$$

$$A = \frac{s + 0.7}{\cancel{(s + 0.7 - j2)}\,(s + 0.7 + j2)} \times \cancel{(s + 0.7 - j2)}\Bigg|_{s = -0.7 + j2}$$

$$= \frac{-0.7 + j2 + 0.7}{-0.7 + j2 + 0.7 + j2} = \frac{j2}{j4} = 0.5$$

$$A^* = (0.5)^* = 0.5$$

$$\therefore\ H(s) = \frac{0.5}{(s + 0.7 - j2)} + \frac{0.5}{(s + 0.7 + j2)}$$

The roots of the quadratic $s^2 + 1.4s + 4.49 = 0$ are

$$s = \frac{-1.4 \pm \sqrt{1.4^2 - 4 \times 4.49}}{2}$$

$$= \frac{-1.4}{2} \pm \frac{1}{2}\sqrt{-16} = -0.7 \pm j2$$

$$\therefore\ (s^2 + 1.4s + 4.49)$$

$$= (s - (-0.7 + j2)(s - (-0.7 - j2))$$

$$= (s + 0.7 - j2)(s + 0.7 + j2)$$

By impulse invariant transformation we know that,

$$\frac{A_i}{s + p_i} \xrightarrow[\text{(is transformed to)}]{} \frac{A_i}{1 - e^{-p_i T}\, z^{-1}} \quad \text{and let, } T = 1$$

$$\therefore\ H(z) = \frac{0.5}{1 - e^{-(0.7 - j2)T}\, z^{-1}} + \frac{0.5}{1 - e^{-(0.7 + j2)T}\, z^{-1}} = \frac{0.5}{1 - e^{-0.7}\, e^{j2}\, z^{-1}} + \frac{0.5}{1 - e^{-0.7}\, e^{-j2}\, z^{-1}}$$

$$= \frac{0.5\left(1 - e^{-0.7}\, e^{-j2}\, z^{-1}\right) + 0.5\left(1 - e^{-0.7}\, e^{j2}\, z^{-1}\right)}{\left(1 - e^{-0.7}\, e^{j2}\, z^{-1}\right)\left(1 - e^{-0.7}\, e^{-j2}\, z^{-1}\right)}$$

$$= \frac{0.5 - 0.5e^{-0.7}\, e^{-j2}\, z^{-1} + 0.5 - 0.5e^{-0.7}\, e^{j2}\, z^{-1}}{1 - e^{-0.7}\, e^{-j2}\, z^{-1} - e^{-0.7}\, e^{j2}\, z^{-1} + e^{-0.7}\, e^{j2}\, e^{-0.7}\, e^{-j2}\, z^{-2}}$$

$$= \frac{1 - 0.5e^{-0.7}\, z^{-1}\left(e^{j2} + e^{-j2}\right)}{1 - e^{-0.7}\, z^{-1}\left(e^{j2} + e^{-j2}\right) + e^{-1.4}\, z^{-2}} = \frac{1 - 0.5 \times (2\cos 2)e^{-0.7}\, z^{-1}}{1 - e^{-0.7}\, z^{-1}(2\cos 2) + e^{-1.4}\, z^{-2}}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$= \frac{1 - (\cos 2)e^{-0.7}\, z^{-1}}{1 - 2(\cos 2)e^{-0.7}\, z^{-1} + e^{-1.4}\, z^{-2}} = \frac{1 + 0.2067z^{-1}}{1 + 0.4133z^{-1} + 0.2466z^{-2}}$$

Note : Evaluate cos q by keeping calculator in radian mode.

Alternatively,

$$H(z) = \frac{1 + 0.2067z^{-1}}{1 + 0.4133z^{-1} + 0.2466z^{-2}} = \frac{1 + 0.2067z^{-1}}{z^{-2}(z^2 + 0.4133z + 0.2466)} = \frac{z^2 + 0.2067z}{z^2 + 0.4133z + 0.2466}$$

E7.3. Using impulse invariant transformation convert the following analog filter transfer function to digital filter transfer function by taking sampling time, T = 0.5 second.

$$H(s) = \frac{1}{s^2 + 2s + 10}$$

Solution

Given that, $H(s) = \dfrac{1}{s^2 + 2s + 10} = \dfrac{1}{(s + 1 - j3)(s + 1 + j3)}$

By partial fraction expansion H(s) can be expressed as,

$$H(s) = \frac{1}{(s + 1 - j3)\,(s + 1 + j3)} = \frac{A}{s + 1 - j3} + \frac{A^*}{s + 1 + j3}$$

The roots of the quadratic $s^2 + 2s + 10 = 0$ are

$$s = \frac{-2 \pm \sqrt{2^2 - 4 \times 10}}{2}$$

$$= \frac{-2 \pm j6}{2} = -1 \pm j3$$

$$\therefore\ (s^2 + 2s + 10)$$

$$= (s - (-1 + j3))(s - (-1 - j3))$$

$$= (s + 1 - j3)(s + 1 + j3)$$

$$A = \frac{1}{\cancel{(s+1-j3)}\,(s+1+j3)} \times \cancel{(s+1-j3)}\Bigg|_{s=-1+j3} = \frac{1}{-1+j3+1+j3} = \frac{1}{j6} = -j0.1667$$

$$A^* = (-j0.1667)^* = j0.1667$$

$$\therefore\ H(s) = \frac{-j0.1667}{s+1-j3} + \frac{j0.1667}{s+1+j3}$$

By impulse invariant transformation we know that,

$$\frac{A_i}{s+p_i} \xrightarrow{\text{(is transformed to)}} \frac{A_i}{1-e^{-p_iT}\,z^{-1}} \quad \text{and let, } T = 0.5$$

$$\therefore\ H(z) = \frac{-j0.1667}{1-e^{-(1-j3)T}\,z^{-1}} + \frac{j0.1667}{1-e^{-(1+j3)T}\,z^{-1}} = \frac{-j0.1667}{1-e^{-0.5}\,e^{j1.5}\,z^{-1}} + \frac{j0.1667}{1-e^{-0.5}\,e^{-j1.5}\,z^{-1}}$$

$$= \frac{-j0.1667\left(1-e^{-0.5}\,e^{-j1.5}\,z^{-1}\right) + j0.1667\left(1-e^{-0.5}\,e^{j1.5}\,z^{-1}\right)}{\left(1-e^{-0.5}\,e^{j1.5}\,z^{-1}\right)\left(1-e^{-0.5}\,e^{-j1.5}\,z^{-1}\right)}$$

$$= \frac{-j0.1667 + j0.1667\,e^{-0.5}\,e^{-j1.5}\,z^{-1} + j0.1667 - j0.1667\,e^{-0.5}\,e^{j1.5}\,z^{-1}}{1-e^{-0.5}\,e^{-j1.5}\,z^{-1} - e^{-0.5}\,e^{j1.5}\,z^{-1} + e^{-0.5}\,e^{j1.5}\,e^{-0.5}\,e^{-j1.5}\,z^{-2}}$$

$$= \frac{-j0.1667\,e^{-0.5}\,z^{-1}\left(e^{j1.5} - e^{-j1.5}\right)}{1-e^{-0.5}\,z^{-1}\left(e^{j1.5}+e^{-j1.5}\right) + e^{-1}z^{-2}}$$

$$= \frac{-j0.1667\,e^{-0.5}\,(2j\sin 1.5)\,z^{-1}}{1-e^{-0.5}\,z^{-1}(2\cos 1.5) + e^{-1}z^{-2}} = \frac{-j0.1667\,(2\sin 1.5)\,e^{-0.5}\,z^{-1}}{1-(2\cos 1.5)\,e^{-0.5}\,z^{-1} + e^{-1}z^{-2}}$$

$$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}\ ,\ \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}}$$

$$= \frac{0.2017\,z^{-1}}{1-0.0858\,z^{-1} + 0.3678\,z^{-2}}$$

Note : Evalutate $\cos\theta$ and $\sin\theta$ by keeping calculator in radian mode.

Alternatively,

$$H(z) = \frac{0.2017\,z^{-1}}{1-0.0858\,z^{-1}+0.3678\,z^{-2}} = \frac{0.2017\,z^{-1}}{z^{-2}(z^2-0.0858\,z+0.3678)} = \frac{0.2017\,z}{z^2-0.0858\,z+0.3678}$$

Since T < 1, we can compute magnitude normalized transfer function, $H_N(z)$

$$H_N(z) = T \times H(z) = \frac{0.5 \times 0.2017z^{-1}}{1-0.0858z^{-1}+0.3678z^{-2}} = \frac{0.1009\,z^{-1}}{1-0.0858z^{-1}+0.3678z^{-2}}$$

Alternatively,

$$H_N(z) = T \times H(z) = \frac{0.5 \times 0.2017z}{z^2-0.0858z+0.3678} = \frac{0.1009\,z}{z^2-0.0858z+0.3678}$$

E7.4. For the analog transfer function, $H(s) = \dfrac{0.8}{s^2+1.6s+9.64}$***, determine H(z) using bilinear transformation if (a) T = 1 second and (b) T = 0.6 second.***

Solution

Given that, $H(s) = \dfrac{0.8}{s^2+1.6s+9.64}$

Put, $s = \dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}$ in H(s) to get H(z).

$$\therefore\ H(z) = \frac{0.8}{\left(\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.6\left(\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}\right) + 9.64}$$

$$= \frac{0.8}{\dfrac{4\left(1-z^{-1}\right)^2}{T^2\left(1+z^{-1}\right)^2} + \dfrac{3.2\left(1-z^{-1}\right)}{T\left(1+z^{-1}\right)} + 9.64}$$

$$= \frac{0.8}{\dfrac{4\left(1-z^{-1}\right)^2 + 3.2T\left(1-z^{-1}\right)\left(1+z^{-1}\right) + 9.64T^2\left(1+z^{-1}\right)^2}{T^2\left(1+z^{-1}\right)^2}}$$

$$= \frac{0.8T^2\left(1+z^{-1}\right)^2}{4\left(1-z^{-1}\right)^2 + 3.2T\left(1-z^{-2}\right) + 9.64T^2\left(1+z^{-1}\right)^2}$$

$$\boxed{(a+b)\,(a-b) = a^2-b^2}$$

(a) T = 1 second

$$\therefore\ H(z) = \frac{0.8\left(1+z^{-1}\right)^2}{4\left(1-z^{-1}\right)^2+3.2\left(1-z^{-2}\right)+9.64\left(1+z^{-1}\right)^2}$$

$$= \frac{0.8\left(1+2z^{-1}+z^{-2}\right)}{4\left(1-2z^{-1}+z^{-2}\right)+3.2\left(1-z^{-2}\right)+9.64\left(1+2z^{-1}+z^{-2}\right)}$$

$$= \frac{0.8+1.6z^{-1}+0.8z^{-2}}{16.84+11.28z^{-1}+10.44z^{-2}} = \frac{0.8+1.6z^{-1}+0.8z^{-2}}{16.84\left(1+\frac{11.28}{16.84}z^{-1}+\frac{10.44}{16.84}z^{-2}\right)}$$

$$= \frac{\frac{0.8}{16.84}+\frac{1.6}{16.84}z^{-1}+\frac{0.8}{16.84}z^{-2}}{1+\frac{11.28}{16.84}z^{-1}+\frac{10.44}{16.84}z^{-2}} = \frac{0.0475+0.0950z^{-1}+0.0475z^{-2}}{1+0.6698z^{-1}+0.6199z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.0475+0.0950z^{-1}+0.0475z^{-2}}{1+0.6698z^{-1}+0.6199z^{-2}} = \frac{z^{-2}(0.0475z^2+0.0950z+0.0475)}{z^{-2}\left(z^2+0.6698z+0.6199\right)} = \frac{0.0475z^2+0.0950z+0.0475}{z^2+0.6698z+0.6199}$$

(b) T = 0.6 second

$$H(z) = \frac{0.8\times 0.6^2\left(1+z^{-1}\right)^2}{4\left(1-z^{-1}\right)^2+3.2\times 0.6\left(1-z^{-2}\right)+9.64\times 0.6^2\left(1+z^{-1}\right)^2}$$

$$= \frac{0.288\left(1+2z^{-1}+z^{-2}\right)}{4\left(1-2z^{-1}+z^{-2}\right)+1.92\left(1-z^{-2}\right)+3.4704\left(1+2z^{-1}+z^{-2}\right)}$$

$$= \frac{0.288+0.576z^{-1}+0.288z^{-2}}{9.3904-1.0592z^{-1}+5.5504z^{-2}}$$

$$= \frac{\frac{0.288}{9.3904}+\frac{0.576}{9.3904}z^{-1}+\frac{0.288}{9.3904}z^{-2}}{1-\frac{1.0592}{9.3904}z^{-1}+\frac{5.5504}{9.3904}z^{-2}} = \frac{0.0307+0.0613z^{-1}+0.0307z^{-2}}{1-0.1128z^{-1}+0.5911z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.0307+0.0613z^{-1}+0.0307z^{-2}}{1-0.1128z^{-1}+0.5911z^{-2}} = \frac{z^{-2}(0.0307z^2+0.0613z+0.0307)}{z^{-2}(z^2-0.1128z+0.5911)} = \frac{0.0307z^2+0.0613z+0.0307}{z^2-0.1128z+0.5911}$$

E7.5. Obtain H(z) from H(s) when T = 1 second and $H(s) = \dfrac{4s}{(s+0.5)(s+4)}$.

Solution

Given that, $H(s) = \dfrac{4s}{(s+0.5)(s+4)} = \dfrac{4s}{s^2+4s+0.5s+2} = \dfrac{4s}{s^2+4.5s+2}$

Put, $s = \dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}$ in H(s) to get H(z).

$$\therefore\ H(z) = \frac{4\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2+4.5\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)+2} = \frac{\frac{8(1-z^{-1})}{1+z^{-1}}}{\frac{4(1-z^{-1})^2}{(1+z^{-1})^2}+\frac{9(1-z^{-1})}{1+z^{-1}}+2}$$

Put, T = 1

$$= \frac{\frac{8(1-z^{-1})}{(1+z^{-1})}}{\frac{4(1-z^{-1})^2+9(1-z^{-1})\,(1+z^{-1})+2(1+z^{-1})^2}{(1+z^{-1})^2}} = \frac{8(1-z^{-1})(1+z^{-1})}{4(1-z^{-1})^2+9(1-z^{-1})\,(1+z^{-1})+2(1+z^{-1})^2}$$

$$\therefore\ H(z) = \frac{8(1-z^{-2})}{4(1-2z^{-1}+z^{-2})+9(1-z^{-2})+2(1+2z^{-1}+z^{-2})} = \frac{8-8z^{-2}}{15-4z^{-1}-3z^{-2}}$$

$(a+b)\,(a-b) = a^2-b^2$

$(a+b)^2 = a^2+2ab+b^2$

$(a-b)^2 = a^2-2ab+b^2$

$$= \frac{\frac{8}{15}-\frac{8}{15}z^{-2}}{1-\frac{4}{15}z^{-1}-\frac{3}{15}z^{-2}} = \frac{0.5333-0.5333z^{-2}}{1-0.2667z^{-1}-0.2z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.5333-0.5333z^{-2}}{1-0.2667z^{-1}-0.2z^{-2}} = \frac{z^{-2}(0.5333z^2-0.5333)}{z^{-2}(z^2-0.2667z-0.2)} = \frac{0.5333z^2-0.5333}{z^2-0.2667z-0.2}$$

E7.6. Obtain H(z) from H(s) when T = 0.1 second, and $H(s) = \dfrac{0.6s^3}{s^3 + 4s^2 + 0.9s + 1}$.

Solution

Given that, $H(s) = \dfrac{0.6s^3}{s^3 + 4s^2 + 0.9s + 1}$

Put, $s = \dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}$ in H(s) to get H(z).

$$\therefore\ H(z) = \frac{0.6\left(\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}\right)^3}{\left(\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}\right)^3 + 4\left(\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.9\left(\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}\right) + 1} \qquad \boxed{\text{Put, } T = 0.1}$$

$$= \frac{\dfrac{4800(1-z^{-1})^3}{(1+z^{-1})^3}}{\dfrac{8000(1-z^{-1})^3}{(1+z^{-1})^3} + \dfrac{1600(1-z^{-1})^2}{(1+z^{-1})^2} + \dfrac{18(1-z^{-1})}{1+z^{-1}} + 1}$$

$$= \frac{\dfrac{4800(1-z^{-1})^3}{\cancel{(1+z^{-1})^3}}}{\dfrac{8000(1-z^{-1})^3 + 1600(1-z^{-1})^2(1+z^{-1}) + 18(1-z^{-1})(1+z^{-1})^2 + (1+z^{-1})^3}{\cancel{(1+z^{-1})^3}}}$$

$$= \frac{4800(1-3z^{-1}+3z^{-2}-z^{-3})}{8000(1-3z^{-1}+3z^{-2}-z^{-3}) + 1600(1-2z^{-1}+z^{-2})(1+z^{-1}) + 18(1-z^{-1})(1+2z^{-1}+z^{-2}) + (1+3z^{-1}+3z^{-2}+z^{-3})}$$

$$= \frac{(4800 - 14400z^{-1} + 14400z^{-2} - 4800z^{-3})}{8000(1-3z^{-1}+3z^{-2}-z^{-3}) + 1600(1-z^{-1}-z^{-2}+z^{-3}) + 18(1+z^{-1}-z^{-2}-z^{-3}) + (1+3z^{-1}+3z^{-2}+z^{-3})}$$

$$= \frac{4800 - 14400z^{-1} + 14400z^{-2} - 4800z^{-3}}{9619 - 25579z^{-1} + 22385z^{-2} - 6417z^{-3}} = \frac{\dfrac{4800}{9619} - \dfrac{14400}{9619}z^{-1} + \dfrac{14400}{9619}z^{-2} - \dfrac{4800}{9619}z^{-3}}{1 - \dfrac{25579}{9619}z^{-1} + \dfrac{25385}{9619}z^{-2} - \dfrac{6417}{9619}z^{-3}}$$

$$= \frac{0.4990 - 1.4970z^{-1} + 1.4970z^{-2} - 0.4990z^{-3}}{1 - 2.6592z^{-1} + 2.3272z^{-2} - 0.6671z^{-3}}$$

Alternatively,

$$H(z) = \frac{0.4990 - 1.4970z^{-1} + 1.4970z^{-2} + 0.4990z^{-3}}{1 - 2.6592z^{-1} + 2.3272z^{-2} - 0.6671z^{-3}} = \frac{z^{-3}\left[0.4990z^3 - 1.4970z^2 + 1.4970z + 0.4990\right]}{z^{-3}\left[z^3 - 2.6592z^2 + 2.3272z - 0.6671\right]}$$

$$= \frac{0.4990z^3 - 1.4970z^2 + 1.4970z + 0.4990}{z^3 - 2.6592z^2 + 2.3272z - 0.6671}$$

E7.7. Convert the analog filter with system function H(s) into digital filter using bilinear transformation.

$$H(s) = \frac{(s+0.1)}{(s+0.1)^2 + 5} \quad ; \quad \textbf{Take, } T = 0.2$$

Solution

Given that, $H(s) = \dfrac{s+0.1}{(s+0.1)^2 + 5} = \dfrac{s+0.1}{s^2 + 0.2s + 0.01 + 5} = \dfrac{s+0.1}{s^2 + 0.2s + 5.01}$

Put, $s = \dfrac{2}{T}\dfrac{1 - z^{-1}}{1 + z^{-1}}$ in H(s) to get H(z).

$$\therefore\ H(z) = \frac{\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}} + 0.1}{\left(\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.2\left(\dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}\right) + 5.01} = \frac{\dfrac{2(1-z^{-1})}{T(1+z^{-1})} + 0.1}{\dfrac{4(1-z^{-1})^2}{T^2(1+z^{-1})^2} + \dfrac{0.4(1-z^{-1})}{T(1+z^{-1})} + 5.01}$$

$$= \frac{\dfrac{2(1-z^{-1}) + 0.1T(1+z^{-1})}{T(1+z^{-1})}}{\dfrac{4(1-z^{-1})^2 + 0.4T(1-z^{-1})(1+z^{-1}) + 5.01T^2(1+z^{-1})^2}{T^2(1+z^{-1})^2}} = \frac{\left[2(1-z^{-1}) + 0.1T(1+z^{-1})\right]T(1+z^{-1})}{4(1-z^{-1})^2 + 0.4T(1-z^{-2}) + 5.01T^2(1+z^{-1})^2}$$

$$= \frac{\left[2(1-z^{-1}) + 0.1\times 0.2(1+z^{-1})\right]0.2(1+z^{-1})}{4(1-z^{-1})^2 + 0.4\times 0.2(1-z^{-2}) + 5.01\times 0.2^2(1+z^{-1})^2} = \frac{0.4(1-z^{-1})(1+z^{-1}) + 0.004(1+z^{-1})^2}{4(1-z^{-1})^2 + 0.08(1-z^{-2}) + 0.2004(1+z^{-1})^2} \qquad \boxed{\text{Put, } T = 0.2}$$

$$\therefore\ H(z) = \frac{0.4(1-z^{-2}) + 0.004(1+2z^{-1}+z^{-2})}{4(1-2z^{-1}+z^{-2}) + 0.08(1-z^{-2}) + 0.2004(1+2z^{-1}+z^{-2})}$$

$$= \frac{0.404 + 0.008z^{-1} - 0.396z^{-2}}{4.2804 - 7.5992z^{-1} + 4.1204z^{-2}} = \frac{\dfrac{0.404}{4.2804} + \dfrac{0.008}{4.2804}z^{-1} - \dfrac{0.396}{4.2804}z^{-2}}{1 - \dfrac{7.5992}{4.2804}z^{-1} + \dfrac{4.1204}{4.2804}z^{-2}}$$

$$= \frac{0.0943 + 0.0018z^{-1} - 0.0925z^{-2}}{1 - 1.7735z^{-1} + 0.9626z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.0943 + 0.0018z^{-1} - 0.0925z^{-2}}{1 - 1.7735z^{-1} + 0.9626z^{-2}} = \frac{z^{-2}(0.0943z^2 + 0.0018z - 0.0925)}{z^{-2}(z^2 - 1.7735z + 0.9626)} = \frac{0.0943z^2 + 0.0018z - 0.0925}{z^2 - 1.7735z + 0.9626}$$

E7.8. Design a Butterworth digital IIR lowpass filter using bilinear transformation by taking T = 0.2second, to satisfy the following specifications.

$$0.8 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.4\pi$$

$$|H(e^{j\omega})| \le 0.3 \quad ; \quad \text{for } 0.7\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternatively,

Passband ripple $\le$ 1.9382 dB

Stopband attenuation $\ge$ 10.4576 dB

Passband edge frequency = 0.4π rad/sample

Stopband edge frequency = 0.7π rad/sample

The above specifications can be converted to A_p and A_s as shown below.

$$A_p = 10^{(-\delta_{p,dB}/20)} = 10^{(-1.9382/20)} = 0.8$$

$$A_s = 10^{(-\alpha_{s,dB}/20)} = 10^{(-10.4576/20)} = 0.3$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.4\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.7\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.8$

Gain in normal value at stopband edge, $A_s = 0.3$

Sampling time, T = 0.2second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.8$

Gain in normal value at stopband edge, $A_s = 0.3$

Gain is same in analog and digital filter.

For bilinear transformation,

Passband edge analog frequency, $\Omega_p = \dfrac{2}{T}\tan\dfrac{\omega_p}{2}$ — Using equation (7.53).

$$= \frac{2}{0.2}\tan\frac{0.4\pi}{2} = 7.2654 \text{ rad / second}$$

Stopband edge analog frequency, $\Omega_s = \dfrac{2}{T}\tan\dfrac{\omega_s}{2}$ — Using equation (7.54).

$$= \frac{2}{0.2}\tan\frac{0.7\pi}{2} = 19.6261 \text{ rad / second}$$

Order of the filter

$$N_1 = \frac{1}{2}\,\frac{\log\left[\dfrac{(1/A_s^2)-1}{(1/A_p^2)-1}\right]}{\log\dfrac{\Omega_s}{\Omega_p}} = \frac{1}{2}\,\frac{\log\left[\dfrac{(1/0.3^2)-1}{(1/0.8^2)-1}\right]}{\log\dfrac{19.6261}{7.2654}} = \frac{1}{2}\,\frac{\log\left[\dfrac{10.1111}{0.5625}\right]}{\log\dfrac{19.6261}{7.2654}} = 1.4536$$

Using equation (7.57).

Choose order N, such that $N \ge N_1$ and N is an integer.

Let, order, N = 2.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For even N,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$ Using equation (7.58).

where, $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$ Using equation (7.60).

Here, $N = 2, \therefore k = \frac{N}{2} = \frac{2}{2} = 1$

$$\therefore H(s_n) = \frac{1}{s_n^2 + b_1 s_n + 1}$$

When k = 1 ; $b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times 2}\right] = 1.4142$

Calculate sinq using calculator in radian mode.

$$\therefore H(s_n) = \frac{1}{s_n^2 + 1.4142\,s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2) - 1\right]^{\frac{1}{2N}}} = \frac{19.6261}{\left[(1/0.3^2) - 1\right]^{\frac{1}{4}}} = 11.0061\ \text{rad / sec}$$ Using equation (7.61).

$$\therefore H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{s_n^2 + 1.4142\,s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{1}{\frac{s^2}{\Omega_c^2} + 1.4142\frac{s}{\Omega_c} + 1} = \frac{1}{\frac{s^2 + 1.4142\,\Omega_c s + \Omega_c^2}{\Omega_c^2}}$$

$$= \frac{\Omega_c^2}{s^2 + 1.4142\,\Omega_c s + \Omega_c^2} = \frac{11.0061^2}{s^2 + 1.4142 \times 11.0061 s + 11.0061^2}$$

$$= \frac{121.1342}{s^2 + 15.5648\,s + 121.1342}$$

Digital IIR lowpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{121.1342}{s^2 + 15.5648\,s + 121.1342}\Bigg|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{121.1342}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 15.5648\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + 121.1342} = \frac{121.1342}{\frac{4(1-z^{-1})^2}{T^2(1+z^{-1})^2} + \frac{31.1296(1-z^{-1})}{T(1+z^{-1})} + 121.1342}$$

$$= \frac{121.1342}{\frac{4(1-z^{-1})^2 + 31.1296\,T(1-z^{-1})(1+z^{-1}) + 121.1342\,T^2(1+z^{-1})^2}{T^2(1+z^{-1})^2}}$$

$$= \frac{121.1342 \times 0.2^2(1+2z^{-1}+z^{-2})}{4(1-2z^{-1}+z^{-2}) + 31.1296 \times 0.2(1-z^{-2}) + 121.1342 \times 0.2^2(1+2z^{-1}+z^{-2})}$$ Put, T = 0.2

$$= \frac{4.8454(1+2z^{-1}+z^{-2})}{4(1-2z^{-1}+z^{-2}) + 6.2259(1-z^{-2}) + 4.8454(1+2z^{-1}+z^{-2})}$$

$$= \frac{4.8454 + 9.6908\,z^{-1} + 4.8454\,z^{-2}}{15.0713 + 1.6908\,z^{-1} + 2.6195\,z^{-2}} = \frac{\frac{4.8454}{15.0713} + \frac{9.6908}{15.0713}z^{-1} + \frac{4.8454}{15.0713}z^{-2}}{1 + \frac{1.6908}{15.0713}z^{-1} + \frac{2.6195}{15.0713}z^{-2}}$$

$$= \frac{0.3215 + 0.643z^{-1} + 0.3215\,z^{-2}}{1 + 0.1122\,z^{-1} + 0.1738\,z^{-2}}$$

Alternatively,

$$H(z)=\frac{0.3215+0.643\,z^{-1}+0.3215\,z^{-2}}{1+0.1122\,z^{-1}+0.1738\,z^{-2}}=\frac{z^{-2}\left(0.3215\,z^{2}+0.643\,z+0.3215\right)}{z^{-2}\left(z^{2}+0.1122\,z+0.1738\right)}=\frac{0.3215\,z^{2}+0.643\,z+0.3215}{z^{2}+0.1122\,z+0.1738}$$

Direct form-I structure of digital IIR lowpass filter

Let, $H(z)=\dfrac{Y(z)}{X(z)}=\dfrac{0.3215+0.643\,z^{-1}+0.3215\,z^{-2}}{1+0.1122\,z^{-1}+0.1738\,z^{-2}}$

On cross multiplying the above equation we get,

$Y(z) + 0.1122z^{-1}Y(z) + 0.1738z^{-2}Y(z) = 0.3215\,X(z) + 0.643z^{-1}X(z) + 0.3215z^{-2}X(z)$

$\therefore$ $Y(z) = 0.3215X(z) + 0.643z^{-1}X(z) + 0.3215z^{-2}X(z) - 0.1122z^{-1}Y(z) - 0.1738z^{-2}Y(z)$ .....(1)

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

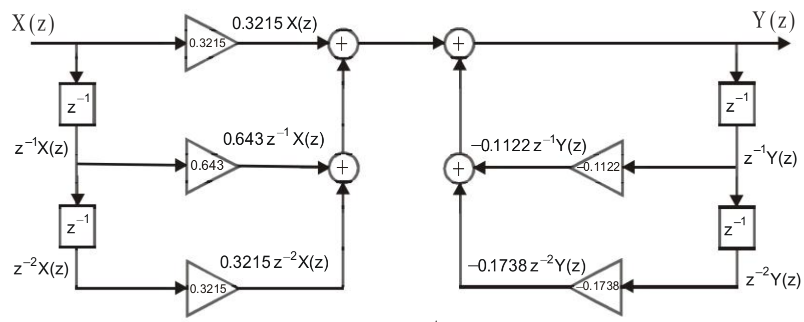


***Fig 1 :** Direct form-I structure of 2^{nd} order digital IIR lowpass filter.*

Direct form-II structure of digital IIR lowpass filter

Let, $H(z)=\dfrac{Y(z)}{X(z)}=\dfrac{W(z)}{X(z)}\times\dfrac{Y(z)}{W(z)}=\dfrac{0.3215+0.643\,z^{-1}+0.3215\,z^{-2}}{1+0.1122\,z^{-1}+0.1738\,z^{-2}}$

where, $\dfrac{W(z)}{X(z)}=\dfrac{1}{1+0.1122\,z^{-1}+0.1738\,z^{-2}}$ .....(2)

$\dfrac{Y(z)}{W(z)}=0.3215+0.643\,z^{-1}+0.3215\,z^{-2}$ .....(3)

On cross multiplying equation (2) we get,

$W(z) + 0.1122z^{-1}W(z) + 0.1738z^{-2}W(z) = X(z)$

$\therefore$ $W(z) = X(z) - 0.1122z^{-1}W(z) - 0.1738z^{-2}W(z)$ .....(4)

On cross multiplying equation (3) we get,

$Y(z) = 0.3215W(z) + 0.643z^{-1}W(z) + 0.3215z^{-2}W(z)$ .....(5)

Using equations (4) and (5), the direct form-II structure is drawn as shown in fig 2.

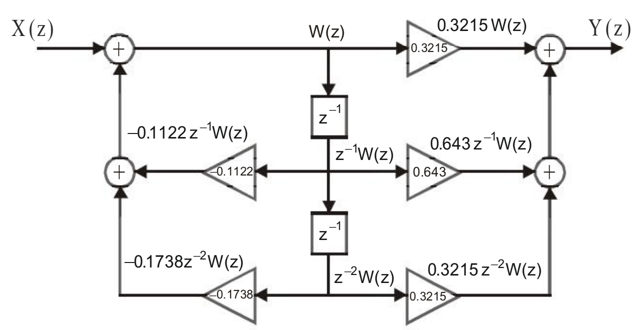


***Fig 2 :** Direct form-II structure of 2^{nd} order digital IIR lowpass filter.*

Frequency Response, $H(e^{j\omega})$

$$H\left(e^{j\omega}\right)=H(z)\Big|_{z=e^{j\omega}}=\frac{0.3215+0.643\,z^{-1}+0.3215\,z^{-2}}{1+0.1122\,z^{-1}+0.1738\,z^{-2}}\bigg|_{z=e^{j\omega}}=\frac{0.3215+0.643\,e^{-j\omega}+0.3215\,e^{-j2\omega}}{1+0.1122\,e^{-j\omega}+0.1738\,e^{-j2\omega}}$$

$$=\frac{0.3215+0.643\left(\cos\omega-j\sin\omega\right)+0.3215\left(\cos2\omega-j\sin2\omega\right)}{1+0.1122\left(\cos\omega-j\sin\omega\right)+0.1738\left(\cos2\omega-j\sin2\omega\right)}$$

$$=\frac{\left(0.3215+0.643\cos\omega+0.3215\cos2\omega\right)+j\left(-0.643\sin\omega-0.3215\sin2\omega\right)}{\left(1+0.1122\cos\omega+0.1738\cos2\omega\right)+j\left(-0.1122\sin\omega-0.1738\sin2\omega\right)}$$

$$\text{Let, } H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} = \frac{(0.3215 + 0.643\cos\omega + 0.3215\cos 2\omega) + j(-0.643\sin\omega - 0.3215\sin 2\omega)}{(1 + 0.1122\cos\omega + 0.1738\cos 2\omega) + j(-0.1122\sin\omega - 0.1738\sin 2\omega)}$$

$$\text{where, } H_N(e^{j\omega}) = (0.3215 + 0.643\cos\omega + 0.3215\cos 2\omega) + j(-0.643\sin\omega - 0.3215\sin 2\omega)$$

$$H_D(e^{j\omega}) = (1 + 0.1122\cos\omega + 0.1738\cos 2\omega) + j(-0.1122\sin\omega - 0.1738\sin 2\omega)$$

The frequency response $H(e^{j\omega})$ and hence the magnitude response $|H(e^{j\omega})|$ are calculated for various values of ω and listed in table 1. Using the values listed in table 1, the magnitude response of lowpass filter is sketched as shown in fig 3.

TABLE 1: $H(e^{j\omega})$ and $|H(e^{j\omega})|$ for various values of ω.

ω	$H_N(e^{j\omega})$	$H_D(e^{j\omega})$	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
$\frac{0\times\pi}{16}$	1.286 + j0	1.286 + j0	1 + j0	1.0000
$\frac{1\times\pi}{16}$	1.2492 – j0.2485	1.2706 – j0.0884	0.9920 – j0.1266	1.0000
$\frac{2\times\pi}{16}$	1.1429 – j0.4734	1.2265 – j0.1658	0.9664 – j0.2553	0.9995
$\frac{3\times\pi}{16}$	0.9792 – j0.6543	1.1598 – j0.2229	0.9188 – j0.3878	0.9973
$\frac{4\times\pi}{16}$	0.7762 – j0.7762	1.0793 – j0.2531	0.8415 – j0.5218	0.9902
$\frac{5\times\pi}{16}$	0.5557 – j0.8317	0.9958 – j0.2539	0.7239 – j0.6506	0.9733
$\frac{6\times\pi}{16}$	0.3402 – j0.8214	0.9200 – j0.2265	0.5559 – j0.7559	0.9384
$\frac{7\times\pi}{16}$	0.1499 – j0.7537	0.8613 – j0.1765	0.3391 – j0.8056	0.8740
$\frac{8\times\pi}{16}$	0 – j0.643	0.8262 – j0.1122	0.1038 – j0.7642	0.7712
$\frac{9\times\pi}{16}$	–0.1009 – j0.5076	0.8175 – j0.0435	–0.0901 – j0.6257	0.6322
$\frac{10\times\pi}{16}$	–0.1519 – j0.3667	0.8341 + j0.0192	–0.1921 – j0.4352	0.4757
$\frac{11\times\pi}{16}$	–0.1588 – j0.2376	0.8712 + j0.0673	–0.2021 – j0.2571	0.3271
$\frac{12\times\pi}{16}$	–0.1332 – j0.1332	0.9207 + j0.0945	–0.1578 – j0.1285	0.2035
$\frac{13\times\pi}{16}$	–0.0901 – j0.0602	0.9732 + j0.0982	–0.0978 – j0.0519	0.1108
$\frac{14\times\pi}{16}$	–0.0452 – j0.0187	1.0192 + j0.0799	–0.0455 – j0.0148	0.0478
$\frac{15\times\pi}{16}$	–0.0121 – j0.0024	1.0505 + j0.0446	–0.0116 – j0.0018	0.0117
$\frac{16\times\pi}{16}$	0 + j0	1.0616 + j0	0 + j0	0

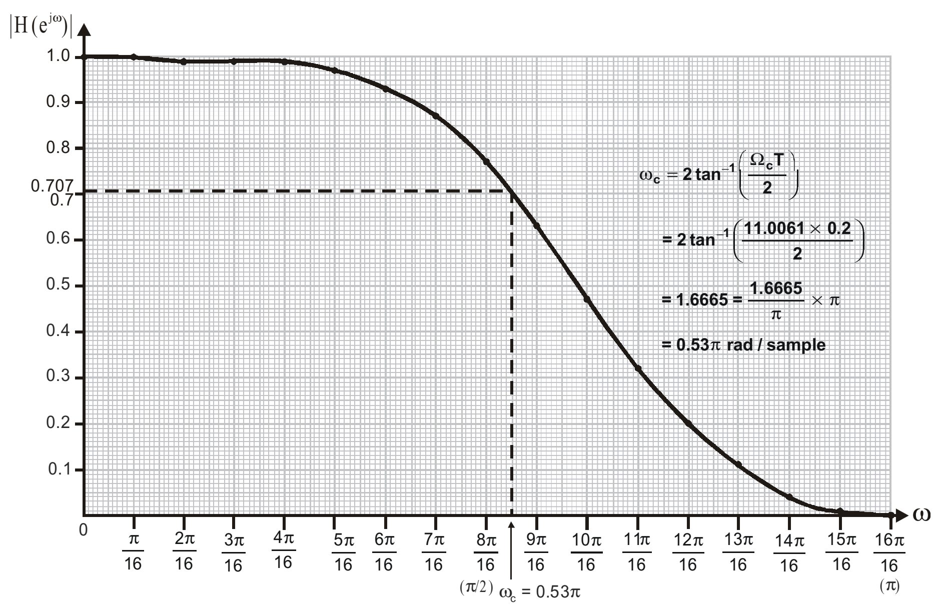


Fig 3 : Frequency response of 2nd order digital Butterworth IIR lowpass filter.

E7.9. Design a Butterworth digital IIR highpass filter using bilinear transformation by taking T = 0.2second, to satisfy the following specifications.

$$0.8 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0.7\pi \le \omega \le \pi$$

$$|H(e^{j\omega})| \le 0.3 \quad ; \quad \text{for } 0 \le \omega \le 0.4\pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternatively,

Passband ripple ≤ 1.9382 ***dB***

Stopband attenuation ≥ 10.4576 ***dB***

Passband edge frequency = 0.7π ***rad/sample***

Stopband edge frequency = 0.4π ***rad/sample***

The above specifications can be converted to A_p ***and*** A_s ***as shown below.***

$$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{\left(-1.9382/20\right)} = 0.8$$

$$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{\left(-10.4576/20\right)} = 0.3$$

Solution

Specifications of digital IIR highpass filter

Passband edge digital frequency, $\omega_p = 0.7\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.4\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.8$

Gain in normal value at stopband edge, $A_s = 0.3$

Sampling time, T = 0.2second

The highpass filter is designed via lowpass filter using frequency transformation technique. Hence the given specifications of IIR highpass filter are converted to corresponding specification of IIR lowpass filter.

Specifications of digital IIR lowpass filter

The specification of lowpass filter is obtained by taking passband edge of highpass as stopband edge of lowpass and stopband edge of highpass as passband edge of lowpass. The gain of passband and stopband remain same.

$\therefore$ Passband edge digital frequency, $\omega_p = 0.4\pi$ rad/sample

$\therefore$ Stopband edge digital frequency, $\omega_s = 0.7\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.8$

Gain in normal value at stopband edge, $A_s = 0.3$

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.8$

Gain in normal value at stopband edge, $A_s = 0.3$

Gain is same in analog and digital filter.

For bilinear transformation,

Passband edge analog frequency, $\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$

$$= \frac{2}{0.2}\tan\frac{0.4\pi}{2} = 7.2654 \text{ rad / second}$$

Using equation (7.53).

Stopband edge analog frequency, $\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$

$$= \frac{2}{0.2}\tan\frac{0.7\pi}{2} = 19.6261 \text{ rad / second}$$

Using equation (7.54).

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\frac{(1/A_s^2)-1}{(1/A_p^2)-1}\right]}{\log\frac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\frac{(1/0.3^2)-1}{(1/0.8^2)-1}\right]}{\log\frac{19.6261}{7.2654}} = \frac{1}{2}\frac{\log\left[\frac{10.1111}{0.5625}\right]}{\log\frac{19.6261}{7.2654}} = 1.4536$$

Using equation (7.57).

Choose order N, such that $N \ge N_1$ and N is an integer.

Let, order, N = 2.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For even N,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.58).

where, $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$

Using equation (7.60).

Here, N = 2, $\therefore\ k = \frac{N}{2} = \frac{2}{2} = 1$

$$\therefore\ H(s_n) = \frac{1}{s_n^2 + b_1 s_n + 1}$$

When k = 1 ; $b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times 2}\right] = 1.4142$

> Calculate $\sin\theta$ using calculator in radian mode.

$$\therefore\ H(s_n) = \frac{1}{s_n^2 + 1.4142\, s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth highpass filter

The highpass filter with cutoff frequency, Ω_c can be obtained from normalized lowpass filter using the transformation $s_n \to \Omega_c/s$.

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{\Omega_c}{s}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[\left(1/A_s^2\right) - 1\right]^{\frac{1}{2N}}} = \frac{19.6261}{\left[\left(1/0.3^2\right) - 1\right]^{\frac{1}{4}}} = 11.0061\ \text{rad / sec}$$

> Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{\Omega_c}{s}} = \frac{1}{s_n^2 + 1.4142\, s_n + 1}\Bigg|_{s_n = \frac{\Omega_c}{s}}$$

$$= \frac{1}{\frac{\Omega_c^2}{s^2} + 1.4142\frac{\Omega_c}{s} + 1} = \frac{1}{\frac{\Omega_c^2 + 1.4142\,\Omega_c s + s^2}{s^2}} = \frac{s^2}{s^2 + 1.4142\,\Omega_c s + \Omega_c^2}$$

$$= \frac{s^2}{s^2 + 1.4142 \times 11.0061 s + 11.0061^2} = \frac{s^2}{s^2 + 15.5648 + 121.1342}$$

Digital IIR highpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{s^2}{s^2 + 15.5648\, s + 121.1342}\Bigg|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 15.5648\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + 121.1342} = \frac{\frac{4\left(1-z^{-1}\right)^2}{T^2\left(1+z^{-1}\right)^2}}{\frac{4\left(1-z^{-1}\right)^2}{T^2\left(1+z^{-1}\right)^2} + \frac{31.1296\left(1-z^{-1}\right)}{T\left(1+z^{-1}\right)} + 121.1342}$$

$$= \frac{\frac{4\left(1-z^{-1}\right)^2}{\cancel{T^2\left(1+z^{-1}\right)^2}}}{\frac{4\left(1-z^{-1}\right)^2 + 31.1296\,T\left(1-z^{-1}\right)\left(1+z^{-1}\right) + 121.1342\,T^2\left(1+z^{-1}\right)^2}{\cancel{T^2\left(1+z^{-1}\right)^2}}}$$

$$= \frac{4\left(1-z^{-1}\right)^2}{4\left(1-z^{-1}\right)^2 + 31.1296\,T\left(1-z^{-1}\right)\left(1+z^{-1}\right) + 121.1342\,T^2\left(1+z^{-1}\right)^2}$$

> Put, T = 0.2

$$= \frac{4\left(1-2z^{-1}+z^{-2}\right)}{4\left(1-2z^{-1}+z^{-2}\right) + 31.1296 \times 0.2\left(1-z^{-2}\right) + 121.1342 \times 0.2^2\left(1+2z^{-1}+z^{-2}\right)}$$

$$= \frac{4\left(1-2z^{-1}+z^{-2}\right)}{4\left(1-2z^{-1}+z^{-2}\right) + 6.2259\left(1-z^{-2}\right) + 4.8453\left(1+2z^{-1}+z^{-2}\right)} = \frac{4 - 8z^{-1} + 4z^{-2}}{15.0712 + 1.6906\,z^{-1} + 2.6194\,z^{-2}}$$

$$= \frac{\frac{4}{15.0712} - \frac{8}{15.0712}z^{-1} + \frac{4}{15.0712}z^{-2}}{1 + \frac{1.6906}{15.0712}z^{-1} + \frac{2.6194}{15.0712}z^{-2}} = \frac{0.2654 - 0.5308\,z^{-1} + 0.2654z^{-2}}{1 + 0.1122\,z^{-1} + 0.1738\,z^{-2}}$$

> Alternatively,
>
> $$H(z) = \frac{0.2654 - 0.5308\,z^{-1} + 0.2654\,z^{-2}}{1 + 0.1122\,z^{-1} + 0.1738\,z^{-2}} = \frac{z^{-2}\left(0.2654\,z^2 - 0.5308\,z + 0.2654\right)}{z^{-2}\left(z^2 + 0.1122\,z + 0.1738\right)} = \frac{0.2654\,z^2 - 0.5308\,z + 0.2654}{z^2 + 0.1122\,z + 0.1738}$$

Direct form-I structure of digital IIR highpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{0.2654 - 0.5308\,z^{-1} + 0.2654\,z^{-2}}{1 + 0.1122\,z^{-1} + 0.1738\,z^{-2}}$$

On cross multiplying the above equation we get,

$$Y(z) + 0.1122z^{-1}Y(z) + 0.1738z^{-2}Y(z) = 0.2654X(z) - 0.5308z^{-1}X(z) + 0.2654z^{-2}X(z)$$

$$\therefore\ Y(z) = 0.2654X(z) - 0.5308z^{-1}X(z) + 0.2654z^{-2}X(z) - 0.1122z^{-1}Y(z) - 0.1738z^{-2}Y(z) \quad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

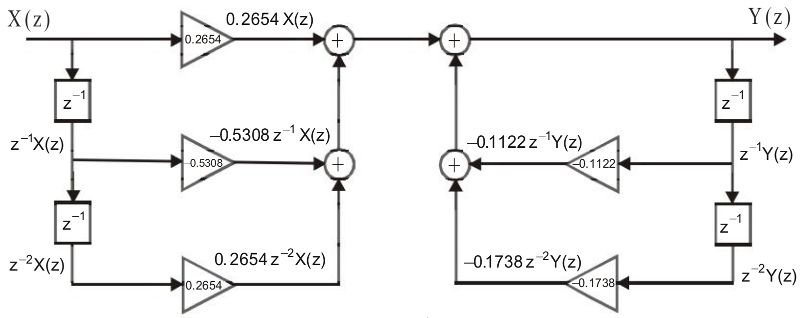


Fig 1 : Direct form-I structure of 2nd order digital IIR highpass filter.

Direct form-II structure of digital IIR highpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.2654 - 0.5308\,z^{-1} + 0.2654\,z^{-2}}{1 + 0.1122\,z^{-1} + 0.1738\,z^{-2}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1 + 0.1122\,z^{-1} + 0.1738\,z^{-2}} \quad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.2654 - 0.5308\,z^{-1} + 0.2654\,z^{-2} \quad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) + 0.1122z^{-1}W(z) + 0.1738z^{-2}W(z) = X(z)$$

$$\therefore\ W(z) = X(z) - 0.1122z^{-1}W(z) - 0.1738z^{-2}W(z) \quad .....(4)$$

On cross multiplying equation (3) we get,

$$Y(z) = 0.2654W(z) - 0.5308z^{-1}W(z) + 0.2654z^{-2}W(z) \quad .....(5)$$

Using equations (4) and (5), the direct form-II structure is drawn as shown in fig 2.

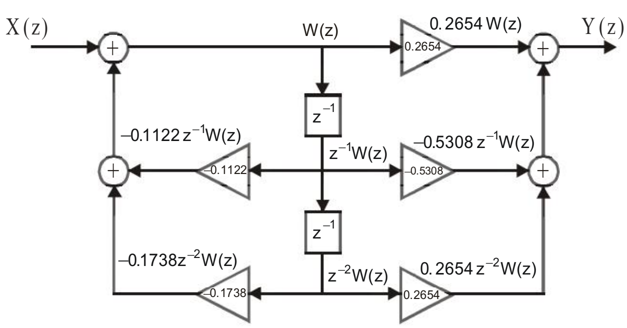


Fig 2 : Direct form-II structure of 2nd order digital IIR highpass filter.

Frequency Response, $H(e^{j\omega})$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \left.\frac{0.2654 - 0.5308\,z^{-1} + 0.2654\,z^{-2}}{1 + 0.1122\,z^{-1} + 0.1738\,z^{-2}}\right|_{z=e^{j\omega}}$$

$$= \frac{0.2654 - 0.5308\,e^{-j\omega} + 0.2654\,e^{-j2\omega}}{1 + 0.1122\,e^{-j\omega} + 0.1738\,e^{-j2\omega}} = \frac{0.2654 - 0.5308(\cos\omega - j\sin\omega) + 0.2654(\cos 2\omega - j\sin 2\omega)}{1 + 0.1122(\cos\omega - j\sin\omega) + 0.1738(\cos 2\omega - j\sin 2\omega)}$$

$$= \frac{(0.2654 - 0.5308\cos\omega + 0.2654\cos 2\omega) + j(0.5308\sin\omega - 0.2654\sin 2\omega)}{(1 + 0.1122\cos\omega + 0.1738\cos 2\omega) + j(-0.1122\sin\omega - 0.1738\sin 2\omega)}$$

$$\text{Let, } H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} = \frac{(0.2654 - 0.5308\cos\omega + 0.2654\cos 2\omega) + j(0.5308\sin\omega - 0.2654\sin 2\omega)}{(1 + 0.1122\cos\omega + 0.1738\cos 2\omega) + j(-0.1122\sin\omega - 0.1738\sin 2\omega)}$$

$$\text{where, } H_N(e^{j\omega}) = (0.2654 - 0.5308\cos\omega + 0.2654\cos 2\omega) + j(0.5308\sin\omega - 0.2654\sin 2\omega)$$

$$H_D(e^{j\omega}) = (1 + 0.1122\cos\omega + 0.1738\cos 2\omega) + j(-0.1122\sin\omega - 0.1738\sin 2\omega)$$

The frequency response $H(e^{j\omega})$ and hence the magnitude response $|H(e^{j\omega})|$ are calculated for various values of ω and listed in table 1. Using the values listed in table 1, the magnitude response of highpass filter is sketched as shown in fig 3.

TABLE 1: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w.

w	$H_N(e^{jw})$	$H_D(e^{jw})$	$H(e^{jw})$	$\|H(e^{jw})\|$
$\frac{0\times\pi}{16}$	0 + j0	1.286 + j0	0 + j0	0
$\frac{1\times\pi}{16}$	–0.0100 + j0.0019	1.2706 – j0.0884	–0.0079 + j0.0009	0.0079
$\frac{2\times\pi}{16}$	–0.0373 + j0.0155	1.2266 – j0.1658	–0.0315 + j0.0084	0.0326
$\frac{3\times\pi}{16}$	–0.0744 + j0.0497	1.1598 – j0.2229	–0.0698 + j0.0294	0.0757
$\frac{4\times\pi}{16}$	–0.1099 + j0.1099	1.0793 – j0.2531	–0.1191 + j0.0739	0.1402
$\frac{5\times\pi}{16}$	–0.1310 + j0.1961	0.9958 – j0.2538	–0.1707 + j0.1534	0.2295
$\frac{6\times\pi}{16}$	–0.1254 + j0.3027	0.9200 – j0.2265	–0.2048 + j0.2786	0.3458
$\frac{7\times\pi}{16}$	–0.0834 + j0.4190	0.8613 – j0.1765	–0.1886 + j0.4478	0.4859
$\frac{8\times\pi}{16}$	0 + j0.5308	0.8262 – j0.1122	–0.0857 + j0.6308	0.6366
$\frac{9\times\pi}{16}$	0.1237 + j0.6222	0.8175 – j0.0435	0.1105 + j0.7669	0.7749
$\frac{10\times\pi}{16}$	0.2809 + j0.6781	0.8341 + j0.0192	0.3553 + j0.8048	0.8797
$\frac{11\times\pi}{16}$	0.4587 + j0.6865	0.8711 + j0.0672	0.5838 + j0.7430	0.9450
$\frac{12\times\pi}{16}$	0.6407 + j0.6407	0.9206 + j0.0944	0.7593 + j0.6181	0.9791
$\frac{13\times\pi}{16}$	0.8083 + j0.5401	0.9732 + j0.0982	0.8776 + j0.4664	0.9939
$\frac{14\times\pi}{16}$	0.9435 + j0.3908	1.0192 + j0.0799	0.9499 + j0.3089	0.9989
$\frac{15\times\pi}{16}$	1.0312 + j0.2051	1.0505 + j0.0446	0.9881 + j0.1533	0.9999
$\frac{16\times\pi}{16}$	1.0616 + j0	1.0616 + j0	1 + j0	1.0000

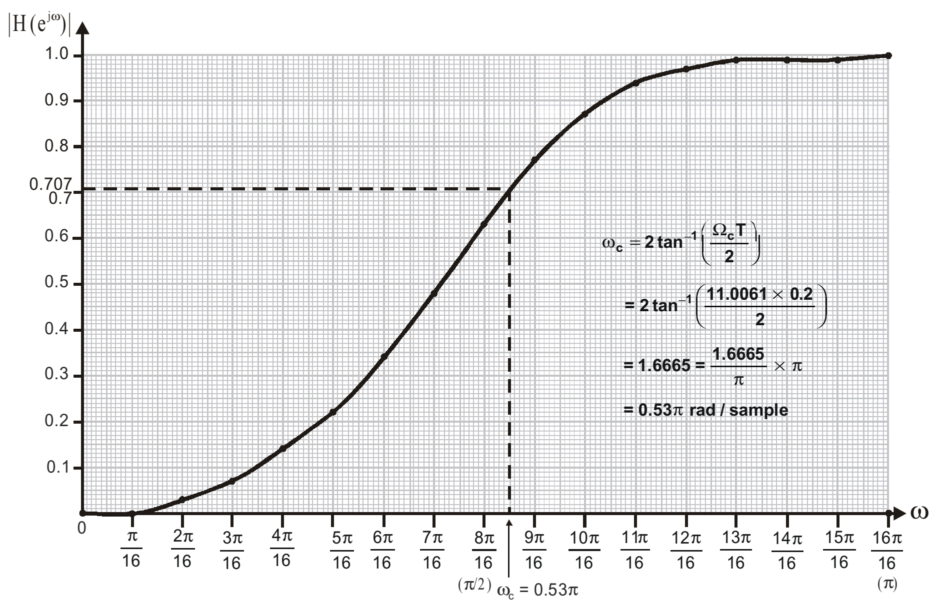


Fig 3 : Frequency response of 2^{nd} order digital Butterworth IIR highpass filter.

E7.10. Design a Butterworth digital IIR lowpass filter using bilinear transformation by taking T = 0.3second, to satisfy the following specifications.

0.45 £ $|H(e^{jw})|$ £ 1.0 ; for 0 £ w £ 0.675p

$|H(e^{jw})|$ £ 0.15 ; for 0.8p £ w £ p

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternatively,

Passband ripple £ 6.9357 dB

Stopband attenuation ³ 16.4781 dB

Passband edge frequency = 0.675p rad/sample

Stopband edge frequency = 0.8p rad/sample

The above specifications can be converted to A_p and A_s as shown below.

$$A_p = 10^{(-\delta_{p,dB}/20)} = 10^{(-6.9357/20)} = 0.45$$

$$A_s = 10^{(-\alpha_{s,dB}/20)} = 10^{(-16.4781/20)} = 0.15$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.675\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.8\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.45$

Gain in normal value at stopband edge, $A_s = 0.15$

Sampling time, T = 0.3second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.45$

Gain in normal value at stopband edge, $A_s = 0.15$

Gain is same in analog and digital filter.

For bilinear transformation,

$$\text{Passband edge analog frequency, } \Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$$

Using equation (7.53).

$$= \frac{2}{0.3}\tan\frac{0.675\pi}{2} = 11.9042 \text{ rad / second}$$

$$\text{Stopband edge analog frequency, } \Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$$

Using equation (7.54).

$$= \frac{2}{0.3}\tan\frac{0.8\pi}{2} = 20.5179 \text{ rad / second}$$

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\frac{(1/A_s^2)-1}{(1/A_p^2)-1}\right]}{\log\frac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\frac{(1/0.15^2)-1}{(1/0.45^2)-1}\right]}{\log\frac{20.5179}{11.9042}} = \frac{1}{2}\frac{\log\left[\frac{43.4445}{3.9383}\right]}{\log\frac{20.5179}{11.9042}} = 2.2049$$

Using equation (7.57).

Choose order N, such that $N \geq N_1$ and N is an integer.

Let, order, N = 3.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For odd N,

$$H(s_n) = \frac{1}{s_n+1}\prod_{k=1}^{\frac{N-1}{2}}\frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.59).

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Using equation (7.60).

$$\text{Here, } N = 3, \ \therefore\ k = \frac{N-1}{2} = \frac{3-1}{2} = 1$$

$$\therefore\ H(s_n) = \frac{1}{s_n+1} \times \frac{1}{s_n^2 + b_1 s_n + 1}$$

$$\text{When } k = 1\ ;\ \ b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times 3}\right] = 1$$

Calculate $\sin\theta$ using calculator in radian mode.

$$\therefore\ H(s_n) = \frac{1}{(s_n+1)\,(s_n^2 + s_n + 1)} = \frac{1}{s_n^3 + s_n^2 + s_n + s_n^2 + s_n + 1}$$

$$= \frac{1}{s_n^3 + 2s_n^2 + 2s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2)-1\right]^{\frac{1}{2N}}} = \frac{20.5179}{\left[(1/0.15^2)-1\right]^{\frac{1}{6}}} = 10.9432 \text{ rad / second}$$

Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{(s_n+1)(s_n^2 + s_n + 1)}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$H(s)=\frac{1}{\left(\frac{s}{\Omega_c}+1\right)\left(\frac{s^2}{\Omega_c^2}+\frac{s}{\Omega_c}+1\right)}=\frac{1}{\left(\frac{s+\Omega_c}{\Omega_c}\right)\left(\frac{s^2+s\Omega_c+\Omega_c^2}{\Omega_c^2}\right)}$$

$$=\frac{\Omega_c^3}{(s+\Omega_c)(s^2+s\Omega_c+\Omega_c^2)}=\frac{(10.9432)^3}{(s+10.9432)(s^2+10.9432s+10.9432^2)}$$

$$=\frac{1310.4879}{(s+10.9432)(s^2+10.9432s+119.7536)}$$

$$=\frac{1310.4879}{s^3+21.8864s^2+239.5072s+1310.4876}$$

Digital IIR lowpass filter transfer function, H(z)

For bilinear transformation,

$$H(z)=H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}=\frac{1310.4879}{s^3+21.8864s^2+239.5072s+1310.4876}\Bigg|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$=\frac{1310.4879}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3+21.8864\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2+239.5072\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)+1310.4876}$$

$$=\frac{1310.4879}{\frac{8(1-z^{-1})^3}{T^3(1+z^{-1})^3}+\frac{87.5456(1-z^{-1})^2}{T^2(1+z^{-1})^2}+\frac{479.0144(1-z^{-1})}{T(1+z^{-1})}+1310.4876}$$

$$=\frac{1310.4879}{\frac{8(1-z^{-1})^3+87.5456T(1+z^{-1})(1-z^{-1})^2+479.0144(1-z^{-1})T^2(1+z^{-1})^2+1310.4876T^3(1+z^{-1})^3}{T^3(1+z^{-1})^3}}$$

Put, T = 0.3

$$=\frac{1310.4879T^3(1+z^{-1})^3}{8(1-z^{-1})^3+87.5456T(1+z^{-1})(1-z^{-1})^2+479.0144(1-z^{-1})T^2(1+z^{-1})^2+1310.4876T^3(1+z^{-1})^3}$$

$$=\frac{1310.4879\times 0.3^2(1+z^{-1})^3}{8(1-z^{-1})^3+87.5456\times 0.3(1-z^{-1})^2(1+z^{-1})+479.0144\times 0.3^2(1-z^{-1})(1+z^{-1})^2+1310.4876\times 0.3^3(1+z^{-1})^3}$$

$$=\frac{35.3832(1+3z^{-1}+3z^{-2}+z^{-3})}{8(1-3z^{-1}+3z^{-2}-z^{-3})+26.2636(1-z^{-2})(1-z^{-1})+43.1113(1+z^{-1})(1-z^{-2})+35.3832(1+3z^{-1}+3z^{-2}+z^{-3})}$$

(a + b) (a – b) = $a^2 - b^2$
$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$
$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$

$$=\frac{35.3832+106.1496z^{-1}+106.1496z^{-2}+35.3832z^{-3}}{8(1-3z^{-1}+3z^{-2}-z^{-3})+26.2636(1-z^{-1}-z^{-2}+z^{-3})+43.1113(1+z^{-1}-z^{-2}-z^{-3})+35.3832(1+3z^{-1}+3z^{-2}+z^{-3})}$$

$$=\frac{35.3832+106.1496z^{-1}+106.1496z^{-2}+35.3832z^{-3}}{112.7581+98.9973z^{-1}+60.7747z^{-2}+10.5355z^{-3}}=\frac{\frac{35.3832}{112.7581}+\frac{106.1496}{112.7581}z^{-1}+\frac{106.1496}{112.7581}z^{-2}+\frac{35.3832}{112.7581}z^{-3}}{1+\frac{98.9973}{112.7581}z^{-1}+\frac{60.7747}{112.7581}z^{-2}+\frac{10.5355}{112.7581}z^{-3}}$$

$$=\frac{0.3138+0.9414z^{-1}+0.9414z^{-2}+0.3138z^{-3}}{1+0.8779z^{-1}+0.5389z^{-2}+0.0934z^{-3}}$$

Alternatively,

$$H(z)=\frac{0.3138+0.9414z^{-1}+0.9414z^{-2}+0.3138z^{-3}}{1+0.8779z^{-1}+0.5389z^{-2}+0.0934z^{-3}}=\frac{z^{-3}\left(0.3138z^3+0.9414z^2+0.9414z+0.3138\right)}{z^{-3}\left(z^3+0.8779z^2+0.5389z+0.0934\right)}$$

$$=\frac{0.3138z^3+0.9414z^2+0.9414z+0.3138}{z^3+0.8779z^2+0.5389z+0.0934}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=\frac{0.3138+0.9414z^{-1}+0.9414z^{-2}+0.3138z^{-3}}{1+0.8779z^{-1}+0.5389z^{-2}+0.0934z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z)+0.8779z^{-1}Y(z)+0.5389z^{-2}Y(z)+0.0934z^{-3}Y(z)=0.3138X(z)+0.9414z^{-1}X(z)+0.9414z^{-2}X(z)+0.3138z^{-3}X(z)$$

$$\therefore Y(z)=0.3138X(z)+0.9414z^{-1}X(z)+0.9414z^{-2}X(z)+0.3138z^{-3}X(z)-0.8779z^{-1}Y(z)-0.5389z^{-2}Y(z)-0.0934z^{-3}Y(z) \quad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

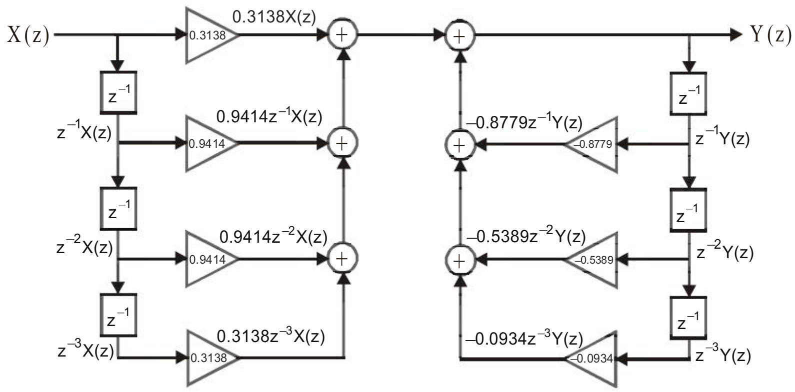


Fig 1 : *Direct form-I structure of 3rd order digital IIR lowpass filter.*

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.3138 + 0.9414\,z^{-1} + 0.9414\,z^{-2} + 0.3138\,z^{-3}}{1 + 0.8779\,z^{-1} + 0.5389\,z^{-2} + 0.0934\,z^{-3}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1 + 0.8779\,z^{-1} + 0.5389\,z^{-2} + 0.0934\,z^{-3}} \qquad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.3138 + 0.9414\,z^{-1} + 0.9414\,z^{-2} + 0.3138\,z^{-3} \qquad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) + 0.8779z^{-1}W(z) + 0.5389z^{-2}W(z) + 0.0934z^{-3}W(z) = X(z)$$

\ $W(z) = X(z) - 0.8779z^{-1}W(z) - 0.5389z^{-2}W(z) - 0.0934z^{-3}W(z)$ .....(4)

On cross multiplying equation (3) we get,

$$Y(z) = 0.3138W(z) + 0.9414z^{-1}W(z) + 0.9414z^{-2}W(z) + 0.3138z^{-3}W(z) \qquad .....(5)$$

Using equations (4) and (5), the direct form-II structure is drawn as shown in fig 2.

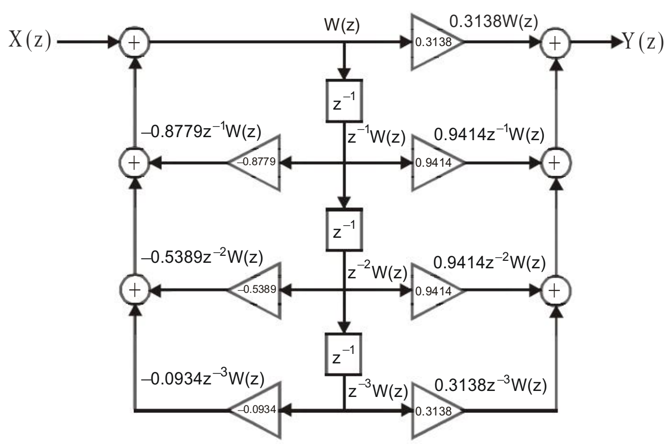


Fig 2 : *Direct form-II structure of 3rd order digital IIR lowpass filter.*

Frequency Response, $H(e^{jw})$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \left.\frac{0.3138 + 0.9414\,z^{-1} + 0.9414\,z^{-2} + 0.3138z^{-3}}{1 + 0.8779\,z^{-1} + 0.5389\,z^{-2} + 0.0934\,z^{-3}}\right|_{z=e^{j\omega}}$$

$$= \frac{0.3138 + 0.9414\,e^{-j\omega} + 0.9414\,e^{-j2\omega} + 0.3138\,e^{-j3\omega}}{1 + 0.8779\,e^{-j\omega} + 0.5389\,e^{-j2\omega} + 0.0934\,e^{-j3\omega}}$$

$$= \frac{0.3138 + 0.9414(\cos\omega - j\sin\omega) + 0.9414(\cos 2\omega - j\sin 2\omega) + 0.3138(\cos 3\omega - j\sin 3\omega)}{1 + 0.8779(\cos\omega - j\sin\omega) + 0.5389(\cos 2\omega - j\sin 2\omega) + 0.0934(\cos 3\omega - j\sin 3\omega)}$$

$$= \frac{(0.3138 + 0.9414\cos\omega + 0.9414\cos 2\omega + 0.3138\cos 3\omega) + j(-0.9414\sin\omega - 0.9414\sin 2\omega - 0.3138\sin 3\omega)}{(1 + 0.8779\cos\omega + 0.5389\cos 2\omega + 0.0934\cos 3\omega) + j(-0.8779\sin\omega - 0.5389\sin 2\omega - 0.0934\sin 3\omega)}$$

$$\text{Let, } H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} = \frac{\begin{array}{r}(0.3138 + 0.9414\cos\omega + 0.9414\cos 2\omega + 0.3138\cos 3\omega) \\ + j(-0.9414\sin\omega - 0.9414\sin 2\omega - 0.3138\sin 3\omega)\end{array}}{\begin{array}{r}(1 + 0.8779\cos\omega + 0.5389\cos 2\omega + 0.0934\cos 3\omega) \\ + j(-0.8779\sin\omega - 0.5389\sin 2\omega - 0.0934\sin 3\omega)\end{array}}$$

$$\text{where, } H_N(e^{jw}) = (0.3138 + 0.9414\cos w + 0.9414\cos 2w + 0.03138\cos 3w) + j(-0.9414\sin w - 0.9414\sin 2w - 0.3138\sin 3w)$$

$$H_D(e^{jw}) = (1 + 0.8779\cos w + 0.5389\cos 2w + 0.0934\cos 3w) + j(-0.8779\sin w - 0.5389\sin 2w - 0.0934\sin 3w)$$

The frequency response $H(e^{jw})$ and hence the magnitude response $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the values listed in table 1, the magnitude response of lowpass filter is sketched as shown in fig 3.

TABLE 1: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w.

w	$H_N(e^{jw})$	$H_D(e^{jw})$	$H(e^{jw})$	$\|H(e^{jw})\|$
$\frac{0\times\pi}{16}$	2.5102 + j0	2.5102 + j0	1 + j0	1.0000
$\frac{1\times\pi}{16}$	2.3678 – j0.7183	2.4366 – j0.4294	0.9929 – j0.1198	1.0000
$\frac{2\times\pi}{16}$	1.9693 – j1.3158	2.2279 – j0.8033	0.9707 – j0.2406	1.0000
$\frac{3\times\pi}{16}$	1.3956 – j1.7005	1.9179 – j1.0772	0.9317 – j0.3633	1.0000
$\frac{4\times\pi}{16}$	0.7576 – j1.8289	1.5547 – j1.2257	0.8724 – j0.4885	0.9999
$\frac{5\times\pi}{16}$	0.1688 – j1.7137	1.1899 – j1.2460	0.7870 – j0.6161	0.9995
$\frac{6\times\pi}{16}$	–0.2815 – j1.4153	0.8686 – j1.1564	0.6655 – j0.7433	0.9977
$\frac{7\times\pi}{16}$	–0.5466 – j1.0227	0.6215 – j0.9896	0.4924 – j0.8616	0.9923
$\frac{8\times\pi}{16}$	–0.6276 – j0.6276	0.4611 – j0.7845	0.2451 – j0.9441	0.9754
$\frac{9\times\pi}{16}$	–0.5653 – j0.3021	0.3827 – j0.5771	–0.0876 – j0.9215	0.9256
$\frac{10\times\pi}{16}$	–0.4222 – j0.0839	0.3693 – j0.3943	–0.4209 – j0.6766	0.7968
$\frac{11\times\pi}{16}$	–0.2617 + j0.0258	0.3976 – j0.2503	–0.5006 – j0.2503	0.5597
$\frac{12\times\pi}{16}$	–0.1299 + j0.0538	0.4453 – j0.1479	–0.2989 + j0.0216	0.2996
$\frac{13\times\pi}{16}$	–0.0475 + j0.0389	0.4945 – j0.0815	–0.1061 + j0.0612	0.1225
$\frac{14\times\pi}{16}$	–0.0103 + j0.0154	0.5342 – j0.0412	–0.0214 + j0.0272	0.0346
$\frac{15\times\pi}{16}$	–0.0007 + j0.0023	0.5592 – j0.0169	–0.0014 + j0.0041	0.0043
$\frac{16\times\pi}{16}$	0 + j0	0.5676 + j0	0 + j0	0

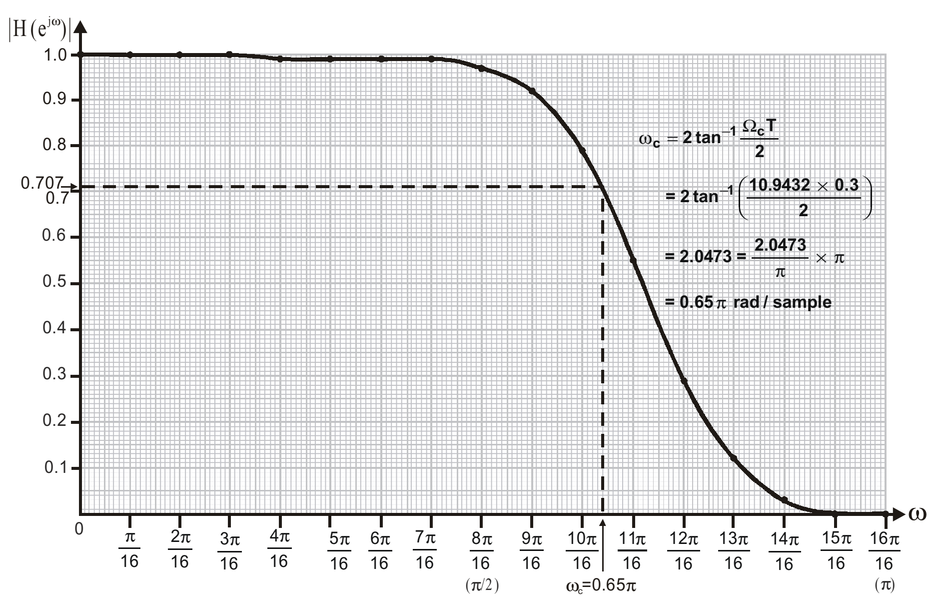


***Fig 3 :** Frequency response of 3[rd] order digital Butterworth IIR lowpass filter.*

E7.11. *Design a Butterworth digital IIR highpass filter using bilinear transformation by taking T = 0.3second, to satisfy the following specifications.*

$$0.45 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0.8\pi \le \omega \le \pi$$

$$|H(e^{j\omega})| \le 0.15 \quad ; \quad \text{for } 0 \le \omega \le 0.675\pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternatively,

Passband ripple $\le$ 6.9357 dB

Stopband attenuation $\ge$ 16.4781 dB

Passband edge frequency = 0.8π rad/sample

Stopband edge frequency = 0.675π rad/sample

The above specifications can be converted to A_p and A_s as shown below.

$$A_p = 10^{(-\delta_{p,dB}/20)} = 10^{(-6.9357/20)} = 0.45$$

$$A_s = 10^{(-\alpha_{s,dB}/20)} = 10^{(-16.4781/20)} = 0.15$$

Solution

Specifications of digital IIR highpass filter

Passband edge digital frequency, $\omega_p = 0.8\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.675\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.45$

Gain in normal value at stopband edge, $A_s = 0.15$

The highpass filter is designed via lowpass filter using frequency transformation technique. Hence the given specifications of IIR highpass filter are converted to corresponding specification of IIR lowpass filter.

Specifications of digital IIR lowpass filter

The specification of lowpass filter is obtained by taking passband edge of highpass as stopband edge of lowpass and stopband edge of highpass as passband edge of lowpass. The gain of passband and stopband remain same.

Passband edge digital frequency, $\omega_p = 0.675\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.8\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.45$

Gain in normal value at stopband edge, $A_s = 0.15$

Sampling time, T = 0.3second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.45$

Gain in normal value at stopband edge, $A_s = 0.15$

Gain is same in analog and digital filter.

For bilinear transformation,

Passband edge analog frequency, $\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$

$$= \frac{2}{0.3}\tan\frac{0.675\pi}{2} = 11.9042 \text{ rad / second}$$

Using equation (7.53).

Stopband edge analog frequency, $\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$

$$= \frac{2}{0.3}\tan\frac{0.8\pi}{2} = 20.5179 \text{ rad / second}$$

Using equation (7.54).

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\frac{(1/A_s^2)-1}{(1/A_p^2)-1}\right]}{\log\frac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\frac{(1/0.15^2)-1}{(1/0.45^2)-1}\right]}{\log\frac{20.5179}{11.9042}} = \frac{1}{2}\frac{\log\left[\frac{43.4445}{3.9383}\right]}{\log\frac{20.5179}{11.9042}} = 2.2049$$

Using equation (7.57).

Choose order N, such that $N \ge N_1$ and N is an integer.

Let, order, N = 3.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For odd N,

$$H(s_n) = \frac{1}{s_n + 1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.59).

where, $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$

Using equation (7.60).

Here, $N = 3$, $\therefore\ k = \frac{N-1}{2} = \frac{3-1}{2} = 1$

$$\therefore\ H(s_n) = \frac{1}{s_n + 1} \times \frac{1}{s_n^2 + b_1 s_n + 1}$$

When $k = 1$; $b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times 3}\right] = 1$

Calculate sinq using calculator in radian mode.

$$\therefore\ H(s_n) = \frac{1}{(s_n+1)\,(s_n^2 + s_n + 1)} = \frac{1}{s_n^3 + s_n^2 + s_n + s_n^2 + s_n + 1} = \frac{1}{s_n^3 + 2s_n^2 + 2s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth highpass filter

The highpass filter with cutoff frequency, Ω_c can be obtained from normalized lowpass filter using the transformation, $s_n \rightarrow \Omega_c/s$.

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{\Omega_c}{s}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2) - 1\right]^{\frac{1}{2N}}} = \frac{20.5179}{\left[(1/0.15^2) - 1\right]^{\frac{1}{6}}} = 10.9432 \text{ rad / second}$$

Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{\Omega_c}{s}} = \frac{1}{s_n^3 + 2s_n^2 + 2s_n + 1}\Bigg|_{s_n = \frac{\Omega_c}{s}}$$

$$= \frac{1}{\left(\frac{\Omega_c}{s}\right)^3 + 2\left(\frac{\Omega_c}{s}\right)^2 + 2\frac{\Omega_c}{s} + 1} = \frac{1}{\frac{\Omega_c^3 + 2\Omega_c^2 s + 2\Omega_c s^2 + s^3}{s^3}} = \frac{s^3}{s^3 + 2\Omega_c s^2 + 2\Omega_c^2 s + \Omega_c^3}$$

$$= \frac{s^3}{s^3 + 2\times 10.9432\ s^2 + 2\times 10.9432^2 s + 10.9432^3} = \frac{s^3}{s^3 + 21.8864 s^2 + 239.5073 s + 1310.4879}$$

Digital IIR highpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{s^3}{s^3 + 21.8864 s^2 + 239.5073 s + 1310.4879}\Bigg|_{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + 21.8864\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 239.5073\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + 1310.4879}$$

$$= \frac{\frac{8(1-z^{-1})^3}{\cancel{T^3(1+z^{-1})^3}}}{\frac{8(1-z^{-1})^3 + 21.8864\times 4T(1-z^{-1})^2(1+z^{-1}) + 239.5073\times 2T^2(1-z^{-1})(1+z^{-1})^2 + 1310.4879\times T^3(1+z^{-1})^3}{\cancel{T^3(1+z^{-1})^3}}}$$

Put, T = 0.3

$$= \frac{8(1-z^{-1})^3}{8(1-z^{-1})^3 + 21.8864\times 4\times 0.3(1-z^{-1})^2\ (1+z^{-1}) + 239.5073\times 2\times\ 0.3^2(1-z^{-1})(1+z^{-1})^2 + 1310.4879\times 0.3^3(1+z^{-1})^3}$$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
$(a + b)\ (a - b) = a^2 - b^2$

$$= \frac{8(1-z^{-1})^3}{8(1-z^{-1})^3 + 26.2637(1-z^{-1})^2(1+z^{-1}) + 43.1113(1-z^{-1})(1+z^{-1})^2 + 35.3832(1+z^{-1})^3}$$

$$= \frac{8\left(1 - 3z^{-1} + 3z^{-2} - z^{-3}\right)}{8\left(1 - 3z^{-1} + 3z^{-2} - z^{-3}\right) + 26.2637(1-z^{-2})(1-z^{-1}) + 43.1113(1+z^{-1})(1-z^{-2}) + 35.3832(1 + 3z^{-1} + 3z^{-2} + z^{-3})}$$

$$\therefore\ H(z) = \frac{8 - 24z^{-1} + 24z^{-2} - 8z^{-3}}{8\left(1 - 3z^{-1} + 3z^{-2} - z^{-3}\right) + 26.2637(1 - z^{-1} - z^{-2} + z^{-3}) + 43.1113(1 + z^{-1} - z^{-2} - z^{-3}) + 35.3832(1 + 3z^{-1} + 3z^{-2} + z^{-3})}$$

$$= \frac{8 - 24z^{-1} + 24z^{-2} - 8z^{-3}}{112.7582 + 98.9972z^{-1} + 60.7746z^{-2} + 10.5356z^{-3}}$$

$$= \frac{\dfrac{8}{112.7582} - \dfrac{24}{112.7582}z^{-1} + \dfrac{24}{112.7582}z^{-2} - \dfrac{8}{112.7582}z^{-3}}{1 + \dfrac{98.9972}{112.7582}z^{-1} + \dfrac{60.7746}{112.7582}z^{-2} + \dfrac{10.5356}{112.7582}z^{-3}}$$

$$= \frac{0.0709 - 0.2128z^{-1} + 0.2128z^{-2} - 0.0709z^{-3}}{1 + 0.8779z^{-1} + 0.5389z^{-2} + 0.0934z^{-3}}$$

Alternatively,

$$H(z) = \frac{0.0709 - 0.2128z^{-1} + 0.2128z^{-2} - 0.0709z^{-3}}{1 + 0.8779z^{-1} + 0.5389z^{-2} + 0.0934z^{-3}} = \frac{z^{-3}\left(0.0709z^{3} - 0.2128z^{2} + 0.2128z - 0.0709\right)}{z^{-3}\left(z^{3} + 0.8779z^{2} + 0.5389z + 0.0934\right)}$$

$$= \frac{0.0709z^{3} - 0.2128z^{2} + 0.2128z - 0.0709}{z^{3} + 0.8779z^{2} + 0.5389z + 0.0934}$$

Direct form-I structure of digital IIR highpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{0.0709 - 0.2128z^{-1} + 0.2128z^{-2} - 0.0709z^{-3}}{1 + 0.8779z^{-1} + 0.5389z^{-2} + 0.0934z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z) + 0.8779z^{-1}Y(z) + 0.5389z^{-2}Y(z) + 0.0934z^{-3}Y(z) = 0.0709X(z) - 0.2128z^{-1}X(z) + 0.2128z^{-2}X(z) - 0.0709z^{-3}X(z)$$

$$\therefore\ Y(z) = 0.0709X(z) - 0.2128z^{-1}X(z) + 0.2128z^{-2}X(z) - 0.0709z^{-3}X(z) - 0.8779z^{-1}Y(z) - 0.5389z^{-2}Y(z) - 0.0934z^{-3}Y(z) \quad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

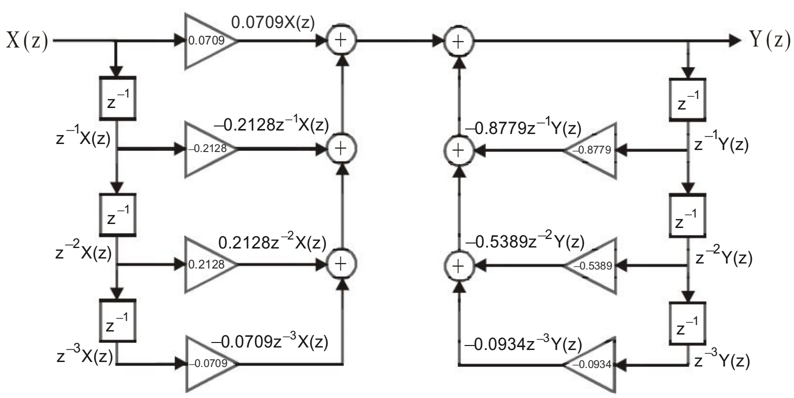


Fig 1 : Direct form-I structure of 3rd order digital IIR highpass filter.

Direct form-II structure of digital IIR highpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.0709 - 0.2128z^{-1} + 0.2128z^{-2} - 0.0709z^{-3}}{1 + 0.8779z^{-1} + 0.5389z^{-2} + 0.0934z^{-3}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1 + 0.8779z^{-1} + 0.5389z^{-2} + 0.0934z^{-3}} \quad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.0709 - 0.2128z^{-1} + 0.2128z^{-2} - 0.0709z^{-3} \quad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) + 0.8779z^{-1}W(z) + 0.5389z^{-2}W(z) + 0.0934z^{-3}W(z) = X(z)$$

$$\therefore\ W(z) = X(z) - 0.8779z^{-1}W(z) - 0.5389z^{-2}W(z) - 0.0934z^{-3}W(z) \quad .....(4)$$

On cross multiplying equation (3) we get,

$$Y(z) = 0.0709W(z) - 0.2128z^{-1}W(z) + 0.2128z^{-2}W(z) - 0.0709z^{-3}W(z) \quad .....(5)$$

Using equation (4) and (5), the direct form-II structure is drawn as shown in fig 2.

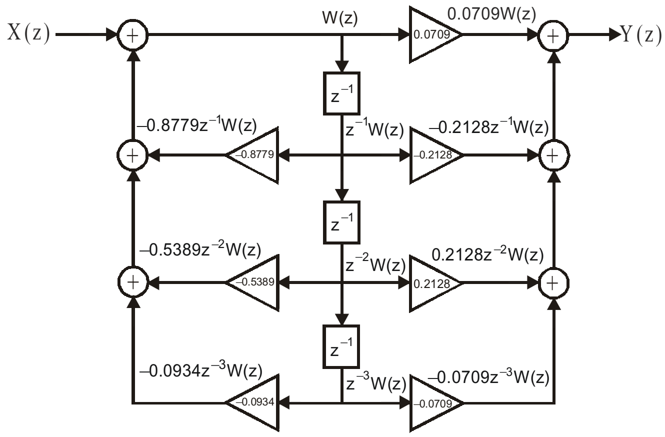


***Fig 2 :** Direct form-II structure of 3rd order digital IIR highpass filter.*

Frequency Response, $H(e^{jw})$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{0.0709 - 0.2128\,z^{-1} + 0.2128\,z^{-2} - 0.0709\,z^{-3}}{1 + 0.8779\,z^{-1} + 0.5389\,z^{-2} + 0.0934\,z^{-3}}\Bigg|_{z=e^{j\omega}}$$

$$= \frac{0.0709 - 0.2128\,e^{-j\omega} + 0.2128\,e^{-j2\omega} - 0.0709\,e^{-j3\omega}}{1 + 0.8779\,e^{-j\omega} + 0.5389\,e^{-j2\omega} + 0.0934\,e^{-j3\omega}}$$

$$= \frac{0.0709 - 0.2128(\cos\omega - j\sin\omega) + 0.2128(\cos 2\omega - j\sin 2\omega) - 0.0709(\cos 3\omega - j\sin 3\omega)}{1 + 0.8779(\cos\omega - j\sin\omega) + 0.5389(\cos 2\omega - j\sin 2\omega) + 0.0934(\cos 3\omega - j\sin 3\omega)}$$

$$= \frac{(0.0709 - 0.2128\cos\omega + 0.2128\cos 2\omega - 0.0709\cos 3\omega) + j(0.2128\sin\omega - 0.2128\sin 2\omega + 0.0709\sin 3\omega)}{(1 + 0.8779\cos\omega + 0.5389\cos 2\omega + 0.0934\cos 3\omega) + j(-0.8779\sin\omega - 0.5389\sin 2\omega - 0.0934\sin 3\omega)}$$

Let, $$H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} = \frac{(0.0709 - 0.2128\cos\omega + 0.2128\cos 2\omega - 0.0709\cos 3\omega) + j(0.2128\sin\omega - 0.2128\sin 2\omega + 0.0709\sin 3\omega)}{(1 + 0.8779\cos\omega + 0.5389\cos 2\omega + 0.0934\cos 3\omega) + j(-0.8779\sin\omega - 0.5389\sin 2\omega - 0.0934\sin 3\omega)}$$

where, $H_N(e^{jw}) = (0.0709 - 0.2128\cos w + 0.2128\cos 2w - 0.0709\cos 3w) + j(0.2128\sin w - 0.2128\sin 2w + 0.0709\sin 3w)$

$H_D(e^{jw}) = (1 + 0.8779\cos w + 0.5389\cos 2w + 0.0934\cos 3w) + j(-0.8779\sin w - 0.5389\sin 2w - 0.0934\sin 3w)$

The frequency response $H(e^{jw})$ and hence the magnitude response $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the values listed in table 1, the magnitude response of highpass filter is sketched as shown in fig 3.

TABLE 1: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w.

w	$H_N(e^{jw})$	$H_D(e^{jw})$	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	0 + j0	2.5102 + j0	0 + j0	0
$\frac{1\times\pi}{16}$	–0.0002 – j0.0005	2.4366 – j0.4294	–0.00004 – j0.0002	0.0002
$\frac{2\times\pi}{16}$	–0.0024 – j0.0035	2.2279 – j0.8033	–0.0005 – j0.0017	0.0018
$\frac{3\times\pi}{16}$	–0.0108 – j0.0088	1.9179 – j1.0772	–0.0023 – j0.0059	0.0063
$\frac{4\times\pi}{16}$	–0.0294 – j0.0122	1.5547 – j1.2257	–0.0078 – j0.0140	0.0160
$\frac{5\times\pi}{16}$	–0.0592 – j0.0058	1.1899 – j1.2460	–0.0213 – j0.0272	0.0345
$\frac{6\times\pi}{16}$	–0.0956 + j0.0189	0.8686 – j1.1564	–0.0501 – j0.0450	0.0674
$\frac{7\times\pi}{16}$	–0.1278 + j0.0683	0.6215 – j0.9896	–0.1076 – j0.0615	0.1240
$\frac{8\times\pi}{16}$	–0.1419 + j0.1419	0.4611 – j0.7845	–0.2135 – j0.0554	0.2205
$\frac{9\times\pi}{16}$	–0.1236 + j0.2312	0.3827 – j0.5771	–0.3769 + j0.0358	0.3786
$\frac{10\times\pi}{16}$	–0.0636 + j0.3199	0.3693 – j0.3943	–0.5126 + j0.3189	0.6037
$\frac{11\times\pi}{16}$	0.0382 + j0.3874	0.3976 – j0.2503	–0.3705 + j0.7411	0.8286
$\frac{12\times\pi}{16}$	0.1712 + j0.4134	0.4453 – j0.1479	0.0685 + j0.9511	0.9536
$\frac{13\times\pi}{16}$	0.3154 + j0.3844	0.4945 – j0.0815	0.4962 + j0.8591	0.9921
$\frac{14\times\pi}{16}$	0.4451 + j0.2974	0.5342 – j0.0412	0.7856 + j0.6173	0.9991
$\frac{15\times\pi}{16}$	0.5352 + j0.1623	0.5592 – j0.0169	0.9474 + j0.3189	0.9996
$\frac{16\times\pi}{16}$	0.5676 + j0	0.5676 + j0	1 + j0	1.0000

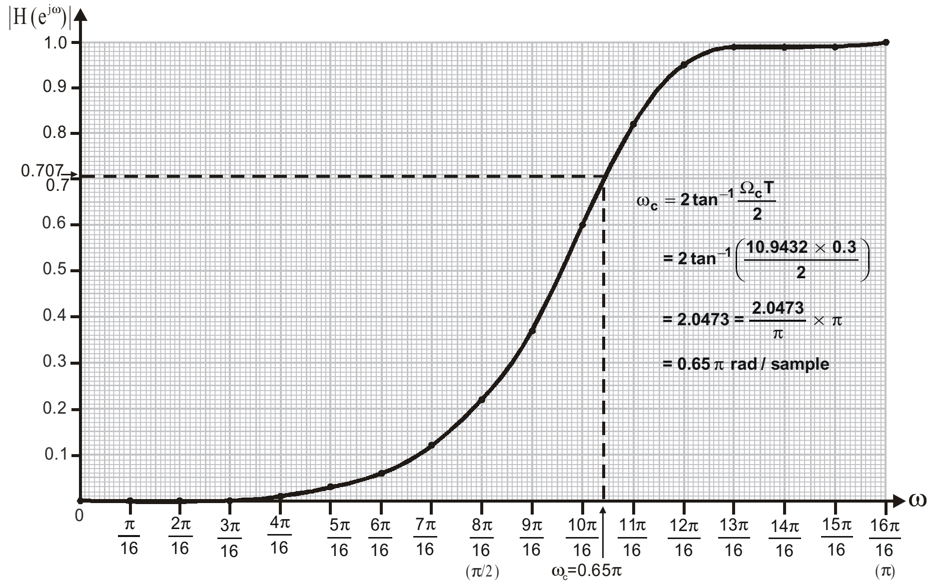


Fig 3 : Frequency response of 3rd order digital Butterworth IIR highpass filter.

E7.12. Design a Butterworth digital IIR lowpass filter using impulse invariant transformation by taking T = 0.8second, to satisfy the following specifications.

$$0.8 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.3\pi$$

$$|H(e^{j\omega})| \le 0.3 \quad ; \quad \text{for } 0.7\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

> **Alternatively,**
>
> **Passband ripple $\le$ 1.9382 dB**
>
> **Stopband attenuation $\ge$ 10.4576 dB**
>
> **Passband edge frequency = 0.3π rad/sample**
>
> **Stopband edge frequency = 0.7π rad/sample**
>
> **The above specifications can be converted to A_p and A_s as shown below.**
>
> $$A_p = 10^{(-\delta_{p,dB}/20)} = 10^{(-1.9382/20)} = 0.8$$
>
> $$A_s = 10^{(-\alpha_{s,dB}/20)} = 10^{(-10.4576/20)} = 0.3$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, ω_p = 0.3π rad/sample

Stopband edge digital frequency, ω_s = 0.7π rad/sample

Gain in normal value at passband edge, A_p = 0.8

Gain in normal value at stopband edge, A_s = 0.3

Sampling time, T = 0.8second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, A_p = 0.8

Gain in normal value at stopband edge, A_s = 0.3

Gain is same in analog and digital filter.

For impulse invariant transformation,

Passband edge analog frequency, $\Omega_p = \frac{\omega_p}{T} = \frac{0.3\pi}{0.8} = 1.1781$ rad / second

Using equation (7.55).

Stopband edge analog frequency, $\Omega_s = \frac{\omega_s}{T} = \frac{0.7\pi}{0.8} = 2.7489$ rad / second

Using equation (7.56).

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\frac{(1/A_s^2)-1}{(1/A_p^2)-1}\right]}{\log\frac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\frac{(1/0.3^2)-1}{(1/0.8^2)-1}\right]}{\log\frac{2.7489}{1.1781}}$$

Using equation (7.57).

$$= \frac{1}{2}\frac{\log\left[\frac{10.1111}{0.5625}\right]}{\log\frac{2.7489}{1.1781}} = \frac{1}{2}\left(\frac{1.2546}{0.3679}\right) = 1.7047$$

Choose order N, such that N ³ N_1 and N is an integer.

Let, order, N = 2.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For even N,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.58).

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Using equation (7.60).

Here, N = 2, $\therefore$ k = $\frac{N}{2} = \frac{2}{2} = 1$

$$\therefore H(s_n) = \frac{1}{s_n^2 + b_1 s_n + 1}$$

When k = 1 ; $b_k = b_1 = 2\sin\left[\frac{(2-1)\pi}{2\times 2}\right] = 1.4142$

Calculate sinq using calculator in radian mode.

$$\therefore H(s_n) = \frac{1}{s_n^2 + 1.4142\,s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, W c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2)-1\right]^{\frac{1}{2N}}} = \frac{2.7489}{\left[(1/0.3^2)-1\right]^{\frac{1}{4}}} = 1.5416 \text{ rad / second}$$

Using equation (7.61).

$$\therefore H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{s_n^2 + 1.4142\,s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{1}{\frac{s^2}{\Omega_c^2} + 1.4142\frac{s}{\Omega_c} + 1} = \frac{1}{\frac{s^2 + 1.4142\,\Omega_c s + \Omega_c^2}{\Omega_c^2}}$$

$$= \frac{\Omega_c^2}{s^2 + 1.4142\,\Omega_c s + \Omega_c^2} = \frac{1.5416^2}{s^2 + 1.4142 \times 1.5416\,s + 1.5416^2}$$

$$= \frac{2.3765}{s^2 + 2.1801s + 2.3765}$$

To convert the analog transfer function to digital transfer function using impulse invariant transformation, the above equation can be simplified as follows.

$$H(s) = \frac{2.3765}{s^2 + 1.0901\times 2s + 1.0901^2 - 1.0901^2 + 2.3765} = \frac{2.3765}{(s+1.0901)^2 + 1.1882}$$

$$= \frac{2.3765}{1.09} \times \frac{1.09}{(s+1.0901)^2 + 1.09^2} = 2.1803 \times \frac{1.09}{(s+1.0901)^2 + 1.09^2}$$

Digital IIR lowpass filter transfer function, H(z)

For impulse invariant transformation,

$$\frac{b}{(s+a)^2 + b^2} \xrightarrow{\text{is transformed to}} \frac{e^{-aT}(\sin bT)z^{-1}}{1 - 2e^{-aT}(\cos bT)z^{-1} + e^{-2aT}z^{-2}}$$

Using equation (7.19).

T = 0.8 second,

$$H(z) = 2.1803 \times \frac{e^{-1.0901\times 0.8}\sin(1.09\times 0.8)z^{-1}}{1-2e^{-1.0901\times 0.8}\cos(1.09\times 0.8)z^{-1}+e^{-2\times 1.0901\times 0.8}z^{-2}}$$

$$= 2.1803 \times \frac{0.4181\sin(0.8720)z^{-1}}{1-2\times 0.4181\cos(0.8720)z^{-1}+0.1748z^{-2}}$$

$$= \frac{0.6979z^{-1}}{1-0.5379z^{-1}+0.1748z^{-2}}$$

Alternatively,

$$H(z) = \frac{0.6979z^{-1}}{1-0.5379z^{-1}+0.1748z^{-2}} = \frac{0.6979z^{-1}}{z^{-2}(z^{2}-0.5379z^{-1}+0.1748)} = \frac{0.6979z}{z^{2}-0.5379z+0.1748}$$

Since T < 1, we can compute magnitude normalized transfer function, $H_N(z)$

$$H_N(z) = T\times H(z) = \frac{0.8\times 0.6979z^{-1}}{1-0.5379z^{-1}+0.1748z^{-2}} = \frac{0.5583z^{-1}}{1-0.5379z^{-1}+0.1748z^{-2}}$$

Alternatively,

$$H_N(z) = T\times H(z) = \frac{0.8\times 0.6979z}{z^{2}-0.5379z+0.1748} = \frac{0.5583z}{z^{2}-0.5379z+0.1748}$$

Direct form-I structure of digital IIR lowpass filter

Let, $H(z) = \frac{Y(z)}{X(z)} = \frac{0.5583z^{-1}}{1-0.5379z^{-1}+0.1748z^{-2}}$

On cross multiplying the above equation we get,

$Y(z) - 0.5379z^{-1}Y(z) + 0.1748z^{-2}Y(z) = 0.5583z^{-1}X(z)$

\ $Y(z) = 0.5583z^{-1}X(z) + 0.5379z^{-1}Y(z) - 0.1748z^{-2}Y(z)$ .....(1)

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

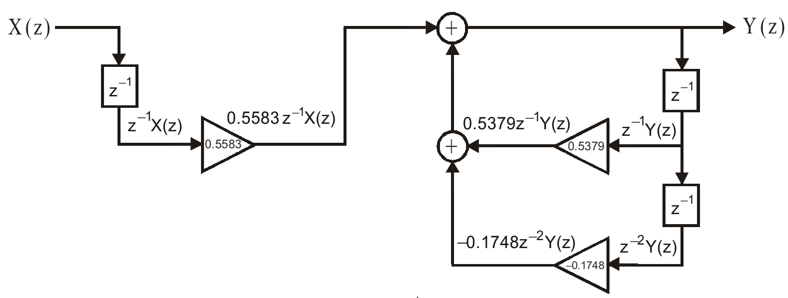


***Fig 1 :** Direct form-I structure of 2^{nd} order digital IIR lowpass filter.*

Direct form-II structure of digital IIR lowpass filter

Let, $H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)}\times\frac{Y(z)}{W(z)} = \frac{0.5583z^{-1}}{1-0.5379z^{-1}+0.1748z^{-2}}$

where, $\frac{W(z)}{X(z)} = \frac{1}{1-0.5379z^{-1}+0.1748z^{-2}}$ .....(2)

$\frac{Y(z)}{W(z)} = 0.5583z^{-1}$ .....(3)

On cross multiplying equation (2) we get,

$W(z) - 0.5379z^{-1}W(z) + 0.1748z^{-2}W(z) = X(z)$

\ $W(z) = X(z) + 0.5379z^{-1}W(z) - 0.1748z^{-2}W(z)$ .....(4)

On cross multiplying equation (3) we get,

$Y(z) = 0.5583z^{-1}W(z)$ .....(5)

Using equation (4) and (5), the direct form-II structure is drawn as shown in fig 2.

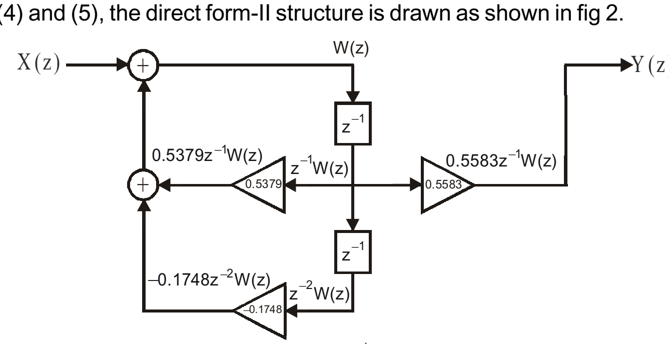


***Fig 2 :** Direct form-II structure of 2^{nd} order digital IIR lowpass filter.*

Frequency Response, $H(e^{j\omega})$

$$H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}} = \left.\frac{0.5583\,z^{-1}}{1-0.5379\,z^{-1}+0.1748\,z^{-2}}\right|_{z=e^{j\omega}}$$

$$= \frac{0.5583\,e^{-j\omega}}{1-0.5379\,e^{-j\omega}+0.1748\,e^{-j2\omega}}$$

$$= \frac{0.5583(\cos\omega - j\sin\omega)}{1-0.5379(\cos\omega - j\sin\omega)+0.1748(\cos 2\omega - j\sin 2\omega)}$$

$$= \frac{0.5583\cos\omega - j0.5583\sin\omega}{(1-0.5379\cos\omega + 0.1748\cos 2\omega)+j(0.5379\sin\omega - 0.1748\sin 2\omega)}$$

$$\text{Let, } H\left(e^{j\omega}\right) = \frac{H_N\left(e^{j\omega}\right)}{H_D\left(e^{j\omega}\right)} = \frac{0.5583\cos\omega - j0.5583\sin\omega}{(1-0.5379\cos\omega + 0.1748\cos 2\omega)+j(0.5379\sin\omega - 0.1748\sin 2\omega)}$$

where, $H_N(e^{j\omega}) = 0.5583\cos\omega - j\,0.5583\sin\omega$

$H_D(e^{j\omega}) = (1 - 0.5379\cos\omega + 0.1748\cos 2\omega) + j(0.5379\sin\omega - 0.1748\sin 2\omega)$

The frequency response $H(e^{j\omega})$ and hence the magnitude response $|H(e^{j\omega})|$ are calculated for various values of ω and listed in table 1. Using the values listed in table 1, the magnitude response of lowpass filter is sketched as shown in fig 3.

TABLE 1: $H(e^{j\omega})$ and $|H(e^{j\omega})|$ for various values of ω.

ω	$H_N(e^{j\omega})$	$H_D(e^{j\omega})$	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
$\frac{0\times\pi}{16}$	0.5583 + j0	0.6369 + j0	0.8766 + j0	0.8766
$\frac{1\times\pi}{16}$	0.5476 – j0.1089	0.6339 + j0.0380	0.8505 – j0.2228	0.8792
$\frac{2\times\pi}{16}$	0.5158 – j0.2137	0.6266 + j0.0822	0.7653 – j0.4414	0.8835
$\frac{3\times\pi}{16}$	0.4642 – j0.3102	0.6196 + j0.1373	0.6084 – j0.6355	0.8797
$\frac{4\times\pi}{16}$	0.3948 – j0.3948	0.6196 + j0.2056	0.3835 – j0.7644	0.8553
$\frac{5\times\pi}{16}$	0.3102 – j0.4642	0.6343 + j0.2858	0.1324 – j0.7915	0.8025
$\frac{6\times\pi}{16}$	0.2137 – j0.5158	0.6705 + j0.3734	–0.0837 – j0.7227	0.7275
$\frac{7\times\pi}{16}$	0.1089 – j0.5476	0.7336 + j0.4606	–0.2297 – j0.6022	0.6446
$\frac{8\times\pi}{16}$	0 – j0.5583	0.8252 + j0.5379	–0.3095 – j0.4748	0.5668
$\frac{9\times\pi}{16}$	–0.1089 – j0.5476	0.9434 + j0.5944	–0.3444 – j0.3634	0.5007
$\frac{10\times\pi}{16}$	–0.2137 – j0.5158	1.0822 + j0.6206	–0.3543 – j0.2735	0.4475
$\frac{11\times\pi}{16}$	–0.3102 – j0.4642	1.2319 + j0.6087	–0.3520 – j0.2029	0.4063
$\frac{12\times\pi}{16}$	–0.3948 – j0.3948	1.3804 + j0.5552	–0.3452 – j0.1472	0.3753
$\frac{13\times\pi}{16}$	–0.4642 – j0.3102	1.5141 + j0.4603	–0.3377 – j0.1022	0.3528
$\frac{14\times\pi}{16}$	–0.5158 – j0.2137	1.6205 + j0.3294	–0.2799 + j0.1888	0.3376
$\frac{15\times\pi}{16}$	–0.5476 – j0.1089	1.6891 + j0.1718	–0.3274 – j0.0312	0.3288
$\frac{16\times\pi}{16}$	–0.5583 + j0	1.7127 + j0	–0.3260 + j0	0.3260

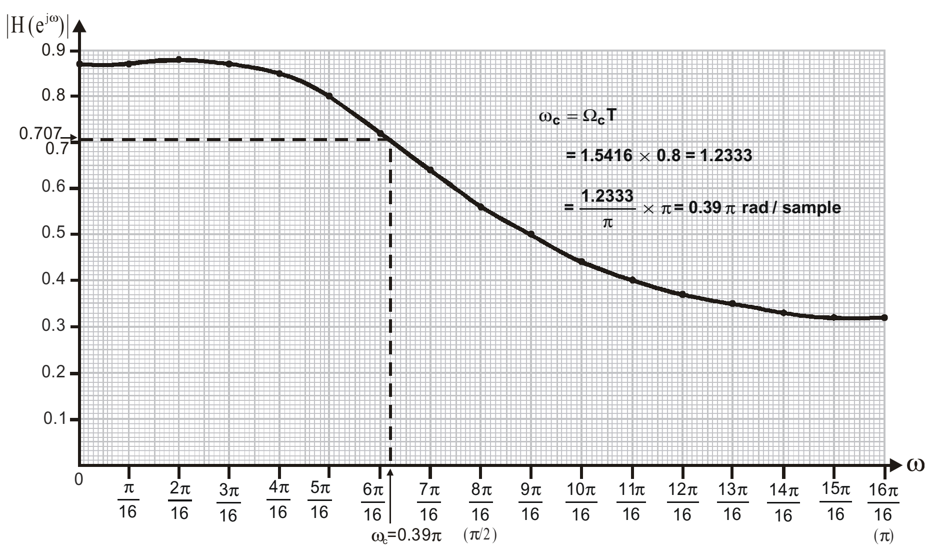


***Fig 3 :** Frequency response of digital Butterworth IIR lowpass filter.*

E7.13. Design a Butterworth digital IIR lowpass filter using impulse invariant transformation by taking T = 1second, to satisfy the following specifications.

$$0.45 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad 0 \le \omega \le 0.5\pi$$

$$|H(e^{j\omega})| \le 0.15 \quad ; \quad 0.8\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

Alternatively,

Passband ripple $\le$ 6.9357 dB

Stopband attenuation $\ge$ 16.4781 dB

Passband edge frequency = 0.5π rad/sample

Stopband edge frequency = 0.8π rad/sample

The above specifications can be converted to A_p and A_s as shown below.

$$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{\left(-6.9357/20\right)} = 0.45$$

$$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{\left(-16.4781/20\right)} = 0.15$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.5\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.7\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.45$

Gain in normal value at stopband edge, $A_s = 0.15$

Sampling time, T = 1second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.45$

Gain in normal value at stopband edge, $A_s = 0.15$

Gain is same in analog and digital filter.

For impulse invariant transformation,

$$\text{Passband edge analog frequency, } \Omega_p = \frac{\omega_p}{T} = \frac{0.5\pi}{1} = 1.5708 \text{ rad / second}$$

Using equation (7.55).

$$\text{Stopband edge analog frequency, } \Omega_s = \frac{\omega_s}{T} = \frac{0.8\pi}{1} = 2.5133 \text{ rad / second}$$

Using equation (7.56).

Order of the filter

$$N = \frac{1}{2}\frac{\log\left[\frac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]}{\log\frac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\frac{\left(1/0.15^2\right)-1}{\left(1/0.45^2\right)-1}\right]}{\log\frac{2.5133}{1.5708}}$$

Using equation (7.57).

$$= \frac{1}{2}\frac{\log\left[\frac{43.4444}{3.9383}\right]}{\log\frac{2.5133}{1.5708}} = \frac{1}{2}\left[\frac{1.0426}{0.2041}\right] = 2.5539$$

Choose order N, such that $N \ge N_1$ and N is an integer.

Let, order, N = 3.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For odd N,

$$H(s_n) = \frac{1}{s_n+1}\prod_{k=1}^{\frac{N-1}{2}}\frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.59).

$$\text{where, } b_k = 2\sin\left[\tfrac{(2k-1)\pi}{2N}\right]$$

Using equation (7.60).

$$\text{Here, } N = 3, \ \therefore\ k = \tfrac{N-1}{2} = \tfrac{3-1}{2} = 1$$

$$\therefore\ H(s_n) = \frac{1}{s_n+1}\ \frac{1}{s_n^2 + b_1 s_n + 1}$$

$$\text{When } k = 1\ ; \quad b_k = b_1 = 2\sin\left[\tfrac{(2-1)\pi}{2\times 3}\right] = 1$$

Calculate $\sin\theta$ using calculator in radian mode.

$$\therefore\ H(s_n) = \frac{1}{(s_n+1)\ (s_n^2+s_n+1)} = \frac{1}{s_n^3 + 2s_n^2 + 2s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2) - 1\right]^{\frac{1}{2N}}} = \frac{2.5133}{\left[(1/0.15^2) - 1\right]^{\frac{1}{6}}} = 1.3405 \text{ rad / second}$$

Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{(s_n+1)(s_n^2+s_n+1)}\Big|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{1}{\left(\frac{s}{\Omega_c}+1\right)\left(\frac{s^2}{\Omega_c^2}+\frac{s}{\Omega_c}+1\right)} = \frac{1}{\left(\frac{s+\Omega_c}{\Omega_c}\right)\left(\frac{s^2+\Omega_c s+\Omega_c^2}{\Omega_c^2}\right)}$$

$$= \frac{\Omega_c^3}{(s+\Omega_c)(s^2+\Omega_c s+\Omega_c^2)} = \frac{1.3405^3}{(s+1.3405)(s^2+1.3405s+1.3405^2)}$$

$$= \frac{2.4088}{(s+1.3405)(s^2+1.3405s+1.7969)} = \frac{2.4088}{s^3+2.681s^2+3.5939s+2.4088}$$

To convert the analog transfer function to digital transfer function using impulse invariant transformation, the above equation can be simplified as follows.

$$H(s) = \frac{2.4088}{(s+1.3405)\ (s^2+1.3405s+1.7969)}$$

$$\frac{2.4088}{(s+1.3405)(s^2+1.3405s+1.7969)} = \frac{A}{(s+1.3405)} + \frac{Bs+C}{(s^2+1.3405s+1.7969)} \quad .....(1)$$

On cross multiplying equation (1) we get,

$$2.4088 = As^2 + 1.3405As + 1.7969A + Bs^2 + 1.3405Bs + Cs + 1.3405C \quad .....(2)$$

On equating coefficients of s^2 in equation (2) we get,	On equating coefficients of s in equation (2) we get,	On equating constants of equation (2) we get,
$A + B = 0$	$1.3405A + 1.3405B + C = 0$	$1.7969A + 1.3405C = 2.4088$
$B = -A$	$B = -A$	$1.7969A + 0 = 2.4088$
	$1.3405 - 1.3405A + C = 0$	$A = \frac{2.4088}{1.7969} = 1.3405$
	$C = 0$	

$$\therefore\ H(s) = \frac{1.3405}{s+1.3405} + \frac{-1.3405s}{s^2+1.3405s+1.7969}$$

$$= \frac{1.3405}{s+1.3405} - \frac{1.3405s}{s^2+2s\times 0.6703+0.6703^2-0.6703^2+1.7969}$$

$$= \frac{1.3405}{s+1.3405} - \frac{1.3405s}{(s+0.6703)^2+1.3476}$$

$$= \frac{1.3405}{s+1.3405} - 1.3405\left[\frac{s+0.6703-0.6703}{(s+0.6703)^2+1.1609^2}\right]$$

$$= \frac{1.3405}{s+1.3405} - 1.3405\left[\frac{s+0.6703}{(s+0.6703)^2+1.1609^2}\right] + 1.3405\left[\frac{0.6703}{(s+0.6307)^2+1.1609^2}\right]$$

$$= \frac{1.3405}{s+1.3405} - 1.3405\left[\frac{s+0.6703}{(s+0.6703)^2+1.1609^2}\right] + \frac{0.8985}{1.1609}\left[\frac{1.1609}{(s+0.6307)^2+1.1609^2}\right]$$

Digital IIR lowpass filter transfer function, H(z)

For impulse invariant transformation,

$$\frac{A_i}{s+p_i} \xrightarrow{\text{is transformed to}} \frac{A_i}{1-e^{-p_i T}z^{-1}}$$

$$\frac{(s+a)}{(s+a)^2+b^2} \xrightarrow{\text{is transformed to}} \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \xrightarrow{\text{is transformed to}} \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

Using equation (7.17), (7.18) and (7.19).

$$H(z) = \frac{1.3405}{1-e^{-1.3405}z^{-1}} - 1.3405\left[\frac{1-e^{-0.6703}\cos(1.1609)z^{-1}}{1-2e^{-0.6703}\cos(1.1609)z^{-1}+e^{-2\times0.6703}z^{-2}}\right]$$

$$+0.7740\left[\frac{e^{-0.6703}\sin(1.1609)z^{-1}}{1-2e^{-0.6703}\cos(1.1609)z^{-1}+e^{-2\times0.6703}z^{-2}}\right]$$

$\boxed{\text{Put, T = 1}}$

$$= \frac{1.3405}{1-0.2617z^{-1}} - 1.3405\left[\frac{1-0.5116\times\cos(1.1609)z^{-1}}{1-1.0232\times\cos(1.1609)z^{-1}+0.2617z^{-2}}\right]$$

$$+0.7740\left[\frac{0.5116\times\sin(1.1609)z^{-1}}{1-1.0232\times\cos(1.1609)z^{-1}+0.2617z^{-2}}\right]$$

$$= \frac{1.3405}{1-0.2617z^{-1}} + \frac{-1.3405+0.2733z^{-1}+0.3631z^{-1}}{1-0.4078z^{-1}+0.2617z^{-2}}$$

$$= \frac{1.3405}{1-0.2617z^{-1}} + \frac{-1.3405+0.6364z^{-1}}{1-0.4078z^{-1}+0.2617z^{-2}}$$

$$= \frac{1.3405\,(1-0.4078z^{-1}+0.2617z^{-2})+(-1.3405+0.6364z^{-1})\,(1-0.2617z^{-1})}{(1-0.2617z^{-1})\,(1-0.4078z^{-1}+0.2617z^{-2})}$$

$$= \frac{1.3405\ -0.5467z^{-1}+0.3508z^{-2}-1.3405+0.6364z^{-1}+0.3508z^{-1}-0.1665z^{-2}}{1-0.4078z^{-1}+0.2617z^{-2}-0.2617z^{-1}+0.1067z^{-2}-0.06851z^{-3}}$$

$$= \frac{0.4405z^{-1}+0.1843z^{-2}}{1-0.6695z^{-1}+0.3684z^{-2}-0.0685z^{-3}}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{0.4405z^{-1}+0.1843z^{-2}}{1-0.6695z^{-1}+0.3684z^{-2}-0.0685z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.6695z^{-1}Y(z) + 0.3684z^{-2}Y(z) - 0.0685z^{-3}Y(z) = 0.4405z^{-1}X(z) + 0.1843z^{-2}X(z)$$

$$\backslash\ \ Y(z) = 0.4405z^{-1}X(z) + 0.1843z^{-2}X(z) + 0.6695z^{-1}Y(z) - 0.3684z^{-1}Y(z) + 0.0685z^{-3}Y(z) \quad .....(3)$$

Using equation (3), the direct form-I structure is drawn as shown in fig 1.

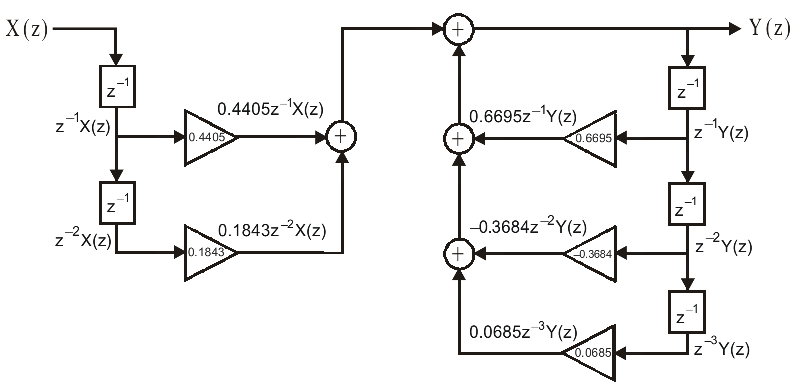


Fig 1 : Direct form-I structure of 3rd order digital IIR lowpass filter.

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.4405z^{-1}+0.1843z^{-2}}{1-0.6695z^{-1}+0.3684z^{-2}-0.0685z^{-3}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1-0.6695z^{-1}+0.3684z^{-2}-0.0685z^{-3}} \quad .....(4)$$

$$\frac{Y(z)}{W(z)} = 0.4405z^{-1}+0.1843z^{-2} \quad .....(5)$$

On cross multiplying equation (4) we get,

$$W(z) - 0.6695z^{-1}W(z) + 0.3684z^{-2}W(z) - 0.0685\,z^{-3}W(z) = X(z)$$

$$\backslash\ \ W(z) = X(z) + 0.6695z^{-1}W(z) - 0.3684z^{-2}W(z) + 0.0685z^{-3}W(z) \quad .....(6)$$

On cross multiplying equation (5) we get,

$$Y(z) = 0.4405z^{-1}W(z) + 0.1843z^{-2}W(z) \quad .....(7)$$

Using equation (6) and (7), the direct form-II structure is drawn as shown in fig 2.

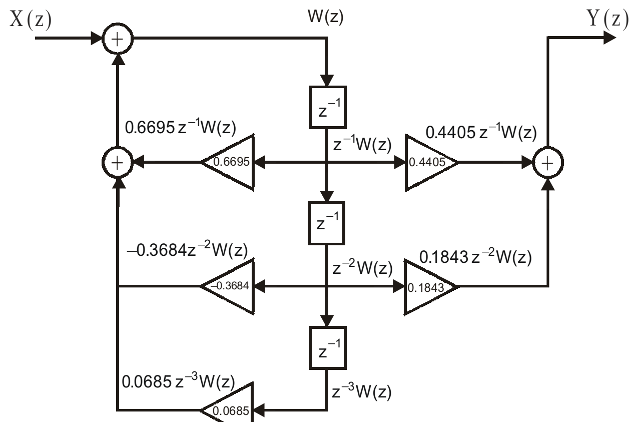


Fig 2 : *Direct form-II structure of 3rd order digital IIR lowpass filter.*

Frequency Response, $H(e^{jw})$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \left.\frac{0.4405z^{-1} + 0.1843z^{-2}}{1 - 0.6695z^{-1} + 0.3684z^{-2} - 0.0685z^{-3}}\right|_{z=e^{j\omega}}$$

$$= \frac{0.4405e^{-j\omega} + 0.1843e^{-j2\omega}}{1 - 0.6695e^{-j\omega} + 0.3684e^{-j2\omega} - 0.0685e^{-j3\omega}}$$

$$= \frac{0.4405(\cos\omega - j\sin\omega) + 0.1843(\cos 2\omega - j\sin 2\omega)}{1 - 0.6695(\cos\omega - j\sin\omega) + 0.3684(\cos 2\omega - j\sin 2\omega) - 0.0685(\cos 3\omega - j\sin 3\omega)}$$

$$= \frac{(0.4405\cos\omega + 0.1843\cos 2\omega) + j(-0.4405\sin\omega - 0.1843\sin 2\omega)}{(1 - 0.6695\cos\omega + 0.3684\cos 2\omega - 0.0685\cos 3\omega) + j(0.6695\sin\omega - 0.3684\sin 2\omega + 0.0685\sin 3\omega)}$$

$$\text{Let, } H(e^{j\omega}) = \frac{H_N(e^{j\omega})}{H_D(e^{j\omega})} = \frac{(0.4405\cos\omega + 0.1843\cos 2\omega) + j(-0.4405\sin\omega - 0.1843\sin 2\omega)}{(1 - 0.6695\cos\omega + 0.3684\cos 2\omega - 0.0685\cos 3\omega) + j(0.6695\sin\omega - 0.3684\sin 2\omega + 0.0685\sin 3\omega)}$$

where, $H_N(e^{jw}) = (0.4405\cos w + 0.1843\cos 2w) + j(-0.4405\sin w - 0.1843\sin 2w)$

$H_D(e^{jw}) = (1 - 0.6695\cos w + 0.3684\cos 2w - 0.0685\cos 3w) + j(0.6695\sin w - 0.3684\sin 2w + 0.0685\sin 3w)$

The frequency response $H(e^{jw})$ and hence the magnitude response $|H(e^{jw})|$ are calculated for various values of w and listed in table 1. Using the values listed in table 1, the magnitude response of lowpass filter is sketched as shown in fig 3.

TABLE 1: $H(e^{jw})$ and $|H(e^{jw})|$ for various values of w.

w	$H_N(e^{jw})$	$H_D(e^{jw})$	$H(e^{jw})$	$\lvert H(e^{jw})\rvert$
$\frac{0\times\pi}{16}$	0.6248 + j0	0.6304 + j0	0.9911 + j0	0.9911
$\frac{1\times\pi}{16}$	0.6128 – j0.1219	0.6268 + j0.0276	0.9672 – j0.2371	0.9959
$\frac{2\times\pi}{16}$	0.5772 – j0.2391	0.6157 + j0.0589	0.8922 – j0.4737	1.0101
$\frac{3\times\pi}{16}$	0.5195 – j0.3471	0.5978 + j0.0987	0.7526 – j0.7049	1.0312
$\frac{4\times\pi}{16}$	0.4418 – j0.4418	0.5750 + j0.1534	0.5259 – j0.9087	1.0499
$\frac{5\times\pi}{16}$	0.3471 – j0.5195	0.5542 + j0.2296	0.2031 – j1.0215	1.0415
$\frac{6\times\pi}{16}$	0.2391 – j0.5772	0.5466 + j0.3318	–0.1488 – j0.9657	0.9771
$\frac{7\times\pi}{16}$	0.1219 – j0.6128	0.5671 + j0.4587	–0.3984 – j0.7583	0.8566
$\frac{8\times\pi}{16}$	0 – j0.6248	0.6316 + j0.6101	–0.4943 – j0.5117	0.7115
$\frac{9\times\pi}{16}$	–0.1219 – j0.6128	0.7522 + j0.7407	–0.4896 – j0.3326	0.5919
$\frac{10\times\pi}{16}$	–0.2391 – j0.5772	0.9324 + j0.8528	–0.4479 – j0.2094	0.4944
$\frac{11\times\pi}{16}$	–0.3471 – j0.5195	1.1637 + j0.9104	–0.4161 – j0.1209	0.4333
$\frac{12\times\pi}{16}$	–0.4418 – j0.4418	1.4249 + j0.8902	–0.3623 – j0.0837	0.3719
$\frac{13\times\pi}{16}$	–0.5195 – j0.3471	1.6842 + j0.7794	–0.3326 – j0.0522	0.3367
$\frac{14\times\pi}{16}$	–0.5572 – j0.2391	1.9052 + j0.5799	–0.3026 – j0.0334	0.3045
$\frac{15\times\pi}{16}$	–0.6128 – j0.1219	2.0539 + j0.3096	–0.3005 – j0.0141	0.3008
$\frac{16\times\pi}{16}$	–0.6248 + j0	2.1064 + j0	–0.2966 + j0	0.2966

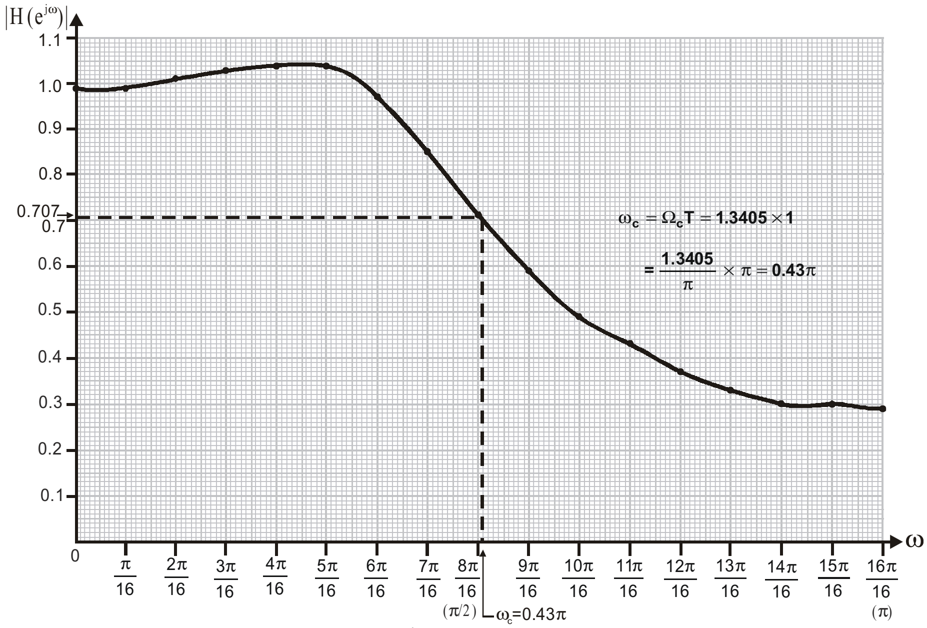


Fig 3 : Frequency response of 3rd order digital Butterworth IIR lowpass filter.

E7.14. Design a Butterworth digital IIR lowpass filter using impulse invariant transformation by taking T = 1second, to satisfy the following specifications.

$$0.9 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.25\pi$$

$$|H(e^{j\omega})| \le 0.35 \quad ; \quad \text{for } 0.3981\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter. Verify the design by sketching the frequency response.

> ***Alternatively,***
>
> ***Passband ripple $\le$ 0.9151 dB***
>
> ***Stopband attenuation $\ge$ 9.1186 dB***
>
> ***Passband edge frequency = 0.25π rad/sample***
>
> ***Stopband edge frequency = 0.3981π rad/sample***
>
> ***The above specifications can be converted to A_p and A_s as shown below.***
>
> $$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{\left(-0.9151/20\right)} = 0.9$$
>
> $$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{\left(-13.97/20\right)} = 0.35$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, ω_p = 0.25π rad/sample

Stopband edge digital frequency, ω_s = 0.3981π rad/sample

Gain in normal value at passband edge, A_p = 0.9

Gain in normal value at stopband edge, A_s = 0.35

sampling time, T = 1second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, A_p = 0.9

Gain in normal value at stopband edge, A_s = 0.35

Gain is same in analog and digital filter.

For impulse invariant transformation,

Passband edge analog frequency, $\Omega_p = \dfrac{\omega_p}{T}$

Using equation (7.55).

$$= \frac{0.25\pi}{1} = 0.7854 \text{ rad / second}$$

Stopband edge analog frequency, $\Omega_s = \dfrac{\omega_s}{T}$

Using equation (7.56).

$$= \frac{0.3981\pi}{1} = 1.2507 \text{ rad / second}$$

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\frac{(1/A_s^2)-1}{(1/A_p^2)-1}\right]}{\log\frac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\frac{(1/0.35^2)-1}{(1/0.9^2)-1}\right]}{\log\frac{1.2507}{0.7854}} = \frac{1}{2}\frac{\log\left[\frac{7.1633}{0.2346}\right]}{\log\frac{1.2507}{0.7854}} = \frac{1}{2}\frac{1.4848}{0.2021} = 3.6734$$

Using equation (7.57).

Choose order N, such that N ³ N_1 and N is an integer.

Let, order, N = 4.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For even N,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.58).

$$\text{where, } b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

Using equation (7.60).

Here, $N = 4$, $\therefore\ k = \frac{N}{2} = \frac{4}{2} = 2$

$$\therefore\ H(s_n) = \frac{1}{s_n^2 + b_1 s_n + 1} \times \frac{1}{s_n^2 + b_2 s_n + 1}$$

Calculate sinq using calculator in radian mode.

When k = 1 ; $b_k = b_1 = 2\sin\left[\frac{(2\times1-1)\pi}{2\times4}\right] = 0.7654$

When k = 2 ; $b_k = b_2 = 2\sin\left[\frac{(2\times2-1)\pi}{2\times4}\right] = 1.8478$

$$H(s_n) = \frac{1}{(s_n^2 + 0.7654\,s_n + 1)\ (s_n^2 + 1.8478\,s_n + 1)}$$

$$= \frac{1}{s_n^4 + 1.8478\,s_n^3 + s_n^2 + 0.7654\,s_n^3 + 1.4143\,s_n^2 + 0.7654\,s_n + s_n^2 + 1.8478 + 1}$$

$$= \frac{1}{s_n^4 + 2.6132\,s_n^3 + 3.4143 s_n^2 + 2.6132\,s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[(1/A_s^2) - 1\right]^{\frac{1}{2N}}} = \frac{1.2507}{\left[(1/0.35^2) - 1\right]^{\frac{1}{8}}} = 0.9778 \text{ rad / second}$$

Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{(s_n^2 + 0.7654\,s_n + 1)(s_n^2 + 1.8478\,s_n + 1)}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{1}{\left(\frac{s^2}{\Omega_c^2} + 0.7654\frac{s}{\Omega_c} + 1\right)\left(\frac{s^2}{\Omega_c^2} + 1.8478\frac{s}{\Omega_c} + 1\right)}$$

$$= \frac{1}{\left(\frac{s^2 + 0.7654\,\Omega_c\,s + \Omega_c^2}{\Omega_c^2}\right)\left(\frac{s^2 + 1.8478\,\Omega_c\,s + \Omega_c^2}{\Omega_c^2}\right)}$$

$$= \frac{\Omega_c^4}{(s^2 + 0.7654\,\Omega_c s + \Omega_c^2)(s^2 + 1.8478\,\Omega_c\,s + \Omega_c^2)}$$

$$= \frac{(0.9778)^4}{(s^2 + 0.7654\times0.9778\,s + 0.9778^2)(s^2 + 1.8478\times0.9778\,s + 0.9778^2)}$$

$$= \frac{0.9141}{(s^2 + 0.7484s + 0.9561)(s^2 + 1.8068s + 0.9561)}$$

$$= \frac{0.9141}{s^4 + 2.5552s^3 + 3.2644\,s^2 + 2.443\,s + 0.9141}$$

To convert the analog transfer function to digital transfer function using impulse invariant transformation, the above equation is simplified as follows.

$$H(s)=\frac{0.9141}{s^4+2.5552s^3+3.2644s^2+2.443s+0.9141}$$

$$=\frac{0.9141}{\left(s^2+0.7484s+0.9561\right)\left(s^2+1.8068s+0.9561\right)}$$

$$=\frac{0.9141}{(s+0.3742-j0.9034)(s+0.3742+j0.9034)(s+0.9034-j0.3741)(s+0.9034+j0.3741)}$$

The roots of the quadratic $s^2+0.7484s+0.9561=0$ are

$$s=\frac{-0.7484\ \pm\ \sqrt{0.7484^2-4\times0.9561}}{2}$$

$$=\frac{-0.7484\pm j1.8067}{2}=-0.3742\pm j0.9034$$

$$=(s-(-0.3742+j0.9034))\ (s-(-0.3742-j0.9034))$$

$$=(s+0.3742-j0.9034)\ (s+0.3742+j0.9034)$$

By partial fraction expansion H(s) can be expressed as

$$H(s)=\frac{A_1}{s+0.3742-j0.9034}+\frac{A_1^*}{s+0.3742+j0.9034}+\frac{A_2}{s+0.9034-0.3741}+\frac{A_2^*}{s+0.9034+0.3741}$$

where, A_1, A_1^*, A_2, A_2^* are residues

The roots of the quadratic $s^2+1.8068s+0.9561=0$ are

$$s=\frac{-1.8068\ \pm\ \sqrt{1.8068^2-4\times0.9561}}{2}$$

$$=\frac{-1.8068\pm j0.7482}{2}$$

$$=-0.9034\pm j0.3741$$

$$=(s-(-0.9034+j0.3741))\ (s-(-0.9034-j0.3741))$$

$$=(s+0.9034-j0.3741)\ (s+0.9034+j0.3741)$$

$$A_1=\left.\frac{0.9141\times \cancel{s+0.3742-j0.9034}}{\cancel{(s+0.3742-j0.9034)}(s+0.3742+j0.9034)\left(s^2+1.8068s+0.9561\right)}\right|_{s=-0.3742+j0.9034}$$

$$=\frac{0.9141}{(-0.3742+j0.9034+0.3742+j0.9034)\ [(-0.3742+j0.9034)^2+1.8068(-0.3742+j0.9034)+0.9561]}$$

$$=\frac{0.9141}{(j1.8068)(-0.3961+j0.9562)}=-0.4516+j0.1871$$

$$A_1^*=\text{Conjugate of }A_1=-0.4516-j0.1871$$

$$A_2=\left.\frac{0.9141\times \cancel{s+0.9034-j0.3741}}{\left(s^2+0.7484s+0.9561\right)\cancel{(s+0.9034-j0.3741)}(s+0.9034s+j0.3741)}\right|_{s=-0.9034+j0.3741}$$

$$=\frac{0.9141}{\left[(-0.9034+j0.3741)^2+0.7484(-0.9034+j0.3741)+0.9561\right](-0.9034+j0.3741+0.9034+j0.3741)}$$

$$=\frac{0.9141}{(0.9562-j0.3959)(j0.7482)}=0.4517-j1.0907$$

$$A_2^*=\text{Conjugate of }A_2=0.4517+j1.0907$$

$$H(s)=\frac{-0.4516+j0.1871}{s+0.3742-j0.9034}+\frac{-0.4516-j0.1871}{s+0.3742+j0.9034}+\frac{0.4517-j1.0907}{s+0.9034-j0.3741}+\frac{0.4517+j1.0907}{s+0.9034+j0.3741}$$

Digital IIR lowpass filter transfer function, H(z)

For impulse invariant transformation,

$$\frac{A_i}{s+p_i}\xrightarrow{\text{is transformed to}}\frac{A_i}{1-e^{-p_iT}\ z^{-1}}$$

Using equation (7.17).

Using the above transformation, the H(s) can be transformed to H(z) as shown below.

Put, T = 1sec

$$H(z)=\frac{-0.4516+j0.1871}{1-e^{-(0.3742-j0.9034)}z^{-1}}+\frac{-0.4516-j0.1871}{1-e^{-(0.3742+j0.9034)}z^{-1}}+\frac{0.4517-j1.0907}{1-e^{-(0.9034-j0.3741)}z^{-1}}+\frac{0.4517+j1.0907}{1-e^{-(0.9034+j0.3741)}z^{-1}}$$

$$=\frac{(-0.4516+j0.1871)\left(1-e^{-(0.3742-j0.9034)}z^{-1}\right)+(-0.4516-j0.1871)\left(1-e^{-(0.3742-j0.9034)}z^{-1}\right)}{\left(1-e^{-(0.3742-j0.9034)}z^{-1}\right)\left(1-e^{-(0.3742+j0.9034)}z^{-1}\right)}$$

$$+\frac{(0.4517-j1.0907)\left(1-e^{-(0.9034+j0.3741)}z^{-1}\right)+(0.4517+j1.0907)\left(1-e^{-(0.9034-j0.3741)}z^{-1}\right)}{\left(1-e^{-(0.9034-j0.3741)}z^{-1}\right)\left(1-e^{-(0.9034+j0.3741)}z^{-1}\right)}$$

$$=\frac{\begin{array}{c}-0.4516+0.4516e^{-0.3742}e^{-j0.9034}z^{-1}+j0.1871-j0.1871e^{-0.3742}e^{-j0.9034}z^{-1}\\-0.4516+0.4516e^{-0.3742}e^{j0.9034}z^{-1}-j0.1871+j0.1871e^{-0.3742}e^{j0.9034}z^{-1}\end{array}}{1-e^{-0.3742}e^{j0.9034}z^{-1}-e^{-0.3742}e^{-j0.9034}z^{-1}+e^{-0.3742\times2}z^{-2}}$$

$$+\frac{\begin{array}{c}0.4517-0.4517e^{-0.9034}e^{-j0.3741}z^{-1}-j1.0907+j1.0907e^{-0.9034}e^{-j0.3741}z^{-1}\\0.4517-0.4517e^{-0.9034}e^{j0.3741}z^{-1}+j0.0907-j1.0907e^{-0.9034}e^{-j0.3741}z^{-1})\end{array}}{1-e^{-0.9034}e^{-j0.3741}z^{-1}-e^{-0.9034}e^{j0.3741}z^{-1}+e^{-2\times0.9034}z^{-1}}$$

$$\therefore\ H(z)=\frac{-0.9032+0.4516\,e^{-0.3742}\left(e^{j0.9034}+e^{-j0.9034}\right)z^{-1}+j0.1871e^{-0.3742}\left(e^{j0.9034}-e^{-j0.9034}\right)z^{-1}}{1-e^{-0.3742}\left(e^{j0.9034}+e^{-j0.9034}\right)z^{-1}+e^{-0.7484}\,z^{-1}}$$

$$+\frac{0.9034-0.4517\,e^{-0.9034}\left(e^{j0.3741}+e^{-j0.3741}\right)z^{-1}-j1.0907\,e^{-0.9034}\left(e^{j0.3741}-e^{-j0.3741}\right)z^{-1}}{1-e^{-0.9034}\left(e^{j0.3741}+e^{-j0.3741}\right)z^{-1}+e^{-1.8068}\,z^{-1}}$$

$$=\frac{-0.9032+0.4516\,e^{-0.3742}\,(2\cos 0.9034)\,z^{-1}+j0.1871e^{-0.3742}\,(2j\sin 0.9034)\,z^{-1}}{1-e^{-0.3742}\,(2\cos 0.9034)\,z^{-1}+e^{-0.7484}\,z^{-1}}$$

$$+\frac{0.9034-0.4517\,e^{-0.9034}\,(2\cos 0.3741)\,z^{-1}-j1.0907\,e^{-0.9034}\,(2j\sin 0.3741)\,z^{-1}}{1-e^{-0.9034}\,(2\cos 0.3741)\,z^{-1}+e^{-1.8068}\,z^{-1}}$$

$$=\frac{-0.9032+0.3845z^{-1}-0.2022z^{-1}}{1-0.8515z^{-1}+0.4731z^{-2}}+\frac{0.9034-0.3407z^{-1}+0.323z^{-1}}{1-0.7543z^{-1}+0.1642z^{-2}}$$

$$=\frac{-0.9032+0.1823z^{-1}}{1-0.8515z^{-1}+0.4731z^{-2}}+\frac{0.9034-0.0177z^{-1}}{1-0.7543z^{-1}+0.1642z^{-2}}$$

$$=\frac{\left(-0.9032+0.1823z^{-1}\right)\left(1-0.7543z^{-1}+0.1642z^{-2}\right)+\left(0.9034-0.0177z^{-1}\right)\left(1-0.8515z^{-1}+0.4731z^{-2}\right)}{\left(1-0.8515z^{-1}+0.4731z^{-2}\right)\left(1-0.7543z^{-1}+0.1642z^{-2}\right)}$$

$$=\frac{\begin{array}{r}-0.9032+0.6813z^{-1}-0.1483z^{-2}+0.1823z^{-1}-0.1375z^{-2}+0.0299z^{-3}+0.9034-0.7692z^{-1}+0.4274z^{-2}\\ -0.0177z^{-1}+0.0151z^{-2}-0.0084z^{-3}\end{array}}{1-0.7543z^{-1}+0.1642z^{-2}-0.8515z^{-1}+0.6423z^{-2}-0.1398z^{-3}+0.4731z^{-2}-0.3569z^{-3}+0.0777z^{-4}}$$

$$=\frac{0.0767z^{-1}+0.1567z^{-2}+0.0215z^{-3}}{1-1.6058z^{-1}+1.2796z^{-2}-0.4967z^{-3}+0.0777z^{-4}}$$

Alternatively,

$$H(z)=\frac{0.0767z^{-1}+0.1567z^{-2}+0.0215z^{-3}}{1-1.6058z^{-1}+1.2796z^{-2}-0.4967z^{-3}+0.0777z^{-4}}=\frac{0.0767z^{-1}+0.1567z^{-2}+0.0215z^{-3}}{z^{-4}\left(z^{4}-1.6058z^{3}+1.2796z^{2}-0.4967z+0.0777\right)}$$

$$=\frac{0.0767z^{3}+0.1567z^{2}+0.0215z}{z^{4}-1.6058z^{3}+1.2796z^{2}-0.4967z+0.0777}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=\frac{0.0767z^{-1}+0.1567z^{-2}+0.0215z^{-3}}{1-1.6058z^{-1}+1.2796z^{-2}-0.4967z^{-3}+0.0777z^{-4}}$$

On cross multiplying the above equation we get,

$$Y(z)-1.6058z^{-1}Y(z)+1.2796z^{-2}Y(z)-0.4967z^{-3}Y(z)+0.0777z^{-4}Y(z)=0.0767z^{-1}X(z)+0.1567z^{-2}X(z)+0.0215z^{-3}X(z)$$

$$\therefore\ Y(z)=0.0767z^{-1}X(z)+0.1567z^{-2}X(z)+0.0215z^{-3}X(z)+1.6058z^{-1}Y(z)$$
$$-1.2796z^{-2}Y(z)+0.4967z^{-3}Y(z)-0.0777z^{-4}Y(z) \qquad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

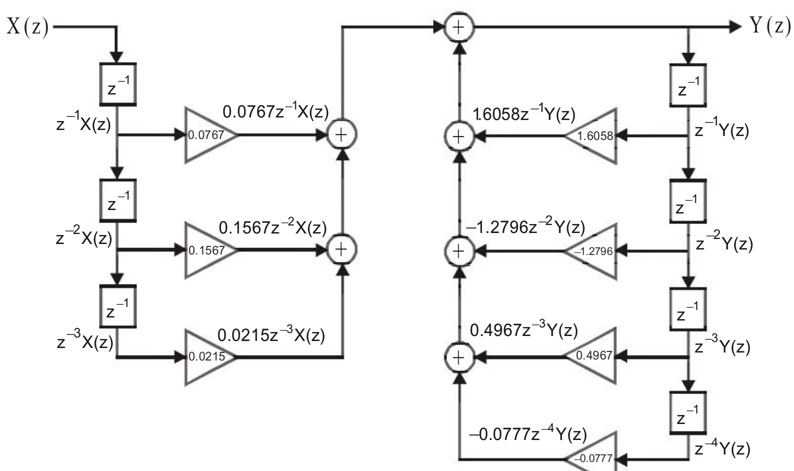


***Fig 1 :** Direct form-I structure of 4th order digital IIR lowpass filter.*

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=\frac{W(z)}{X(z)}\times\frac{Y(z)}{W(z)}=\frac{0.0767z^{-1}+0.1567z^{-2}+0.0215z^{-3}}{1-1.6058z^{-1}+1.2796z^{-2}-0.4967z^{-3}+0.0777z^{-4}}$$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 - 1.6058\,z^{-1} + 1.2796\,z^{-2} - 0.4967\,z^{-3} + 0.0777z^{-4}} \qquad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.0767\,z^{-1} + 0.1567\,z^{-2} + 0.0215\,z^{-3} \qquad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) - 1.6058z^{-1}W(z) + 1.2796z^{-2}W(z) - 0.4967\,z^{-3}W(z) + 0.0777z^{-4}W(z) = X(z)$$

$$\therefore\; W(z) = X(z) + 1.6058z^{-1}W(z) - 1.2796z^{-2}W(z) + 0.4967z^{-3}W(z) - 0.0777z^{-4}W(z) \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$Y(z) = 0.0767z^{-1}W(z) + 0.1567z^{-2}W(z) + 0.0215z^{-3}W(z) \qquad .....(5)$$

Using equation (4) and (5), the direct form-II structure is drawn as shown in fig 2.

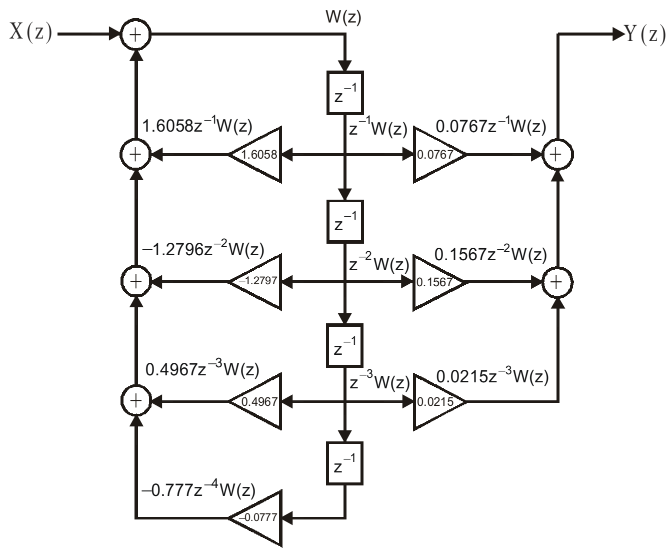


Fig 2 : *Direct form-II structure of 4th order digital IIR lowpass filter.*

E7.15. Design a Butterworth digital IIR lowpass filter using bilinear transformation by taking T = 0.6second, to satisfy the following specifications.

$$0.6 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad 0 \le \omega \le 0.3\pi$$

$$|H(e^{j\omega})| \le 0.02 \quad ; \quad 0.575\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter.

> ***Alternatively,***
>
> ***Passband ripple $\le$ 4.4370 dB***
>
> ***Stopband attenuation $\ge$ 33.9794 dB***
>
> ***Passband edge frequency = 0.3π rad/sample***
>
> ***Stopband edge frequency = 0.575π rad/sample***
>
> ***The above specifications can be converted to A_p and A_s as shown below.***
>
> $$A_p = 10^{(-\delta_{p,dB}/20)} = 10^{(-4.4370/20)} = 0.6$$
>
> $$A_s = 10^{(-\alpha_{s,dB}/20)} = 10^{(-33.9794/20)} = 0.02$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.3\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.575\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.6$

Gain in normal value at stopband edge, $A_s = 0.02$

Sampling time, T = 0.6second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.6$

Gain in normal value at stopband edge, $A_s = 0.02$

Gain is same in analog and digital filter.

For bilinear transformation,

Passband edge analog frequency, $\Omega_p = \dfrac{2}{T}\tan\dfrac{\omega_p}{2} = \dfrac{2}{0.6}\tan\dfrac{0.3\pi}{2} = 1.6984\ \text{rad / second}$ — Using equation (7.53).

Stopband edge analog frequency, $\Omega_s = \dfrac{2}{T}\tan\dfrac{\omega_s}{2} = \dfrac{2}{0.6}\tan\dfrac{0.575\pi}{2} = 4.2283\ \text{rad / second}$ — Using equation (7.54).

Order of the filter

$$N_1 = \frac{1}{2}\frac{\log\left[\dfrac{\left(1/A_s^2\right)-1}{\left(1/A_s^2\right)-1}\right]}{\log\dfrac{\Omega_s}{\Omega_p}} = \frac{1}{2}\frac{\log\left[\dfrac{\left(1/0.02^2\right)-1}{\left(1/0.6^2\right)-1}\right]}{\log\dfrac{4.2283}{1.6984}} = \frac{1}{2}\frac{\log\left[\dfrac{2499}{1.7778}\right]}{\log\dfrac{4.2283}{1.6984}} = \frac{1}{2}\,\frac{3.1479}{0.3961} = 3.9736$$

Using equation (7.57).

Choose order N_1 such that N ³ N_1 and N is an integer.

Let, order, N = 4.

Normalized transfer function, $H(s_n)$ of Butterworth lowpass filter

For even N,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

Using equation (7.58).

where, $b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$ — Using equation (7.60).

Here, $N = 4,\ \therefore\ k = \frac{N}{2} = \frac{4}{2} = 2$

Calculate sinq using calculator in radian mode.

When k = 1, $b_k = b_1 = 2\sin\left[\dfrac{(2\times 1-1)\pi}{2\times 4}\right] = 0.7654$

When k = 2, $b_k = b_2 = 2\sin\left[\dfrac{(2\times 2-1)\pi}{2\times 4}\right] = 1.8478$

$$\therefore\ H(s_n) = \frac{1}{(s_n^2 + 0.7654\,s_n + 1)\,(s_n^2 + 1.8478\,s_n + 1)}$$

$$= \frac{1}{s_n^4 + 1.8478\,s_n^3 + s_n^2 + 0.7654\,s_n^3 + 1.4143\,s_n^2 + 0.7654\,s_n + s_n^2 + 1.8478 s_n + 1}$$

$$= \frac{1}{s_n^4 + 2.6132\,s_n^3 + 3.4143\,s_n^2 + 2.6132\,s_n + 1}$$

Unnormalized transfer function, H(s) of Butterworth lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, W$_c$ = Cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{\left[\left(1/A_s^2\right)-1\right]^{\frac{1}{2N}}} = \frac{4.2283}{\left[\left(1/0.02^2\right)-1\right]^{\frac{1}{8}}} = 1.5902\ \text{rad / second}$$

Using equation (7.61).

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{1}{s_n^4 + 2.6132\,s_n^3 + 3.4143\,s_n^2 + 2.6132\,s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$\therefore\ H(s) = \frac{1}{\dfrac{s^4}{\Omega_c^4} + 2.6132\left(\dfrac{s}{\Omega_c}\right)^3 + 3.4143\left(\dfrac{s}{\Omega_c}\right)^2 + 2.6132\left(\dfrac{s}{\Omega_c}\right) + 1}$$

$$= \frac{1}{\dfrac{s^4}{\Omega_C^4} + 2.6132\dfrac{s^3}{\Omega_c^3} + 3.4143\dfrac{s^2}{\Omega_c^2} + 2.6132\dfrac{s}{\Omega_c} + 1}$$

$$= \frac{\Omega_c^4}{s^4 + 2.6132\,\Omega_c\,s^3 + 3.4143\,\Omega_c^2\,s^2 + 2.6132\,\Omega_c^3\,s + \Omega_c^4}$$

$$= \frac{1.5902^4}{s^4 + 2.6132\times 1.5902 s^3 + 3.4143\times 1.5902^2\,s^2 + 2.6132\times 1.5902^3\,s + 1.5902^4}$$

$$= \frac{6.3945}{s^4 + 4.1555\,s^3 + 8.6339\,s^2 + 10.5082\,s + 6.3945}$$

Digital IIR lowpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{6.3945}{s^4 + 4.1555\,s^3 + 8.6339\,s^2 + 10.5082\,s + 6.3945}\Bigg|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{6.3945}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^4 + 4.1555\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + 8.6339\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 10.5082\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + 6.3945}$$

$$= \frac{6.3945}{\frac{16(1-z^{-1})^4}{T^4(1+z^{-1})^4} + \frac{33.244(1-z^{-1})^3}{T^3(1+z^{-1})^3} + \frac{34.5356(1-z^{-1})^2}{T^2(1+z^{-1})^2} + \frac{21.0164(1-z^{-1})}{T(1+z^{-1})} + 6.3945}$$

$$= \frac{6.3945}{\frac{16(1-z^{-1})^4 + 33.244\,T(1-z^{-1})^3(1+z^{-1}) + 34.5356\,T^2(1-z^{-1})^2(1+z^{-1})^2 + 21.0164\,T^3(1-z^{-1})(1+z^{-1})^3 + 6.3945\,T^4(1+z^{-1})^4}{T^4(1+z^{-1})^4}}$$

$$= \frac{6.3945\,T^4(1+z^{-1})^4}{16(1-z^{-1})^4 + 33.244\,T(1-z^{-1})^3(1+z^{-1}) + 34.5356\,T^2(1-z^{-1})^2(1+z^{-1})^2 + 21.0164\,T^3(1-z^{-1})(1+z^{-1})^3 + 6.3945\,T^4(1+z^{-1})^4}$$

Put, T = 0.6

$$= \frac{6.3845 \times 0.6^4(1+z^{-1})^4}{16(1-z^{-1})^4 + 33.244 \times 0.6(1-z^{-1})^3(1+z^{-1}) + 34.5356 \times 0.6^2(1-z^{-1})^2(1+z^{-1})^2 + 21.0164 \times 0.6^3(1-z^{-1})(1+z^{-1})^3 + 6.3945 \times 0.6^4(1+z^{-1})^4}$$

$$= \frac{0.8287(1+z^{-1})^2(1+z^{-1})^2}{16(1-z^{-1})^2(1-z^{-1})^2 + 19.9464\,(1-3z^{-1}+3z^{-2}-z^{-3})(1+z^{-1}) + 12.4328(1-z^{-1})^2(1+z^{-1})^2 + 4.5395(1-z^{-1})(1+3z^{-1}+3z^{-2}+z^{-3}) + 0.8287(1+z^{-1})^2(1+z^{-1})^2}$$

$$= \frac{0.8287(1+2z^{-1}+z^{-2})(1+2z^{-1}+z^{-2})}{16(1-2z^{-1}+z^{-2})(1-2z^{-1}+z^{-2}) + 19.9464(1-2z^{-1}+2z^{-3}-z^{-4}) + 12.4328(1-2z^{-1}+z^{-2})(1+2z^{-1}+z^{-2}) + 4.5395(1+2z^{-1}-2z^{-3}-z^{-4}) + 0.8287(1+2z^{-1}+z^{-2})(1+2z^{-1}+z^{-2})}$$

$$= \frac{0.8287(1+4z^{-1}+6z^{-2}+4z^{-3}+z^{-4})}{16(1-4z^{-1}+6z^{-2}-4z^{-3}+z^{-4}) + 19.9464(1-2z^{-1}+2z^{-3}-z^{-4}) + 12.4328(1-2z^{-2}+z^{-4}) + 4.5395(1+2z^{-1}-2z^{-3}-z^{-4}) + 0.8287(1+4z^{-1}+6z^{-2}+4z^{-3}+z^{-4})}$$

$$= \frac{0.8287 + 3.3148 + 4.9722z^{-2} + 3.3148z^{-3} + 0.8287z^{-4}}{53.7474 - 91.499z^{-1} + 76.1066z^{-2} - 29.8714z^{-3} + 4.7756z^{-4}}$$

$$= \frac{\frac{0.8287}{53.7474} + \frac{3.3148}{53.7474}z^{-1} + \frac{4.9722}{53.7474}z^{-2} + \frac{3.3148}{53.7474}z^{-3} + \frac{0.8287}{53.7474}z^{-4}}{1 - \frac{91.499}{53.7474}z^{-1} + \frac{76.1066}{53.7474}z^{-2} - \frac{29.8714}{53.7474}z^{-3} + \frac{4.7756}{53.7474}z^{-4}}$$

$$= \frac{0.0154 + 0.0617z^{-1} + 0.0925z^{-2} + 0.0617z^{-3} + 0.0154z^{-4}}{1 - 1.7024z^{-1} + 1.4160z^{-2} - 0.5558z^{-3} + 0.0889z^{-4}}$$

Alternatively,

$$H(z) = \frac{0.0154 + 0.0617z^{-1} + 0.0925z^{-2} + 0.0617z^{-3} + 0.0154z^{-4}}{1 - 1.7024z^{-1} + 1.4160z^{-2} - 0.5558z^{-3} + 0.0889z^{-4}}$$

$$= \frac{z^{-4}(0.0154z^4 + 0.0617z^3 + 0.0925z^2 + 0.0617z + 0.0154)}{z^{-4}(z^4 - 1.7024z^3 + 1.4160z^2 - 0.5558z + 0.0889)}$$

$$= \frac{0.0154z^4 + 0.0617z^3 + 0.0925z^2 + 0.0617z + 0.0154}{z^4 - 1.7024z^3 + 1.4160z^2 - 0.5558z + 0.0889}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{0.0154 + 0.0617z^{-1} + 0.0925z^{-2} + 0.0617z^{-3} + 0.0154z^{-4}}{1 - 1.7024z^{-1} + 1.4160z^{-2} - 0.5558z^{-3} + 0.0889z^{-4}}$$

On cross multiplying the above equation we get,

$$Y(z) - 1.7024z^{-1}Y(z) + 1.4160z^{-2}Y(z) - 0.5558z^{-3}Y(z) + 0.0889\,z^{-4}Y(z)$$
$$= 0.01054X(z) + 0.0617z^{-1}X(z) + 0.0925z^{-2}X(z) + 0.0617z^{-3}X(z) + 0.0154z^{-4}X(z)$$

$$\therefore\ Y(z) = 0.0154X(z) + 0.0617z^{-1}X(z) + 0.0925z^{-2}X(z) + 0.0617z^{-3}X(z) + 0.0154z^{-4}X(z) + 1.7024z^{-1}Y(z) - 1.4160z^{-2}Y(z) + 0.5558\,z^{-3}Y(z) - 0.0889\,z^{-4}Y(z) \quad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

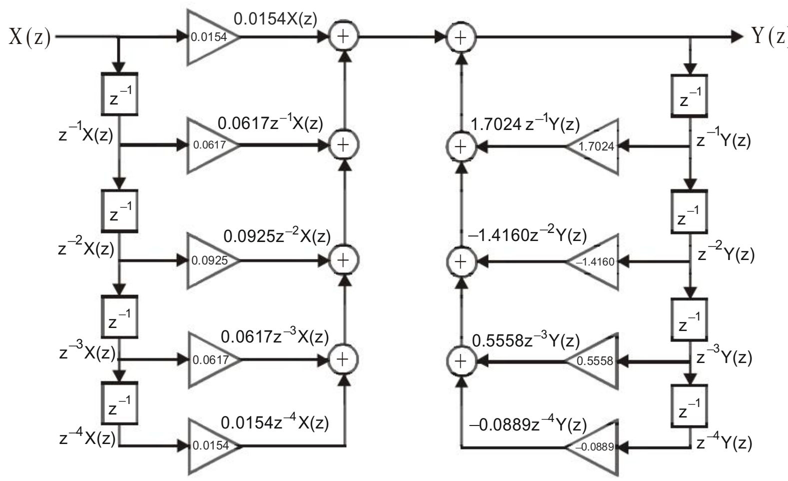


Fig 1 : *Direct form-I structure of 4^{th} order digital IIR lowpass filter.*

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.0154 + 0.0617\,z^{-1} + 0.0925\,z^{-2} + 0.0617\,z^{-3} + 0.0154\,z^{-4}}{1 - 1.7024\,z^{-1} + 1.4160\,z^{-2} - 0.5558\,z^{-3} + 0.0889\,z^{-4}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1 - 1.7024\,z^{-1} + 1.4160\,z^{-2} - 0.5558\,z^{-3} + 0.0889\,z^{-4}} \quad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.0154 + 0.0617\,z^{-1} + 0.0925\,z^{-2} + 0.0617\,z^{-3} + 0.0154\,z^{-4} \quad .....(3)$$

On cross multiplying equation (2) we get,

$$W(z) - 1.7024z^{-1}W(z) + 1.4160z^{-2}W(z) - 0.5558\,z^{-3}W(z) + 0.0889\,z^{-4}W(z) = X(z)$$

$$\therefore\ W(z) = X(z) + 1.7024z^{-1}W(z) - 1.4160z^{-2}W(z) + 0.5558z^{-3}W(z) - 0.0889\,z^{-4}W(z) \quad .....(4)$$

On cross multiplying equation (3) we get,

$$Y(z) = 0.0154W(z) + 0.0617z^{-1}W(z) + 0.0925z^{-2}W(z) + 0.0617z^{-3}W(z) + 0.0154z^{-4}W(z) \quad .....(5)$$

Using equations (4) and (5), the direct form-II structure is drawn as shown in fig 2.

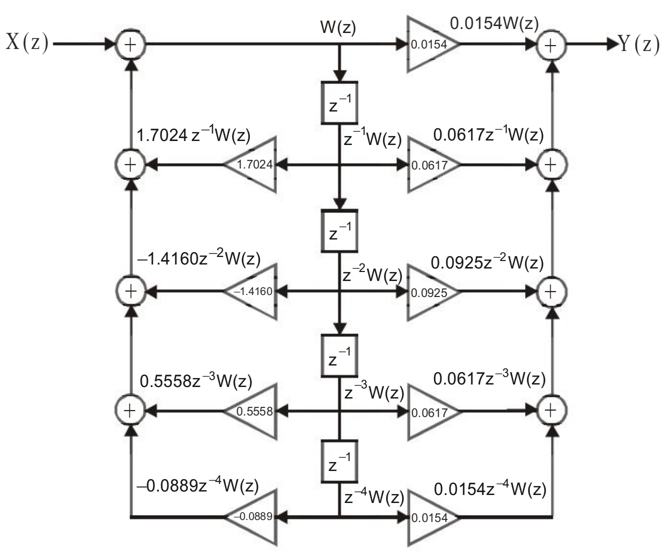


Fig 2 : *Direct form-II structure of 4^{th} order digital IIR lowpass filter.*

E7.16. Design a Chebyshev digital IIR lowpass filter using impulse invariant transformation by taking T = 1second, to satisfy the following specifications.

$$0.87 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.25\pi$$

$$|H(e^{j\omega})| \le 0.35 \quad ; \quad \text{for } 0.375\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter.

Alternatively,

Passband ripple ≤ 1.2096 *dB*

Stopband attenuation ≥ 9.1136 *dB*

Passband edge frequency $= 0.25\pi$ *rad/sample*

Stopband edge frequency $= 0.375\pi$ *rad/sample*

The above specifications can be converted to A_p *and* A_s *as shown below.*

$$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{\left(-1.2096/20\right)} = 0.87$$

$$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{\left(-9.1186/20\right)} = 0.35$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.25\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.375\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.87$

Gain in normal value at stopband edge, $A_s = 0.35$

Sampling time, T = 1second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.87$

Gain in normal value at stopband edge, $A_s = 0.35$

Gain is same in analog and digital filter.

For bilinear transformation,

$$\text{Passband edge analog frequency, } \Omega_p = \frac{\omega_p}{T} = \frac{0.25\pi}{1} = 0.7854 \text{ rad / second}$$

Using equation (7.85).

$$\text{Stopband edge analog frequency, } \Omega_s = \frac{\omega_s}{T} = \frac{0.375\pi}{1} = 1.1781 \text{ rad / second}$$

Using equation (7.86).

Order of the filter

$$N_1 = \frac{\cosh^{-1}\left[\frac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]^{\frac{1}{2}}}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1}\left[\frac{\left(1/0.35^2\right)-1}{\left(1/0.87^2\right)-1}\right]^{\frac{1}{2}}}{\cosh^{-1}\frac{1.1781}{0.7854}} = \frac{\cosh^{-1}\left[\frac{7.1633}{0.3212}\right]^{\frac{1}{2}}}{\frac{1.1781}{0.7854}} = \frac{2.2341}{0.9624} = 2.3214$$

Using equation (7.87).

Choose order N_1 such that $N \ge N_1$ and N is an integer.

Let, order, N = 3.

Normalized transfer function, $H(s_n)$ of Chebyshev lowpass filter

For odd N,

$$H(s_n) = \frac{B_0}{s_n + c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s_n^2 + b_k s_n + c_k}$$

Using equation (7.89).

$$\text{Here, } N = 3, \ \therefore \ k = \tfrac{N-1}{2} = \tfrac{3-1}{2} = 1$$

$$\therefore \ H(s_n) = \frac{B_0}{s_n + c_0} \times \frac{B_1}{s_n^2 + b_1 s_n + c_1}$$

$$\epsilon = \left[\left(1/A_p^2\right) - 1\right]^{\frac{1}{2}}$$

$$= \left[\left(1/0.87^2\right) - 1\right]^{\frac{1}{2}} = 0.5667$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right\}$$

$$= \frac{1}{2}\left[\left[\left(\frac{1}{0.5667^2}+1\right)^{\frac{1}{2}} + \frac{1}{0.5667}\right]^{\frac{1}{3}} - \left[\left(\frac{1}{0.5667^2}+1\right)^{\frac{1}{2}} + \frac{1}{0.5667}\right]^{-\frac{1}{3}}\right]$$

Using equation (7.93).

$$= \frac{1}{2}\left[1.5595 - 0.6412\right] = 0.4591$$

$$c_0 = y_N = 0.4591$$

Using equation (7.94).

$$b_k = 2\,y_N \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

When $k = 1;\ b_k = b_1 = 2 \times 0.4591 \sin\left[\dfrac{(2-1)\pi}{2\times 3}\right] = 0.4591$ Using equation (7.90).

$$c_k = y_N^2 + \cos^2\left[\frac{(2k-1)\pi}{2N}\right]$$ Using equation (7.91).

$$c_k = c_1 = 0.4591^2 + \cos^2\left[\frac{(2-1)\pi}{2\times 3}\right]$$

$$= 0.4591^2 + \left[\frac{1+\cos\left(\frac{2\pi}{6}\right)}{2}\right]$$ $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$= 0.2108 + 0.75 = 0.9608$$

$$\therefore\ H(s_n) = \frac{B_0}{s_n + c_0} \times \frac{B_k}{s_n^2 + b_1 s_n + c_1} = \frac{B_0}{(s_n + 0.4591)} \times \frac{B_1}{s_n^2 + 0.4591\, s_n + 0.9608}$$

To evaluate B_0 and B_1,

Let, $H(s_n)\big|_{s_n = 0} = 1,$

When $s_n = 0$, $H(s_n) = \dfrac{B_0 B_1}{(0.4591)\ (0.9608)} = 2.2670 B_0 B_1$

$$\therefore\ 2.2670\ B_0 B_1 = 1\ \Rightarrow\ B_0 B_1 = \frac{1}{2.2670}\quad \Rightarrow\ B_0 B_1 = 0.4411$$

Let, $B_0 = B_1$; $\therefore\ B_0^2 = 0.4411\ \Rightarrow\ B_0 = \sqrt{0.4411} = 0.6642$

$\therefore\ B_1 = B_0 = 0.6642$

$$H(s_n) = \frac{B_0}{(s_n + 0.4591)} \times \frac{B_1}{(s_n^2 + 0.4591 s_n + 0.9608)} = \frac{0.6642}{(s_n + 0.4591)} \times \frac{0.6642}{(s_n^2 + 0.4591 s_n + 0.9608)}$$

$$= \frac{0.4412}{(s_n + 0.4591)(s_n^2 + 0.4591 s_n + 0.9608)}$$

$$= \frac{0.4412}{s_n^3 + 0.9182 s_n^2 + 1.1716 s_n + 0.4411}$$

Unnormalized transfer function, H(s) of Chebyshev lowpass filter

$$H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$$

where, W_c = Cutoff frequency.

Here, $W_c = W_p$ = 0.7854 rad/second.

$$\therefore\ H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}} = \frac{0.4412}{(s_n + 0.4591)(s_n^2 + 0.4591 s_n + 0.9608)}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{0.4412}{\left(\dfrac{s}{\Omega_c} + 0.4591\right)\left(\dfrac{s^2}{\Omega_c^2} + 0.4591 \dfrac{s}{\Omega_c} + 0.9608\right)} = \frac{0.4412}{\left(\dfrac{s + 0.4591\Omega_c}{\Omega_c}\right)\left(\dfrac{s^2 + 0.4591\Omega_c\, s + 0.9608\Omega_c^2}{\Omega_c^2}\right)}$$

$$= \frac{0.4412\,\Omega_c^3}{(s + 0.4591\Omega_c)(s^2 + 0.4591\Omega_c s + 0.9608\Omega_c^2)}$$

$$= \frac{0.4412 \times 0.7854^3}{(s + 0.4591 \times 0.7854)(s^2 + 0.4591 \times 0.7854 + 0.9608 \times 0.7854^2)}$$

$$= \frac{0.2138}{(s + 0.3606)(s^2 + 0.3606 s + 0.5927)} \qquad .....(1)$$

$$= \frac{0.2138}{s^3 + 0.7212 s^2 + 0.7227 s + 0.2137}$$

To convert the analog transfer function to digital transfer function using impulse invariant transformation, the equation (1) is simplified as follows.

By partial fraction expansion H(s) can be expressed as,

$$H(s) = \frac{0.2138}{(s + 0.3606)(s^2 + 0.3606 s + 0.5927)} = \frac{A}{s + 0.3606} + \frac{Bs + C}{s^2 + 0.3606\, s + 0.5927} \qquad ..... (2)$$

On cross multiplying the equation (2) we get,

$$0.2138 = A(s^2 + 0.3606s + 0.5927) + (Bs + C)\,(s + 0.3607)$$

$$0.2138 = As^2 + 0.3606\,As + 0.5927\,A + Bs^2 + 0.3606\,Bs + Cs + 0.3606\,C \qquad ..... (3)$$

On equating coefficients of s^2 in equation (3) we get,

$$A + B = 0$$

$$B = -A$$

On equating coefficient of s in equation (3) we get,

$$0.3606\,A + 0.3606\,B + C = 0$$

Put, $B = -A$

\ $0.3606A - 0.3606B + C = 0$

\ $C = 0$

On equating constants of equation (3) we get,

$$0.5927\,A + 0.3606\,C = 0.2138$$

Put, $C = 0$

\ $0.5927A = 0.2138$

$$A = \frac{0.2138}{0.5927} = 0.3607$$

$$B = -A = -0.3607$$

$$\therefore\ H(s) = \frac{A}{(s+0.3606)} + \frac{Bs+C}{(s^2+0.3606s+0.5927)}$$

$$= \frac{0.3607}{s+0.3606} - \frac{0.3607s}{s^2+0.3606s+0.5927}$$

$$(s+a)^2 = s^2+2as+a^2$$

$$2s = 0.3606 \Rightarrow a = \frac{0.3606}{2} = 0.1803$$

$$= \frac{0.3607}{s+\ 0.3606} - \frac{0.3607\,s}{(s^2+\ 2s\times 0.\,1803+0.\,1803^2)+\left(\sqrt{0.5927-0.1803^2}\right)^2}$$

$$= \frac{0.3607}{s+\ 0.3606} - \frac{0.3607s}{(s+\ 0.1803)^2+\ 0.7485^2} = \frac{0.3607}{s+\ 0.3606} - \ 0.3607\left[\frac{s+0.1803-0.1803}{(s+0.1803)^2+0.7485^2}\right]$$

$$= \frac{0.3607}{s+0.3606} - \ 0.3607\left[\frac{s+0.1803}{(s+0.1803)^2+0.7485^2}\right] + 0.3607\left[\frac{0.1803}{(s+0.1803)^2+0.7485^2}\right]$$

$$= \frac{0.3607}{s+0.3606} - 0.3607\left[\frac{s+0.1803}{(s+0.1803)^2+0.7845^2}\right] + \frac{0.0650}{0.7845}\left[\frac{0.7845}{(s+0.1803)^2+0.7845^2}\right]$$

$$= \frac{0.3607}{s+0.3606} - 0.3607\left[\frac{s+0.1803}{(s+0.1803)^2+0.7845^2}\right] + 0.0829\left[\frac{0.7845}{(s+0.1803)^2+0.7845^2}\right]$$

Digital IIR lowpass filter transfer function, H(z)

For impulse invariant transformation,

Using equation (7.17), (7.18) and (7.19).

$$\frac{1}{s+p_i} \xrightarrow{\text{is transformed to}} \frac{1}{1-e^{-p_iT}\,z^{-1}}$$

$$\frac{s+a}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-ze^{-aT}\,(\cos bT)z^{-1}+e^{-2aT}\,z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT}\,(\sin bT)z^{-1}}{1-2e^{-aT}\,(\cos bT)z^{-1}+e^{-2aT}\,z^{-2}}$$

Using the above transformation, the H(s) can be transformed to H(z) as shown below.

Put, T = 1

$$\therefore\ H(z) = \frac{0.3607}{1-e^{-0.3606}\,z^{-1}} - 0.3607\left[\frac{1-e^{-0.1803}\cos(0.7845)z^{-1}}{1-2\,e^{-0.1803}\cos(0.7845)\,z^{-1}+e^{-2\times0.1803}\,z^{-2}}\right]$$

$$+0.0829\left[\frac{e^{-0.1803}\sin(0.7845)z^{-1}}{1-2\,e^{-0.1803}\cos(0.7845)z^{-1}+e^{-2\,\times\,0.1803}\,z^{-2}}\right]$$

$$= \frac{0.3607}{1-0.6973z^{-1}} + \frac{-0.3607+0.2132z^{-1}}{1-1.1820z^{-1}+0.6973z^{-2}} + \frac{0.0490z^{-1}}{1-1.1820z^{-1}+0.6973z^{-2}}$$

$$= \frac{0.3607}{1-0.6973z^{-1}} + \frac{-0.3607+0.2622z^{-1}}{1-1.1820z^{-1}+0.6973z^{-2}}$$

$$= \frac{0.3607(1-1.1820z^{-1}+0.6973z^{-2}) + (-0.3607+0.2622z^{-1})(1-0.6973z^{-1})}{(1-0.6973z^{-1})(1-1.1820z^{-1}+0.6973z^{-2})}$$

$$= \frac{0.3607-0.4263z^{-1}+0.2515z^{-2}-0.3607+0.2515z^{-1}+0.2622z^{-1}-0.1828z^{-2}}{1-1.1820z^{-1}+0.6973z^{-2}-0.6973z^{-1}+0.8242z^{-2}-0.4862z^{-3}}$$

$$= \frac{0.0874z^{-1}+0.0687z^{-2}}{1-1.8793z^{-1}+1.5215z^{-2}-0.4862z^{-3}}$$

Alternatively,

$$H(z)=\frac{0.0874\,z^{-1}+0.0687\,z^{-2}}{1-1.8793\,z^{-1}+1.5215\,z^{-2}-0.4862\,z^{-3}}=\frac{z^{-3}(0.0874\,z^{2}+0.0687z)}{z^{-3}(z^{3}-1.8793z^{2}+1.5215z-0.4862)}$$

$$=\frac{0.0874z^{2}+0.0687z}{z^{3}-1.8793z^{2}+1.5215z-0.4862}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=\frac{0.0874\,z^{-1}+0.0687\,z^{-2}}{1-1.8793\,z^{-1}+1.5215\,z^{-2}-0.4862\,z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z)-1.8793z^{-1}Y(z)+1.5215z^{-2}Y(z)-0.4862z^{-3}Y(z)=0.0874z^{-1}X(z)+0.0687z^{-1}X(z)$$

$$\backslash\ \ Y(z)=0.0874z^{-1}X(z)+0.0687z^{-1}X(z)+1.8793z^{-1}Y(z)-1.5215z^{-2}Y(z)+0.4862z^{-3}Y(z) \quad .....(4)$$

Using equation (4), the direct form -I structure is drawn as shown in fig 1.

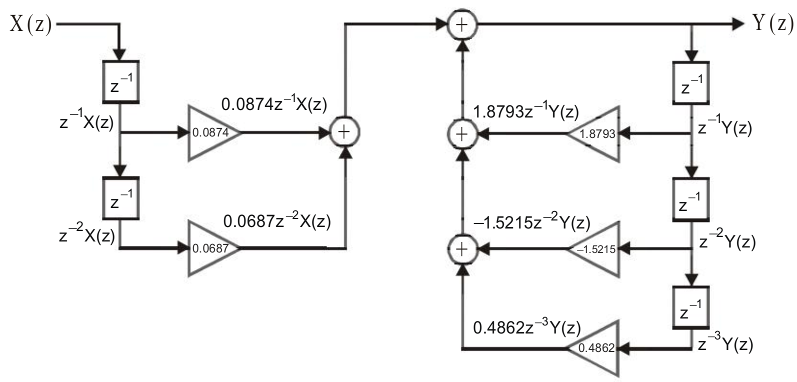


***Fig 1 :** Direct form-I structure of digital IIR lowpass filter.*

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=\frac{W(z)}{X(z)}\times\frac{Y(z)}{W(z)}=\frac{0.0874\,z^{-1}+0.0687\,z^{-2}}{1-1.8793\,z^{-1}+1.5215\,z^{-2}-0.4862\,z^{-3}}$$

$$\text{where, } \frac{W(z)}{X(z)}=\frac{1}{1-1.8793z^{-1}+1.5215z^{-2}-0.4862z^{-2}} \quad .....(5)$$

$$\frac{Y(z)}{W(z)}=0.0874z^{-1}+0.0687z^{-2} \quad .....(6)$$

On cross multiplying equation (5) we get,

$$W(z)-1.8793z^{-1}W(z)+1.5215z^{-2}W(z)-0.4862z^{-3}W(z)=X(z)$$

$$\backslash\ \ W(z)=X(z)+1.8793z^{-1}W(z)-1.5215z^{-2}W(z)+0.4862z^{-3}W(z) \quad .....(7)$$

On cross multiplying equation (6) we get,

$$Y(z)=0.0874z^{-1}X(z)+0.0687z^{-2}W(z) \quad .....(8)$$

Using equation (7) and (8), the direct form-II structure is drawn as shown in fig 2.

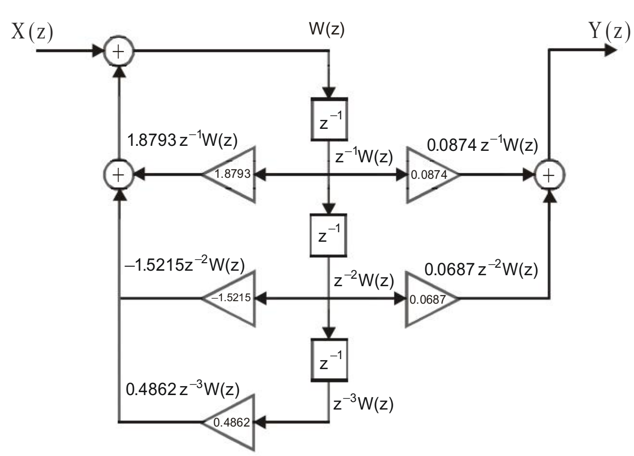


***Fig 2 :** Direct form-II structure of digital IIR lowpass filter.*

E7.17.* *Design a Chebyshev digital IIR lowpass filter using bilinear transformation by taking $T = 0.5$second, to satisfy the following specifications.

$$0.9 \le |H(e^{j\omega})| \le 1.0 \quad ; \quad \text{for } 0 \le \omega \le 0.25\pi$$

$$|H(e^{j\omega})| \le 0.35 \quad ; \quad \text{for } 0.375\pi \le \omega \le \pi$$

Draw direct form-I and II structure of the filter.

Alternatively,

Passband ripple $\le$ 0.9151 dB

Stopband attenuation $\ge$ 9.1186 dB

Passband edge frequency = 0.25π rad/sample

Stopband edge frequency = 0.375π rad/sample

The above specifications can be converted to A_p and A_s as shown below.

$$A_p = 10^{\left(-\delta_{p,dB}/20\right)} = 10^{\left(-0.9151/20\right)} = 0.9$$

$$A_s = 10^{\left(-\alpha_{s,dB}/20\right)} = 10^{\left(-9.1186/20\right)} = 0.35$$

Solution

Specifications of digital IIR lowpass filter

Passband edge digital frequency, $\omega_p = 0.25\pi$ rad/sample

Stopband edge digital frequency, $\omega_s = 0.375\pi$ rad/sample

Gain in normal value at passband edge, $A_p = 0.9$

Gain in normal value at stopband edge, $A_s = 0.35$

Sampling time, $T = 0.5$second

Specifications of analog IIR lowpass filter

Gain in normal value at passband edge, $A_p = 0.9$

Gain in normal value at stopband edge, $A_s = 0.35$

Gain is same in analog and digital filter.

For bilinear transformation,

Passband edge analog frequency, $\Omega_p = \dfrac{2}{T}\tan\dfrac{\omega_p}{2}$

$$= \frac{2}{0.5}\tan\frac{0.25\pi}{2} = 1.6569 \text{ rad / second}$$

Using equation (7.83).

Stopband edge analog frequency, $\Omega_s = \dfrac{2}{T}\tan\dfrac{\omega_s}{2}$

$$= \frac{2}{0.5}\tan\frac{0.375\pi}{2} = 2.6727 \text{ rad / second}$$

Using equation (7.84).

Order of the filter

$$N_1 = \frac{\cosh^{-1}\left[\dfrac{\left(1/A_s^2\right)-1}{\left(1/A_p^2\right)-1}\right]^{\frac{1}{2}}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1}\left[\dfrac{\left(1/0.35^2\right)-1}{\left(1/0.9^2\right)-1}\right]^{\frac{1}{2}}}{\cosh^{-1}\dfrac{2.6727}{1.6569}} = \frac{\cosh^{-1}5.5258}{\cosh^{-1}1.6131} = 2.2643$$

Using equation (7.87).

Choose order N_1 such that $N \ge N_1$ and N is an integer.

Let, order, $N = 3$.

Normalized transfer function, $H(s_n)$ of Chebyshev lowpass filter

For odd N,

$$H(s_n) = \frac{B_0}{s_n + c_0}\prod_{k=1}^{\frac{N-1}{2}}\frac{B_k}{s_n^2 + b_k s_n + c_k}$$

Using equation (7.89).

Here, $N = 3$, $\quad \therefore\ k = \frac{N-1}{2} = \frac{3-1}{2} = 1$

$$\therefore\ H(s_n) = \frac{B_0}{s_n + c_0} \times \frac{B_1}{s_n^2 + b_1 s_n + c_1}$$

$$\epsilon = \left[\left(1/A_p^2\right) - 1\right]^{\frac{1}{2}}$$

$$= \left[\left(1/0.9^2\right) - 1\right]^{\frac{1}{2}} = 0.4843$$

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{\frac{1}{2}}+\frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right\}$$

$$= \frac{1}{2}\left[\left[\left(\frac{1}{0.4843^2}+1\right)^{\frac{1}{2}}+\frac{1}{0.4843}\right]^{\frac{1}{3}} - \left[\left(\frac{1}{0.4843^2}+1\right)^{\frac{1}{2}}+\frac{1}{0.4843}\right]^{-\frac{1}{3}}\right]$$ Using equation (7.93).

$$= \frac{1}{2}\,[1.6335 - 0.6122] = 0.5107$$ Using equation (7.94).

$$c_0 = y_N = 0.5107$$

$$b_k = 2\,y_N \sin\left[\tfrac{(2k-1)\pi}{2N}\right]$$

When k = 1 ; $b_k = b_1 = 2\times 0.5107 \sin\left(\tfrac{(2-1)\pi}{2\times 3}\right) = 0.5107$ Using equation (7.90).

$$c_k = y_N^2 + \cos^2\left[\tfrac{(2k-1)\pi}{2N}\right]$$

$$c_k = c_1 = 0.5107^2 + \cos^2\left(\tfrac{(2-1)\pi}{6}\right) = 0.5107^2 + \cos^2\frac{\pi}{6}$$ Using equation (7.91).

$$= 0.5107^2 + \left[\frac{1+\cos\frac{2\pi}{6}}{2}\right]$$ $\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$

$$= 0.2608 + 0.75 = 1.0108$$

$$\therefore\ H(s_n) = \frac{B_0}{s_n + c_0}\times\frac{B_1}{s_n^2 + b_1 s_n + c_1} = \frac{B_0}{(s+0.5107)}\times\frac{B_1}{s_n^2 + 0.5107\,s_n + 1.0108}$$

To evaluate B_0 and B_1,

Let, $H(s_n)\big|_{s_n=0} = 1,$

When $s_n = 0$; $H(s_n) = \dfrac{B_0 B_1}{(0.5107)\,(1.0108)} = 1.9372 B_0 B_1$

$$\therefore\ 1.9372\,B_0 B_1 = 1 \Rightarrow B_0 B_1 = \frac{1}{1.9372} \Rightarrow B_0 B_1 = 0.5162$$

Let, $B_0 = B_1$; $\therefore\ B_0^2 = 0.5162 \Rightarrow B_0 = \sqrt{0.5162} = 0.7185$

$$\therefore\ B_1 = B_0 = 0.7185$$

$$H(s_n) = \frac{B_0}{(s_n+0.5107)}\times\frac{B_1}{(s_n^2+0.5107 s_n + 1.0108)} = \frac{0.7185}{(s_n+0.5107)}\times\frac{0.7185}{(s_n^2+0.5107 s_n+1.0108)}$$

$$= \frac{0.5162}{(s_n+0.5107)\,(s_n^2+0.5107 s_n+1.0108)}$$

$$= \frac{0.5162}{s_n^3 + 1.0214 s_n^2 + 1.2716 s_n + 0.5162}$$

Unnormalized transfer function, H(s) of Chebyshev lowpass filter

$$H(s) = H(s_n)\Big|_{s_n=\frac{s}{\Omega_c}}$$

where, Ω_c = Cutoff frequency.

$\Omega_c = \Omega_p$ = 1.6569 rad/second.

$$\therefore\ H(s) = H(s_n)\Big|_{s_n=\frac{s}{\Omega_c}} = \frac{0.5162}{s_n^3 + 1.0214 s_n^2 + 1.2716 s_n + 0.5162)}\Big|_{s_n=\frac{s}{\Omega_c}} = \frac{0.5162}{\dfrac{s^3}{\Omega_c^3} + 1.0214\dfrac{s^2}{\Omega_c^2} + 1.2716\dfrac{s}{\Omega_c} + 0.5162}$$

$$= \frac{0.5162}{\dfrac{s^3 + 1.0214\Omega_c s^2 + 1.2716\Omega_c^2 s + 0.5162\Omega_c^3}{\Omega_c^3}} = \frac{0.5162\,\Omega_c^3}{s^3 + 1.0214\,\Omega_c s^2 + 1.2716\,\Omega_c^2 s + 0.5162\Omega_c^3}$$

$$= \frac{0.5162\times 1.6569^3}{s^3 + 1.0214\times 1.6569 s^2 + 1.2716\times 1.6569^2 s + 0.5162\times 1.6569^3} = \frac{2.3480}{s^3 + 1.6924 s^2 + 3.4909 s + 2.3480}$$

Digital IIR lowpass filter transfer function, H(z)

For bilinear transformation,

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{2.3480}{s^3 + 1.6924\,s^2 + 3.4909\,s + 2.3480}\Bigg|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$= \frac{2.3480}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^3 + 1.6924\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 3.4909\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right) + 2.3480}$$

$$= \frac{2.3480}{\frac{8(1-z^{-1})^3}{T^3(1+z^{-1})^3} + \frac{6.7696(1-z^{-1})^2}{T^2(1+z^{-1})^2} + \frac{6.9818(1-z^{-1})}{T(1+z^{-1})} + 2.3480}$$

$$= \frac{2.3480}{\frac{8(1-z^{-1})^3 + 6.7696\,T(1-z^{-1})^2(1+z^{-1}) + 6.9818\,T^2(1-z^{-1})(1+z^{-1})^2 + 2.3480\,T^3(1+z^{-1})^3}{T^3(1+z^{-1})}}$$

$$= \frac{2.3480\,T^3(1+z^{-1})^3}{8(1-z^{-1})^3 + 6.7696\,T(1-z^{-1})^2(1+z^{-1}) + 6.9818\,T^2(1-z^{-1})(1+z^{-1})^2 + 2.3480\,T^3(1+z^{-1})^3}$$

Put, T = 0.5

$$= \frac{2.3480 \times 0.5^3(1+z^{-1})^3}{8(1-z^{-1})^3 + 6.7696 \times 0.5(1-z^{-1})^2(1+z^{-1}) + 6.9818 \times 0.5^2(1-z^{-1})(1+z^{-1})^2 + 2.3480 \times 0.5^3(1+z^{-1})^3}$$

$$= \frac{0.2935(1+3z^{-1}+3z^{-2}+z^{-3})}{8(1-3z^{-1}+3z^{-2}-z^{-3}) + 3.3848(1-z^{-2})(1-z^{-1}) + 1.7455(1+z^{-1})(1-z^{-2}) + 0.2935(1+3z^{-1}+3z^{-2}+z^{-3})}$$

(a + b) (a – b) = a² – b²
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$$= \frac{0.2935 + 0.8805\,z^{-1} + 0.8805\,z^{-2} + 0.2935\,z^{-3}}{8(1-3z^{-1}+3z^{-2}-z^{-3}) + 3.3848(1-z^{-1}-z^{-2}+z^{-3}) + 1.7455(1+z^{-1}-z^{-2}-z^{-3}) + 0.2935(1+3z^{-1}+3z^{-2}+z^{-3})}$$

$$= \frac{0.2935 + 0.8805\,z^{-1} + 0.8805\,z^{-2} + 0.2935\,z^{-3}}{13.4238 - 24.7588\,z^{-1} + 19.7502\,z^{-2} - 6.0672\,z^{-3}} = \frac{\frac{0.2935}{13.4238} + \frac{0.8805}{13.4238}z^{-1} + \frac{0.8805}{13.4238}z^{-2} + \frac{0.2935}{13.4238}z^{-3}}{1 - \frac{24.7588}{13.4238}z^{-1} + \frac{19.7502}{13.4238}z^{-2} - \frac{6.0672}{13.4238}z^{-3}}$$

$$= \frac{0.0219 + 0.0656\,z^{-1} + 0.0656\,z^{-2} + 0.0219\,z^{-3}}{1 - 1.8444\,z^{-1} + 1.4713\,z^{-2} - 0.4519\,z^{-3}}$$

Alternatively,

$$H(z) = \frac{0.0219 + 0.0656\,z^{-1} + 0.0656\,z^{-2} + 0.0219\,z^{-3}}{1 - 1.8444\,z^{-1} + 1.4713\,z^{-2} - 0.4519\,z^{-3}} = \frac{z^{-3}(0.0219\,z^3 + 0.0656\,z^2 + 0.0656\,z + 0.0219}{z^{-3}(z^3 - 1.8444\,z^2 + 1.4713\,z - 0.4519)}$$

$$= \frac{0.0219\,z^3 + 0.0656\,z^2 + 0.0656\,z + 0.0219}{z^3 - 1.8444\,z^2 + 1.4713\,z - 0.4519}$$

Direct form-I structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{0.0219 + 0.0656\,z^{-1} + 0.0656\,z^{-2} + 0.0219\,z^{-3}}{1 - 1.8444\,z^{-1} + 1.4713\,z^{-2} - 0.4519\,z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z) - 1.8444z^{-1}Y(z) + 1.4713z^{-2}Y(z) - 0.4519\,Y(z) = 0.0219\,X(z) + 0.0656z^{-1}X(z) + 0.0656z^{-2}X(z) + 0.0219z^{-3}X(z)$$

$$\therefore\ Y(z) = 0.0219X(z) + 0.0656z^{-1}X(z) + 0.0656z^{-2}X(z) + 0.0219z^{-3}X(z) + 1.8444z^{-1}Y(z) - 1.4713z^{-2}Y(z) + 0.4519z^{-3}Y(z) \quad .....(1)$$

Using equation (1), the direct form-I structure is drawn as shown in fig 1.

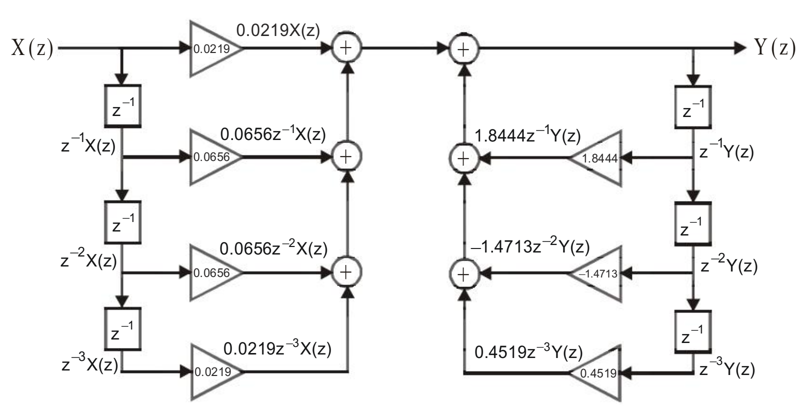


***Fig 1 :** Direct form-I structure of 3rd order digital IIR lowpass filter.*

Direct form-II structure of digital IIR lowpass filter

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.0219 + 0.0656\,z^{-1} + 0.0656\,z^{-2} + 0.0219\,z^{-3}}{1 - 1.8444\,z^{-1} + 1.4713\,z^{-2} - 0.4519\,z^{-3}}$$

$$\text{where, } \frac{W(z)}{X(z)} = \frac{1}{1 - 1.8444\,z^{-1} + 0.4713\,z^{-2} - 0.4519\,z^{-3}} \quad .....(2)$$

$$\frac{Y(z)}{W(z)} = 0.0219 + 0.0656z^{-1} + 0.0656\,z^{-2} + 0.0219\,z^{-3} \quad .....(3)$$

On cross multiplying equation (2) we get,

$W(z) - 1.8444z^{-1}W(z) + 1.4713z^{-2}\,W(z) - 0.4519\,z^{-3}W(z) = X(z)$

\ $W(z) = X(z) + 1.8444z^{-1}W(z) - 1.4713z^{-2}\,W(z) + 0.4519z^{-3}W(z)$ .....(4)

On cross multiplying equation (3) we get,

$Y(z) = 0.0219W(z) + 0.0656z^{-1}W(z) + 0.0656z^{-2}W(z) + 0.0219z^{-3}W(z)$ .....(5)

Using equation (4) and (5), the direct form-II structure is drawn as shown in fig 2.

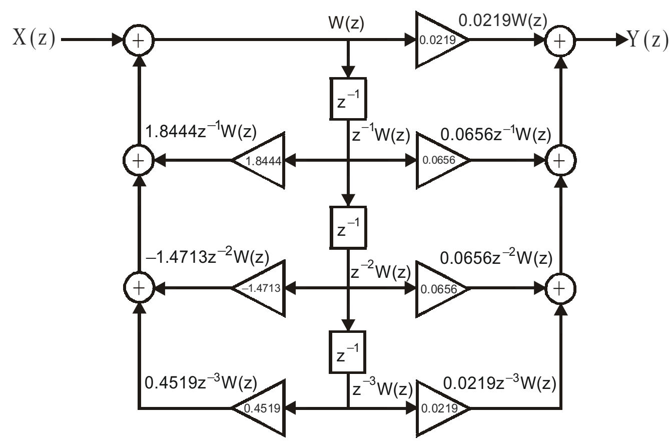


Fig 2 : *Direct form-II structure of 3rd order digital IIR lowpass filter.*

Chapter 8

Finite Word Length Effects In Digital Filters

8.1 Introduction

The fundamental operations in the various computational procedure like convolution, spectral estimation, etc., in DSP (Digital Signal Processing) are multiplication and addition. These operations are performed using the samples of input sequence, samples of impulse response and the coefficients of the difference equation governing the system. The informations (or numbers) used for computation are called input data and the results of computation are called output data. The input and output data are stored in registers in a ***digital system***.

The ***registers*** are the basic storage device in digital system. The maximum size of the binary information (or data) that can be stored in a register is called ***register word length***. For example, when a register stores an 8-bit data then its word length is 8-bit. For storing the input data in registers they have to be quantized and coded in binary. The quantization and coding depends on the register word length. For example, when the register word length is 8-bit, we can generate $2^8 = 256$ binary codes and so we have 256 quantized levels. Any analog value of the input data has to be fitted into one of the 256 quantized levels in an 8-bit representation. This quantization and coding will introduce error in input data, because the analog data has infinite precision but the digital equivalent has finite precision.

While performing computations the size of the result may be exceeding the size of the register used for storing the result. For example the result of the addition of two eight bit data may be 8 or 9 bits and the result of the multiplication of two eight bit data may go up to 16-bits. In this case if the register used to store the result is 8-bit, then the result has to be truncated or rounded to accommodate in the register. This makes the system nonlinear, and leads to limit cycle behaviour. The effect of truncation or rounding can be represented in terms of an additive error signal, which is called ***roundoff noise***.

In general the effects due to finite precision representation of numbers in a digital system are commonly referred to as ***finite word length effects***. The following are some of the finite word length effects in digital filters.

1. Errors due to quantization of input data by A/D (Analog-to-Digital) converter.
2. Errors due to quantization of filter coefficients.
3. Errors due to rounding the product in multiplication.
4. Errors due to overflow in addition.
5. Limit cycles.

8.2 Representation of Numbers in Digital Systems

8.2.1 Binary Codes

The ***binary codes*** are framed using the numeric symbols “0” and “1”. Each digit of the binary code is called ***bit***. The size of the binary code is specified in terms of number of bits. In digital system the binary codes are used to represent any information like text, image, numbers, etc.

In general, using n-bits it is possible to frame 2^n binary codes.

<table>
<tr><td>When n = 1 ; $2^n = 2^1 = 2$
Binary codes : 0
1</td><td rowspan="3">When n = 4 ; $2^n = 2^4 = 16$
Binary codes : 0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
0 1 0 0
0 1 0 1
0 1 1 0
0 1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1</td></tr>
<tr><td>When n = 2 ; $2^n = 2^2 = 4$
Binary codes : 0 0
0 1
1 0
1 1</td></tr>
<tr><td>When n = 3 ; $2^n = 2^3 = 8$
Binary codes : 0 0 0
0 0 1
0 1 0
0 1 1
1 0 0
1 0 1
1 1 0
1 1 1</td></tr>
</table>

When decimal numbers are represented in binary codes, the size of the code will decide the range of numbers that can be represented in binary. The 4-bit binary codes that can be used to represent different types of decimal numbers are listed in table 8.1.

When 4-bit binary is used to represent unsigned decimal integers then the range is,

$$0 \text{ to } 2^4 - 1 \quad \Rightarrow \quad 0 \text{ to } 15_{10}.$$

When 4-bit binary is used to represent signed decimal integers in sign-magnitude format then the range is,

$$-(2^3 - 1) \text{ to } +(2^3 - 1) \quad \Rightarrow \quad -7_{10} \text{ to } +7_{10}.$$

When 4-bit binary is used to represent unsigned decimal fraction in fixed point representation then the range is,

$$0 \text{ to } 1 - 2^{-4} \quad \Rightarrow \quad 0 \text{ to } \frac{15}{16} \quad \Rightarrow \quad 0 \text{ to } 0.9375_{10}$$

When 4-bit binary is used to represent signed decimal fraction in fixed point sign-magnitude format then the range is,

$$-(1-2^{-3}) \text{ to } +(1-2^{-3}) \quad \Rightarrow \quad -\frac{7}{8} \text{ to } +\frac{7}{8} \quad \Rightarrow \quad -0.875_{10} \text{ to } +0.875_{10}$$

Table 8.1 : Binary Representation of Decimal Numbers

Binary Code	Unsigned decimal integer	Signed decimal integer	Unsigned decimal fraction	Signed decimal fraction
0 0 0 0	0	0	0/16 = 0	0/8 = 0
0 0 0 1	1	1	1/16 = 0.0625	1/8 = 0.125
0 0 1 0	2	2	2/16 = 0.1250	2/8 = 0.250
0 0 1 1	3	3	3/16 = 0.1875	3/8 = 0.375
0 1 0 0	4	4	4/16 = 0.2500	4/8 = 0.500
0 1 0 1	5	5	5/16 = 0.3125	5/8 = 0.625
0 1 1 0	6	6	6/16 = 0.3750	6/8 = 0.750
0 1 1 1	7	7	7/16 = 0.4375	7/8 = 0.875
1 0 0 0	8	–0	8/16 = 0.5000	–0/8 = –0
1 0 0 1	9	–1	9/16 = 0.5625	–1/8 = –0.125
1 0 1 0	10	–2	10/16 = 0.6250	–2/8 = –0.250
1 0 1 1	11	–3	11/16 = 0.6875	–3/8 = –0.375
1 1 0 0	12	–4	12/16 = 0.7500	–4/8 = –0.500
1 1 0 1	13	–5	13/16 = 0.8125	–5/8 = –0.625
1 1 1 0	14	–6	14/16 = 0.8750	–6/8 = –0.750
1 1 1 1	15	–7	15/16 = 0.9375	–7/8 = –0.875

8.2.2 Radix Number System

In radix number representation the numbers can be represented by a summation relation as shown in the following examples.

$$\begin{aligned} 178.25_{10} &= (1 \times 10^2) + (7 \times 10^1) + (8 \times 10^0) + (2 \times 10^{-1}) + (5 \times 10^{-2}) \\ &= (d_{-2} \times 10^2) + (d_{-1} \times 10^1) + (d_0 \times 10^0) + (d_1 \times 10^{-1}) + (d_2 \times 10^{-2}) \\ &= \sum_{i=-2}^{2} d_i\, r^{-i} \quad ; \text{where, } r = 10 \end{aligned}$$

$$\begin{aligned} 111.11_2 &= (1 \times 2^2) + (1 \times 2^1) + (1 \times 2^0) + (1 \times 2^{-1}) + (1 \times 2^{-2}) \\ &= (d_{-2} \times 2^2) + (d_{-1} \times 2^1) + (d_0 \times 2^0) + (d_1 \times 2^{-1}) + (d_2 \times 2^{-2}) \\ &= \sum_{i=-2}^{2} d_i\, r^{-i} \quad ; \text{where, } r = 2 \end{aligned}$$

In general any number can be represented as,

$$\text{Number, } N = \sum_{i=-A}^{B} d_i \, r^{-i} \qquad .....(8.1)$$

where, A = Number of integer digits

B = Number of fraction digits

r = Radix or Base

d_i = i^{th} digit of the number

In a radix number system the possible values for d_i will be in the range 0 £ d_i £ (r – 1).

> **Example :** When r = 2, d_i = 0 or 1.
>
> When r = 10, d_i = 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9.

In digital systems the numbers are represented in binary, in which the radix r = 2. Hence for a binary number the equation (8.1) can be written as,

$$\text{Binary number, } N = \sum_{i=-A}^{B} d_i \, 2^{-i} \qquad .....(8.2)$$

The binary digit d_{-A} is called the ***Most Significant Digit*** (MSD) and the binary digit d_B is called the ***Least Significant Digit*** (LSD) of the binary number N. The binary point between the digits d_0 and d_1 does not exist physically in the digital system. The binary digit is also known as bit.

In the various computation procedures of DSP we use fraction format because mixed numbers (i.e., numbers with integer and fraction parts) are difficult to multiply and the number of digits representing an integer cannot be reduced by truncation or rounding. For the fraction format of binary numbers the equation (8.2) can be modified as shown in equation (8.3).

$$\text{Binary fraction number, } N = \pm \sum_{i=1}^{B} d_i \, 2^{-i}$$

$$\text{or Binary fraction number, } N = \sum_{i=0}^{B} d_i \, 2^{-i} \qquad .....(8.3)$$

where, d_0 is used to represent the sign of the number.

The two major methods of representing binary numbers are fixed point representation and floating point representation. They are discussed in the following sections.

In ***fixed point representation*** the digits allotted for integer part and fraction part are fixed, and so the position of binary point is fixed. Since the number of digits is fixed it is impossible to represent too large and too small numbers by fixed point representation. Therefore the range of numbers that can be represented in fixed point representation for a given binary word size is less when compared to floating point representation.

In ***floating point representation*** the binary point can be shifted to desired position so that number of digits in the integer part and fraction part of a number can be varied. This leads to larger range of number that can be represented in floating point representation.

8.2.3 Fixed Point Representation

In fixed point representation there are three different formats for representing negative binary fraction numbers. They are,

1. Sign-magnitude format
2. One's complement format
3. Two's complement format

In fixed point representation there is only one unique way of representing positive binary fraction number as shown in equation (8.4).

$$\text{Positive binary fraction number, } N_p = 0.d_1d_2.....d_B$$
$$= (0 \times 2^0) + (d_1 \times 2^{-1}) + (d_2 \times 2^{-2}) + ... + (d_B \times 2^{-B})$$
$$= \sum_{i=0}^{B} d_i\, 2^{-i} \quad ;\text{where } d_0 = 0 \qquad .....(8.4)$$
$$= (0 \times 2^0) + \sum_{i=1}^{B} d_i\, 2^{-i} \qquad .....(8.5)$$

In equation (8.4) the most significant digit d_0 is set to zero to represent the positive sign. In all the three formats for negative numbers the most significant digit d_0 is one to represent the negative sign.

> *Note : The binary point between d_0 and d_1 is not mandatory because it does not exist physically in a digital system.*

Sign-magnitude Format

In sign magnitude format the negative value of a given number differ only in sign bit (i.e., digit d_0). The sign digit d_0 is zero for positive number and one for negative number. Except the sign bit all other digits of the negative of a given number are same as that of its positive representation.

$$\therefore \text{ Negative binary fraction number, } N_N = (1 \times 2^0) + \sum_{i=1}^{B} d_i\, 2^{-i} \qquad .....(8.6)$$

The range of decimal fraction numbers that can be represented in B-bit fixed point sign-magnitude format is,

$$-\left[1-2^{-(B-1)}\right] \text{ to } +\left[1-2^{-(B-1)}\right] \quad ; \text{with step size } = \frac{1}{2^{B-1}}$$

When B = 4,

$$\text{Range} = -\left[1-2^{-(4-1)}\right] \text{ to } +\left[1-2^{-(4-1)}\right] = -\left[1-\tfrac{1}{8}\right] \text{ to } +\left[1-\tfrac{1}{8}\right] = -\tfrac{7}{8} \text{ to } +\tfrac{7}{8}$$
$$= -0.875_{10} \text{ to } +0.875_{10}$$

$$\text{Step size} = \frac{1}{2^{4-1}} = \frac{1}{2^3} = \frac{1}{8} = 0.125_{10}$$

The 4-bit fixed point sign-magnitude binary representation of decimal fractions are listed in table 8.2.

Table 8.2 : Decimal Equivalents of 4-bit Binary Numbers in Fixed Point Representation

Binary number in fixed point representation			Decimal
Sign-magnitude	**One's complement**	**Two's complement**	**Equivalent**
0 0 0 0	0 0 0 0	0 0 0 0	0
0 0 0 1	0 0 0 1	0 0 0 1	1/8 = 0.125
0 0 1 0	0 0 1 0	0 0 1 0	2/8 = 0.250
0 0 1 1	0 0 1 1	0 0 1 1	3/8 = 0.375
0 1 0 0	0 1 0 0	0 1 0 0	4/8 = 0.500
0 1 0 1	0 1 0 1	0 1 0 1	5/8 = 0.625
0 1 1 0	0 1 1 0	0 1 1 0	6/8 = 0.750
0 1 1 1	0 1 1 1	0 1 1 1	7/8 = 0.875
1 0 0 0	1 1 1 1	-------	–0
1 0 0 1	1 1 1 0	1 1 1 1	–1/8 = –0.125
1 0 1 0	1 1 0 1	1 1 1 0	–2/8 = –0.250
1 0 1 1	1 1 0 0	1 1 0 1	–3/8 = –0.375
1 1 0 0	1 0 1 1	1 1 0 0	–4/8 = –0.500
1 1 0 1	1 0 1 0	1 0 1 1	–5/8 = –0.625
1 1 1 0	1 0 0 1	1 0 1 0	–6/8 = –0.750
1 1 1 1	1 0 0 0	1 0 0 1	–7/8 = –0.875
		1 0 0 0	–8/8 = –1.000

Example 8.1

Convert $+0.125_{10}$ and -0.125_{10} to sign-magnitude format of binary and verify the result by converting the binary to decimal.

Solution

Decimal to binary conversion

$$+.125_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.001 \xrightarrow[\text{sign bit}]{\text{Append}} 0.001 \xrightarrow[\text{dot}]{\text{Remove}} 0001_2$$

$$-.125_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -.001 \xrightarrow[\text{sign bit}]{\text{Append}} 1.001 \xrightarrow[\text{dot}]{\text{Remove}} 1001_2$$

$$\therefore \; +0.125_{10} = 0001_2$$
$$-0.125_{10} = 1001_2$$

Decimal to binary conversion

$$\begin{array}{r} .125 \\ \times\ 2 \\ \hline \boxed{0}.250 \\ \times\ 2 \\ \hline \boxed{0}.500 \\ \times\ 2 \\ \hline \boxed{1}.000 \end{array}$$

$.001_2$

Binary to decimal conversion

$$0001_2 \xrightarrow[\text{sign bit}]{\text{Remove}} +.001 \xrightarrow[\text{to decimal}]{\text{Convert}} +.125_{10}$$

$$1001_2 \xrightarrow[\text{sign bit}]{\text{Remove}} -.001 \xrightarrow[\text{to decimal}]{\text{Convert}} -.125_{10}$$

$$\therefore \; 0001_2 = +0.125_{10}$$
$$1001_2 = -0.125_{10}$$

Binary to decimal conversion

$$+.001_2 = +(0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3})$$
$$= +(0 + 0 + .125) = +.125_{10}$$
$$-.001_2 = -(0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3})$$
$$= -(0 + 0 + .125) = -.125_{10}$$

One's Complement Format

The positive number is same in all the formats of fixed point representation and it is given by equation (8.5). In one's complement format the negative of the given number is obtained by bit by bit complement of its positive representation given by equation (8.5). The complement of a digit d_i can be obtained by subtracting the digit from one.

$$\therefore \text{ Complement of } d_i = \overline{d}_i = (1 - d_i) \quad .....(8.7)$$

In equation (8.5) if we set the sign bit to one and replace d_i by $(1 - d_i)$ we get the one's complement format for negative number.

$$\therefore \left.\begin{array}{l}\text{Negative binary fraction} \\ \text{number in one's complement}\end{array}\right\} N_{1c} = (1 \times 2^0) + \sum_{i=1}^{B} (1 - d_i)2^{-i} \quad .....(8.8)$$

The range of decimal fraction numbers that can be represented in B-bit fixed point one's complement format is same as that of sign-magnitude format. The 4-bit fixed point one's complement binary representation of decimal fraction are listed in table 8.2.

Example 8.2

Convert $+0.125_{10}$ and -0.125_{10} to one's complement format of binary and verify the result by converting the binary to decimal.

Solution

Decimal to binary conversion

$$+.125_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.001 \xrightarrow[\text{sign bit}]{\text{Append}} 0.001 \xrightarrow[\text{dot}]{\text{Remove}} 0001_2$$

$$-.125_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -.001 \xrightarrow[\text{fraction part}]{\text{Complement}} -.110 \xrightarrow[\text{sign bit}]{\text{Append}} 1.110 \xrightarrow[\text{dot}]{\text{Remove}} 1110_2$$

$\therefore$ $+0.125_{10} = 0001_2$

$-0.125_{10} = 1110_2$

Refer example 8.1 for decimal to binary conversion of $.125_{10}$

Binary to decimal conversion

$$0001_2 \xrightarrow[\text{sign bit}]{\text{Remove}} +.001 \xrightarrow[\text{to decimal}]{\text{Convert}} +0.125_{10}$$

$$1110_2 \xrightarrow[\text{sign bit}]{\text{Remove}} -.110 \xrightarrow[\text{fraction part}]{\text{Complement}} -.001 \xrightarrow[\text{to decimal}]{\text{Convert}} -.125_{10}$$

$\therefore$ $0001_2 = +0.125_{10}$

$1110_2 = -0.125_{10}$

Refer example 8.1for binary to decimal conversion of $+.001_2$ and $-.001_2$

Two's Complement Format

The positive number is same in all the formats of fixed point representation and it is given by equation (8.5). In two's complement format the negative of the given number is obtained by taking one's complement of its positive representation and then adding one to the least significant bit. Hence in equation (8.8) if we add 1×2^{-B} then we get two's complement format for negative numbers.

$$\therefore \left.\begin{array}{l}\text{Negative binary fraction} \\ \text{number in two's complement}\end{array}\right\} N_{2c} = (1 \times 2^0) + \sum_{i=1}^{B} (1 - d_i)2^{-i} + (1 \times 2^{-B}) \quad .....(8.9)$$

The two's complement format provides single representation for zero, whereas the sign-magnitude and one's complement format has two representation for zero. Hence, the two's complement format of binary number system is practically used in all digital systems.

The range of decimal fraction numbers that can be represented in B-bit fixed point two's complement format is,

$$-1 \text{ to } +\left[1-2^{-(B-1)}\right] \quad ;\text{with step size } = \frac{1}{2^{B-1}}$$

When B = 4,

$$\text{Range} = -1 \text{ to } +\left[1-2^{-(4-1)}\right] = -1 \text{ to } +\left[1-\tfrac{1}{8}\right] = -1 \text{ to } +\tfrac{7}{8} = -1 \text{ to } +0.875_{10}$$

$$\text{Step size } = \frac{1}{2^{4-1}} = \frac{1}{2^3} = \frac{1}{8} = 0.125_{10}$$

The 4-bit fixed point two's complement binary representation of decimal fractions are listed in table 8.2.

Example 8.3

Convert $+0.125_{10}$ and -0.125_{10} to two's complement format of binary and verify the result by converting the binary to decimal.

Solution

Decimal to binary conversion

$$+.125_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.001 \xrightarrow[\text{sign bit}]{\text{Append}} 0.001 \xrightarrow[\text{dot}]{\text{Remove}} 0001_2$$

$$-.125_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -.001 \xrightarrow[\text{fraction part}]{\text{Complement}} -.110 \xrightarrow[\text{to LSD}]{\text{Add 1}} -.111 \xrightarrow[\text{sign bit}]{\text{Append}} 1.111 \xrightarrow[\text{dot}]{\text{Remove}} 1111_2$$

$\therefore \; +0.125_{10} = 0001_2$

$-0.125_{10} = 1111_2$

Refer example 8.1 for decimal to binary conversion of $.125_{10}$

Binary to decimal conversion

$$0001_2 \xrightarrow[\text{sign bit}]{\text{Remove}} +.001 \xrightarrow[\text{to decimal}]{\text{Convert}} +.125_{10}$$

$$1111_2 \xrightarrow[\text{sign bit}]{\text{Remove}} -.111 \xrightarrow[\text{fraction part}]{\text{Complement}} -.000 \xrightarrow[\text{to LSD}]{\text{Add 1}} -.001 \xrightarrow[\text{to decimal}]{\text{Convert}} -.125_{10}$$

$\therefore \; 0001_2 = +0.125_{10}$

$1111_2 = -0.125_{10}$

Refer example 8.1for binary to decimal conversion of $+.001_2$ and $-.001_2$

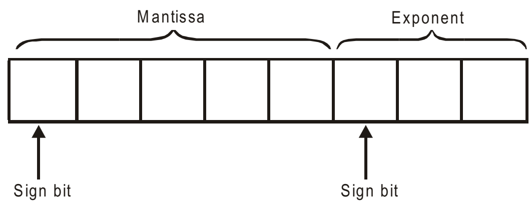
8.2.4 Floating Point Representation

The floating point representation is employed to represent larger range of numbers in a given binary word size. The ***floating point number*** is represented as,

$$\text{Floating point number, } N_f = M \times 2^E \qquad \text{..... (8.10)}$$

In equation (8.10), M is called ***mantissa*** and it will be in binary fraction format. The value of M will be in the range $0.5 \le M < 1$. In equatioin (8.10), E is called ***exponent*** and it is either a positive or negative integer. In floating point representation both mantissa and exponent uses one bit for representing sign. Usually the leftmost bit in mantissa and exponent is used to represent the sign. A "1" in the leftmost bit position represents negative sign and a "0" in the leftmost bit position represents positive sign.

The floating point representation is explained by considering a five bit mantissa and three bit exponent, with a total data size of eight bits. In mantissa the leftmost bit is used to represent the sign and other four bits are used to represent a binary fraction number. In exponent the leftmost bit is used to represent the sign and the other two bits are used to represent a binary integer number.



The range of numbers that can be represented by this floating point format is from $\pm\,[2^{-4} \times 2^{-3}]$ to $\pm\,[(2-2^{-4}) \times 2^{3}]$ i.e., from $\pm\,7.8125\;10^{-3}$ to $\pm\,15.5$.

> *Note : In the range of floating point format the "4" in 2^{-4} represents the 4-bit alloted for fractional binary number in mantissa and the "3" in 2^{-3} or 2^{+3} represents the maximum size of integer that can be represented using 2-bits in exponent.*

Let us represent +5, –5, +0.125 and –0.125 using the floating point format discussed above. Let us use sign-magnitude format for representing mantissa and exponent. First the given decimal number is converted to binary and then the binary point is moved to a position such that the most significant bit of mantissa is one and the exponent is adjusted accordingly. This form of floating point number is called normalized form.

$$+5_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +101_2 \xrightarrow[\text{exponent}]{\text{Add}} +101.0 \times 2^0 \xrightarrow{\text{Normalize}} +.1010 \times 2^{+3_{10}} \xrightarrow[\text{to binary}]{\text{Convert exponent}} +.1010 \times 2^{+11_2} \xrightarrow[\text{sign bit}]{\text{Append}} 0.1010 \times 2^{011_2} \xrightarrow[\text{dot}]{\text{Remove}} 01010 \times 2^{011_2}$$

$$\therefore \quad +5_{10} = 0101\;0011_2$$

$$-5_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -101_2 \xrightarrow[\text{exponent}]{\text{Add}} -101.0 \times 2^0 \xrightarrow{\text{Normalize}} -.1010 \times 2^{+3_{10}} \xrightarrow[\text{to binary}]{\text{Convert exponent}} -.1010 \times 2^{+11_2} \xrightarrow[\text{sign bit}]{\text{Append}} 1.1010 \times 2^{011_2} \xrightarrow[\text{dot}]{\text{Remove}} 11010 \times 2^{011_2}$$

$$\therefore \quad -5_{10} = 1101\;0011_2$$

$$+0.125_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.001_2 \xrightarrow[\text{exponent}]{\text{Add}} +.001\times 2^0 \xrightarrow{\text{Normalize}} +.1000\times 2^{-2_{10}} \xrightarrow[\text{to binary}]{\text{Convert exponent}}$$

$$01000\times 2^{110_2} \xleftarrow[\text{dot}]{\text{Remove}} 0.1000\times 2^{110_2} \xleftarrow[\text{sign bit}]{\text{Append}} +.1000\times 2^{-10_2}$$

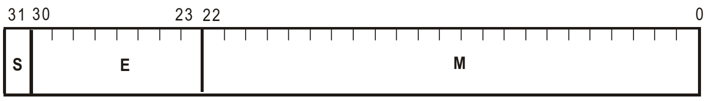
$\therefore \quad +0.125_{10} = 0100\ 0110_2$

$$-0.125_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -.001_2 \xrightarrow[\text{exponent}]{\text{Add}} -.001\times 2^0 \xrightarrow{\text{Normalize}} -.1000\times 2^{-2_{10}} \xrightarrow[\text{to binary}]{\text{Convert exponent}}$$

$$11000\times 2^{110_2} \xleftarrow[\text{dot}]{\text{Remove}} 1.1000\times 2^{110_2} \xleftarrow[\text{sign bit}]{\text{Append}} -.1000\times 2^{-10_2}$$

$\therefore \quad -0.125_{10} = 1100\ 0110_2$

In various digital systems or computers, a variety of formats are employed for floating point representation. The IEEE (Institute of Electrical and Electronic Engineers) has proposed a standard format for floating point representation, which is widely followed in digital computers. The IEEE-754 standard format for 32-bit single precision floating point number is shown in fig 8.1.



S = 1-bit field for sign of number.
E = 8-bit field for exponent.
M = 23-bit field for mantissa.

***Fig 8.1 :** IEEE-754 format for 32-bit floating point number.*

The floating point number, N shown in fig 8.1 can be interpreted as follows.

<u>When E = 1 to 254</u>

$$N = (-1)^S \times 1.M \times 2^{E-127} \qquad .....(8.11)$$

<u>When E = 0</u>

If, M = 0, then $N = (-1)^S \times 0$

If, $M \neq 0$, then $N = (-1)^S \times 0.M \times 2^{-126}$

<u>When E = 255</u>

If, M = 0, then $N = (-1)^S \times \infty$

If, $M \neq 0$, then N is not a number.

The range of the decimal numbers that can be represented by 32-bit IEEE-754 format is from $\pm[2^{-23} \times 2^{-126}]$ to $\pm[(2-2^{-23}) \times 2^{127}]$ i.e., from $\pm 1.4 \times 10^{-45}$ to $\pm 3.40 \times 10^{38}$.

Example 8.4

Convert $+25_{10}$ and -25_{10} to 32-bit IEEE-754 format of binary and verify the result by converting the binary to decimal.

Solution

<u>Decimal to IEEE-754 binary format conversion</u>

$$+25_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +11001_2 \xrightarrow[\text{in 1.M format}]{\text{Represent}} 1.1001_2\times 2^{4_{10}} \xrightarrow[\text{to E-127 format}]{\text{Convert exponent}} 1.1001_2\times 2^{131_{10}-127_{10}}$$

The number, N in IEEE-754 format is,

$$N = (-1)^S \times 1.M \times 2^{E-127}$$ Using equation (8.11).

$$\therefore \quad +25_{10} = (-1)^0 \times 1.1001_2 \times 2^{131_{10}-127_{10}}$$

$$-25_{10} = (-1)^1 \times 1.1001_2 \times 2^{131_{10}-127_{10}}$$

$$\therefore \quad 1.M = 1.1001 \xrightarrow[\text{part to 23-bits}]{\text{Convert fraction}} 1.1001\ 0000\ 0000\ 0000\ 0000\ 000$$

$$E = 131_{10} \xrightarrow[\text{to binary}]{\text{Convert}} 1000\ 0011_2$$

$$\therefore +25_{10} = \underset{\substack{\text{1-bit}\\ \text{sign}}}{0}\ \underbrace{1000\ 0011}_{\substack{\text{8-bit}\\ \text{exponent}}}\ \underbrace{1001\ 0000\ 0000\ 0000\ 0000\ 000}_{\substack{\text{23-bit}\\ \text{mantissa}}}$$

$$-25_{10} = \underset{\substack{\text{1-bit}\\ \text{sign}}}{1}\ \underbrace{1000\ 0011}_{\substack{\text{8-bit}\\ \text{exponent}}}\ \underbrace{1001\ 0000\ 0000\ 0000\ 0000\ 000}_{\substack{\text{23-bit}\\ \text{mantissa}}}$$

$$\therefore \quad +25_{10} = 0100\ 0001\ \ 1100\ \ 1000\ \ 0000\ \ 0000\ \ 0000\ \ 0000_2$$

$$-25_{10} = 1100\ \ 0001\ \ 1100\ \ 1000\ \ 0000\ \ 0000\ \ 0000\ \ 0000_2$$

IEEE - 754 binary to decimal conversion

$$0100\ \ 0001\ \ 1100\ \ 1000\ \ 0000\ \ 0000\ \ 0000\ 0000_2$$

$$\Downarrow$$

$$\underset{S}{0}\ \underbrace{1000\ \ 0011}_{E}\ \underbrace{1001\ 0000\ \ 0000\ \ 0000\ \ 0000\ \ 000}_{M}$$

$$\therefore \quad 0100\ \ 0001\ 1100\ 1000\ \ 0000\ \ 0000\ \ 0000\ \ 0000_2 = (-1)^0 \times 2^{1000\ 0011_2 - 127_{10}} \times 1.1001$$

$$= +2^{131_{10}-127_{10}} \times 1.1001$$

$$= +2^4 \times 1.1001 = +11001_2$$

$$+11001_2 = +(1\times 2^4 + 1\times 2^3 + 0\times 2^2 + 0\times 2^1 + 1\times 2^0) = +(16+8+0+0+1) = +25_{10}$$

$$1100\ \ 0001\ 1100\ 1000\ \ 0000\ \ 0000\ \ 0000\ \ 0000_2$$

$$\Downarrow$$

$$\underset{S}{1}\ \underbrace{1000\ \ 0011}_{E}\ \underbrace{1001\ 0000\ \ 0000\ \ 0000\ \ 0000\ \ 000}_{M}$$

$$\therefore \quad 1100\ \ 0001\ 1100\ 1000\ \ 0000\ \ 0000\ \ 0000\ \ 0000_2 = (-1)^1 \times 2^{1000\ 0011_2 - 127_{10}} \times 1.1001$$

$$= -2^{131_{10}-127_{10}} \times 1.1001$$

$$= -2^4 \times 1.1001 = -11001_2$$

$$-11001_2 = -(1\times 2^4 + 1\times 2^3 + 0\times 2^2 + 0\times 2^1 + 1\times 2^0) = -(16+8+0+0+1) = -25_{10}$$

$$\therefore \quad 0100\ \ 0001\ \ 1100\ \ 1000\ \ 0000\ \ 0000\ \ 0000\ \ 0000_2 = +25_{10}$$

$$1100\ \ 0001\ \ 1100\ \ 1000\ \ 0000\ \ 0000\ \ 0000\ \ 0000_2 = -25_{10}$$

8.3 Types of Arithmetic in Digital Systems

The types of arithmetic in digital systems generally depends on the representation of the binary numbers. Hence the arithmetic can also be classified into two broad classes: fixed point arithmetic and floating point arithmetic. The fixed point number system has three types of representation for negative numbers. Hence we have three types of fixed point arithmetic. They are sign-magnitude arithmetic, one's complement arithmetic and two's complement arithmetic. The sign-magnitude arithmetic is generally avoided in general purpose digital systems due to inherent difficulty in handling negative numbers during additions.

The fundamental arithmetic operation in digital system is addition. The subtraction is treated as addition of positive and negative numbers. Generally the multiplication is performed in terms of successive addition and division is performed in terms of successive subtraction except in case of special purpose hardware.

8.3.1 One's Complement Addition

In one's complement addition the numbers are represented in one's complement format and then the addition is performed. The carry generated in addition is added to the least significant digit (LSD) to get the actual sum. If the carry is zero after addition then the sum is negative and if the carry is one then the sum is positive. Two examples of one's complement addition one with positive sum and the other with negative sum are presented here.

Example 8.5

Add +0.375 and –0.625 by one's complement addition.

Solution

The one's complement representation of the given numbers are shown below.

$$+.375_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.011_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.011_2 \xrightarrow[\text{dot}]{\text{Remove}} 0011_2$$

$$-.625_{10} \xrightarrow[\text{binary}]{\text{Convert to}} -.101_2 \xrightarrow[\text{bit}]{\text{Add sign}} 1.101_2 \xrightarrow[\text{fraction part}]{\text{Complement}} 1.010_2 \xrightarrow[\text{dot}]{\text{Remove}} 1010_2$$

$$\begin{array}{r} 0011 \\ +\ 1010 \\ \hline \end{array}$$

$$\text{Carry} \rightarrow \boxed{0}\,\boxed{1101} \leftarrow \text{sum}$$

Decimal to binary conversion

$$\begin{array}{ll} .375 & .625 \\ \times\ 2 & \times\ 2 \\ \boxed{0}.750 & \boxed{1}.250 \\ \times\ 2 & \times\ 2 \\ \boxed{1}.500 & \boxed{0}.500 \\ \times\ 2 & \times\ 2 \\ \boxed{1}.000 & \boxed{1}.000 \\ .011_2 & .101_2 \end{array}$$

Since the carry is zero the sum is negative. The sum can be converted to decimal as shown below.

$$1101_2 \xrightarrow[\text{Sign bit}]{\text{Extract}} -.101_2 \xrightarrow[\text{fraction part}]{\text{Complement}} -.010_2 \xrightarrow[\text{Decimal}]{\text{Convert to}} -.25_{10}$$

Binary to decimal conversion

$$.010_2 = (0 \times 2^{-1}) + (1 \times 2^{-2}) + (0 \times 2^{-3})$$
$$= 0.25_{10}$$

In summary,

$$+0.375_{10} \Rightarrow 0011_2$$

$$-0.625_{10} \Rightarrow 1010_2$$

$$(+0.375_{10}) + (-0.625_{10}) \Rightarrow \boxed{1101_2} \Rightarrow -.25_{10}$$

Example 8.6

Add +0.625 and –0.375 by one's complement addition.

Solution

The one's complement representation of the given numbers are shown below.

$$+.625_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.101_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.101_2 \xrightarrow[\text{dot}]{\text{Remove}} 0101_2$$

$$-.375_{10} \xrightarrow[\text{binary}]{\text{Convert to}} -.011_2 \xrightarrow[\text{bit}]{\text{Add sign}} 1.011_2 \xrightarrow[\text{fraction part}]{\text{Complement}} 1.100_2 \xrightarrow[\text{dot}]{\text{Remove}} 1100_2$$

```
              0101
            + 1100
Carry ® [1] [0001] ¬  sum
               ® 1   (add carry to LSD)
            [0010] ¬ Final sum
```

Refer example 8.5 for decimal to binary conversion of $.375_{10}$ and $.625_{10}$.

Since the carry is one the sum is positive. The final sum can be obtained by adding the carry to least significant digit (LSD) of the sum. The final sum can be converted to decimal as shown below.

$$0010_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.010_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.25_{10}$$

In summary,

$+.625_{10}$	Þ	0101_2		
$-.375_{10}$	Þ	1100_2		
$(+.625_{10}) + (-.375_{10})$	Þ	0010_2	Þ	$+.25_{10}$

Refer example 8.5 for binary to decimal conversion of $.010_2$.

8.3.2 Two's Complement Addition

In two's complement addition the numbers are represented in two's complement format and then the addition is performed. The carry generated in addition is discarded. If the carry is zero after addition then the sum is negative and if the carry is one then the sum is positive. Two examples of two's complement addition one with positive sum and the other with negative sum are presented here.

Example 8.7

Add +0.375 and –0.625 by two's complement addition.

Solution

The two's complement representation of the given numbers are shown below.

$$+.375_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.011_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.011_2 \xrightarrow[\text{dot}]{\text{Remove}} 0011_2$$

$$-.625_{10} \xrightarrow[\text{binary}]{\text{Convert to}} -.101_2 \xrightarrow[\text{bit}]{\text{Add sign}} 1.101_2 \xrightarrow[\text{fraction part}]{\text{Complement}} 1.010_2 \xrightarrow[\text{to LSD}]{\text{Add one}} 1.011_2 \xrightarrow[\text{dot}]{\text{Remove}} 1011_2$$

```
             0011₂
            +1011₂
Carry ® [0] [1110] ¬  sum
```

Refer example 8.5 for decimal to binary conversion of $.375_{10}$ and $.625_{10}$.

Since the carry is zero the sum is negative. The sum can be converted to decimal as shown below.

$$1110_2 \xrightarrow[\text{sign bit}]{\text{Extract}} -.110_2 \xrightarrow[\text{fraction part}]{\text{Complement}} -.001_2 \xrightarrow[\text{to LSD}]{\text{Add one}} -.010_2 \xrightarrow[\text{decimal}]{\text{Convert to}} -.25_{10}$$

In summary,

$+.375_{10}$ Þ 0011_2

$-.625_{10}$ Þ 1011_2

$(+.375_{10}) + (-.625)_{10})$ Þ 1110_2 Þ $-.25_{10}$

Refer example 8.5 for binary to decimal conversion of $.010_2$.

Example 8.8

Add $+0.625_{10}$ and -0.375_{10} by two's complement addition.

Solution

The two's complement representation of the given numbers are shown below.

$$+0.625_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.101_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.101_2 \xrightarrow[\text{dot}]{\text{Remove}} 0101_2$$

$$-0.375_{10} \xrightarrow[\text{binary}]{\text{Convert to}} -.011_2 \xrightarrow[\text{bit}]{\text{Add sign}} 1.011_2 \xrightarrow[\text{fraction part}]{\text{Complement}} 1.100_2 \xrightarrow[\text{to LSD}]{\text{Add one}} 1.101_2 \xrightarrow[\text{dot}]{\text{Remove}} 1101_2$$

0101_2

$+1101_2$

Carry ® $\boxed{1}\boxed{0010}$ ¬ sum

Refer example 8.5 for decimal to binary conversion of $.375_{10}$ and $.625_{10}$.

Since the carry is one the sum is positive. The carry is discarded in two's complement addition. The sum can be converted to decimal as shown below.

$$0010_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.010_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.25_{10}$$

In summary,

$+.625_{10}$ Þ 0101_2

$-.375_{10}$ Þ 1101_2

$(+.625_{10}) + (-.375_{10})$ Þ 0010_2 Þ $+.25_{10}$

Refer example 8.5 for binary to decimal conversion of $.010_2$.

8.3.3 Floating Point Addition

For performing floating point addition the numbers are represented in the desired floating point format. The addition can be performed only when the exponents of both the numbers are equal. Hence the exponent of the smaller number is changed to equal the exponent of the larger number and then addition is performed. In floating point addition if the sum is in the unnormalized form then it has to be normalized to represent in proper (or correct) floating point format.

Example 8.9

Add $+5_{10}$ and $+0.25_{10}$ by floating point addition. Choose 10-bit floating point format with 7-bits for mantissa and 3-bits for exponent.

Solution

Let us convert the given numbers to floating point format. For simplicity we can use sign-magnitude representation for exponent and mantissa. The leftmost bit in mantissa and exponent is used to represent the sign.

$$+5_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +101_2 \xrightarrow[\text{exponent}]{\text{Add}} +101.000 \times 2^0 \xrightarrow{\text{Normalize}} +.101000 \times 2^{+3_{10}} \xrightarrow[\text{to binary}]{\text{Convert exponent}}$$

$$0101000 \times 2^{011_2} \xleftarrow[\text{dot}]{\text{Remove}} 0.101000 \times 2^{011_2} \xleftarrow[\text{sign bit}]{\text{Append}} +.101000 \times 2^{+11_2}$$

$$+.25_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.01_2 \xrightarrow[\text{exponent}]{\text{Add}} +.0100000 \times 2^0 \xrightarrow{\text{Normalize}} +.100000 \times 2^{-1_{10}} \xrightarrow[\text{to binary}]{\text{Convert exponent}}$$

$$0100000 \times 2^{101_2} \xleftarrow[\text{dot}]{\text{Remove}} 0.100000 \times 2^{101_2} \xleftarrow[\text{sign bit}]{\text{Append}} +.100000 \times 2^{-01_2}$$

$\therefore \quad +5_{10} = 01\ 0100\ 0011_2$

$+0.25_{10} = 01\ 0000\ 0101_2$

Since the exponents of +5 and +0.25, are not equal, the exponent of +0.25 is unnormalized to make its exponent equal to that of +5.

$$\therefore \ +.25_{10} = 0.100000 \times 2^{101_2} = 0.100000 \times 2^{-1} \xrightarrow{\text{unnormalizing}} = 0.000010 \times 2^3 = 0.000010 \times 2^{011_2}$$

Now the unnormalized mantissa of $+0.25_{10}$ is added to the mantissa of $+5_{10}$ to get the sum of mantissa. The exponent of the sum is same as that of the exponents of the numbers added.

$$\begin{array}{ll} +5_{10} & 0101000 \times 2^{011} \\ +.25_{10} & \underline{0000010 \times 2^{011}} \\ (5_{10} + .25_{10}) & \underline{0101010 \times 2^{011}} \quad \Rightarrow \quad +5.25_{10} \end{array}$$

$$\therefore \quad 5_{10} + .25_{10} = 0.101010 \times 2^{011} = 0101010\ 011_2$$

The sum in floating point format can be converted to decimal as shown below.

$$0101010 \times 2^{011} \xrightarrow[\text{sign bit}]{\text{Remove}} +.101010 \times 2^{+11_2} \xrightarrow[\text{to decimal}]{\text{Convert exponent}} +.101010 \times 2^{+3_{10}} = +101.010_2$$

$$+101.010 = +(1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3})$$

$$= +(4 + 0 + 1 + 0 + .25 + 0) = +5.25_{10}$$

8.3.4 Floating Point Multiplication

For performing floating point multiplication the numbers are represented in the desired floating point format. The product is obtained by multiplying the mantissa and adding the exponents. The sign bits of mantissa should be added separately to determine the sign of product of mantissa. For multiplication of two floating point numbers the exponents need not be same. In floating point multiplication if the product is in unnormalized form then it has to be normalized to represent the product in proper (or correct) floating point format.

Example 8.10

Multiply Add $+5_{10}$ and $+0.25_{10}$ by floating point multiplication. Choose 10-bit floating point format with 7-bits for mantissa and 3-bits for exponent.

Solution

Let us convert $+5_{10}$ and $+0.25_{10}$ to floating point format.

$\therefore \ +5_{10} = 0101000 \times 2^3 = 0.101000 \times 2^3$

$+.25_{10} = 0100000 \times 2^{-1} = 0.100000 \times 2^{-1}$

Using result of example 8.9.

The floating point multiplication is performed as shown below.

$$5_{10} \times .25_{10} = (0 + 0) \ . \ (101000 \times 100000) \times 2^{3+(-1)} = 0.010100 \times 2^2$$

$$0.010100 \times 2^2 \xrightarrow{\text{Normalizing}} 0.10100 \times 2^1 \xrightarrow[\text{to binary}]{\text{Convert exponent}} 0.10100 \times 2^{001_2} \xrightarrow[\text{dot}]{\text{Remove}} 010100 \times 2^{001_2}$$

$$\therefore \ 5_{10} \times .25_{10} = 010100 \times 2^{001} = 010100001_2$$

The product in floating point format can be converted to decimal as shown below.

$$010100 \times 2^{001} \xrightarrow[\text{sign bit}]{\text{Remove}} +.10100 \times 2^{+1_2} = +1.01_2$$

$$+1.01_2 = +(1\times 2^0 + 0\times 2^{-1} + 1\times 2^{-2}) = 1 + 0 + .25 = 1.25_{10}$$

8.3.5 Comparison of Fixed Point and Floating Point Arithmetic

The floating point number system can accommodate a large range of numbers and so in floating point arithmetic higher accuracy in processing can be achieved. But the hardware implementation for floating point arithmetic is costlier and the speed of processing is low due to double calculations i.e., separate calculation for mantissa and exponent. Therefore, the floating point arithmetic is preferred for non-real time applications on general purpose digital systems (computers) in which the cost and speed are not significant.

In floating point arithmetic the truncation and rounding errors occur both for multiplication and addition, whereas in fixed point arithmetic such errors occur only for multiplication.

The addition in fixed point arithmetic leads to overflow, but the overflow is a rare phenomena in floating point arithmetic due to larger dynamic range. In general, for real time applications in DSP fixed point arithmetic is preferred due to the reduced cost of the hardware and high speed of processing.

8.4 Quantization by Truncation and Rounding

In fixed point or floating point arithmetic the size of the result of an operation (sum or product) may be exceeding the size of binary used in the number system. In such cases the low order bits has to be eliminated in order to store the result. The two methods of eliminating these low order bits are truncation and rounding. This process is also referred to as ***quantization*** via truncation and rounding. The effect of rounding and truncation is to introduce an error whose value depends on the number of bits eliminated. The characteristics of the errors introduced through either truncation or rounding depend on the type of number representation.

8.4.1 Quantization Steps

The decimal numbers that are encountered as filter coefficients, sum, product, etc., in DSP applications will usually lie in the range of –1 to +1. When "B" bit binary is selected to represent the decimal numbers, then 2^B binary codes are possible. Hence the range of decimal numbers has to be divided into 2^B steps and each step is represented by a binary code. Each step of decimal number is also called ***quantization step***.

$$\therefore \text{ Quantization step size, } q = \frac{R}{2^B} = \frac{1-(-1)}{2^B} = \frac{2}{2^B} = \frac{1}{2^B - 2^{-1}}$$

$$= \frac{1}{2^{B-1}} = \frac{1}{2^b} = 2^{-b} \qquad .....(8.12)$$

Where, R = Range of decimal number

B = Size of binary including sign bit

b = B – 1 = Size of binary excluding sign bit

Therefore, the quantization steps of decimal numbers that lies in the range –1 to +1 are,

$$-1,\ ......,\ -3\times 2^{-b},\ -2\times 2^{-b},\ -1\times 2^{-b},\ 0\times 2^{-b},\ 1\times 2^{-b},\ 2\times 2^{-b},\ 3\times 2^{-b},\ ......,\ +1$$

The quantization steps of decimal numbers that lies in the range –1 to +1 for b = 2 and b = 3 are listed in table 8.3 and 8.4 respectively.

Table 8.3 : Quantization Steps for B = 3 and b = B – 1 = 2

Binary Code	Quantization Steps		
	Sign-magnitude	**One's complement**	**Two's complement**
000	$+0\times 2^{-2}=+0\times\frac{1}{4}=+0$	$+0\times 2^{-2}=+0\times\frac{1}{4}=+0$	$+0\times 2^{-2}=+0\times\frac{1}{4}=+0$
001	$+1\times 2^{-2}=+1\times\frac{1}{4}=+0.25$	$+1\times 2^{-2}=+1\times\frac{1}{4}=+0.25$	$+1\times 2^{-2}=+1\times\frac{1}{4}=+0.25$
010	$+2\times 2^{-2}=+2\times\frac{1}{4}=+0.50$	$+2\times 2^{-2}=+2\times\frac{1}{4}=+0.50$	$+2\times 2^{-2}=+2\times\frac{1}{4}=+0.50$
011	$+3\times 2^{-2}=+3\times\frac{1}{4}=+0.75$	$+3\times 2^{-2}=+3\times\frac{1}{4}=+0.75$	$+3\times 2^{-2}=+3\times\frac{1}{4}=+0.75$
100	$-0\times 2^{-2}=-0\times\frac{1}{4}=-0$	$-3\times 2^{-2}=-3\times\frac{1}{4}=-0.75$	$-4\times 2^{-2}=-4\times\frac{1}{4}=-1.00$
101	$-1\times 2^{-2}=-1\times\frac{1}{4}=-0.25$	$-2\times 2^{-2}=-2\times\frac{1}{4}=-0.50$	$-3\times 2^{-2}=-3\times\frac{1}{4}=-0.75$
110	$-2\times 2^{-2}=-2\times\frac{1}{4}=-0.50$	$-1\times 2^{-2}=-1\times\frac{1}{4}=-0.25$	$-2\times 2^{-2}=-2\times\frac{1}{4}=-0.50$
111	$-3\times 2^{-2}=-3\times\frac{1}{4}=-0.75$	$-0\times 2^{-2}=-0\times\frac{1}{4}=-0$	$-1\times 2^{-2}=-1\times\frac{1}{4}=-0.25$

Table 8.4 : Quantization Steps for B = 4 and b = B – 1 = 3

Binary Code	Quantization Steps		
	Sign-magnitude	**One's complement**	**Two's complement**
0000	$+0\times 2^{-3}=+0\times\frac{1}{8}=+0$	$+0\times 2^{-3}=+0\times\frac{1}{8}=+0$	$+0\times 2^{-3}=+0\times\frac{1}{8}=+0$
0001	$+1\times 2^{-3}=+1\times\frac{1}{8}=+0.125$	$+1\times 2^{-3}=+1\times\frac{1}{8}=+0.125$	$+1\times 2^{-3}=+1\times\frac{1}{8}=+0.125$
0010	$+2\times 2^{-3}=+2\times\frac{1}{8}=+0.250$	$+2\times 2^{-3}=+2\times\frac{1}{8}=+0.250$	$+2\times 2^{-3}=+2\times\frac{1}{8}=+0.250$
0011	$+3\times 2^{-3}=+3\times\frac{1}{8}=+0.375$	$+3\times 2^{-3}=+3\times\frac{1}{8}=+0.375$	$+3\times 2^{-2}=+3\times\frac{1}{4}=+0.75$
0100	$+4\times 2^{-3}=+4\times\frac{1}{8}=+0.500$	$+4\times 2^{-3}=+4\times\frac{1}{8}=+0.500$	$+4\times 2^{-3}=+4\times\frac{1}{8}=+0.500$
0101	$+5\times 2^{-3}=+5\times\frac{1}{8}=+0.625$	$+5\times 2^{-3}=+5\times\frac{1}{8}=+0.625$	$+5\times 2^{-3}=+5\times\frac{1}{8}=+0.625$
0110	$+6\times 2^{-3}=+6\times\frac{1}{8}=+0.750$	$+6\times 2^{-3}=+6\times\frac{1}{8}=+0.750$	$+6\times 2^{-3}=+6\times\frac{1}{8}=+0.750$
0111	$+7\times 2^{-3}=+7\times\frac{1}{8}=+0.875$	$+7\times 2^{-3}=+7\times\frac{1}{8}=+0.875$	$+7\times 2^{-3}=+7\times\frac{1}{8}=+0.875$
1000	$-0\times 2^{-3}=-0\times\frac{1}{8}=-0$	$-7\times 2^{-3}=-7\times\frac{1}{8}=-0.875$	$-8\times 2^{-3}=-8\times\frac{1}{8}=-1.000$
1001	$-1\times 2^{-3}=-1\times\frac{1}{8}=-0.125$	$-6\times 2^{-3}=-6\times\frac{1}{8}=-0.750$	$-7\times 2^{-3}=-7\times\frac{1}{8}=-0.875$
1010	$-2\times 2^{-3}=-2\times\frac{1}{8}=-0.250$	$-5\times 2^{-3}=-5\times\frac{1}{8}=-0.625$	$-6\times 2^{-3}=-6\times\frac{1}{8}=-0.750$
1011	$-3\times 2^{-3}=-3\times\frac{1}{8}=-0.375$	$-4\times 2^{-3}=-4\times\frac{1}{8}=-0.500$	$-5\times 2^{-3}=-5\times\frac{1}{8}=-0.625$
1100	$-4\times 2^{-3}=-4\times\frac{1}{8}=-0.500$	$-3\times 2^{-3}=-3\times\frac{1}{8}=-0.375$	$-4\times 2^{-3}=-4\times\frac{1}{8}=-0.500$
1101	$-5\times 2^{-3}=-5\times\frac{1}{8}=-0.625$	$-2\times 2^{-3}=-2\times\frac{1}{8}=-0.250$	$-3\times 2^{-3}=-3\times\frac{1}{8}=-0.375$
1110	$-6\times 2^{-3}=-6\times\frac{1}{8}=-0.750$	$-1\times 2^{-3}=-1\times\frac{1}{8}=-0.125$	$-2\times 2^{-3}=-2\times\frac{1}{8}=-0.250$
1111	$-7\times 2^{-3}=-7\times\frac{1}{8}=-0.875$	$-0\times 2^{-3}=-0\times\frac{1}{8}=-0$	$-1\times 2^{-3}=-1\times\frac{1}{8}=-0.125$

8.4.2 Truncation

The ***truncation*** is the process of reducing the size of binary number (or reducing the number of bits in a binary number) by discarding all bits less significant than the least significant bit that is retained. In the truncation of a binary number to b bits, all the less significant bits beyond b^{th} bit are discarded.

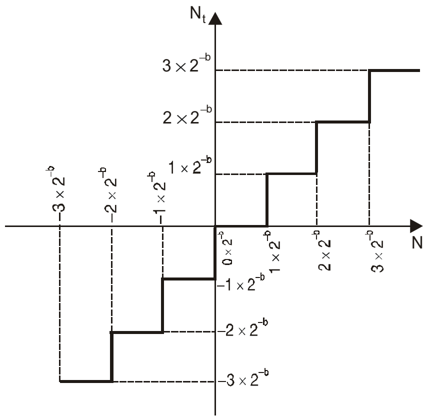


Fig 8.2a : *Truncation characteristics of two's complement quantizer.*

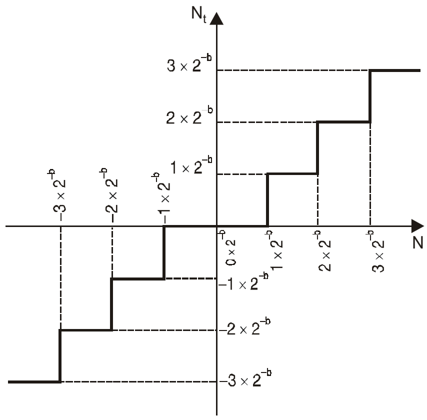


Fig 8.2b : *Truncation characteristics of sign-magnitude or one's complement quantizer.*

Fig 8.2 : *Input-output characteristics of quantizer used for truncation.*

The input-output characteristics of the quantizer used for truncation is shown in fig 8.2. In fig 8.2. the quantization steps are marked on the y-axis. The range of unquantized numbers are marked on x-axis. The characteristics shown in fig 8.2 can be interpreted as follows.

1. Any positive unquantized number in the range, $0 \le N < (1 \times 2^{-b})$, will be assigned the quantization step, 0×2^{-b}.
2. Any positive unquantized number in the range, $(1 \times 2^{-b}) \le N < (2 \times 2^{-b})$, will be assigned the quantization step, 1×2^{-b} and so on.
3. In sign-magnitude and one's complement quantizer, any negative unquantized number in the range, $(-1 \times 2^{-b}) < N \le 0$, will be assigned the quantization step, 0×2^{-b}.
4. In sign-magnitude and one's complement quantizer, any negative unquantized number in the range, $(-2 \times 2^{-b}) < N \le (-1 \times 2^{-b})$, will be assigned the quantization step, -1×2^{-b} and so on.
5. In two's complement quantizer, any negative unquantized number in the range, $(-1 \times 2^{-b}) \le N < 0$, will be assigned the quantization step, -1×2^{-b}.
6. In two's complement quantizer, any negative unquantized number in the range, $(-2 \times 2^{-b}) \le N < (-1 \times 2^{-b})$, will be assigned the quantization step, -2×2^{-b} and so on.

In fixed point number system there are three different types of number representation. The effect of truncation on positive numbers are same in all the three representations (because the format for positive number is same in all the three representations). The error due to truncation of negative number depends on the type of representation of the number.

Let, N = Unquantized fixed point binary number.

N_t = Fixed point binary number quantized by truncation.

The quantization error in fixed point number due to truncation is defined as,

$$\text{Truncation error, } e_t = N_t - N \qquad .....(8.13)$$

Case i : Positive number

The unquantized positive number in the range,

$$(1\times 2^{-b}) \leq N < (2\times 2^{-b}) \xrightarrow[\text{truncated to}]{\text{is}} N_t = 1\times 2^{-b}$$

$$\therefore \text{ Minimum error } = 1\times 2^{-b} - 2\times 2^{-b} = -2^{-b}$$

$$\text{Maximum error } = 1\times 2^{-b} - 1\times 2^{-b} = 0$$

$$\therefore \text{ Range of error } = \ -2^{-b} < e \leq 0$$

Case ii : Sign-magnitude and one's complement negative number

The unquantized negative number in the range,

$$(-2\times 2^{-b}) < N \leq (-1\times 2^{-b}) \xrightarrow[\text{truncated to}]{\text{is}} N_t = -1\times 2^{-b}$$

$$\therefore \text{ Minimum error } = -1\times 2^{-b} - (-1\times 2^{-b}) = 0$$

$$\text{Maximum error } = -1\times 2^{-b} - (-2\times 2^{-b}) = 2^{-b}$$

$$\therefore \text{ Range of error } = \ 0 \leq e < 2^{-b}$$

Case iii : Two's complement negative number

The unquantized negative number in the range,

$$(-1\times 2^{-b}) < N \leq (-2\times 2^{-b}) \xrightarrow[\text{truncated to}]{\text{is}} N_t = -2\times 2^{-b}$$

$$\therefore \text{ Minimum error } = -2\times 2^{-b} - (-1\times 2^{-b}) = -2^{-b}$$

$$\text{Maximum error } = -2\times 2^{-b} - (-2\times 2^{-b}) = 0$$

$$\therefore \text{ Range of error } = \ -2^{-b} < e \leq 0$$

The range of errors for different types of number representation are summarized in table 8.5. The truncation of a positive number results in a number that is smaller than the unquantized number. Hence the truncation error is always negative when positive number is truncated.

Table 8.5 : Range of Errors in Truncation of Fixed Point Numbers

Number and its representation	**Range of error when truncated to b bits**
Positive number	$-2^{-b} < e \leq 0$
Sign - magnitude negative number	$0 \leq e < 2^{-b}$
One's complement negative number	$0 \leq e < 2^{-b}$
Two's complement negative number	$-2^{-b} < e \leq 0$

For the truncation of negative numbers represented in sign magnitude and one's complement format the error is always positive because the truncation basically reduces the magnitude of the numbers. In the two's complement representation, the effect of truncation on a negative number is to increase the magnitude of the negative number and so the truncation error is always negative.

In floating point representation the mantissa of the number alone is truncated. The truncation error in a floating point number is proportional to the number being quantized.

Let, N_f = Unquantized floating point binary number.

N_{tf} = Truncated floating point binary number.

Now, $N_{tf} = N_f + N_f e_t$ ..... (8.14)

where, e_t is the ***relative error*** due to truncation of a floating point number.

$$\therefore \text{ Relative error due to truncation, } \varepsilon_t = \frac{N_{tf} - N_f}{N_f} \quad \text{..... (8.15)}$$

The range of errors for different types of representation for mantissa of floating point numbers are shown in table 8.6.

Table 8.6 : Range of Errors in Truncation of Floating Point Numbers

Type of representation for mantissa	Range of error when mantissa is truncated to b bits
Two's complement positive mantissa	$-2 \times 2^{-b} < \varepsilon_t \le 0$
Two's complement negative mantissa	$0 \le e_t < 2^{-b} \times 2$
One's complement positive and negative mantissa	$-2 \times 2^{-b} < \varepsilon_t \le 0$
Sign-magnitude positive and negative mantissa	$-2 \times 2^{-b} < \varepsilon_t \le 0$

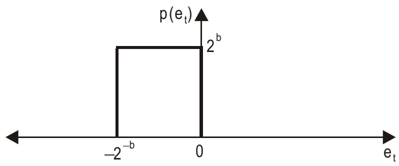


Fig 8.3a : Fixed point-two's complement.

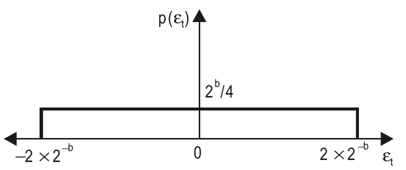


Fig 8.3b : Floating point-when mantissa in two's complement.

In truncation of binary number the range of error is known but the probability of obtaining an error within the range is not known. Hence it is assumed that the errors occur uniformly throughout the interval and with this assumption the probability density functions for truncation of fixed point and floating point numbers are shown in fig 8.3.

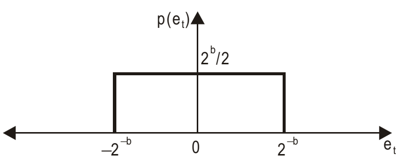


Fig 8.3c : Fixed point-one's complement or sign-magnitude.

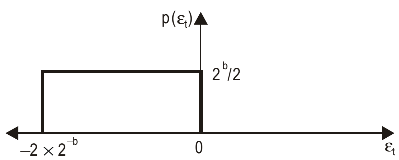


Fig 8.3d : Floating point-when mantissa in one's complement or in sign magnitude.

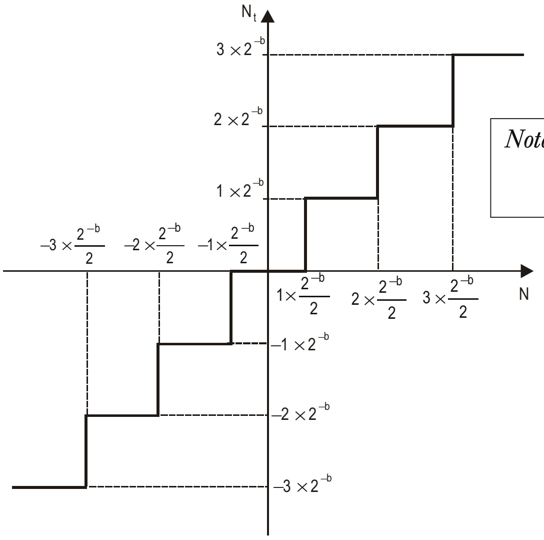
Fig 8.3 : Quantization noise probability density functions for truncation.

8.4.3 Rounding

Rounding is the process of reducing the size of a binary number to finite word size of b-bits such that the rounded b-bit number is closest to the original unquantized number. The rounding process consists of truncation and addition. In rounding of a number to b-bits, first the unquantized number is truncated to b-bits by retaining the most significant b-bits. Then a zero or one is added to the least significant bit of the truncated number depending on the bit that is next to the least significant bit that is retained.

If the bit next to the least significant bit that is retained is zero then zero is added to the least significant bit of the truncated number. If the bit next to the least significant bit that is retained is one then one is added to the least significant bit of the truncated number. (Here adding one is called rounding up).

The input-output characteristics of the quantizer used for rounding is shown in fig 8.4. In fig 8.4. the quantization steps are marked on y-axis. The range of unquantized numbers are marked on x-axis. The characteristics shown in fig 8.4 can be interpreted as follows.



Note : Quantization step size = 2^{-b}
Half of quantization step size = $\frac{2^{-b}}{2}$

Fig 8.4 : *Input-output characteristics of quantizer used for rounding.*

1. Any positive unquantized number in the range, $\left(1 \times \frac{2^{-b}}{2}\right) \leq N < \left(2 \times \frac{2^{-b}}{2}\right)$, will be assigned the quantization step, 1×2^{-b}.
2. Any positive unquantized number in the range, $\left(2 \times \frac{2^{-b}}{2}\right) \leq N < \left(3 \times \frac{2^{-b}}{2}\right)$, will be assigned the quantization step, 2×2^{-b}, and so on.
3. Any negative unquantized number in the range, $\left(-2 \times \frac{2^{-b}}{2}\right) < N \leq \left(-1 \times \frac{2^{-b}}{2}\right)$, will be assigned the quantization step, -1×2^{-b}.
4. Any negative unquantized number in the range, $\left(-3 \times \frac{2^{-b}}{2}\right) < N \leq \left(-2 \times \frac{2^{-b}}{2}\right)$, will be assigned the quantization step, -2×2^{-b}, and so on.

Let, N = Unquantized fixed point binary number.

N_r = Fixed point binary number quantized by rounding.

The quantization error in fixed point number due to rounding is defined as,

Rounding error, $e_r = N_r - N$ .....(8.16)

The range of error due to rounding for all the three formats (i.e., one's complement, two's complement and sign-magnitude) of fixed point representation is same.

In fixed point representation the range of error made by rounding a number to b bits is,

$$\boxed{-\frac{2^{-b}}{2} \le e_r \le \frac{2^{-b}}{2}}$$

Let, N_f = Unquantized floating point binary number.

N_{rf} = Rounded floating point binary number.

Now, $N_{rf} = N_f + N_f \varepsilon_r$ ..... (8.17)

where ε_r is the relative error due to rounding of a floating point number.

$\therefore$ Relative error due to rounding, $\varepsilon_r = \dfrac{N_{rf} - N_f}{N_f}$ ..... (8.18)

The range of error due to rounding for all the three formats (i.e., one's complement, two's complement and sign-magnitude) of the mantissa is same. In floating point representation the range of error made by rounding a number to b-bits is given by, $-2^{-b} \le \varepsilon_r \le 2^{-b}$.

In rounding of binary number the range of error is known but the probability of obtaining an error within the range is not known. Hence it is assumed that the errors occur uniformly throughout the interval and with this assumption the probability density functions for rounding of fixed point and floating point numbers are shown in fig 8.5.

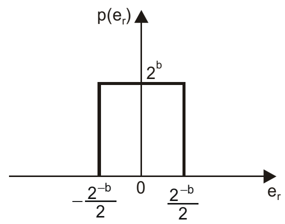


Fig 8.5a : *Rounding - fixed point.*

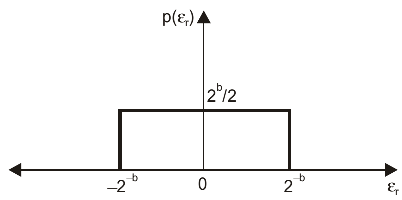


Fig 8.5b : *Rounding - floating point.*

Fig 8.5 : *Quantization noise probability density functions for rounding.*

8.5 Quantization of Input Data

For processing of analog signal using a digital system the analog signal has to be digitized by A/D (Analog to Digital) converter. The A/D converter consists of sampler and quantizer. The ***sampler*** will sample the value of analog signal at uniform intervals to produce a sequence of unquantized values of the signal. The ***quantizer*** will quantize the analog value and produce the corresponding binary codes. The process of assigning binary number to quantized analog value is also called ***coding***.

The two types of errors that are produced by A/D conversion process are quantization errors and saturation errors. The ***quantization error*** is due to representation of the sampled signal by a fixed number of digital levels (quantization levels). The ***saturation error*** occurs when the analog signal exceed the dynamic range of A/D converter.

In analog to digital conversion, when B-bits binary code (including sign bit) is selected, we can generate 2^B different binary numbers. If the range of analog signal to be quantized is R then the quantization step size q is given by,

$$\text{Quantization step size,}\ q = \frac{R}{2^B} = \frac{R}{2^{b+1}} \quad .....(8.19)$$

where, B = Size of binary including sign bit

b = B – 1 = Size of binary excluding sign bit.

Usually the analog signal is scaled such that the magnitude of quantized signal is less than or equal to one. In such case the range of analog signal to be quantized is –1 to +1, therefore R = 2.

Let, x(n) = Unquantized sample of the signal

and $x_q(n)$ = Quantized sample of the signal

Now the quantization error is defined as,

$$\text{Quantization error, } e(n) = x_q(n) - x(n) \quad .....(8.20)$$

In A/D converters the quantization can be performed by truncation or rounding. But the quantization by rounding is preferred in A/D converters due to zero mean value of quantization error and low variance when compared to truncation.

The quantization error for rounding will be in the range of –q/2 to +q/2 (Refer section 8.4.3 for the characteristics of quantizer with rounding). Also we assume that all errors are equiprobable and so the mean value of error is zero. The error due to rounding is treated as a random variable.

For a uniformly distributed random variable "x" in the interval, (x_1, x_2), the expected value (or mean value) and variance are given by,

$$\text{Expected value, } E\{x\} = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} x\,dx$$

$$\text{Variance, } \sigma^2 = E\{x^2\} - E^2\{x\}$$

If the random variable x is uniform in the interval (–c, c) then E{x} = 0 i.e., mean value is zero and so variance, $s^2 = E\{x^2\}$

Let, E{e} = Expected value (or mean value) of error signal.

$$\therefore\ E\{e\} = \frac{1}{\frac{q}{2} - \left(-\frac{q}{2}\right)}\int_{-q/2}^{+q/2} e\,de = \frac{1}{q}\left[\frac{e^2}{2}\right]_{-q/2}^{+q/2}$$

$$= \frac{1}{2q}\left[\left(\frac{q}{2}\right)^2 - \left(-\frac{q}{2}\right)^2\right] = 0 \quad .....(8.21)$$

$$\text{Variance of error signal,} \sigma_e^2 = E\{e^2\} - E^2\{e\} = E\{e^2\}$$

$$= \frac{1}{\frac{q}{2} - \left(-\frac{q}{2}\right)}\int_{-q/2}^{+q/2} e^2 de = \frac{1}{q}\left[\frac{e^3}{3}\right]_{-q/2}^{+q/2}$$

$$\therefore \text{ Variance of error signal}, \sigma_e^2 = \frac{1}{3q}\left[\left(\frac{q}{2}\right)^3 - \left(\frac{-q}{2}\right)^3\right] = \frac{1}{3q}\left[\frac{q^3}{8} + \frac{q^3}{8}\right]$$

$$= \frac{1}{3q} \times \frac{2q^3}{8} = \frac{q^2}{12} \qquad .....(8.22)$$

Using equation (8.19) in equation (8.22) we get,

$$\text{Variance of error signal}, \sigma_e^2 = \frac{1}{12}\left(\frac{R}{2^B}\right)^2 = \frac{R^2}{12} 2^{-2B} \qquad .....(8.23)$$

$$\text{When } R = 2,\ \sigma_e^2 = \frac{2^2}{12} 2^{-2B} = \frac{2^{-2B}}{3} \qquad .....(8.24)$$

where, B = size of binary including sign bit.

The variance of error signal is also called ***steady state noise power*** due to input quantization. From equation (8.23) we can say that the steady state noise power tends to zero as B tends to infinity. The value of B is infinite only if A/D converter has infinite precision, which is not practically possible.

Another important point to be noted here is that the analog signals are also corrputed by some form of noise. When a large number of bits are used to digitize such a signal, then the analog noise are well represented on the digitized signal. Hence we can say that by increasing the number of bits in A/D converter beyond a certain limit merely increases the accuracy by which an analog noise is represented. Therefore the word length of an A/D converter also depends on the type of signal to be converted.

Steady State Output Noise Variance (Power) Due to the Quantization Error Signal

The quantized input signal of a digital system can be represented as a sum of unquantized signal x(n) and error signal e(n) as shown in fig 8.6. [From equation (8.20) we get $x_q(n) = x(n) + e(n)$].

Fig 8.6a : *LTI system with unquantized input.*

Fig 8.6b : *LTI system with quantized input.*

Fig 8.6 : *Representation of input quantization noise in an LTI system.*

In fig 8.6, h(n) is the impulse response of the system and $y'(n)$ is the response or output of the system due to input and error signal. The response of the system is given by convolution of input and impulse response. For linear systems using distributive property of convolution the response $y'(n)$ can be written as shown in equation (8.25).

$$\begin{aligned} y'(n) &= x_q(n) * h(n) \\ &= [x(n) + e(n)] * h(n) \\ &= [x(n) * h(n)] + [e(n) * h(n)] \qquad .....(8.25) \end{aligned}$$

Let, $\quad y'(n) = y(n) + \varepsilon(n) \qquad .....(8.26)$

where, $y(n) = x(n) * h(n)$ = Output due to input signal x(n).

$\varepsilon(n) = e(n) * h(n)$ = Output due to error signal e(n).

The variance of the signal e(n) is called ***output noise power*** or ***steady state output noise power*** (or variance) due to the quantization error signal. Using autocorrelation function and the definition for variance of a discrete time signal, the expression for output noise power shown in equation (8.27) can be derived.

$$\left.\begin{array}{l}\text{Steady state output noise power}\\ \text{due to input quantization errors}\end{array}\right\}\sigma_{eoi}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n) \qquad .....(8.27)$$

In equation (8.27) the variance of error signal σ_e^2 can be evaluated using equation (8.23) or (8.24) and the summation of $h^2(n)$ can be evaluated using Parseval's theorem.

$$\therefore\ \sigma_{eoi}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n) = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z)\, H(z^{-1})\, z^{-1}\, dz \qquad .....(8.28)$$

where, $\oint_c$ denote integration around unit circle $|z| = 1$, in the anticlockwise direction.

The closed contour integration of equation (8.28) can be evaluated using residue theorem of $\mathcal{Z}$-transform.

$$\therefore\ \sigma_{eoi}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z)H(z^{-1})\, z^{-1}\, dz$$

$$= \sigma_e^2 \sum_{i=1}^{N} \mathrm{Res}\left[H(z)\, H(z^{-1})\, z^{-1}\right]_{z = p_i}$$

$$= \sigma_e^2 \sum_{i=1}^{N} \left[(z - p_i)\, H(z)\, H(z^{-1})\, z^{-1}\right]_{z = p_i} \qquad .....(8.29)$$

where, $p_1, p_2, ..., p_N$ are poles of $H(z)\, H(z^{-1})\, z^{-1}$.

Since the closed contour integration in equation (8.29) is around the unit circle $|z| = 1$, only the residues for the poles that lie inside the unit circle in z-plane are considered.

Example 8.11

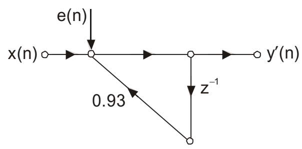


Fig 1.

For the recursive filter shown in fig 1, the input x(n) has a peak value of 10 V, represented by 6 bits. Compute the variance of output due to A/D conversion process.

Solution

Let us assume that the input is positive and so the 6-bits are used to represent only positive numbers.

$$\therefore\ \text{Quantization step size, } q = \frac{R}{2^B}$$

Given that, R = 10 and B = 6

$$\therefore\ q = \frac{10}{2^6} = 0.15625$$

$$\text{Variance of error signal, } \sigma_e^2 = \frac{q^2}{12} = \frac{0.15625^2}{12} = 2.0345 \times 10^{-3} \qquad .....(1)$$

Consider the given LTI system without error e(n) as shown in fig 2. The difference equation of the system is,

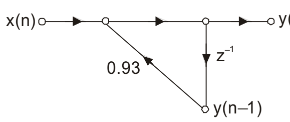


Fig 2.

$$y(n) = 0.93\, y(n-1) + x(n)$$

On taking $\mathcal{Z}$-transform of above equation we get,

$$Y(z) = 0.93\, z^{-1}\, Y(z) + X(z)$$

$$Y(z) - 0.93\, z^{-1}\, Y(z) = X(z)$$

$$Y(z)\,[1 - 0.93\, z^{-1}] = X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.93\, z^{-1}}$$

We know that the transfer function, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore H(z) = \frac{1}{1 - 0.93\, z^{-1}}$$

$$\therefore H(z)\, H(z^{-1})\, z^{-1} = \frac{1}{1 - 0.93 z^{-1}} \times \frac{1}{1 - 0.93 z} \times z^{-1}$$

$$= \frac{z^{-1}}{\left(1 - \dfrac{0.93}{z}\right)(-0.93)\left(z - \dfrac{1}{0.93}\right)} = \frac{-1.0753\, z^{-1}}{\left(\dfrac{z - 0.93}{z}\right)(z - 1.0753)}$$

$$= \frac{-1.0753}{(z - 0.93)(z - 1.0753)}$$

Now, poles of $H(z)\, H(z^{-1})\, z^{-1}$ are $p_1 = 0.93$, $p_2 = 1.0753$.

Here, $p_1 = 0.93$ is the only pole that lies inside the unit circle in z-plane.

The steady state output noise power (or variance) due to input quantization error signal is given by,

$$\left.\begin{matrix}\text{Output noise power}\\ \text{due to A / D process}\end{matrix}\right\}\sigma_{eoi}^2 = \sigma_e^2\, \frac{1}{2\pi j} \oint_c H(z)\, H(z^{-1})\, z^{-1}\, dz$$

$$= \sigma_e^2 \sum_{i=1}^{N} \text{Res}\left[H(z) H(z^{-1})\, z^{-1}\right]\Big|_{z = p_i}$$

$$= \sigma_e^2 \sum_{i=1}^{N} \left[(z - p_i)\, H(z)\, H(z^{-1})\, z^{-1}\right]\Big|_{z = p_i}$$ $\boxed{\text{Using equation (8.29)}}$

where, $p_1, p_2, \ldots\ldots p_N$ are poles of $H(z)\, H(z^{-1})\, z^{-1}$, that lies inside the unit circle in z-plane.

$$\therefore \sigma_{eoi}^2 = \sigma_e^2 \times \cancel{(z - 0.93)} \times \frac{-1.0753}{\cancel{(z - 0.93)}(z - 1.0753)}\Bigg|_{z = 0.93}$$

$$= \sigma_e^2 \times \frac{-1.0753}{0.93 - 1.0753} = 7.4006\, \sigma_e^2$$

$$= 7.4006 \times 2.0345 \times 10^{-3}$$ $\boxed{\text{Using equation (1)}}$

$$= 0.0151$$

Example 8.12

An LTI system is characterized by the difference equation, $y(n) = 0.68\, y(n-1) + 0.15x(n)$. The input signal $x(n)$ has a range of –5 V to +5 V, represented by 8-bits. Find the quantization step size, variance of the error signal and variance of the quantization noise at the output.

Solution

Given that,

Range, $R = -5$ to $+5 = 5 - (-5) = 10$.

Size of binary, B = 8 bits (including sign bit).

$$\therefore \text{ Quantization step size, } q = \frac{R}{2^B} = \frac{10}{2^8} = 0.0390625$$

$$\text{Variance of error signal, } \sigma_e^2 = \frac{q^2}{12} = \frac{0.0390625^2}{12} = 1.2716 \times 10^{-4} \qquad .....(1)$$

The difference equation governing the LTI system is,

$$y(n) = 0.68y(n-1) + 0.15x(n)$$

On taking Z-transform of above equation we get,

$$Y(z) = 0.68z^{-1}Y(z) + 0.15X(z)$$

$$Y(z) - 0.68z^{-1}Y(z) = 0.15X(z)$$

$$Y(z)\,[1 - 0.68z^{-1}] = 0.15X(z)$$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{0.15}{1-0.68z^{-1}}$$

We know that the transfer function, $H(z) = \dfrac{Y(z)}{X(z)}$.

$$\therefore \ H(z) = \frac{0.15}{1-0.68z^{-1}}$$

$$\therefore \ H(z)\,H(z^{-1})\,z^{-1} = \frac{0.15}{1-0.68\,z^{-1}} \times \frac{0.15}{1-0.68\,z} \times z^{-1}$$

$$= \frac{0.0225z^{-1}}{\left(1-\frac{0.68}{z}\right)(-0.68)\left(z-\frac{1}{0.68}\right)} = \frac{-0.0331z^{-1}}{\left(\frac{z-0.68}{z}\right)(z-1.4706)}$$

$$= \frac{-0.0331}{(z-0.68)(z-1.4706)}$$

Now, poles of $H(z)\,H(z^{-1})\,z^{-1}$ are $p_1 = 0.68$, $p_2 = 1.4706$.

Here, $p_1 = 0.68$ is the only pole that lies inside the unit circle in z-plane.

$$\therefore \left.\begin{array}{l}\text{Variance of the input quantization}\\ \text{noise at the output}\end{array}\right\} \sigma_{eoi}^2 = \sigma_e^2\,\frac{1}{2\pi j}\oint_c H(z)\,H(z^{-1})\,z^{-1}\,dz$$

$$= \sigma_e^2 \sum_{i=1}^{N} \text{Res}\left[H(z)H(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma_e^2 \sum_{i=1}^{N} \left[(z-p_i)\,H(z)\,H(z^{-1})\,z^{-1}\right]\Big|_{z=p_i} \qquad \boxed{\text{Using equation (8.29).}}$$

where, $p_1, p_2, \ldots\ldots p_N$ are poles of $H(z)\,H(z^{-1})\,z^{-1}$, that lies inside the unit circle in z-plane.

$$\therefore \ \sigma_{eoi}^2 = \sigma_e^2 \times \cancel{(z-0.68)} \times \left.\frac{-0.0331}{\cancel{(z-0.68)}(z-1.4706)}\right|_{z=0.68}$$

$$= \sigma_e^2 \times \frac{-0.0331}{0.68-1.4706} = 0.0419\,\sigma_e^2$$

$$= 0.0419 \times 1.2716 \times 10^{-4} \qquad \boxed{\text{Using equation (1)}}$$

$$= 5.328 \times 10^{-6}$$

Example 8.13

The output of an A/D converter is applied to a digital filter with the system function $H(z) = \dfrac{0.45z}{z - 0.72}$. Find the output noise power for the digital filter, when the input signal is quantized to 7 bits.

Solution

The range of input signal is not specified.

Therefore, let us assume that input varies from –1 to +1.

$\therefore$ Range, R = –1 to +1 = 1 – (–1) = 2

Size of binary, B = 7 bits (including sign bit).

$$\therefore \text{ Quantization step size, } q = \frac{R}{2^B} = \frac{2}{2^7} = 0.015625$$

$$\text{Variance of error signal, } \sigma_e^2 = \frac{q^2}{12} = \frac{0.015625^2}{12}$$

$$= 2.0345 \times 10^{-5} \qquad .....(1)$$

Given that,

$$H(z) = \frac{0.45z}{z - 0.72}$$

$$\therefore \ H(z)\,H(z^{-1})\,z^{-1} = \frac{0.45z}{z - 0.72} \times \frac{0.45z^{-1}}{z^{-1} - 0.72} \times z^{-1}$$

$$= \frac{0.45^2\,z^{-1}}{(z - 0.72)\left(\dfrac{1}{z} - 0.72\right)} = \frac{0.2025z^{-1}}{(z - 0.72)\left(\dfrac{1 - 0.72z}{z}\right)}$$

$$= \frac{0.2025\,z^{-1}z}{(z - 0.72)(-0.72)\left(z - \dfrac{1}{0.72}\right)} = \frac{-0.28125}{(z - 0.72)(z - 1.3889)}$$

Now, poles of $H(z)\,H(z^{-1})\,z^{-1}$ are $p_1 = 0.72$, $p_2 = 1.3889$.

Here, $p_1 = 0.72$ is the only pole that lies inside the unit circle in z-plane.

$$\left.\begin{array}{l}\therefore \text{ Output noise power due} \\ \quad \text{to input quantization}\end{array}\right\} \sigma_{eoi}^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z)\,H(z^{-1})\,z^{-1}\,dz$$

$$= \sigma_e^2 \sum_{i=1}^{N} \operatorname{Res}\left[H(z)H(z^{-1})\,z^{-1}\right]\Big|_{z = p_i}$$

$$= \sigma_e^2 \sum_{i=1}^{N} \left[(z - p_i)\,H(z)\,H(z^{-1})\,z^{-1}\right]\Big|_{z = p_i} \qquad \boxed{\text{Using equation (8.29)}}$$

where, $p_1, p_2, ... p_N$ are poles of $H(z)\,H(z^{-1})\,z^{-1}$, that lies inside the unit circle in z-plane.

$$\therefore\ \sigma_{eoi}^2 = \sigma_e^2 \times \cancel{(z-0.72)} \times \left.\frac{-0.28125}{\cancel{(z-0.72)}(z-1.3889)}\right|_{z=0.72}$$

$$= \sigma_e^2 \times \frac{-0.28125}{0.72 - 1.3889} = 0.4205\,\sigma_e^2$$

$$= 0.4205 \times 2.0345 \times 10^{-5}$$ Using equation (1).

$$= 8.5551 \times 10^{-6}$$

8.6 Quantization of Filter Coefficients

In the realization of FIR and IIR filters in hardware or in software, the accuracy with which filter coefficients can be specified is limited by the word length of the register used to store the coefficients. Usually the filter coefficients are quantized to the word size of the register used to store them either by truncation or by rounding.

The location (or the value) of poles and zeros of the digital filters directly depends on the value of filter coefficients. The quantization of the filter coefficients will modify the value of poles and zeros, and so the location of the poles and zeros will be shifted from the desired location. This will create deviations in the frequency response of the system. Hence we obtain a filter having a frequency response that is different from the frequency response of the filter with unquantized coefficients.

The sensitivity of the filter frequency response characteristics to quantization of the filter coefficients is minimized by realizing the filter having a large number of poles and zeros as an interconnection of second-order sections. This leads to the parallel form and cascade form realization in which the basic building blocks are first-order and second-order sections. It is possible to prove that the coefficient quantization has less effect in cascade realization when compared to parallel realization.

Example 8.14

For second-order IIR filter, $H(z) = \dfrac{1}{(1-0.5\,z^{-1})\,(1-0.45\,z^{-1})}$

Study the effect of shift in pole location with 3-bit coefficient representation in direct and cascade form.

Solution

$$\text{Given that, } H(z) = \frac{1}{(1-0.5\,z^{-1})\,(1-0.45\,z^{-1})} = \frac{1}{z^{-1}(z-0.5)\;z^{-1}(z-0.45)}$$

$$= \frac{z^2}{(z-0.5)\,(z-0.45)}$$

The roots of the denominator of H(z) are the original poles of H(z). Let the original poles of H(z) be p_1 and p_2.

Here, $p_1 = 0.5$ and $p_2 = 0.45$

Case (i) : Direct form Realization

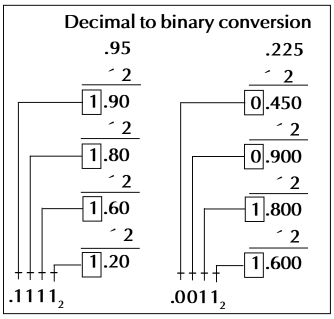
$$H(z) = \frac{1}{(1-0.5\,z^{-1})\,(1-0.45\,z^{-1})}$$

$$= \frac{1}{1-0.5\,z^{-1}-0.45\,z^{-1}+0.225\,z^{-2}} = \frac{1}{1-0.95\,z^{-1}+0.225\,z^{-2}}$$

Let us quantize the coefficients by truncation.

$$.95_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .1111_2 \xrightarrow[\text{3-bits}]{\text{Truncate to}} .111_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .875_{10}$$

$$.225_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .0011_2 \xrightarrow[\text{3-bits}]{\text{Truncate to}} .001_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .125_{10}$$



Binary to decimal conversion

$.111_2 = (1 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3}) = .875_{10}$

$.001_2 = (0 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3}) = .125_{10}$

Let $\overline{H}(z)$ be the transfer function of the IIR system after quantizing the coefficients.

$$\therefore \quad \overline{H}(z) = \frac{1}{1 - 0.875\, z^{-1} + 0.125\, z^{-2}}$$

$$\text{Let,} \quad \overline{H}(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.875 z^{-1} + 0.125 z^{-2}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.875z^{-1}Y(z) + 0.125z^{-2}Y(z) = X(z)$$

$$\therefore \quad Y(z) = X(z) + 0.875z^{-1}Y(z) - 0.125z^{-2}Y(z)$$

Using the above equation the direct form structure is drawn as shown in fig 1.

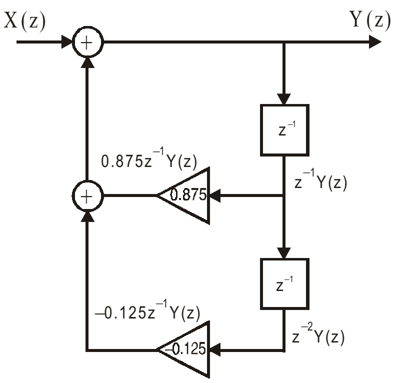


***Fig 1 :** Direct form Realization of $\overline{H}(z)$.*

Let us examine the poles of the system after coefficient quantization.

$$\text{Let,} \quad \overline{H}(z) = \frac{1}{z^{-2}(z^2 - 0.875\, z + 0.125)}$$

$$= \frac{z^2}{z^2 - 0.875\, z + 0.125} = \frac{z^2}{(z - 0.695)\,(z - 0.18)}$$

The roots of the quadratic,

$z^2 - 0.875z + 0.125 = 0$, are given by,

$$z = \frac{0.875 \pm \sqrt{0.875^2 - 4 \times 0.125}}{2}$$

$$= 0.695 \text{ or } 0.18$$

The poles of $\overline{H}(z)$ are given by roots of the denominator polynomial of $\overline{H}(z)$. Let the poles of $\overline{H}(z)$ be $\overline{p}_{d1}$ and $\overline{p}_{d2}$.

$$\therefore \quad \overline{p}_{d1} = 0.625 \quad \text{and} \quad \overline{p}_{d2} = 0.18$$

If we compare the poles of H(z) and $\overline{H}(z)$ we can observe that the poles of $\overline{H}(z)$ deviate very much from the original poles.

Case (ii) : Cascade Realization

Given that, $H(z) = \dfrac{1}{(1 - 0.5\,z^{-1})\,(1 - 0.45\,z^{-1})}$

In cascade realization the system can be realized as cascade of first order sections.

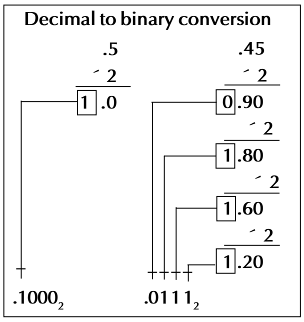
$$\therefore\; H(z) = H_1(z)\,H_2(z)$$

$$\text{where, } H_1(z) = \frac{1}{1 - 0.5\,z^{-1}} \text{ and } H_2(z) = \frac{1}{1 - 0.45\,z^{-1}}$$

Let us quantize the coefficients of $H_1(z)$ and $H_2(z)$ by truncation.

$$.5_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .1000_2 \xrightarrow[\text{3 bits}]{\text{Truncate to}} .100_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .5_{10}$$

$$.45_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .0111_2 \xrightarrow[\text{3 bits}]{\text{Truncate to}} .011_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .375_{10}$$



Binary to decimal conversion

$.100_2 = (1 \times 2^{-1}) + (0 \times 2^{-2}) + (0 \times 2^{-3})$
$= .5_{10}$

$.011_2 = (0 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3})$
$= .375_{10}$

Let, $\overline{H}_1(z)$ and $\overline{H}_2(z)$ be the transfer function of the first-order sections after quantizing the coefficients.

$$\therefore\; \overline{H}_1(z) = \frac{1}{1 - 0.5\,z^{-1}}$$

$$\overline{H}_2(z) = \frac{1}{1 - 0.375\,z^{-1}}$$

$$\text{Let, } \overline{H}_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1 - 0.5\,z^{-1}}$$

On cross multiplying the above equation we get,

$$Y_1(z) - 0.5z^{-1}Y_1(z) = X(z)$$

$$\therefore\; Y_1(z) = X(z) + 0.5z^{-1}Y_1(z) \qquad .....(1)$$

$$\text{Let, } \overline{H}_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{1 - 0.375\,z^{-1}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.375z^{-1}Y(z) = Y_1(z)$$

$$\therefore\; Y(z) = Y_1(z) + 0.375z^{-1}Y(z) \qquad .....(2)$$

Using equations (1) and (2) the cascade structure of the system is drawn as shown in fig 2.

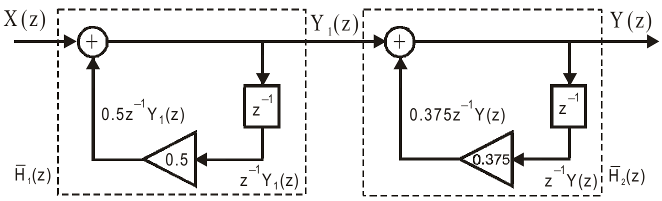


Fig 2 : Cascade realization of the system.

Let us examine the poles of the cascade system.

Let the poles of the cascade system be $\overline{p}_{c1}$ and $\overline{p}_{c2}$ which are given by the roots of the denominator polynomials of $\overline{H}_1(z)$ and $\overline{H}_2(z)$.

$$\text{Let,}\quad \overline{H}_1(z) = \frac{1}{1-0.5\,z^{-1}} = \frac{1}{z^{-1}(z-0.5)} = \frac{z}{z-0.5}$$

$$\overline{H}_2(z) = \frac{1}{1-0.375\,z^{-1}} = \frac{1}{z^{-1}(z-0.375)} = \frac{z}{z-0.375}$$

$$\therefore\ \overline{p}_{c1} = 0.5 \ \text{ and } \ \overline{p}_{c2} = 0.375$$

On comparing the poles of the cascade system with original poles we can say that one of the pole is same and other pole is very close to original pole.

Example 8.15

Discuss the effect of coefficient quantization on pole locations of the following IIR system, when it is realized in direct form-I and in cascade form. Assume a word length of 4-bits through truncation.

$$H(z) = \frac{1}{1-0.7\,z^{-1}+0.12\,z^{-2}}$$

Solution

$$\text{Given that,}\ H(z) = \frac{1}{1-0.7\,z^{-1}+0.12\,z^{-2}}$$

$$= \frac{1}{z^{-2}(z^2-0.7z+0.12)} = \frac{z^2}{z^2-0.7z+0.12}$$

$$= \frac{z^2}{(z-0.4)(z-0.3)} \qquad .....(1)$$

The roots of the quadratic $z^2 - 0.7z + 0.12 = 0$, are given by,

$$z = \frac{0.7\ \pm\ \sqrt{0.7^2\ -\ 4\ \times\ 0.12}}{2}$$

$$= \frac{0.7 \pm 0.1}{2} = 0.4,\ 0.3$$

The roots of the denominator of H(z) are the original poles of H(z). Let the original poles of H(z) be p_1 and p_2.

Here, $p_1 = 0.4$ and $p_2 = 0.3$.

Case(i) : Direct form-I Realization

$$\text{Given that,}\ H(z) = \frac{1}{1-0.7z^{-1}+0.12z^{-2}}$$

Let us quantize the coefficients by truncation.

Decimal to binary conversion

$$\begin{array}{rl} .7 & \\ \times\ 2 & \\ \hline \boxed{1}.4 & \\ \times\ 2 & \\ \hline \boxed{0}.8 & \\ \times\ 2 & \\ \hline \boxed{1}.6 & \\ \times\ 2 & \\ \hline \boxed{1}.2 & \\ \times\ 2 & \\ \hline \boxed{0}.4 & \end{array} \qquad .1\,0\,1\,1\,0$$

$$\begin{array}{rl} .12 & \\ \times\ 2 & \\ \hline \boxed{0}.24 & \\ \times\ 2 & \\ \hline \boxed{0}.48 & \\ \times\ 2 & \\ \hline \boxed{0}.96 & \\ \times\ 2 & \\ \hline \boxed{1}.92 & \\ \times\ 2 & \\ \hline \boxed{1}.84 & \end{array} \qquad .0\,0\,0\,1\,1$$

Binary to decimal conversion

$$.1011_2 = (1\times2^{-1})+(0\times2^{-2})+(1\times2^{-3})+(1\times2^{-4}) = .6875_{10}$$

$$.0001_2 = (0\times2^{-1})+(0\times2^{-2})+(0\times2^{-3})+(1\times2^{-4}) = .0625_{10}$$

$$.7_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .10110_2 \xrightarrow[\text{4-bits}]{\text{Truncate to}} .1011_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .6875_{10}$$

$$.12_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .00011_2 \xrightarrow[\text{4-bits}]{\text{Truncate to}} .0001_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .0625_{10}$$

Let, $\overline{H}(z)$ be the transfer function of the IIR system after quantizing the coefficients.

$$\therefore \quad \overline{H}(z) = \frac{1}{1 - 0.6875\,z^{-1} + 0.0625\,z^{-2}}$$

$$\text{Let, } \overline{H}(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.6875\,z^{-1} + 0.0625\,z^{-2}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.6875z^{-1}Y(z) + 0.0625z^{-2}Y(z) = X(z)$$

\ $Y(z) = X(z) + 0.6875z^{-1}Y(z) - 0.0625z^{-2}Y(z)$

Using the above equation the direct form-I structure of IIR system is drawn as shown in fig 1.

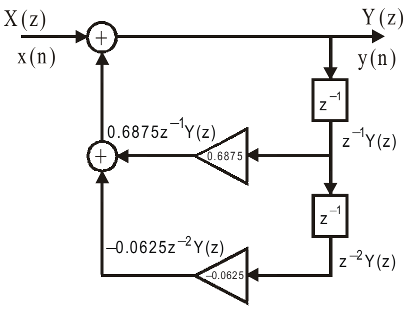


Fig 1 : Direct form-I realization of $\overline{H}(z)$.

Let us examine the poles of the system, after coefficient quantization.

$$\text{Let, } \quad \overline{H}(z) = \frac{1}{z^{-2}(z^2 - 0.6875\,z + 0.0625)} = \frac{z^2}{z^2 - 0.6875z + 0.0625}$$

$$= \frac{z^2}{(z - 0.5797)(z - 0.1078)}$$

The roots of the quadratic,
$z^2 - 0.6875z + 0.0625 = 0$ are,

$$z = \frac{0.6875 \pm \sqrt{0.6875^2 - 4 \times 0.0625}}{2}$$

$$= \frac{0.6875 \pm 0.4719}{2}$$

$$= 0.5797, \; 0.1078$$

The poles of $\overline{H}(z)$ are given by roots of the denominator polynomial of $\overline{H}(z)$.

Let the poles of $\overline{H}(z)$ be $\overline{p}_{d1}$ and $\overline{p}_{d2}$.

$$\therefore \quad \overline{p}_{d1} = 0.5797 \quad \text{and} \quad \overline{p}_{d2} = 0.1078$$

If we compare the poles of H(z) and $\overline{H}(z)$ we can observe that the poles of $\overline{H}(z)$ deviate very much from the original pole.

Case(ii) : Cascade Realization

$$\text{Given that, } H(z) = \frac{1}{1 - 0.7\,z^{-1} + 0.12\,z^{-2}} = \frac{z^2}{(z - 0.4)(z - 0.3)} = \frac{z}{z - 0.4} \times \frac{z}{z - 0.3} \qquad \boxed{\text{Using equation (1).}}$$

$$= \frac{z}{z(1 - 0.4\,z^{-1})} \times \frac{z}{z(1 - 0.3\,z^{-1})}$$

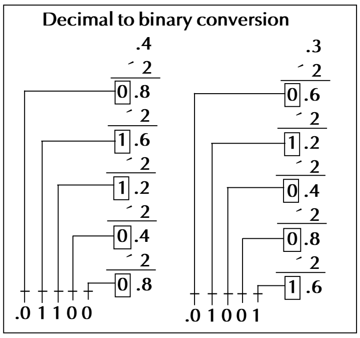
$$= \frac{1}{1 - 0.4\,z^{-1}} \times \frac{1}{1 - 0.3\,z^{-1}} = H_1(z) \times H_2(z)$$

$$\text{where,}\quad H_1(z) = \frac{1}{1-0.4\,z^{-1}}$$

$$H_2(z) = \frac{1}{1-0.3\,z^{-1}}$$

In cascade realization the system can be realized as cascade of first-order sections.

Let us quantize the coefficients of $H_1(z)$ and $H_2(z)$ by truncation.



Binary to decimal conversion

$.0110_2 = (0\times2^{-1})+(1\times2^{-2})+(1\times2^{-3})+(0\times2^{-4})$
$= .375_{10}$

$.0100_2 = (0\times2^{-1})+(1\times2^{-2})+(0\times2^{-3})+(0\times2^{-4})$
$= .25_{10}$

$$.4_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .01100_2 \xrightarrow[\text{4-bits}]{\text{Truncate to}} .0110_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .375_{10}$$

$$.3_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .01001_2 \xrightarrow[\text{4-bits}]{\text{Truncate to}} .0100_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .25_{10}$$

Let, $\overline{H}_1(z)$ and $\overline{H}_2(z)$ be the transfer function of the first-order sections after quantizing the coefficients.

$$\therefore\ \overline{H}_1(z) = \frac{1}{1-0.375\,z^{-1}} \quad ; \quad \overline{H}_2(z) = \frac{1}{1-0.25z^{-1}}$$

$$\text{Let,}\quad \overline{H}_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-0.375\,z^{-1}}$$

On cross multiplying the above equation we get,

$$Y_1(z) - 0.375z^{-1}Y_1(z) = X(z)$$

$$\backslash\ \ Y_1(z) = X(z) + 0.375z^{-1}Y_1(z) \qquad .....(2)$$

$$\text{Let,}\quad \overline{H}_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{1-0.25\,z^{-1}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.25z^{-1}Y(z) = Y_1(z)$$

$$\backslash\ \ Y(z) = Y_1(z) + 0.25z^{-1}Y(z) \qquad .....(3)$$

Using equations (2) and (3) the cascade structure of the system is drawn as shown in fig 2.

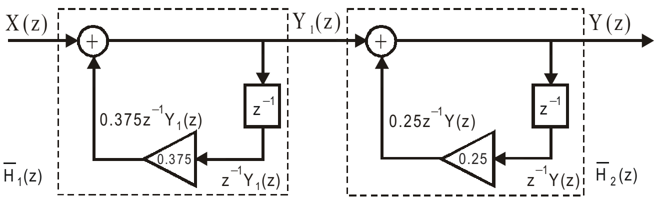


Fig 2 : Cascade realization of the system.

Let us examine the poles of the cascade system.

Let, the poles of the cascade system be $\overline{p}_{c1}$ and $\overline{p}_{c2}$ which are given by the roots of the denominator polynomials of $\overline{H}_1(z)$ and $\overline{H}_2(z)$.

$$\text{Let,}\quad \overline{H}_1(z) = \frac{1}{1 - 0.375\,z^{-1}} = \frac{1}{z^{-1}(z - 0.375)} = \frac{z}{z - 0.375}$$

$$H_2(z) = \frac{1}{1 - 0.25\,z^{-1}} = \frac{1}{z^{-1}(z - 0.25)} = \frac{z}{z - 0.25}$$

$$\therefore\ \overline{p}_{c1} = 0.375 \quad \text{and} \quad \overline{p}_{c2} = 0.25$$

On comparing the poles of the cascade system with original poles we can say that both the poles are very close to original poles of the system. Also we can observe that the deviation of poles of cascaded system is less when compared to deviation of poles in direct form realization.

Example 8.16

Consider the LTI system governed by the equation, $y(n) + 0.8301y(n-1) + 0.7348y(n-2) = x(n-2)$. Discuss the effect of coefficient quantization on pole locations, when the coefficients are quantized by,

(i) 3-bits by truncation (ii) 4-bits by truncation

Solution

Given that, $y(n) + 0.8301y(n-1) + 0.7348y(n-2) = x(n-2)$

On taking $\mathcal{Z}$-transform of the given equation we get,

$$Y(z) + 0.8301z^{-1}Y(z) + 0.7348z^{-2}Y(z) = z^{-2}X(z)$$

$$[z^2 + 0.8301z + 0.7348]z^{-2}Y(z) = z^{-2}X(z)$$

$$\therefore\ \text{Transfer function,}\ H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{z^{-2}\left[z^2 + 0.8301z + 0.7348\right]}$$

$$= \frac{1}{z^2 + 0.8301z + 0.7348}$$

$$= \frac{1}{(z + 0.415 - j0.75)(z + 0.415 + j0.75)}$$

The roots of the quadratic, $z^2 + 0.8301z + 0.7348 = 0$ are,

$$z = \frac{-0.8301 \ \pm\ \sqrt{0.8301^2 - 4 \times 0.7348}}{2}$$

$$= \frac{0.8301 \pm j1.5}{2}$$

$$= 0.415 \pm j0.75$$

The poles of the given system are roots of denominator polynomial of H(z).

Let the poles be p_1 and p_2.

\ $p_1 = -0.415 + j0.75$

$p_2 = -0.415 - j0.75$

Case(i) : Quantization of coefficients to 3-bits by truncation

The coefficients to be quantized are 0.8301_{10} and 0.7348_{10}.

Decimal to binary conversion

.8301	.7348
´ 2	´ 2
1.6602	1.4696
´ 2	´ 2
1.3204	0.9392
´ 2	´ 2
0.6408	1.8784
´ 2	´ 2
1.2816	1.7568
´ 2	´ 2
0.5632	1.5136
.1 1 0 1 0	.1 0 1 1 1

Binary to decimal conversion

$.110_2 = (1 ´ 2^{-1}) + (1 ´ 2^{-2}) + (0 ´ 2^{-3})$
$= .75_{10}$

$.101_2 = (1 ´ 2^{-1}) + (0 ´ 2^{-2}) + (1 ´ 2^{-3})$
$= .625_{10}$

$$.8301_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .11010 \xrightarrow[\text{3-bits}]{\text{Truncate to}} .110_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .75_{10}$$

$$.7348_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .10111 \xrightarrow[\text{3-bits}]{\text{Truncate to}} .101_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .625_{10}$$

Let, $\overline{H}_3(z)$ be the transfer function of the IIR system after quantizing the coefficients to 3-bits by truncation.

$$\therefore\ \overline{H}_3(z) = \frac{1}{z^2 + 0.75\,z + 0.625}$$

$$= \frac{1}{(z + 0.375 - j0.696)(z + 0.375 + j0.696)}$$

The roots of the quadratic, $z^2 + 0.75z + 0.625 = 0$ are,

$$z = \frac{-0.75 \pm \sqrt{0.75^2 - 4 \times 0.625}}{2} = \frac{-0.75 \pm j1.3919}{2}$$

$$= -0.375 \pm j0.696$$

The poles of $\overline{H}_3(z)$ are given by roots of the denominator polynomial of $\overline{H}_3(z)$. Let the poles of $\overline{H}_3(z)$ be p_{13} and p_{23}.

$$\therefore\ p_{13} = -0.375 + j0.696 \quad ; \quad p_{23} = -0.375 - j0.696$$

Case(ii) : Quantization of coefficients to 4-bits by truncation

The coefficients to be quantized are 0.8301_{10} and 0.7348_{10}.

$$.8301_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .11010_2 \xrightarrow[\text{4-bits}]{\text{Truncate to}} .1101_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .8125_{10}$$

$$.7348_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .10111_2 \xrightarrow[\text{4-bits}]{\text{Truncate to}} .1011_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .6875_{10}$$

Let, $\overline{H}_4(z)$ be the transfer function of the IIR system after quantizing the coefficients to 4-bits by truncation.

Note : The decimal to binary conversion is same as that of case(i).

Binary to decimal conversion

$.1101_2 = (1 \times 2^{-1}) + (1 \times 2^{-2}) + (0 \times 2^{-3}) + (1 \times 2^{-4})$
$= .8125_{10}$

$.1011_2 = (1 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3}) + (1 \times 2^{-4})$
$= .6875_{10}$

$$\therefore\ \overline{H}_4(z) = \frac{1}{z^2 + 0.8125\,z + 0.6875}$$

$$= \frac{1}{(z + 0.4063 - j0.7228)(z + 0.4063 + j0.7228)}$$

The roots of the quadratic, $z^2 + 0.8125z + 0.6875 = 0$ are,

$$z = \frac{-0.8125 \pm \sqrt{0.8125^2 - 4 \times 0.6875}}{2} = \frac{-0.8125 \pm j1.4456}{2}$$

$$= -0.4063 \pm j0.7228$$

The poles of $\overline{H}_4(z)$ are given by roots of the denominator polynomial of $\overline{H}_4(z)$. Let the poles of $\overline{H}_4(z)$ be p_{14} and p_{24}.

$$\therefore\ p_{14} = -0.4063 + j0.7228 \quad ; \quad p_{24} = -0.4063 - j0.7228$$

Conclusion

The quantization of coefficients result in deviation of pole locations. The deviation is lesser, when the quantization is performed with higher size binary.

8.7 Product Quantization Error

In realization structures of IIR system, multipliers are used to multiply the signal by constants. The output of the multipliers i.e, the products are quantized to finite word length in order to store them in registers and to be used in subsequent calculations. In fixed point arithmetic, the muliplication of two b-bit numbers results in a product of length 2b-bits. If the word length of the register used to store the result is b-bits then it is necessary to quantize the product (result) to b-bits. The error due to the quantization of the output of multiplier is referred to as ***product quantization error***.

In digital system the product quantization is performed by rounding due to the following desirable characteristics of rounding.

1. In rounding the error signal is independent of the type of arithmetic employed.
2. The mean value of error signal due to rounding is zero.
3. The variance of error signal due to rounding is the least.

The analysis of product quantization error is similar to the analysis of quantization error due to A/D process. But in product quantization error analysis it is necessary to define the noise transfer function, which depends on the structure of the digital network.

The ***Noise Transfer Function (NTF)*** is defined as transfer function from the noise source to the filter output (i.e., NTF is the transfer function obtained by treating the noise source as actual input).

The model of the multiplier of a digital network using fixed point arithmetic is shown in fig 8.7. The multiplier is considered as an infinite precision multiplier. Using an adder the error signal is added to the output of multiplier so that the output of adder is equal to the quantized product. Therefore the output of finite word length multiplier can be expressed as,

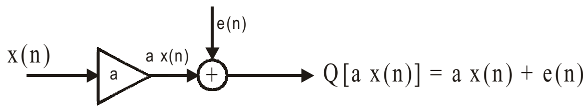


Fig 8.7 : *Statistical model of fixed point product quantization.*

Quantized product = $Q[a\,x(n)] = a\,x(n) + e(n)$ ..... (8.30)

where, $a\,x(n)$ = Unquantized product

$e(n)$ = Product quantization error signal

The product quantization error signal is treated as a random process with uniform probability density function. In general the following assumptions are made regarding the statistical independence of the various noise sources in the digital filter.

1. Any two different samples from the same noise source are uncorrelated.
2. Any two different noise sources, when considered as random processes are uncorrelated.
3. Each noise source is uncorrelated with the input sequence.

The ***product quantization noise models*** for first-order and second-order IIR systems using direct form-I and direct form-II structures are shown in fig 8.8. The product quantization noise models for IIR systems using cascade structures are shown in fig 8.9. In these models each finite precision multiplier is replaced by an ideal multiplier and an additive roundoff noise. The noise signal is added to the output of ideal multiplier.

In each model shown in fig 8.8 and fig 8.9 there are a number of noise sources. The output noise variance (power) due to each source is computed separately by considering one noise source at a time. The total output noise variance (power) is given by sum of the output noise variance (power) of all the noise sources.

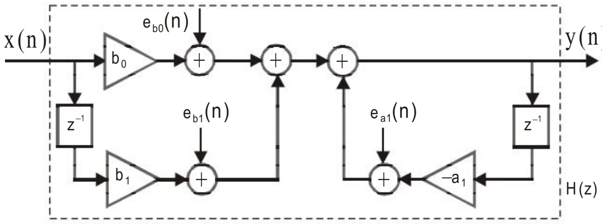


***Fig 8.8a :** First-order direct form-I.*

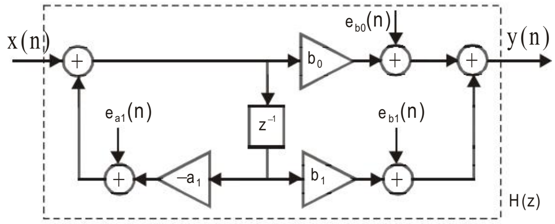


***Fig 8.8b :** First-order direct form-II.*

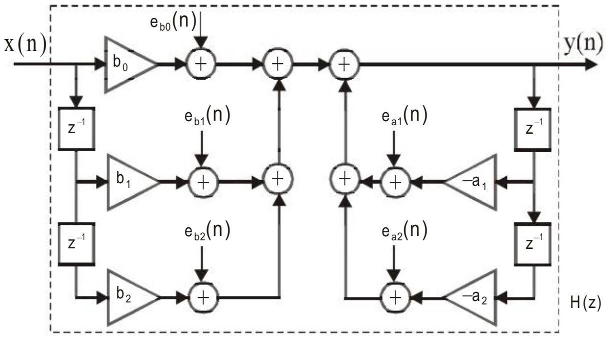


***Fig 8.8c :** Second-order direct form-I.*

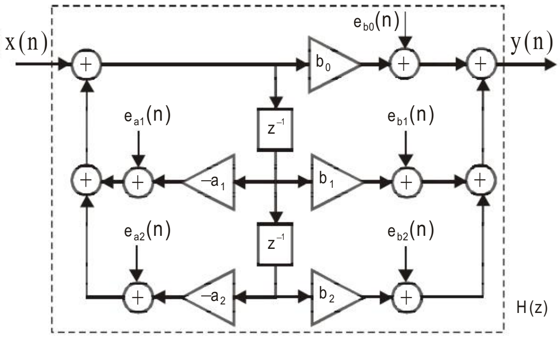


***Fig 8.8d :** Second-order direct form-II.*

***Fig 8.8 :** Product quantization noise models of IIR systems for direct form realization.*

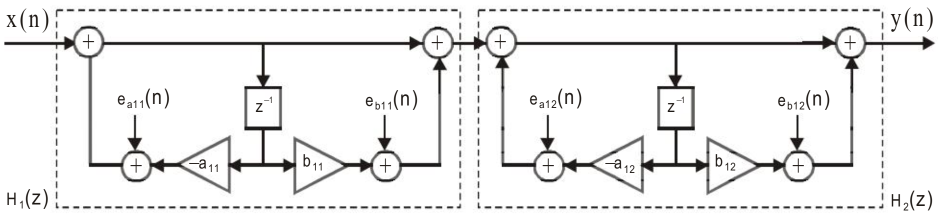


***Fig 8.9a :** Cascading of two first-order sections.*

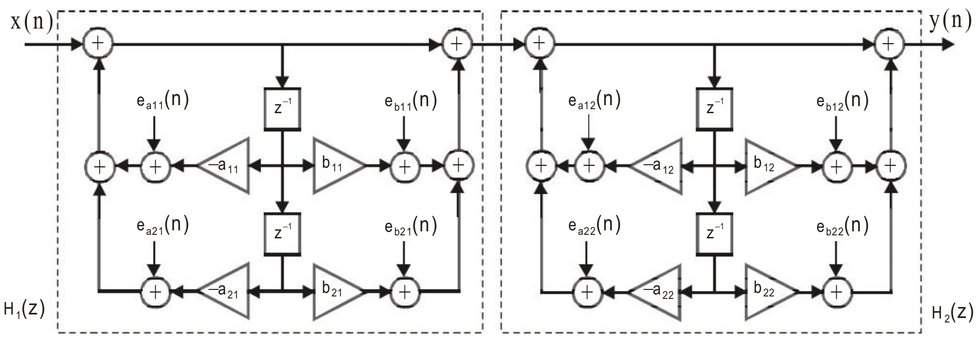


***Fig 8.9b :** Cascading of two second-order sections.*

***Fig 8.9 :** Product quantization noise models of IIR systems for cascade form realization.*

The equations used for computing the steady state output noise variance (power) due to quantization error in A/D conversion process can be used to compute the output noise variance due to product quantization, because in both cases the quantization is performed by rounding. But the transfer function seen by each noise source is different. Therefore for each noise source, the Noise Transfer Function (NTF) has to be determined by treating the noise source as input (and the output being the output of the system). With reference to fig 8.8 and fig 8.9 some examples of Noise Transfer Functions are given below.

NTF for noise source $e_{a1}(n)$ in fig 8.8b $= H(z)$

NTF for noise source $e_{a1}(n)$ and $e_{a2}(n)$ in fig 8.8d $= H(z)$

NTF for noise source $e_{a11}(n)$ in fig 8.9a $= H_1(z)\,H_2(z)$

NTF for noise source $e_{b11}(n)$ and $e_{a12}(n)$ in fig 8.9b $= H_2(z)$

Output Noise Power (Roundoff Noise Power) Due to Product Quantization

Let, $e_k(n)$ = Error signal from k^{th} noise source.

$h_k(n)$ = Impulse response for k^{th} noise source.

$T_k(z) = \mathcal{Z}\{h_k(n)\}$ = Noise Transfer Function (NTF) for k^{th} noise source.

σ_{ek}^2 = Variance of k^{th} noise source.

σ_{ekop}^2 = Output noise power or variance due to k^{th} noise source.

Refer equations (8.22) and (8.24).

$$\text{Now, Variance of } k^{th} \text{ noise source, } \sigma_{ek}^2 = \frac{q^2}{12} \text{ or } \frac{2^{-2B}}{3} \qquad \text{..... (8.31)}$$

$$\text{Now, Output noise power due to } k^{th} \text{ noise source, } \sigma_{ekop}^2 = \sigma_{ek}^2 \sum_{n=0}^{\infty} h_k^2(n) \qquad \text{..... (8.32)}$$

In equation (8.32) the summation of $h_k(n)$ can be evaluated using Parseval's theorem.

$$\therefore\ \sigma_{ekop}^2 = \sigma_{ek}^2 \frac{1}{2\pi j} \oint_c T_k(z)\, T_k(z^{-1})\, z^{-1}\, dz \qquad \text{..... (8.33)}$$

where, $\oint_c$ denote integration around unit circle $|z| = 1$, in the anticlockwise direction.

The closed contour integration of equation (8.33) can be evaluated using residue theorem of $\mathcal{Z}$-transform as shown below.

$$\therefore\ \sigma_{ekop}^2 = \sigma_{ek}^2 \sum_{i=1}^{N} \text{Res}\left[T_k(z)\, T_k(z^{-1})\, z^{-1}\right]\Big|_{z = p_i}$$

$$= \sigma_{ek}^2 \sum_{i=1}^{N} \left[(z - p_i)\, T_k(z)\, T_k(z^{-1})\, z^{-1}\right]\Big|_{z = p_i} \qquad \text{..... (8.34)}$$

where $p_1, p_2, \ldots, p_N$ are poles of $T_k(z)\,T_k(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

Let the number of noise sources in a digital system (or filter) be M. The total steady state noise variance at the output of the system due to product quantization errors is given by the sum of the output noise variances due to all the noise sources.

Let, σ_{eTop}^2 = Total output noise power due to product quantization error (or Total roundoff noise power)

$$\therefore\ \sigma_{eTop}^2 = \sigma_{e1op}^2 + \sigma_{e2op}^2 + \ldots + \sigma_{eMop}^2 \qquad \text{..... (8.35)}$$

Example 8.17

In the IIR system given below the products are rounded to 4-bits (including sign bit).

$$H(z) = \frac{1}{(1 - 0.35\,z^{-1})\,(1 - 0.62\,z^{-1})}$$

Find the output roundoff noise power in **a)** direct form realization **b)** cascade realization.

Solution

Given that the products are rounded to 4 bits. Therefore, B = 4. (Including sign bit)

Let us assume that the range of product is –1 to +1.

\ Range, R = –1 to +1 = 1 – (–1) = 2

$$\text{Quantization step size, } q = \frac{R}{2^B} = \frac{2}{2^4} = 0.125$$

$$\text{Variance of noise source, } \sigma_e^2 = \frac{q^2}{12} = \frac{0.125^2}{12} = 1.3021 \times 10^{-3}$$
(due to rounding)

a) Direct form Realization

$$\text{Given that, } H(z) = \frac{1}{(1 - 0.35\,z^{-1})\,(1 - 0.62\,z^{-1})}$$

$$= \frac{1}{1 - 0.62\,z^{-1} - 0.35\,z^{-1} + 0.217\,z^{-2}} = \frac{1}{1 - 0.97\,z^{-1} + 0.217\,z^{-2}}$$

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.97\,z^{-1} + 0.2172\,z^{-2}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.97z^{-1}Y(z) + 0.217z^{-2}Y(z) = X(z)$$

$$\backslash\ \ Y(z) = X(z) + 0.97z^{-1}Y(z) - 0.217z^{-2}Y(z) \quad .....(1)$$

Using equation (1), the direct form structure of the given system is drawn as shown in fig 1.

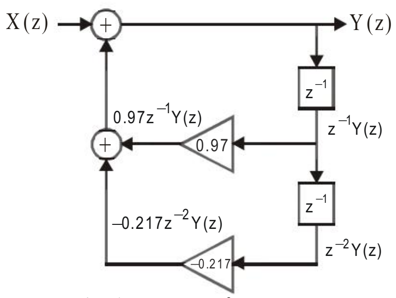


Fig 1 : Direct form structure.

The direct form product quantization noise model of the given system is shown in fig 2.

The direct form structure has two constant multipliers. Hence a noise source is introduced at the output of each multiplier as shown in fig 2.

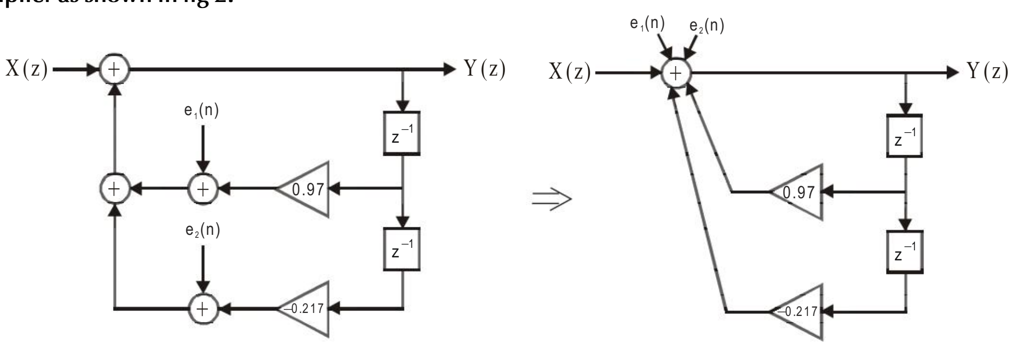


Fig 2 : Direct form structure quantization noise model of H(z).

In fig 2 it can be observed that both the noise sources are at the input node of the system. Hence NTF (Noise Transfer Function) for both the noise sources is the system transfer function H(z).

$$\therefore \text{ NTF for noise signal } e_1(n) = T_1(z) = H(z) = \frac{1}{(1-0.35\,z^{-1})\,(1-0.62\,z^{-1})}$$

$$\text{NTF for noise signal } e_2(n) = T_2(z) = H(z) = \frac{1}{(1-0.35\,z^{-1})\,(1-0.62\,z^{-1})}$$

Let, $\sigma^2_{e1op,\,d}$ = Output noise power due to error signal $e_1(n)$ in direct form realization.

$\sigma^2_{e2op,\,d}$ = Output noise power due to error signal $e_2(n)$ in direct form realization.

Now, $\sigma^2_{e1op,\,d} = \sigma^2_e \dfrac{1}{2\pi j} \oint\limits_c T_1(z)\,T_1(z^{-1})\,z^{-1}\,dz$ $\boxed{\text{Using equation (8.29).}}$

$$= \sigma^2_e \sum_{i=1}^{N} \text{Res}\left[T_1(z)\,T_1(z^{-1})\,z^{-1}\right]\Big|_{z=p_i} \quad = \sigma^2_e \sum_{i=1}^{N} \left[(z-p_i)\,T_1(z)\,T_1(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

where $p_1, p_2 \,.....,\, p_N$ are poles of $T_1(z)\,T_1(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

$$T_1(z)\,T_1(z^{-1})\,z^{-1} = \frac{1}{(1-0.35\,z^{-1})(1-0.62\,z^{-1})} \times \frac{1}{(1-0.35\,z)(1-0.62z)} \times z^{-1}$$

$$= \frac{z^{-1}}{\left(1-\dfrac{0.35}{z}\right)\left(1-\dfrac{0.62}{z}\right)(-0.35)\left(z-\dfrac{1}{0.35}\right)(-0.62)\left(z-\dfrac{1}{0.62}\right)}$$

$$= \frac{z^{-1}}{0.217\left(\dfrac{z-0.35}{z}\right)\left(\dfrac{z-0.62}{z}\right)(z-2.8571)\,(z-1.6129)}$$

$$= \frac{4.6083\,z}{(z-0.35)\,(z-0.62)\,(z-2.8571)\,(z-1.6129)}$$

The poles of $T_1(z)\,T_1(z^{-1})\,z^{-1}$, which are lying inside the unit circle are $p_1 = 0.35$ and $p_2 = 0.62$. (Here the other two poles $p_3 = 2.8571$ and $p_4 = 1.6129$ are lying outside the unit circle. For calculation of residues only the poles lying inside the unit circle are to be considered).

$$\therefore \sigma^2_{e1op,\,d} = \sigma^2_e \times \sum_{i=1}^{N} \left[(z-p_i)\,T_1(z)\,T_1(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma^2_e \times \left[\cancel{(z-0.35)} \frac{4.6083\,z}{\cancel{(z-0.35)}\,(z-0.62)\,(z-2.8571)\,(z-1.6129)}\Bigg|_{z=0.35} + \cancel{(z-0.62)} \frac{4.6083\,z}{(z-0.35)\,\cancel{(z-0.62)}\,(z-2.8571)\,(z-1.6129)}\Bigg|_{z=0.62}\right]$$

$$= \sigma^2_e \times \left[\frac{4.6083 \times 0.35}{(0.35-0.62)\,(0.35-2.8571)\,(0.35-1.6129)} + \frac{4.6083 \times 0.62}{(0.62-0.35)\,(0.62-2.8571)\,(0.62-1.6129)}\right]$$

$$= 1.3021 \times 10^{-3} \times [-1.8867 + 4.7641] = 3.7467 \times 10^{-3} \qquad .....(2)$$

Here, $\sigma^2_{e1op,\,d} = \sigma^2_{e2op,\,d} = 3.7467 \times 10^{-3}$

Let, $\sigma^2_{eTop,\,d}$ = Total output noise power due to all the noise sources in direct form realization.

$\therefore$ Total output noise power, $\sigma^2_{eTop,\,d} = \sigma^2_{e1op,\,d} + \sigma^2_{e2op,\,d} = 2 \times \sigma^2_{e1op,\,d}$

$$= 2 \times 3.7467 \times 10^{-3} = 7.4934 \times 10^{-3}$$

b) Cascade Realization

Given that, $H(z) = \dfrac{1}{(1 - 0.35\,z^{-1})\,(1 - 0.62\,z^{-1})}$

Let, $H(z) = H_1(z)\,H_2(z)$

where, $H_1(z) = \dfrac{1}{1 - 0.35\,z^{-1}}$ and $H_2(z) = \dfrac{1}{1 - 0.62\,z^{-1}}$

Let, $H_1(z) = \dfrac{Y_1(z)}{X(z)} = \dfrac{1}{1 - 0.35\,z^{-1}}$

On cross multiplying the above equation we get,

$Y_1(z) - 0.35z^{-1}Y_1(z) = X(z)$

\ $Y_1(z) = X(z) + 0.35z^{-1}Y_1(z)$ .....(3)

Using equation (3) the direct form structure of $H_1(z)$ is drawn as shown in fig 3.

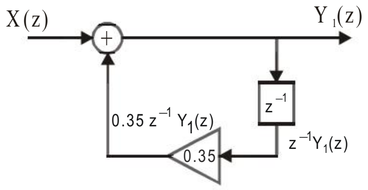


Fig 3 : Direct form structure of $H_1(z)$.

Let, $H_2(z) = \dfrac{Y(z)}{Y_1(z)} = \dfrac{1}{1 - 0.62\,z^{-1}}$

On cross multiplying the above equation we get,

$Y(z) - 0.62z^{-1}Y(z) = Y_1(z)$

\ $Y(z) = Y_1(z) + 0.62z^{-1}Y(z)$ .....(4)

Using equation (4) the direct form structure of $H_2(z)$ is drawn as shown in fig 4.

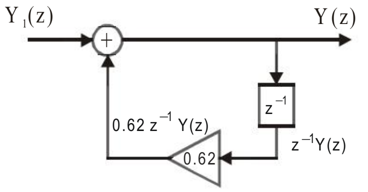


Fig 4 : Direct form structure of $H_2(z)$.

The system can be cascaded in two different ways. They are $H_1(z)H_2(z)$ and $H_2(z)H_1(z)$ as shown in fig 5. The product quantization noise will depend on the order of cascading. Let us estimate the output noise power for both the cases.

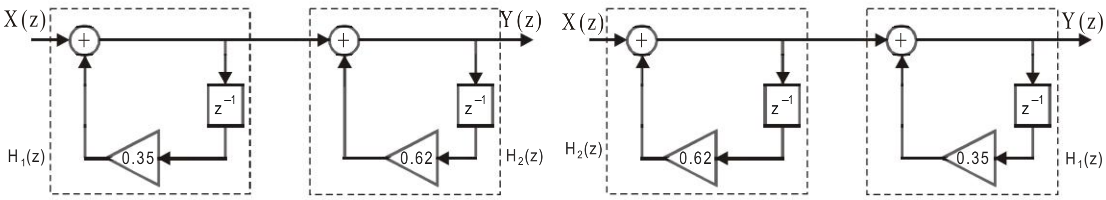


Fig 5a : Order of cascading is $H_1(z)H_2(z)$. *Fig 5b : Order of cascading is $H_2(z)H_1(z)$.*

Fig 5 : Two ways of cascading H(z).

Case (i) : The order of cascading is $H_1(z)\,H_2(z)$

The cascade form product quantization noise model of $H_1(z)H_2(z)$ is shown in fig 6.

From fig 6 it can be observed that the system has two multipliers and so there are two noise sources. One of the noise source, i.e., $e_1(n)$ is at the input of first section and so its NTF is $H_1(z)\,H_2(z) = H(z)$. The second noise source, i.e., $e_2(n)$ is at the input of second section and so its NTF is $H_2(z)$.

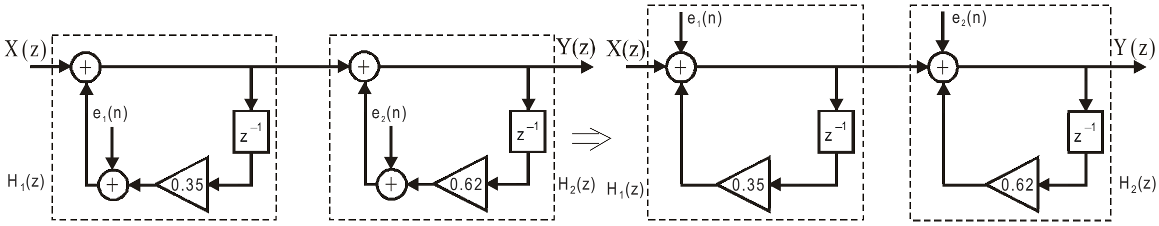


Fig 6 : *Cascade form product quantization noise model of $H_1(z)H_2(z)$.*

$\therefore$ NTF for noise signal $e_1(n) = T_1(z) = H(z) = \dfrac{1}{(1-0.35\,z^{-1})\,(1-0.62\,z^{-1})}$

NTF for noise signal $e_2(n) = T_2(z) = H_2(z) = \dfrac{1}{(1-0.62\,z^{-1})}$

Let, $\sigma^2_{e1op,\,c1}$ = Output noise power due to error signal $e_1(n)$ in cascade realization $H_1(z)\,H_2(z)$.

$\sigma^2_{e2op,\,c1}$ = Output noise power due to error signal $e_2(n)$ in cascade realization $H_1(z)\,H_2(z)$.

In this cascade realization the output noise power due to noise signal $e_1(n)$ will be same as that of direct form realization, because the NTF for $e_1(n)$ is same in both the realizations.

$\therefore\ \sigma^2_{e1op,\,c1} = 3.7467 \times 10^{-3}$ From equation (2).

Now, $\sigma^2_{e2op,\,c1} = \sigma^2_e \dfrac{1}{2\pi j} \oint_c T_2(z)\,T_2(z^{-1})\,z^{-1}\,dz$ Using equation (8.29).

$$= \sigma^2_e \sum_{i=1}^{N} \text{Res}\left[T_2(z) T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_i} = \sigma^2_e \sum_{i=1}^{N} \left[(z-p_i)\,T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

where, $p_1, p_2, \ldots, p_N$ are poles of $T_2(z)\,T_2(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

Here, $T_2(z)\,T_2(z^{-1})\,z^{-1} = \dfrac{1}{1-0.62z^{-1}} \times \dfrac{1}{1-0.62z} \times z^{-1}$

$$= \frac{z^{-1}}{\left(1-\dfrac{0.62}{z}\right)(-0.62)\left(z-\dfrac{1}{0.62}\right)} = \frac{-1.6129\,z^{-1}}{\left(\dfrac{z-0.62}{z}\right)(z-1.6129)}$$

$$= \frac{-1.6129}{(z-0.62)(z-1.6129)}$$

The function $T_2(z)\,T_2(z^{-1})\,z^{-1}$ has only one pole inside the unit circle, i.e., $p_1 = 0.62$

$$\therefore\ \sigma^2_{e2op,\,c1} = \sigma^2_e \times \sum_{i=1}^{N} \left[(z-p_i)\,T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Bigg|_{z=p_i}$$

$$= \sigma^2_e \times \left[\cancel{(z-0.62)}\frac{-1.6129}{\cancel{(z-0.62)}\,(z-1.6129)}\right]_{z=0.62}$$

$$= 1.3021 \times 10^{-3} \times \frac{-1.6129}{0.62-1.6129} = 2.1152 \times 10^{-3}$$

Let, $s_{eTop,\, c1}$ = Total output noise power due to all noise sources in cascade realization $H_1(z)\, H_2(z)$.

$$\text{Total output noise power, } \sigma^2_{eT0p,\, c1} = \sigma^2_{e10p,\, c1} + \sigma^2_{e20p,\, c1}$$

$$= 3.7467 \times 10^{-3} + 2.1152 \times 10^{-3}$$

$$= 5.8619 \times 10^{-3}$$

Case (ii): The order of cascading is $H_2(z)\, H_1(z)$

The cascade form product quantization noise model of $H_2(z)H_1(z)$ is shown in fig 7.

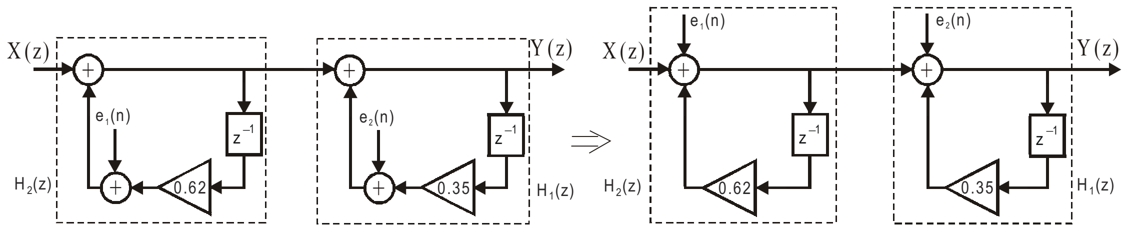


Fig 7 : Cascade form product quantization noise model of $H_2(z)H_1(z)$.

From fig 7 it can be observed that the system has two multipliers and so there are two noise sources. One of the noise source i.e., $e_1(n)$ is at the input of first section and so its NTF is $H_2(z)\, H_1(z) = H(z)$. The second noise source i.e., $e_2(n)$ is at the input of second section and so its NTF is $H_1(z)$

$$\therefore \text{ NTF for noise signal } e_1(n) = T_1(z) = H(z) = \frac{1}{(1 - 0.35\, z^{-1})\, (1 - 0.62\, z^{-1})}$$

$$\text{NTF for noise signal } e_2(n) = T_2(z) = H_1(z) = \frac{1}{(1 - 0.35\, z^{-1})}$$

Let, $\sigma^2_{e1op,\, c2}$ = Output noise power due to error signal $e_1(n)$ in cascade realization $H_2(z)\, H_1(z)$.

$\sigma^2_{e2op,\, c2}$ = Output noise power due to error signal $e_2(n)$ in cascade realization $H_2(z)\, H_1(z)$.

In this cascade realization the output noise power due to noise signal $e_1(n)$ will be same as that of direct form realization, because the NTF for $e_1(n)$ is same in both the realizations.

$$\therefore\ \sigma^2_{e1op,c2} = 3.7467 \times 10^{-3}$$ From equation (2).

$$\text{Now, } \sigma^2_{e2op,c2} = \sigma^2_e \frac{1}{2\pi j} \oint_c T_2(z)\, T_2(z^{-1})\, z^{-1}\, dz$$ Using equation (8.29).

$$= \sigma^2_e \sum_{i=1}^{N} \text{Res}\left[T_2(z) T_2(z^{-1})\, z^{-1}\right]\Big|_{z = p_i} = \sigma^2_e \sum_{i=1}^{N} \left[(z - p_i) T_2(z) T_2(z^{-1})\, z^{-1}\right]\Big|_{z = p_i}$$

where $p_1, p_2, \ldots, p_N$ are poles of $T_2(z)\, T_2(z^{-1})\, z^{-1}$, that lie inside the unit circle in z-plane.

$$T_2(z)\, T_2(z^{-1})\, z^{-1} = \frac{1}{1 - 0.35z^{-1}} \times \frac{1}{1 - 0.35z} \times z^{-1} = \frac{z^{-1}}{\left(1 - \frac{0.35z}{z}\right)(-0.35)\left(z - \frac{1}{0.35z}\right)}$$

$$= \frac{-2.8571z^{-1}}{\left(\frac{z - 0.35}{z}\right)(z - 2.8571)}$$

$$= \frac{-2.8571}{(z - 0.35)(z - 2.8571)}$$

The function $T_2(z)\,T_2(z^{-1})\,z^{-1}$ has only one pole inside the unit circle, i.e., $p_1 = 0.35$.

$$\therefore\ \sigma^2_{e2op,\,c2} = \sigma_e^2 \times \sum_{i=1}^{N} \left[(z-p_i)T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma_e^2\,\cancel{(z-0.35)}\,\frac{-2.8571}{\cancel{(z-0.35)}\,(z-2.8571)}\Bigg|_{z=0.35}$$

$$= 1.3021\times 10^{-3}\ \frac{-2.8571}{(0.35-2.8571)}$$

$$= 1.4839\times 10^{-3}$$

Let, $s^2_{eTop,\,c2}$ = Total output noise power due to all noise sources in cascade realization $H_2(z)\,H_1(z)$.

$$\text{Total output noise power, } \sigma^2_{eTop,c2} = \sigma^2_{e1op,c2} + \sigma^2_{e2op,c2}$$

$$= 3.7467 \times 10^{-3} + 1.4839 \times 10^{-3}$$

$$= 5.2306 \times 10^{-3}$$

Conclusion

1. In direct form realization the product roundoff noise power is higher than cascade realization.
2. In the cascade realization the product roundoff noise power is less in case (ii) when compared to case (i).

Example 8.18

Find the output roundoff noise power, when the products are rounded to 5-bits (including sign bit) in two different ways of cascade realization of the following IIR system.

$$H(z) = \frac{1}{(1-0.41z^{-1})(1-0.59z^{-1})}$$

Solution

Given that the products are rounded to 5-bits. Therefore, B = 5. (Including sign bit)

Let us assume that the range of product is –1 to +1.

\ Range, R = –1 to +1 = 1 – (–1) = 2

$$\text{Quantization step size, } q = \frac{R}{2^B} = \frac{2}{2^5} = 0.0625$$

$$\text{Variance of noise source, } \sigma_e^2 = \frac{q^2}{12} = \frac{0.0625^2}{12} = 3.2552 \times 10^{-4}$$

(due to rounding)

$$\text{Given that, } H(z) = \frac{1}{(1-0.41z^{-1})(1-0.59z^{-1})}$$

Let, $H(z) = H_1(z)\,H_2(z)$

$$\text{where, } H_1(z) = \frac{1}{1-0.41z^{-1}} \text{ and } H_2(z) = \frac{1}{1-0.59z^{-1}}$$

$$\text{Let, } H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-0.41z^{-1}}$$

On cross multiplying the above equation we get,

$$Y_1(z) - 0.41z^{-1}Y_1(z) = X(z)$$

$$\therefore\ Y_1(z) = X(z) + 0.41z^{-1}Y_1(z) \qquad .....(1)$$

Using equation (1), the direct form structure of $H_1(z)$ is drawn as shown in fig 1.

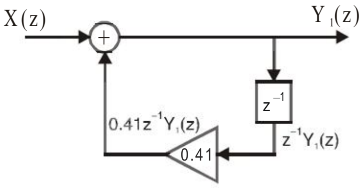


Fig 1 : *Direct form structure of $H_1(z)$.*

$$\text{Let, } H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{1 - 0.59z^{-1}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.59z^{-1}Y(z) = Y_1(z)$$

$$\therefore\ Y(z) = Y_1(z) + 0.59z^{-1}Y(z) \qquad .....(2)$$

Using equation (2), the direct form structure of $H_2(z)$ is drawn as shown in fig 2.

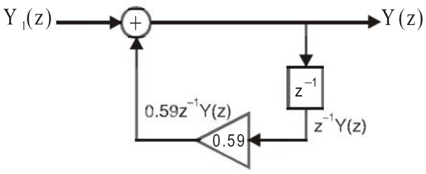


Fig 2 : *Direct form structure of $H_2(z)$.*

The given system H(z) can be cascaded in two different ways. They are $H_1(z)\ H_2(z)$ and $H_2(z)\ H_1(z)$ as shown in fig 3. The product quantization noise will depend on the order of cascading. Let us estimate the output noise power for both the cases.

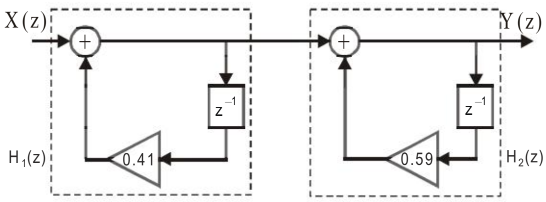


Fig 3a : *Order of cascading is $H_1(z)H_2(z)$.*

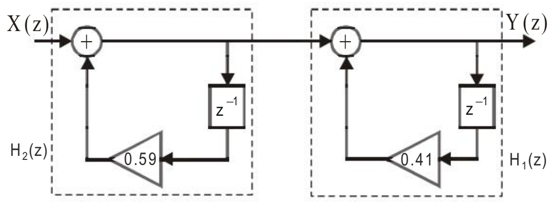


Fig 3b : *Order of cascading is $H_2(z)H_1(z)$.*

Fig 3 : *Two ways of cascading H(z).*

Case(i) : The order of cascading is $H_1(z)\ H_2(z)$

The cascade form product quantization noise model of $H_1(z)\ H_2(z)$ is shown in fig 4.

From fig 6 it can be observed that the system has two multipliers and so there are two noise sources. One of the noise source, i.e., $e_1(n)$ is at the input of first section and so its NTF is $H_1(z)\ H_2(z) = H(z)$. The second noise source, i.e., $e_2(n)$ is at the input of second section and so its NTF is $H_2(z)$.

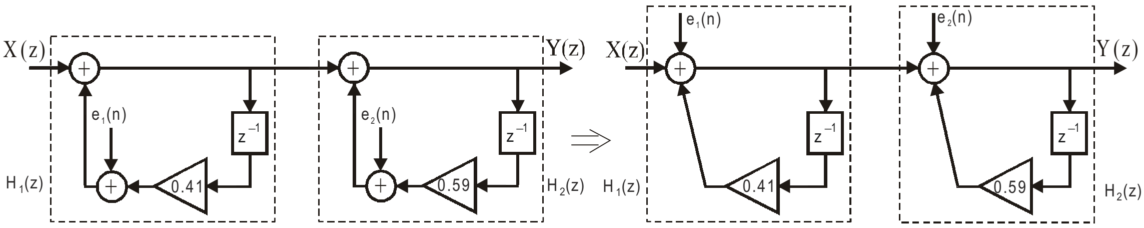


Fig 4 : *Cascade form product quantization noise model of $H_1(z)H_2(z)$.*

$\therefore$ NTF for noise signal $e_1(n) = T_1(z) = H(z) = \dfrac{1}{(1-0.41z^{-1})(1-0.59z^{-1})}$

NTF for noise signal $e_2(n) = T_2(z) = H_2(z) = \dfrac{1}{1-0.59z^{-1}}$

Let, $\sigma^2_{e1op,\,c1}$ = Output noise power due to error signal $e_1(n)$ in cascade realization $H_1(z)\ H_2(z)$.

Now, $\sigma^2_{e1op,\,c1} = \sigma^2_e \dfrac{1}{2\pi j} \oint\limits_c T_1(z)\ T_2(z^{-1})\ z^{-1}\,dz$ $\boxed{\text{From equation (8.29).}}$

$$= \sigma_e^2 \sum_{i=1}^{N} \text{Res}\left[T_1(z)\ T_1(z^{-1})\ z^{-1}\right]\Bigg|_{z=p_i} = \sigma_e^2 \sum_{i=1}^{N} \left[(z-p_i)\,T_1(z)\ T_1(z^{-1})\ z^{-1}\right]\Bigg|_{z=p_i}$$

where p_1, p_2, ...p_N are poles of $T_1(z)\ T_1(z^{-1})\ z^{-1}$, that lie inside the unit circle in z-plane.

Here, $T_1(z)\ T_1(z^{-1})\ z^{-1} = \dfrac{1}{(1-0.41z^{-1})\ (1-0.59z^{-1})} \times \dfrac{1}{(1-0.41z)\ (1-0.59z)} \times z^{-1}$

$$= \frac{z^{-1}}{\left(1-\frac{0.41}{z}\right)\left(1-\frac{0.59}{z}\right)(-0.41)\left(z-\frac{1}{0.41}\right)(-0.59)\left(z-\frac{1}{0.59}\right)}$$

$$= \frac{4.1339\,z^{-1}}{\left(\frac{z-0.41}{z}\right)\left(\frac{z-0.59}{z}\right)(z-2.439)\ (z-1.6949)}$$

$$= \frac{4.1339\,z}{(z-0.41)(z-0.59)\ (z-2.439)\ (z-1.6949)}$$

The function $T_1(z)\ T_1(z^{-1})\ z^{-1}$ has two poles inside the unit circle and they are i.e., $p_1 = 0.41$ and $p_2 = 0.59$.

$$\therefore\ \sigma^2_{e1op,\,c1} = \sigma_e^2 \times \sum_{i=1}^{N} \left[(z-p_i)\,T_1(z)\ T_1(z^{-1})\ z^{-1}\right]\Bigg|_{z=p_i}$$

$$= \sigma_e^2 \times \left[\cancel{(z-0.41)}\ \frac{4.1339\,z}{\cancel{(z-0.41)}\ (z-0.59)\ (z-2.439)\ (z-1.6949)}\Bigg|_{z=0.41} + \cancel{(z-0.59)}\ \frac{4.1339\,z}{(z-0.41)\ \cancel{(z-0.59)}\ (z-2.439)\ (z-1.6949)}\Bigg|_{z=0.59}\right]$$

$$\therefore\ \sigma^2_{e1op,\ c1} = \sigma^2_e\left[\frac{4.1339\times 0.41}{(0.41-0.59)\ (0.41-2.439)\ (0.41-1.6949)}\right.$$

$$\left.+\frac{4.1339\times 0.59}{(0.59-0.41)\ (0.59-2.439)\ (0.59-1.6949)}\right]$$

$$= 3.2552\times 10^{-4}\left[-3.6118+6.6325\right]$$

$$= 9.833\times 10^{-4} \qquad .....(3)$$

Let, $\sigma^2_{e2op,\ c1}$ = Output noise power due to error signal $e_2(n)$ in cascade realization $H_1(z)\ H_2(z)$.

$$\text{Now, } \sigma^2_{e2op,\ c1} = \sigma^2_e\ \frac{1}{2\pi j}\oint_c T_2(z)\ T_2(z^{-1})\ z^{-1}\,dz = \sigma^2_e \sum_{k=1}^{M} \text{Res}\left[T_2(z)\ T_2(z^{-1})\ z^{-1}\right]\Big|_{z=p_k}$$

$$= \sigma^2_e \sum_{k=1}^{M}\left[(z-p_k)\,T_2(z)\ T_2(z^{-1})\ z^{-1}\right]\Big|_{z=p_k} \qquad \boxed{\text{From equation (8.29).}}$$

where $p_1, p_2, ...p_M$ are poles of $T_2(z)\ T_2(z^{-1})\ z^{-1}$, that lie inside the unit circle in z-plane.

$$\text{Here, } T_2(z)\ T_2(z^{-1})\ z^{-1} = \frac{1}{1-0.59z^{-1}}\times\frac{1}{1-0.59z}\times z^{-1} = \frac{z^{-1}}{\left(1-\dfrac{0.59}{z}\right)(-0.59)\left(z-\dfrac{1}{0.59}\right)}$$

$$= \frac{-1.6949\,z^{-1}}{\left(\dfrac{z-0.59}{z}\right)(z-1.6949)} = \frac{-1.6949}{(z-0.59)\ (z-1.6949)}$$

The function $T_2(z)\ T_2(z^{-1})\ z^{-1}$ has only one pole inside the unit circle, i.e., $p_1 = 0.59$.

$$\therefore\ \sigma^2_{e2op,\ c1} = \sigma^2_e\ \times \sum_{k=1}^{M}\left[(z-p_k)\,T_2(z)\ T_2(z^{-1})\ z^{-1}\right]\Big|_{z=p_k}$$

$$= \sigma^2_e\times \cancel{(z-0.59)}\ \frac{-1.6949}{\cancel{(z-0.59)}\ (z-1.6949)}\Bigg|_{z=0.59}$$

$$= 3.2552\times 10^{-4}\times\frac{-1.6949}{0.59-1.6949} = 4.9934\ \times 10^{-4}$$

Let, $\sigma^2_{eTop,\ c1}$ = Total output noise power due to all noise sources in cascade realization $H_1(z)\ H_2(z)$.

$$\therefore\ \text{Total output noise power, } \sigma^2_{eTop,\ c1} = \sigma^2_{e1op,\ c1} + \sigma^2_{e2op,\ c1}$$

$$= 9.833\times 10^{-4} + 4.9934\times 10^{-4}$$

$$= 14.8264\times 10^{-4} = 1.48264\times 10^{-3}$$

Case(ii) : The order of cascading is $H_2(z)\ H_1(z)$

The cascade form product quantization noise model of $H_2(z)\ H_1(z)$ is shown in fig 5.

From fig 5 it can be observed that the system has two multipliers and so there are two noise sources. One of the noise source, i.e., $e_1(n)$ is at the input of first section and so its NTF is $H_2(z)\ H_1(z) = H(z)$. The second noise source, i.e., $e_2(n)$ is at the input of second section and so its NTF is $H_1(z)$.

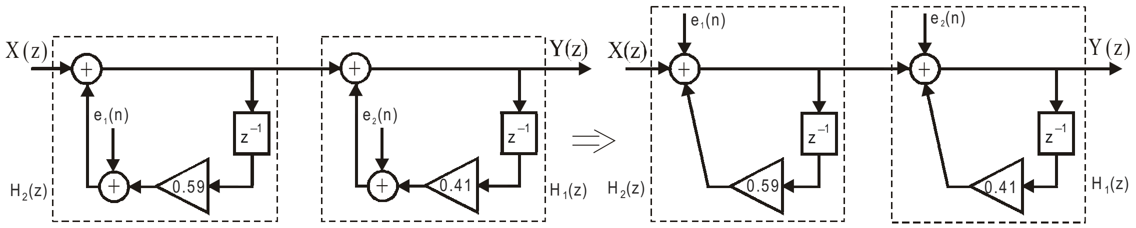


Fig 5 : Cascade form product quantization noise model of $H_2(z)H_1(z)$.

$$\therefore \ \text{NTF for noise signal } e_1(n) = T_1(z) = H(z) = \frac{1}{(1-0.41z^{-1})(1-0.59\,z^{-1})}$$

$$\text{NTF for noise signal } e_2(n) = T_2(z) = H_1(z) = \frac{1}{1-0.41z^{-1}}$$

Let, $\sigma^2_{e1op,\ c2}$ = Output noise power due to error signal $e_1(n)$ in cascade realization $H_2(z)\ H_1(z)$.

Since, the NTF for $e_1(n)$ is same in both the case of cascade realization, the output noise power due to noise signal $e_1(n)$ will also be same in both the case of cascade realization.

$$\therefore \ \sigma^2_{e1op,\ c2} = 9.833 \times 10^{-4}$$ From equation (3).

Let, $\sigma^2_{e2op,\ c2}$ = Output noise power due to error signal $e_2(n)$ in cascade realization $H_2(z)\ H_1(z)$.

$$\text{Now, } \sigma^2_{e2op,\ c2} = \sigma^2_e \, \frac{1}{2\pi j} \oint_c T_2(z)\ T_2(z^{-1})\ z^{-1}\, dz = \sigma^2_e \sum_{k=1}^{M} \text{Res}\left[T_2(z)\ T_2(z^{-1})\ z^{-1}\right]\Bigg|_{z\,=\,p_k}$$ Using equation (8.29).

$$= \sigma^2_e \sum_{k=1}^{M} \left[(z-p_k)\,T_2(z)\ T_2(z^{-1})\ z^{-1}\right]\Bigg|_{z\,=\,p_k}$$

where $p_1, p_2, \ldots\ldots p_M$ are poles of $T_2(z)\ T_2(z^{-1})\ z^{-1}$, that lie inside the unit circle in z-plane.

$$\text{Here, } T_2(z)\ T_2(z^{-1})\ z^{-1} = \frac{1}{1-0.41z^{-1}} \times \frac{1}{1-0.41z} \times z^{-1} = \frac{z^{-1}}{\left(1-\dfrac{0.41}{z}\right)(-0.41)\left(z-\dfrac{1}{0.41}\right)}$$

$$= \frac{-2.439\,z^{-1}}{\left(\dfrac{z-0.41}{z}\right)(z-2.439)} = \frac{-2.439}{(z-0.41)\ (z-2.439)}$$

The function $T_2(z)\ T_2(z^{-1})\ z^{-1}$ has only one pole inside the unit circle, i.e., $p_1 = 0.41$.

$$\therefore \ \sigma^2_{e2op,\ c2} = \sigma^2_e \times \sum_{k=1}^{M} \left[(z-p_k)\,T_2(z)\ T_2(z^{-1})\ z^{-1}\right]\Bigg|_{z\,=\,p_k}$$

$$= \sigma^2_e \times \left[\cancel{(z-0.41)}\ \frac{-2.439}{\cancel{(z-0.41)}\ (z--2.439)}\right]\Bigg|_{z\,=\,0.41}$$

$$= 3.2552 \times 10^{-4} \times \frac{-2.439}{0.41-2.439} = 3.913 \ \times 10^{-4}$$

Let, $\sigma_{eTop,\ c2}$ = Total output noise power due to all noise sources in cascade realization $H_2(z)\ H_1(z)$.

$\therefore$ Total output noise power, $\sigma^2_{eTop,\ c2} = \sigma^2_{e1op,\ c2} + \sigma^2_{e2op,\ c2}$

$$= 9.833 \times 10^{-4} + 3.913 \times 10^{-4}$$

$$= 13.746 \times 10^{-4} = 1.3746 \times 10^{-3}$$

Conclusion

In cascade realization the product roundoff noise power is less in case(ii) when compared to case(i).

Example 8.19

Given that, $H(z) = \dfrac{1}{(1-0.15z^{-1})\,(1-0.3z^{-1})\,(1-0.4z^{-1})}$. Determine the output roundoff noise power in the direct form realization of the above system.

(a) When the products are rounded to 4-bits (including sign bit).

(b) When the products are rounded to 8-bits (including sign bit). Comment on the result.

Solution

$$\text{Given that, } H(z) = \frac{1}{(1-0.15z^{-1})\,(1-0.3z^{-1})\,(1-0.4z^{-1})} = \frac{1}{(1-0.45z^{-1}+0.045z^{-2})\,(1-0.4z^{-1})}$$

$$= \frac{1}{1-0.45z^{-1}+0.045z^{-2}-0.4z^{-1}+0.18z^{-2}-0.018z^{-3}}$$

$$= \frac{1}{1-0.85z^{-1}+0.225z^{-2}-0.018z^{-3}}$$

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.85z^{-1}+0.225z^{-2}-0.018z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.85z^{-1}Y(z) + 0.225z^{-2}Y(z) - 0.018z^{-3}Y(z) = X(z)$$

$$\backslash\ Y(z) = X(z) + 0.85z^{-1}Y(z) - 0.225z^{-2}Y(z) + 0.018z^{-3}Y(z) \qquad .....(1)$$

Using equation (1), the direct form structure of given system is drawn as shown in fig 1.

The direct form structure has three constant multipliers. Hence a noise source is introduced at the output of each multiplier as shown in fig 2.

From fig 2, it can be observed that all the three noise sources are at the input node of the system. Hence, NTF (Noise Transfer Function) for all the three noise sources is the system transfer function H(z).

$$\therefore\ \text{NTF} = T(z) = H(z)$$

$$= \frac{1}{(1-0.85\,z^{-1}+0.225\,z^{-2}-0.018z^{-3})}$$

$$= \frac{1}{(1-0.15\,z^{-1})\,(1-0.3z^{-1})\,(1-0.4z^{-1})}$$

Let, s^2_{eop} = Output noise power due to one noise source.

s^2_{eTop} = Total output noise power

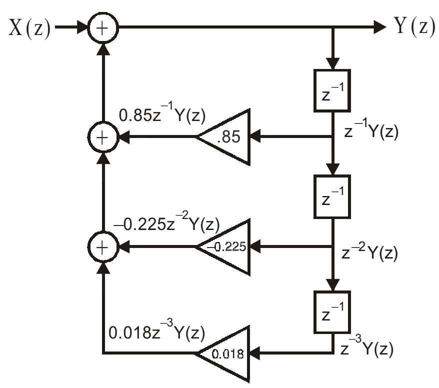


Fig 1 : Direct form structure of H(z).

Now, $\sigma^2_{eTop} = 3\,\sigma^2_{eop}$

Here, $$\sigma^2_{eop} = \sigma^2_e \times \frac{1}{2\pi j}\oint_c T(z)\,T(z^{-1})\,z^{-1}\,dz = \sigma^2_e \times \sum_{i=1}^{N} \text{Res}\left[T(z)T(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma^2_e \times \sum_{i=1}^{N}\left[(z-p_i)T(z)T(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$ $\boxed{\text{From equation (8.29).}}$

where $p_1, p_2 \ldots\ldots, p_N$ are poles of $T(z)\,T(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

$$T(z)\,T(z^{-1})\,z^{-1} = \frac{1}{(1-0.15z^{-1})\,(1-0.3z^{-1})\,(1-0.4z^{-1})} \times \frac{1}{(1-0.15z)\,(1-0.3z)\,(1-0.4z)} \times z^{-1}$$

$$= \frac{z^{-1}}{\left(1-\frac{0.15}{z}\right)\left(1-\frac{0.3}{z}\right)\left(1-\frac{0.4}{z}\right)(-0.15)\left(z-\frac{1}{0.15}\right)(-0.3)\left(z-\frac{1}{0.3}\right)(-0.4)\left(z-\frac{1}{0.4}\right)}$$

$$= \frac{-55.5556\,z^{-1}}{\left(\frac{z-0.15}{z}\right)\left(\frac{z-0.3}{z}\right)\left(\frac{z-0.4}{z}\right)(z-6.6667)\,(z-3.3333)\,(z-2.5)}$$

$$= \frac{-55.5556\,z^2}{(z-0.15)\,(z-0.3)\,(z-0.4)\,(z-6.6667)\,(z-3.3333)\,(z-2.5)}$$

The poles of $T(z)\,T(z^{-1})\,z^{-1}$ that are lying inside the unit circle are, $z_1 = 0.15$, $z_2 = 0.3$, $z_3 = 0.4$.

$$\therefore\ \sigma^2_{eop} = \sigma^2_e \times \sum_{i=1}^{N}\left[(z-p_i)T(z)\,T(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma^2_e\Bigg[\cancel{(z-0.15)}\,\frac{-55.5556\,z^2}{\cancel{(z-0.15)}\,(z-0.3)\,(z-0.4)\,(z-6.6667)\,(z-3.3333)\,(z-2.5)}\Bigg|_{z=0.15}$$

$$+\cancel{(z-0.3)}\,\frac{-55.5556\,z^2}{(z-0.15)\,\cancel{(z-0.3)}\,(z-0.4)\,(z-6.6667)\,(z-3.3333)\,(z-2.5)}\Bigg|_{z=0.3}$$

$$+\cancel{(z-0.4)}\,\frac{-55.5556\,z^2}{(z-0.15)\,(z-0.3)\,\cancel{(z-0.4)}\,(z-6.6667)\,(z-3.3333)\,(z-2.5)}\Bigg|_{z=0.4}\Bigg]$$

$$= \sigma^2_e\Bigg[\frac{-55.5556\times 0.15^2}{(0.15-0.3)\,(0.15-0.4)\,(0.15-6.6667)\,(0.15-3.3333)\,(0.15-2.5)}$$

$$+\frac{-55.5556\times 0.3^2}{(0.3-0.15)\,(0.3-0.4)\,(0.3-6.6667)\,(0.3-3.3333)\,(0.3-2.5)}$$

$$+\frac{-55.5556\times 0.4^2}{(0.4-0.15)\,(0.4-0.3)\,(0.4-6.6667)\,(0.4-3.3333)\,(0.4-2.5)}\Bigg]$$

$$= \sigma^2_e\left[0.6838 - 7.8456 + 11.1033\right] = 3.9415\,\sigma^2_e$$

$$\therefore\ \sigma^2_{eTop} = 3\,\sigma^2_{eop} = 3 \times 3.9415\,\sigma^2_e = 11.8245\,\sigma^2_e$$

Case (a) : When products are rounded to 4-bits

Given that the products are rounded to 4 bits. Therefore, B = 4. (Including sign bit).

Let us assume that the range of product is –1 to +1,

\ Range, R = –1 to +1 = 1 – (–1) = 2

Quantization step size, $q = \frac{R}{2^B} = \frac{2}{2^4} = 0.125$

Let, $s_e^2 = s_{e4}^2$ = Variance of noise source when the products are rounded to 4-bits.

$$\therefore \ \sigma_{e4}^2 = \frac{q^2}{12} = \frac{0.125^2}{12} = 1.3021 \times 10^{-3}$$

Let, $s_{eTop,4}^2$ = Total output roundoff noise power when products are rounded to 4-bits.

$$\therefore \ \sigma_{eT0p,4}^2 = 11.8245\sigma_{e4}^2 = 11.8245 \times 1.3021 \times 10^{-3}$$

$$= 15.3967 \times 10^{-3}$$

$$= 1.5397 \times 10^{-2}$$

Case (b) : When products are rounded to 8-bits

Given that the products are rounded to 8-bits. Therefore, B = 8. (Including sign bit).

Let us assume that the range of product is –1 to +1,

\ Range, R = –1 to +1 = 1 – (–1) = 2

Quantization step size, $q = \frac{R}{2^B} = \frac{2}{2^8} = 7.8125 \times 10^{-3}$

Let, $s_e^2 = s_{e8}^2$ = Variance of noise source when the products are rounded to 8-bits.

$$\therefore \ \sigma_{e8}^2 = \frac{q^2}{12} = \frac{(7.8125 \times 10^{-3})^2}{12} = 5.0863 \times 10^{-6}$$

Let, $s_{eTop,8}^2$ = Total output roundoff noise power when products are rounded to 8-bits.

$$\therefore \ \sigma_{eT0p,8}^2 = 11.8245\ \sigma_{e8}^2 = 11.8245 \times 5.0863 \times 10^{-6}$$

$$= 60.143 \times 10^{-6} = 6.0143 \times 10^{-5} = 0.0060143 \times 10^{-2}$$

Comment

The output roundoff noise power reduces when the products are rounded to higher bit size.

$$\% \text{ reduction in output noise power} = \frac{\sigma_{eTop,4}^2 - \sigma_{eTop,8}^2}{\sigma_{eTop,4}^2} \times 100$$

$$= \frac{1.5397 \times 10^{-2} - 0.006 \times 10^{-2}}{1.5397 \times 10^{-2}} \times 100$$

$$= 99.61\%$$

8.8 Limit Cycles in Recursive Systems

8.8.1 Zero Input Limit Cycles

In recursive systems, when the input is zero or some nonzero constant value, the nonlinearities due to finite precision arithmetic operations may cause periodic oscillations in the output. During periodic oscillations, the output y(n) of a system will oscillate between a finite positive and negative value for increasing n or the output will become constant for increasing n. Such oscillations are called ***limit cycles***. These oscillations are due to round-off errors in multiplication and overflow in addition.

In recursive systems, if the system output enters a limit cycle, it will continue to remain in limit cycle even when the input is made zero. Hence these limit cycles are also called ***zero input limit cycles***. The system output remains in limit cycle until another input of sufficient magnitude is applied to drive the system out of limit cycle.

Consider the difference equation of first order system with only pole as shown in equation (8.36).

$$y(n) = a\,y(n-1) + x(n) \qquad \text{..... (8.36)}$$

The system has one product [a y(n–1)]. If the product is quantized to finite word length then the response y(n) will deviate from actual value. Let y'(n) be the response of the system when the product is quantized in each recursive realization. Now the equation (8.42) can be written as shown in equation (8.37).

$$y'(n) = Q[a\,y'(n-1)] + x(n) \qquad \text{..... (8.37)}$$

where, Q[ ] stands for quantization operation.

$$Q\,[ay'(n-1)] = \text{Quantized value of the product } ay'(n-1).$$

The structures of the systems described by equations (8.36) and (8.37) are shown in fig 8.10. In the first-order system with only pole, the coefficient "a" will be the pole of the system. Let us examine the nature of response of first-order system for an impulse input and various values of poles. For simplicity let us choose sign-magnitude representation for binary product and response. Let the product be quantized to four bit binary (excluding sign bit) by upward rounding.

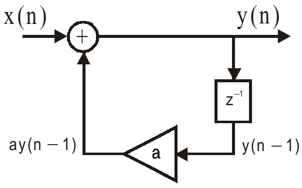


Fig 8.10a : *Ideal system.*

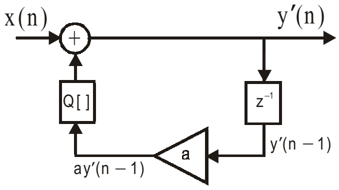


Fig 8.10b : *Nonlinear system due to product quantization.*

Fig 8.10 : *First-order recursive system.*

Let, $y'(n) = 0$; for $n < 0$ | $x(n) = \frac{15}{16}$; for $n = 0$, $= 0$; for $n \neq 0$ | $a = \frac{1}{2}$

When $n = 0,\ y'(n) = y'(0) = Q\big[a\, y'(n-1)\big] + x(n) = Q\big[a\, y'(-1)\big] + x(0)$ $\quad\boxed{\text{Using equation (8.37).}}$

$$= Q\left[\frac{1}{2} \times 0\right] + \frac{15}{16} = Q[0] + \frac{15}{16} = 0 + \frac{15}{16}$$

$$= 0.9375_{10} = +\frac{15}{16}$$

$$+.9375_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.11110_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.11110_2 \xrightarrow[\text{to 4-bits}]{\substack{\text{Round}\\ \text{magnitude}}} 0.1111_2$$

$$0.1111_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.1111_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.9375_{10} = +\frac{15}{16}$$

When $n = 1,\ y'(n) = y'(1) = Q\big[a\, y'(n-1)\big] + x(n) = Q\big[a\, y'(0)\big] + x(1)$ $\quad\boxed{\text{Using equation (8.37).}}$

$$= Q\left[\frac{1}{2} \times \frac{15}{16}\right] + 0 = Q\big[0.46875\big] = 0.5_{10} = +\frac{8}{16}$$

$$+.46875_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.01111_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.01111_2 \xrightarrow[\text{to 4-bits}]{\substack{\text{Round}\\ \text{magnitude}}} 0.1000_2$$

$$0.1000_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.1000_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.5_{10} = +\frac{1}{2} = +\frac{8}{16}$$

When $n = 2,\ y'(n) = y'(2) = Q\big[a\, y'(n-1)\big] + x(n) = Q\big[a\, y'(1)\big] + x(2)$ $\quad\boxed{\text{Using equation (8.37).}}$

$$= Q\left[\frac{1}{2} \times \frac{8}{16}\right] + 0 = Q\big[0.25\big] = 0.25_{10} = +\frac{4}{16}$$

$$+.25_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.01000_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.01000_2 \xrightarrow[\text{to 4-bits}]{\substack{\text{Round}\\ \text{magnitude}}} 0.0100_2$$

$$0.0100_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.0100_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.25_{10} = +\frac{1}{4} = +\frac{4}{16}$$

Similarly, the y'(n) can be calculated for other values of n.

Here, when $n = 3,\ y'(n) = y'(3) = \frac{2}{16} = 0.0010_2$

when $n = 4,\ y'(n) = y'(4) = \frac{1}{16} = 0.0001_2$

when $n = 5,\ y'(n) = y'(5) = \frac{1}{16} = 0.0001_2$

For all values of n ³ 4, the $y'(n) = 1/16 = 0.0001_2$

Hence the system output becomes constant for n ³ 4. Also for n ³ 4, the input x(n) is zero. Therefore the system enters a limit cycle for n ³ 4 even though the input becomes zero.

The limit cycles of the first-order system described above have been calculated for various values of poles and listed in table 8.7.

Table 8.7 : Limit Cycles of First-Order System

n	x(n)	y'(n)							
		a = 1/2		a = –1/2		a = 3/4		a = –3/4	
		Binary	Decimal	Binary	Decimal	Binary	Decimal	Binary	Decimal
0	15/16	0.1111	$\frac{15}{16}$	0.1111	$+\frac{15}{16}$	0.1011	$\frac{11}{16}$	0.1011	$+\frac{11}{16}$
1	0	0.1000	$\frac{8}{16}$	1.1000	$-\frac{8}{16}$	0.1000	$\frac{8}{16}$	1.1000	$-\frac{8}{16}$
2	0	0.0100	$\frac{4}{16}$	0.0100	$+\frac{4}{16}$	0.0110	$\frac{6}{16}$	0.0110	$+\frac{6}{16}$
3	0	0.0010	$\frac{2}{16}$	1.0010	$-\frac{2}{16}$	0.0101	$\frac{5}{16}$	1.0101	$-\frac{5}{16}$
4	0	0.0001	$\frac{1}{16}$	0.0001	$+\frac{1}{16}$	0.0100	$\frac{4}{16}$	0.0100	$+\frac{4}{16}$
5	0	0.0001	$\frac{1}{16}$	1.0001	$-\frac{1}{16}$	0.0011	$\frac{3}{16}$	1.0011	$-\frac{3}{16}$
6	0	0.0001	$\frac{1}{16}$	0.0001	$+\frac{1}{16}$	0.0010	$\frac{2}{16}$	0.0010	$+\frac{2}{16}$
7	0	0.0001	$\frac{1}{16}$	1.0001	$-\frac{1}{16}$	0.0010	$\frac{2}{16}$	1.0010	$-\frac{2}{16}$
8	0	0.0001	$\frac{1}{16}$	0.0001	$+\frac{1}{16}$	0.0010	$\frac{2}{16}$	0.0010	$+\frac{2}{16}$

The limit cycles shown in table 8.7 are due to quantization of the product by rounding. It can be shown that most of the limit cycles (not all) can be eliminated if quantization is performed by truncation, but truncation is not preferred in product quantization, due to the biased errors it may introduce in the output.

In a limit cycle the amplitudes of the output are confined to a range of values, which is called the ***dead band*** of the filter.

For a first-order system described by the equation, y(n) = a y(n–1) + x(n), the dead band is given by,

$$\text{Dead band} = \pm\frac{2^{-B}}{1-|a|} = -\frac{2^{-B}}{1-|a|} \text{ to } +\frac{2^{-B}}{1-|a|} \qquad \text{..... (8.38)}$$

where, B = Number of binary bits (including sign bit) used to represent the product.

For a second-order system described by the equation, $y(n) = a_1\, y(n-1) + a_2\, y(n-2) + x(n)$, the dead band of the filter is given by,

$$\text{Dead band} = \pm\frac{2^{-B}}{1-|a_2|} = -\frac{2^{-B}}{1-|a_2|} \text{ to } +\frac{2^{-B}}{1-|a_2|} \qquad \text{..... (8.39)}$$

Example 8.20

Explain the characteristics of a limit cycle oscillation with respect to the system described by the equation

y(n) = 0.95 y(n – 1) + x(n),

when the product is quantized to 5-bits by rounding. The system is excited by an input x(n) = 0.75 for n = 0 and x(n) = 0 for n ¹ 0.

Also, determine the dead band of the filter.

Solution

Given that, $y(n) = 0.95\, y(n-1) + x(n)$.

The recursive realization of the given system involves the product $0.95\, y(n-1)$. Let $y'(n)$ be the response of the system when the product is quantized by rounding (upward rounding).

$$\therefore\; y'(n) = Q[0.95\, y(n-1)] + x(n)$$

where, $Q[\,]$ stands for quantization of product.

Given that the products are quantized to 5-bits

Let us choose 5-bit sign-magnitude binary representation to represent the quantized product with 4-bits for magnitude and1-bit for sign.

Given that, $x(n) = 0.75$; for $n = 0$

$\qquad\qquad = 0$; for $n \neq 0$

Let, $y'(n) = 0$; for $n < 0$

$\therefore$ When $n = -1$, $y'(n) = y'(-1) = 0$

When $n = 0$,

$$\begin{aligned} y'(n) = y'(0) &= Q[0.95\, y'(n-1)] + x(n) \\ &= Q[0.95\, y'(-1)] + x(0) \\ &= Q[0.95 \times 0] + 0.75 \\ &= Q[0] + 0.75 \\ &= 0.75_{10} = 0.1100_2 \end{aligned}$$

Binary to decimal conversion

$$.1100_2 = (1\times 2^{-1}) + (1 \times 2^{-2}) + (0\times 2^{-3}) + (0 \times 2^{-4}) = .75_{10}$$

Decimal to binary conversion

$$\begin{array}{r} .75 \\ \times\ 2 \\ \hline \boxed{1}.50 \\ \times\ 2 \\ \hline \boxed{1}.00 \end{array}$$

$.11_2$

$.11000_2$

$$+.75_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.11000_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.11000_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 0.1100_2$$

$$0.1100_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.1100_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.75_{10}$$

When $n = 1$,

$$\begin{aligned} y'(n) = y'(1) &= Q[0.95\, y'(n-1)] + x(n) \\ &= Q[0.95\, y'(0)] + x(1) \\ &= Q[0.95 \times 0.75] + 0 \\ &= Q[0.7125] \\ &= 0.6875_{10} = 0.1011_2 \end{aligned}$$

Binary to decimal conversion

$$.1011_2 = (1\times 2^{-1}) + (0\times 2^{-2}) + (1\times 2^{-3}) + (1\times 2^{-4}) = .6875_{10}$$

Decimal to binary conversion

$$\begin{array}{r} .7125 \\ \times\ 2 \\ \hline \boxed{1}.4250 \\ \times\ 2 \\ \hline \boxed{0}.8500 \\ \times\ 2 \\ \hline \boxed{1}.7000 \\ \times\ 2 \\ \hline \boxed{1}.4000 \\ \times\ 2 \\ \hline \boxed{0}.8000 \end{array}$$

$.1\ 0\ 1\ 1\ 0_2$

$$+.7125_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.10110_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.10110_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 0.1011_2$$

$$0.1011_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.1011_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.6875_{10}$$

When n = 2,

$$y'(n) = y'(2) = Q[0.95\ y'(n-1)] + x(n)$$
$$= Q[0.95\ y'(1)] + x(2)$$
$$= Q[0.95 \times 0.6875] + 0$$
$$= Q[0.653125]$$
$$= 0.625_{10} = 0.1010_2$$

Binary to decimal conversion

$$.1010_2 = (1 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3}) + (0 \times 2^{-4}) = .625_{10}$$

Decimal to binary conversion

$$\begin{array}{r} .653125 \\ \times\ 2 \\ \hline \boxed{1}.306250 \\ \times\ 2 \\ \hline \boxed{0}.612500 \\ \times\ 2 \\ \hline \boxed{1}.225000 \\ \times\ 2 \\ \hline \boxed{0}.450000 \\ \times\ 2 \\ \hline \boxed{0}.900000 \end{array}$$

$$.10100_2$$

$$+.653125_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.10100_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.10100_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 0.1010_2$$

$$0.1010_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.1010_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.625_{10}$$

When n = 3,

$$y'(n) = y'(3) = Q[0.95\ y'(n-1)] + x(n)$$
$$= Q[0.95\ y'(2)] + x(3)$$
$$= Q[0.95 \times 0.625] + 0$$
$$= Q[0.59375]$$
$$= 0.625_{10} = 0.1010_2$$

Decimal to binary conversion

$$\begin{array}{r} .59375 \\ \times\ 2 \\ \hline \boxed{1}.18750 \\ \times\ 2 \\ \hline \boxed{0}.37500 \\ \times\ 2 \\ \hline \boxed{0}.75000 \\ \times\ 2 \\ \hline \boxed{1}.50000 \\ \times\ 2 \\ \hline \boxed{1}.0000 \end{array}$$

$$.10011_2$$

$$.59375_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .10011_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.10011_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 0.1010_2$$

From the above calculations it can be observed that y′(n) for n = 3, is same as that for n = 2, i.e., y′(3) = y′(2). Hence for all values of n, where $n \geq 2$, the output y′(n) will remain same as 0.625_{10} (or 0.1010_2). Therefore the system enters a limit cycle when n = 2. The limit cycle of the system is shown in table 1.

For the first-order system with only poles the dead band is given by,

$$\text{Dead band} = \pm \frac{2^{-B}}{1 - |a|}$$

Here, B = 5, |a| = 0.95

$$\therefore\ \text{Dead band} = \pm \frac{2^{-5}}{1 - 0.95} = \pm 0.625$$
$$= -0.625 \text{ to } +0.625$$

Table 1

n	x(n)	y'(n) Decimal	y'(n) Binary
0	0.75	0.75	0.1100
1	0	0.6875	0.1011
2	0	0.625	0.1010
3	0	0.625	0.1010
4	0	0.625	0.1010
5	0	0.625	0.1010
.	.	.	.
.	.	.	.
.	.	.	.

Example 8.21

Study the limit cycle behaviour of the system described by w(n) = Q[a w(n – 1)] + x(n) where w(n) is the output of the system and Q[·] is quantization. Assume that, a = –0.875, x(0) = 0.75 and x(n) = 0 for n > 0. choose 4-bits for quantization.

Solution

Given that, $w(n) = Q[a\, w(n-1)] + x(n)$

The recursive realization of the given system involves the product $[a\, w(n-1)]$ and the product is quantized. Let us assume that the product is quantized by rounding (upward rounding). Here $w(n)$ is the response of the system when the product is quantized. Let us choose a 4-bit sign-magnitude binary representation to represent the quantized product with 3-bits for magnitude and 1-bit for sign.

Given that, $a = -0.875_{10}$ | $x(n) = 0.75$; $n = 0$
$\quad = 0$; $n \neq 0$.

Let, $w(n) = 0$ for $n < 0$.

When n = 0,

$$w(n) = w(0) = Q[-0.875\, w(n-1)] + x(n)$$
$$= Q[-0.875\, w(-1)] + x(0)$$
$$= Q[-0.875 \times 0] + 0.75$$
$$= Q[0] + 0.75$$
$$= 0.75_{10} = 0.110_2$$

$$+.75_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.1100_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.1100_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.110_2$$

$$0.110_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.110_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.75_{10}$$

Binary to decimal conversion

$.110_2 = (1 \times 2^{-1}) + (1 \times 2^{-2}) + (0 \times 2^{-3}) = .75_{10}$

Decimal to binary conversion

$.75 \times 2 = \boxed{1}.50$

$.50 \times 2 = \boxed{1}.00$

$.11_2$

$.1100_2$

When n = 1,

$$w(n) = w(1) = Q[-0.875\, w(n-1)] + x(n)$$
$$= Q[-0.875\, w(0)] + x(1)$$
$$= Q[-0.875 \times 0.75] + 0$$
$$= Q[-0.65625]$$
$$= -0.625_{10} = 1.101_2$$

$$-.65625_{10} \xrightarrow[\text{binary}]{\text{Convert to}} -.1010_2 \xrightarrow[\text{bit}]{\text{Add sign}} 1.1010_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 1.101_2$$

$$1.101_2 \xrightarrow[\text{sign bit}]{\text{Extract}} -.101_2 \xrightarrow[\text{decimal}]{\text{Convert to}} -.625_{10}$$

Binary to decimal conversion

$.101_2 = (1 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3}) = .625_{10}$

Decimal to binary conversion

$.65625 \times 2 = \boxed{1}.31250$

$.31250 \times 2 = \boxed{0}.62500$

$.62500 \times 2 = \boxed{1}.25000$

$.25000 \times 2 = \boxed{0}.50000$

$.1\,0\,1\,0$

When n = 2,

$$w(n) = w(2) = Q[-0.875\, w(n-1)] + x(n)$$
$$= Q[-0.875\, w(1)] + x(2)$$
$$= Q[-0.875 \times -0.625] + 0$$
$$= Q[0.546875]$$
$$= 0.5_{10} = 0.100_2$$

$$+.546875_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.1000_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.1000_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.100_2$$

$$0.100_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.100_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.5_{10}$$

Binary to decimal conversion

$.100_2 = (1 \times 2^{-1}) + (0 \times 2^{-2}) + (0 \times 2^{-3}) = .5_{10}$

Decimal to binary conversion

$.546875 \times 2 = \boxed{1}.093750$

$.093750 \times 2 = \boxed{0}.187500$

$.187500 \times 2 = \boxed{0}.375000$

$.375000 \times 2 = \boxed{0}.750000$

$.1\,0\,0\,0_2$

When n = 3,

$$\begin{aligned} w(n) = w(3) &= Q[-0.875\ w(n-1)] + x(n) \\ &= Q[-0.875\ w(2)] + x(3) \\ &= Q[-0.875 \times 0.5] + 0 \\ &= Q[-0.4375] \\ &= -0.5_{10} = 1.100_2 \end{aligned}$$

Binary to decimal conversion

$.100_2 = (1 \times 2^{-1}) + (0 \times 2^{-2}) + (0 \times 2^{-3}) = .5_{10}$

Decimal to binary conversion

$$\begin{array}{r} .4375 \\ \times\ 2 \\ \hline \boxed{0}.8750 \\ \times\ 2 \\ \hline \boxed{1}.7500 \\ \times\ 2 \\ \hline \boxed{1}.5000 \\ \times\ 2 \\ \hline \boxed{1}.0000 \end{array}$$

$.0\,1\,1\,1_2$

$$-.4375_{10} \xrightarrow[\text{binary}]{\text{Convert to}} -.0111_2 \xrightarrow[\text{bit}]{\text{Add sign}} 1.0111_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 1.100_2$$

$$1.100_2 \xrightarrow[\text{sign bit}]{\text{Extract}} -.100_2 \xrightarrow[\text{decimal}]{\text{Convert to}} -.5_{10}$$

When n = 4,

$$\begin{aligned} w(n) = w(4) &= Q[-0.875\ w(n-1)] + x(n) = Q[-0.875\ w(3)] + x(3) \\ &= Q[-0.875 \times -0.5] + 0 = Q[0.4375] = 0.5_{10} \end{aligned}$$

When n = 5,

$$\begin{aligned} w(n) = w(5) &= Q[-0.875\ w(n-1)] + x(n) = Q[-0.875\ w(4)] + x(4) \\ &= Q[-0.875 \times 0.5] + 0 = Q[-0.4375] = -0.5_{10} \end{aligned}$$

From the above calculations it can be observed that w(n) for n = 3, is negative of w(n) for n = 2, i.e., w(3) = –w(2). Hence for all values of n, where $n \geq 2$, the output w(n) will alternate between $+.5_{10}$ and $-.5_{10}$. Therefore the system enters a limit cycle when n = 2. The limit cycle of the system is shown in table 1.

For the first-order system with only poles the dead band is given by,

$$\text{Dead band} = \pm \frac{2^{-B}}{1 - |a|}$$

Here, B = 4, |a| = 0.875

$$\therefore \text{ Dead band} = \pm \frac{2^{-4}}{1 - 0.875} = \pm 0.5 = -0.5 \text{ to } +0.5$$

Table 1

n	x(n)	w(n)	
		Decimal	Binary
0	0.75	0.75	0.110
1	0	0.625	0.101
2	0	0.5	0.100
3	0	–0.5	1.100
4	0	0.5	0.100
5	0	–0.5	1.100
.	.	.	.
.	.	.	.
.	.	.	.

Example 8.22

An LTI system is characterized by the difference equation, y(n) = 0.87y(n – 1) + x(n).

Determine the limit cycle behaviour and the deadband of the system when x(n) = 0 and y(–1) = 0.61. Assume that the product is quantized to 4-bits by rounding.

Solution

Given that, y(n) = 0.87 y(n – 1)] + x(n)

The recursive realization of the given system involves the product "0.87 y(n–1)" and the product is quantized to 4-bits by rounding.

The equation of the system with quantized product is,

$$y(n) = Q[0.87\ y(n-1)] + x(n)$$

where, Q[0.87 y(n – 1)] represents quantized product.

Let us choose a 4-bit sign-magnitude binary representation to represent the quantized product with 3-bits for magnitude and 1-bit for sign.

Given that, y(–1) = 0.61 and x(n) = 0.

When n = 0,

$$\begin{aligned} y(n) = y(0) &= Q[0.87\ y(n-1)] + x(n) \\ &= Q[0.87\ y(-1)] + x(0) \\ &= Q[0.87 \times 0.61] + 0 \\ &= Q[0.5307] \\ &= 0.5_{10} \end{aligned}$$

Binary to decimal conversion

$.100_2 = (1 \times 2^{-1}) + (0 \times 2^{-2}) + (0 \times 2^{-3}) = .5_{10}$

Decimal to binary conversion

$$\begin{array}{r} .5307 \\ \times\ 2 \\ \hline \boxed{1}.0614 \\ \times\ 2 \\ \hline \boxed{0}.1228 \\ \times\ 2 \\ \hline \boxed{0}.2456 \\ \times\ 2 \\ \hline \boxed{0}.4912 \end{array}$$

$.1000_2$

$$+.5307_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.1000_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.1000_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.100_2$$

$$0.100_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.100_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.5_{10}$$

When n = 1,

$$\begin{aligned} y(n) = y(1) &= Q[0.87\ y(n-1)] + x(n) \\ &= Q[0.87\ y(0)] + x(0) \\ &= Q[0.87 \times 0.5] + 0 \\ &= Q[0.435] \\ &= 0.375_{10} \end{aligned}$$

Binary to decimal conversion

$.011_2 = (0 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3}) = .375_{10}$

Decimal to binary conversion

$$\begin{array}{r} .435 \\ \times\ 2 \\ \hline \boxed{0}.87 \\ \times\ 2 \\ \hline \boxed{1}.74 \\ \times\ 2 \\ \hline \boxed{1}.48 \\ \times\ 2 \\ \hline \boxed{0}.96 \end{array}$$

$.0110_2$

$$+.435_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.0110_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.0110_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.011_2$$

$$0.011_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.011_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.375_{10}$$

When n = 2,

$$\begin{aligned} y(n) = y(2) &= Q[0.87\ y(n-1)] + x(n) \\ &= Q[0.87\ y(1)] + x(2) \\ &= Q[0.87 \times 0.375] + 0 \\ &= Q[0.32625] \\ &= 0.375_{10} \end{aligned}$$

Binary to decimal conversion

$.011_2 = (0 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3}) = .375_{10}$

Decimal to binary conversion

$$\begin{array}{r} .32625 \\ \times\ 2 \\ \hline \boxed{0}.6525 \\ \times\ 2 \\ \hline \boxed{1}.305 \\ \times\ 2 \\ \hline \boxed{0}.61 \\ \times\ 2 \\ \hline \boxed{1}.24 \end{array}$$

$.0101_2$

$$+.32625_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.0101_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.0101_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.011_2$$

$$0.011_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.011_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +0.375_{10}$$

From the above calculations it can be observed that y(n) for n = 2 is same as that for n = 1, i.e., y(2) = y(1). Hence for all values of n, where $n \geq 1$, the output y(n) will remain same as 0.375_{10} (or 0.011_2). Therefore, the system enters a limit cycle when n = 1. The limit cycle of the system is shown in table 1.

For the first-order system with only poles the dead band is given by,

$$\text{Dead band} = \pm \frac{2^{-B}}{1-|a|}$$

Here, B = 4, |a| = 0.87

$$\therefore \text{ Dead band} = \pm \frac{2^{-4}}{1-0.87} = \pm 0.4808 = +0.4808 \text{ to } -0.4808$$

Note : $Q[0.87 \times 0.4808] = Q[0.418296] = 0.375_{10}$.

Table 1

n	x(n)	y(n) Decimal	y(n) Binary
–1	0	0.61	....
0	0	0.5	0.100
1	0	0.375	0.011
2	0	0.375	0.011
3	0	0.375	0.011
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.

8.8.2 Overflow Limit Cycle

In fixed point addition of two binary numbers the overflow occurs when the sum exceeds the finite word length of the register used to store the sum. The overflow in addition may lead to oscillations in the output which is referred to as ***overflow limit cycles***.

The overflow occurs when the sum exceeds the dynamic range of number system. When binary fraction format is used for computing, the dynamic range is (–1, 1). The overflow is explained by considering 4-bit binary fraction number in two's complement representation. The possible numbers for 4-bit representation are shown in table 8.8.

Let us add $+\frac{3}{8}$ and $+\frac{5}{8}$ in two's complement addition

$$+\frac{3}{8} \Rightarrow 0.011$$

$$+\frac{5}{8} \Rightarrow 0.101$$

$$\frac{3}{8} + \frac{5}{8} \Rightarrow \overline{1.000} \Rightarrow -\frac{8}{8} = -1$$

The actual sum of $+\frac{3}{8}$ and $+\frac{5}{8}$ is +1 but due to overflow it becomes –1. The input-output (transfer) characteristics of two's complement adder is shown in fig 8.11. In fig 8.11, the x represents the actual sum and f(x) represents two's complement sum.

Table 8.8 : Four-bit Two's Complement number

Binary	Two's complement
0	0.000
1/8	0.001
2/8	0.010
3/8	0.011
4/8	0.100
5/8	0.101
6/8	0.110
7/8	0.111
–1 = –8/8	1.000
–7/8	1.001
–6/8	1.010
–5/8	1.011
–4/8	1.100
–3/8	1.101
–2/8	1.110
–1/8	1.111

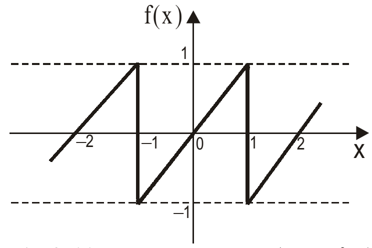


Fig 8.11 : *Input - output (transfer) characteristics of two's complement adder.*

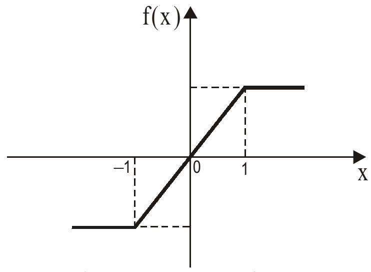


Fig 8.12 : *Characteristics of saturation adder.*

The overflow oscillations can be eliminated if saturation arithmetic is performed. The characteristics of saturation adder is shown in fig 8.12. In ***saturation arithmetic***, when an overflow is sensed, the output (sum) is set equal to maximum allowable value and when an underflow is sensed, the output (sum) is set equal to minimum allowable value. The saturation arithmetic introduce nonlinearity in the adder and the signal distortion due to this nonlinearity is small if the saturation occurs infrequently.

8.8.3 Scaling to Prevent Overflow

The two methods of preventing overflow are saturation arithmetic and scaling the input signal to the adder. In saturation arithmetic, undesirable signal distortion is introduced. In order to limit the signal distortion due to frequent overflows, the input signal to the adder can be scaled such that the overflow becomes a rare event.

Let, $x(n)$ = Input to the system.

$h_k(n)$ = Impulse response between the input and output of node-k.

$y_k(n)$ = Response of the system at node-k.

$$\text{Now, } y_k(n) = h_k(n) * x(n) = \sum_{m=-\infty}^{+\infty} h_k(m)\, x(n-m) \qquad \text{..... (8.40)}$$

Using equation (2.33) of chapter - 2.

On taking absolute value of equation (8.40) we get,

$$|y_k(n)| = \left| \sum_{m=-\infty}^{+\infty} h_k(m)\, x(n-m) \right| = \sum_{m=-\infty}^{+\infty} |h_k(m)|\, |x(n-m)| \qquad \text{..... (8.41)}$$

Let the maximum value of input be A_x such that $|x(n-m)|$ £ A_x.

On substituting A_x for $|x(n-m)|$ in equation (8.41) we get,

$$|y_k(n)| = \sum_{m=-\infty}^{+\infty} |h_k(m)|\, A_x = A_x \sum_{m=-\infty}^{+\infty} |h_k(m)| \qquad \text{..... (8.42)}$$

If the dynamic range of the digital system is (–1, 1), then $|y_k(n)| < 1$.

$$\therefore\ A_x \sum_{m=-\infty}^{\infty} |h_k(m)| < 1 \quad \text{or} \quad A_x < \frac{1}{\sum\limits_{m=-\infty}^{\infty} |h_k(m)|} \qquad \text{..... (8.43)}$$

The maximum value of the input should satisfy the condition of equation (8.43) in order to avoid overflow. The scaling using equation (8.43) will be severe so that the precision used to represent the input signal will be lost.

Another approach to scaling is to scale the input so that,

$$\sum_{n=-\infty}^{\infty} |y_k(n)|^2 \le S^2 \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad ; \quad \text{where, S is the scaling factor.} \qquad \text{..... (8.44)}$$

Using Parseval's theorem and residue theorem the expression for scaling factor shown in equation (8.45) can be determined.

$$S^2 = \frac{1}{\sum\limits_{n=-\infty}^{+\infty} |h_k(n)|^2} = \frac{1}{\frac{1}{2\pi j} \oint_c T_s(z)\, T_s(z^{-1})\, z^{-1}\, dz}$$

$$= \frac{1}{\sum\limits_{i=1}^{N} \text{Res}\left[T_s(z)\, T_s(z^{-1})\, z^{-1}\right]\Big|_{z=p_i}} = \frac{1}{\sum\limits_{i=1}^{N} \left[(z-p_i)\, T_s(z)\, T_s(z^{-1})\, z^{-1}\right]\Big|_{z=p_i}} \qquad \text{..... (8.45)}$$

where, p_1, p_2, ... p_N are poles of $T_s(z)\, T_s(z^{-1})\, z^{-1}$ that lie inside the unit circle in z-plane.
and, $T_s(z)$ is the transfer function between system input and output of node-k.

<u>*Proof :*</u>

$$\text{Let, } \sum_{n=-\infty}^{+\infty} |y_k(n)|^2 = S^2 \sum_{n=-\infty}^{+\infty} |x(n)|^2 \quad \boxed{\text{Using equation (8.44).}} \quad .....(8.46)$$

Let, the maximum value of input be A_x such that $|x(n)| £ A_x$.

$$\therefore \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=-\infty}^{+\infty} A_x^2 = C \text{ (constant)} \quad .....(8.47)$$

From equations (8.46) and (8.47) we get,

$$\sum_{n=-\infty}^{+\infty} |y_k(n)|^2 = S^2 C$$

$$\therefore S^2 = \frac{C}{\sum_{n=-\infty}^{+\infty} |y_k(n)|^2} = \frac{C}{\sum_{n=-\infty}^{+\infty} \left| \sum_{m=-\infty}^{+\infty} h_k(m)\, x(n-m) \right|^2}$$

$$= \frac{C}{\sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |h_k(m)|^2 \, |x(n-m)|^2} = \frac{C}{\sum_{m=-\infty}^{+\infty} |h_k(m)|^2 \sum_{n=-\infty}^{+\infty} |x(n-m)|^2}$$

$$= \frac{C}{\sum_{m=-\infty}^{+\infty} |h_k(m)|^2 \, C} \quad \boxed{\text{Using equation (8.47).}}$$

$$= \frac{1}{\sum_{n=-\infty}^{+\infty} |h_k(n)|^2} \quad \boxed{\text{Let, } m = n}$$

Some examples of transfer function between input of the system and output of summing nodes are given below.

Consider the second-order direct form-II structure of IIR system shown in fig 8.13. In the system shown in fig 8.13, a scaling multiplier with scale factor S is introduced to avoid overflow in adder-A (or register A). Here x(n) is the input to the system (when there is no scaling multiplier) and w(n) is the output of the adder. Therefore the transfer function between the input to the system and output of adder-A is given by,

$$T_s(z) = \frac{W(z)}{X(z)}$$

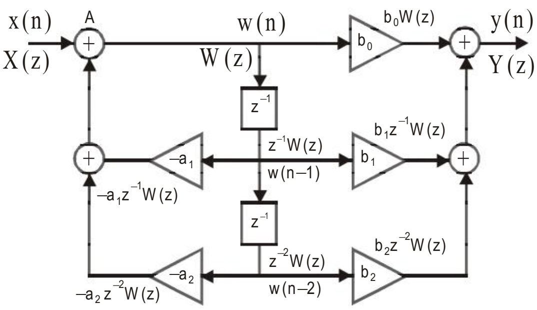


Fig 8.13a : *System without scaling multiplier.*

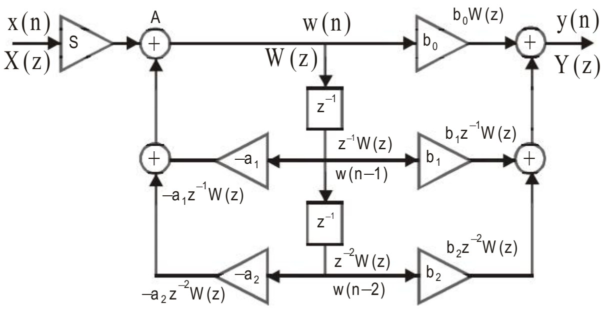


Fig 8.13b : *System with input scaling multiplier.*

Fig 8.13 : *Second-order direct form-II structure of IIR system.*

In fig 8.13a, on equating the input signals to the adder-A to the output signal of adder-A we get,

$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z)$$

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) = X(z)$$

$$W(z)\,[1 + a_1 z^{-1} + a_2 z^{-2}] = X(z)$$

$$\therefore\ T_s(z) = \frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} \qquad .....(8.48)$$

The equation (8.48) is the transfer function seen between input to the system and output of adder-A, for the system shown in fig 8.13.

Consider the cascaded structure of IIR system shown in fig 8.14. In the system shown in fig 8.14b, two scaling multipliers are introduced. The scaling multiplier at the input of $H_1(z)$ with scale factor S_A is provided to avoid overflow in adder A. The scaling multiplier at the input of $H_2(z)$ with scale factor S_B is provided to avoid overflow in adder B.

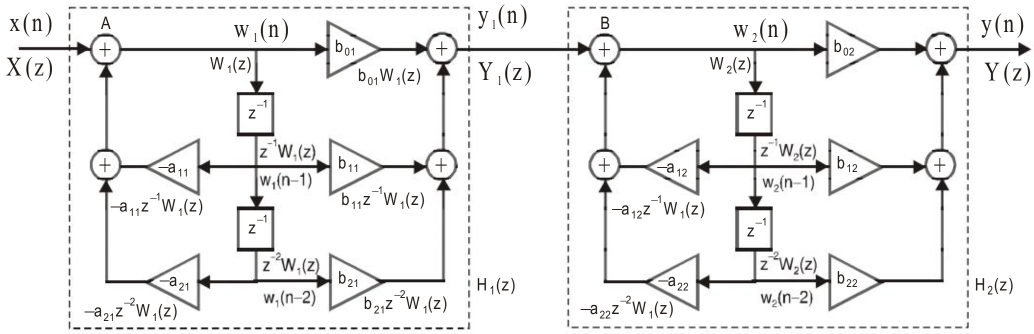


Fig 8.14a : *System without scaling multiplier.*

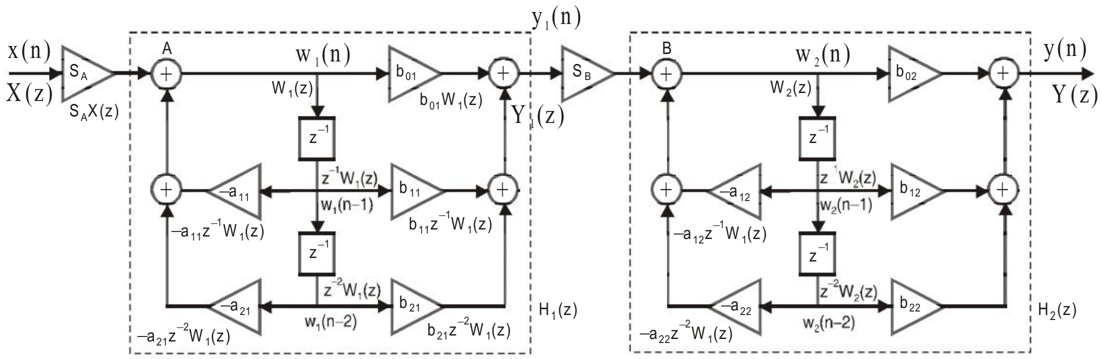


Fig 8.14b : *system with input scaling multiplier.*

Fig 8.14 : *Cascade structure of IIR system with two second-order direct form-II structure in cascade.*

In fig 8.14a, x(n) is the input to the system and $w_1(n)$ is the output of adder - A. Therefore, the transfer function seen between input to the system and output of adder-A is given by,

$$T_{SA}(z) = \frac{W_1(z)}{X(z)}$$

In fig 8.14a, on equating the input signals to the adder-A to the output signal of adder-A we get,

$$W_1(z) = X(z) - a_{11}z^{-1}\,W_1(z) - a_{21}z^{-2}\,W_1(z)$$

$$W_1(z) + a_{11}z^{-1}\,W_1(z) + a_{21}z^{-2}\,W_1(z) = X(z)$$

$$W_1(z)\,[1 + a_{11}z^{-1} + a_{21}z^{-2}] = X(z) \quad \text{.....(8.49)}$$

$$\therefore\; T_{SA}(z) = \frac{W_1(z)}{X(z)} = \frac{1}{1 + a_{11}\,z^{-1} + a_{21}\,z^{-2}} \quad \text{.....(8.50)}$$

In fig 8.14, x(n) is the input to the system and $w_2(n)$ is the output of adder-B. Therefore the transfer function seen between input to the system and output of adder-B is given by,

$$T_{SB}(z) = \frac{W_2(z)}{X(z)} = \frac{Y_1(z)}{X(z)} \times \frac{W_2(z)}{Y_1(z)} \qquad \text{(When } S_A \text{ is absent)} \quad \text{.....(8.51)}$$

$$T_{SB}(z) = \frac{W_2(z)}{S_A X(z)} = \frac{1}{S_A} \times \frac{Y_1(z)}{X(z)} \times \frac{W_2(z)}{Y_1(z)} \qquad \text{(When } S_A \text{ is present)} \quad \text{.....(8.52)}$$

In fig 8.14a, an equation for the output $Y_1(z)$ of system $H_1(z)$ can be obtained as shown below.

$$Y_1(z) = b_{01}\,W_1(z) + b_{11}z^{-1}\,W_1(z) + b_{21}z^{-2}\,W_1(z)$$

$$\therefore\; Y_1(z) = W_1(z)\,[b_{01} + b_{11}z^{-1} + b_{21}z^{-2}] \quad \text{.....(8.53)}$$

From equations (8.49) and (8.53) we get,

$$\frac{Y_1(z)}{X(z)} = \frac{W_1(z)\left[b_{01} + b_{11}z^{-1} + b_{21}z^{-2}\right]}{W_1(z)\left[1 + a_{11}z^{-1} + a_{21}z^{-2}\right]} = \frac{b_{01} + b_{11}z^{-1} + b_{21}z^{-2}}{1 + a_{11}z^{-1} + a_{21}z^{-2}} \quad \text{.....(8.54)}$$

In fig 8.14a, on equating the input signals to the adder-B to the output signal of adder-B we get,

$$W_2(z) = Y_1(z) - a_{12}z^{-1}\,W_2(z) - a_{22}z^{-2}\,W_2(z)$$

$$W_2(z) + a_{12}z^{-1}\,W_2(z) + a_{22}z^{-2}\,W_2(z) = Y_1(z)$$

$$W_2(z)\,[1 + a_{12}z^{-1} + a_{22}z^{-2}] = Y_1(z)$$

$$\frac{W_2(z)}{Y_1(z)} = \frac{1}{1 + a_{12}\,z^{-1} + a_{22}\,z^{-2}} \quad \text{.....(8.55)}$$

Using equations (8.54) and (8.55), the $T_{SB}(z)$ can be written as shown below.

When S_A is absent,

$$T_{SB}(z) = \frac{Y_1(z)}{X(z)} \times \frac{W_2(z)}{Y_1(z)} = \frac{b_{01} + b_{11}z^{-1} + b_{21}z^{-2}}{1 + a_{11}z^{-1} + a_{21}z^{-2}} \times \frac{1}{1 + a_{12}z^{-1} + a_{22}z^{-2}}$$

$$= \frac{b_{01} + b_{11}z^{-1} + b_{21}z^{-2}}{\left(1 + a_{11}z^{-1} + a_{21}z^{-2}\right)\left(1 + a_{12}z^{-1} + a_{22}z^{-2}\right)}$$

When S_A is present,

$$T_{SB}(z) = \frac{1}{S_A} \times \frac{Y_1(z)}{X(z)} \times \frac{W_2(z)}{Y_1(z)} = \frac{1}{S_A} \times \frac{b_{01} + b_{11}z^{-1} + b_{21}z^{-2}}{1 + a_{11}z^{-1} + a_{21}z^{-2}} \times \frac{1}{1 + a_{12}z^{-1} + a_{22}z^{-2}}$$

$$= \frac{b_{01} + b_{11}z^{-1} + b_{21}z^{-2}}{S_A\left(1 + a_{11}z^{-1} + a_{21}z^{-2}\right)\left(1 + a_{12}z^{-1} + a_{22}z^{-2}\right)}$$

Example 8.23

For the digital network shown in fig 1, find the transfer function, H(z) and scale factor S_0 to avoid overflow in register A_1.

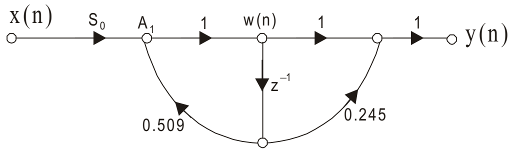


Fig 1 .

Solution

To find transfer function

The digital network without scale factor S_0 is shown in fig 2. The digital network has direct relation between time domain and z-domain as shown in fig 2.

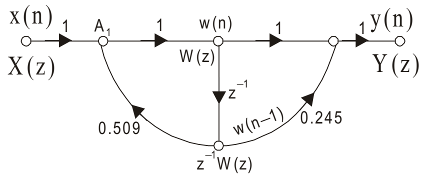


Fig 2.

In fig 2, on equating the sum of incoming signals of A_1 to outgoing signal of A_1 we get,

$X(z) + 0.509\, z^{-1}\, W(z) = W(z)$

\ $X(z) = W(z) - 0.509\, z^{-1}\, W(z)$

$= W(z)\ [1 - 0.509\, z^{-1})$ .....(1)

In fig 2, on equating the sum of incoming signals to output node to Y(z) we get,

$Y(z) = W(z) + 0.245\, z^{-1} W(z)$

\ $Y(z) = W(z)\ [1 + 0.245z^{-1}]$ .....(2)

The transfer function of the system, H(z) is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)\left[1 + 0.245\, z^{-1}\right]}{W(z)\left[1 - 0.509\, z^{-1}\right]}$$

$$= \frac{1 + 0.245\, z^{-1}}{1 - 0.509\, z^{-1}}$$ Using equation (1) and equation (2).

To find scale factor S_0

Let, $T_s(z)$ be the transfer function seen between the input to the system and output of register A_1.

$$\text{Now, } T_s(z) = \frac{W(z)}{X(z)} = \frac{1}{1 - 0.509\, z^{-1}}$$ Using equation (1).

The scale factor S_0 can be evaluated using equation (8.43).

$$\therefore\ S_0^2 = \frac{1}{\frac{1}{2\pi j}\oint_c T_s(z)\,T_s(z^{-1})\,z^{-1}\,dz} = \frac{1}{\sum_{i=1}^{N} \text{Res}\left[T_s(z)\,T_s(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}}$$

$$= \frac{1}{\sum_{i=1}^{N}\left[(z-p_i)T_s(z)\,T_s(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}}$$

where $p_1, p_2, \ldots\, p_N$ are poles of $T_s(z)\,T_s(z^{-1})\,z^{-1}$ that lie inside the unit circle in z-plane.

$$\text{Here, } T_s(z)\,T_s(z^{-1})z^{-1} = \frac{1}{1-0.509\,z^{-1}} \times \frac{1}{1-0.509\,z}\,z^{-1} = \frac{z^{-1}}{\left(1-\frac{0.509}{z}\right)(-0.509)\left(z-\frac{1}{0.509}\right)}$$

$$= \frac{-1.9646\,z^{-1}}{\left(\frac{z-0.509}{z}\right)(z-1.9646)} = \frac{-1.9646}{(z-0.509)\,(z-1.9646)}$$

Now, the poles of $T_s(z)\,T_s(z^{-1})\,z^{-1}$ are $p_1 = 0.509$, $p_2 = 1.9646$.

Here, $p_1 = 0.509$ is the only pole that lies inside the unit circle in z-plane.

$$\therefore\ S_0^2 = \frac{1}{\sum_{i=1}^{N}\left[(z-p_i)T_s(z)\,T_s(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}} = \frac{1}{\cancel{(z-0.509)} \times \frac{-1.9646}{\cancel{(z-0.509)}\,(z-1.9646)}\Big|_{z=0.509}}$$

$$= \frac{1}{\frac{-1.9646}{0.509-1.9646}} = \frac{1}{1.3497} = 0.7409$$

$$\therefore\ \text{Scale factor, } S_0 = \sqrt{S_0^2} = \sqrt{0.7409} = 0.8608$$

Example 8.24

For the digital network shown in fig 1, find the scale factor, S to avoid overflow in register A.

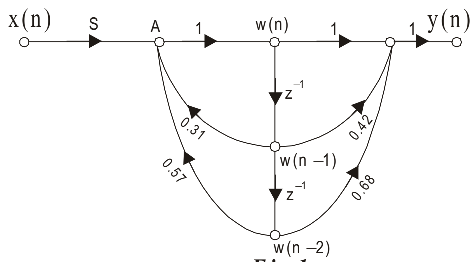


Fig 1.

Solution

The digital network without scale factor S is shown in fig 2. The digital network has direct relation between time domain and z-domain as shown in fig 2.

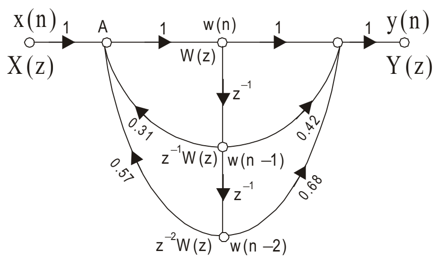


Fig 2.

Now, the scale factor S can be evaluated using the following equation.

$$S^2 = \frac{1}{\frac{1}{2\pi j}\oint_c T_s(z)\, T_s(z^{-1})\, z^{-1}\, dz} = \frac{1}{\sum_{i=1}^{N} \text{Res}\left[T_s(z)\, T_s(z^{-1})\, z^{-1}\right]\Big|_{z=p_i}}$$

Using equation (8.51).

$$= \frac{1}{\sum_{i=1}^{N} \left[(z-p_i) T_s(z)\, T_s(z^{-1})\, z^{-1}\right]\Big|_{z=p_i}}$$

where, $T_s(z) = \frac{W(z)}{X(z)}$ = Transfer function seen between the input to the system and output of register A.

and $p_1, p_2, \ldots\ldots p_N$ are poles of the function $T_s(z)\, T_s(z^{-1})\, z^{-1}$ that lie inside the unit circle in z-plane.

In order to determine $T_s(z)$, let us form an equation by equating the sum of incoming signals to A, to the outgoing signal of A as shown below.

$$X(z) + 0.31z^{-1}\, W(z) + 0.57z^{-2}\, W(z) = W(z)$$

$$\therefore \quad X(z) = W(z) - 0.31z^{-1} W(z) - 0.57z^{-2} W(z)$$

$$X(z) = W(z)\, [1 - 0.31z^{-1} - 0.57z^{-2}]\, W(z)$$

$$\therefore \quad T_s(z) = \frac{W(z)}{X(z)} = \frac{1}{1 - 0.31\, z^{-1} - 0.57\, z^{-2}}$$

Also, $T_s(z) = \frac{1}{z^{-2}(z^2 - 0.31\, z - 0.57)} = \frac{1}{z^{-2}(z - 0.9597)\,(z + 0.6497)}$

$$= \frac{1}{(1 - 0.9597\, z^{-1})\,(1 + 0.6497\, z^{-1})}$$

The roots of quadratic $z^2 - 0.31z - 0.57 = 0$ are,

$$z = \frac{0.31 \pm \sqrt{0.31 + 4 \times 0.57}}{2}$$

$$= \frac{0.31 \pm 1.6093}{2}$$

$$= 0.9597, \quad -0.6497$$

Now, $T_s(z)\, T_s(z^{-1}) z^{-1} = \frac{1}{(1 - 0.9597\, z^{-1})\,(1 + 0.6497\, z^{-1})} \times \frac{1}{(1 - 0.9597\, z)\,(1 + 0.6497\, z)} \times z^{-1}$

$$= \frac{z^{-1}}{\left(1 - \frac{0.9597}{z}\right)\left(1 + \frac{0.6497}{z}\right)(-0.9597)\left(z - \frac{1}{0.9597}\right)(0.6497)\left(z + \frac{1}{0.6497}\right)}$$

$$= \frac{-1.6038\, z^{-1}}{\left(\frac{z - 0.9597}{z}\right)\left(\frac{z + 0.6497}{z}\right)(z - 1.042)\,(z + 1.5392)}$$

$$= \frac{-1.6038\, z}{(z - 0.9597)\,(z + 0.6497)\,(z - 1.042)\,(z + 1.5392)}$$

Now, the poles of $T_s(z)\, T_s(z^{-1})\, z^{-1}$ are $p_1 = 0.9597$, $p_2 = -0.6497$, $p_3 = 1.042$ and $p_4 = -1.5392$.

Here, p_1 and p_2 are the two poles that lie inside the unit circle in z-plane.

$$\therefore \sum_{i=1}^{N} \left[(z - p_i)T_s(z)\ T_s(z^{-1})\ z^{-1}\right]\Bigg|_{z=p_i}$$

$$= \cancel{(z-0.9597)} \times \frac{-1.6038\,z}{\cancel{(z-0.9597)}\ (z+0.6497)\ (z-1.042)\ (z+1.5392)}\Bigg|_{z\,=\,0.9597}$$

$$+ \cancel{(z+0.6497)} \times \frac{-1.6038\,z}{(z-0.9597)\ \cancel{(z+0.6497)}\ (z-1.042)\ (z+1.5392)}\Bigg|_{z\,=\,-0.6497}$$

$$= \frac{-1.6038 \times 0.9597}{(0.9597+0.6497)\ (0.9597-1.042)\ (0.9597+1.5392)}$$

$$+ \frac{-1.6038 \times (-0.6497)}{(-0.6497-0.9597)\ (-0.6497-1.042)\ (-0.6497+1.5392)}$$

$$= 4.6502 + 0.4303 = 5.0805$$

$$\therefore\ S^2 = \frac{1}{\displaystyle\sum_{i=1}^{N} \left[(z-p_i)T_s(z)\ T_s(z^{-1})\ z^{-1}\right]\Bigg|_{z=p_i}} = \frac{1}{5.0805} = 0.1968$$

$$\therefore\ \text{Scale factor, } S = \sqrt{S^2} = \sqrt{0.1968} = 0.4436$$

Example 8.25

For the digital network shown in fig 1, find the scale factor, S_A to avoid overflow in register A. Then find the scale factor S_B to avoid overflow in register-B,

a) When S_A is present and **b)** When S_A is absent

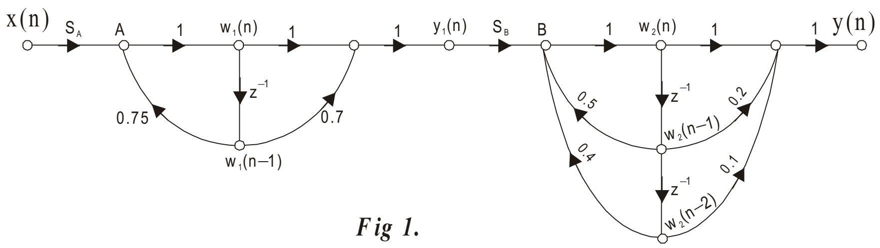


Fig 1.

Solution

To find scale factor, S_A

A part of digital network without scale factor is shown in fig 2. The digital network has direct relation between time domain and z-domain as shown in fig 2.

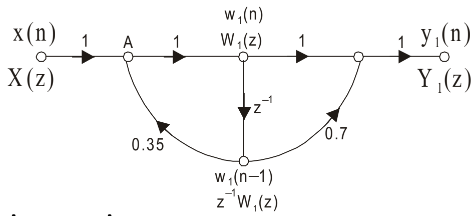


Fig 2.

Now, the scale factor S_A can be evaluated using the following equation.

$$\therefore\ S_A^2 = \frac{1}{2\pi j \oint_c T_{SA}(z)\ T_{SA}(z^{-1})\,z^{-1}\,dz} = \frac{1}{\displaystyle\sum_{i=1}^{N} \text{Res}\left[T_{SA}(z)\ T_{SA}(z^{-1})\,z^{-1}\right]\Bigg|_{z=p_i}} \qquad \boxed{\text{Using equation (8.51).}}$$

$$= \frac{1}{\displaystyle\sum_{i=1}^{N} \left[(z-p_i)T_{SA}(z)\ T_{SA}(z^{-1})\ z^{-1}\right]\Bigg|_{z=p_i}}$$

where, $T_{SA}(z) = \dfrac{W_1(z)}{X(z)}$ = Transfer function seen between the input to the system and output of register A.

and $p_1, p_2, \ldots\ldots p_N$ are poles of the function $T_{SA}(z)\, T_{SA}(z^{-1})\, z^{-1}$ that lie inside the unit circle in z-plane.

In order to determine $T_{SA}(z)$ let us form an equation by equating the sum of incoming signals to A, to the outgoing signal of A as shown below.

$$X(z) + 0.35z^{-1}\, W_1(z) = W_1(z)$$

\ $X(z) = W_1(z) - 0.35z^{-1} W_1(z)$

$$X(z) = [1 - 0.35z^{-1}]\, W_1(z)$$

$$\therefore\ T_{SA}(z) = \frac{W_1(z)}{X(z)} = \frac{1}{1 - 0.35\, z^{-1}} \qquad .....(1)$$

$$\therefore\ T_{SA}(z)\, T_{SA}(z^{-1}) z^{-1} = \frac{1}{1 - 0.35\, z^{-1}} \times \frac{1}{1 - 0.35\, z} \times z^{-1}$$

$$= \frac{z^{-1}}{\left(1 - \dfrac{0.35}{z}\right)\ (-0.35)\left(z - \dfrac{1}{0.35}\right)}$$

$$= \frac{-2.8571 z^{-1}}{\left(\dfrac{z - 0.35}{z}\right)(z - 2.8571)} = \frac{-2.8571}{(z - 0.35)\ (z - 2.8571)}$$

Now, the poles of $T_{SA}(z)\, T_{SA}(z^{-1})\, z^{-1}$ are $p_1 = 0.35$, $p_2 = 2.8571$.

Here, $p_1 = 0.35$ is the only pole that lies inside the unit circle in z-plane.

$$\therefore\ S_A^2 = \frac{1}{\displaystyle\sum_{i=1}^{N} \left[(z - p_i) T_{SA}(z)\ T_{SA}(z^{-1})\ z^{-1}\right]\Bigg|_{z = p_i}} = \frac{1}{\cancel{(z - 0.35)} \times \dfrac{-2.8571}{\cancel{(z - 0.35)}\ (z - 2.8571)}\Bigg|_{z=0.35}}$$

$$= \frac{1}{\dfrac{-2.8571}{0.35 - 2.8571}} = \frac{1}{1.1396} = 0.8775$$

$$\therefore\ \text{Scale factor, } S_A = \sqrt{S_A^2} = \sqrt{0.8775} = 0.9367 \qquad .....(2)$$

To Find Scale Factor, S_B

Case a : When S_A is present

The digital network without scale factor S_B is shown in fig 3. The digital network has direct relation between time domain and z-domain as shown in fig 3.

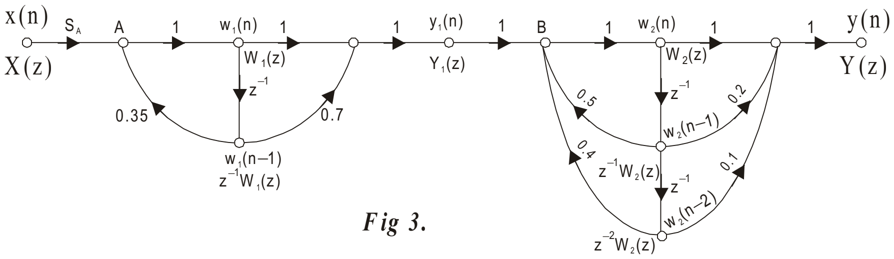


Fig 3.

Now, the scale factor S_B can be evaluated using the following equation.

$$\therefore\ S_B^2 = \frac{1}{2\pi j \oint_c T_{SB}(z)\, T_{SB}(z^{-1})\, z^{-1}\, dz} = \frac{1}{\sum_{i=1}^{N} \text{Res}\left[T_{SB}(z)\, T_{SB}(z^{-1})\, z^{-1}\right]\Big|_{z=p_i}}$$

Using equation (8.51).

$$= \frac{1}{\sum_{i=1}^{N} \left[(z-p_i) T_{SB}(z)\, T_{SB}(z^{-1})\, z^{-1}\right]\Big|_{z=p_i}}$$

where, $T_{SB}(z) = \dfrac{W_2(z)}{S_A\ X(z)}$ = Transfer function seen between the input to the system and output of register B.

and $p_1, p_2, \ldots\ldots p_N$ are poles of the function $T_{SB}(z)\, T_{SB}(z^{-1})\, z^{-1}$ that lie inside the unit circle in z-plane.

The transfer function $T_{SB}(z)$ can be obtained as shown below.

$$\text{Let, } T_{SB}(z) = \frac{W_2(z)}{S_A\, X(z)} = \frac{Y_1(z)}{S_A\, X(z)} \times \frac{W_2(z)}{Y_1(z)} = \frac{1}{S_A} \times \frac{W_1(z)}{X(z)} \times \frac{Y_1(z)}{W_1(z)} \times \frac{W_2(z)}{Y_1(z)}$$

Here, $S_A = 0.9367$ — From equation (2).

$$\frac{W_1(z)}{X(z)} = \frac{1}{1-0.35\,z^{-1}}$$

From equation (1).

The function $Y_1(z)/W_1(z)$ can be obtained by forming an equation for node with signal $Y_1(z)$ as shown below.

$$Y_1(z) = W_1(z) + 0.7z^{-1}\, W_1(z)$$

$$\backslash\ \ Y_1(z) = W_1(z)\ [1 + 0.7z^{-1}]$$

$$\therefore\ \frac{Y_1(z)}{W_1(z)} = 1 + 0.7z^{-1} \qquad .....(3)$$

The function $W_2(z)/Y_1(z)$ can be obtained by forming an equation for node B as shown below.

$$W_2(z) = Y_1(z) + 0.5z^{-1}\, W_2(z) + 0.4z^{-2}\, W_2(z)$$

$$\backslash\ \ W_2(z) - 0.5z^{-1} W_2(z) - 0.4z^{-2} W_2(z) = Y_1(z)$$

$$W_2(z)\ [1 - 0.5z^{-1} - 0.4z^{-2}] = Y_1(z)$$

$$\therefore\ \frac{W_2(z)}{Y_1(z)} = \frac{1}{1-0.5\,z^{-1}-0.4\,z^{-2}}$$

$$= \frac{1}{z^{-2}(z^2 - 0.5\,z - 0.4)}$$

$$= \frac{1}{z^{-2}(z-0.9301)\,(z+0.4301)}$$

$$= \frac{1}{(1-0.9301z^{-1})\,(1+0.4301z^{-1})}$$

The roots of quadratic, $z^2 - 0.5z - 0.4 = 0$ are,

$$z = \frac{0.5 \pm \sqrt{0.5^2 + 4 \times 0.4}}{2}$$

$$= \frac{0.5 \pm 1.3601}{2}$$

$$= 0.9301,\ \ -0.4301$$

$$\text{Now, } T_{SB}(z) = \frac{1}{S_A} \times \frac{W_1(z)}{X(z)} \times \frac{Y_1(z)}{X(z)} \times \frac{W_2(z)}{Y_1(z)}$$

$$= \frac{1}{0.9367} \times \frac{1}{1-0.35z^{-1}} \times (1+0.7z^{-1}) \times \frac{1}{\left(1-0.9301z^{-1}\right)\left(1+0.4301z^{-1}\right)}$$

$$= \frac{1.0676(1+0.7z^{-1})}{\left(1-0.35z^{-1}\right)\left(1-0.9301z^{-1}\right)\left(1+0.4301z^{-1}\right)}$$

$$\therefore\ T_{SB}(z)T_{SB}(z^{-1})z^{-1} = \frac{1.0676(1+0.7z^{-1})}{\left(1-0.35z^{-1}\right)\left(1-0.9301z^{-1}\right)\left(1+0.4301z^{-1}\right)} \times \frac{1.0676(1+0.7z)}{(1-0.35z)(1-0.9301z)\,(1+0.4301z)} \times z^{-1}$$

$$= \frac{1.0676\left(1+\dfrac{0.7}{z}\right) \times 1.0676 \times 0.7\left(z+\dfrac{1}{0.7}\right)\dfrac{1}{z}}{\left(1-\dfrac{0.35}{z}\right)\left(1-\dfrac{0.9301}{z}\right)\left(1+\dfrac{0.4301}{z}\right)(-0.35)\left(z-\dfrac{1}{0.35}\right)(-0.9301)\left(z-\dfrac{1}{0.9301}\right)(0.4301)\left(z+\dfrac{1}{0.4301}\right)}$$

$$= \frac{5.6983\left(\dfrac{z+0.7}{z}\right)(z+1.4286)\dfrac{1}{z}}{\left(\dfrac{z-0.35}{z}\right)\left(\dfrac{z-0.9301}{z}\right)\left(\dfrac{z+0.4301}{z}\right)(z-2.8571)(z-1.0752)(z+2.325)}$$

$$= \frac{5.6983\,z(z+0.7)(z+1.4286)}{(z-0.35)(z-0.9301)(z+0.4301)(z-2.8571)(z-1.0752)(z+2.325)}$$

Now, the poles of $T_{SB}(z)\ T_{SB}(z^{-1})\ z^{-1}$ are,

$$p_1 = 0.35,\ p_2 = 0.9301,\ p_3 = -0.4301,\ p_4 = 2.8571,\ p_5 = 1.0752,\ p_6 = -2.325$$

Here, p_1, p_2 and p_3 are the three poles that lie inside the unit circle in z-plane.

$$\therefore\ \sum_{i=1}^{N}\left[(z-p_i)T_{SB}(z)\ T_{SB}(z^{-1})\ z^{-1}\right]\Bigg|_{z=p_i}$$

$$= \cancel{(z-0.35)} \times \frac{5.6983\,z(z+0.7)(z+1.4286)}{\cancel{(z-0.35)}\ (z-0.9301)\ (z+0.4301)\ (z-2.8571)\ (z-1.0752)(z+2.325)}\Bigg|_{z\,=\,0.35}$$

$$+ \cancel{(z-0.9301)} \times \frac{5.6983\,z(z+0.7)(z+1.4286)}{(z-0.35)\ \cancel{(z-0.9301)}\ (z+0.4301)\ (z-2.8571)\ (z-1.0752)(z+2.325)}\Bigg|_{z\,=\,0.9301}$$

$$+ \cancel{(z+0.4301)} \times \frac{5.6983\,z(z+0.7)(z+1.4286)}{(z-0.35)\ (z-0.9301)\ \cancel{(z+0.4301)}\ (z-2.8571)\ (z-1.0752)(z+2.325)}\Bigg|_{z\,=\,-0.4301}$$

$$= \frac{5.6983 \times 0.35(0.35+0.7)(0.35+1.4286)}{(0.35-0.9301)\ (0.35+0.4301)\ (0.35-2.8571)(0.35-1.0752)\ (0.35+2.325)}$$

$$+ \frac{5.6983 \times 0.9301(0.9301+0.7)(0.9301+1.4286)}{(0.9301-0.35)\ (0.9301+0.4301)\ (0.9301-2.8571)(0.9301-1.0752)\ (0.9301+2.325)}$$

$$+ \frac{5.6983 \times (-0.4301)(-0.4301+0.7)(-0.4301+1.4286)}{(-0.4301-0.35)\ (-0.4301-0.9301)\ (-0.4301-2.8571)(-0.4301-1.0752)\ (-0.4301+2.325)}$$

$$\therefore \sum_{i=1}^{N} \left[(z-p_i)T_{SB}(z)\,T_{SB}(z^{-1})\,z^{-1}\right]\Big|_{z=p_i} = \frac{3.7246}{-2.2009} + \frac{20.378}{0.7182} + \frac{-0.6605}{9.9492}$$

$$= -1.6923 + 28.3737 - 0.0664 = 26.615 \qquad .....(4)$$

$$\therefore\ S_B^2 = \frac{1}{\sum_{i=1}^{N} \left[(z-p_i)T_{SB}(z)\,T_{SB}(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}} = \frac{1}{26.615} = 0.0376$$

$$\therefore\ \text{Scale factor, } S_B = \sqrt{S_B^2} = \sqrt{0.0376} = 0.1939$$

Case b : When S_A is absent

Let, $T_{SB2}(z)$ = Transfer function seen between the input to the system and output of register B when S_A is absent.

$$\text{Now, } T_{SB2}(z) = \frac{W_2(z)}{X(z)}$$

From the analysis in case-a we get,

$$T_{SB}(z) = \frac{W_2(z)}{S_A\,X(z)}$$

$$\therefore\ T_{SB2}(z) = S_A\,T_{SB}$$

$$\therefore\ T_{SB2}(z)\,T_{SB2}(z^{-1})\,z^{-1} = S_A T_{SB}(z)\,S_A T_{SB}(z^{-1})\,z^{-1} = S_A^2 T_{SB}(z)\,S_A T_{SB}(z^{-1})\,z^{-1}$$

$$\therefore \sum_{i=1}^{N} \left[(z-p_i)T_{SB2}(z)\,T_{SB2}(z^{-1})\,z^{-1}\right]\Big|_{z=p_i} = \sum_{i=1}^{N} \left[(z-p_i)\,S_A^2 T_{SB}(z)\,T_{SB}(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= S_A^2 \sum_{i=1}^{N} \left[(z-p_i)T_{SB}(z)\,T_{SB}(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= S_A^2 \times 26.615 \qquad \boxed{\text{Using equation (4).}}$$

$$= 0.9367^2 \times 26.615 \qquad \boxed{\text{Using equation (2).}}$$

$$= 23.3522$$

Now the scale factor S_B when S_A is absent can be computed as shown below.

$$S_B^2 = \frac{1}{\sum_{i=1}^{N} \left[(z-p_i)T_{SB2}(z)\,T_{SB2}(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}} = \frac{1}{23.3522} = 0.0428$$

$$\therefore\ \text{Scale factor, } S_B = \sqrt{S_B^2} = \sqrt{0.0428} = 0.2069$$

8.9 Summary of Important Concepts

1. The fundamental operations in all DSP computations are multiplication and addition.
2. The registers are basic storage devices in a digital system.
3. The maximum size of binary that can be stored in a register is called register word length.
4. In DSP computations, the input data and the results are quantized to finite size and stored in registers.
5. The effects due to finite size (or finite precision) representation of binary in a digital system are called finite word length effects.

6. Using n-bits it is possible to frame 2^n binary codes.
7. The size of binary code will decide the range of numbers that can be represented in binary.
8. In a binary number the rightmost digit is called LSD (Least Significant Digit).
9. In a binary number the leftmost digit is called MSD (Most Significant Digit).
10. The binary digit is also called bit.
11. The two major methods of representing binary numbers are fixed point representation and floating point representation.
12. In fixed point representation the position of binary point is fixed and so the digits allotted for integer and fraction parts are fixed.
13. In a floating point representation the position of binary point can be shifted to the desired position.
14. The three different formats for representing binary numbers are sign-magnitude, one's complement and two's complement formats.
15. In all the three formats of binary numbers the positive number representation is same.
16. In all the three formats of binary numbers, a one in the MSD position represents negative number and a zero represents positive number.
17. In sign-magnitude format the negative of a given number differs only in sign bit.
18. In one's complement format the negative of a given number is obtained by bit by bit complement of its positive representations.
19. In two's complement format the negative of the given number is obtained by taking one's complement of its positive representation and then adding one to least significant bit.
20. The two's complement format provides single representation for zero and so it is practically used in all digital systems.
21. The sign-magnitude and one's complement format has two representations for zero.
22. The floating point representation is used to represent a large range of numbers.
23. In the normalized form of floating point representation, the most significant bit of mantissa is always one.
24. The IEEE-754 format of floating point number has 23-bit mantissa, 8-bit exponent and 1-bit sign field.
25. In one's complement addition, the carry is added to sum to get final sum, whereas in two's complement addition, the carry is discarded.
26. For floating point addition of two numbers, the exponents of both the numbers should be equal.
27. In floating point addition, only the mantissa of two numbers are added and the exponent of the sum is same that of numbers added.
28. In floating point multiplication, the mantissa is multiplied and exponents are added.
29. The two methods of quantization of binary numbers are truncation and rounding.
30. In quantization to B-bits, the range of decimal numbers is divided into 2^B steps called quantization steps.
31. The decimal numbers that are encountered in DSP applications will usually lie in the range –1 to +1.
32. In quantization to B-bits (including sign bit), if R is the range of decimal numbers, then quantization step size, $q = R/2^B$.
33. The truncation is the process of quantization by discarding all bits less significant than the least significant bit that is retained.

34. In truncation of positive numbers the error is same in all the three representation of binary numbers.
35. If N is unquantized number and N_t is the number obtained by truncation then truncation error, $e_t = N_t - N$.
36. In truncation of positive numbers, the truncation error is always negative.
37. In truncation of negative numbers in sign-magnitude and one's complement form, the truncation error is positive.
38. In truncation of negative numbers in two's complement form, the truncation error is negative.
39. The rounding is the process of quantization, which involves truncation and addition.
40. In rounding to b-bits, first the number is truncated to b-bits, then the most significant bit that is discarded is added to least significant bit that is retained.
41. If N is unquantized number and N_t is the number obtained by rounding then rounding error, $e_r = N_r - N$.
42. The error due to rounding is same in all the three formats of binary number representation.
43. Since the rounding error is same in all the formats of binary reperesentation, the rounding is the most preferred method of quantization in digital systems.
44. The variance of error signal due to quantization is also called steady state noise power due to quantization.
45. If e(n) is error in quantization of input, and $\varepsilon(n)$ is output of LTI system due to e(n), then the variance of $\varepsilon(n)$ is called output steady state noise power due to input quantization.
46. The quantization of filter coefficients will result in shift in pole-zero locations which in turn produce deviation in frequency response.
47. The sensitivity of frequency response to quantization of filter coefficients will be less in cascade realization when compared to direct form realization.
48. The error due to the quantization of the output of multiplier is called product quantization error.
49. The noise transfer function (NTF) is defined as the transfer function from the noise source to the filter output.
50. Since product quantization is performed by rounding, the system output noise power due to product quantization error is called roundoff noise power.
51. In product quantization noise model, the noise signal is added to the output of ideal multiplier.
52. In recursive system, the limit cycle is a condition of the system in which the output attains a finite value or oscillates between a finite positive and finite negative value for increasing value of n.
53. In recursive systems, the limit cycle exist even if the input is made zero and so they are called zero input limit cycle.
54. When a recursive system enter a limit cycle, it can be brought out of limit cycle only with an input higher than its deadband.
55. The deadband is the finite value of the output when a recursive system enters the limit cycle.
56. The overflow limit cycle are oscillations in the output due to overflow in addition.
57. The overflow in addition occurs when the sum exceeds the range of decimal number that can be represented in the binary format for computation.
58. The overflow in addition can be eliminated either by using saturation arithmetic or by introducing scaling multipliers at the input of adders.
59. In saturation arithmetic, when a overflow occurs then the sum is set equal to maximum allowable value.
60. In saturation arithmetic, when an underflow occurs then the sum is set equal to minimum allowable value.

8.10. Short Questions and Answers

Q8.1 What is meant by finite word length effects in digital filters?

The fundamental operation in digital filters are multiplication and addition. When these operations are performed in a digital system the input data as well as the product and sum (output data) have to be represented in finite word length, which depends on the size (length) of the register used to store the data. In digital computation the input and output data (sum and product) are quantized by rounding or truncation to convert them to a finite word size. This creates error (noise) in the output or creates oscillations (limit cycles) in the output. These effects due to finite precision representation of numbers in a digital system are called finite word length effects.

Q8.2 List some of the finite word length effects in digital filters.

i. Errors due to quantization of input data.

ii. Errors due to quantization of filter coefficients.

iii. Errors due to rounding the product in multiplication.

iv. Limit cycles due to product quantization and overflow in addition.

Q8.3 Explain the fixed point representation of binary numbers.

In fixed point representation of binary numbers in a given word size, the bits alloted for integer part and fraction part of the numbers are fixed. Therefore the position of binary point is fixed. The most significant bit is used to represent the sign of the number.

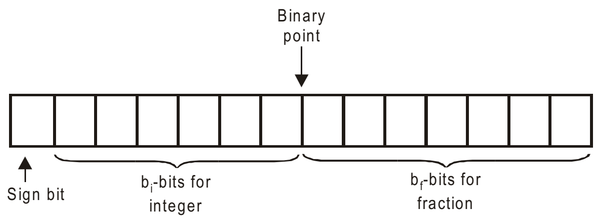


Fig Q8.3 : *Fixed point representation of binary numbers.*

Q8.4 What are the different formats of fixed point representation?

In fixed point representation, there are three different formats for representing binary numbers.

i. Sign-magnitude format.

ii. One's complement format.

iii. Two's complement format.

In all the three formats, the positive number is same but they differ only in representation of negative numbers.

Q8.5 Explain the floating point representation of binary numbers.

The floating number will have a mantissa part and exponent part. In a given word size the bits alloted for mantissa and exponent are fixed. The mantissa is used to represent a binary fraction number and the exponent is a positive or negative binary integer. The value of the exponent can be adjusted to move the position of binary point in mantissa. Hence this representation is called floating point. The floating point number can be expressed as,

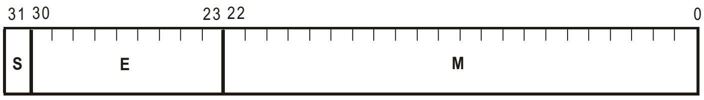
Floating point number, $N_f = M \times 2^E$

where M = Mantissa and E = Exponent.

Q8.6 Give the IEEE - 754 standard format for 32-bit floating point numbers.

The IEEE -754 standard for 32-bit single precision floating point number is given by

Floating point numbers, $N_f = (-1) \times 2^{E-127} \times M$



S = 1-bit field for sign of number.
E = 8-bit field for exponent.
M = 23-bit field for mantissa.

Fig Q8.6 : *IEEE-754 format for 32-bit floating point number.*

Q.8.7 What are the types of arithmetic used in digital computers?

The floating point arithmetic and two's complement arithmetic are the two types of arithmetic employed in digital systems.

Q8.8 Compare the fixed point and floating point number representations.

Fixed point representation	Floating point representation
1. In a b-bit binary the range of numbers represented is less when compared to floating point representation.	1. In a b-bit binary the range of numbers represented is large when compared to fixed point representation.
2. The position of binary point is fixed.	2. The position of binary point is variable.
3. The resolution is uniform throughout the range.	3. The resolution is variable.

Q8.9 Compare the fixed point and floating point arithmetic

Fixed point arithmetic	Floating point arithmetic
1. The accuracy of the result is less due to smaller dynamic range.	1. The accuracy of the results will be higher due to larger dynamic range.
2. Speed of processing is high.	2. Speed of processing is low.
3. Hardware implementation is cheaper.	3. Hardware implementation is costlier.
4. Fixed point arithmetic can be used for real time computations.	4. Floating point arithmetic cannot be used for real time computations.
8. Quantization error occurs only in multiplication.	8. Quantization error occurs in both multiplication and addition.

Q8.10 What are the two types of quantization employed in a digital system?

The two types of quantization in a digitial system are truncation and rounding.

Q8.11 What is truncation?

Truncation is the process of reducing the size of binary number by discarding all bits less significant than the least significant bit that is retained. (In the truncation of a binary number to b bits all the less significant bits beyond b^{th} bit are discarded).

Q8.12 Sketch the characteristics of the quantizer used for truncation.

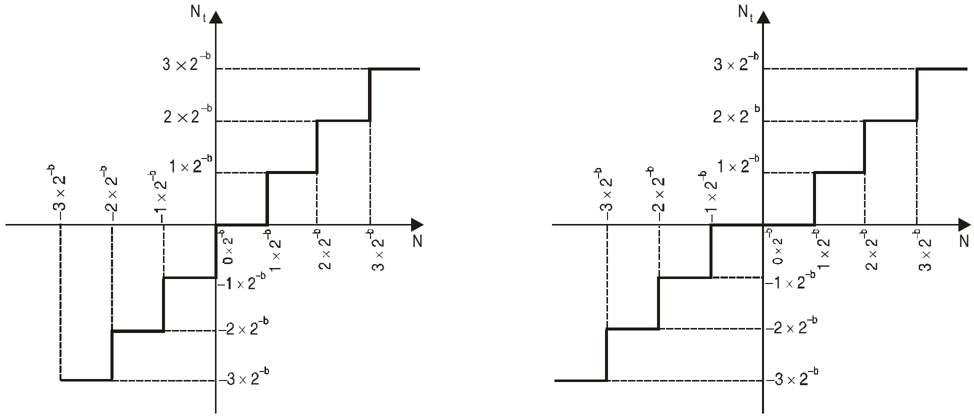


Fig a : Truncation characteristics of two's complement quantizer.

Fig b : Truncation characteristics of sign-magnitude and one's complement quantizer.

Fig Q8.12 : Input-output characteristics of quantizer used for truncation.

Q8.13 What is rounding?.

Rounding is one of the qunatization method in which the number is truncated to required size and then the most significant bit of discarded part is added to least significant bit of retained part.

Q8.14 What is the range of error in rounding?

The rounding error is same in all the three types of fixed point representation. The range of rounding error is $-\dfrac{2^{-b}}{2}$ to $+\dfrac{2^{-b}}{2}$.

Q8.15 Sketch the noise probability density functions for rounding.

Fig a : Rounding-fixed point.

Fig b : Rounding-floating point.

Fig Q8.15 : Quantization noise probability density functions for rounding.

Q8.16 What are the errors generated by A/D process?.

The A/D process generates two types of errors. They are quantization error and saturation error. The quantization error is due to representation of the sampled signal by a fixed number of digital levels (quantization levels). The saturation error occurs when the analog signal exceeds the dynamic range of A/D converter.

Q8.17 What is quantization step size?

In digital systems, the numbers are represented in binary. With B-bit binary (including sign bit) we can generate 2^B different binary codes. Any range of analog value to be represented in binary should be divided into 2^B levels with equal increment. The 2^B levels are called quantization levels and the increment in each level is called quantization step size. It R is the range of analog signal then,

Quantization step size, $q = R/2^B$

Q8.18 Why errors are created in A/D process?

In A/D process the analog signals are sampled and converted to binary. The sampled analog signal will have infinite precision. In binary representation of b-bits (excluding sign bit) we have 2^b different values with finite precision. The 2^b binary values are called quantization levels. Hence the samples of analog signal are quantized in order to fit into any one of the quantized levels. This quantization process introduce errors in the signal.

Q8.19 How the input quantization noise is represented in LTI system?

The quantized input signal of a digital system can be represented as a sum of unquantized signal x(n) and error signal e(n) as shown in fig Q8.19.

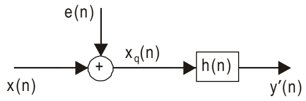


Fig Q8.19 : *Representation of input quantization noise in an LTI system.*

Q8.20 What is steady state output noise power due to input quantization?

The input signal to digital system can be considered as a sum of unquantized signal and error signal due to input quantization. The response of the system can be expressed as a summation of response due to unquantized input and error signal.

The response of the system due to error signal is given by convolution of error signal and impulse response. The variance of response of the system for error signal is called steady state output noise power.

Q8.21 What is meant by coefficient inaccuracy?

In digital computation the filter coefficients are represented in binary. With b-bit (excluding sign bit) binary we can generate only 2^b different binary numbers and they are called quantization levels. Any filter coefficient has to be fitted into any one of the quantization levels. Hence the filter coefficients are quantized to represent in binary and the quantization introduces errors in filter coefficients. Therefore the coefficients cannot be accurately represented in a digital system and this problem is referred to as coefficient inaccuracy.

Q8.22 How the digital filter is affected by quantization of filter coefficients?

The quantization of the filter coefficients will modify the value of poles and zeros which in turn shift the location of poles and zeros. This will create deviations in the frequency response of the system. Hence the resultant filter will have a frequency response different from that of the filter with unquantized coefficients.

Q8.23 How the sensitivity of frequency response to quantization of filter coefficients is minimized?

The sensitivity of the filter frequency response to quantization of the filter coefficients is minimized by realizing the filter having a large number of poles and zeros as an interconnection of second order sections. Hence the filter can be realized in cascade or parallel form, in which the basic building blocks are first-order and second-order sections.

Q8.24 What is meant by product quantization error?

In digital computations, the output of multipliers i.e, the products are quantized to finite word length in order to store them in registers and to be used in subsequent calculations. The error due to the quantization of the output of multiplier is referred to as product quantization error.

Q8.25 Why rounding is preferred for quantizing the product?

In digital system the product quantization is performed by rounding due to the following desirable characteristics of rounding.

i. The rounding error is independent of the type of arithmetic.

ii. The mean value of rounding error signal is zero.

iii. The variance of the rounding error signal is least.

Q8.26 Define noise transfer function (NTF)?

The Noise Transfer Function (NTF) is defined as the transfer function from the noise source to the filter output. The NTF depends on the structure of the digital network.

Q8.27 Draw the statistical model of fixed point product quantization.

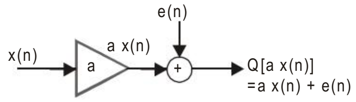


Fig Q8.27 : *Product quantization noise model.*

The multiplier is considered as an infinite precision multiplier. Using an adder the error signal is added to the output of multiplier so that the output of adder is equal to the quantized product as shown in fig Q8.27.

Q8.28 What are the assumptions made regarding the statistical independence of the various noise sources in the digital filter?

The assumptions made regarding the statistical independence of the noise sources are,

i. Any two different samples from the same noise source are uncorrelated.

ii. Any two different noise source, when considered as random processes are uncorrelated.

iii. Each noise source is uncorrelated with the input sequence.

Q8.29 Draw the product quantization noise model of second-order IIR system.

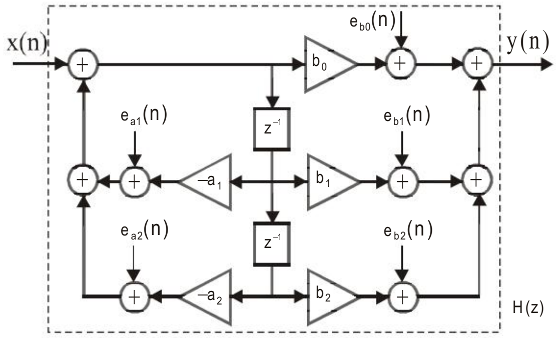
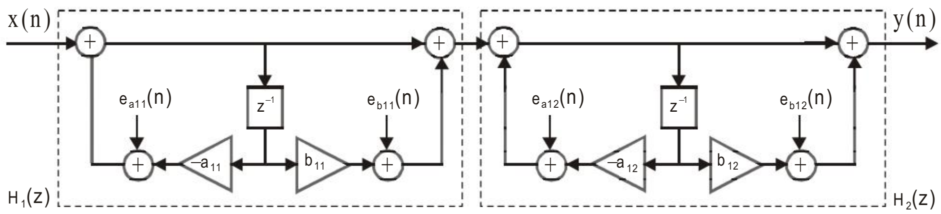


Fig Q8.29 : *Product quantization noise model of second-order direct form-II.*

Q8.30 Draw the product quantization noise model of IIR system with two first-order sections in cascade.



FigQ8.30 : *Product quantization noise model of cascading of two first-order sections.*

Q8.31 What are limit cycles?

In recursive systems when the input is zero or some nonzero constant value, the nonlinearities due to finite precision arithmetic operations may cause periodic oscillations in the output. These oscillations are called limit cycles.

Q8.32 What are the two types of limit cycles?

The two types of limit cycles are zero input limit cycles and overflow limit cycles.

Q8.33 What is zero input limit cycle?

In recursive system, the product quantization may create periodic oscillations in the output. These oscillations are called limit cycles. If the system output enters a limit cycle, it will continue to remain in limit cycle even when the input is made zero. Hence these limit cycles are also called zero input limit cycles.

Q8.34 What is dead band?

In a limit cycle the amplitudes of the output are confined to a range of value and this range of value is called dead band of the filter.

Q8.35 How the system output can be brought out of limit cycle?

The system output can be brought out of limit cycle by applying an input of large magnitude, which is sufficient to drive the system out of limit cycle.

Q8.36 Draw the transfer characteristics of two's complement adder.

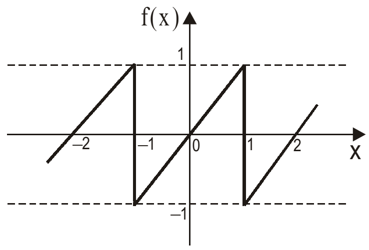


Fig Q8.36 : *Input - output (transfer) characteristics of two's complement adder.*

Q8.37 What is saturation arithmetic?

In saturation arithmetic when the result of an arithmetic operation exceeds the dynamic range of number system, then the result is set to maximum or minimum possible value. If the upper limit is exceeded then the result is set to maximum possible value. If the lower limit is exceeded then the result is set to minimum possible value.

Q8.38 What is overflow limit cycle?

In fixed point addition the overflow occurs when the sum exceeds the finite word length of the register used to store the sum. The overflow in addition may lead to oscillations in the output which is called overflow limit cycle.

Q8.39 How overflow limit cycles can be eliminated?

The overflow limit cycles can be eliminated either by using saturation arithmetic or by scaling the input signal to the adder.

Q8.40 What is the drawback in saturation arithmetic?

The saturation arithmetic introduces nonlinearity in the adder which creates signal distortion.

8.10. Exercises

I. Fill in the blanks with appropriate words

1. The maximum size of the binary information that can be stored in a register is called_______ .
2. The effects due to finite precision representation of numbers is called _______ effects.
3. The two major methods of representing binary numbers are _______ and _______ representation.
4. In _______ point representation of numbers _______ point is fixed.
5. In binary number system the most significant digit is always _______.
6. For real time computations _______ arithmetic is preferred.
7. The two types of quantization are _______ and _______ .
8. The effect of _______ on positive numbers are same in all the three fixed point representations.
9. In two's complement representation the truncation error is always _______.
10. In truncation and rounding the probability of error is assumed to be _______ throughout the range.
11. The range of error due to _______ is same for all the formats of fixed point representation.
12. In _______ the mean value of quantization error is _______.
13. The quantization by _______ is preferred in digital system due to its low variance and zero mean error.
14. The two types of errors produced by A/D conversion process are _______ and _______ errors.
15. The process of assigning binary numbers to quantized analog value is called _______.
16. If R is the range of analog signal and B is the binary word size then quantization step size is _______.
17. The _______ of error signal is called steady state noise power.
18. The variance of the response of the system due to error signal is called _______.
19. The quantization of filter coefficients will modify the _______ of the system.
20. The sensitivity of frequency response to quantization of filter coefficients is minimized in _______ realization.
21. The error due to the quantization of the output of multiplier is called _______ error.
22. The _______ is defined as the transfer function from the noise source to filter output.
23. The product quantization error is also called as _______ noise.
24. In addition the _______ occurs when the sum exceeds the dynamic range of number system.
25. The _______ limit cycles are avoided by scaling the input signal.

Answers

1. register word length	9. negative	17. variance
2. finite word length	10. equal	18. output noise power
3. fixed point, floating point	11. rounding	19. frequency response
4. fixed, binary	12. rounding, zero	20. cascade
5. sign bit	13. rounding	21. product quantization
6. fixed point	14. quantization, saturation	22. noise transfer function
7. truncation, rounding	15. coding	23. product roundoff
8. truncation	16. $R/2^B$	24. overflow
		25. overflow

II. State whether the following statements are True/False

1. The effects of truncation or rounding are represented in terms of additive error signals.
2. In representation of numbers in digital systems the binary point does not exist physically in the system.
3. For a given word size, the dynamic range of fixed point representation is greater than the floating point representation.
4. In floating point representation the binary point is fixed.
5. In floating point representation the size of mantissa and exponent are variable.
6. In all types (formats) of binary number representation the most significant digit is used to indicate the sign of the number.
7. In fixed point representation there is only one format for representing positive number.
8. The sign-magnitude arithmetic is avoided in digital computers.
9. The fundamental arithmetic operation in digital system is addition.
10. The overflow in addition occurs frequently in floating point arithmetic.
11. The characteristics of the errors introduced by quantization depend on the type of number representation.
12. The effect of truncation of negative number is same in all the fixed point representations.
13. In floating point number representation, only the mantissa is truncated or rounded.
14. The rounding process consists of truncation and addition.
15. The quantization by truncation is preferred in digital system because the process is simple to implement.
16. In digital system the coefficient accuracy is decided by the word length of the register.
17. The coefficient quantization will have less effect in cascade realization when compared to parallel realization.
18. The noise transfer function depends on the structure of digital network.
19. The rounding error is independent of the type of arithmetic.
20. The quantization error signals cannot be treated as random process.

Answers

1.	True	5.	False	9.	True	13.	True	17.	True
2.	True	6.	True	10.	False	14.	True	18.	True
3.	False	7.	True	11.	True	15.	False	19.	True
4.	False	8.	True	12.	False	16.	True	20.	False

III. Choose the right answer for the following questions

1. The finite word length effects are due to,

a) Quantization of input.

b) Quantization of coefficients.

c) Quantization of product.

d) All of the above.

2. With n-bit binary the possible binary codes are,

a) 2^{n-1} **b)** 2^{n+1}

c) 2^{n} **d)** $2^{n/2}$

3. Which of the following is true in fixed point binary representation?

a) Only positive number can be represented.

b) Integers cannot be represented.

c) The position of binary point is fixed.

d) None of the above.

4. Which of the following is false in sign-magnitude format of fixed point representation?

a) The negative of a given number differs only in sign bit.

b) The fractions cannot be represented.

c) The position of binary point is fixed.

d) The MSD is sign bit.

5. Which of the following is true in two's complement format of fixed point representation?

a) Single representation for zero.

b) The range of positive and negative numbers are same.

c) Addition of two's complement number will never generate carry.

d) None of the above.

6. Which of the following is true in floating point representation?

a) The position of binary point is movable.

b) The bits allotted for mantissa and exponent are fixed.

c) The MSD is sign bit.

d) All of the above.

7. ***In quantization to b-bits (excluding sign bit), if R is the range, then quantization step size q is,***

   a) $\frac{R}{2^b}$ **b)** $\frac{R}{2^{b+1}}$ **c)** $\frac{R}{2^{b-1}}$ **d)** $R\,2^{b+1}$

8. ***In quantization by truncation to b-bits (excluding sign bit) any positive unquantized number, N in the range (1 ´ 2^–b^) £ N < (2 ´ 2^–b^) wil be assigned the quantization step,***

   a) 1 ´ $2^{-b/2}$ **b)** 1 ´ 2^{-b}

   c) 2 ´ 2^{-b} **d)** 2 ´ $2^{-b/2}$

9. ***The quantization step size assigned by one's and two's complement quantizer, in quantization by truncation to b-bits (excluding sign bit) for any negative unquantized number N in the range –2 ´ 2^{-b} < N £ –1 ´ 2^{-b} are respectively,***

   a) –1 ´ 2^{-b}, –1 ´ 2^{-b} **b)** –1 ´ 2^{-b}, –2 ´ 2^{-b}

   c) –2 ´ 2^{-b}, –1 ´ 2^{-b} **d)** –2 ´ 2^{-b}, –2 ´ 2^{-b}

10. ***If N is unquantized number and N_t is the number quantized by truncation then truncation error is defined as,***

    a) $N_t - N$ **b)** $N - N_t$

    c) $(N_t - N)/2$ **d)** $(N_t + N)/2$

11. ***Which of the following is false with respect to truncation error?***

    a) The truncation error of a positive number is always negative.

    b) The truncation error of a negative number in one's complement form is always positive.

    c) The truncation error of a negative number in sign-magnitude form is always negative.

    d) The truncation error of a negative number in two's complement form is always negative.

12. ***If N is unquantized number and N_t is the number quantized by rounding, then rounding error is defined as,***

    a) $N_r - N$ **b)** $N - N_r$

    c) $(N_r - N)/2$ **d)** $(N_r + N)/2$

13. ***In quantization by rounding to b-bits (excluding sign bit) any unquantized number N in the range $\left(1\times\frac{2^{-b}}{2}\right) \le N < \left(2\times\frac{2^{-b}}{2}\right)$ will be assigned the step size,***

    a) 1×2^{-b} **b)** $1\times\frac{2^{-b}}{2}$ **c)** 2×2^{-b} **d)** $2\times\frac{2^{-b}}{2}$

14. ***In quantization by rounding to b-bits (excluding sign bit) any unquantized number N in the range $\left(-2\times\frac{2^{-b}}{2}\right) < N \le \left(-1\times\frac{2^{-b}}{2}\right)$ will be assigned the step size,***

    a) -1×2^{-b} **b)** $-1\times\frac{2^{-b}}{2}$ **c)** -2×2^{-b} **d)** $-2\times\frac{2^{-b}}{2}$

15. Which of the following is true with respect to rounding error?

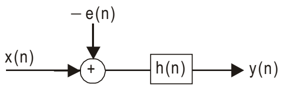
a) Rounding error of a positve number is always negative.

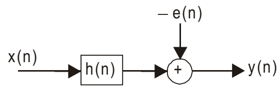
b) Rounding error of a negative number is always positive.

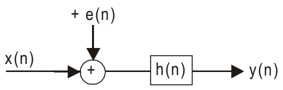
c) Rounding error is same in all the three formats of fixed point representation.

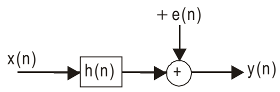
d) None of the above.

16. If x(n), y(n) and h(n) are input, output and impulse response of an LTI system respectively and if e(n) is the error in input quantization, then the input quantization noise model of the system is,

a) 

b) 

c) 

d) 

17. Which of the following are true with respect to quantization of filter coefficients?

a) Shift in location of poles and zeros.

b) Deviation of impulse response.

c) Deviation of frequency response.

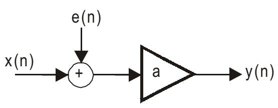
d) All of the above.

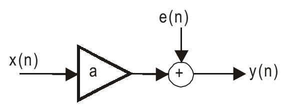
18. For the transfer function, $H(z) = \dfrac{1}{1-0.45z^{-1}}$ when the coefficient is quantized to 3-bits by truncation the original and shifted poles are respectively,

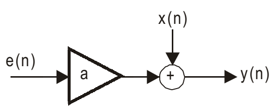
a) 0.45, 0.5 **b)** 0.45, 0.375

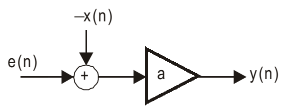
c) –0.45, –0.5 **d)** –0.45, –0.375

19. If a x(n) is a product in an LTI system then the product quantization noise model is,

a) 

b) 

c) 

d) 

20. Which of the following is false with respect to the limit cycle in a recursive system?

a) Limit cycles are due to product quantization.

b) During limit cycle, the output is finite or oscillate between finite values.

c) Limit cycle exists even if the input is very much larger than the dead band.

d) During limit cycle, the output is finite even if the input is zero.

Answers

1. d	5. a	9. b	13. a	17. d
2. c	6. d	10. a	14. a	18. b
3. c	7. b	11. c	15. c	19. b
4. b	8. b	12. a	16. c	20. c

IV. Answer the following questions

1. Discuss in detail, the various finite word length effects in digital filters.
2. Explain the three different fixed point formats of binary fraction numbers, with your own examples.
3. Explain the floating point representation of decimal numbers with an example.
4. Explain the IEEE-754 standard 32-bit format of floating point number representation with an example.
5. Explain the one's complement addition with an example.
6. Explain the two's complement addition with an example.
7. How is floating point addition performed? Explain with an example.
8. How is floating point multiplication performed? Explain with an example.
9. What are quantization steps? Tabulate the quantization steps in 3-bit binary representation of all the three fixed point formats.
10. Draw the truncation characteristics of all the three fixed point quantizer and explain. Also discuss the possible range of errors in all the fixed point formats.
11. Draw the rounding characteristics of fixed point quantizer and explain. Also, discuss the possible range of errors.
12. Discuss in detail the effects of quantization of input (or A/D process).
13. Discuss in detail the effects of quantization of filter coefficients with your own example.
14. Discuss the various aspects of product quantization noise in digital filters. Also draw some product quantization noise models of typical LTI systems.
15. Explain the limit cycle in a recursive system with your own example.

V. Solve the following problems

E8.1. Convert +0.0625$_{10}$ and –0.0625$_{10}$ to sign-magnitude format of binary and verify the result by converting the binary to decimal.

E8.2. Convert +0.0625$_{10}$ and –0.0625$_{10}$ to one's complement format of binary and verify the result by converting the binary to decimal.

E8.3. Convert +0.0625$_{10}$ and –0.0625$_{10}$ to two's complement format of binary and verify the result by converting the binary to decimal.

E8.4. Convert +31$_{10}$ and –31$_{10}$ to 32-bit IEEE-754 format of binary and verify the result by converting the binary to decimal.

E8.5. Add +0.45 and –0.575 by one's complement addition.

E8.6. Add +0.575 and –0.45 by one's complement addition.

E8.7. Add +0.45 and –0.575 by two's complement addition.

E8.8. Add $+0.575_{10}$ and -0.45_{10} by two's complement addition.

E8.9. Add $+4_{10}$ and $+0.375_{10}$ by floating point addition. Choose 10-bit floating point format with 7-bits for mantissa and 3-bits for exponent.

E8.10. Multiply $+4_{10}$ and $+.375_{10}$ by floating point multiplication. Choose 10-bit floating point format with 7-bits for mantissa and 3-bits for exponent.

E8.11. For the recursive filter shown in fig 1 the input $x(n)$ has a peak value of 12 V, represented by 5 bits. Compute the variance of output due to A/D conversion process.

E8.12. An LTI system is characterized by the difference equation, $y(n) = 0.75y(n - 1) + 0.3x(n)$. The input signal $x(n)$ has a range of -4 V to $+4$ V, represented by 9-bits. Find the quantization step size, variance of the error signal and variance of the quantization noise at the output.

E8.13. The output of an A/D converter is applied to a digital filter with the system function $H(z) = \dfrac{0.29z}{z - 0.64}$. Find the output noise power for the digital filter, when the input signal is quantized to 11 bits.

E8.14. For second order IIR filter, $H(z) = \dfrac{1}{(1 - 0.1\,z^{-1})(1 - 0.79\,z^{-1})}$. Study the effect of shift in pole location with 4-bit coefficient in direct and cascade form.

E8.15. Discuss the effect of coefficient quantization on pole locations of the following IIR system, when it is realized in direct form-I and in cascade form. Assume a word length of 5 bits through truncation.

E8.16. Consider the LTI system governed by the equation $y(n) + 0.92y(n - 1) + 0.35y(n - 2) = x(n - 2)$. Discuss the effect of coefficient quantization on pole locations, when the coefficients are quantized by (i) 3-bits by truncation (ii) 4-bits by truncation

E8.17. In the IIR system given below the products are rounded to 4-bits (including sign bit).

$$H(z) = \frac{1}{(1 - 0.15\,z^{-1})(1 - 0.43\,z^{-1})}$$

Find the output round off noise power in (a) Direct form realization (b) Cascade realization

E8.18. Find the output roundoff noise power, where the products are rounded to 5-bits (including sign bit) in the two different ways of cascade realization of the following IIR system.

$$H(z) = \frac{1}{(1 - 0.29\,z^{-1})(1 - 0.58\,z^{-1})}$$

E8.19. Given that, $H(z) = \dfrac{1}{(1 - 0.18\,z^{-1})\,(1 - 0.34\,z^{-1})\,(1 - 0.42\,z^{-1})}$. Determine the output roundoff noise power in the direct form realization of the above system.

(a) When the products are rounded to 4-bits (including sign bit).

(b) When the products are rounded to 7-bits (including sign bit). Comment on the result.

E8.20. Explain the characteristics of a limit cycle oscillation with respect to the system described by the equation,

$$y(n) = 0.82\,y(n - 1) + x(n).$$

When the product is quantized to 4-bits by rounding. The system is excited by an input $x(n) = 0.875$ for $n = 0$ and $x(n) = 0$ for $n \neq 0$. Also, determine the dead band of the filter.

E8.21. ***Study the limit cycle behaviour of the system described by $w(n) = Q[a\, w(n-1)] + x(n)$ where $w(n)$ is the output of the system and $Q[.]$ is quantization. Assume that, $a = -0.752$, $x(0) = 0.625$ and $x(n) = 0$ for $n > 0$. Choose 5-bits for quantization.***

E8.22. ***An LTI system is characterized by the difference equation, $y(n) = 0.91y(n-1) + x(n)$.***

Determine the limit cycle behaviour and the deadband of the system when $x(n) = 0$ and $y(-1) = 0.78$. Assume that the product is quantized to 4-bits by rounding.

E8.23. ***For the digital network shown in fig E8.23, find the transfer function, $H(z)$ and scale factor S_0 to avoid overflow in register A_1.***

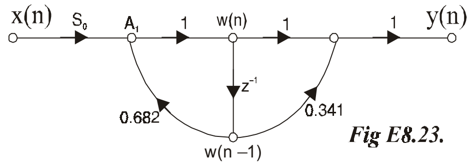


Fig E8.23.

E8.24. ***For the digital network shown in fig E8.24, find the scale factor, S to avoid overflow in register A.***

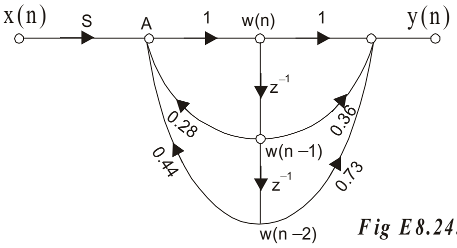


Fig E8.24.

E8.25. ***For the digital network shown in fig E8.25, find the scale factor, S_A to avoid overflow in register A, then find the scale factor S_B to avoid overflow in register-B,***

a) When S_A is present and b) When S_A is absent

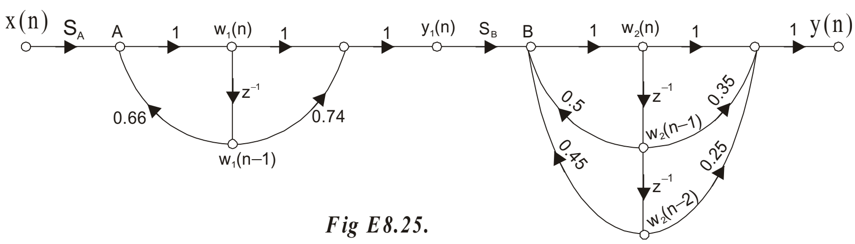


Fig E8.25.

Answers

E8.1. $+0.0625_{10} = 00001_2 \quad ; \quad -0.0625_{10} = 10001_2$

E8.2. $+0.0625_{10} = 00001_2 \quad ; \quad -0.0625_{10} = 11110_2$

E8.3. $+0.0625_{10} = 00001_2 \quad ; \quad -0.0625_{10} = 11111_2$

E8.4.

$$+31_{10} = \underset{\substack{\downarrow \\ \text{1-bit} \\ \text{sign}}}{0} \quad \underbrace{1000\ 0011}_{\substack{\downarrow \\ \text{8-bit} \\ \text{exponent}}} \quad \underbrace{1111\ 0000\ 0000\ 0000\ 0000\ 000}_{\substack{\downarrow \\ \text{23-bit Mantissa}}}$$

$$-31_{10} = \underset{\substack{\downarrow \\ \text{1-bit} \\ \text{sign}}}{1} \quad \underbrace{1000\ 0011}_{\substack{\downarrow \\ \text{8-bit} \\ \text{exponent}}} \quad \underbrace{1111\ 0000\ 0000\ 0000\ 0000\ 000}_{\substack{\downarrow \\ \text{23-bit Mantissa}}}$$

E8.5. $(+0.45_{10})+(-0.575_{10})=1110_2=-0.125_{10}$

E8.6. $(+0.575_{10})+(-0.45_{10})=0001_2=+0.125_{10}$

E8.7. $(+0.45_{10})+(-0.575_{10})=1111_2=-0.125_{10}$

E8.8. $(+0.575_{10})+(-0.45_{10})=0001_2=+0.125_{10}$

E8.9. $4_{10}+0.375_{10}=\ 0.100011\times 2^{011}=010\ 0011\ 011_2$

E8.10. $4_{10}\times 0.375_{10}=011000\times 2^{001}=011000001_2$

E8.11. $\sigma_{eoi}^2=0.0325$

E8.12. $q=0.015625,\ \ \sigma_e^2=2.0345\times 10^{-5},\ \ \sigma_{eoi}^2=4.1850\times 10^{-6}$

E8.13. $\sigma_{eoi}^2=1.1317\times 10^{-8}$

E8.14. ***Direct form Realization***

$$\overline{H}(z)=\frac{1}{1-0.875z^{-1}+0.0625z^{-2}} \quad ; \quad p_{d1}=0.7965 \text{ and } p_{d2}=0.0785$$

Cascade Realization

$$\overline{H}_1(z)=\frac{1}{1-0.0625z^{-1}} \quad ; \quad \overline{H}_2(z)=\frac{1}{1-0.75z^{-1}}$$

$\overline{p}_{c1}=0.0625$ and $\overline{p}_{c2}=0.75$

E8.15. ***Direct form-I Realization***

$$\overline{H}(z)=\frac{1}{1-0.5313z^{-1}+0.0625z^{-2}} \quad ; \quad \overline{p}_{d1}=0.3555 \text{ and } \overline{p}_{d2}=0.1758$$

Cascade Realization

$$\overline{H}_1(z)=\frac{1}{1-0.3438z^{-1}} \quad ; \quad \overline{H}_2(z)=\frac{1}{1-0.1875z^{-1}}$$

$\overline{p}_{c1}=0.3438$ and $\overline{p}_{c2}=0.1875$

E8.16. ***a)*** $\overline{H}_1(z)=\dfrac{1}{z^2+0.875z+0.25}$

$p_{11}=-0.375+j0.2421$

$p_{21}=-0.4375-j0.2421$

b) $\overline{H}_2(z)=\dfrac{1}{z^2+0.875z+0.5625}$

$p_{12}=-0.4375+j0.6092$

$p_{22}=-0.4375-j0.6092$

E8.17. *a)* $\sigma^2_{eTop,\,d} = 3.7194 \times 10^{-3}$

b) *The order of cascading is $H_1(z)\ H_2(z)$*

$\sigma^2_{eTop,\,c1} = 3.4572 \times 10^{-3}$

The order of cascading is $H_2(z)\ H_1(z)$

$\sigma^2_{eTop,\,c2} = 3.1918 \times 10^{-3}$

E8.18. *a)* $\sigma^2_{eTop,\,c1} = 1.2428 \times 10^{-3}$

b) $\sigma^2_{eTop,\,c2} = 1.1076 \times 10^{-3}$

E8.19. *a)* $\sigma^2_{eTop,4} = 9.2986 \times 10^{-3}$

b) $\sigma^2_{eT0p,7} = 1.4529 \times 10^{-4}$

E8.20. ***Output***

n	x(n)	y'(n)	
		Decimal	**Binary**
0	0.875	0.875	0.111
1	0	0.75	0.110
2	0	0.625	0.101
3	0	0.5	0.100
4	0	0.375	0.011
5	0	0.25	0.010
6	0	0.25	0.010
.	.	.	.
.	.	.	.

Dead band $= \pm 0.3472$

Note : 0.3472 ´ 0.82 = Q[0.284704] = .010$_2$ = 0.25$_{10}$.

E8.21. ***Output***

n	x(n)	w(n)	
		Decimal	**Binary**
0	0.625	0.625	0.1010
1	0	–0.375	1.0110
2	0	0.3125	0.0101
3	0	–0.25	1.0100
4	0	0.1875	0.0011
5	0	–0.125	1.0010
6	0	0.125	0.0010
7	0	–0.125	1.0010
8	0	0.125	0.0010
.	.	.	.
.	.	.	.

Dead band $= \pm 0.126$

Note : 0.126 = Q[0.126 ´ 0.752] = Q[0.09475] = 0.125$_{10}$.

E8.22. ***Output***

n	x(n)	y'(n) Decimal	y'(n) Binary
0	0	0.625	0.101
1	0	0.625	0.101
2	0	0.625	0.101
3	0	.	.
4	0	.	.
5	0	.	.

Dead band = ±0.6944

Note : $Q[0.91 \times 0.6944] = Q[0.6319] = 0.625_{10}$.

E8.23. $S_0 = 0.7314$

E8.24. $S = 0.9508$

E8.25. $S_A = 0.7513$

(a) $S_B = 0.0174$

(b) $S_{B2} = 0.0231$

Solution for Exercise Problems

E8.1. ***Convert +0.0625$_{10}$ and –0.0625$_{10}$ to sign-magnitude format of binary and verify the result by converting the binary to decimal.***

Solution

Decimal to binary conversion

$$+.0625_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.0001 \xrightarrow[\text{sign bit}]{\text{Append}} 0.0001 \xrightarrow[\text{dot}]{\text{Remove}} 00001_2$$

$$-.0625_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -.0001 \xrightarrow[\text{sign bit}]{\text{Append}} 1.0001 \xrightarrow[\text{dot}]{\text{Remove}} 10001_2$$

$$\therefore\ +.0625_{10} = 00001_2$$
$$-.0625_{10} = 10001_2$$

Decimal to binary conversion

$$\begin{array}{r} .0625 \\ \times 2 \\ \hline \boxed{0}.125 \\ \times 2 \\ \hline \boxed{0}.250 \\ \times 2 \\ \hline \boxed{0}.500 \\ \times 2 \\ \hline \boxed{1}.000 \end{array}$$

$.0\,0\,0\,1_2$

Binary to decimal conversion

$$00001_2 \xrightarrow[\text{sign bit}]{\text{Remove}} +.0001 \xrightarrow[\text{to decimal}]{\text{Convert}} +.0625_{10}$$

$$10001_2 \xrightarrow[\text{sign bit}]{\text{Remove}} -.0001 \xrightarrow[\text{to decimal}]{\text{Convert}} -.0625_{10}$$

$$\therefore\ 00001_2 \ \rightarrow\ +.0625_{10}$$
$$10001_2 \ \rightarrow\ -.0625_{10}$$

Binary to decimal conversion

$$+.0001_2 = +(0\times 2^{-1} + 0\times 2^{-2} + 0\times 2^{-3} + 1\times 2^{-4})$$
$$= +.0625_{10}$$
$$-.0001_2 = -(0\times 2^{-1} + 0\times 2^{-2} + 0\times 2^{-3} + 1\times 2^{-4})$$
$$= -.0625_{10}$$

E8.2. ***Convert +0.0625$_{10}$ and –0.0625$_{10}$ to one's complement format of binary and verify the result by converting the binary to decimal.***

Solution

Decimal to binary conversion

$$+.0625_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.0001 \xrightarrow[\text{sign bit}]{\text{Append}} 0.0001 \xrightarrow[\text{dot}]{\text{Remove}} 00001_2$$

$$-.0625_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -.0001 \xrightarrow[\text{fraction part}]{\text{Complement}} -.1110 \xrightarrow[\text{sign bit}]{\text{Append}} 1.1110 \xrightarrow[\text{dot}]{\text{Remove}} 11110_2$$

$$\therefore\ +.0625_{10} = 00001_2$$
$$-.0625_{10} = 11110_2$$

Decimal to binary conversion

$$\begin{array}{r} .0625 \\ \times 2 \\ \hline \boxed{0}.125 \\ \times 2 \\ \hline \boxed{0}.250 \\ \times 2 \\ \hline \boxed{0}.500 \\ \times 2 \\ \hline \boxed{1}.000 \end{array}$$

$.0\,0\,0\,1_2$

Binary to decimal conversion

$$00001_2 \xrightarrow[\text{sign bit}]{\text{Remove}} +.0001_2 \xrightarrow[\text{to decimal}]{\text{Convert}} +.0625_{10}$$

$$11110_2 \xrightarrow[\text{sign bit}]{\text{Remove}} -.1110_2 \xrightarrow[\text{fraction part}]{\text{Complement}} -.0001_2 \xrightarrow[\text{to decimal}]{\text{Convert}} -.0625_{10}$$

$$\therefore\ 0001_2 \ \rightarrow\ +.0625_{10}$$
$$11110_2 \ \rightarrow\ -.0625_{10}$$

Binary to decimal conversion

$$+.0001_2 = +(0\times 2^{-1} + 0\times 2^{-2} + 0\times 2^{-3} + 1\times 2^{-4})$$
$$= +.0625_{10}$$
$$-.0001_2 = -(0\times 2^{-1} + 0\times 2^{-2} + 0\times 2^{-3} + 1\times 2^{-4})$$
$$= -.0625_{10}$$

E8.3. ***Convert +0.0625$_{10}$ and –0.0625$_{10}$ to two's complement format of binary and verify the result by converting the binary to decimal.***

Solution

Decimal to binary conversion

$$+.0625_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.0001 \xrightarrow[\text{sign bit}]{\text{Append}} 0.0001 \xrightarrow[\text{dot}]{\text{Remove}} 00001_2$$

$$-.0625_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -.0001 \xrightarrow[\text{fraction part}]{\text{Complement}} -.1110 \xrightarrow[\text{to LSD}]{\text{Add 1}} .1111 \xrightarrow[\text{sign bit}]{\text{Append}} 1.1111 \xrightarrow[\text{dot}]{\text{Remove}} 11111_2$$

$$\therefore\ +.0625_{10} = 00001_2$$
$$-.0625_{10} = 11111_2$$

Decimal to binary conversion

$$\begin{array}{r} .0625 \\ \times 2 \\ \hline \boxed{0}.125 \\ \times 2 \\ \hline \boxed{0}.250 \\ \times 2 \\ \hline \boxed{0}.500 \\ \times 2 \\ \hline \boxed{1}.000 \end{array}$$

$.0\,0\,0\,1$

Binary to decimal conversion

$$00001_2 \xrightarrow[\text{sign bit}]{\text{Remove}} +.0001 \xrightarrow[\text{to decimal}]{\text{Convert}} +.0625_{10}$$

$$11111_2 \xrightarrow[\text{sign bit}]{\text{Remove}} -.1111 \xrightarrow[\text{fraction part}]{\text{Complement}} -.0000 \xrightarrow[\text{to LSD}]{\text{Add 1}} -.0001 \xrightarrow[\text{to decimal}]{\text{Convert}} -.0625_{10}$$

$$\therefore\ 00001_2 \ \rightarrow\ +.0625_{10}$$
$$11111_2 \ \rightarrow\ -.0625_{10}$$

Binary to decimal conversion

$$+.0001_2 = +(0\times 2^{-1} + 0\times 2^{-2} + 0\times 2^{-3} + 1\times 2^{-4})$$
$$= +.0625_{10}$$
$$-.0001_2 = -(0\times 2^{-1} + 0\times 2^{-2} + 0\times 2^{-3} + 1\times 2^{-4})$$
$$= -.0625_{10}$$

E8.4. Convert +31$_{10}$ and –31$_{10}$ to 32-bit IEEE-754 format of binary and verify the result by converting the binary to decimal.

Solution

Decimal to IEEE-754 binary format conversion

$$31_{10} \xrightarrow[\text{to binary}]{\text{Convert}} 11111_2 \xrightarrow[\text{in 1.M format}]{\text{Represent}} 1.1111_2 \times 2^{4_{10}} \xrightarrow[\text{to E-127 format}]{\text{Convert exponent}} 1.1111_2 \times 2^{131_{10}-127_{10}}$$

The number, N in IEEE-754 format is,

Using equation (8.11).

$$N = (-1)^s \times 1.M \times 2^{E-127}$$

$$\therefore\ +31 = (-1)^0 \times 1.1111_2 \times 2^{131_{10}-127_{10}}$$

$$-31 = (-1)^1 \times 1.1111_2 \times 2^{131_{10}-127_{10}}$$

$$\therefore\ 1.M = 1.1111 \xrightarrow[\substack{\text{fraction part}\\ \text{to 23-bits}}]{\text{Convert}} 1.1111\ 0000\ 0000\ 0000\ 0000\ 000$$

$$E = 131_{10} \xrightarrow[\text{to binary}]{\text{Convert}} 1000\ 0011_2$$

$$\therefore\ +31_{10} = \underset{\substack{\text{1-bit}\\ \text{sign}}}{\underset{\downarrow}{0}}\ \ \underbrace{1000\ 0011}_{\substack{\text{8-bit}\\ \text{exponent}}}\ \ \underbrace{1111\ 0000\ 0000\ 0000\ 0000\ 000}_{\text{23-bit Mantissa}}$$

$$\therefore\ -31_{10} = \underset{\substack{\text{1-bit}\\ \text{sign}}}{\underset{\downarrow}{1}}\ \ \underbrace{1000\ 0011}_{\substack{\text{8-bit}\\ \text{exponent}}}\ \ \underbrace{1111\ 0000\ 0000\ 0000\ 0000\ 000}_{\text{23-bit Mantissa}}$$

$$\backslash\ \ +31_{10} = 0100\ 0001\ 1111\ 1000\ 0000\ 0000\ 0000\ 0000_2$$

$$-31_{10} = 1100\ 0001\ 1111\ 1000\ 0000\ 0000\ 0000\ 0000_2$$

IEEE-754 binary to decimal format conversion

$$0100\ 0001\ 1111\ 1000\ 0000\ 0000\ 0000\ 0000_2$$

$$\Downarrow$$

$$\underset{S}{\underset{\downarrow}{0}}\ \ \underbrace{1000\ 0011}_{E}\ \ \underbrace{1111\ 0000\ 0000\ 0000\ 0000\ 000}_{M}$$

$$0100\ 0001\ 1111\ 1000\ 0000\ 0000\ 0000\ 0000_2 = (-1)^0 \times 2^{10000011_2 - 127_{10}} \times 1.1111$$

$$= +2^{137_{10}-127_{10}} \times 1.1111$$

$$= +2^4 \times 1.1111 = +11111_2$$

$$+11111_2 = +(1\times 2^4 + 1\times 2^3 + 1\times 2^2 + 1\times 2^1 + 1\times 2^0) = +(16+8+4+2+1) = +31_{10}$$

$$1100\ 0001\ 1111\ 1000\ 0000\ 0000\ 0000\ 0000_2$$

$$\Downarrow$$

$$\underset{S}{\underset{\downarrow}{1}}\ \ \underbrace{1000\ 0011}_{E}\ \ \underbrace{1111\ 0000\ 0000\ 0000\ 0000\ 000}_{M}$$

$$\therefore\ 1100\ 0001\ 1111\ 1000\ 0000\ 0000\ 0000\ 0000_2 = (-1)^1 \times 2^{10000011_2 - 127_{10}} \times 1.1111$$

$$= -2^{137_{10}-127_{10}} \times 1.1111 = -2^4 \times 1.1111$$

$$= -11111_2$$

$$-11111_2 = -(1\times 2^4 + 1\times 2^3 + 1\times 2^2 + 1\times 2^1 + 1\times 2^0) = -(16+8+4+2+1) = -31_{10}$$

E8.5. Add +0.45 and –0.575 by one's complement addition.

Solution

The one's complement representation of the given numbers are shown below.

$$+.45_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.011_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.011_2 \xrightarrow[\text{dot}]{\text{Remove}} 0011_2$$

$$-.575_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -.100_2 \xrightarrow[\text{sign bit}]{\text{Add}} 1.100_2 \xrightarrow[\text{fraction part}]{\text{Complement}} 1.011_2 \xrightarrow[\text{dot}]{\text{Remove}} 1011_2$$

$$\begin{array}{r} 0\,0\,1\,1 \\ +\ 1\,0\,1\,1 \\ \hline \end{array}$$

Carry ® $\boxed{0}\boxed{1\,1\,1\,0}$ ¬ sum

Decimal to binary conversion

$$\begin{array}{ll} .45 & .575 \\ \times 2 & \times 2 \\ \boxed{0}.90 & \boxed{1}.15 \\ \times 2 & \times 2 \\ \boxed{1}.80 & \boxed{0}.3 \\ \times 2 & \times 2 \\ \boxed{1}.60 & \boxed{0}.6 \\ .0\,1\,1_2 & .1\,0\,0_2 \end{array}$$

Since the carry is zero the sum is negative. The sum can be converted to decimal as shown below.

$$1110_2 \xrightarrow[\text{sign bit}]{\text{Extract}} -.110_2 \xrightarrow[\text{fraction part}]{\text{Complement}} -.001_2 \xrightarrow[\text{to decimal}]{\text{Convert}} -0.125_{10}$$

In summary,

$+.45_{10}$ Þ 0011_2

$-.575_{10}$ Þ 1011_2

$(+.45_{10}) + (-.575_{10})$ Þ 1110_2 Þ $-.125_{10}$

Binary to decimal conversion,
$.001_2 = (0 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3})$
$= .125_{10}$

E8.6. ***Add +0.575 and –0.45 by one's complement addition.***

Solution

The one's complement representation of the given numbers are shown below.

$$+.575_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.100_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.100_2 \xrightarrow[\text{dot}]{\text{Remove}} 0100_2$$

$$-.45_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -.011_2 \xrightarrow[\text{sign bit}]{\text{Add}} 1.011_2 \xrightarrow[\text{fraction part}]{\text{Complement}} 1.100_2 \xrightarrow[\text{dot}]{\text{Remove}} 1100_2$$

```
          0 1 0 0
        + 1 1 0 0
Carry ® [1][0 0 0 0] ¬  sum
              ® 1  (add carry to LSD)
          [0 0 0 1] ¬  Final sum
```

Decimal to binary conversion

.45	.575
×2	×2
[0].90	[1].15
×2	×2
[1].80	[0].3
×2	×2
[1].60	[0].6
$.011_2$	$.100_2$

Since the carry is one the sum is positive. The final sum can be obtained by adding the carry to least significant digit (LSD) of the sum. The final sum can be converted to decimal as shown below.

$$0001_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.001_2 \xrightarrow[\text{to decimal}]{\text{Convert}} +.125_{10}$$

In summary,

$+.575_{10}$ Þ 0100_2

$-.45_{10}$ Þ 1100_2

$(+.575_{10}) + (-.45_{10})$ Þ 0001_2 Þ $+.125_{10}$

Binary to decimal conversion,
$.001_2 = (0 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3})$
$= .125_{10}$

E8.7. ***Add +0.45 and –0.575 by two's complement addition.***

Solution

The two's complement representation of the given numbers are shown below.

$$+.45 \xrightarrow[\text{to binary}]{\text{Convert}} +.011_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.011_2 \xrightarrow[\text{dot}]{\text{Remove}} 0011_2$$

$$-.575 \xrightarrow[\text{to binary}]{\text{Convert}} -.100_2 \xrightarrow[\text{sign bit}]{\text{Add}} 1.100_2 \xrightarrow[\text{fraction part}]{\text{Complement}} 1.011_2 \xrightarrow[\text{to LSD}]{\text{Add one}} 1.100_2 \xrightarrow[\text{dot}]{\text{Remove}} 1100_2$$

```
          0 0 1 1
        + 1 1 0 0
Carry ® [0][1 1 1 1] ¬  sum
```

Decimal to binary conversion

.45	.575
×2	×2
[0].90	[1].15
×2	×2
[1].80	[0].3
×2	×2
[1].60	[0].6
$.011_2$	$.100_2$

Since the carry is zero the sum is negative. The sum can be converted to decimal as shown below.

$$1111_2 \xrightarrow[\text{sign bit}]{\text{Extract}} -.111_2 \xrightarrow[\text{fraction part}]{\text{Complement}} -.000_2 \xrightarrow[\text{to LSD}]{\text{Add one}} -.001_2 \xrightarrow[\text{to decimal}]{\text{Convert}} -.125_{10}$$

In summary,

$+.45_{10}$ Þ 0011_2

$-.575_{10}$ Þ 1100_2

$(+.45_{10}) + (-.575_{10})$ Þ 1111_2 Þ $-.125_{10}$

Binary to decimal conversion,
$.001_2 = (0 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3})$
$= .125_{10}$

E8.8. ***Add +0.575$_{10}$ and –0.45$_{10}$ by two's complement addition.***

Solution

The two's complement representation of the given numbers are shown below.

$$+.575_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.100_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.100_2 \xrightarrow[\text{dot}]{\text{Remove}} 0100_2$$

$$-.45_{10} \xrightarrow[\text{to binary}]{\text{Convert}} -.011_2 \xrightarrow[\text{sign bit}]{\text{Add}} 1.011_2 \xrightarrow[\text{fraction part}]{\text{Complement}} 1.100_2 \xrightarrow[\text{to LSD}]{\text{Add one}} 1.101_2 \xrightarrow[\text{dot}]{\text{Remove}} 1101_2$$

```
          0 1 0 0
        + 1 1 0 1
Carry ® [1][0 0 0 1] ¬  sum
```

Decimal to binary conversion

.45	.575
×2	×2
[0].90	[1].15
×2	×2
[1].80	[0].3
×2	×2
[1].60	[0].6
$.011_2$	$.100_2$

Since the sum is positive. The carry is discarded in two's complement addition. The sum can be converted to decimal as shown below.

$$0001_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.001_2 \xrightarrow[\text{to decimal}]{\text{Convert}} +.125_{10}$$

In summary,

$$+.575_{10} \Rightarrow 0100_2$$
$$-.45_{10} \Rightarrow 1101_2$$
$$(+.575_{10}) + (-.45_{10}) \Rightarrow \overline{\underline{0001_2}} \Rightarrow +.125_{10}$$

Binary to decimal conversion,

$$.001_2 = (0 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3})$$
$$= .125_{10}$$

E8.9. Add $+4_{10}$ and $+0.375_{10}$ by floating point addition. Choose 10-bit floating point format with 7-bits for mantissa and 3-bits for exponent.

Solution

Let us convert the given numbers to floating point format. For simplicity we can use sign-magnitude representation for exponent and mantissa. The leftmost bit in mantissa and exponent is used to represent the sign.

$$+4_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +100_2 \xrightarrow[\text{exponent}]{\text{Add}} +100.000 \times 2^0 \xrightarrow{\text{Normalize}} +.100000 \times 2^{+3_{10}} \xrightarrow[\text{to binary}]{\text{Convert exponent}}$$
$$0100000 \times 2^{011_2} \xleftarrow[\text{dot}]{\text{Remove}} +0.100000 \times 2^{011_2} \xleftarrow[\text{sign bit}]{\text{Append}} +.100000 \times 2^{+11_2}$$

$$+.375_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.011_2 \xrightarrow[\text{exponent}]{\text{Add}} +.0110000 \times 2^0 \xrightarrow{\text{Normalize}} +.110000 \times 2^{-1_{10}} \xrightarrow[\text{to binary}]{\text{Convert exponent}}$$
$$0110000 \times 2^{101_2} \xleftarrow[\text{dot}]{\text{Remove}} 0.110000 \times 2^{101_2} \xleftarrow[\text{sign bit}]{\text{Append}} +.110000 \times 2^{-01_2}$$

Decimal to binary conversion,

$$\begin{array}{r} .375 \\ \times 2 \\ \hline \boxed{0}.75 \\ \times 2 \\ \hline \boxed{1}.50 \\ \times 2 \\ \hline \boxed{1}.00 \end{array}$$

$.0\,1\,1_2$

$$\therefore \quad +4_{10} = 01\ 0000\ 0011_2$$
$$+0.375_{10} = 00\ 0001\ 1101_2$$

Since the exponents of $+4_{10}$ and $+0.375_{10}$ are not equal, the exponent of +0.375 is unnormalized to make its exponent equal to that of +4.

$$\therefore \quad +.375_{10} = 0.110000 \times 2^{101_2} = 0.110000 \times 2^{-1} \xrightarrow{\text{unnormalizing}} = 0.000011 \times 2^3 = 0.000011 \times 2^{011_2}$$

Now the unnormalized mantissa of $+0.375_{10}$ is added to the mantissa of $+4_{10}$ to get the sum of mantissa. The exponent of the sum is same as that of the exponents of the numbers added.

$$+4_{10} \Rightarrow 0100000 \times 2^{011}$$
$$+.375_{10} \Rightarrow 0000011 \times 2^{011}$$
$$(4_{10} + .375_{10}) \Rightarrow \overline{\underline{0100011 \times 2^{011}}} \Rightarrow +4.375_{10}$$
$$\therefore \quad 4_{10} + .375_{10} = 0.100011 \times 2^{011} = 010\ 0011\ 011_2$$

The sum in floating point format can be converted to decimal as shown below.

$$0100011 \times 2^{011} \xrightarrow[\text{sign bit}]{\text{Remove}} +.100011 \times 2^{+11_2} \xrightarrow[\text{to decimal}]{\text{Convert exponent}} +.100011 \times 2^{+3_{10}} = +100.011_2$$

$$+100.011 = +(1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3})$$
$$= +(4 + 0 + 0 + 0 + 0.25 + 0.125) = +4.375_{10}$$

E8.10. Multiply $+4_{10}$ and $+.375_{10}$ by floating point multiplication. Choose 10-bit floating point format with 7-bits for mantissa and 3-bits for exponent.

Solution

Let us convert $+4_{10}$ and $+0.375_{10}$ to floating point format.

$$+4_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +100_2 \xrightarrow[\text{exponent}]{\text{Add}} +100.000 \times 2^0 \xrightarrow{\text{Normalize}} +.100000 \times 2^{+3_{10}} \xrightarrow[\text{to binary}]{\text{Convert exponent}}$$
$$0100000 \times 2^{011_2} \xleftarrow[\text{dot}]{\text{Remove}} +0.100000 \times 2^{011_2} \xleftarrow[\text{sign bit}]{\text{Append}} +.100000 \times 2^{+11_2}$$

$$+.375_{10} \xrightarrow[\text{to binary}]{\text{Convert}} +.011_2 \xrightarrow[\text{exponent}]{\text{Add}} +.0110000 \times 2^0 \xrightarrow{\text{Normalize}} +.110000 \times 2^{-1_{10}} \xrightarrow[\text{to binary}]{\text{Convert exponent}}$$
$$0110000 \times 2^{101_2} \xleftarrow[\text{dot}]{\text{Remove}} 0.110000 \times 2^{101_2} \xleftarrow[\text{sign bit}]{\text{Append}} +.110000 \times 2^{-01_2}$$

Decimal to binary conversion,

$$\begin{array}{r} .375 \\ \times 2 \\ \hline \boxed{0}.75 \\ \times 2 \\ \hline \boxed{1}.50 \\ \times 2 \\ \hline \boxed{1}.00 \end{array}$$

$.0\,1\,1_2$

$$\therefore \quad +4_{10} = 01\ 0000\ 0011_2$$
$$+0.375_{10} = 00\ 0001\ 1101_2$$
$$\therefore \quad +4_{10} = 0100000 \times 2^3 = 0.100000 \times 2^3$$
$$+.375_{10} = 0110000 \times 2^{-1} = 0.110000 \times 2^{-1}$$

The floating point multiplication is performed as shown below.

$$4_{10} \times .375_{10} = (0 + 0)\,.\,(100000 \times 110000) \times 2^{3+(-1)} = 0.011000 \times 2^{2}$$

$$0.011000 \times 2^2 \xrightarrow{\text{Normalizing}} 0.11000 \times 2^1 \xrightarrow[\text{to binary}]{\text{Convert exponent}} 0.11000 \times 2^{001_2} \xrightarrow[\text{dot}]{\text{Remove}} 011000 \times 2^{001_2}$$

$$\therefore\ 4_{10} \times .375_{10} = 011000 \times 2^{001} = 011000001_2$$

The product in floating point format can be converted to decimal as shown below.

$$011000\ \times\ 2^{001} \xrightarrow[\text{sign bit}]{\text{Remove}} +.11000\ \times\ 2^{+1_2} = +1.10_2$$

$$+1.10_2 = +(1\times 2^0 + 1\times 2^{-1} + 0\times 2^{-2}) = (1 + 0.5 + 0) = +1.5_{10}$$

E8.11. ***For the recursive filter shown in fig 1 the input x(n) has a peak value of 12 V, represented by 5 bits. Compute the variance of output due to A/D conversion process.***

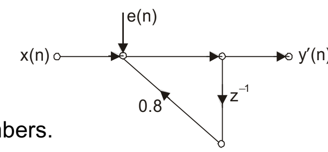


Fig 1.

Solution

Let us assume that the input is positive and so 5-bits are used to represent only positive numbers.

$$\therefore\ \text{Quantization step size, } q = \frac{R}{2^B}$$

Given that, R = 12 and B = 5

$$\therefore\ q = \frac{12}{2^5} = 0.375$$

$$\text{Variance of error signal, } \sigma_e^2 = \frac{q^2}{12} = \frac{0.375^2}{12} = 0.0117 \qquad .....(1)$$

Consider the given LTI system without error e(n) as shown in fig 2. The difference equation of the system is,

$$y(n) = 0.8y(n-1) + x(n)$$

On taking z-transform of above equation we get,

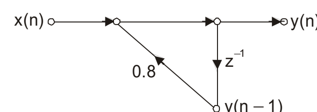


Fig 2.

$$Y(z) = 0.8z^{-1}\,Y(z) + X(z)$$

$$Y(z)\,(1 - 0.8z^{-1}) = X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{1}{1-0.8z^{-1}}$$

$$\text{We know that the transfer function, } H(z) = \frac{Y(z)}{X(z)}$$

$$\therefore\ H(z) = \frac{1}{1-0.8z^{-1}} = \frac{1}{z^{-1}(z-0.8)} = \frac{z}{z-0.8}$$

$$\therefore\ H(z)\,H(z^{-1})z^{-1} = \frac{z}{z-0.8} \times \frac{z^{-1}}{z^{-1}-0.8} \times z^{-1} = \frac{z^{-1}}{(z-0.8)\left(\dfrac{1}{z} - 0.8\right)}$$

$$= \frac{z^{-1}}{(z-0.8)\left(\dfrac{1-0.8z}{z}\right)} = \frac{z z^{-1}}{(z-0.8)\,(-0.8)\left(z - \dfrac{1}{0.8}\right)}$$

$$= \frac{-1.25}{(z-0.8)(z-1.25)}$$

Now, poles of $H(z)\,H(z^{-1})\,z^{-1}$ are $p_1 = 0.8$, $p_2 = 1.25$.

Here, $p_1 = 0.8$ is the only pole that lies inside the unit circle in z-plane.

The steady state output noise power (or variance) due to input quantization error signal is given by,

$$\left.\begin{matrix}\text{Output noise power}\\ \text{due to A / D process}\end{matrix}\right\}\sigma_{eoi}^2 = \sigma_e^2\,\frac{1}{2\pi j}\oint_c H(z)\,H(z^{-1})z^{-1}\,dz = \sigma_e^2 \sum_{i=1}^{N} \text{Res}\left[H(z)H(z^{-1})\,z^{-1}\right]\Big|_{z\,=\,p_i}$$

$\boxed{\text{Using equation (8.29).}}$

$$= \sigma_e^2 \sum_{i=1}^{N} \left[(z-p_i)\,H(z)\,H(z^{-1})\,z^{-1}\right]\Big|_{z\,=\,p_i}$$

where, $p_1, p_2, ...\ p_N$ are poles of $H(z)\,H(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

$$\therefore\ \sigma_{eoi}^2 = \sigma_e^2 \times \cancel{(z-0.8)} \times \frac{-1.25}{\cancel{(z-0.8)}(z-1.25)}\Bigg|_{z\,=\,0.8} = \sigma_e^2 \times \frac{-1.25}{0.8-1.25} = 2.7778\,\sigma_e^2$$

$$= 2.7778 \times 0.0117 \qquad \boxed{\text{Using equation (1).}}$$

$$= 0.0325$$

E8.12. An LTI system is characterized by the difference equation, y(n) = 0.75y(n – 1) + 0.3x(n). The input signal x(n) has a range of –4 V to +4 V, represented by 9-bits. Find the quantization step size, variance of the error signal and variance of the quantization noise at the output.

Solution

Given that,

Range, R = –4 to +4 = 4 – (–4) = 8.

Size of binary, B = 9 bits (including sign bit).

$$\therefore \text{ Quantization step size, } q = \frac{R}{2^B} = \frac{8}{2^9} = 0.015625$$

$$\text{Variance of error signal, } \sigma_e^2 = \frac{q^2}{12} = \frac{0.015625^2}{12}$$
$$= 2.0345 \times 10^{-5} \qquad .....(1)$$

The difference equation governing the LTI system is,

$$y(n) = 0.75y(n-1) + 0.3x(n)$$

On taking z-transform of above equation we get,

$$Y(z) = 0.75z^{-1}Y(z) + 0.3X(z)$$
$$Y(z) - 0.75z^{-1}Y(z) = 0.3X(z)$$
$$Y(z)\,[1 - 0.75z^{-1}] = 0.3X(z)$$
$$\therefore \frac{Y(z)}{X(z)} = \frac{0.3}{1 - 0.75z^{-1}}$$

We know that the transfer function, $H(z) = \frac{Y(z)}{X(z)}$.

$$\therefore H(z) = \frac{0.3}{1 - 0.75z^{-1}} = \frac{0.3}{z^{-1}(z - 0.75)} = \frac{0.3z}{z - 0.75}$$

$$\therefore H(z)\,H(z^{-1})z^{-1} = \frac{0.3z}{z - 0.75} \times \frac{0.3z^{-1}}{z^{-1} - 0.75} \times z^{-1} = \frac{0.3^2\,z^{-1}}{(z - 0.75)\left(\frac{1}{z} - 0.75\right)}$$

$$= \frac{0.09z^{-1}}{(z - 0.75)\left(\frac{1 - 0.75z}{z}\right)} = \frac{0.09zz^{-1}}{(z - 0.75)(-0.75)\left(z - \frac{1}{0.75}\right)}$$

$$= \frac{-0.12}{(z - 0.75)(z - 1.3333)}$$

Now, poles of $H(z)\,H(z^{-1})\,z^{-1}$ are $p_1 = 0.75$, $p_2 = 1.3333$.

Here, $p_1 = 0.75$ is the only pole that lies inside the unit circle in z-plane.

$$\therefore \left.\begin{array}{l}\text{Variance of the input quantization}\\ \text{noise at the output}\end{array}\right\}\sigma_{eoi}^2 = \sigma_e^2\,\frac{1}{2\pi j}\oint_c H(z)\,H(z^{-1})z^{-1}dz = \sigma_e^2\sum_{i=1}^{N}\text{Res}\left[H(z)H(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma_e^2\sum_{i=1}^{N}\left[(z - p_i)\,H(z)\,H(z^{-1})\,z^{-1}\right]\Big|_{z=p_i} \qquad \boxed{\text{Using equation (8.29).}}$$

where, $p_1, p_2, \ldots\ldots\, p_N$ are poles of $H(z)\,H(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

$$\therefore \sigma_{eoi}^2 = \sigma_e^2 \times \cancel{(z - 0.75)} \times \frac{-0.12}{\cancel{(z - 0.75)}(z - 1.3333)}\Bigg|_{z=0.75}$$

$$= \sigma_e^2 \times \frac{-0.12}{0.75 - 1.3333} = 0.2057\,\sigma_e^2$$

$$= 0.2057 \times 2.0345 \times 10^{-5} \qquad \boxed{\text{Using equation (1).}}$$

$$= 4.1850 \times 10^{-6}$$

E8.13. The output of an A/D converter is applied to a digital filter with the system function $H(z) = \frac{0.29z}{z - 0.64}$. Find the output noise power for the digital filter, when the input signal is quantized to 11 bits.

Solution

The range of input signal is not specified.

Therefore, let us assume that input varies from –1 to +1.

$\therefore$ Range, R = –1 to +1 = 1 – (–1) = 2

Size of binary, B = 11 bits (including sign bit).

$$\therefore \text{ Quantization step size, } q = \frac{R}{2^B} = \frac{2}{2^{11}} = 9.7656 \times 10^{-4}$$

$$\text{Variance of error signal, } \sigma_e^2 = \frac{q^2}{12} = \frac{(9.7656 \times 10^{-4})^2}{12}$$

$$= 7.9472 \times 10^{-8} \qquad .....(1)$$

Given that,

$$H(z) = \frac{0.29z}{z - 0.64}$$

$$\therefore\ H(z)\,H(z^{-1})\,z^{-1} = \frac{0.29z}{z-0.64} \times \frac{0.29z^{-1}}{z^{-1}-0.64} \times z^{-1} = \frac{0.29^2\,z^{-1}}{(z-0.64)\left(\dfrac{1}{z}-0.64\right)}$$

$$= \frac{0.0841z^{-1}}{(z-0.64)\left(\dfrac{1-0.64z}{z}\right)} = \frac{0.0841z^{-1}z}{(z-0.64)(-0.64)\left(z-\dfrac{1}{0.64}\right)}$$

$$= \frac{-0.13141}{(z-0.64)(z-1.5625)}$$

Now, poles of $H(z)\,H(z^{-1})\,z^{-1}$ are $p_1 = 0.64$, $p_2 = 1.5625$.

Here, $p_1 = 0.64$ is the only pole that lies inside the unit circle in z-plane.

$$\left.\begin{array}{l}\therefore \text{ Output noise power due} \\ \quad \text{to input quantization}\end{array}\right\} \sigma_{eoi}^2 = \sigma_e^2\,\frac{1}{2\pi j}\oint_c H(z)\,H(z^{-1})z^{-1}\,dz = \sigma_e^2 \sum_{i=1}^{N} \text{Res}\left[H(z)H(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma_e^2 \sum_{i=1}^{N} \left[(z-p_i)\,H(z)\,H(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

Using equation (8.29).

where, $p_1, p_2, ... p_N$ are poles of $H(z)\,H(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

$$\therefore\ \sigma_{eoi}^2 = \sigma_e^2 \times \cancel{(z-0.64)} \times \frac{-0.13141}{\cancel{(z-0.64)}(z-1.5625)}\Bigg|_{z=0.64}$$

$$= \sigma_e^2 \times \frac{-0.13141}{0.64-1.5625} = 0.1424\,\sigma_e^2$$

$$= 0.1424 \times 7.9472 \times 10^{-8}$$

Using equation (1).

$$= 1.1317 \times 10^{-8}$$

E8.14. For second order IIR filter, $H(z) = \dfrac{1}{(1-0.1z^{-1})(1-0.79z^{-1})}$. Study the effect of shift in pole location with 4-bit coefficient in direct and cascade form.

Solution

$$\text{Given that, } H(z) = \frac{1}{(1-0.1z^{-1})(1-0.79z^{-1})} = \frac{1}{z^{-1}(z-0.1)\,z^{-1}(z-0.79)}$$

$$= \frac{z^2}{(z-0.1)(z-0.79)}$$

The roots of the denominator of H(z) are the original poles of H(z). Let the original poles of H(z) be p_1 and p_2.

Here, $p_1 = 0.1$ and $p_2 = 0.79$.

Case(i) : Direct form Realization

$$H(z) = \frac{1}{(1-0.1z^{-1})(1-0.79z^{-1})} = \frac{1}{1-0.79z^{-1}-0.1z^{-1}+0.079z^{-2}}$$

$$= \frac{1}{1-0.89z^{-1}+0.079z^{-2}}$$

Let us quantize the coefficient by truncation.

$$.89_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .11100_2 \xrightarrow[\text{4 bits}]{\text{Truncate to}} .1110_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .875_{10}$$

$$.079_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .00010_2 \xrightarrow[\text{4 bits}]{\text{Truncate to}} .0001_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .0625_{10}$$

Decimal to binary conversion

.89	.079
×2	×2
$\boxed{1}$.78	$\boxed{0}$.158
×2	×2
$\boxed{1}$.56	$\boxed{0}$.316
×2	×2
$\boxed{1}$.12	$\boxed{0}$.632
×2	×2
$\boxed{0}$.24	$\boxed{1}$.264
×2	×2
$\boxed{0}$.48	$\boxed{0}$.528
$.11100_2$	$.00010_2$

Let, $\overline{H}(z)$ be the transfer function of the IIR system after quantizing the coefficients.

$$\therefore\ \overline{H}(z) = \frac{1}{1-0.875z^{-1}+0.0625z^{-2}}$$

$$\text{Let, } \overline{H}(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.875z^{-1}+0.0625z^{-2}}$$

On cross multiplying the above equation we get,

$Y(z) - 0.875z^{-1}Y(z) + 0.0625z^{-2}Y(z) = X(z)$

\ $Y(z) = X(z) + 0.875z^{-1}Y(z) - 0.0625z^{-2}Y(z)$

Binary to decimal conversion,

$.1110_2 = (1\times 2^{-1})+(1\times 2^{-2})+(1\times 2^{-3})+(0\times 2^{-4})$
$= .875$

$.0001_2 = (0\times 2^{-1})+(0\times 2^{-2})+(0\times 2^{-3})+(1\times 2^{-4})$
$= .0625$

Using the above equation the direct form structure is drawn as shown in fig 1.

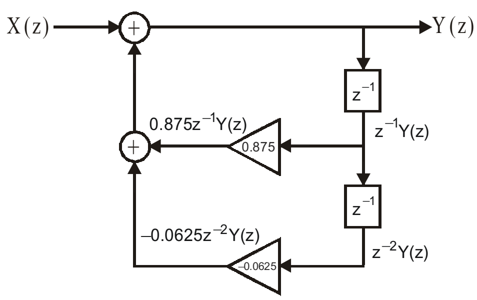


Fig 1 : Direct form Realization.

Let us examine the poles of the system, after coefficient quantization.

$$\text{Let, } \overline{H}(z) = \frac{1}{z^{-2}(z^2-0.875z+0.0625)}$$

$$= \frac{z^2}{z^2-0.875z+0.0625} = \frac{z^2}{(z-0.7965)(z-0.0785)}$$

The roots of the quadratic $z^2 - 0.875z + 0.0625 = 0$ are given by,

$$z = \frac{0.875 \pm \sqrt{0.875^2 - 4\times 0.0625}}{2}$$
$= 0.7965$ (or) 0.0785

The poles of $\overline{H}(z)$ are given by roots of the denominator polynomial of $\overline{H}(z)$. Let the poles of $\overline{H}(z)$ be p_{d1} and p_{d2}.

\ $p_{d1} = 0.7965$ and $p_{d2} = 0.0785$

If we compare the poles of H(z) and $\overline{H}(z)$ we can observe that both the poles of $\overline{H}(z)$ deviate very much from the original pole.

Case(ii) : Cascade Realization

$$\text{Given that, } H(z) = \frac{1}{(1-0.1z^{-1})(1-0.79z^{-1})}$$

In cascade realization the system can be realized as cascade of first order sections.

\ $H(z) = H_1(z)\, H_2(z)$

$$\text{where, } H_1(z) = \frac{1}{1-0.1z^{-1}} \text{ and } H_2(z) = \frac{1}{1-0.79z^{-1}}$$

$$.1_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .00011_2 \xrightarrow[\text{4 bits}]{\text{Truncate to}} .0001_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .0625_{10}$$

$$.79_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .11001_2 \xrightarrow[\text{4 bits}]{\text{Truncate to}} .1100_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .75_{10}$$

Decimal to binary conversion

.1	.79
×2	×2
$\boxed{0}$.2	$\boxed{1}$.58
×2	×2
$\boxed{0}$.4	$\boxed{1}$.16
×2	×2
$\boxed{0}$.8	$\boxed{0}$.32
×2	×2
$\boxed{1}$.6	$\boxed{0}$.64
×2	×2
$\boxed{1}$.2	$\boxed{1}$.28
$.0\,0\,0\,1\,1_2$	$.1\,1\,0\,0\,1_2$

Let $\overline{H}_1(z)$ and $\overline{H}_2(z)$ be the transfer function of the first order sections after quantizing the coefficients.

$$\therefore\ \overline{H}_1(z) = \frac{1}{1-0.0625z^{-1}} \quad ; \quad \overline{H}_2(z) = \frac{1}{1-0.75z^{-1}}$$

Binary to decimal conversion,

$.0001_2 = (0\times 2^{-1})+(0\times 2^{-2})+(0\times 2^{-3})+(1\times 2^{-4})$
$= .0625_{10}$

$.1100_2 = (1\times 2^{-1})+(1\times 2^{-2})+(0\times 2^{-3})+(0\times 2^{-4})$
$= .75_{10}$

$$\text{Let, } \overline{H}_1(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.0625z^{-1}}$$

On cross multiplying the above equation we get,

$Y_1(z) - 0.0625z^{-1}Y_1(z) = X(z)$

\ $Y_1(z) = X(z) + 0.0625z^{-1}Y_1(z)$ .....(1)

$$\text{Let, } \overline{H}_2(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.75z^{-1}}$$

On cross multiplying the above equation we get,

$Y(z) - 0.75z^{-1}Y(z) = Y_1(z)$

\ $Y(z) = Y_1(z) + 0.75z^{-1}Y(z)$ .....(2)

Using equation (1) and (2) the cascade structure of the system is drawn as shown in fig 2.

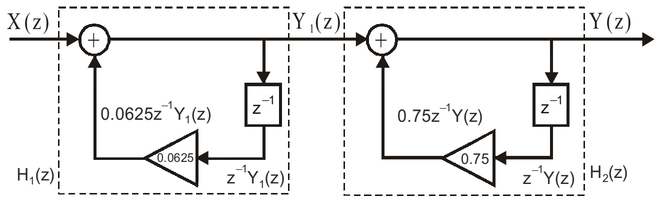


Fig 2 : Cascade realization of the system.

Let us examine the poles of the cascade system.

Let the poles of the cascade system be $\overline{p}_{c1}$ and $\overline{p}_{c2}$ which are given by the roots of the denominator polynomials of $\overline{H}_1(z)$ and $\overline{H}_2(z)$.

$$\text{Let,}\quad \overline{H}_1(z) = \frac{1}{1-0.0625\,z^{-1}} = \frac{1}{z^{-1}(z-0.0625)} = \frac{z}{z-0.0625}$$

$$H_2(z) = \frac{1}{1-0.75\,z^{-1}} = \frac{1}{z^{-1}(z-0.75)} = \frac{z}{z-0.75}$$

$$\therefore\ \overline{p}_{c1} = 0.0625 \quad\text{and}\quad \overline{p}_{c2} = 0.75$$

On comparing the poles of the cascade system with original poles we can say that one pole is much deviate from the original and other pole is very close to original pole.

E8.15. ***Discuss the effect of coefficient quantization on pole locations of the following IIR system, when it is realized in direct form-I and in cascade form. Assume a word length of 5 bits through truncation.***

$$H(z) = \frac{1}{1-0.55\,z^{-1}+0.07\,z^{-2}}$$

Solution

$$\text{Given that,}\ H(z) = \frac{1}{1-0.55z^{-1}+0.07z^{-2}} = \frac{1}{z^{-2}(z^2-0.55z+0.07)}$$

$$= \frac{z^2}{z^2-0.55z+0.07} = \frac{z^2}{(z-0.35)(z-0.2)} \qquad .....(1)$$

The roots of the quadratic $z^2 - 0.55z + 0.07 = 0$ are given by,

$$z = \frac{0.55 \pm \sqrt{0.55^2 - 4\times 0.07}}{2} = \frac{0.55 \pm 0.15}{2} = 0.2,\ 0.35$$

The roots of the denominator of H(z) are the original poles of H(z). Let the original poles of H(z) be p_1 and p_2.

Here, $p_1 = 0.35$ and $p_2 = 0.2$.

Case(i) : Direct form-I Realization

$$\text{Given that,}\ H(z) = \frac{1}{1-0.55z^{-1}+0.07z^{-2}}$$

Let us quantize the coefficients by truncation.

$$.55_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .100011_2 \xrightarrow[\text{5 bits}]{\text{Truncate to}} .10001_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .5313_{10}$$

$$.07_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .000100_2 \xrightarrow[\text{5 bits}]{\text{Truncate to}} .00010_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .0625_{10}$$

Decimal to binary conversion,

.55	.07
× 2	× 2
$\boxed{1}$.10	$\boxed{0}$.14
× 2	× 2
$\boxed{0}$.2	$\boxed{0}$.28
× 2	× 2
$\boxed{0}$.4	$\boxed{1}$.56
× 2	× 2
$\boxed{0}$.8	$\boxed{1}$.12
× 2	× 2
$\boxed{1}$.6	$\boxed{0}$.24
× 2	× 2
$\boxed{1}$.2	$\boxed{0}$.48
$1\,0\,0\,0\,1\,1_2$	$0\,0\,1\,1\,0\,0_2$

Binary to decimal conversion,

$$.10001_2 = (1\times 2^{-1})+(0\times 2^{-2})+(0\times 2^{-3})+(0\times 2^{-4})+(1\times 2^{-5}) = .5313_{10}$$

$$.00010_2 = (0\times 2^{-1})+(0\times 2^{-2})+(0\times 2^{-3})+(1\times 2^{-4})+(0\times 2^{-5}) = .0625_{10}$$

Let, $\overline{H}(z)$ be the transfer function of the IIR system after quantizing the coefficients.

$$\therefore\ \overline{H}(z) = \frac{1}{1-0.5313\,z^{-1}+0.0625\,z^{-2}}$$

$$\text{Let,}\ \overline{H}(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.5313\,z^{-1}+0.0625\,z^{-2}}$$

On crossing multiplying the above equation we get,

$$Y(z) - 0.5313z^{-1}Y(z) + 0.0625z^{-2}Y(z) = X(z)$$

$$\backslash\ Y(z) = X(z) + 0.5313z^{-1}Y(z) - 0.0625z^{-2}Y(z)$$

Using the above equation the direct form-I structure of IIR system is drawn as shown in fig 1.

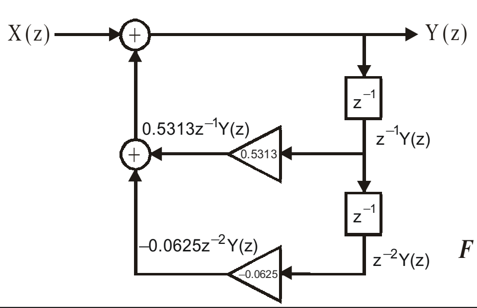


Fig 1 : Direct form-I realization of the system.

Let us examine the poles of the system, after coefficient quantization.

$$\text{Let,}\quad \overline{H}(z)=\frac{1}{z^{-2}(z^2-0.5313z+0.0625)}=\frac{z^2}{z^2-0.5313z+0.0625}$$

$$=\frac{z^2}{(z-0.3555)(z-0.1758)}$$

The roots of the quadratic $z^2 - 0.5313z + 0.0625 = 0$ are given by,

$$z=\frac{0.5313\ \pm\ \sqrt{0.5313^2-4\times 0.0625}}{2}$$

$$=\frac{0.5313 \pm 0.1797}{2}$$

$$=0.3555,\ 0.1758$$

The poles of $\overline{H}(z)$ are given by roots of the denominator polynomial of $\overline{H}(z)$. Let the poles of $\overline{H}(z)$ be $\overline{p}_{d1}$ and $\overline{p}_{d1}$.

$$\therefore\ \overline{p}_{d1}=0.3555\quad\text{and}\quad \overline{p}_{d2}=0.1758$$

If we compare the poles of H(z) and $\overline{H}(z)$ we can observe that the poles of $\overline{H}(z)$ are very close to the original pole.

Case(ii) : Cascade Realization

$$\text{Given that,}\ H(z)=\frac{1}{1-0.55z^{-1}+0.07z^{-2}}=\frac{z^2}{(z-0.35)(z-0.2)}$$

Using equation (1).

$$=\frac{z}{z-0.35}\times\frac{z}{z-0.2}=\frac{z}{z(1-0.35z^{-1})}\times\frac{z}{z(1-0.2z^{-1})}$$

$$=\frac{1}{1-0.35z^{-1}}\times\frac{1}{1-0.2z^{-1}}=H_1(z)\times H_2(z)$$

$$\text{where,}\quad H_1(z)=\frac{1}{1-0.35z^{-1}}\quad\text{and}\quad H_2(z)=\frac{1}{1-0.2z^{-1}}$$

In cascade realization the system can be realized as cascade of first order sections.

Let us quantize the coefficients of $H_1(z)$ and $H_2(z)$ by truncation.

$$.35_{10}\xrightarrow[\text{binary}]{\text{Convert to}}.010110_2\xrightarrow[\text{5 bits}]{\text{Truncate to}}.01011_2\xrightarrow[\text{decimal}]{\text{Convert to}}.3438_{10}$$

$$.2_{10}\xrightarrow[\text{binary}]{\text{Convert to}}.001100_2\xrightarrow[\text{4 bits}]{\text{Truncate to}}.00110_2\xrightarrow[\text{decimal}]{\text{Convert to}}.1875_{10}$$

Decimal to binary conversion,

.35	.2
×2	×2
$\boxed{1}$70	$\boxed{0}$.4
×2	×2
$\boxed{1}$.4	$\boxed{0}$.8
×2	×2
$\boxed{0}$.8	$\boxed{1}$.6
×2	×2
$\boxed{1}$.6	$\boxed{1}$.2
×2	×2
$\boxed{1}$.2	$\boxed{0}$.4
×2	×2
$\boxed{0}$.4	$\boxed{0}$.8
$.010110_2$	$.001100_2$

Let $\overline{H}_1(z)$ and $\overline{H}_2(z)$ be the transfer function of the first order sections after quantizing the coefficients.

$$\therefore\ \overline{H}_1(z)=\frac{1}{1-0.3438z^{-1}}\quad;\quad \overline{H}_2(z)=\frac{1}{1-0.1875z^{-1}}$$

$$\text{Let,}\quad \overline{H}_1(z)=\frac{Y_1(z)}{X(z)}=\frac{1}{1-0.3438z^{-1}}$$

On cross multiplying the above equation we get,

$Y_1(z) - 0.3438z^{-1}Y_1(z) = X(z)$

$\therefore\ Y_1(z) = X(z) + 0.3438z^{-1}Y_1(z)$ .....(2)

Binary to decimal conversion,

$$.01011_2=(0\times 2^{-1})+(1\times 2^{-2})+(0\times 2^{-3})+(1\times 2^{-4})+(1\times 2^{-5})$$
$$=.3438_{10}$$
$$.00110_2=(0\times 2^{-1})+(0\times 2^{-2})+(1\times 2^{-3})+(1\times 2^{-4})+(0\times 2^{-5})$$
$$=.1875$$

$$\text{Let,}\ \overline{H}_2(z)=\frac{Y(z)}{Y_1(z)}=\frac{1}{1-0.1875z^{-1}}$$

On cross multiplying the above equation we get,

$Y(z) - 0.1875z^{-1}Y(z) = Y_1(z)$

$\therefore\ Y(z) = Y_1(z) + 0.1875z^{-1}Y(z)$ .....(3)

Using equation (2) and (3) the cascade structure of the system is drawn as shown in fig 2.

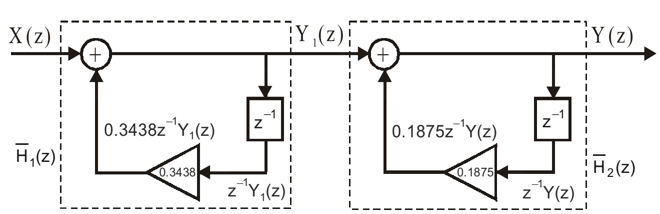


Fig 2 : Cascade realization of the system.

Let us examine the poles of the cascade system.

Let, the poles of the cascade system be $\overline{p}_{c1}$ and $\overline{p}_{c2}$ which are given by the roots of the denominator polynomials of $\overline{H}_1(z)$ and $\overline{H}_2(z)$.

$$\text{Let,}\quad \overline{H}_1(z) = \frac{1}{1-0.3438z^{-1}} = \frac{1}{z^{-1}(z-0.3438)} = \frac{z}{z-0.3438}$$

$$H_2(z) = \frac{1}{1-0.1875z^{-1}} = \frac{1}{z^{-1}(z-0.1875)} = \frac{z}{z-0.1875}$$

$$\therefore\ \overline{p}_{c1} = 0.3438 \ \text{ and } \ \overline{p}_{c2} = 0.1875$$

On comparing the poles of the cascade system with original poles we can say that the poles are very close to original poles of the system. Also we can observe that the deviation of poles of cascaded system is less when compare to deviation of poles in direct form realization.

E8.16. ***Consider the LTI system governed by the equation y(n) + 0.92y(n – 1) + 0.35y(n – 2) = x(n – 2). Discuss the effect of coefficient quantization on pole locations, when the coefficients are quantized by***

(i) 3-bits by truncation (ii) 4-bits by truncation

Solution

Given that, y(n) + 0.92y(n – 1) + 0.35y(n – 2) = x(n – 2)

On taking Z-transform of the given equation we get,

$$Y(z) + 0.92z^{-1}Y(z) + 0.35z^{-2}Y(z) = z^{-2}X(z)$$

$$[z^2 + 0.92z + 0.35]z^{-2}Y(z) = z^{-2}X(z)$$

$$\therefore\ \text{Transfer function, } H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{z^{-2}\left[z^2+0.92z+0.35\right]} = \frac{1}{z^2+0.92z+0.35}$$

$$= \frac{1}{(z+0.46-j0.372)(z+0.46+j0.372)}$$

The roots of the quadratic $z^2 + 0.92z + 0.35 = 0$ are,

$$z = \frac{-0.92 \pm \sqrt{0.92^2-4\times 0.35}}{2} = \frac{-0.92 \pm j0.744}{2} = -0.46 \pm j0.372$$

The poles of the given system are roots of denominator polynomial of H(z). Let the poles be p_1 and p_2.

\ $p_1 = -0.46 + j0.372$

$p_2 = -0.46 - j0.372$

Case(i) : Quantization of coefficient to 3-bits by truncation

The coefficients to be quantized are 0.92_{10} and 0.35_{10}.

$$.92_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .11101_2 \xrightarrow[\text{3 bits}]{\text{Truncate to}} .111_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .875_{10}$$

$$.35_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .01011_2 \xrightarrow[\text{3 bits}]{\text{Truncate to}} .010_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .25_{10}$$

Decimal to binary conversion,

.92	.35
$\times 2$	$\times 2$
$\boxed{1}.84$	$\boxed{0}.7$
$\times 2$	$\times 2$
$\boxed{1}.68$	$\boxed{1}.4$
$\times 2$	$\times 2$
$\boxed{1}.36$	$\boxed{0}.8$
$\times 2$	$\times 2$
$\boxed{0}.72$	$\boxed{1}.6$
$\times 2$	$\times 2$
$\boxed{1}.44$	$\boxed{1}.2$
$.1\,1\,1\,0\,1_2$	$.0\,1\,0\,1\,1_2$

Binary to decimal conversion,

$$.111_2 = (1\times 2^{-1})+(1\times 2^{-2})+(1\times 2^{-3}) = 0.875$$

$$.010_2 = (0\times 2^{-1})+(1\times 2^{-2})+(0\times 2^{-3}) = 0.25$$

Let, $\overline{H}_1(z)$ be the transfer function of the IIR system after quantizing the coefficient to 3-bits by truncation.

$$\therefore\ \overline{H}_1(z) = \frac{1}{z^2+0.875z+0.25}$$

$$= \frac{1}{(z+0.4375-j0.2421)(z+0.4375+j0.2421)}$$

The roots of the quadratic $z^2 + 0.875z + 0.25 = 0$ are,

$$z = \frac{-0.875 \pm \sqrt{0.875^2-4\times 0.25}}{2} = \frac{-0.875 \pm j0.4842}{2} = -0.4375 \pm j0.2421$$

The poles of $\overline{H}_1(z)$ are given by roots of the denominator polynomial of $\overline{H}_1(z)$. Let the poles of $\overline{H}_3(z)$ be p_{11} and p_{21}.

\ $p_{11} = -0.375 + j0.2421$

$p_{21} = -0.4375 - j0.2421$

Case(ii) : Quantization of coefficient to 4-bits by truncation

The coefficients to be quantized are 0.92_{10} and 0.35_{10}.

$$.92_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .11101_2 \xrightarrow[\text{4 bits}]{\text{Truncate to}} .1110_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .875_{10}$$

$$.35_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .01011_2 \xrightarrow[\text{4 bits}]{\text{Truncate to}} .0101_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .5625_{10}$$

Note : The decimal to binary conversion is same as that of case(i).

Binary to decimal conversion,

$$.1110_2 = (1\times 2^{-1})+(1\times 2^{-2})+(1\times 10^{-3})+(0\times 10^{-0}) = .875$$

$$.0101_2 = (0\times 2^{-1})+(1\times 2^{-2})+(0\times 2^{-3})+(1\times 2^{-4}) = .5625$$

Let, $\overline{H}_2(z)$ be the transfer function of the IIR system after quantizing the coefficients to 4-bits by truncation.

$$\therefore\ \overline{H}_2(z) = \frac{1}{z^2+0.875z+0.5625}$$

$$= \frac{1}{(z+0.4375-j0.6092)(z+0.4375+j0.6092)}$$

The poles of $\overline{H}_2(z)$ are given by roots of the denominator polynomial of $\overline{H}_2(z)$. Let the poles of $\overline{H}_2(z)$ be p_{12} and p_{22}.

\ $p_{12} = -0.4375 + j0.6092$

$p_{22} = -0.4375 - j0.6092$

The roots of the quadratic,
$z^2 + 0.875z + 0.5625 = 0$ are,

$$z = \frac{-0.875 \pm \sqrt{0.875^2 - 4 \times 0.5625}}{2}$$
$$= \frac{-0.875 \pm j1.2183}{2}$$
$$= -0.4375 \pm j0.6092$$

Conclusion

The quantization of coefficients result in deviation of pole locations. The deviation is lesser when, the quantization is performed with higher size binary.

E8.17. In the IIR system given below the products are rounded to 4-bits (including sign bit).

$$H(z) = \frac{1}{(1 - 0.15\,z^{-1})(1 - 0.43\,z^{-1})}$$

Find the output round off noise power in (a) Direct form realization (b) Cascade realization

Solution

Given that the products are rounded to 4 bits. Therefore, B = 4. (including sign bit)

Let us assume that the range of product is –1 to +1.

\ Range, R = –1 to +1 = 1 – (–1) = 2

Quantization step size, $q = \frac{R}{2^B} = \frac{2}{2^4} = 0.125$

Variance of noise source, $\sigma_e^2 = \frac{q^2}{12} = \frac{0.125^2}{12} = 1.3021 \times 10^{-3}$
(due to rounding)

(a) Direct form Realization

Given that, $H(z) = \dfrac{1}{(1-0.15z^{-1})(1-0.43z^{-1})} = \dfrac{1}{1-0.43z^{-1}-0.15z^{-1}+0.0645z^{-2}}$

$$= \frac{1}{1-0.58z^{-1}+0.0645z^{-2}}$$

Let, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1-0.58z^{-1}+0.0645z^{-2}}$

On cross multiplying the above equation we get,

$Y(z) - 0.58z^{-1}Y(z) + 0.0645z^{-2}Y(z) = X(z)$

\ $Y(z) = X(z) + 0.58z^{-1}Y(z) - 0.0645z^{-2}Y(z)$ .....(1)

Using the above equation the direct form structure is drawn as shown in fig 1.

The direct form product quantization noise model of the given system is shown in fig 2.

The direct form structure has two constant multipliers. Hence a noise source is introduced at the output of each multiplier as shown in fig 2.

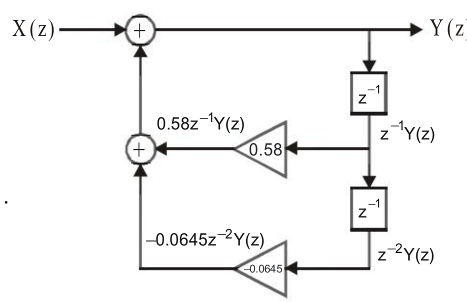


Fig 1 : Direct form structure.

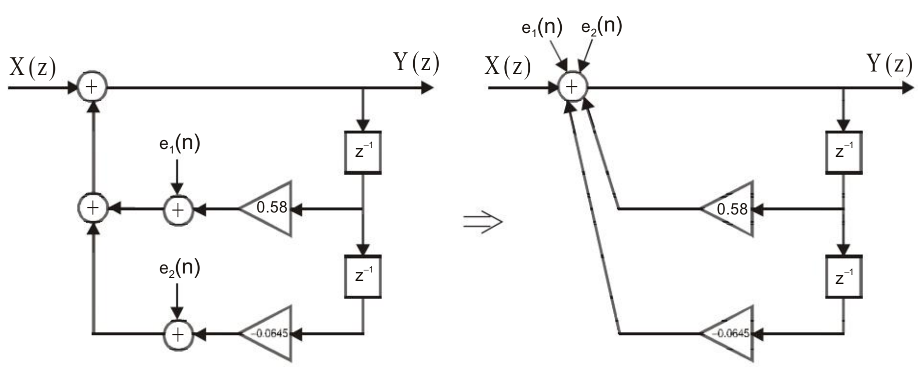


Fig 2 : Direct form product quantization noise model of H(z).

In fig 2 it can be observed that both the noise sources are at the input node of the system. Here, NTF (Noise Transfer Function) for both the noise sources is the system transfer function H(z).

$\therefore$ NTF for noise signal $e_1(n) = T_1(z) = H(z) = \dfrac{1}{(1-0.15z^{-1})(1-0.43z^{-1})}$

NTF for noise signal $e_2(n) = T_2(z) = H(z) = \dfrac{1}{(1-0.15z^{-1})(1-0.43z^{-1})}$

Let, $\sigma^2_{e1op,\,d}$ = Output noise power due to error signal $e_1(n)$ in direct form realization.

$\sigma^2_{e2op,\,d}$ = Output noise power due to error signal $e_2(n)$ in direct form realization.

$$\sigma^2_{e1op,\,d} = \sigma^2_e \frac{1}{2\pi j}\oint_c T_1(z)\,T_1(z^{-1})\,z^{-1}dz = \sigma^2_e \sum_{i=1}^{N} \text{Res}\left[T_1(z)\,T_1(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma^2_e \sum_{i=1}^{N}\left[(z-p_i)\,T_1(z)\,T_1(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$ Using equation (8.29).

where, $p_1, p_2, \ldots\ldots p_N$ are poles of $T_1(z)\,T_1(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

$$T_1(z)T_1(z^{-1})z^{-1} = \frac{1}{(1-0.15z^{-1})(1-0.43z^{-1})} \times \frac{1}{(1-0.15z)(1-0.43z)} \times z^{-1}$$

$$= \frac{z^{-1}}{\left(1-\dfrac{0.15}{z}\right)\left(1-\dfrac{0.43}{z}\right)(-0.15)\left(z-\dfrac{1}{0.15}\right)(-0.43)\left(z-\dfrac{1}{0.43}\right)}$$

$$= \frac{z^{-1}}{0.0645\left(\dfrac{z-0.15}{z}\right)\left(\dfrac{z-0.43}{z}\right)(z-6.6667)\,(z-2.3256)}$$

$$= \frac{15.5039z}{(z-0.15)\,(z-0.43)\,(z-6.6667)\,(z-2.3256)}$$

The poles of $T_1(z)\,T_1(z^{-1})\,z^{-1}$, which are lying inside the unit circle are $p_1 = 0.15$ and $p_2 = 0.43$. (Here, the other two poles $p_3 = 6.6667$ and $p_4 = 2.3256$ are lying outside the unit circle. For calculation of residues only the poles lying inside the unit circle are to be considered).

$$\sigma^2_{e1op,d} = \sigma^2_e \times \sum_{i=1}^{N}\left[(z-p_i)T_1(z)T_1(z^{-1})z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma^2_e \times \left[\cancel{(z-0.15)}\frac{15.5039z}{\cancel{(z-0.15)}(z-0.43)(z-6.6667)(z-2.3256)}\Bigg|_{z=0.15}\right.$$

$$\left. + \cancel{(z-0.43)}\frac{15.5039z}{(z-0.15)\cancel{(z-0.43)}(z-6.6667)(z-2.3256)}\Bigg|_{z=0.43}\right]$$

$$= \sigma^2_e \times \left[\frac{15.5039\times 0.15}{(0.15-0.43)(0.15-6.6667)\,(0.15-2.3256)} + \frac{15.5039\times 0.43}{(0.43-0.15)\,(0.43-6.6667)\,(0.43-2.3256)}\right]$$

$$= 1.3021\times 10^{-3} \times \left[-0.5858+2.014\right] = 1.8597\times 10^{-3} \qquad .....(2)$$

Here, $\sigma^2_{e1op,\,d} = \sigma^2_{e2op,\,d} = 1.8597\times 10^{-3}$

Let, $\sigma^2_{eTop,\,d}$ = Total output noise power due to all the source in direct form realization.

Total output noise power, $\sigma^2_{eTop,\,d} = \sigma^2_{e1op,\,d} + \sigma^2_{e2op,\,d} = 2\times\sigma^2_{e1op,\,d}$

$$= 2\times 1.8597\times 10^{-3} = 3.7194\times 10^{-3}$$

(b) Cascade Realization

Given that, $H(z) = \dfrac{1}{(1-0.15z^{-1})(1-0.43z^{-1})}$

Let, $H(z) = H_1(z)\,H_2(z)$

where, $H_1(z) = \dfrac{1}{1-0.15z^{-1}}$ and $H_2(z) = \dfrac{1}{1-0.43z^{-1}}$

Let, $H_1(z) = \dfrac{Y_1(z)}{X(z)} = \dfrac{1}{1-0.15z^{-1}}$

On cross multiplying the above equation we get,

$Y_1(z) - 0.15z^{-1}Y_1(z) = X(z)$

$\therefore\ Y_1(z) = X(z) + 0.15z^{-1}Y_1(z)$ .....(3)

Using equation (3) the direct form structure of $H_1(z)$ is drawn as shown in fig 3.

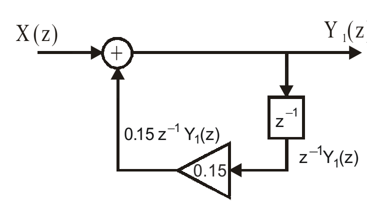


Fig 3 : Direct form structure of $H_1(z)$.

Let, $H_2(z) = \dfrac{Y(z)}{Y_1(z)} = \dfrac{1}{1-0.43z^{-1}}$

On cross multiplying the above equation we get,

$$Y(z) - 0.43z^{-1}Y(z) = Y_1(z)$$

\ $Y(z) = Y_1(z) + 0.43z^{-1}Y(z)$ .....(4)

Using equation (4) the direct form structure of $H_2(z)$ is drawn as shown in fig 4.

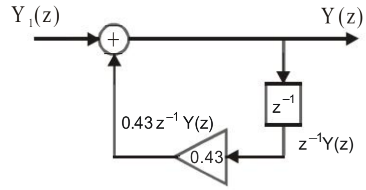


Fig 4 : *Direct form structure of $H_2(z)$.*

The system can be cascaded in two different ways. They are $H_1(z)\ H_2(z)$ and $H_2(z)\ H_1(z)$ as shown in fig 5. The product quantization noise will depend on the order of cascading.

Let us estimate the output noise power for both the cases.

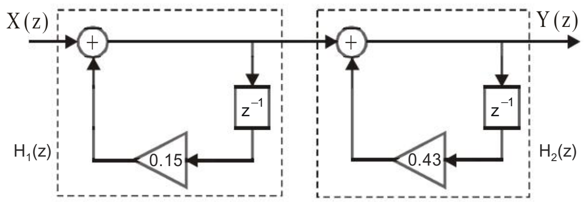


Fig 5a : *Order of cascading is $H_1(z)\ H_2(z)$.*

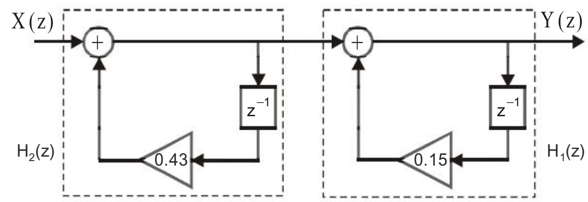


Fig 5b : *Order of cascading is $H_2(z)\ H_1(z)$.*

Fig 5 : *Two ways of cascading $H(z)$.*

Case(i) : The order of cascading is $H_1(z)\ H_2(z)$

The cascade form product quantization noise model of $H_1(z)\ H_2(z)$ is shown in fig 6.

From fig 6 it can be observed that the system has two multipliers and so there are two noise sources. One of the noise source, i.e., $e_1(n)$ is at the input of first section and so its NTF is $H_1(z)\ H_2(z) = H(z)$. The second noise source, i.e., $e_2(n)$ is at the input of second section and so its NTF is $H_2(z)$.

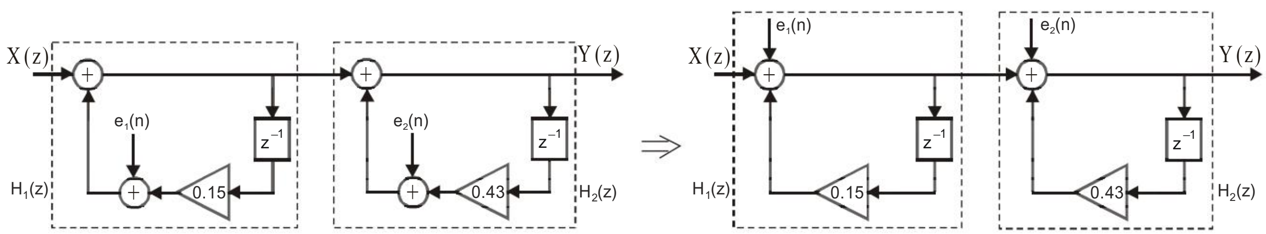


Fig 6 : *Cascade form product quantization noise model of $H_1(z)\ H_2(z)$.*

$$\therefore \text{ NTF for noise signal } e_1(n) = T_1(z) = H(z) = \frac{1}{(1-0.15z^{-1})(1-0.43z^{-1})}$$

$$\text{NTF for noise signal } e_2(n) = T_2(z) = H(z) = \frac{1}{1-0.43z^{-1}}$$

Let, $\sigma^2_{e1op,\ c1}$ = Output noise power due to error signal $e_1(n)$ in cascade realization $H_1(z)\ H_2(z)$.

$\sigma^2_{e2op,\ c1}$ = Output noise power due to error signal $e_2(n)$ in cascade realization $H_1(z)\ H_2(z)$.

In this cascade realization the output noise power due to noise signal will be same as that of direct form realization, because of NTF for $e_1(n)$ is same in both the realizations.

$$\therefore\ \sigma^2_{e1op,\ c1} = 1.8597 \times 10^{-3}$$

From equation (2).

$$\text{Now, } \sigma^2_{e1op,\ c1} = \sigma_e^2\ \frac{1}{2\pi j}\oint_c T_2(2)\ T_2(z^{-1})\ z^{-1}dz = \sigma_e^2 \sum_{i=1}^{N} \text{Res}\left[T_2(z)\ T_2(z^{-1})\ z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma_e^2 \sum_{i=1}^{N} \text{Res}\left[(z-p_i)T_2(z)\ T_2(z^{-1})\ z^{-1}\right]\Big|_{z=p_i}$$

Using equation (8.29).

where $p_1, p_2, ...p_N$ are poles of $T_2(z)\ T_2(z^{-1})\ z^{-1}$, that lie inside the unit circle in z-plane.

$$T_2(z)\ T_2(z^{-1})\ z^{-1} = \frac{1}{1-0.43z^{-1}} \times \frac{1}{1-0.43z} \times z^{-1} = \frac{z^{-1}}{\left(1-\dfrac{0.43}{z}\right)(-0.43)\left(z-\dfrac{1}{0.43}\right)}$$

$$= \frac{-2.3256z^{-1}}{\left(\dfrac{z-0.43}{z}\right)(z-2.3256)} = \frac{-2.3256}{(z-0.43)(z-2.3256)}$$

The function $T_2(z)\,T_2(z^{-1})\,z^{-1}$ has only one pole inside the unit circle, i.e., $p_1 = 0.43$.

$$\therefore\ \sigma^2_{e2op,\,c1} = \sigma^2_e \sum_{i=1}^{N} \left[(z-p_i)\,T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma^2_e \times \left[\cancel{(z-0.43)}\ \frac{-2.3256}{\cancel{(z-0.43)}\,(z-2.3256)}\Bigg|_{z=0.43}\right]$$

$$= 1.3021 \times 10^{-3} \times \frac{-2.3256}{0.43-2.3256} = 1.5975 \times 10^{-3}$$

Let, $s_{eTop,\,c1}$ = Total output noise power due to all noise sources in cascade realization $H_1(z)\,H_2(z)$.

Total output noise power, $\sigma^2_{eTop,\,c1} = \sigma^2_{e1op,\,c1} + \sigma^2_{e2op,\,c1}$

$$= 1.8597 \times 10^{-3} + 1.5975 \times 10^{-3}$$

$$= 3.4572 \times 10^{-3}$$

Case(ii) : The order of cascading is $H_2(z)\,H_1(z)$

The cascade form product quantization noise model of $H_2(z)\,H_1(z)$ is shown in fig 7.

From fig 7 it can be observed that the system has two multipliers and so there are two noise sources. One of the noise source, i.e., $e_1(n)$ is at the input of first section and so its NTF is $H_2(z)\,H_1(z) = H(z)$. The second noise source, i.e., $e_2(n)$ is at the input of second section and so its NTF is $H_1(z)$.

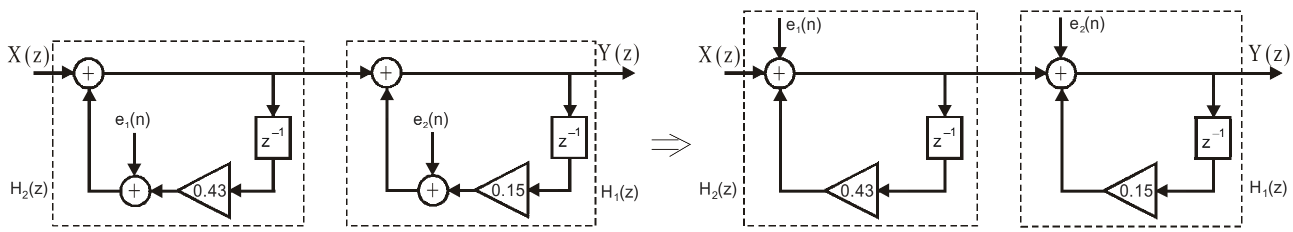


Fig 7 : Cascade form product quantization noise model of $H_2(z)\,H_1(z)$.

$$\therefore\ \text{NTF for noise signal } e_1(n) = T_1(z) = H(z) = \frac{1}{(1-0.15z^{-1})(1-0.43z^{-1})}$$

$$\text{NTF for noise signal } e_2(n) = T_2(z) = H_1(z) = \frac{1}{1-0.15z^{-1}}$$

Let, $\sigma^2_{e1op,\,c2}$ = Output noise power due to error signal $e_1(n)$ in cascade realization $H_2(z)\,H_1(z)$.

$\sigma^2_{e2op,\,c2}$ = Output noise power due to error signal $e_2(n)$ in cascade realization $H_2(z)\,H_1(z)$.

In this cascade realization the output noise power due to noise signal $e_1(n)$will be same as that of direct form realization, because the NTF for $e_1(n)$ is same in both the realizations.

$$\therefore\ \sigma^2_{e1op,\,c1} = 1.8597 \times 10^{-3} \qquad \boxed{\text{Using equation (2).}}$$

$$\text{Now, } \sigma^2_{e20p,\,c1} = \sigma^2_e\,\frac{1}{2\pi j}\oint_c T_2(z)\,T_2(z^{-1})\,z^{-1}dz = \sigma^2_e \sum_{i=1}^{N} \text{Res}\left[T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma^2_e \sum_{i=1}^{N} \left[(z-p_i)\,T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_i} \qquad \boxed{\text{Using equation (8.29).}}$$

where $p_1, p_2, ...p_N$ are poles of $T_2(z)\,T_2(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

$$T_2(z)\,T_2(z^{-1})\,z^{-1} = \frac{1}{1-0.15z^{-1}} \times \frac{1}{1-0.15z} \times z^{-1} = \frac{z^{-1}}{\left(1-\dfrac{0.15z}{z}\right)(-0.15)\left(z-\dfrac{1}{0.15z}\right)}$$

$$= \frac{-6.6667\,z^{-1}}{\left(\dfrac{z-0.15}{z}\right)(z-6.6667)} = \frac{-6.6667}{(z-0.15)(z-6.6667)}$$

The function $T_2(z)\,T_2(z^{-1})\,z^{-1}$ has only one pole inside the unit circle, i.e., $p_1 = 0.15$.

$$\therefore\ \sigma^2_{e2op,\,c1} = \sigma^2_e \times \sum_{i=1}^{N} \left[(z-p_i)\,T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_i} = \sigma^2_e \times \left[\cancel{(z-0.15)}\ \frac{-6.6667}{\cancel{(z-0.15)}\,(z-6.6667)}\Bigg|_{z=0.15}\right]$$

$$= 1.3021 \times 10^{-3} \times \frac{-6.6667}{0.15-6.6667} = 1.3321 \times 10^{-3}$$

Let, $s^2_{eTop,\,c2}$ = Total output noise power due to all noise sources in cascade realization $H_2(z)\,H_1(z)$.

$\therefore$ Total output noise power, $\sigma^2_{eTop,\,c2} = \sigma^2_{e1op,\,c2} + \sigma^2_{e2op,\,c2}$

$$= 1.8597 \times 10^{-3} + 1.3321 \times 10^{-3}$$

$$= 3.1918 \times 10^{-3}$$

Conclusion

1. In direct form realization the product roundoff noise power is higher than cascade realization.
2. In the cascade realization the product roundoff noise power is less in case(ii) when compared to case(i).

E8.18. Find the output roundoff noise power, where the products are rounded to 5-bits (including sign bit) in the two different ways of cascade realization of the following IIR system.

$$H(z) = \frac{1}{(1-0.29\,z^{-1})(1-0.58\,z^{-1})}$$

Solution

Given that the products are rounded to 5 bits. Therefore, B = 5. (Including sign bit)

Let us assume that the range of product is –1 to +1.

\ Range, R = –1 to +1 = 1 – (–1) = 2

$$\text{Quantization step size, } q = \frac{R}{2^B} = \frac{2}{2^5} = 0.0625$$

$$\text{Variance of noise source, } \sigma_e^2 = \frac{q^2}{12} = \frac{0.0625^2}{12} = 3.2552 \times 10^{-4}$$

(due to rounding)

$$\text{Given that, } H(z) = \frac{1}{(1-0.29z^{-1})(1-0.58z^{-1})}$$

Let, $H(z) = H_1(z)\,H_2(z)$

$$\text{where, } H_1(z) = \frac{1}{1-0.29z^{-1}} \text{ and } H_2(z) = \frac{1}{1-0.58z^{-1}}$$

$$\text{Let, } H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-0.29z^{-1}}$$

On cross multiplying the above equation we get,

$Y_1(z) - 0.29z^{-1}Y_1(z) = X(z)$

\ $Y_1(z) = X(z) + 0.29z^{-1}Y_1(z)$ .....(1)

Using equation (1), the direct form structure of $H_1(z)$ is drawn as shown in fig 1.

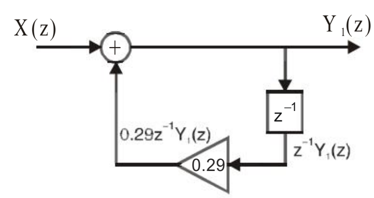


Fig 1 : Direct form structure of $H_1(z)$.

$$\text{Let, } H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1}{1-0.58z^{-1}}$$

On cross multiplying the above equation we get,

$Y(z) - 0.58z^{-1}Y(z) = Y_1(z)$

\ $Y(z) = Y_1(z) + 0.58z^{-1}Y(z)$ .....(2)

Using equation (2), the direct form structure of $H_2(z)$ is drawn as shown in fig 2.

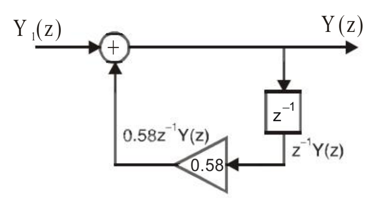


Fig 2 : Direct form structure of $H_2(z)$.

The given system H(z) can be cascaded in two different ways. They are $H_1(z)$ $H_2(z)$ and $H_2(z)$ $H_1(z)$ as shown in fig 3.

The product quantization noise will depend on the order of cascading. Let us estimate the output noise power for both the cases.

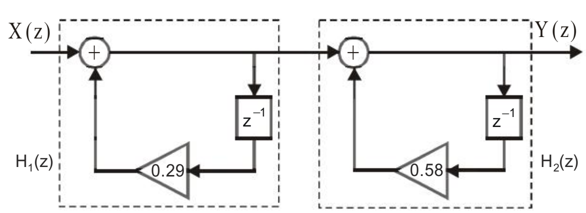


Fig 3a : Order of cascading is $H_1(z)$ $H_2(z)$.

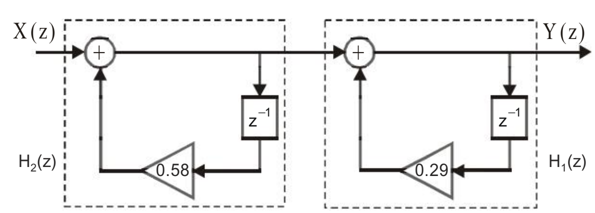


Fig 3b : Order of cascading is $H_2(z)$ $H_1(z)$.

Fig 3 : Two ways of cascading H(z).

Case(i) : The order of cascading is $H_1(z)$ $H_2(z)$

The cascade form product quantization noise model of $H_1(z)$ $H_2(z)$ is shown in fig 4.

From fig 4 it can be observed that the system has two multipliers and so there are two noise sources. One of the noise source, i.e., $e_1(n)$ is at the input of first section and so its NTF is $H_1(z)\,H_2(z) = H(z)$. The second noise source, i.e., $e_2(n)$ is at the input of second section and so its NTF is $H_2(z)$.

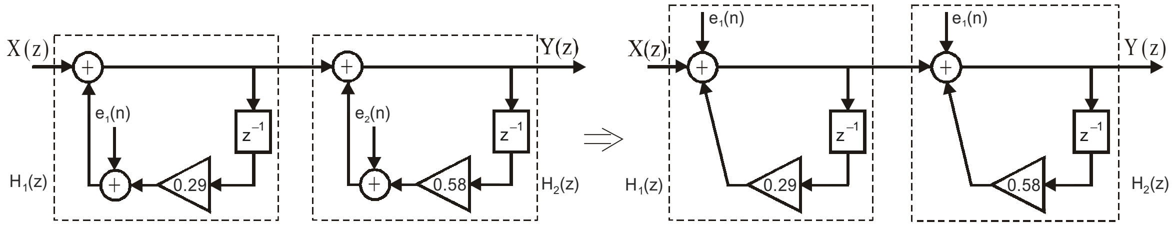


Fig 4 : Cascade form product quantization noise model of $H_1(z)\,H_2(z)$.

$\therefore$ NTF for noise signal $e_1(n) = T_1(z) = H(z) = \dfrac{1}{(1-0.29z^{-1})(1-0.58z^{-1})}$

NTF for noise signal $e_2(n) = T_2(z) = H_2(z) = \dfrac{1}{1-0.58z^{-1}}$

Let, $\sigma^2_{e1op,\,c1}$ = Output noise power due to error signal $e_1(n)$ in cascade realization $H_1(z)\,H_2(z)$.

Now, $\sigma^2_{e1op,\,c1} = \sigma^2_e \dfrac{1}{2\pi j}\oint_c T_1(z)\,T_1(z^{-1})\,z^{-1}dz = \sigma^2_e \sum_{i=1}^{N} \mathrm{Res}\left[T_1(z)\,T_1(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$

Using equation (8.29).

$$= \sigma^2_e \sum_{i=1}^{N}\left[(z-p_i)\,T_1(z)\,T_1(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

where $p_1, p_2, ...p_N$ are poles of $T_1(z)\,T_1(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

Here, $T_1(z)\,T_1(z^{-1})\,z^{-1} = \dfrac{1}{(1-0.29z^{-1})\,(1-0.58z^{-1})} \times \dfrac{1}{(1-0.29z)\,(1-0.58z)} \times z^{-1}$

$$= \frac{z^{-1}}{\left(1-\dfrac{0.29}{z}\right)\left(1-\dfrac{0.58}{z}\right)(-0.29)\left(z-\dfrac{1}{0.29}\right)(-0.58)\left(z-\dfrac{1}{0.58}\right)}$$

$$= \frac{5.9453\,z^{-1}}{\left(\dfrac{z-0.29}{z}\right)\left(\dfrac{z-0.58}{z}\right)(z-3.4483)\,(z-1.7241)}$$

$$= \frac{5.9453\,z}{(z-0.29)(z-0.58)\,(z-3.4483)\,(z-1.7241)}$$

The function $T_1(z)\,T_1(z^{-1})\,z^{-1}$ has two poles inside the unit circle and they are $p_1 = 0.29$ and $p_2 = 0.58$.

$$\therefore\ \sigma^2_{e1op,\,c1} = \sigma^2_e \times \sum_{i=1}^{N}\left[(z-p_i)\,T_1(z)\,T_1(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma^2_e \times \left[\cancel{(z-0.29)}\,\frac{5.9453\,z}{\cancel{(z-0.29)}\,(z-0.58)\,(z-3.4483)\,(z-1.7241)}\Bigg|_{z=0.29}\right.$$

$$\left. + \cancel{(z-0.58)}\,\frac{5.9453\,z}{(z-0.29)\,\cancel{(z-0.58)}\,(z-3.4483)\,(z-1.7241)}\Bigg|_{z=0.58}\right]$$

$$= \sigma^2_e\left[\frac{5.9453\times 0.29}{(0.29-0.58)\,(0.29-3.4483)\,(0.29-1.7241)} + \frac{5.9453\times 0.58}{(0.58-0.29)\,(0.58-3.4483)\,(0.58-1.7241)}\right]$$

$$= 3.2552\times 10^{-4}\left[-1.3126 + 3.6234\right]$$

$$= 7.5221\times 10^{-4} \qquad .....(3)$$

Let, $\sigma^2_{e2op,\,c1}$ = Output noise power due to error signal $e_2(n)$ in cascade realization $H_1(z)\,H_2(z)$.

Now, $\sigma^2_{e2op,\,c1} = \sigma^2_e \dfrac{1}{2\pi j}\oint_c T_2(z)\,T_2(z^{-1})\,z^{-1}dz = \sigma^2_e \sum_{k=1}^{M} \mathrm{Res}\left[T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_k}$

$$= \sigma^2_e \sum_{k=1}^{M}\left[(z-p_k)\,T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_k}$$

From equation (8.29).

where $p_1, p_2, ...p_M$ are poles of $T_2(z)\,T_2(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

$$\text{Here, } T_2(z)\,T_2(z^{-1})\,z^{-1} = \frac{1}{1-0.58z^{-1}} \times \frac{1}{1-0.58z} \times z^{-1} = \frac{z^{-1}}{\left(1-\frac{0.58}{z}\right)(-0.58)\left(z-\frac{1}{0.58}\right)}$$

$$= \frac{-1.7241z^{-1}}{\left(\frac{z-0.58}{z}\right)(z-1.7241)} = \frac{-1.7241}{(z-0.58)\,(z-1.7241)}$$

The function $T_2(z)\,T_2(z^{-1})\,z^{-1}$ has only one pole inside the unit circle, i.e., $p_1 = 0.58$.

$$\therefore\ \sigma^2_{e2op,\,c1} = \sigma_e^2 \times \sum_{k=1}^{M}\left[(z-p_k)T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_k} = \sigma_e^2 \times \cancel{(z-0.58)}\,\frac{-1.7241}{\cancel{(z-0.58)}\,(z-1.7241)}\Bigg|_{z=0.58}$$

$$= 3.2552 \times 10^{-4} \times \frac{-1.7241}{0.58-1.7241} = 4.9054 \times 10^{-4}$$

Let, $\sigma^2_{eTop,\,c1}$ = Total output noise power due to all noise sources in cascade realization $H_1(z)\,H_2(z)$.

$$\therefore\ \text{Total output noise power, } \sigma^2_{eTop,\,c1} = \sigma^2_{e1op,\,c1} + \sigma^2_{e2op,\,c1}$$

$$= 7.5221 \times 10^{-4} + 4.9054 \times 10^{-4}$$

$$= 12.4275 \times 10^{-4} = 1.2428 \times 10^{-3}$$

Case(ii) : The order of cascading is $H_2(z)\,H_1(z)$

The cascade form product quantization noise model of $H_2(z)\,H_1(z)$ is shown in fig 5.

From fig 5 it can be observed that the system has two multipliers and so there are two noise sources. One of the noise source, i.e., $e_1(n)$ is at the input of first section and so its NTF is $H_2(z)\,H_1(z) = H(z)$. The second noise source, i.e., $e_2(n)$ is at the input of second section and so its NTF is $H_1(z)$.

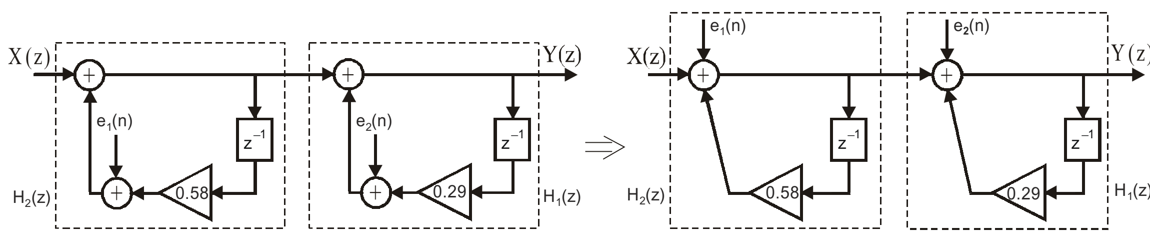


Fig 5 : Cascade form product quantization noise model of $H_2(z)\,H_1(z)$.

$$\therefore\ \text{NTF for noise signal } e_1(n) = T_1(z) = H(z) = \frac{1}{(1-0.29z^{-1})(1-0.58z^{-1})}$$

$$\text{NTF for noise signal } e_2(n) = T_2(z) = H_2(z) = \frac{1}{1-0.29z^{-1}}$$

Let, $\sigma^2_{e1op,\,c2}$ = Output noise power due to error signal $e_1(n)$ in cascade realization $H_2(z)\,H_1(z)$.

Since, the NTF for $e_1(n)$ is same in both the case of cascade realization, the output noise power due to noise signal $e_1(n)$ will also be same in both the case of cascade realization.

$$\therefore\ \sigma^2_{e1op,\,c2} = 7.5221 \times 10^{-4}$$ $\boxed{\text{From equation (3).}}$

Let, $\sigma^2_{e2op,\,c2}$ = Output noise power due to error signal $e_2(n)$ in cascade realization $H_2(z)\,H_1(z)$.

$$\text{Now, } \sigma^2_{e2op,\,c1} = \sigma_e^2\,\frac{1}{2\pi j}\oint_c T_2(z)\,T_2(z^{-1})\,z^{-1}dz = \sigma_e^2 \sum_{k=1}^{M} \text{Res}\left[T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_k}$$ $\boxed{\text{From equation (8.29).}}$

$$= \sigma_e^2 \sum_{k=1}^{M}\left[(z-p_k)T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Big|_{z=p_k}$$

where $p_1, p_2, \ldots\ldots p_M$ are poles of $T_2(z)\,T_2(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

$$\text{Here, } T_2(z)\,T_2(z^{-1})\,z^{-1} = \frac{1}{1-0.29z^{-1}} \times \frac{1}{1-0.29z} \times z^{-1} = \frac{z^{-1}}{\left(1-\frac{0.29}{z}\right)(-0.29)\left(z-\frac{1}{0.29}\right)}$$

$$= \frac{-3.4483z^{-1}}{\left(\frac{z-0.29}{z}\right)(z-3.4483)} = \frac{-3.4483}{(z-0.29)\,(z-3.4483)}$$

The function $T_2(z)\,T_2(z^{-1})\,z^{-1}$ has only one pole inside the unit circle, i.e., $p_1 = 0.29$.

$$\therefore\ \sigma^2_{e2op,\,c2} = \sigma^2_e \times \sum_{k=1}^{M}\left[(z-p_k)\,T_2(z)\,T_2(z^{-1})\,z^{-1}\right]\Bigg|_{z=p_k} = \sigma^2_e \times \left[\cancel{(z-0.29)}\,\frac{-3.4483}{\cancel{(z-0.29)}\,(z-3.4483)}\right]\Bigg|_{z=0.29}$$

$$= 3.2552\times10^{-4}\times\frac{-3.4483}{0.29-3.4483} = 3.5541\times10^{-4}$$

Let, $\sigma^2_{eTop,\,c2}$ = Total output noise power due to all noise sources in cascade realization $H_2(z)\,H_1(z)$.

$$\therefore\ \text{Total output noise power, } \sigma^2_{eTop,\,c2} = \sigma^2_{e1op,\,c2} + \sigma^2_{e2op,\,c2}$$

$$= 7.5221\times10^{-4} + 3.5541\times10^{-4}$$

$$= 11.0762\times10^{-4} = 1.1076\times10^{-3}$$

Conclusion

In cascade realization the product roundoff noise power is less in case(ii) when compared to case(i).

E8.19. Given that, $H(z) = \dfrac{1}{(1-0.18\,z^{-1})\,(1-0.34\,z^{-1})\,(1-0.42\,z^{-1})}$. Determine the output roundoff noise power in the direct form realization of the above system.

(a) When the products are rounded to 4-bits (including sign bit).

(b) When the products are rounded to 7-bits (including sign bit). Comment on the result.

Solution

$$\text{Given that, } H(z) = \frac{1}{(1-0.18z^{-1})\,(1-0.34z^{-1})\,(1-0.42z^{-1})} = \frac{1}{(1-0.52z^{-1}+0.0612z^{-2})\,(1-0.42z^{-1})}$$

$$= \frac{1}{1-0.52z^{-1}+0.0612z^{-2}-0.42z^{-1}+0.2184z^{-2}-0.025z^{-3}}$$

$$= \frac{1}{1-0.94z^{-1}+0.2796z^{-2}-0.025z^{-3}}$$

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.94z^{-1}+0.2796z^{-2}-0.025z^{-3}}$$

On cross multiplying the above equation we get,

$$Y(z) - 0.94z^{-1}Y(z) + 0.2796z^{-2}Y(z) - 0.025z^{-3}Y(z) = X(z)$$

\ $Y(z) = X(z) + 0.94z^{-1}Y(z) - 0.2796z^{-2}Y(z) + 0.025z^{-3}Y(z)$ .....(1)

Using equation (1), the direct form structure of given system is drawn as shown in fig 1.

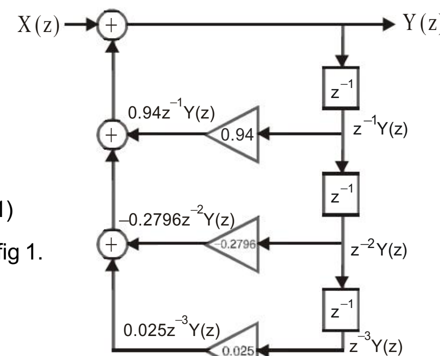


***Fig 1 :** Direct form structure of H(z).*

The direct form structure has three constant multipliers. Hence a noise source is introduced at the output of each multiplies as shown in fig 2.

From fig 2, it can be observed that all the three noise sources are at the input node of the system. Hence, NTF (Noise Transfer Function) for all the three noise sources is the system transfer function H(z).

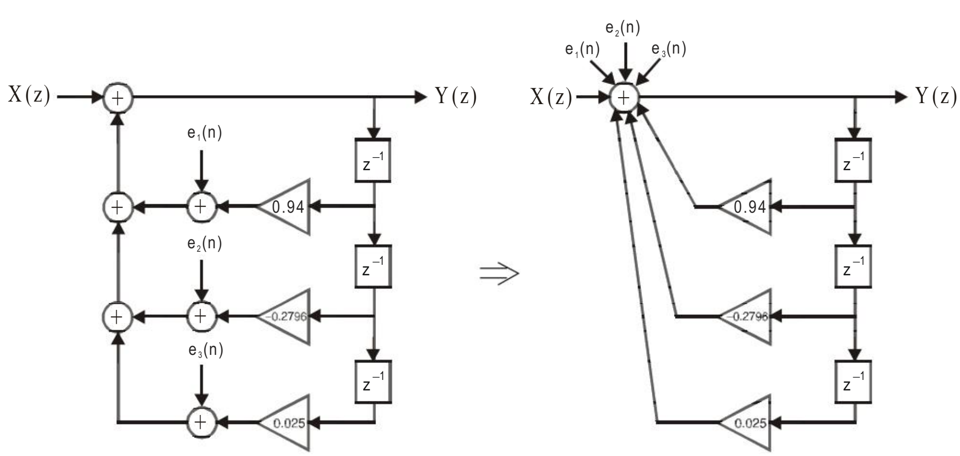


***Fig 2 :** Direct form product quantization noise model of H(z).*

$$\therefore\ \text{NTF} = T(z) = H(z) = \frac{1}{1-0.94z^{-1}+0.2796z^{-2}-0.025z^{-3}}$$

$$= \frac{1}{(1-0.18\,z^{-1})\,(1-0.34z^{-1})\,(1-0.42z^{-1})}$$

Let, s^2_{eop} = Output noise power due to one noise source.

s^2_{eTop} = Total output noise power

Now, $\sigma^2_{eTop} = 3\,\sigma^2_{eop}$

Here, $$\sigma^2_{eop} = \sigma^2_e \times \frac{1}{2\pi j}\oint_c T(z)\,T(z^{-1})\,z^{-1}\,dz = \sigma^2_e \times \sum_{i=1}^{N} \mathrm{Res}\left[T(z)\,T(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma^2_e \times \sum_{i=1}^{N}\left[(z-p_i)T(z)T(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$ $\boxed{\text{Using equation (8.29).}}$

where p_1, p_2 ....., p_N are poles of $T(z)\,T(z^{-1})\,z^{-1}$, that lie inside the unit circle in z-plane.

$$T(z)\,T(z^{-1})\,z^{-1} = \frac{1}{(1-0.18z^{-1})\,(1-0.34z^{-1})\,(1-0.42z^{-1})} \times \frac{1}{(1-0.18z)\,(1-0.34z)\,(1-0.42z)} \times z^{-1}$$

$$= \frac{z^{-1}}{\left(1-\frac{0.18}{z}\right)\left(1-\frac{0.34}{z}\right)\left(1-\frac{0.42}{z}\right)(-0.18)\left(z-\frac{1}{0.18}\right)(-0.34)\left(z-\frac{1}{0.34}\right)(-0.42)\left(z-\frac{1}{0.42}\right)}$$

$$= \frac{-38.9045\,z^{-1}}{\left(\frac{z-0.18}{z}\right)\left(\frac{z-0.34}{z}\right)\left(\frac{z-0.42}{z}\right)(z-5.5556)\,(z-2.9412)\,(z-2.39)}$$

$$= \frac{-38.9045\,z^{2}}{(z-0.18)\,(z-0.34)\,(z-0.42)\,(z-5.5556)\,(z-2.9412)\,(z-2.39)}$$

The poles of $T(z)\,T(z^{-1})\,z^{-1}$ that are lying inside the unit circle are, $p_1 = 0.18$, $p_2 = 0.34$, $p_3 = 0.42$.

$$\therefore\ \sigma^2_{eop} = \sigma^2_e \times \sum_{i=1}^{N}\left[(z-p_i)\,T(z)\,T(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}$$

$$= \sigma^2_e\Bigg[\cancel{(z-0.18)}\,\frac{-38.9045\,z^2}{\cancel{(z-0.18)}\,(z-0.34)\,(z-0.42)\,(z-5.5556)\,(z-2.9412)\,(z-2.39)}\Bigg|_{z=0.18}$$
$$+\cancel{(z-0.34)}\,\frac{-38.9045\,z^2}{(z-0.18)\,\cancel{(z-0.34)}\,(z-0.42)\,(z-5.5556)\,(z-2.9412)\,(z-2.39)}\Bigg|_{z=0.34}$$
$$+\cancel{(z-0.42)}\,\frac{-38.9045\,z^2}{(z-0.18)\,(z-0.34)\,\cancel{(z-0.42)}\,(z-5.5556)\,(z-2.9412)\,(z-2.39)}\Bigg|_{z=0.42}\Bigg]$$

$$= \sigma^2_e\Bigg[\frac{-38.9045 \times 0.18^2}{(0.18-0.34)\,(0.18-0.42)\,(0.18-5.5556)\,(0.18-2.9412)\,(0.18-2.39)}$$
$$+\frac{-38.9045 \times 0.34^2}{(0.34-0.18)\,(0.34-0.42)\,(0.34-5.5556)\,(0.34-2.9412)\,(0.34-2.39)}$$
$$+\frac{-38.9045 \times 0.42^2}{(0.42-0.18)\,(0.42-0.34)\,(0.42-5.5556)\,(0.42-2.9412)\,(0.42-2.39)}\Bigg]$$

$$= \sigma^2_e\left[1.0007 - 12.6333 + 14.0130\right] = 2.3804\,\sigma^2_e$$

\ $s^2_{eTop} = 3s^2_{eop} = 3$ ´ $2.3804s^2_e = 7.1412s^2_e$

Case (a) : When products are rounded to 4-bits

Given that the products are rounded to 4 bits. Therefore, B = 4. (including sign bit).

Let us assume that the range of product is –1 to +1.

\ Range, R = –1 to +1 = 1 – (–1) = 2

Quantization step size, $q = \frac{R}{2^B} = \frac{2}{2^4} = 0.125$

Let, $s^2_e = s^2_{e4}$ = Variance of noise source when the product are rounded to 4-bits.

$$\therefore\ \sigma^2_{e4} = \frac{q^2}{12} = \frac{0.125^2}{12} = 1.3021 \times 10^{-3}$$

Let, $s^2_{eTop,4}$ = Total output roundoff noise power when products are rounded to 4-bits.

$$\therefore\ \sigma^2_{eTop,4} = 7.1412\,\sigma^2_{e4} = 7.1412 \times 1.3021 \times 10^{-3}$$
$$= 9.2986 \times 10^{-3}$$

Case (b) : When products are rounded to 7-bits

Given that the products are rounded to 7 bits. Therefore, B = 7. (including sign bit).

Let us assume that the range of product is –1 to +1.

$\therefore$ Range, R = –1 to +1 = 1 – (–1) = 2

$$\text{Quantization step size, } q = \frac{R}{2^B} = \frac{2}{2^7} = 0.015625$$

Let, $\sigma_e^2 = \sigma_{e7}^2$ = Variance of noise source when the product are rounded to 7-bits.

$$\therefore\ \sigma_{e7}^2 = \frac{q^2}{12} = \frac{0.015625^2}{12} = 2.0345 \times 10^{-5}$$

Let, $\sigma_{eTop,7}^2$ = Total output roundoff noise power when products are rounded to 7-bits.

$$\therefore\ \sigma_{eT0p,7}^2 = 7.1412 \times 2.0345 \times 10^{-5}$$

$$= 1.4529 \times 10^{-4}$$

Comment

The output roundoff noise power reduces when the products are rounded to higher bit size.

$$\%\text{ reduction in output noise power} = \frac{\sigma_{eTop,4}^2 - \sigma_{eTop,7}^2}{\sigma_{eTop,4}^2} \times 100$$

$$= \frac{9.2986 \times 10^{-3} - 1.4529 \times 10^{4}}{9.2986 \times 10^{-3}} \times 100$$

$$= 98.44\%$$

E8.20. Explain the characteristics of a limit cycle oscillation with respect to the system described by the equation,

$$y(n) = 0.82\, y(n-1) + x(n).$$

When the product is quantized to 4-bits by rounding. The system is excited by an input x(n) = 0.875 for n = 0 and x(n) = 0 for n ≠ 0. Also, determine the dead band of the filter.

Solution

Given that, y(n) = 0.82 y(n – 1) + x(n).

The recursive realization of the given system involves the product 0.82 y(n – 1). Let y'(n) be the response of the system when the product is quantized by rounding (upward rounding).

$\therefore$ y'(n) = Q[0.82 y(n – 1)] + x(n)

where, Q[ ] stands for quantization of product.

Given that the products are quantized to 4-bits

Let us choose 4-bit sign-magnitude binary representation to represent the quantized product with 3-bits for magnitude and1-bit for sign.

Given that, x(n) = 0.875 ; for n = 0

= 0 ; for n ≠ 0

Let y'(n) = 0 ; for n < 0

$\therefore$ When n = –1, y'(n) = y'(–1) = 0

When n = 0,

$$y'(n) = y'(0) = Q[0.82\, y'(n-1)] + x(n)$$
$$= Q[0.82\, y'(-1)] + x(0)$$
$$= Q[0.82 \times 0] + 0.875$$
$$= Q[0] + 0.875$$
$$= 0.875_{10} = .111_2$$

$$+.875_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.1110_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.1110_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.111_2$$

$$0.111_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.111_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.875_{10}$$

Binary to decimal conversion,

$$.111_2 = (1 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3})$$
$$= .875_{10}$$

Decimal to binary conversion,

$$\begin{array}{r} .875 \\ \times 2 \\ \hline \boxed{1}.75 \\ \times 2 \\ \hline \boxed{1}.5 \\ \times 2 \\ \hline \boxed{1}.00 \end{array}$$

$.1\,1\,1\,0_2$

When n = 1,

$y'(n) = y'(1) = Q[0.82\ y'(n-1)] + x(n)$

$= Q[0.82\ y'(0)] + x(1)$

$= Q[0.82$ ´ $0.875] + 0$

$= Q[0.7175]$

$= 0.75_{10} = 0.110_2$

$$+.7175_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.1011_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.1011_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.110_2$$

$$0.110_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.110_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.75_{10}$$

Binary to decimal conversion,

$.110_2 = (1 \times 2^{-1}) + (1 \times 2^{-2}) + (0 \times 2^{-3})$

$= .75_{10}$

Decimal to binary conversion,

$.7175 \times 2 = \boxed{1}.435$

$.435 \times 2 = \boxed{0}.87$

$.87 \times 2 = \boxed{1}.74$

$.74 \times 2 = \boxed{1}.48$

$.1\,0\,1\,1_2$

When n = 2,

$y'(n) = y'(2) = Q[0.82\ y'(n-1)] + x(n)$

$= Q[0.82\ y'(1)] + x(2)$

$= Q[0.82$ ´ $0.75] + 0$

$= Q[0.615]$

$= 0.625_{10} = .101_2$

$$+.615_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.1001_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.1001_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.101_2$$

$$0.101_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.101_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.625_{10}$$

Binary to decimal conversion,

$.101_2 = (1 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3})$

$= .625_{10}$

Decimal to binary conversion,

$.615 \times 2 = \boxed{1}.23$

$.23 \times 2 = \boxed{0}.46$

$.46 \times 2 = \boxed{0}.92$

$.92 \times 2 = \boxed{1}.84$

$.1\,0\,0\,1_2$

When n = 3,

$y'(n) = y'(3) = Q[0.82\ y'(n-1)] + x(n)$

$= Q[0.82\ y'(2)] + x(3)$

$= Q[0.82$ ´ $0.75] + 0$

$= Q[0.5125]$

$= 0.5_{10} = .100_2$

$$.5125_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.1000_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.1000_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.100_2$$

$$0.100_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.100_2 \xrightarrow[\text{to decimal}]{\text{Convert}} +.5_{10}$$

Binary to decimal conversion,

$.100_2 = (1 \times 2^{-1}) + (0 \times 2^{-2}) + (0 \times 2^{-3})$

$= 0.5_{10}$

Decimal to binary conversion,

$.5125 \times 2 = \boxed{1}.025$

$.025 \times 2 = \boxed{0}.05$

$.05 \times 2 = \boxed{0}.1$

$.1 \times 2 = \boxed{0}.2$

$.1\,0\,0\,0_2$

When n = 4,

$y'(n) = y'(4) = Q[0.82\ y'(n-1)] + x(n)$

$= Q[0.82\ y'(3)] + x(4)$

$= Q[0.82$ ´ $0.5] + 0$

$= Q[0.41]$

$= 0.375_{10} = .011_2$

$$+.41_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.0111_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.0111_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.011_2$$

$$0.011_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.011_2 \xrightarrow[\text{to decimal}]{\text{Convert}} +.375_{10}$$

Binary to decimal conversion,

$.011_2 = (0 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3})$

$= .375_{10}$

Decimal to binary conversion,

$0\,.41 \times 2 = \boxed{0}.82$

$.82 \times 2 = \boxed{1}.64$

$.64 \times 2 = \boxed{1}.28$

$.28 \times 2 = \boxed{0}.56$

$.0\,1\,1\,0_2$

When n = 5,

$y'(n) = y'(5) = Q[0.82\ y'(n-1)] + x(n)$

$= Q[0.82\ y'(4)] + x(5)$

$= Q[0.82$ ´ $0.375] + 0$

$= Q[0.3075]$

$= 0.25_{10} = .010_2$

$$+.3075_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.0100_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.0100_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.010_2$$

$$0.010_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.010_2 \xrightarrow[\text{to decimal}]{\text{Convert}} +.25_{10}$$

Binary to decimal conversion,

$.010_2 = (0 \times 2^{-1}) + (1 \times 2^{-2}) + (0 \times 2^{-3})$

$= .25_{10}$

Decimal to binary conversion,

$.3075 \times 2 = \boxed{0}.615$

$.615 \times 2 = \boxed{1}.23$

$.23 \times 2 = \boxed{0}.46$

$.46 \times 2 = \boxed{0}.92$

$.0\,1\,0\,0_2$

When n = 6,

$$y'(n) = y'(6) = Q[0.82\,y'(n-1)] + x(n)$$
$$= Q[0.82\,y'(5)] + x(6)$$
$$= Q[0.82 \times 0.25] + 0$$
$$= Q[0.205]$$
$$= 0.25_{10} = .010_2$$

Binary to decimal conversion,
$.010_2 = (0\times2^{-1}) + (1\times2^{-2}) + (0\times2^{-3})$
$= .25_{10}$

Decimal to binary conversion,

$$\begin{array}{r} .205 \\ \times 2 \\ \hline \boxed{0}.41 \\ \times 2 \\ \hline \boxed{0}.82 \\ \times 2 \\ \hline \boxed{1}.64 \\ \times 2 \\ \hline \boxed{1}.28 \end{array}$$

$.0\,0\,1\,1_2$

$$+.205_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.0011_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.0011_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.010_2$$
$$0.010_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.010_2 \xrightarrow[\text{to decimal}]{\text{Convert}} +.25_{10}$$

From the above calculations it can be observed that y'(n) for n = 6, is same as that for n = 5, i.e., y'(6) = y'(5). Hence for all values of n, where $n \geq 5$, the output y'(n) will remain same as 0.25_{10} (or 0.010_2). Therefore the system enters a limit cycle when n = 5. The limit cycle of the system is shown in table 1.

Table 1

n	x(n)	y'(n)	
		Decimal	**Binary**
0	0.875	0.875	0.111
1	0	0.75	0.110
2	0	0.625	0.101
3	0	0.5	0.100
4	0	0.375	0.011
5	0	0.25	0.010
6	0	0.25	0.010
.	.	.	.
.	.	.	.

For the first-order system with only poles the dead band is given by,

$$\text{Dead band} = \pm\frac{2^{-B}}{1-|a|}$$

Here, B = 4, |a| = 0.82

$$\therefore\ \text{Dead band} = \pm\frac{2^{-4}}{1-0.82} = \pm 0.3472$$
$$= -0.3472 \text{ to } +0.3472$$

Note : $0.3472 \times 0.82 = Q[0.284704] = .010_2 = 0.25_{10}$

E8.21. Study the limit cycle behaviour of the system described by w(n) = Q[a w(n – 1)] + x(n) where w(n) is the output of the system and Q[.] is quantization. Assume that, a = –0.752, x(0) = 0.625 and x(n) = 0 for n > 0. Choose 5-bits for quantization.

Solution

Given that, w(n) = Q[a w(n – 1)] + x(n)

The recursive realization of the given system involves the product [a w(n–1)] and the product is quantized. Let us assume that the product is quantized by rounding (upward rounding). Here w(n) is the response of the system when the product is quantized. Let us choose a 5-bit sign-magnitude binary representation to represent the quantized product with 4-bits for magnitude and 1-bit for sign.

Given that, $a = -0.752_{10}$, $x(n) = 0.625$; $n = 0$
$= 0$; $n \neq 0$.

Let, w(n) = 0 for n < 0.

$\therefore$ When n = –1, w(n) = w(–1) = 0

When n = 0,

$$w(n) = w(0) = Q[-0.752\,w(n-1)] + x(n)$$
$$= Q[-0.752\,w(-1)] + x(0)$$
$$= Q[-0.752 \times 0] + 0.625$$
$$= Q[0] + 0.625$$
$$= 0.625_{10} = 0.1010_2$$

Binary to decimal conversion,
$.1010_2 = (1\times2^{-1}) + (0\times2^{-2}) + (1\times2^{-3}) + (0\times2^{-4})$
$= .625_{10}$

Decimal to binary conversion,

$$\begin{array}{r} .625 \\ \times 2 \\ \hline \boxed{1}.25 \\ \times 2 \\ \hline \boxed{0}.5 \\ \times 2 \\ \hline \boxed{1}.0 \end{array}$$

$.10\,1\,0\,0_2$

$$+.625_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.10100_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.10100_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 0.1010_2$$
$$0.1010_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.1010_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.625_{10}$$

When n = 1,

$$w(n) = w(1) = Q[-0.752\, w(n-1)] + x(n)$$
$$= Q[-0.752\, w(0)] + x(1)$$
$$= Q[-0.752 \times 0.625] + 0$$
$$= Q[-0.47]$$
$$= -0.375_{10} = 1.0110_2$$

$$-.47_{10} \xrightarrow[\text{binary}]{\text{Convert to}} -.01111_2 \xrightarrow[\text{sign bit}]{\text{Add}} 1.01111_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 1.0110_2$$
$$1.0110_2 \xrightarrow[\text{sign bit}]{\text{Extract}} -.0110_2 \xrightarrow[\text{decimal}]{\text{Convert to}} -.375_{10}$$

Binary to decimal conversion,

$$.0110_2 = (0 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3}) + (0 \times 2^{-4})$$
$$= .375_{10}$$

Decimal to binary conversion

$$\begin{array}{r} .47 \\ \times 2 \\ \hline \boxed{0}.94 \\ \times 2 \\ \hline \boxed{1}.88 \\ \times 2 \\ \hline \boxed{1}.76 \\ \times 2 \\ \hline \boxed{1}.52 \\ \times 2 \\ \hline \boxed{1}.04 \end{array}$$

$.0\,1\,1\,1\,1_2$

When n = 2,

$$w(n) = w(2) = Q[-0.752\, w(n-1)] + x(n)$$
$$= Q[-0.752\, w(1)] + x(2)$$
$$= Q[-0.752 \times -0.375] + 0$$
$$= Q[0.282]$$
$$= 0.3125_{10} = 0.0101_2$$

$$+.282_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.01001_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.01001_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 0.0101_2$$
$$0.0101_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.0101_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.3125_{10}$$

Binary to decimal conversion,

$$.0101_2 = (0 \times 2^{-1}) + (1 \times 2^{-2}) + (0 \times 2^{-3}) + (1 \times 2^{-4})$$
$$= .3125_{10}$$

Decimal to binary conversion

$$\begin{array}{r} .282 \\ \times 2 \\ \hline \boxed{0}.564 \\ \times 2 \\ \hline \boxed{1}.128 \\ \times 2 \\ \hline \boxed{0}.256 \\ \times 2 \\ \hline \boxed{0}.512 \\ \times 2 \\ \hline \boxed{1}.024 \end{array}$$

$.0\,1\,0\,0\,1_2$

When n = 3,

$$w(n) = w(3) = Q[-0.752\, w(n-1)] + x(n)$$
$$= Q[-0.752\, w(2)] + x(3)$$
$$= Q[-0.752 \times 0.3125] + 0$$
$$= Q[-0.235]$$
$$= -0.25_{10} = 1.0100_2$$

$$-.235_{10} \xrightarrow[\text{binary}]{\text{Convert to}} -.00111_2 \xrightarrow[\text{sign bit}]{\text{Add}} 1.00111_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 1.0100_2$$
$$1.0100_2 \xrightarrow[\text{sign bit}]{\text{Extract}} -.0100_2 \xrightarrow[\text{decimal}]{\text{Convert to}} -.25_{10}$$

Binary to decimal conversion,

$$.0100_2 = (0 \times 2^{-1}) + (1 \times 2^{-2}) + (0 \times 2^{-3}) + (0 \times 2^{-4})$$
$$= .25_{10}$$

Decimal to binary conversion

$$\begin{array}{r} .235 \\ \times 2 \\ \hline \boxed{0}.47 \\ \times 2 \\ \hline \boxed{0}.94 \\ \times 2 \\ \hline \boxed{1}.88 \\ \times 2 \\ \hline \boxed{1}.76 \\ \times 2 \\ \hline \boxed{1}.52 \end{array}$$

$.0\,0\,1\,1\,1_2$

When n = 4,

$$w(n) = w(4) = Q[-0.752\, w(n-1)] + x(n)$$
$$= Q[-0.752\, w(3)] + x(4)$$
$$= Q[-0.752 \times -0.25] + 0$$
$$= Q[0.188]$$
$$= 0.1875_{10} = 0.0011_2$$

$$.188_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .00110_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.00110_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 0.0011_2$$
$$0.0011_2 \xrightarrow[\text{sign bit}]{\text{Extract}} .0011_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .1875_{10}$$

Binary to decimal conversion,

$$.0011_2 = (0 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3}) + (1 \times 2^{-4})$$
$$= .1875_{10}$$

Decimal to binary conversion

$$\begin{array}{r} .188 \\ \times 2 \\ \hline \boxed{0}.376 \\ \times 2 \\ \hline \boxed{0}.752 \\ \times 2 \\ \hline \boxed{1}.504 \\ \times 2 \\ \hline \boxed{1}.008 \\ \times 2 \\ \hline \boxed{0}.016 \end{array}$$

$.0\,0\,1\,1\,0_2$

When n = 5,

w(n) = w(5) = Q[–0.752 w(n – 1)] + x(n)

= Q[–0.752 w(4)] + x(5)

= Q[–0.752 ´ 0.1875] + 0

= Q[–0.141]

= $-0.125_{10} = 1.0010_2$

$$-.141_{10} \xrightarrow[\text{binary}]{\text{Convert to}} -.00100_2 \xrightarrow[\text{sign bit}]{\text{Add}} 1.00100_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 1.0010_2$$

$$1.0010_2 \xrightarrow[\text{sign bit}]{\text{Extract}} -.0010_2 \xrightarrow[\text{decimal}]{\text{Convert to}} -.125_{10}$$

Binary to decimal conversion,

$$.0010_2 = (0 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3}) + (0 \times 2^{-4}) = .125_{10}$$

Decimal to binary conversion

.141 × 2 = $\boxed{0}$.282
× 2 = $\boxed{0}$.564
× 2 = $\boxed{1}$.128
× 2 = $\boxed{0}$.256
× 2 = $\boxed{0}$.512

$.00100_2$

When n = 6,

w(n) = w(6) = Q[–0.752 w(n – 1)] + x(n)

= Q[–0.752 w(5)] + x(6)

= Q[–0.752 ´ –0.125] + 0

= Q[0.094]

= $0.125_{10} = 0.0010_2$

$$.094_{10} \xrightarrow[\text{binary}]{\text{Convert to}} .00011_2 \xrightarrow[\text{sign bit}]{\text{Add}} 0.00011_2 \xrightarrow[\text{to 4-bits}]{\text{Round magnitude}} 0.0010_2$$

$$0.0010_2 \xrightarrow[\text{sign bit}]{\text{Extract}} .0010_2 \xrightarrow[\text{decimal}]{\text{Convert to}} .125_{10}$$

Binary to decimal conversion,

$$.0010_2 = (0 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3}) + (0 \times 2^{-4}) = .125_{10}$$

Decimal to binary conversion

.094 × 2 = $\boxed{0}$.188
× 2 = $\boxed{0}$.376
× 2 = $\boxed{0}$.752
× 2 = $\boxed{1}$.504
× 2 = $\boxed{1}$.008

$.00011_2$

From the above calculations it can be observed that w(n) for n = 6, is negative of w(n) for n = 5, i.e., w(6) = –w(5). Hence for all values of n, where n = 5, the output w(n) will alternate between $-.125_{10}$ and $+.125_{10}$. Therefore the system enters a limit cycle when n = 5. The limit cycle of the system is shown in table 1.

For the first-order system with only poles the dead band is given by,

$$\text{Dead band} = \pm\frac{2^{-B}}{1 - |a|}$$

Here, B = 5, |a| = 0.752

$$\therefore\ \text{Dead band} = \pm\frac{2^{-5}}{1 - 0.752} = \pm 0.126$$

$$= +0.126 \text{ to } -0.126$$

Note : 0.126 = Q[0.126 ´ 0.752] = Q[0.09475] = 0.125_{10}.

Table 1

n	x(n)	w(n)	
		Decimal	Binary
0	0.625	0.625	0.1010
1	0	–0.375	1.0110
2	0	0.3125	0.0101
3	0	–0.25	1.0100
4	0	0.1875	0.0011
5	0	–0.125	1.0010
6	0	0.125	0.0010
7	0	–0.125	1.0010
8	0	0.125	0.0010
.	.	.	.
.	.	.	.

E8.22. An LTI system is characterized by the difference equation, y(n) = 0.91y(n – 1) + x(n).

Determine the limit cycle behaviour and the deadband of the system when x(n) = 0 and y(–1) = 0.75. Assume that the product is quantized to 4-bits by rounding.

Solution

Given that, y(n) = 0.91 y(n – 1) + x(n)

The recursive realization of the given system involves the product "0.91 y(n–1)" and the product is quantized to 4-bits by rounding.

The equation of the system with quantized product is,

y(n) = Q[0.91 y(n – 1)] + x(n)

where, Q[0.91 y(n – 1)] represents quantized product.

Let us choose a 4-bit sign-magnitude binary representation to represent the quantized product with 3-bits for magnitude and 1-bit for sign.

Given that, y(–1) = 0.75 and x(n) = 0.

When n = 0,

$$y(n) = y(0) = Q[0.91\ y(n-1)] + x(n)$$
$$= Q[0.91\ y(-1)] + x(0)$$
$$= Q[0.91 \times 0.75] + 0$$
$$= Q[0.6825]$$
$$= 0.625_{10} = .101_2$$

$$+.6825_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.1010_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.1010_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.101_2$$
$$0.101_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.101_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.625_{10}$$

Binary to decimal conversion,

$$.101_2 = (1\times 2^{-1}) + (0 \times 2^{-2}) + (1\times 2^{-3})$$
$$= .625_{10}$$

Decimal to binary conversion,

$$\begin{array}{r} .6825 \\ \times 2 \\ \hline \boxed{1}.365 \\ \times 2 \\ \hline \boxed{0}.73 \\ \times 2 \\ \hline \boxed{1}.46 \\ \times 2 \\ \hline \boxed{0}.92 \end{array}$$

$.1\,0\,1\,0_2$

When n = 1,

$$y(n) = y(1) = Q[0.91\ y(n-1)] + x(n)$$
$$= Q[0.91\ y(0)] + x(0)$$
$$= Q[0.91 \times 0.625] + 0$$
$$= Q[0.56875]$$
$$= 0.625_{10} = .101_2$$

$$+.56875_{10} \xrightarrow[\text{binary}]{\text{Convert to}} +.1001_2 \xrightarrow[\text{bit}]{\text{Add sign}} 0.1001_2 \xrightarrow[\text{to 3-bits}]{\text{Round magnitude}} 0.101_2$$
$$0.101_2 \xrightarrow[\text{sign bit}]{\text{Extract}} +.101_2 \xrightarrow[\text{decimal}]{\text{Convert to}} +.625_{10}$$

Binary to decimal conversion,

$$.101_2 = (1\times 2^{-1}) + (0 \times 2^{-2}) + (1\times 2^{-3})$$
$$= .625_{10}$$

Decimal to binary conversion,

$$\begin{array}{r} .56875 \\ \times 2 \\ \hline \boxed{1}.1375 \\ \times 2 \\ \hline \boxed{0}.275 \\ \times 2 \\ \hline \boxed{0}.55 \\ \times 2 \\ \hline \boxed{1}.1 \end{array}$$

$.1\,0\,0\,1_2$

From the above calculations it can be observed that y(n) for n = 1 is same as that for n = 0, i.e., y(1) = y(0). Hence for all values of n, where $n \geq 0$, the output y(n) will remain same as 0.625_{10} (or 0.101_2). Therefore, the system enters a limit cycle when n = 0. The limit cycle of the system is shown in table 1.

Table 1

n	x(n)	y'(n)	
		Decimal	**Binary**
0	0	0.625	0.101
1	0	0.625	0.101
2	0	0.625	0.101
3	0	.	.
4	0	.	.
5	0	.	.

For the first-order system with only poles the dead band is given by,

$$\text{Dead band} = \pm \frac{2^{-B}}{1-|a|}$$

Here, $B = 4,\ |a| = 0.91$

$$\therefore\ \text{Dead band} = \pm \frac{2^{-4}}{1-0.91} = \pm 0.6944$$
$$= +0.6944 \text{ to } -0.6944$$

Note : $Q[0.91 \times 0.6944] = Q[0.6319] = 0.625_{10}$.

E8.23. For the digital network shown in fig 1, find the transfer function, H(z) and scale factor S_0 to avoid overflow in register A_1.

Solution

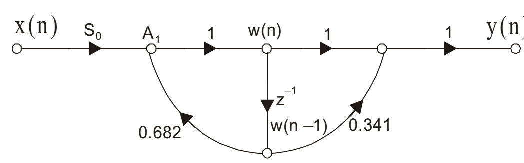


Fig 1 .

For the digital network shown in fig 1, find the transfer function, H(z) and scale factor S_0 to avoid overflow in register A_1.

Solution

To find transfer function

The digital network without scale factor S_0 is shown in fig 2. The digital network has direct relation between time domain and z-domain as shown in fig 2.

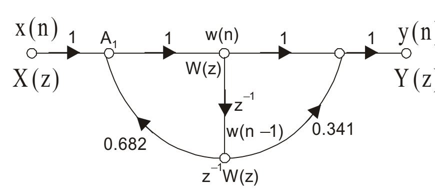


Fig 2.

In fig 2, on equating the sum of incoming signals of A_1 to outgoing signal of A_1 we get,

$$X(z) + 0.682\ z^{-1}\ W(z) = W(z)$$
$$\therefore\quad X(z) = W(z) - 0.682\ z^{-1}\ W(z)$$
$$= W(z)\ [1 - 0.682\ z^{-1}] \qquad .....(1)$$

In fig 2, on equating the sum of incoming signals to output node to Y(z) we get,

$$Y(z) = W(z) + 0.341\ z^{-1}\ W(z)$$
$$\therefore\ Y(z) = W(z)\ [1 + 0.341 z^{-1}] \qquad .....(2)$$

The transfer function of the system, H(z) is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{W(z)\left[1+0.341z^{-1}\right]}{W(z)\left[1-0.682z^{-1}\right]} = \frac{1+0.341z^{-1}}{1-0.682z^{-1}}$$ Using equation (1) and (2).

To find scale factor S_0

Let, $T_s(z)$ be the transfer function seen between the input to the system and output of register A_1.

$$\text{Now, } T_s(z) = \frac{W(z)}{X(z)} = \frac{1}{1-0.682z^{-1}}$$ Using equation (1).

The scale factor S_0 can be evaluated using the following equation.

$$\therefore\ S_0^2 = \frac{1}{\frac{1}{2\pi j}\oint_c T_s(z)\,T_s(z^{-1})\,z^{-1}\,dz} = \frac{1}{\sum_{i=1}^{N} \mathrm{Res}\left[T_s(z)\,T_s(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}}$$ Using equation (8.51).

$$= \frac{1}{\sum_{i=1}^{N} \left[(z-p_i)T_s(z)\,T_s(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}}$$

where $p_1, p_2, \ldots\ p_N$ are poles of $T_s(z)\,T_s(z^{-1})\,z^{-1}$ that lie inside the unit circle in z-plane.

$$\text{Here, } T_s(z)\,T_s(z^{-1})z^{-1} = \frac{1}{1-0.682\,z^{-1}} \times \frac{1}{1-0.682z}z^{-1} = \frac{z^{-1}}{\left(1-\frac{0.682}{z}\right)(-0.682)\left(z-\frac{1}{0.682}\right)}$$

$$= \frac{-1.4663z^{-1}}{\left(\frac{z-0.682}{z}\right)(z-1.4663)} = \frac{-1.4663}{(z-0.682)\,(z-1.4663)}$$

Now, the poles of $T_s(z)\,T_s(z^{-1})\,z^{-1}$ are $p_1 = 0.682$, $p_2 = 1.4663$.

Here, $p_1 = 0.682$ is the only pole that lies inside the unit circle in z-plane.

$$\therefore\ S_0^2 = \frac{1}{\sum_{i=1}^{N} \left[(z-p_i)T_s(z)\,T_s(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}} = \frac{1}{\cancel{(z-0.682)} \times \frac{-1.4663}{\cancel{(z-0.682)}\,(z-1.4663)}\Big|_{z=0.682}}$$

$$= \frac{1}{\frac{-1.4663}{0.682-1.4663}} = \frac{1}{1.8696} = 0.5349$$

$$\therefore\ \text{Scale factor, } S_0 = \sqrt{S_0^2} = \sqrt{0.5349} = 0.7314$$

E8.24. For the digital network shown in fig 1, find the scale factor, S to avoid overflow in register A.

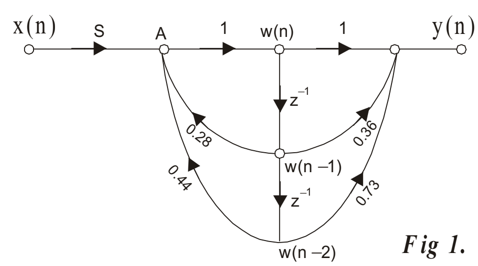


Fig 1.

Solution

The digital network without scale factor S is shown in fig 2. The digital network has direct relation between time domain and z-domain as shown in fig 2.

Now, the scale factor S can be evaluated using the following equation.

$$S^2 = \frac{1}{\frac{1}{2\pi j}\oint_c T_s(z)\,T_s(z^{-1})z^{-1}dz} = \frac{1}{\sum_{i=1}^{N} \mathrm{Res}\left[T_s(z)\,T_s(z^{-1})z^{-1}\right]\Big|_{z=p_i}}$$ Refer equation (8.51).

$$= \frac{1}{\sum_{i=1}^{N} \left[(z-p_i)T_s(z)\,T_s(z^{-1})\,z^{-1}\right]\Big|_{z=p_i}}$$

where, $T_s(z) = \frac{W(z)}{X(z)}$ = Transfer function seen between the input to the system and output of register A.

and $p_1, p_2, \ldots\ p_N$ are poles of the function $T_s(z)\,T_s(z^{-1})\,z^{-1}$ that lie inside the unit circle in z-plane.

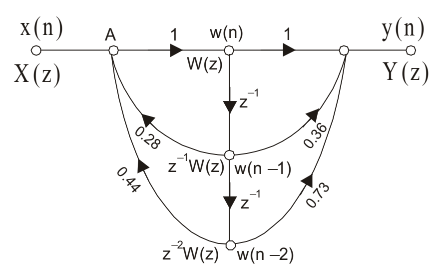


Fig 2.

In order to determine $T_s(z)$, let us form an equation by equating the sum of incoming signals to A, to the outgoing signal of A as shown below.

$$X(z) + 0.28z^{-1}\,W(z) + 0.44z^{-2}\,W(z) = W(z)$$

$$\therefore\; X(z) = W(z) - 0.28z^{-1}\,W(z) - 0.44z^{-2}\,W(z)$$

$$X(z) = W(z)\,[1 - 0.28z^{-1} - 0.44z^{-2}]$$

$$\therefore\; T_s(z) = \frac{W(z)}{X(z)} = \frac{1}{1 - 0.28\,z^{-1} - 0.44\,z^{-2}}$$

$$\text{Also,}\quad T_s(z) = \frac{1}{z^{-2}(z^2 - 0.28z - 0.44)} = \frac{1}{z^{-2}(z - 0.8179)\,(z + 0.5379)} = \frac{1}{(1 - 0.8179z^{-1})\,(1 + 0.5379z^{-1})}$$

The roots of quadratic $z^2 - 0.28z - 0.44 = 0$ are,

$$z = \frac{0.28 \pm \sqrt{0.28^2 + 4 \times 0.44}}{2} = \frac{0.28 \pm 1.8384}{2} = 0.8179,\; -0.5379$$

$$\text{Now,}\quad T_s(z)\,T_s(z^{-1})z^{-1} = \frac{1}{(1 - 0.8179z^{-1})\,(1 + 0.5379z^{-1})} \times \frac{1}{(1 - 0.8179z)\,(1 + 0.5379z)} \times z^{-1}$$

$$= \frac{z^{-1}}{\left(1 - \dfrac{0.8179}{z}\right)\left(1 + \dfrac{0.5379}{z}\right)(-0.8179)\left(z - \dfrac{1}{0.8179}\right)(0.5379)\left(z + \dfrac{1}{0.5379}\right)}$$

$$= \frac{-1.5205\,z^{-1}}{\left(\dfrac{z - 0.8179}{z}\right)\left(\dfrac{z + 0.5379}{z}\right)(z - 1.2226)\,(z + 1.8591)}$$

$$= \frac{-1.5205\,z}{(z - 0.8179)\,(z + 0.5379)\,(z - 1.2226)\,(z + 1.8591)}$$

Now, the poles of $T_s(z)\,T_s(z^{-1})\,z^{-1}$ are $p_1 = 0.8179$, $p_2 = -0.5379$, $p_3 = 1.2226$ and $p_4 = -1.8591$.

Here, p_1 and p_2 are the two poles that lie inside the unit circle in z-plane.

$$\therefore\; \sum_{i=1}^{N}\left[(z - p_i)T_s(z)\,T_s(z^{-1})\,z^{-1}\right]\Bigg|_{z=p_i} = \cancel{(z - 0.8179)} \times \frac{-1.5205z}{\cancel{(z - 0.8179)}\,(z + 0.5379)\,(z - 1.2226)\,(z + 1.8591)}\Bigg|_{z\,=\,0.8179}$$

$$+ \cancel{(z + 0.5379)} \times \frac{-1.5205z}{(z - 0.8179)\,\cancel{(z + 0.5379)}\,(z - 1.2226)\,(z + 1.8591)}\Bigg|_{z\,=\,-0.5379}$$

$$= \frac{-1.5205 \times 0.8179}{(0.8179 + 0.5379)\,(0.8179 - 1.2226)\,(0.8179 + 1.8591)}$$

$$+ \frac{-1.5205 \times (-0.5379)}{(-0.5379 - 0.8179)\,(-0.5379 - 1.2226)\,(-0.5379 + 1.8591)}$$

$$= 0.8467 + 0.2594 = 1.1061$$

$$\therefore\; S^2 = \frac{1}{\displaystyle\sum_{i=1}^{N}\left[(z - p_i)T_s(z)\,T_s(z^{-1})\,z^{-1}\right]\Bigg|_{z=p_i}} = \frac{1}{1.1061} = 0.9041$$

$$\therefore\; \text{Scale factor, } S = \sqrt{S^2} = \sqrt{0.9041} = 0.9508$$

E8.25. For the digital network shown in fig 1, find the scale factor, S_A to avoid overflow in register A. Then find the scale factor S_B to avoid overflow in register-B,

a) When S_A is present and b) When S_A is absent

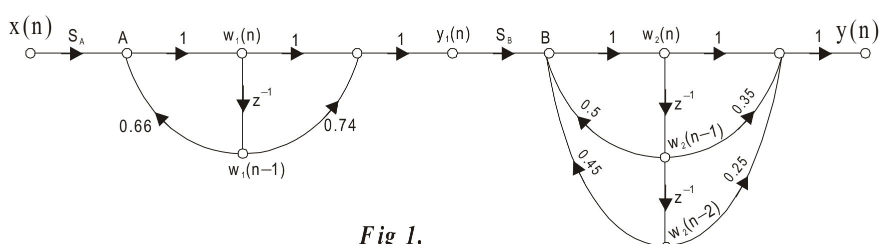


Fig 1.

Solution

To find scale factor, S_A

A part of digital network without scale factor is shown in fig 2. The digital network has direct relation between time domain and z-domain as shown in fig 2.

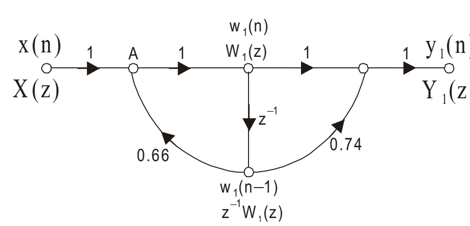


Fig 2.

Now, the scale factor S_A can be evaluated using the following equation.

$$\therefore\; S_A^2 = \frac{1}{2\pi j \oint_c T_{SA}(z)\,T_{SA}(z^{-1})z^{-1}\,dz} = \frac{1}{\displaystyle\sum_{i=1}^{N} \text{Res}\left[T_{SA}(z)\,T_{SA}(z^{-1})z^{-1}\right]\Bigg|_{z=p_i}}$$

Using equation (8.51).

$$\therefore\ S_A^2 = \frac{1}{\sum_{i=1}^{N}\left[(z-p_i)T_{SA}(z)\ T_{SA}(z^{-1})\ z^{-1}\right]\Big|_{z=p_i}}$$

where, $T_{SA}(z) = \dfrac{W_1(z)}{X(z)}$ = Transfer function seen between the input to the system and output of register A.

and $p_1, p_2, \ldots\ldots p_N$ are poles of the function $T_{SA}(z)\ T_{SA}(z^{-1})\ z^{-1}$ that lie inside the unit circle in z-plane.

In order to determine $T_{SA}(z)$ let us form an equation by equating the sum of incoming signals to A, to the outgoing signal of A as shown below.

$$X(z) + 0.66z^{-1}\ W_1(z) = W_1(z)$$

$$\therefore\ X(z) = W_1(z) - 0.66z^{-1}\ W_1(z)$$

$$X(z) = [1 - 0.66z^{-1}]\ W_1(z)$$

$$\therefore\ T_{SA}(z) = \frac{W_1(z)}{X(z)} = \frac{1}{1-0.66\ z^{-1}} \qquad .....(1)$$

$$\therefore\ T_{SA}(z)\ T_{SA}(z^{-1})z^{-1} = \frac{1}{1-0.66z^{-1}} \times \frac{1}{1-0.66z} \times z^{-1} = \frac{z^{-1}}{\left(1-\frac{0.66}{z}\right)\ (-0.66)\left(z-\frac{1}{0.66}\right)}$$

$$= \frac{-1.5152z^{-1}}{\left(\frac{z-0.66}{z}\right)(z-1.5152)} = \frac{-1.5152}{(z-0.66)\ (z-1.5152)}$$

Now, the poles of $T_{SA}(z)\ T_{SA}(z^{-1})\ z^{-1}$ are $p_1 = 0.66$, $p_2 = 1.5152$.

Here, $p_1 = 0.66$ is the only pole that lies inside the unit circle in z-plane.

$$\therefore\ S_A^2 = \frac{1}{\sum_{i=1}^{N}\left[(z-p_i)T_{SA}(z)\ T_{SA}(z^{-1})\ z^{-1}\right]\Big|_{z=p_i}} = \frac{1}{\cancel{(z-0.66)} \times \dfrac{-1.5152}{\cancel{(z-0.66)}\ (z-1.5152)}\Big|_{z=0.66}}$$

$$= \frac{1}{1.7717} = 0.5644$$

$$\therefore\ \text{Scale factor, } S_A = \sqrt{S_A^2} = \sqrt{0.5644} = 0.7513 \qquad .....(2)$$

<u>To Find Scale Factor, S_B</u>

<u>Case a : When S_A is present</u>

The digital network without scale factor S_B is shown in fig 3. The digital network has direct relation between time domain and z-domain as shown in fig 3.

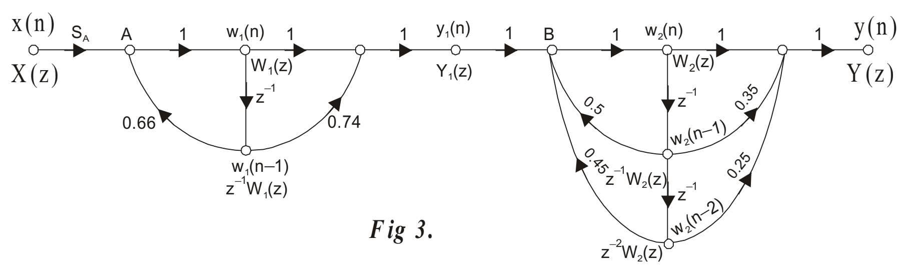


Fig 3.

Now, the scale factor S_B can be evaluated using the following equation.

$$\therefore\ S_B^2 = \frac{1}{2\pi j \oint_c T_{SB}(z)\ T_{SB}(z^{-1})z^{-1}dz} = \frac{1}{\sum_{i=1}^{N} \text{Res}\left[T_{SB}(z)\ T_{SB}(z^{-1})z^{-1}\right]\Big|_{z=p_i}}$$ $\boxed{\text{Using equation (8.51).}}$

$$= \frac{1}{\sum_{i=1}^{N}\left[(z-p_i)T_{SB}(z)\ T_{SB}(z^{-1})\ z^{-1}\right]\Big|_{z=p_i}}$$

where, $T_{SB}(z) = \dfrac{W_2(z)}{S_A\ X(z)}$ = Transfer function seen between the input to the system and output of register B.

and $p_1, p_2, \ldots\ldots p_N$ are poles of the function $T_{SB}(z)\ T_{SB}(z^{-1})\ z^{-1}$ that lie inside the unit circle in z-plane.

The transfer function $T_{SB}(z)$ can be obtained as shown below.

Let, $T_{SB}(z) = \dfrac{W_2(z)}{S_A\,X(z)} = \dfrac{Y_1(z)}{S_A\,X(z)} \times \dfrac{W_2(z)}{Y_1(z)} = \dfrac{1}{S_A} \times \dfrac{W_1(z)}{X(z)} \times \dfrac{Y_1(z)}{W_1(z)} \times \dfrac{W_2(z)}{Y_1(z)}$

Here, $S_A = 0.7513$ Using equation (2).

$\dfrac{W_1(z)}{X(z)} = \dfrac{1}{1-0.66\,z^{-1}}$ Using equation (1).

The function $Y_1(z)/W_1(z)$ can be obtained by forming an equation for node with signal $Y_1(z)$ as shown below.

$$Y_1(z) = W_1(z) + 0.74z^{-1}\,W_1(z)$$

$$\therefore \quad Y_1(z) = W_1(z)\;[1 + 0.74z^{-1}]$$

$$\therefore \quad \frac{Y_1(z)}{W_1(z)} = 1+0.74z^{-1} \qquad .....(3)$$

The function $W_2(z)/Y_1(z)$ can be obtained by forming an equation for node B as shown below.

$$W_2(z) = Y_1(z) + 0.5z^{-1}\,W_2(z) + 0.45z^{-2}\,W_2(z)$$

$$\therefore \quad W_2(z) - 0.5z^{-1}\,W_2(z) - 0.45z^{-2}\,W_2(z) = Y_1(z)$$

$$W_2(z)\,[1 - 0.5z^{-1} - 0.45z^{-2}] = Y_1(z)$$

$$\therefore \quad \frac{W_2(z)}{Y_1(z)} = \frac{1}{1-0.5\,z^{-1}-0.45\,z^{-2}}$$

$$= \frac{1}{z^{-2}(z^2-0.5\,z-0.45)} = \frac{1}{z^{-2}(z-0.975)\;(z+0.475)}$$

$$= \frac{1}{(1-0.975z^{-1})\;(1+0.475z^{-1})}$$

The roots of the quadratic $z^2 - 0.5z + 0.45 = 0$ are given by,

$$z = \frac{0.5 \pm \sqrt{0.5^2 - 4\times 0.45}}{2} = \frac{0.5 \pm 1.45}{2} = 0.975 \text{ (or) } -0.475$$

Now, $T_{SB}(z) = \dfrac{1}{S_A} \times \dfrac{W_1(z)}{X(z)} \times \dfrac{Y_1(z)}{X(z)} \times \dfrac{W_2(z)}{Y_1(z)}$

$$= \frac{1}{0.7513} \times \frac{1}{1-0.66z^{-1}} \times (1+0.74z^{-1}) \times \frac{1}{\left(1-0.975z^{-1}\right)\left(1+0.475z^{-1}\right)}$$

$$= \frac{1.3310(1+0.74z^{-1})}{\left(1-0.66z^{-1}\right)\left(1-0.975z^{-1}\right)\left(1+0.475z^{-1}\right)}$$

$$\therefore \quad T_{SB}(z)T_{SB}(z^{-1})z^{-1} = \frac{1.3310(1+0.74z^{-1})}{\left(1-0.66z^{-1}\right)\left(1-0.975z^{-1}\right)\left(1+0.475z^{-1}\right)} \times \frac{1.3310(1+0.74z)}{(1-0.66z)(1-0.975z)\;(1+0.475z)} \times z^{-1}$$

$$= \frac{1.3310\left(1+\dfrac{0.74}{z}\right) \times 1.3310 \times 0.74\left(z+\dfrac{1}{0.74}\right)\dfrac{1}{z}}{\left(1-\dfrac{0.66}{z}\right)\left(1-\dfrac{0.975}{z}\right)\left(1+\dfrac{0.475}{z}\right)(-0.66)\left(z-\dfrac{1}{0.66}\right)(-0.975)\left(z-\dfrac{1}{0.975}\right)(0.475)\left(z+\dfrac{1}{0.475}\right)}$$

$$= \frac{4.318\left(\dfrac{z+0.74}{z}\right)(z+1.3514)\dfrac{1}{z}}{\left(\dfrac{z-0.66}{z}\right)\left(\dfrac{z-0.975}{z}\right)\left(\dfrac{z+0.475}{z}\right)(z-1.5152)(z-1.0256)(z+2.1053)}$$

$$= \frac{4.318\;z(z+0.74)(z+1.3514)}{(z-0.66)(z-0.975)(z+0.475)(z-5152)(z-1.0256)(z+2.1053)}$$

Now, the poles of $T_{SB}(z)\;T_{SB}(z^{-1})\;z^{-1}$ are,

$p_1 = 0.66$, $p_2 = 0.975$, $p_3 = -0.475$, $p_4 = 1.5152$, $p_5 = 1.025$, $p_6 = -2.1053$

Here, p_1, p_2 and p_3 are the three poles that lie inside the unit circle in z-plane.

$$\therefore \quad \sum_{i=1}^{N}\left[(z-p_i)T_{SB}(z)\;T_{SB}(z^{-1})\;z^{-1}\right]\Bigg|_{z=p_i}$$

$$= \cancel{(z-0.66)} \times \frac{4.318\,z(z+0.74)(z+1.3514)}{\cancel{(z-0.66)}\;(z-0.975)\;(z+0.475)\;(z-1.5152)\;(z-1.0256)(z+2.1053)}\Bigg|_{z\,=\,0.66}$$

$$+ \cancel{(z-0.975)} \times \frac{4.318\,z(z+0.74)(z+1.3514)}{(z-0.66)\;\cancel{(z-0.975)}\;(z+0.475)\;(z-1.5152)\;(z-1.0256)(z+2.1053)}\Bigg|_{z\,=\,0.975}$$

$$+ \cancel{(z+0.475)} \times \frac{4.318\,z(z+0.74)(z+1.3514)}{(z-0.66)\;(z-0.975)\;\cancel{(z+0.475)}\;(z-1.5152)\;(z-1.0256)(z+2.1053)}\Bigg|_{z\,=\,-0.475}$$

$$\therefore \sum_{i=1}^{N}\left[(z-p_i)T_{SB}(z)\ T_{SB}(z^{-1})\ z^{-1}\right]\Big|_{z=p_i}$$

$$= \frac{4.318\times 0.66(0.66+0.74)(0.66+1.3514)}{(0.66-0.975)\ (0.66+0.475)\ (0.6-1.5512)\ (0.66-1.0256)\ (0.66+2.1053)}$$

$$+\frac{4.318\times 0.975(0.975+0.74)(0.975+1.3514)}{(0.975-0.66)\ (0.975+0.475)\ (0.975-1.5152)\ (0.975-1.0256)\ (0.975+2.1053)}$$

$$+\frac{4.318\times -0.475(-0.475+0.74)(-0.475+1.3514)}{(-0.475-0.66)\ (-0.475-0.975)\ (-0.475-1.5152)\ (-0.475-1.0256)\ (-0.475+2.1053)}$$

$$= \frac{8.0251}{-0.3221}+\frac{16.7972}{0.0385}+\frac{-0.4763}{-8.013}$$

$$= -24.9149+436.2909-0.0594$$

$$= 3320.0257 \qquad .....(4)$$

$$\therefore S_B^2 = \frac{1}{\sum_{i=1}^{N}\left[(z-p_i)T_{SB}(z)\ T_{SB}(z^{-1})\ z^{-1}\right]\Big|_{z=p_i}} = \frac{1}{3320.0257} = 3.012\times 10^{-4}$$

$$\therefore \text{Scale factor, } S_B = \sqrt{S_B^2} = \sqrt{3.012\times 10^{-4}} = 0.0174$$

Case b : When S_A is absent

Let, $T_{SB2}(z)$ = Transfer function seen between the input to the system and output of register B when S_A is absent.

$$\text{Now, } T_{SB2}(z) = \frac{W_2(z)}{X(z)}$$

From the analysis in case-a we get,

$$T_{SB}(z) = \frac{W_2(z)}{S_A\, X(z)}$$

$$\therefore T_{SB2}(z) = S_A\, T_{SB}(z)$$

$$\therefore T_{SB2}(z)\ T_{SB2}(z^{-1})\ z^{-1} = S_A T_{SB}(z)\ S_A T_{SB}(z^{-1})\ z^{-1} = S_A^2 T_{SB}(z)\ T_{SB}(z^{-1})\ z^{-1}$$

$$\therefore \sum_{i=1}^{N}\left[(z-p_i)T_{SB2}(z)\ T_{SB2}(z^{-1})\ z^{-1}\right]\Big|_{z=p_i} = \sum_{i=1}^{N}\left[(z-p_i)S_A^2 T_{SB}(z)\ T_{SB}(z^{-1})\ z^{-1}\right]\Big|_{z=p_i}$$

$$= S_A^2\sum_{i=1}^{N}\left[(z-p_i)T_{SB}(z)\ T_{SB}(z^{-1})\ z^{-1}\right]\Big|_{z=p_i}$$

$$= S_A^2 \times 3320.0257 \qquad \boxed{\text{Using equation (4).}}$$

$$= 0.7513^2 \times 3320.0257 \qquad \boxed{\text{Using equation (3).}}$$

$$= 1873.9941$$

Now the scale factor S_B when S_A is absent can be computed as shown below.

$$S_B^2 = \frac{1}{\sum_{i=1}^{N}\left[(z-p_i)T_{SB2}(z)\ T_{SB2}(z^{-1})\ z^{-1}\right]\Big|_{z=p_i}} = \frac{1}{1873.9941} = 5.3362\times 10^{-4}$$

$$\therefore \text{Scale factor, } S_{B2} = \sqrt{S_{B2}^2} = \sqrt{5.3362\times 10^{-4}} = 0.0231$$

Chapter 9

Multirate DSP

9.1 Introduction

The processing of a discrete time signal at different sampling rates in different parts of a system is called ***multirate DSP***. The discrete time systems that employ sampling rate conversion while processing the discrete time signals are called ***multirate DSP systems***. The process of converting a signal from one sampling rate to another sampling rate is called ***sampling rate conversion***.

There are two general methods for sampling rate conversion. In the first method, the discrete signal is converted to analog signal using a D/A converter and then the analog signal is resampled at the desired rate using an A/D converter. The advantage in this method is that the new sampling rate need not have any relation to the old sampling rate. The disadvantage in this method is signal distortion during D/A and A/D process.

In the second method, the sampling rate conversion is entirely performed in the digital domain, using interpolators and decimators. The advantage in rate conversion in the digital domain is that the signal distortion in D/A and A/D process are avoided or eliminated.

There are two ways for sampling rate conversion in the digital domain. They are,

1. Downsampling or decimation
2. Upsampling or interpolation

Downsampling or ***decimation*** is the process of reducing the sampling rate by an integer factor D. ***Upsampling*** or ***interpolation*** is the process of increasing the sampling rate by an integer factor I.

Applications of Multirate DSP Systems

Multirate signal processing is employed in the following systems.

- Sub-band coding of speech signals and image compression
- QMF (Quadrature Mirror Filters) for realizing alias-free LTI multirate systems
- Narrowband FIR and IIR filters for various applications
- Digital transmultiplexers for converting TDM (Time Division Multiplexed) signals to FDM (Frequency Division Multiplexed) signals and vice versa
- Oversampling A/D (Analog-to-Digital) and D/A (Digital-to-Analog) converters for high quality digital audio systems and data loggers (or digital storage systems)
- In digital audio systems the sampling rates of broadcasted signal, CD (Compact Disc), MPEG (Motion Picture Expert Group) standard CD, etc., are different. Hence to access signals from all these devices, sampling rate converters are needed in digital audio systems
- In video broadcasting the American standard NTSC (National Television System Committee) and European standard PAL (Phase Alternating Line) employ different sampling rates. Hence to receive both the signals sampling rate converters are needed in video receivers

Advantages of Multirate Processing

The advantages of multirate processing of discrete time signals are given below.

- The reduction in number of computations
- The reduction in memory requirement (or storage) for filter coefficients and intermediate results.
- The reduction in the order of the system
- The finite word length effects are reduced

9.2 Downsampling (or Decimation)

Downsampling (or ***decimation***) is the process of reducing the samples of the discrete time signal.

Let, $x(n)$ = Discrete time signal

D = Sampling rate reduction factor (and D is an integer)

Now, $x(Dn)$ = Downsampled version of $x(n)$

The device which performs the process of downsampling is called a ***downsampler*** (or ***decimator***). Symbolically, the downsampler can be represented as shown in fig 9.1.

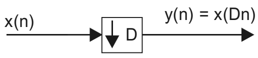


***Fig 9.1** : Decimator.*

Example 9.1

Consider the discrete time signal,

$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Determine the downsampled version of the signals for the sampling rate reduction factors.

a) D = 2 **b)** D = 3 **c)** D = 4.

Solution

Given that,

$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$\therefore$ When n = 0, x(n) = x(0) = 1

When n = 1, x(n) = x(1) = 2

When n = 2, x(n) = x(2) = 3

When n = 3, x(n) = x(3) = 4

When n = 4, x(n) = x(4) = 5

When n = 5, x(n) = x(5) = 6

When n = 6, x(n) = x(6) = 7

When n = 7, x(n) = x(7) = 8

When n = 8, x(n) = x(8) = 9

When n = 9, x(n) = x(9) = 10

When n = 10, x(n) = x(10) = 11

When n = 11, x(n) = x(11) = 12

a) Sampling rate reduction factor, D = 2.

Now, x(Dn) = x(2n) = Discrete time signal decimated by reduction factor 2.

Let, $x(2n) = x_{D2}(n)$

$\therefore$ When n = 0, $x_{D2}(n) = x_{D2}(0) = x(2 \times 0) = x(0) = 1$

When n = 1, $x_{D2}(n) = x_{D2}(1) = x(2 \times 1) = x(2) = 3$

When n = 2, $x_{D2}(n) = x_{D2}(2) = x(2 \times 2) = x(4) = 5$

When n = 3, $x_{D2}(n) = x_{D2}(3) = x(2 \times 3) = x(6) = 7$

When n = 4, $x_{D2}(n) = x_{D2}(4) = x(2 \times 4) = x(8) = 9$

When n = 5, $x_{D2}(n) = x_{D2}(5) = x(2 \times 5) = x(10) = 11$

$\therefore$ $x(2n) = x_{D2}(n) = \{\underset{\uparrow}{1}, 3, 5, 7, 9, 11\}$

b) Sampling rate reduction factor, D = 3.

Now, x(Dn) = x(3n) = Discrete time signal decimated by reduction factor 3.

Let, $x(3n) = x_{D3}(n)$

$\therefore$ When n = 0, $x_{D3}(n) = x_{D3}(0) = x(3 \times 0) = x(0) = 1$

When n = 1, $x_{D3}(n) = x_{D3}(1) = x(3 \times 1) = x(3) = 4$

When n = 2, $x_{D3}(n) = x_{D3}(2) = x(3 \times 2) = x(6) = 7$

When n = 3, $x_{D3}(n) = x_{D3}(3) = x(3 \times 3) = x(9) = 10$

$\therefore$ $x(3n) = x_{D3}(n) = \{\underset{\uparrow}{1}, 4, 7, 10\}$

c) Sampling rate reduction factor, D = 4.

Now, x(Dn) = x(4n) = Discrete time signal decimated by reduction factor 4.

Let, $x(4n) = x_{D4}(n)$

$\therefore$ When n = 0, $x_{D4}(n) = x_{D4}(0) = x(4 \times 0) = x(0) = 1$

When n = 1, $x_{D4}(n) = x_{D4}(1) = x(4 \times 1) = x(4) = 5$

When n = 2, $x_{D4}(n) = x_{D4}(2) = x(4 \times 2) = x(8) = 9$

$\therefore$ $x(4n) = x_{D4}(n) = \{\underset{\uparrow}{1}, 5, 9\}$

Example 9.2

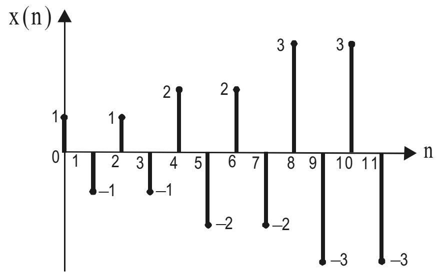


Fig 1.

Consider the discrete time signal shown in fig 1.

Sketch the downsampled version of the signals for the sampling rate reduction factors, **a)** D = 2 **b)** D = 3.

Solution

From fig 1, we can write the samples of the given sequence as shown below.

$$x(n) = \{\underline{1}, -1, 1, -1, 2, -2, 2, -2, 3, -3, 3, -3\}$$

$\therefore$ When n = 0, x(n) = x(0) = 1 | When n = 4, x(n) = x(4) = 2 | When n = 8, x(n) = x(8) = 3

When n = 1, x(n) = x(1) = –1 | When n = 5, x(n) = x(5) = –2 | When n = 9, x(n) = x(9) = –3

When n = 2, x(n) = x(2) = 1 | When n = 6, x(n) = x(6) = 2 | When n = 10, x(n) = x(10) = 3

When n = 3, x(n) = x(3) = –1 | When n = 7, x(n) = x(7) = –2 | When n = 11, x(n) = x(11) = –3

a) Sampling rate reduction factor, D = 2.

Let, $x_{D2}(n)$ = Discrete time signal decimated by reduction factor 2.

Now, $x_{D2}(n) = x(Dn) = x(2n)$

$\therefore$ When n = 0, $x_{D2}(n) = x_{D2}(0) = x(2 \times 0) = x(0) = 1$ | When n = 3, $x_{D2}(n) = x_{D2}(3) = x(2 \times 3) = x(6) = 2$

When n = 1, $x_{D2}(n) = x_{D2}(1) = x(2 \times 1) = x(2) = 1$ | When n = 4, $x_{D2}(n) = x_{D2}(4) = x(2 \times 4) = x(8) = 3$

When n = 2, $x_{D2}(n) = x_{D2}(2) = x(2 \times 2) = x(4) = 2$ | When n = 5, $x_{D2}(n) = x_{D2}(5) = x(2 \times 5) = x(10) = 3$

$$\therefore \; x(2n) = x_{D2}(n) = \{\underline{1}, 1, 2, 2, 3, 3\} \qquad .....(1)$$

Using equation (1), the decimated signal of x(n) by reduction factor 2, is drawn as shown in fig 2.

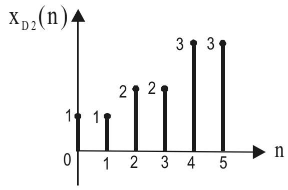


Fig 2 : x(n) decimated by 2.

b) Sampling rate reduction factor, D = 3.

Let, $x_{D3}(n)$ = Discrete time signal decimated by reduction factor 3.

Now, $x_{D3}(n) = x(Dn) = x(3n)$

$\therefore$ When n = 0, $x_{D3}(n) = x_{D3}(0) = x(3 \times 0) = x(0) = 1$

When n = 1, $x_{D3}(n) = x_{D3}(1) = x(3 \times 1) = x(3) = -1$

When n = 2, $x_{D3}(n) = x_{D3}(2) = x(3 \times 2) = x(6) = 2$

When n = 3, $x_{D3}(n) = x_{D3}(3) = x(3 \times 3) = x(9) = -3$

$$\therefore \; x(3n) = x_{D3}(n) = \{\underline{1}, -1, 2, -3\} \qquad .....(2)$$

Using equation (2), the decimated signal of x(n) by reduction factor 3, is drawn as shown in fig 3.

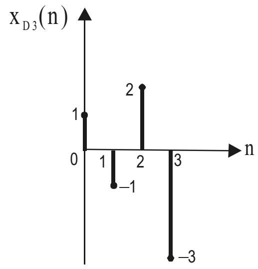


Fig 3 : x(n) decimated by 3.

9.2.1 Spectrum of Downsampler

Let, x(n) be an input signal to the downsampler and y(n) be the output signal.

Let, x'(nD) be a downsampled version of x(n) by an integer factor D.

$$\therefore \; y(n) = x'(nD) \qquad .....(9.1)$$

Consider a unit pulse train defined as,

$$p(n) = 1 \quad ; \text{ for } n = 0, \pm D, \pm 2D, \pm 3D, ...$$
$$= 0 \quad ; \text{ otherwise}$$

Consider the product of $x(n)$ and $p(n)$.

$$x(n)\,p(n) = x(n) \quad ; \text{ for } n = 0, \pm D, \pm 2D, ...$$
$$= 0 \quad ; \text{ otherwise}$$

Now, $x'(n)$ is the signal obtained after removing all zeros from $x(n)\,p(n)$.

$$\therefore\ x'(n) = x(n)\,p(n) \quad ; \text{ for } n = 0, \pm D, \pm 2D, ... \qquad .....(9.2)$$

By definition of Z-transform, $y(n)$ can be expressed as,

$$Y(z) = \sum_{n=-\infty}^{+\infty} y(n)\,z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x'(nD)\,z^{-n}$$

On substituting, $y(n) = x'(nD)$, from equation (9.1).

$$= \sum_{m=-\infty}^{+\infty} x'(m)\,z^{-\frac{m}{D}}$$

Let, $m = nD \Rightarrow n = \frac{m}{D}$
When $n = -\infty$, $m = -\infty$
When $n = +\infty$, $m = +\infty$

$$= \sum_{n=-\infty}^{+\infty} x'(n)\,z^{-\frac{n}{D}}$$

Let, $m = n$

$$= \sum_{n=-\infty}^{+\infty} x(n)\,p(n)\,z^{-\frac{n}{D}}$$

On substituting, $x'(n) = x(n)\,p(n)$, from equation (9.2).

$$= \sum_{n=-\infty}^{+\infty} x(n)\left[\frac{1}{D}\sum_{k=0}^{D-1} e^{\frac{j2\pi kn}{D}}\right] z^{-\frac{n}{D}}$$

Replacing $p(n)$ by its Fourier series representation.

$$= \frac{1}{D}\sum_{k=0}^{D-1}\left[\sum_{n=-\infty}^{+\infty} x(n)\,e^{\frac{j2\pi kn}{D}}\,z^{-\frac{n}{D}}\right]$$

$$= \frac{1}{D}\sum_{k=0}^{D-1}\left[\sum_{n=-\infty}^{+\infty} x(n)\left[e^{\frac{-j2\pi k}{D}}\,z^{\frac{1}{D}}\right]^{-n}\right] \qquad .....(9.3)$$

Fourier series representation of p(n)

One period of $p(n)$ is,

$$p(n) = \{\underset{\substack{\uparrow\\ n=0}}{1},\ 0,\ 0,\ ......\underset{\substack{\uparrow\\ n=D-1}}{0}\}$$

The Fourier coefficients c_k are given by,

$$c_k = \frac{1}{D}\sum_{n=0}^{D-1} p(n)\,e^{\frac{-j2\pi nk}{D}} = \frac{1}{D}$$

The Fourier series representation of $p(n)$ is,

$$p(n) = \sum_{k=0}^{D-1} c_k\,e^{\frac{j2\pi nk}{D}}$$

$$= \sum_{k=0}^{D-1} \frac{1}{D}\,e^{\frac{j2\pi nk}{D}} = \frac{1}{D}\sum_{k=0}^{D-1} e^{\frac{j2\pi nk}{D}}$$

In equation (9.3), the terms inside the bracket is similar to Z-transform of $y(n)$ except that, $z \rightarrow e^{\frac{-j2\pi k}{D}}\,z^{\frac{1}{D}}$, hence $Y(z)$ can be written as shown in equation (9.4).

$$\therefore\ Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{\frac{-j2\pi k}{D}}\,z^{\frac{1}{D}}\right) \qquad .....(9.4)$$

$$\text{where, } X\left(e^{\frac{-j2\pi k}{D}}\,z^{\frac{1}{D}}\right) = \sum_{n=0}^{+\infty} x(n)\left[e^{\frac{-j2\pi k}{D}}\,z^{\frac{1}{D}}\right]^{-n}$$

On substituting, $z = e^{jw}$ in equation (9.4) we get,

$$Y(e^{j\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\, e^{j\omega/D}\right)$$

$$\therefore\ Y(e^{j\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(e^{j(\omega - 2\pi k)/D}\right) \qquad .....(9.5)$$

The equation (9.5) gives the frequency spectrum of the output signal of the decimator, i.e., frequency spectrum of decimated signal.

> *Note : 1. x(n) is the input signal to decimator and $X(e^{jw})$ is the spectrum of input signal.*
>
> *2. y(n) is the output signal of decimator and $Y(e^{jw})$ is the spectrum of output signal of decimator.*

On expanding equation (9.5) we get,

$$Y(e^{j\omega}) = \frac{1}{D} X\left(e^{j\omega/D}\right) + \frac{1}{D} X\left(e^{j(\omega-2\pi)/D}\right) + \frac{1}{D} X\left(e^{j(\omega-4\pi)/D}\right) + ...$$

$$..... + \frac{1}{D} X\left(e^{j(\omega-2\pi(D-1))/D}\right) \qquad .....(9.6)$$

The following observations can be made in equation (9.6).

1. The term, $X\left(e^{j\omega/D}\right)$ is the frequency stretched version of $X(e^{jw})$ stretched by a factor D.

2. The term, $X\left(e^{j(\omega-2\pi)/D)}\right)$ is the shifted version of $X\left(e^{j\omega/D}\right)$ shifted by a factor 2π.

3. In general, the term $X\left(e^{j(\omega-2\pi k)/D)}\right)$ is the shifted version of $X\left(e^{j\omega/D}\right)$ shifted by a factor 2pk.

4. Also it is observed that the magnitude of each component of the spectrum is scaled by a factor 1/D.

Hence we can say that the output spectrum of a decimator is the sum of scaled, stretched and shifted version of the input spectrum.

Since, the output spectrum is the sum of stretched and shifted version of input spectrum, the components of output will overlap and exhibit the phenomena of aliasing, if the input is not bandlimited to p/D.

Therefore, for a signal bandlimited to p/D the spectrum of decimated signal decimated by a factor D is given by the first term of equation (9.6).

$$\therefore\ Y(e^{j\omega}) = \frac{1}{D} X\left(e^{j\omega/D}\right) \qquad .....(9.7)$$

On substituting $z = e^{jw}$ in the above equation we get,

$$Y(z) = \frac{1}{D} X\left(z^{1/D}\right) \qquad .....(9.8)$$

The equations (9.7) and (9.8) can be used to construct the frequency domain and z-domain representation of decimator as shown in fig 9.2 and 9.3.

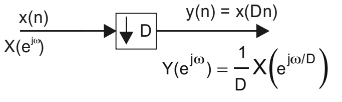


Fig 9.2 : Frequency domain representation of downsampler.

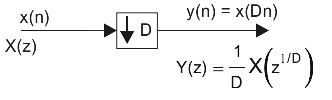


Fig 9.3 : z-domain representation of downsampler.

9.2.2 Anti-aliasing Filter

When the input signal to the decimator is not bandlimited then the spectrum of decimated signal has aliasing (Refer example 9.3 and 9.6). In order to avoid aliasing the input signal should be bandlimited to p/D for decimation by a factor D. Hence the input signal is passed through a lowpass filter with a bandwidth of p/D before decimation. Since this lowpass filter is designed to avoid aliasing in the output spectrum of decimator, it is called ***anti-aliasing filter***.

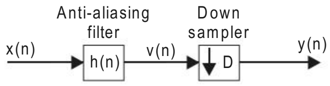


Fig 9.4 : Decimation by a factor D without aliasing.

Example 9.3

Consider a spectrum of input signal $X(e^{jw})$ with a bandwidth of –p/2 to + p/2 as shown in fig 1. When the signal is downsampled by a factor D, sketch the spectrum of a downsampled signal for sampling rate reduction factor D = 2, 3 and 4.

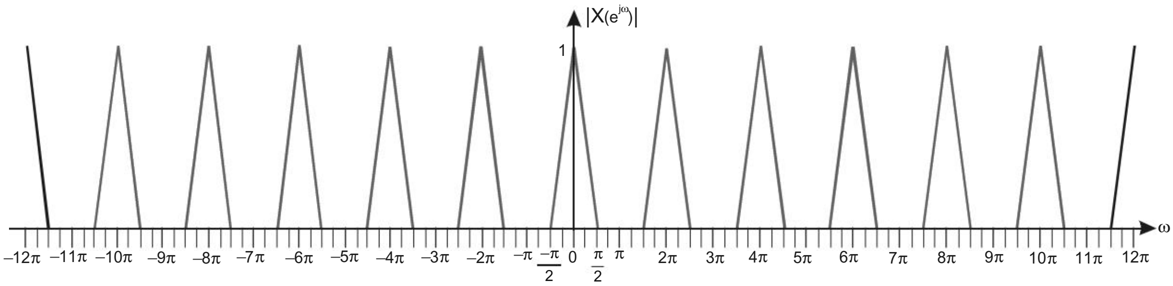


Fig 1.

Solution

Case i : Sampling rate reduction factor, D = 2.

Let, $Y(e^{jw})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 2 is given by,

$$Y(e^{j\omega}) = \frac{1}{2}\sum_{k=0}^{1} X\left(e^{j(\omega - 2\pi k)/2)}\right) \qquad \boxed{\text{Using equation (9.5).}}$$

$$= \frac{1}{2}X\left(e^{j\omega/2}\right) + \frac{1}{2}\, X\left(e^{j(\omega - 2\pi)/2}\right)$$

From the above equation we can say that the spectrum $Y(e^{jw})$ of the decimated signal has two components.

The first component is a frequency-stretched version of input as shown in fig 2. The second component is shifted version of first component, right shifted by 2p as shown in fig 3.

The frequency range of input spectrum is –p/2 to +p/2 and so its bandwidth is p [p/2 – (–p/2) = p]. It can be observed that this bandwidth p is stretched to 2p (p ´ 2 = 2p) for decimation by 2 in each component of spectrum of decimated signal. Therefore, the frequency range of the first component is stretched to –p to +p. Also the magnitude of each component is scaled to 1/2 for decimation by 2.

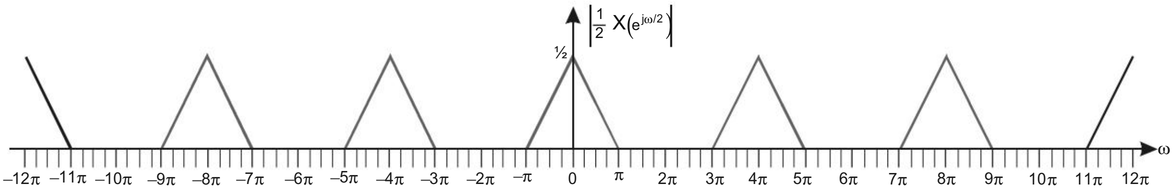


Fig 2.

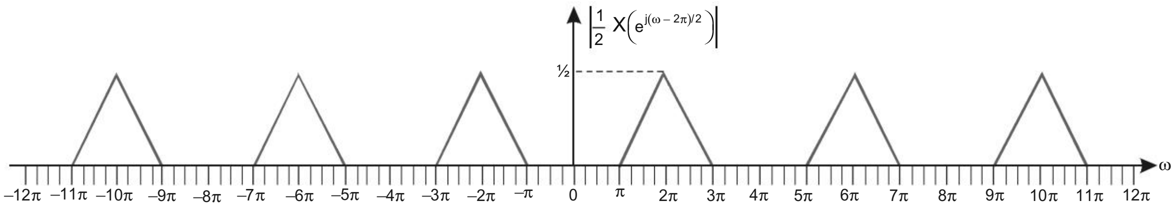


Fig 3.

The spectrum of decimated signal for decimation by 2 is shown in fig 4, which is obtained by adding the components of spectrum shown in fig 2 and fig 3.

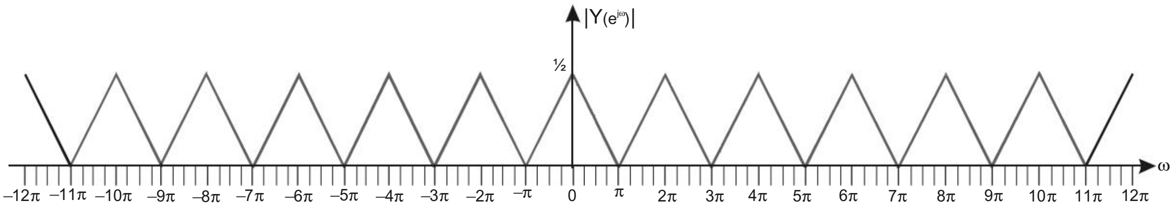


Fig 4.

Case ii : Sampling rate reduction factor, D = 3.

Let, $Y(e^{jw})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 3 is given by,

$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j(\omega - 2\pi k)/3}\right) \qquad \boxed{\text{Using equation (9.5).}}$$

$$= \frac{1}{3}X\left(e^{j\omega/3}\right) + \frac{1}{3}\,X\left(e^{j(\omega - 2\pi)/3}\right) + \frac{1}{3}\,X\left(e^{j(\omega - 4\pi)/3}\right)$$

From the above equation we can say that the spectrum $Y(e^{jw})$ of decimated signal has three components.

The first component is the frequency stretched version of the input as shown in fig 5. The second and third components are shifted version of first component, right shifted by 2π and 4π respectively as shown in fig 6 and fig 7.

It can be observed that the bandwidth π of the input spectrum is stretched to 3π ($\pi \times 3 = 3\pi$) for decimation by 3 in each component of the spectrum of the decimated signal. Therefore, the frequency range of the first component is stretched to $-3\pi/2$ to $+3\pi/2$. Also the magnitude of each component is scaled to 1/3 for decimation by 3.

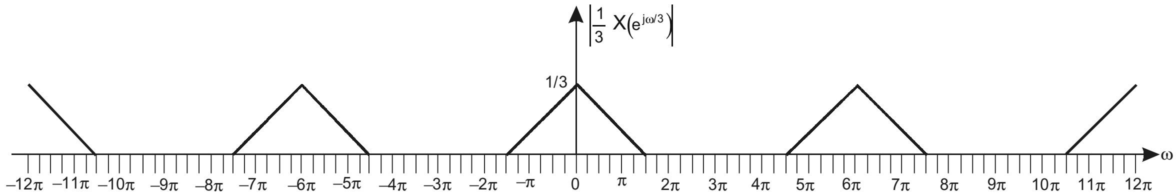


Fig 5.

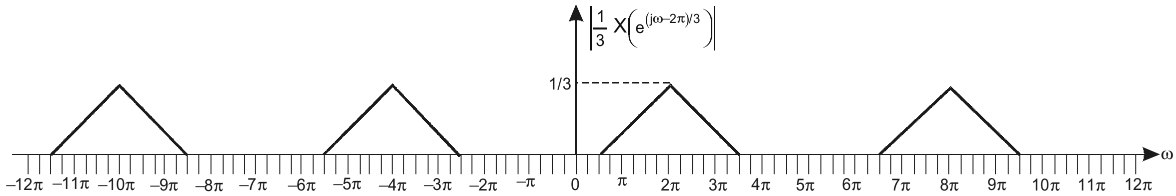


Fig 6.

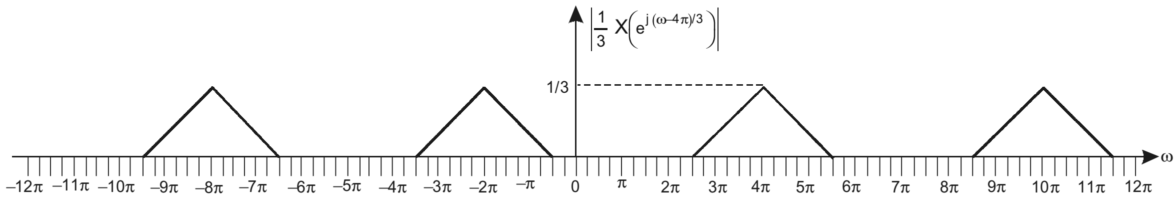


Fig 7.

The spectrum of the decimated signal for decimation by 3 is shown in fig 8, which is obtained by adding the components of spectrum shown in fig 5,6 and fig 7. The shaded parts in fig 8 are aliased portion of the spectrum.

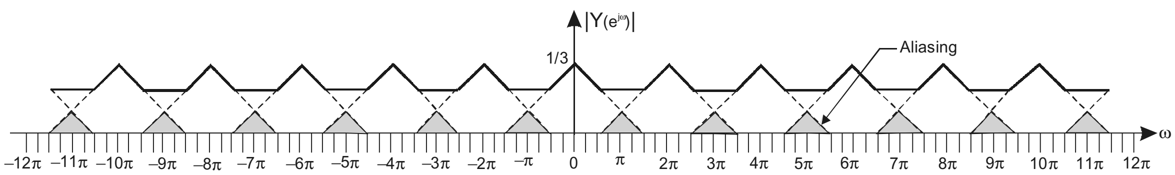


Fig 8.

Case iii : Sampling rate reduction factor, D = 4.

Let, $Y(e^{jw})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 4 is given by,

$$Y(e^{j\omega}) = \frac{1}{4} \sum_{k=0}^{3} X\left(e^{j(\omega - 2\pi k)/4}\right) \qquad \boxed{\text{Using equation (9.5).}}$$

$$= \frac{1}{4} X\left(e^{j\omega/4}\right) + \frac{1}{4} X\left(e^{j(\omega - 2\pi)/4}\right) + \frac{1}{4} X\left(e^{j(\omega - 4\pi)/4}\right) + \frac{1}{4} X\left(e^{j(\omega - 6\pi)/4}\right)$$

From the above equation we can say that the spectrum $Y(e^{jw})$ of decimated signal has four components.

The first component is a frequency-stretched version of the input as shown in fig 9. The second, third and fourth components are shifted versions of first component, right shifted by 2π, 4π, and 6π respectively as shown in fig 10,11 and 12.

It can be observed that the bandwidth π of input spectrum is stretched to 4π ($\pi \times 4 = 4\pi$) for decimation by 4 in each component of the spectrum of the decimated signal. Therefore, the frequency range of first component is stretched to -2π to $+2\pi$. Also, the magnitude of each component is scaled to 1/4 for decimation by 4.

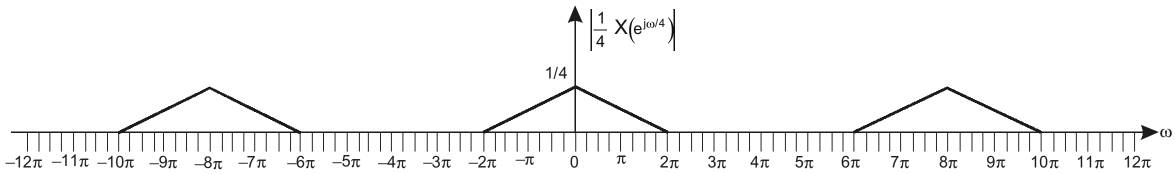


Fig 9.

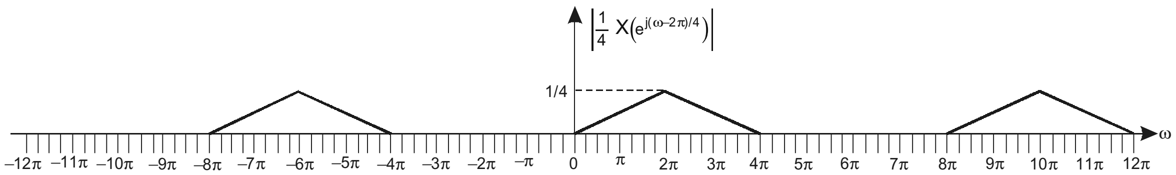


Fig 10.

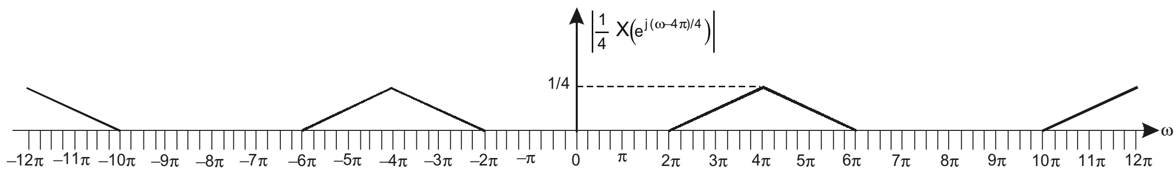


Fig 11.

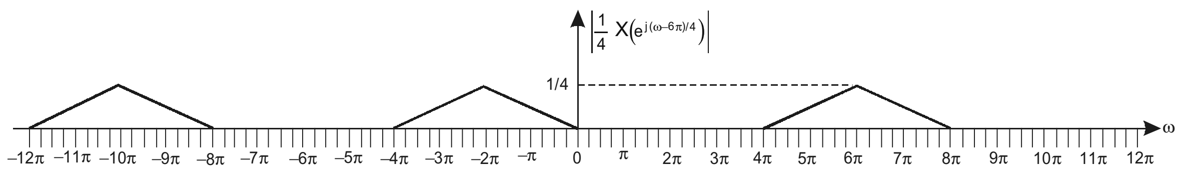


Fig 12.

The spectrum of decimated signal for decimation by 4 is shown in fig 13, which is obtained by adding the components of spectrum shown in fig 9 to 12. The shaded parts in fig 13 are aliased portions of the spectrum.

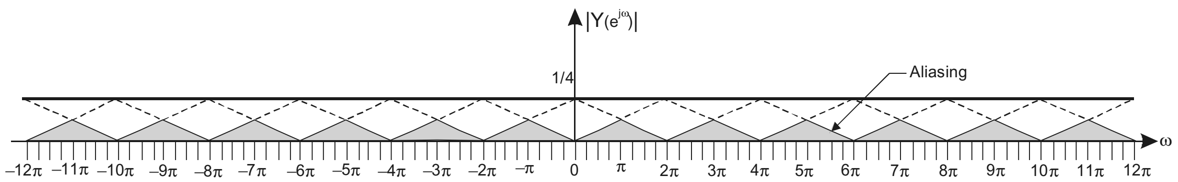


Fig 13.

Conclusion:

From the above three cases of decimation it is observed that, for decimation by a factor D, as long as the input spectrum is bandlimited to $\frac{\pi}{D}$, the spectrum of the decimated signal does not overlap. Hence we can say that, there is no aliasing in the spectrum of the decimated signal if the spectrum of input signal is bandlimited to $\frac{\pi}{D}$.

Example 9.4

The spectrum of a discrete time signal is shown in fig 1. Draw the spectrum of the decimated signal, when the signal is decimated by D = 3.

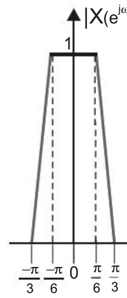


Fig 1.

Solution

The spectrum of a discrete time signal is periodic, with periodicity of 2p. Hence the spectrum of given signal can be drawn as shown in fig 2.

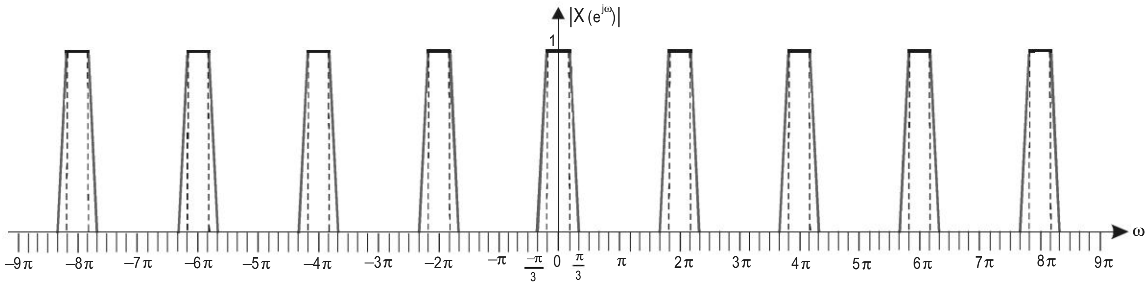


Fig 2.

Let, $Y(e^{jw})$ = Spectrum of decimated signal.

The spectrum of the decimated signal for D = 3 is given by,

$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j(\omega - 2\pi k)/3}\right) \qquad \boxed{\text{Using equation (9.5).}}$$

$$= \frac{1}{3}X\left(e^{j\omega/3}\right) + \frac{1}{3}X\left(e^{j(\omega - 2\pi)/3}\right) + \frac{1}{3}X\left(e^{j(\omega - 4\pi)/3}\right)$$

From the above equation we can say that the spectrum $Y(e^{jw})$ of the decimated signal has three components.

The first component is the frequency-stretched version of inputs as shown in fig 3. The second and third components are shifted versions of first the component, right shifted by 2p and 4p respectively as shown in fig 4 and 5.

The frequency range of input spectrum is –p/3 to +p/3 and so its bandwidth is 2p/3 [p/3 – (–p/3) = 2p/3]. It can be observed that this bandwidth 2p/3 is stretched to 2p (2p/3 ´ 3 = 2p) for decimation by 3 in each component of the spectrum of the decimated signal. Therefore, the frequency range of the first component is stretched to –p to +p. Also the magnitude of each component is scaled to 1/3 for decimation by 3.

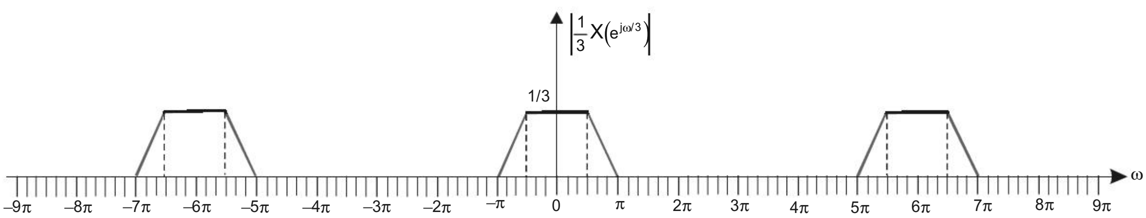


Fig 3.

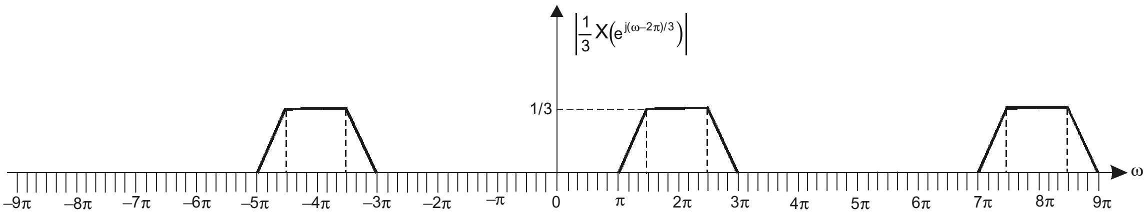


Fig 4.

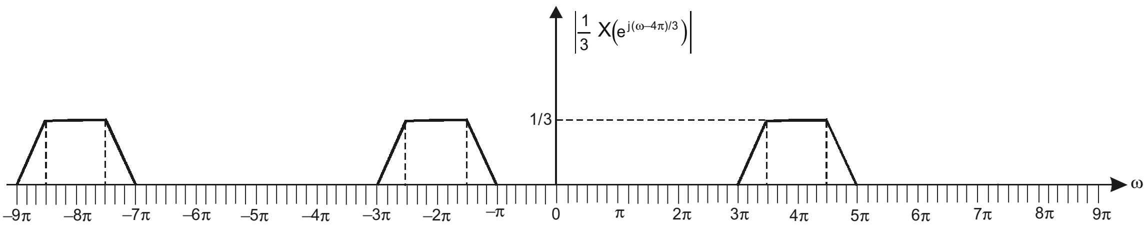


Fig 5.

The spectrum of decimated signal for decimation by 3 is shown in fig 6, which is obtained by adding the components of spectrum shown in fig 3, 4 and 5.

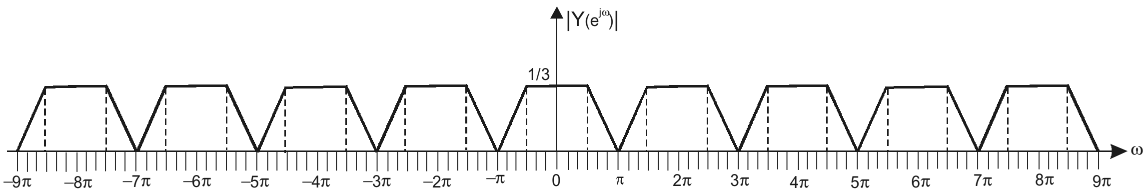


Fig 6.

Example 9.5

The spectrum of a discrete time signal is shown in fig 1. Draw the spectrum of the decimated signal, when the signal is decimated by D = 4.

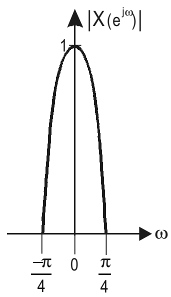


Fig 1.

Solution

The spectrum of a discrete time signal is periodic, with periodicity of 2π. Hence the spectrum of given signal can be drawn as shown in fig 2.

Let, $Y(e^{j\omega})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 4 is given by,

$$Y(e^{j\omega}) = \frac{1}{4}\sum_{k=0}^{3} X\left(e^{j(\omega-2\pi k)/4}\right)$$

Using equation (9.5).

$$= \frac{1}{4}X\left(e^{j\omega/4}\right) + \frac{1}{4}X\left(e^{j(\omega-2\pi)/4}\right) + \frac{1}{4}X\left(e^{j(\omega-4\pi)/4}\right) + \frac{1}{4}X\left(e^{j(\omega-6\pi)/4}\right)$$

From the above equation we can say that the spectrum $Y(e^{j\omega})$ of the decimated signal has four components.

The first component is a frequency-stretched version of inputs as shown in fig 3. The second, third and fourth components are frequency-shifted versions of the first component, right shifted by 2π, 4π and 6π respectively as shown in fig 4, 5 and 6.

The frequency range of the input spectrum is $-\pi/4$ to $+\pi/4$ and so its bandwidth is $\pi/2$ [$\pi/4 - (-\pi/4) = \pi/2$]. It can be observed that this bandwidth $\pi/2$ is stretched to 2π ($\pi/2 \times 4 = 2\pi$) for decimation by 4 in each component of the spectrum of the decimated signal. Therefore, the frequency range of the first component is stretched to $-\pi$ to $+\pi$. Also, the magnitude of each component is scaled to 1/4 for decimation by 4.

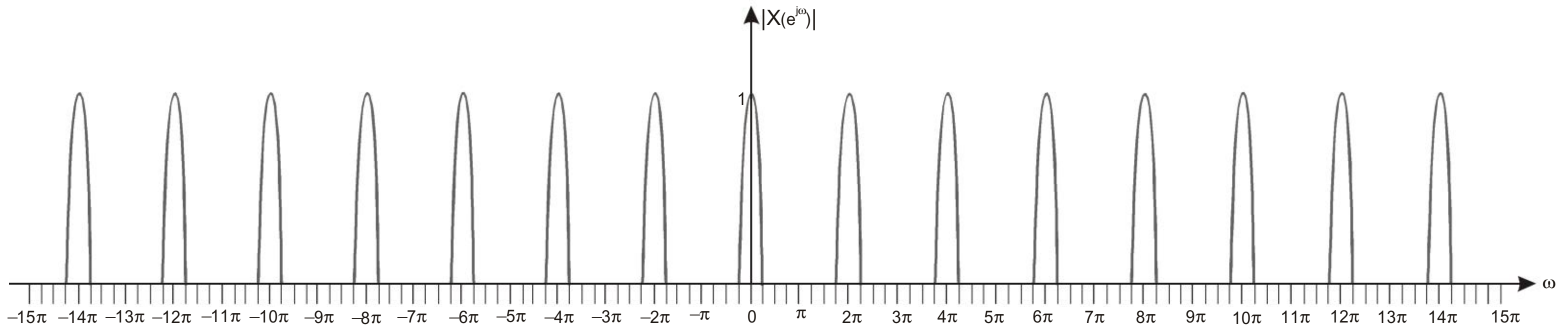


Fig 2.

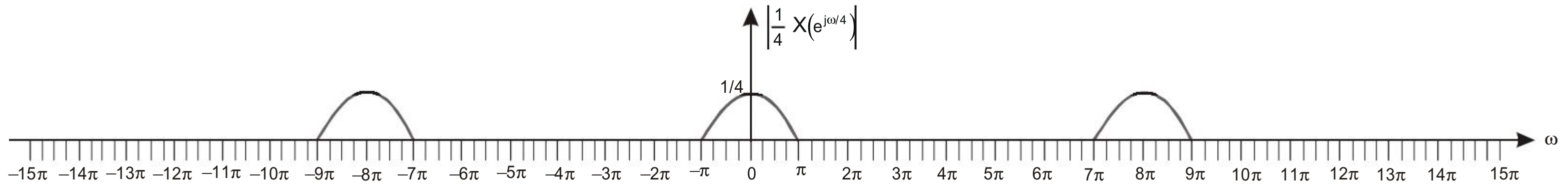


Fig 3.

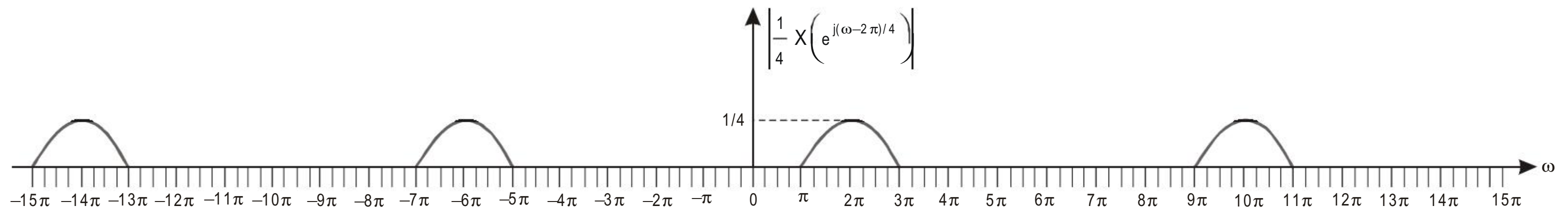


Fig 4.

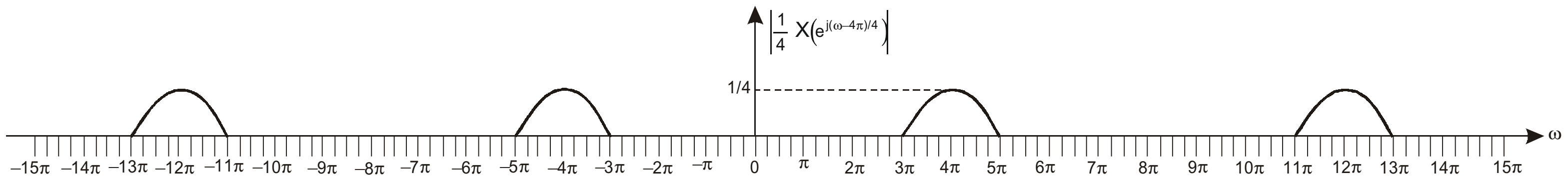


Fig 5.

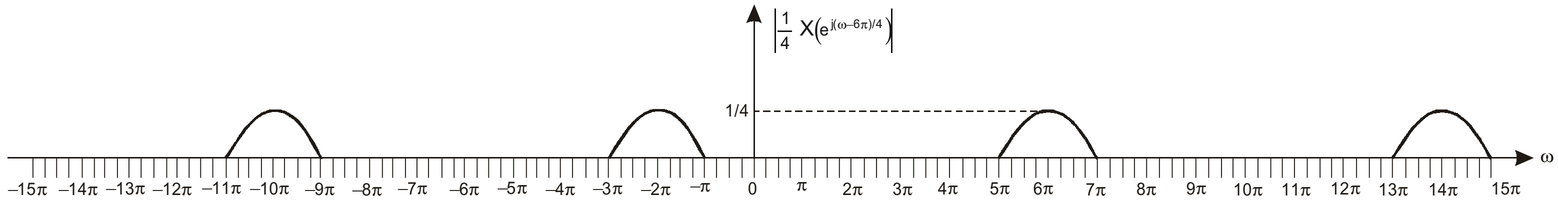


Fig 6.

The spectrum of decimated signal for decimation by 4 is shown in fig 7, which is obtained by adding the components of spectrum shown in fig 3 to 6.

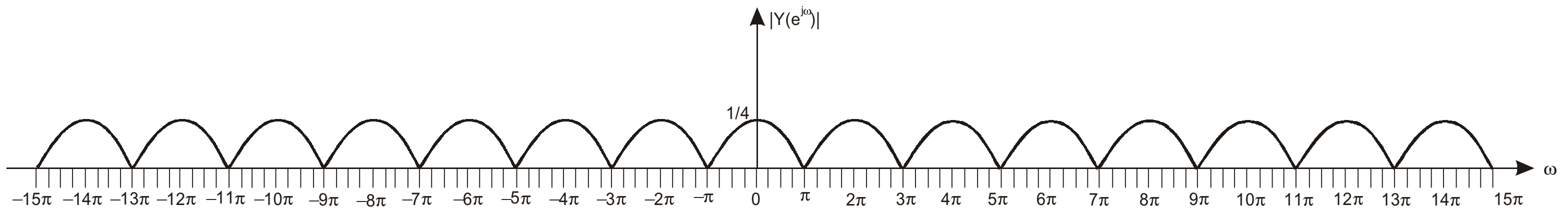


Fig 7.

Example 9.6

The spectrum of a discrete time signal is shown in fig 1. Draw the spectrum of the decimated signal when the signal is decimated by D = 3.

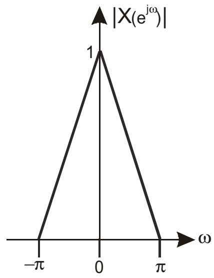


Fig 1.

Solution

The spectrum of a discrete time signal is periodic, with periodicity of 2π. Hence the spectrum of given signal can be drawn as shown in fig 2.

Let, $Y(e^{j\omega})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 3 is given by,

$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j(\omega-2\pi k)/3}\right) \qquad \boxed{\text{Using equation (9.5).}}$$

$$= \frac{1}{3}\,X\left(e^{j\omega/3}\right) + \frac{1}{3}\,X\left(e^{j(\omega-2\pi)/3}\right) + \frac{1}{3}\,X\left(e^{j(\omega-4\pi)/3}\right)$$

From the above equation we can say that the spectrum $Y(e^{j\omega})$ of decimated signal has three components.

The first component is the frequency-stretched version of inputs as shown in fig 3. The second and third components are shifted version of the first component, right shifted by 2π and 4π respectively as shown in fig 4 and 5.

The frequency range of input spectrum is $-\pi$ to $+\pi$ and so its bandwidth is 2π [$\pi - (-\pi) = 2\pi$]. It can be observed that this bandwidth 2π is stretched to 6π ($2\pi \times 3 = 6\pi$) for decimation by 3 in each component of the spectrum of the decimated signal. Therefore, the frequency range of first component is stretched to -3π to $+3\pi$. Also the magnitude of each component is scaled to 1/3 for decimation by 3.

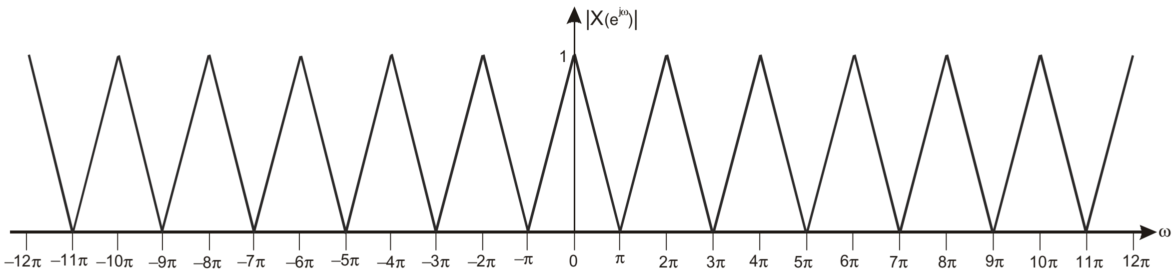


Fig 2.

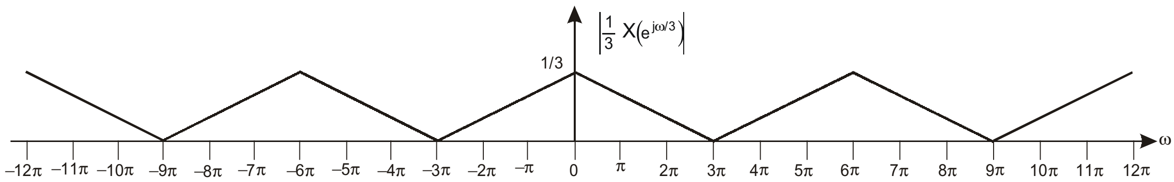


Fig 3.

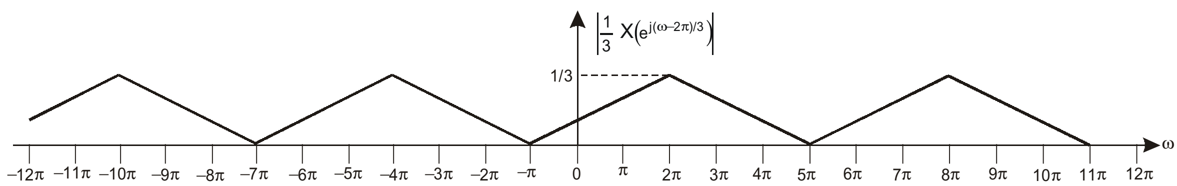


Fig 4.

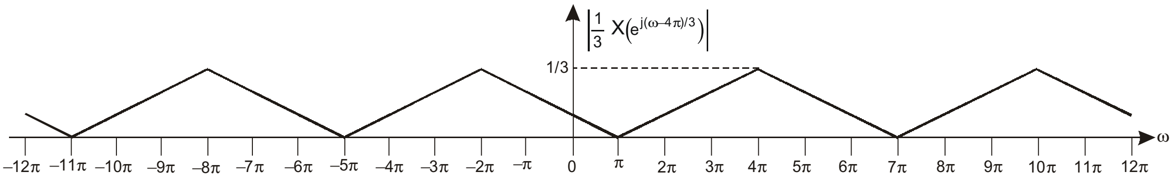


Fig 5.

The spectrum of decimated signal for decimation by 3 is shown in fig 6, which is obtained by adding the components of spectrum shown in fig 3, 4 and 5. The shaded parts in fig 6 are aliased portion of the spectrum.

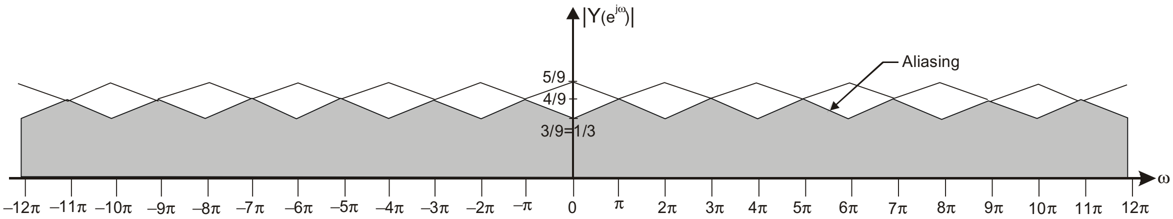


Fig 6.

9.3 Upsampling (or Interpolation)

The ***upsampling*** (or ***interpolation***) is the process of increasing the samples of the discrete time signal.

Let, x(n) = Discrete time signal

I = Sampling rate multiplication factor (and I is an integer).

Now, $x\left(\frac{n}{I}\right)$ = Upsampled version of x(n).

The device which perform the process of upsampling is called ***upsampler*** (or ***interpolator***). Symbolically, the upsampler can be represented as shown in fig 9.5.

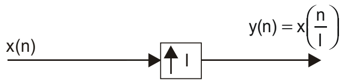


Fig 9.5 : *Interpolator.*

Example 9.7

Consider the discrete time signal,

x(n) = {1, 2, 3, 4}

Determine the upsampled version of the signals for the sampling rate multiplication factor, a) I = 2 b) I = 3 c) I = 4

Solution

Given that,

x(n) = {1, 2, 3, 4}

∴ When n = 0, x(n) = x(0) = 1

When n = 1, x(n) = x(1) = 2

When n = 2, x(n) = x(2) = 3

When n = 3, x(n) = x(3) = 4

a) Sampling rate multiplication factor, I = 2.

Now, $x\left(\frac{n}{I}\right) = x\left(\frac{n}{2}\right)$ = Discrete time signal interpolated by multiplication factor 2.

Let, $x\left(\frac{n}{2}\right) = x_{12}(n)$

$\therefore$ When n = 0, $x_{12}(n) = x_{12}(0) = x\left(\frac{0}{2}\right) = x(0) = 1$

When n = 1, $x_{12}(n) = x_{12}(1) = x\left(\frac{1}{2}\right) = x(0.5) = 0$

When n = 2, $x_{12}(n) = x_{12}(2) = x\left(\frac{2}{2}\right) = x(1) = 2$

When n = 3, $x_{12}(n) = x_{12}(3) = x\left(\frac{3}{2}\right) = x(1.5) = 0$

When n = 4, $x_{12}(n) = x_{12}(4) = x\left(\frac{4}{2}\right) = x(2) = 3$

When n = 5, $x_{12}(n) = x_{12}(5) = x\left(\frac{5}{2}\right) = x(2.5) = 0$

When n = 6, $x_{12}(n) = x_{12}(6) = x\left(\frac{6}{2}\right) = x(3) = 4$

When n = 7, $x_{12}(n) = x_{12}(7) = x\left(\frac{7}{2}\right) = x(3.5) = 0$

$$\therefore\ x\left(\frac{n}{2}\right) = x_{12}(n) = \{\underset{\uparrow}{1},\ 0,\ 2,\ 0,\ 3,\ 0,\ 4,\ 0\}$$

Note : Discrete time signals are defined only for integer values of n. Therefore, the value of discrete time signal for noninteger value of n will be zero.

b) Sampling rate multiplication factor, I = 3.

Now, $x\left(\frac{n}{I}\right) = x\left(\frac{n}{3}\right)$ = Discrete time signal interpolated by multiplication factor 3.

Let, $x\left(\frac{n}{3}\right) = x_{13}(n)$

$\therefore$ When n = 0, $x_{13}(n) = x_{13}(0) = x\left(\frac{0}{3}\right) = x(0) = 1$

When n = 1, $x_{13}(n) = x_{13}(1) = x\left(\frac{1}{3}\right) = x(0.3) = 0$

When n = 2, $x_{13}(n) = x_{13}(2) = x\left(\frac{2}{3}\right) = x(0.7) = 0$

When n = 3, $x_{13}(n) = x_{13}(3) = x\left(\frac{3}{3}\right) = x(1) = 2$

When n = 4, $x_{13}(n) = x_{13}(4) = x\left(\frac{4}{3}\right) = x(1.3) = 0$

When n = 5, $x_{13}(n) = x_{13}(5) = x\left(\frac{5}{3}\right) = x(1.7) = 0$

When n = 6, $x_{13}(n) = x_{13}(6) = x\left(\frac{6}{3}\right) = x(2) = 3$

When n = 7, $x_{13}(n) = x_{13}(7) = x\left(\frac{7}{3}\right) = x(2.3) = 0$

When n = 8, $x_{13}(n) = x_{13}(8) = x\left(\frac{8}{3}\right) = x(2.7) = 0$

When n = 9, $x_{13}(n) = x_{13}(9) = x\left(\frac{9}{3}\right) = x(3) = 4$

When n = 10, $x_{13}(n) = x_{13}(10) = x\left(\frac{10}{3}\right) = x(3.3) = 0$

When n = 11, $x_{13}(n) = x_{13}(11) = x\left(\frac{11}{3}\right) = x(3.7) = 0$

$$\therefore\ x\left(\frac{n}{3}\right) = x_{13}(n) = \{\underset{\uparrow}{1},\ 0,\ 0,\ 2,\ 0,\ 0,\ 3,\ 0,\ 0,\ 4,\ 0,\ 0\}$$

c) Sampling rate multiplication factor, I = 4.

Now, $x\left(\frac{n}{I}\right) = x\left(\frac{n}{4}\right)$ = Discrete time signal interpolated by multiplication factor 4.

Let, $x\left(\frac{n}{4}\right) = x_{14}(n)$

\ When n = 0, $x_{I4}(n) = x_{I4}(0) = x\left(\frac{0}{4}\right) = x(0) = 1$

When n = 1, $x_{I4}(n) = x_{I4}(1) = x\left(\frac{1}{4}\right) = x(0.25) = 0$

When n = 2, $x_{I4}(n) = x_{I4}(2) = x\left(\frac{2}{4}\right) = x(0.5) = 0$

When n = 3, $x_{I4}(n) = x_{I4}(3) = x\left(\frac{3}{4}\right) = x(0.75) = 0$

When n = 4, $x_{I4}(n) = x_{I4}(4) = x\left(\frac{4}{4}\right) = x(1) = 2$

When n = 5, $x_{I4}(n) = x_{I4}(5) = x\left(\frac{5}{4}\right) = x(1.25) = 0$

When n = 6, $x_{I4}(n) = x_{I4}(6) = x\left(\frac{6}{4}\right) = x(1.5) = 0$

When n = 7, $x_{I4}(n) = x_{I4}(7) = x\left(\frac{7}{4}\right) = x(1.75) = 0$

When n = 8, $x_{I4}(n) = x_{I4}(8) = x\left(\frac{8}{4}\right) = x(2) = 3$

When n = 9, $x_{I4}(n) = x_{I4}(9) = x\left(\frac{9}{4}\right) = x(2.25) = 0$

When n = 10, $x_{I4}(n) = x_{I4}(10) = x\left(\frac{10}{4}\right) = x(2.5) = 0$

When n = 11, $x_{I4}(n) = x_{I4}(11) = x\left(\frac{11}{4}\right) = x(2.75) = 0$

When n = 12, $x_{I4}(n) = x_{I4}(12) = x\left(\frac{12}{4}\right) = x(3) = 4$

When n = 13, $x_{I4}(n) = x_{I4}(13) = x\left(\frac{13}{4}\right) = x(3.25) = 0$

When n = 14, $x_{I4}(n) = x_{I4}(14) = x\left(\frac{14}{4}\right) = x(3.5) = 0$

When n = 15, $x_{I4}(n) = x_{I4}(15) = x\left(\frac{15}{4}\right) = x(3.75) = 0$

$$\therefore\ x\left(\frac{n}{4}\right) = x_{I4}(n) = \left\{\underset{\uparrow}{1},\ 0,\ 0,\ 0,\ 2,\ 0,\ 0,\ 0,\ 3,\ 0,\ 0,\ 0,\ 4,\ 0,\ 0,\ 0\right\}$$

Example 9.8

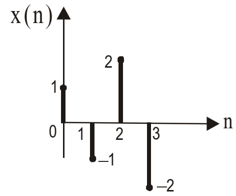


Fig 1.

Consider the discrete time signal shown in fig 1. Sketch the upsampled version of the signals for the sampling rate multiplication factor, **a)** I = 2 **b)** I = 3.

Solution

From fig 1, we can write the samples of given sequence as shown below.

$x(n) = \{1, -1, 2, -2\}$

\ When n = 0, $x(n) = x(0) = 1$

When n = 1, $x(n) = x(1) = -1$

When n = 2, $x(n) = x(2) = 2$

When n = 3, $x(n) = x(3) = -2$

a) Sampling rate multiplication factor, I = 2.

Now, $x\left(\frac{n}{I}\right) = x\left(\frac{n}{2}\right)$ = Discrete time signal interpolated by multiplication factor 2.

Let, $x\left(\frac{n}{2}\right) = x_{I2}(n)$

\ When n = 0, $x_{I2}(n) = x_{I2}(0) = x\left(\frac{0}{2}\right) = x(0) = 1$

When n = 1, $x_{I2}(n) = x_{I2}(1) = x\left(\frac{1}{2}\right) = x(0.5) = 0$

When n = 2, $x_{I2}(n) = x_{I2}(2) = x\left(\frac{2}{2}\right) = x(1) = -1$

When n = 3, $x_{I2}(n) = x_{I2}(3) = x\left(\frac{3}{2}\right) = x(1.5) = 0$

When n = 4, $x_{I2}(n) = x_{I2}(4) = x\left(\frac{4}{2}\right) = x(2) = 2$

When n = 5, $x_{I2}(n) = x_{I2}(5) = x\left(\frac{5}{2}\right) = x(2.5) = 0$

When n = 6, $x_{I2}(n) = x_{I2}(6) = x\left(\frac{6}{2}\right) = x(3) = -2$

When n = 7, $x_{I2}(n) = x_{I2}(7) = x\left(\frac{7}{2}\right) = x(3.5) = 0$

$$\therefore\ x\left(\frac{n}{2}\right) = x_{I2}(n) = \left\{\underset{\uparrow}{1},\ 0,\ -1,\ 0,\ 2,\ 0,\ -2,\ 0\right\} \quad .....(1)$$

> *Note : Discrete time signals are defined only for integer values of n. Therefore, the value of discrete time signal for noninteger value of n will be zero.*

Using equation (1), the interpolated signal of x(n) by multiplication factor 2, is drawn as shown in fig 2.

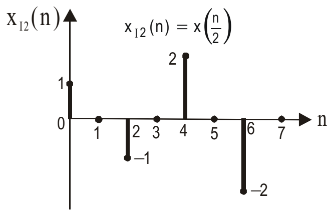


Fig 2 : x(n) interpolated by 2.

b) Sampling rate multiplication factor, I = 3.

Now, $x\left(\frac{n}{I}\right) = x\left(\frac{n}{3}\right)$ = Discrete time signal interpolated by multiplication factor 3.

Let, $x\left(\frac{n}{2}\right) = x_{I3}(n)$

$\therefore$ When n = 0, $x_{I3}(n) = x_{I3}(0) = x\left(\frac{0}{3}\right) = x(0) = 1$

When n = 1, $x_{I3}(n) = x_{I3}(1) = x\left(\frac{1}{3}\right) = x(0.3) = 0$

When n = 2, $x_{I3}(n) = x_{I3}(2) = x\left(\frac{2}{3}\right) = x(0.7) = 0$

When n = 3, $x_{I3}(n) = x_{I3}(3) = x\left(\frac{3}{3}\right) = x(1) = -1$

When n = 4, $x_{I3}(n) = x_{I3}(4) = x\left(\frac{4}{3}\right) = x(1.3) = 0$

When n = 5, $x_{I3}(n) = x_{I3}(5) = x\left(\frac{5}{3}\right) = x(1.7) = 0$

When n = 6, $x_{I3}(n) = x_{I3}(6) = x\left(\frac{6}{3}\right) = x(2) = 2$

When n = 7, $x_{I3}(n) = x_{I3}(7) = x\left(\frac{7}{3}\right) = x(2.3) = 0$

When n = 8, $x_{I3}(n) = x_{I3}(8) = x\left(\frac{8}{3}\right) = x(2.7) = 0$

When n = 9, $x_{I3}(n) = x_{I3}(9) = x\left(\frac{9}{3}\right) = x(3) = -2$

When n = 10, $x_{I3}(n) = x_{I3}(10) = x\left(\frac{10}{3}\right) = x(3.3) = 0$

When n = 11, $x_{I3}(n) = x_{I3}(11) = x\left(\frac{11}{3}\right) = x(3.7) = 0$

$$\therefore\ x\left(\frac{n}{3}\right) = x_{I3}(n) = \{\underset{\uparrow}{1},\ 0,\ 0,\ -1,\ 0,\ 0,\ 2,\ 0,\ 0,\ -2,\ 0,\ 0\} \qquad .....(2)$$

Using equation (2), the interpolated signal of x(n) by multiplication factor 3, is drawn as shown in fig 3.

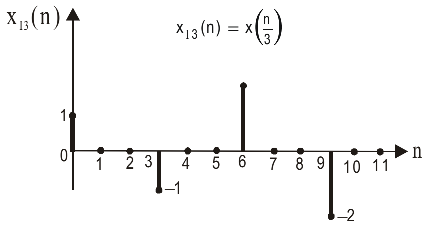


Fig 3 : x(n) interpolated by 3.

9.3.1 Spectrum of Upsampler

Let, x(n) be an input signal to upsampler and y(n) be the output signal.

Let, $x\left(\frac{n}{I}\right)$ be an upsampled version of x(n) by an integer factor I.

$$\therefore\ y(n) = x\left(\frac{n}{I}\right) \qquad .....(9.9)$$

By definition of $\mathcal{Z}$-transform, y(n) can be expressed as,

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{+\infty} y(n)\, z^{-n} \\
&= \sum_{n=-\infty}^{+\infty} x\left(\frac{n}{I}\right) z^{-n} && \boxed{\text{On substituting } y(n) = x\left(\tfrac{n}{I}\right) \text{ from equation (9.9)}} \\
&= \sum_{m=-\infty}^{+\infty} x(m)\, z^{-mI} && \boxed{\begin{array}{l} \text{Let, } m = \frac{n}{I} \;\Rightarrow\; n = mI \\ \text{when } n = -\infty, \quad m = -\infty \\ \text{when } n = +\infty, \quad m = +\infty \end{array}} \\
&= \sum_{n=-\infty}^{+\infty} x(n)\, z^{-nI} && \boxed{\text{Let, } m \rightarrow n} \\
&= \sum_{n=-\infty}^{+\infty} x(n)\, (z^{I})^{-n}
\end{aligned}
$$

The above equation is similar to $\mathcal{Z}$-transform of y(n) except that $z \rightarrow z^I$, hence Y(z) can be written as shown in equation (9.8).

$$\therefore\; Y(z) = X(z^I) \qquad .....(9.10)$$

On substituting, $z = e^{j\omega}$ in equation (9.10) we get,

$$Y(e^{j\omega}) = X(e^{j\omega I}) \qquad .....(9.11)$$

The equation (9.11) is the frequency spectrum of the output signal of the interpolator, i.e., frequency spectrum of upsampled signal.

The term $X(e^{j\omega I})$ is frequency compressed version of $X(e^{j\omega})$ by a factor I. Since the frequency response is periodic with periodicity of 2π the $X(e^{j\omega I})$ will repeat I times in a period of 2π, in the spectrum, of upsampled signal.

The equations (9.11) and (9.10) can be used to construct the frequency domain and z-domain representation of interpolator as shown in fig 9.6 and 9.7.

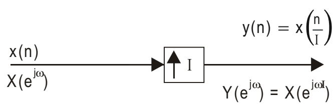


***Fig 9.6 :** Frequency domain representation of upsampler.*

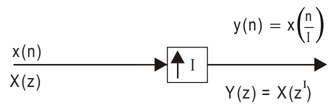


***Fig 9.7 :** z-domain representation of upsampler.*

9.3.2 Anti-imaging Filter

The output spectrum of interpolator is compressed version of the input spectrum, (refer equation 9.11). Therefore, the spectrum of upsampled signal has multiple images in a period of 2π. When upsampled by a factor of I, the output spectrum will have I images in a period of 2π, with each image band limited to π/I. Since the frequency spectrum in the range 0 to $\frac{\pi}{I}$ are unique, we have to filter the other images. Hence the output of upsampler is passed through a lowpass filter with a bandwidth of $\frac{\pi}{I}$ .Since this lowpass filter is designed to avoid multiple images in the output spectrum, it is called ***anti-imaging filter***.

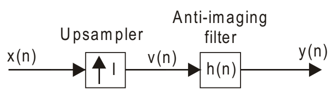


***Fig 9.8 :** Interpolation by a factor I without anti-imaging.*

Example 9.9

The spectrum of a discrete time signal is shown in fig 1. Draw the spectrum of the signal if it is upsampled by I = 2, 3 and 4.

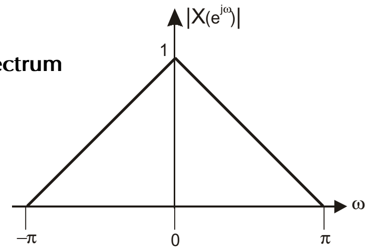


Fig 1.

Solution

Since, the frequency spectrum of a discrete time signal is periodic with periodicity of 2p, the spectrum of given signal can be drawn as shown in fig 2.

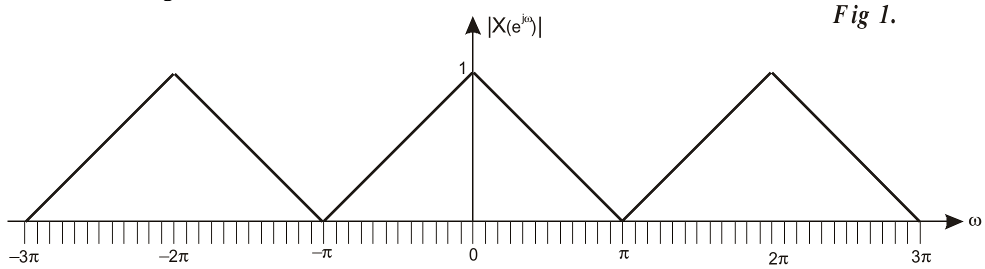


Fig 2.

Case (i) : Upsampling by I = 2

Let, $Y(e^{jw})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{jw})$ of upsampled signal is given by,

$$Y(e^{jw}) = X(e^{jwI}) = X(e^{j2w})$$

Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of two and so the spectrum of upsampled version of the signal consists of two images of the compressed spectrum in a period of 2p as shown in fig 3.

The frequency range of the spectrum of given signal is –p to +p and so its bandwidth is 2p [p – (–p) = 2p]. This bandwidth 2p is compressed to p (2p ´ 1/2 = p) for interpolation by 2, in every image in the spectrum of upsampled signal.

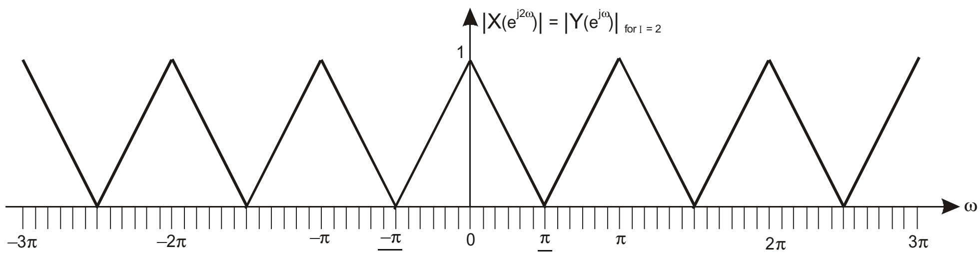


Fig 3.

Case (ii) : Upsampling by I = 3

Let, $Y(e^{jw})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{jw})$ of upsampled signal is given by,

$$Y(e^{jw}) = X(e^{jwI}) = X(e^{j3w})$$

Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of three and so the spectrum of upsampled version of the signal consists of three images of the compressed spectrum in a period of 2p as shown in fig 4.

This bandwidth 2π of a given signal is compressed to $2\pi/3$ ($2\pi \times 1/3 = 2\pi/3$) for interpolation by 3, in every image in the spectrum of upsampled signal.

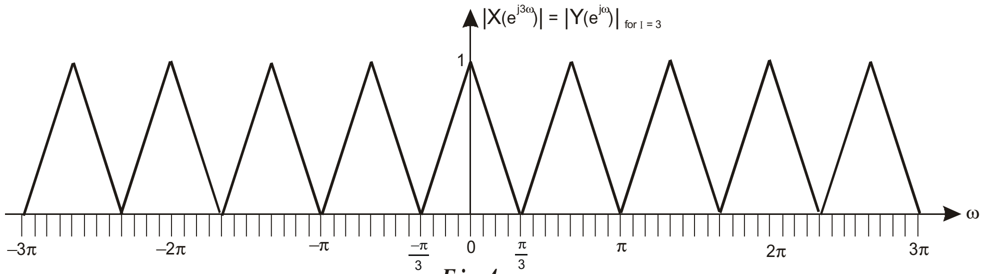


Fig 4.

Case (iii) : Upsampling by I = 4

Let, $Y(e^{j\omega})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{j\omega})$ of upsampled signal is given by,

$$Y(e^{j\omega}) = X(e^{j\omega I}) = X(e^{j4\omega})$$ Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of four and so the spectrum of upsampled version of the signal consists of four images of the compressed spectrum in a period of 2π as shown in fig 5.

This bandwidth 2π of a given signal is compressed to $\pi/2$ ($2\pi \times 1/4 = \pi/2$) for interpolation by 4, in every image in the spectrum of the upsampled signal.

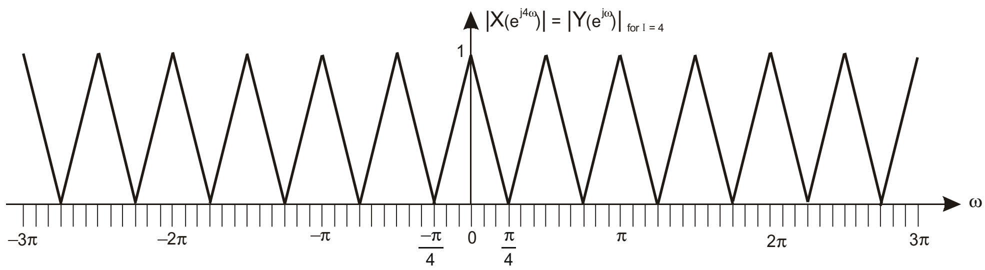


Fig 5.

Example 9.10

The spectrum of a discrete time signal is shown in fig 1. Draw the spectrum of the signal if it is upsampled by I = 2, 3 and 4.

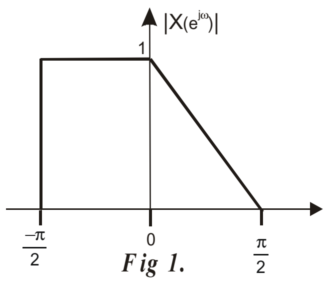


Fig 1.

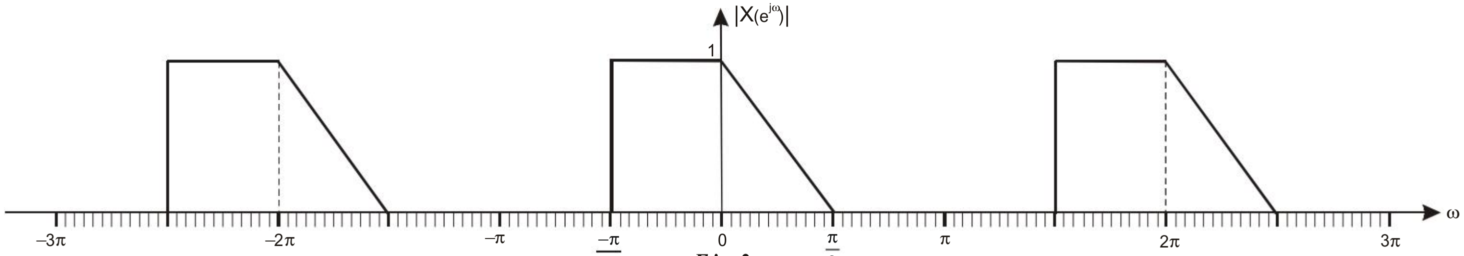


Fig 2.

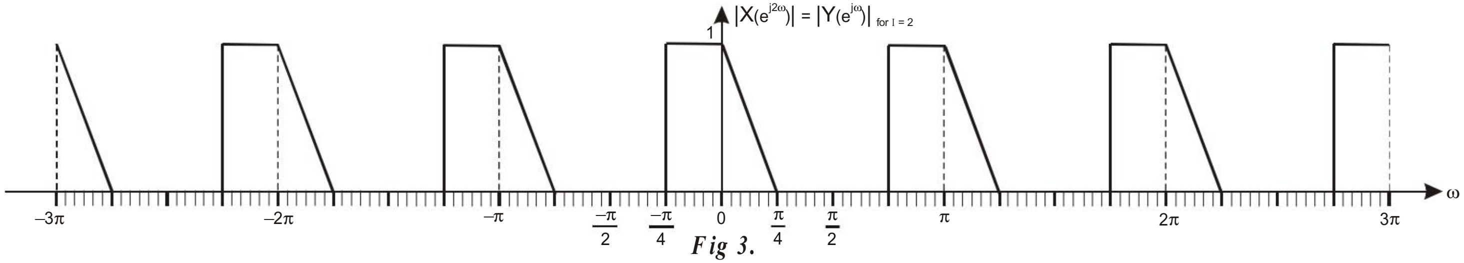


Fig 3.

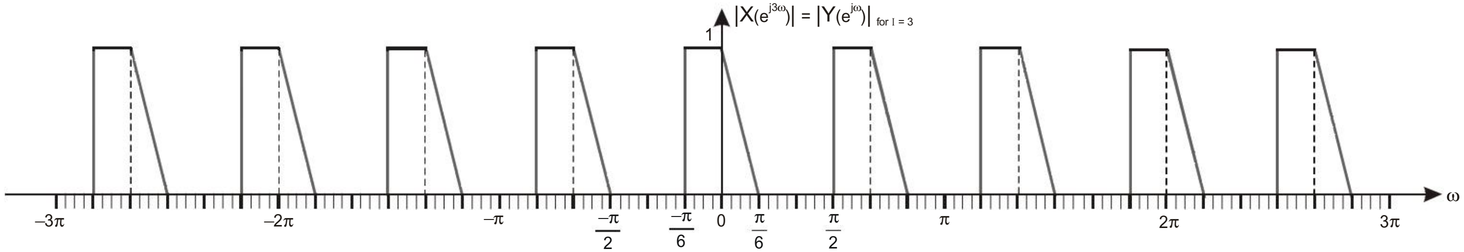


Fig 4.

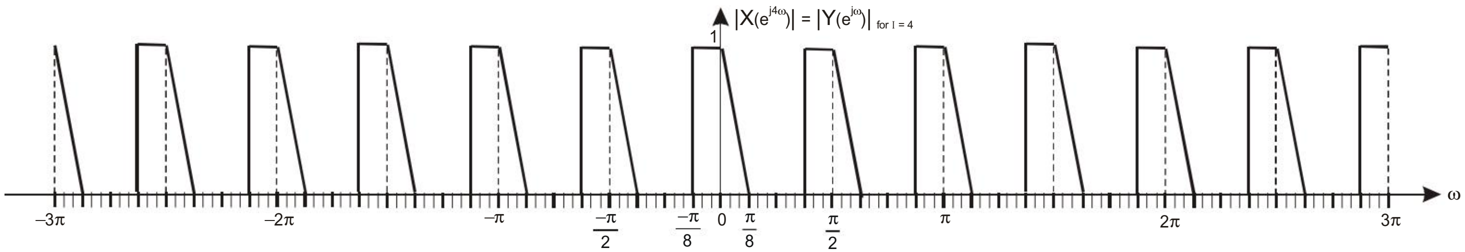


Fig 5.

Solution

Since the frequency spectrum of a discrete time signal is periodic with periodicity of 2π, the spectrum of given signal can be drawn as shown in fig 2.

Case (i) : **Upsampling by I = 2**

Let, $Y(e^{j\omega})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{j\omega})$ of upsampled signal is given by,

$$Y(e^{j\omega}) = X(e^{j\omega I}) = X(e^{j2\omega})$$ Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of two and so the spectrum of the upsampled version of the signal consists of two images of the compressed spectrum in a period of 2π as shown in fig 3.

The frequency range of the spectrum of given signal is $-\pi/2$ to $+\pi/2$ and so its bandwidth is π $[\pi/2 - (-\pi/2) = \pi]$. This bandwidth π is compressed to $\pi/2$ ($\pi \times 1/2 = \pi/2$) for interpolation by 2, in every image in the spectrum of upsampled signal.

Case (ii) : **Upsampling by I = 3**

Let, $Y(e^{j\omega})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{j\omega})$ of upsampled signal is given by,

$$Y(e^{j\omega}) = X(e^{j\omega I}) = X(e^{j3\omega})$$ Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of three and so the spectrum of upsampled version of the signal consists of three images of the compressed spectrum in a period of 2π as shown in fig 4.

The bandwidth π of given signal is compressed to $\pi/3$ ($\pi \times 1/3 = \pi/3$) for interpolation by 3, in every image in the spectrum of upsampled signal.

Case (iii) : **Upsampling by I = 4**

Let, $Y(e^{j\omega})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{j\omega})$ of upsampled signal is given by,

$$Y(e^{j\omega}) = X(e^{j\omega I}) = X(e^{j4\omega})$$ Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of four and so the spectrum of upsampled version of the signal consists of four images of the compressed spectrum in a period of 2π as shown in fig 5.

The bandwidth π of given signal is compressed to $\pi/4$ ($\pi \times 1/4 = \pi/4$) for interpolation by 4, in every image in the spectrum of upsampled signal.

9.4 Sampling Rate Conversion by a Rational Factor $\frac{I}{D}$

In decimation and interpolation, the sampling rate conversion is achieved by integer factor (because D and I are integers). When sampling rate conversion is required by non-integer factor, it is possible to perform sampling rate conversion by a rational factor, $\frac{I}{D}$.

The sampling rate conversion by a factor $\frac{I}{D}$, involves the following steps.

1. Perform interpolation by a factor I.

2. Filter the output of interpolator using a lowpass filter (anti-imaging filter) with bandwidth $\frac{\pi}{I}$.

3. The output of anti-imaging filter is passed through another lowpass filter (anti-aliasing filter), to limit the bandwidth of the signal to $\frac{\pi}{D}$

4. Finally the signal bandlimited to $\frac{\pi}{D}$ is decimated by a factor D.

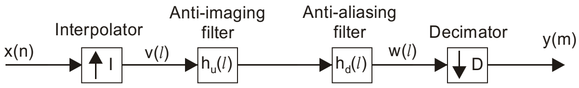


Fig 9.9a.

$\Downarrow$

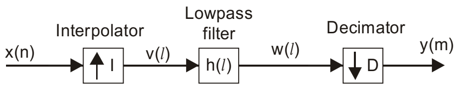


Fig 9.9b.

Fig 9.9 : Sampling rate conversion by a rational factor I/D.

The process of sampling rate conversion by a factor $\frac{I}{D}$ is shown in fig 9.9. Here it is important to note that, in order to preserve the spectral characteristics of x(n), the interpolation has to be performed first and decimation has to be performed next. The two lowpass filters with bandwidth $\frac{\pi}{I}$ and $\frac{\pi}{D}$ can be combined to a single lowpass filter with a bandwidth minimum among $\frac{\pi}{I}$, $\frac{\pi}{D}$ as shown in fig 9.9(b).

9.4.1 Spectrum of Sampling Rate Convertor by a Rational Factor $\frac{I}{D}$

Consider the sampling rate converter shown in fig 9.10, which performs sampling rate conversion by a factor $\frac{I}{D}$. Here, the discrete time signal x(n) is first upsampled by an integer factor I, then bandlimited using a lowpass filter and finally downsampled by an integer factor D.

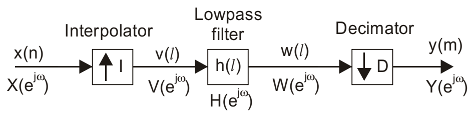


Fig 9.10 : Sampling rate conversion by a factor $\frac{I}{D}$.

Let, x(n) = Input signal.

$x\left(\frac{n}{I}\right) = n(l)$ = Upsampled version of x(n).

w(l) = Bandlimited version of v(l).

w(Dl) = y(m) = Decimated version of w(l).

Let, $X(e^{jw})$, $V(e^{jw})$, $W(e^{jw})$ and $Y(e^{jw})$ are spectrum of the signals x(n), n(l), w(l) and y(m) respectively.

For interpolation by an integer factor I, the frequency spectrum of output signal of interpolator is given by equation (9.11).

$$\backslash \quad V(e^{jw}) = X(e^{jwI}) \qquad .....(9.12)$$

Let, h(l) = Impulse response of the lowpass filter.

$H(e^{jw})$ = Frequency response of lowpass filter.

The lowpass filter is designed to have a cutoff frequency, w_c which is given by minimum among $\frac{\pi}{I}, \frac{\pi}{D}$.

Now, the frequency response, $H(e^{jw})$ of lowpass filter can be defined as,

$$H(e^{jw}) = I \quad ; \text{ for } w = 0 \text{ to } w_c \qquad .....(9.13)$$

$$= 0 \quad ; \text{ otherwise}$$

where, $w_c = \text{Minimum of } \left(\frac{\pi}{I}, \frac{\pi}{D}\right)$.

Therefore, the output spectrum of lowpass filter $W(e^{jw})$ can be written as,

$$W(e^{jw}) = H(e^{jw})\, V(e^{jw}) = H(e^{jw})\, X(e^{jwI}) \qquad \boxed{\text{Using equation (9.12).}}$$

$$= \begin{cases} I\, X(e^{j\omega I}) & ; \text{ for } \omega = 0 \text{ to } \omega_c \\ 0 & ; \text{ otherwise} \end{cases} \qquad \boxed{\text{Using equation (9.13).}} \qquad .....(9.14)$$

For decimation by an integer factor D, the frequency spectrum of output signal of decimator is given by equation (9.5).

$$\therefore\ Y(e^{j\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} W\left(e^{j(\omega - 2\pi k/D)}\right) \qquad .....(9.15)$$

Since there is no aliasing in the output, the equation (9.15) can be evaluated for k = 0 alone.

$$\therefore\ Y(e^{j\omega}) = \frac{1}{D} W\left(e^{j\omega/D}\right) \qquad \boxed{\text{Using equation (9.14).}}$$

$$= \begin{cases} \dfrac{I}{D} X(e^{j\omega I/D}) & ; \text{ for } \omega = 0 \text{ to } \omega_y \\ 0 & ; \text{ otherwise} \end{cases} \qquad .....(9.16)$$

where, w_y is minimum of $\left(\frac{\pi D}{I}, \pi\right)$.

The equation (9.16) is the spectrum of sampling rate convertor by a rational factor $\frac{I}{D}$.

9.5 Multistage Implementation of Sampling Rate Conversion

In sampling rate conversion, when the sampling rate conversion factor I or D is very large then the multistage sampling rate conversion will be a computationally efficient realization.

In interpolation, if sampling rate multiplication factor, I is very large then I can be expressed as a product of positive integer as shown in equation (9.17).

$$I = I_1 \times I_2 \times \ ......\ \times I_L = \prod_{i=1}^{L} I_i \qquad .....(9.17)$$

Now, the interpolation by sampling rate multiplication factors I_1, I_2, ...I_L are implemented separately and then overall interpolation is obtained by cascading the L-stages of interpolation as shown in fig 9.11.

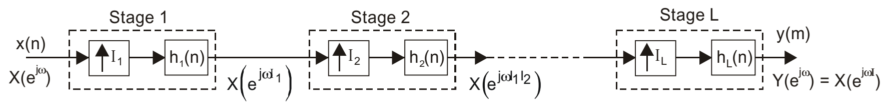


Fig 9.11 : *Multistage implementation of interpolation by a factor I.*

In cascading of interpolators, the anti-imaging filters are introduced at the output of each stage of interpolation in order to eliminate the images introduced due to upsampling in that stage.

In decimation, if sampling rate reduction factor, D is very large then D can be expressed as a product of positive integers as shown in equation (9.18).

$$D = D_1 \times D_2 \times \text{......} \times D_L = \prod_{i=1}^{L} D_L \qquad \text{.....(9.18)}$$

Now, the decimation by sampling rate reduction factors D_1, D_2, ... D_L are implemented separately and then overall decimation is obtained by cascading the L-stages of decimation as shown in fig 9.12.

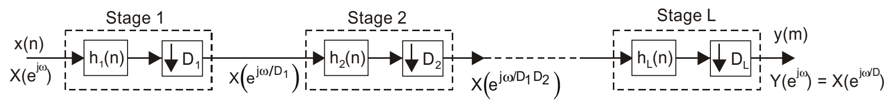


Fig 9.12 : *Multistage implementation of decimation by a factor D.*

In cascading of decimators, the anti-aliasing filters are introduced at the input of each stage to limit the bandwidth of the input signal to decimator in order to avoid aliasing due to decimation in the spectrum of the output signal of that stage.

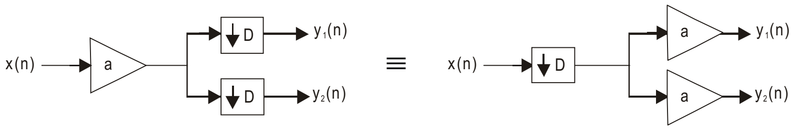
9.6 Identities in Multirate Digital Signal Processing

The useful identities in multirate digital signal processing are given below. The proof of these identities are left as exercise to the readers.

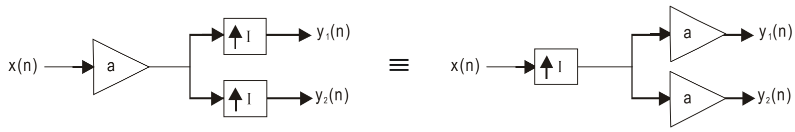
Identity 1

Identity 2

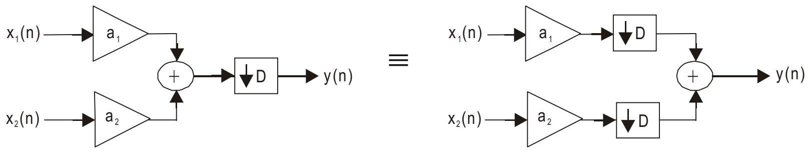
Identity 3



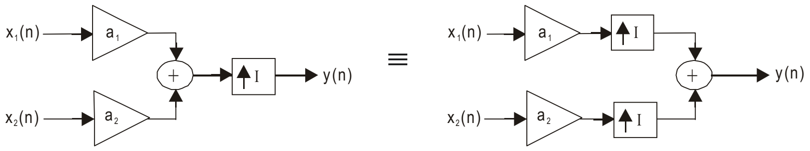
Identity 4



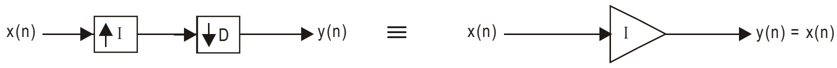
Identity 5



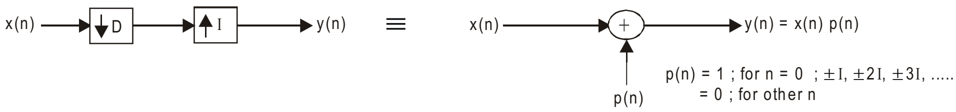
Identity 6



Identity 7



Identity 8



Identity 9

Identity 10

Identity 11

Identity 12

Identity 13

Identity 14

Identity 15

Identity 16

Identity 17

Identity 18

Example 9.11

For the multirate system shown in fig 1, determine y(n) as a function of x(n).

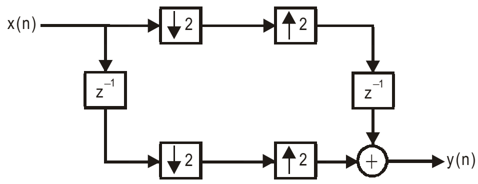


Fig 1.

Solution

Method 1

The z-domain representation of the given system with intermediate signals in z-domain is shown in fig 2.

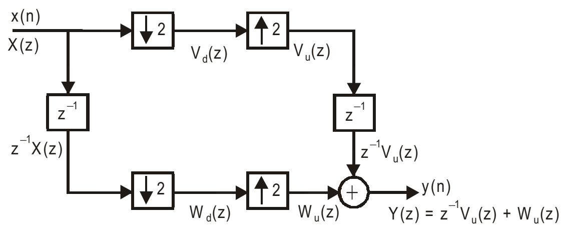


Fig 2.

Using equation (9.8), the z-domain output of decimators are obtained as shown below.

$$V_d(z) = \frac{1}{2}\ X\left(z^{1/2}\right) \qquad .....(1)$$

$$W_d(z) = \frac{1}{2} z^{-1/2}\ X\left(z^{1/2}\right) \qquad .....(2)$$

Using equation (9.10), the z-domain output of interpolators are obtained as shown below.

$$V_u(z) = V_d(z^2) = \frac{1}{2}\ X\left((z^2)^{1/2}\right) \qquad \boxed{\text{Using equation (1).}}$$

$$\therefore\ V_u(z) = \frac{1}{2}\ X(z) \qquad .....(3)$$

$$W_u(z) = W_d(z^2) = \frac{1}{2}(z^2)^{-1/2}\ X\left((z^2)^{1/2}\right) \qquad \boxed{\text{Using equation (2).}}$$

$$\therefore\ W_u(z) = \frac{1}{2} z^{-1}\ X(z) \qquad .....(4)$$

Now, the z-domain output Y(z) of the system is,

$$\begin{aligned} Y(z) &= z^{-1}V_u(z) + W_u(z) \\ &= z^{-1}\frac{1}{2}X(z) + \frac{1}{2}z^{-1}X(z) \\ &= z^{-1}X(z) \end{aligned} \qquad \boxed{\text{Using equations (3) and (4).}} \qquad .....(5)$$

On taking inverse Z-transform of equation (5) we get,

$$y(n) = x(n-1) \qquad .....(6)$$

The equation (6) is the output of system in time domain.

Method 2

The time domain representation of the system with intermediate signals is shown in fig 3.

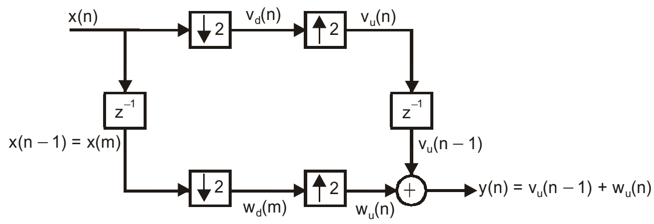


Fig 3.

The input signal x(n) can be expressed as shown below.

$$x(n) = \{ ....., \underset{\underset{n=0}{\uparrow}}{x(0)}, x(1), x(2), x(3), x(4), x(5), x(6), ...\}$$

The downsampled signal $v_d(n)$ in terms of samples of x(n) can be expressed as shown below.

$$v_d(n) = x(2n)$$

Here, $x(2n) = x(n) \quad ; \text{ for } n = 0, \pm 2, \pm 4, \pm 6, ...$

$$\therefore \; v_d(n) = \{....., \underset{\underset{n=0}{\uparrow}}{x(0)}, x(2), x(4), x(6), ...\}$$

The upsampled signal $v_u(n)$ in terms of samples of $v_d(n)$ can be expressed as shown below.

$$v_u(n) = v_d\left(\frac{n}{2}\right)$$

Here, $v_d\left(\frac{n}{2}\right) = v_d(n) \quad ; \text{ for } n = 0, \pm 2, \pm 4, \pm 6, ...$

$\qquad\qquad = 0 \quad ; \text{ for other n}$

$$\therefore \; v_u(n) = \{....., 0, \underset{\underset{n=0}{\uparrow}}{x(0)}, 0, x(2), 0, x(4), 0, x(6), ...\}$$

Now, the shifted signal $v_u(n-1)$ can be written as shown below.

$$\therefore \; v_u(n-1) = \{....., \underset{\underset{n=0}{\uparrow}}{0}, x(0), 0, x(2), 0, x(4), 0, x(6), ...\}$$

The shifted input x(n – 1) can be expressed as shown below.

Let, x(n –1) = x(m)

$$\therefore\ x(m) = \{....., \underset{\substack{\uparrow\\ m=0}}{x(-1)},\ x(0),\ x(1),\ x(2),\ x(3),\ x(4),\ x(5),\ x(6),\ .....\}$$

The downsampled signal $w_d(m)$ in terms of samples of $x(m)$ can be expressed as shown below.

$$w_d(m) = x(2m)$$

Here, $x(2m) = x(m)$; for $m = 0,\ \pm 2,\ \pm 4,\ \pm 6,\ ......$

$$\therefore\ w_d(m) = \{....., \underset{\substack{\uparrow\\ m=0}}{x(-1)},\ x(1),\ x(3),\ x(5),\ .....\}$$

The upsampled signal $w_u(n)$ in terms of samples of $w_d(m)$ can be written as shown below.

Let, $w_d(m) = w_d(n)$

Now, $w_u(n) = w_d\left(\frac{n}{2}\right)$

Here, $w_d\left(\frac{n}{2}\right) = w_d(n)$; for $n = 0,\ \pm 2,\ \pm 4,\ \pm 6,\ ......$

$= 0$; for other n

$$\therefore\ w_u(n) = \{....., \underset{\substack{\uparrow\\ n=0}}{x(-1)},\ 0,\ x(1),\ 0,\ x(3),\ 0,\ x(5),\ 0,\ .....\}$$

Now, The output $y(n)$ is given by sum of $v_u(n-1)$ and $w_u(n)$.

$$\therefore\ y(n) = v_u(n-1) + w_u(n)$$

$$= \{....., \underset{\substack{\uparrow\\ n=0}}{x(-1)}, x(0), x(1), x(2), x(3), x(4), x(5), x(6), ...\}$$

$$= x(n-1)$$

$$\therefore\ y(n) = x(n-1)$$

Example 9.12

Determine the ouput $y(n)$ in terms of input $x(n)$ for the multirate system shown in fig 1.

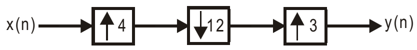


Fig 1.

Solution

Using identity 9, the downsampler with reduction factor 12 can be expressed as cascade of two downsamplers with reduction factor 4 and 3 ($4 \times 3 = 12$) as shown in fig 2.

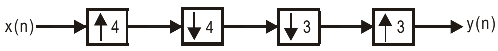


Fig 2.

Using identity 7, the cascade of upsampler with multiplication factor 4 and the downsampler with reduction factor 4 can be replaced by unity gain branch as shown in fig 3.

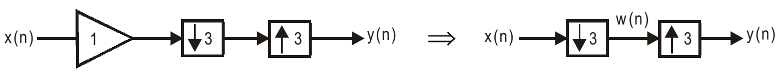


Fig 3.

Using identity 8, the system shown in fig 3 can be replaced by an equivalent system as shown in fig 4.

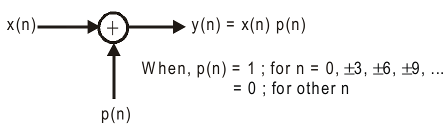


Fig 4.

$\therefore\ y(n) = x(n)\,p(n) = x(n)$; for $n = 0,\ \pm 3,\ \pm 6,\ \pm 9,\ ......$

$= 0$; for other n

$$\therefore\ y(n) = \{....., \underset{\uparrow}{x(0)},\ 0,\ 0,\ x(3),\ 0,\ 0,\ x(6),\ 0,\ 0,\ x(9),\ 0,\ 0,\ .....\}$$

9.7 Implementation of Sampling Rate Conversion in FIR Filters

9.7.1 Implementation of Sampling Rate Conversion Using Decimator in FIR Filters

The transfer functiion of FIR filter is given by,

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore \quad \frac{Y(z)}{X(z)} = \sum_{n=0}^{N-1} h(n)\, z^{-n} \qquad .....(9.19)$$

On expanding the equation (9.19) we get,

$$\frac{Y(z)}{X(z)} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + \,.....\, + h(N-1)\, z^{-(N-1)}$$

$$\therefore \; Y(z) = h(0)\, X(z) + h(1)\, z^{-1}X(z) + h(2)\, z^{-2}X(z) + h(3)\, z^{-3}X(z) + \qquad .....(9.20)$$
$$..... + h(N-1) z^{-(N-1)}\, X(z)$$

Using the equation (9.20) the direct form FIR Filter structure is drawn as shown in fig 9.13.

Let us decimate the output of FIR Filter by introducing a decimator at the output of FIR filter as shown in fig 9.14.

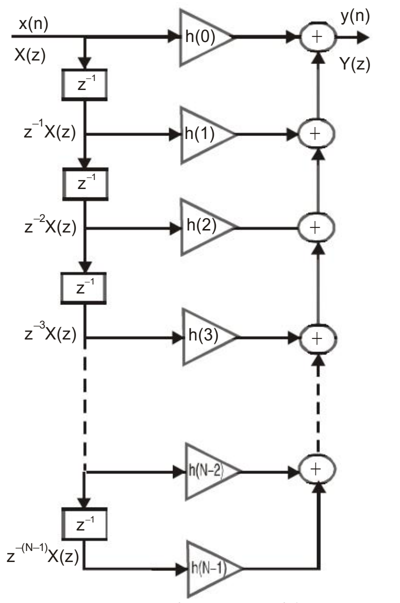


Fig 9.13 : *Direct form FIR filter structure.*

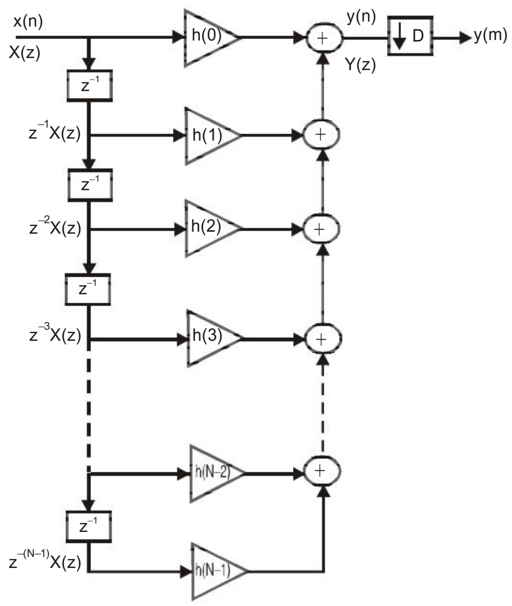


Fig 9.14 : *Direct form FIR filter structure with decimator at the output.*

In the filter structure of fig 9.14, we require only one out of every D samples of the output signal. But actually the system computes all the samples and discard D – 1 samples in every D samples and retain only one sample in every D samples. Hence this structure is an inefficent structure. To overcome the inefficiency in the calculations, (i.e., to avoid unwanted calculations, the decimator can be embedded in the filter structure itself. Therefore the decimator can be introduced before multipliers as shown in fig. 9.15, so that the multiplications and additions can be performed at lower sampling rate (or performed only for the samples to be retained).

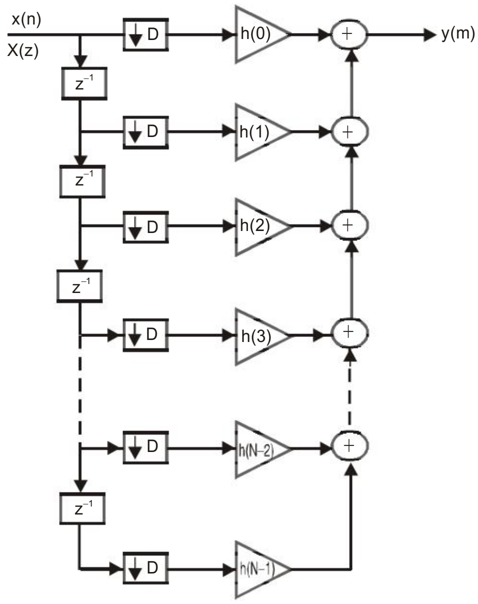


Fig 9.15 : Efficient structure for direct form FIR filter with decimator.

9.7.2 Implementation of Sampling Rate Conversion Using Interpolator in FIR Filters

Consider the direct form FIR filter structure as shown in fig 9.13.

Let us interpolate the input of FIR filter by introducing an upsampler at the input of FIR filter as shown in fig 9.16.

In the filter structure of fig 9.16, the upsampler introduces I–1 zeros inbetween any two samples of x(n) and so a large number of zeros are introduced in the upsampled input signal. Therefore most of the multiplications and additons will be product or sum of zeros. The multiplications and additions of zeros can be eliminated if the interpolator is embedded in the filter structure.

In order to embed the interpolator inside the filter structure, the direct form structure is transposed as shown in fig 9.17 and then the interpolators/upsamplers are introduced after multipliers as shown in fig 9.18, so that zeros are inserted after multiplications, which results in large savings in calculations.

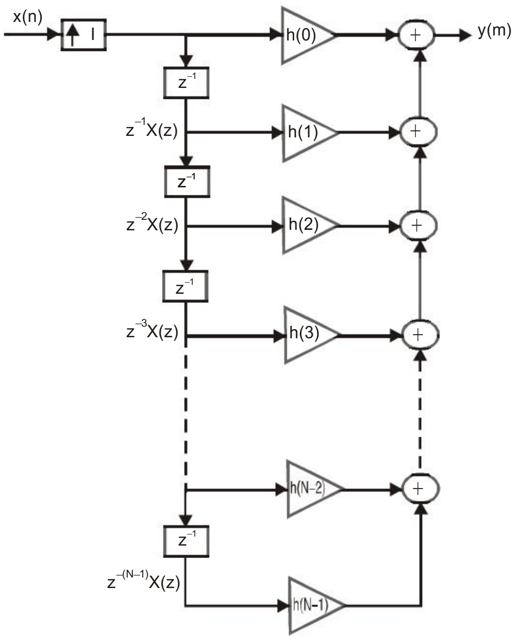


Fig 9.16 : Direct form FIR filter structure with interpolator at the input.

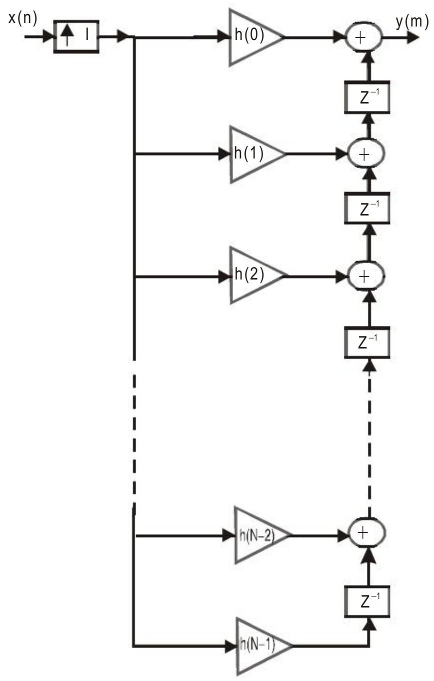


Fig 9.17 : Transposed direct form, FIR filter structure with interpolator at the input.

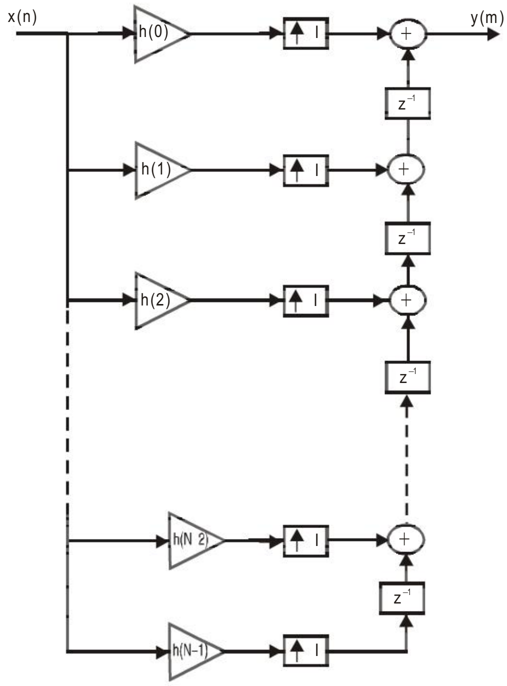


Fig 9.18 : Efficient structure for direct form FIR filter with interpolator.

9.8 Polyphase Decomposition

The ***polyphase decomposition*** is dividing an N^{th} order filter into L-sections of sub-filters that can be realized in parallel. In this decomposition the sub-filters will differ only in phase characteristics and so they are called ***polyphase filters***. Therefore, the process of dividing the filter into sub-filters is called polyphase decomposition.

The polyphase decomposition of filters leads to a realization structure for filters with reduced computations. Hence, the polyphase decomposition can be applied to (anti-aliasing) filters used for decimators and (anti-imaging) filters used for interpolators in order to realize computationally efficient multirate digital signal processing systems.

9.8.1 Polyphase Decomposition of FIR Filters

Type-I Decomposition

Consider the transfer function of the FIR filter shown in equation (9.21).

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{N-1} z^{-n} h(n) \qquad .....(9.21)$$

On expanding the equation (9.21) we get,

$$H(z) = z^{0}\, h(0) + z^{-1}\, h(1) + z^{-2}\, h(2) + z^{-3}\, h(3) + z^{-4}\, h(4) + z^{-5}\, h(5) + z^{-6}\, h(6)$$
$$+ z^{-7}\, h(7) + z^{-8}\, h(8) + z^{-9}\, h(9) + ... \qquad .....(9.22)$$

Let us rearrange the equation (9.22) into two sections as shown below.

$$H(z) = [\,z^0\,h(0) + z^{-2}\,h(2) + z^{-4}\,h(4) + z^{-6}\,h(6) + z^{-8}\,h(8) + ...]$$
$$+ [z^{-1}\,h(1) + z^{-3}\,h(3) + z^{-5}\,h(5) + z^{-7}\,h(7) + z^{-9}\,h(9) + ...]$$

$$= [\,z^0\,h(0) + z^{-2}\,h(2) + z^{-4}\,h(4) + z^{-6}\,h(6) + z^{-8}\,h(8) + ...]$$
$$+ z^{-1}\,[z^0\,h(1) + z^{-2}\,h(3) + z^{-4}\,h(5) + z^{-6}\,h(7) + z^{-8}\,h(9) + ...]$$

$$= [\,(z^2)^0\,h(0) + (z^2)^{-1}\,h(2) + (z^2)^{-2}\,h(4) + (z^2)^{-3}\,h(6) + (z^2)^{-4}\,h(8) + ...]$$
$$+ z^{-1}\,[(z^2)^0\,h(1) + (z^2)^{-1}\,h(3) + (z^2)^{-2}\,h(5) + (z^2)^{-3}\,h(7) + (z^2)^{-4}\,h(9) + ...]$$

$$= E_0(z^2) + z^{-1}\,E_1(z^2) \qquad .....(9.23)$$

where, $E_0(z^2) = (z^2)^0\,h(0) + (z^2)^{-1}\,h(2) + (z^2)^{-2}\,h(4) + (z^2)^{-3}\,h(6) + (z^2)^{-4}\,h(8) + ...$

$E_1(z^2) = (z^2)^0\,h(1) + (z^2)^{-1}\,h(3) + (z^2)^{-2}\,h(5) + (z^2)^{-3}\,h(7) + (z^2)^{-4}\,h(9) + ...$

Note : 1. If, $E_0(z) = h(0) + z^{-1}\,h(2) + z^{-2}\,h(4) + z^{-3}\,h(6) + ...$

then, $E_0(z^2) = h(0) + (z^2)^{-1}\,h(2) + (z^2)^{-2}\,h(4) + (z^2)^{-3}\,h(6) + ...$

2. If, $E_1(z) = h(1) + z^{-1}\,h(3) + z^{-2}\,h(5) + ...$

then, $E_1(z^2) = h(1) + (z^2)^{-1}\,h(3) + (z^2)^{-2}\,h(5) + ...$

The equation (9.23) decomposes the H(z) into two sections, (or two sub-filters), $E_0(z^2)$ and $E_1(z^2)$.

Using equation (9.23), the polyphase realization structure of H(z) as 2-sections (or as two sub-filters) is drawn as shown in fig 9.19.

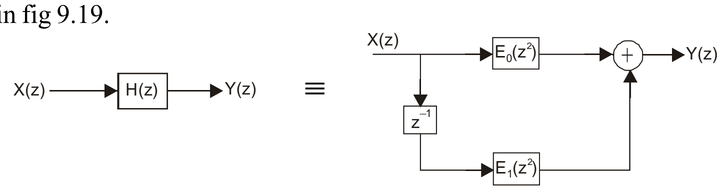


***Fig 9.19 :** Polyphase realization of FIR filter as 2 sections.*

Let us rearrange the equation (9.22) into three sections as shown below.

$$H(z) = [\,z^0\,h(0) + z^{-3}\,h(3) + z^{-6}\,h(6) + ...]$$
$$+ [z^{-1}\,h(1) + z^{-4}\,h(4) + z^{-7}\,h(7) + ...]$$
$$+ [z^{-2}\,h(2) + z^{-5}\,h(5) + z^{-8}\,h(8) + ...]$$

$$= [\,z^0\,h(0) + z^{-3}\,h(3) + z^{-6}\,h(6) + ...]$$
$$+ z^{-1}\,[z^0\,h(1) + z^{-3}\,h(4) + z^{-6}\,h(7) + ...]$$
$$+ z^{-2}\,[z^0\,h(2) + z^{-3}\,h(5) + z^{-6}\,h(8) + ...]$$

$$= [\,(z^3)^0\,h(0) + (z^3)^{-1}\,h(3) + (z^3)^{-2}\,h(6) + ...]$$
$$+ z^{-1}\,[(z^3)^0\,h(1) + (z^3)^{-1}\,h(4) + (z^3)^{-2}\,h(7) + ...]$$
$$+ z^{-2}\,[(z^3)^0\,h(2) + (z^3)^{-1}\,h(5) + (z^3)^{-2}\,h(8) + ...]$$

$$= E_0(z^3) + z^{-1}\,E_1(z^3) + z^{-2}\,E_2(z^3) \qquad .....(9.24)$$

where, $E_0(z^3) = (z^3)^0\,h(0) + (z^3)^{-1}\,h(3) + (z^3)^{-2}\,h(6) + ...$

$E_1(z^3) = (z^3)^0\,h(1) + (z^3)^{-1}\,h(4) + (z^3)^{-2}\,h(7) + ...$

$E_2(z^3) = (z^3)^0\,h(2) + (z^3)^{-1}\,h(5) + (z^3)^{-2}\,h(8) + ...$

> *Note :* 1. If, $E_0(z) = h(0) + z^{-1}\,h(3) + z^{-2}\,h(6) + ...$
> then, $E_0(z^3) = h(0) + (z^3)^{-1}\,h(3) + (z^3)^{-2}\,h(6) + ...$
>
> 2. If, $E_1(z) = h(1) + z^{-1}\,h(4) + z^{-2}\,h(7) + ...$
> then, $E_1(z^3) = h(1) + (z^3)^{-1}\,h(4) + (z^3)^{-2}\,h(7) + ...$
>
> 3. If, $E_2(z) = h(2) + z^{-1}\,h(5) + z^{-2}\,h(8) + ...$
> then, $E_2(z^3) = h(2) + (z^3)^{-1}\,h(5) + (z^3)^{-2}\,h(8) + ...$

The equation (9.24) decomposes the H(z) into three section, (or three sub-filters), $E_0(z^3)$, $E_1(z^3)$ and $E_2(z^3)$.

Using equation (9.24), the polyphase realization structure of H(z) as three-sections (or as three sub-filters) is drawn as shown in fig 9.20.

With reference to equations (9.23) and (9.24), the general equation for polyphase decomposition into L sections can be written as,

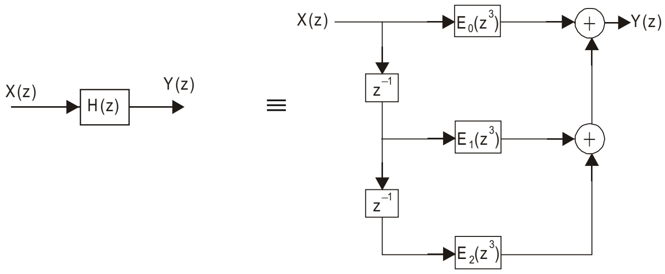


Fig 9.20 : *Polyphase realization of FIR filter as 3 section.*

$$H(z) = z^0\,E_0(z^L) + z^{-1}E_1(z^L) + z^{-2}E_2(z^L) + z^{-3}E_3(z^L) + ... + z^{-(L-1)}E_{L-1}(z^L) \quad .....(9.25)$$

$$\therefore\ H(z) = \sum_{m=0}^{L-1} z^{-m}\,E_m(z^L) \quad .....(9.26)$$

Using equation (9.25), the polyphase realization structure of H(z) as L sections, (or as L sub-filters) is drawn as shown in fig 9.21.

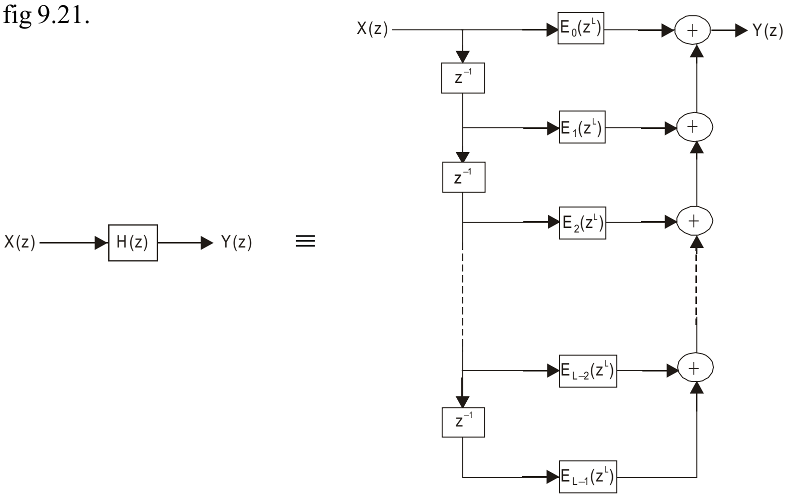


Fig 9.21 : *Type-I polyphase realization of FIR filter as L sections.*

Type-II Decomposition

The transposed realization structure of polyphase realization of fig 9.21 can be obtained as shown in fig 9.22.

The transposed structure of fig 9.22 can be modified as shown in fig 9.23 by using the notation,

$$R_m(z^L) = E_{L-1-m}(z^L) \quad ; \quad \text{for } m = 0 \text{ to } L-1$$

The structure of fig 9.23 is called type-II polyphase decomposition of FIR fitler. From fig 9.23, the general equation for type-II decomposition can be obtained as shown below.

$$H(z) = z^{-(L-1)} R_0(z^L) + z^{-(L-2)} R_1(z^L) + z^{-(L-3)} R_2(z^L) + \ldots\ldots + z^0 R_{L-1}(z^L)$$

$$= z^{-(L-1-0)} R_0(z^L) + z^{-(L-1-1)} R_1(z^L) + z^{-(L-1-2)} R_2(z^L) + \ldots\ldots + z^{-(L-1-(L-1))} R_{L-1}(z^L)$$

$$= \sum_{m=0}^{L-1} z^{-(L-1-m)} R_m(z^L) \qquad \ldots..(9.27)$$

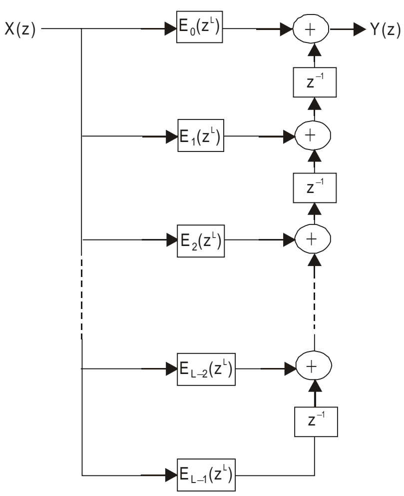


Fig 9.22 : Transposed structure of polyphase realization shown in fig 9.21.

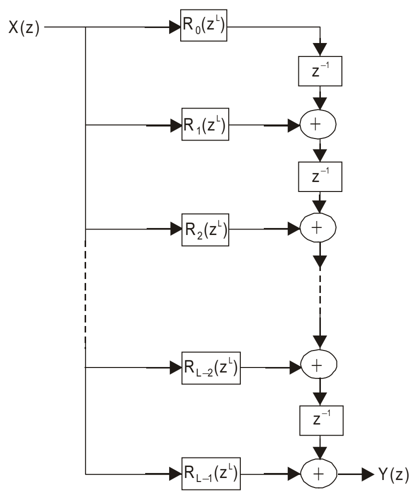


Fig 9.23 : Type-II polyphase realization of FIR filter as L sections.

9.8.2 Polyphase Structure of Decimator

In decimator, a lowpass filter called anti-aliasing filter is employed at the input in order to bandlimit the input signal, so that aliasing is avoided in the output spectrum of decimator.

Consider a decimator with sampling rate reduction factor, L. Let, H(z) be the transfer function of lowpass anti-aliasing FIR filter at the input of decimator as shown in fig 9.24. In order to reduce the computations in FIR filter, polyphase decomposition can be applied to FIR filter to decompose into L sub-filters as shown in fig 9.25.

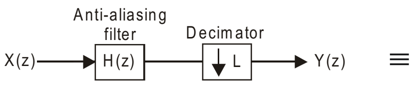


Fig 9.24 : Decimator with anti-aliasing filter.

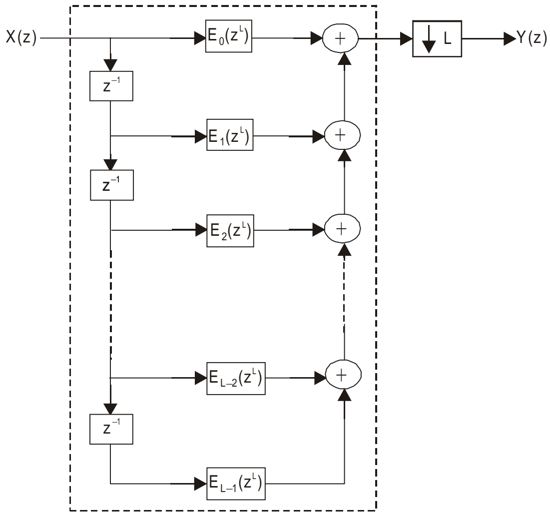
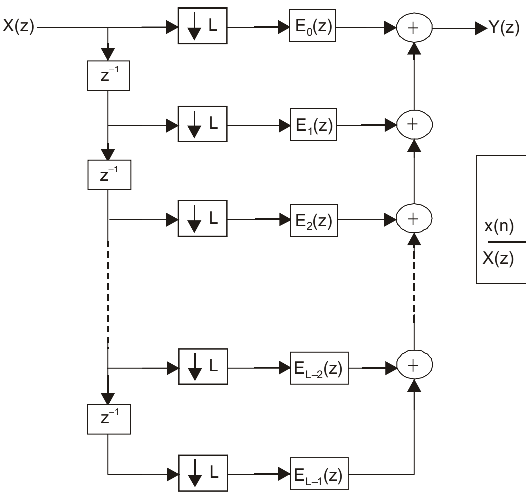


Fig 9.25 : Decimator with type-I decomposition of anti-aliasing filter.

The decimator in the structure of fig 9.25 will select only one sample in every L samples of the output. Therefore, further reduction in computations can be obtained if the decimator is shifted to input of sub-filters (using identity 15) as shown in fig 9.26. The structure shown in fig 9.26 is computationally efficient decimator structure.



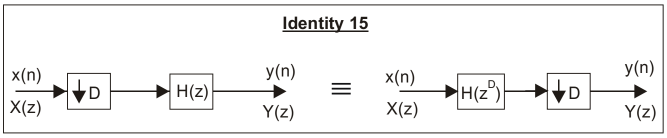


Fig 9.26 : Computationally efficient decimator structure.

9.8.3 Polyphase Structure of Interpolator

In interpolator, a lowpass filter called anti-imaging filter is employed at the output in order to eliminate the multiple images in the output spectrum of interpolator.

Consider an interpolator with sampling rate multiplication factor, L.

Let, H(z) be the transfer function of lowpass anti-imaging FIR filter at the output of interpolator as shown in fig 9.27. In order to reduce the computations in FIR filter, polyphase decomposition can be applied to FIR filter to decompose into L sub-filters as shown in fig 9.28.

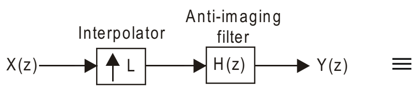


***Fig 9.27 :** Interpolator with anti-imaging filter.*

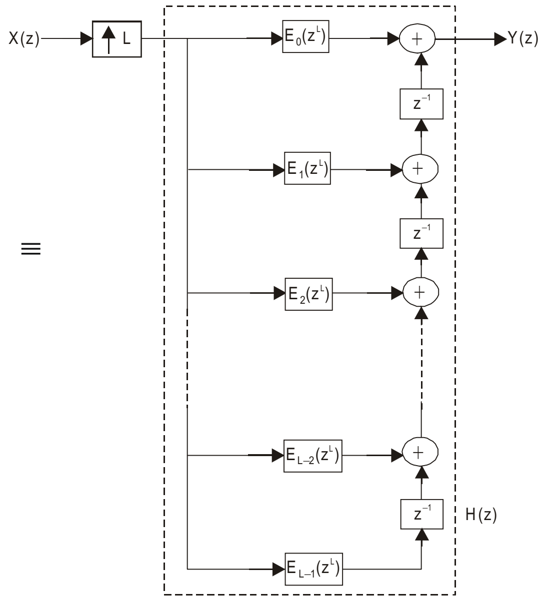
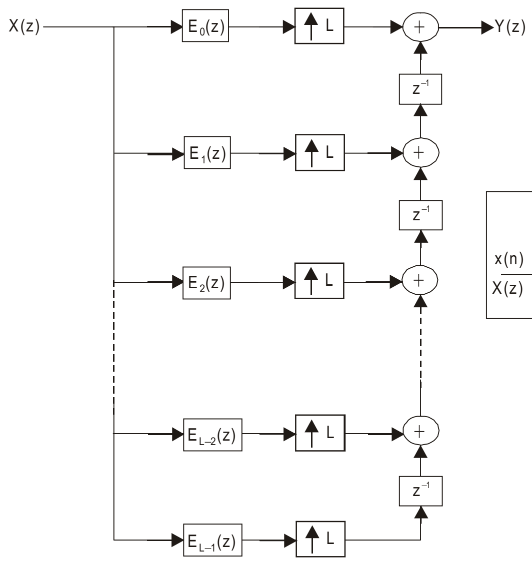


***Fig 9.28 :** Interpolator with type-I decomposition transposed structure of anti-imaging filter.*

The interpolator in the structure of fig 9.28 will introduce large number of zeros in the input signal so that only one in every L samples will be non-zero. Therefore, further reduction in computations can be obtained if the interpolator is shifted to output of sub-filters (using identity 18) as shown in fig 9.29.

The structure shown in fig 9.29 is computationally efficient interpolator structure.



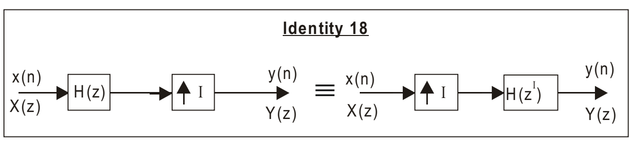


***Fig 9.29 :** Computationally efficient interpolator structure.*

9.8.4 Polyphase Decomposition of IIR Filters

The transfer function, H(z) of IIR filter is given by,

$$H(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{1 + \sum_{m=1}^{N} a_m z^{-m}} = \frac{b_0 z^{-1} + b_1 z^{-2} + \ ...... \ + \ b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ ...... \ + \ a_N z^{-N}}$$

Let, $H(z) = \frac{N(z)}{D(z)}$

where, $N(z)$ = Numerator polynomial of $H(z)$

$= b_0 z^{-1} + b_1 z^{-2} + ... + b_M z^{-M}$

$D(z)$ = Denominator polynomial of $H(z)$

$= 1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}$

In order to obtain the polyphase decomposition of IIR filter transfer function H(z), first multiply and divide H(z) by a suitable polynomial P(z), so that D(z) is converted to the form $D'(z^L)$.

$$\text{i.e., } H(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{D(z)} \times \frac{P(z)}{P(z)} = \frac{N'(z)}{D'(z^L)}$$

where, $N'(z) = N(z)\,P(z)$

$D'(z^L) = D(z)\,P(z)$

Next, the polynomial N'(z) can be decomposed into L sections as shown below.

$$N'(z) = N'_0(z^L) + z^{-1} N'_1(z^L) + z^{-2} N'_2(z^L) + \ ...... \ + N'_{L-1}(z^L)$$

$$\therefore \quad H(z) = \frac{N'(z)}{D'(z^L)} = \frac{N'_0(z^L) + z^{-1} N'_1(z^L) + z^{-2} N'_2(z^L) + \ ...... \ + N'_{L-1}(z^L)}{D'(z^L)}$$

$$= \frac{N'_0(z^L)}{D'(z^L)} + z^{-1} \frac{N'_1(z^L)}{D'(z^L)} + z^{-2} \frac{N'_2(z^L)}{D'(z^L)} + \ ...... \ + \frac{N'_{L-1}(z^L)}{D'(z^L)}$$

$$= E_0(z^L) + z^{-1} E_1(z^L) + z^{-2} E_2(z^L) + \ ...... \ + E_{L-1}(z^L) \qquad .....(9.28)$$

where, $E_0(z^L) = \frac{N'_0(z^L)}{D'(z^L)}$; $E_1(z^L) = \frac{N'_1(z^L)}{D'(z^L)}$; $E_2(z^L) = \frac{N'_2(z^L)}{D'(z^L)}$;

......, $E_{L-1}(z^L) = \frac{N'_{L-1}(z^L)}{D'(z^L)}$

The equation (9.28) is the general equation for decomposition of IIR filters.

Example 9.13

The transfer function of an FIR filter is,

$$H(z) = 0.2 + 0.7\,z^{-1} + 0.8\,z^{-2} + 0.15\,z^{-3} + 0.6\,z^{-4} + 0.32\,z^{-5} + 0.5\,z^{-6} + 0.4\,z^{-7} + 0.9\,z^{-8}$$

Perform polyphase decomposition of H(z) to decompose into, **a)** 2 sections **b)** 3 sections **c)** 4 sections.

Solution

a) Polyphase decomposition into 2 sections

Given that,

$$H(z) = 0.2 + 0.7\,z^{-1} + 0.8\,z^{-2} + 0.15\,z^{-3} + 0.6\,z^{-4} + 0.32\,z^{-5} + 0.5\,z^{-6} + 0.4\,z^{-7} + 0.9\,z^{-8}$$

Let us express H(z) as 2 sections as shown below.

$$\begin{aligned}H(z) &= [0.2 + 0.8\,z^{-2} + 0.6\,z^{-4} + 0.5\,z^{-6} + 0.9\,z^{-8}] \\ &\quad + [0.7z^{-1} + 0.15\,z^{-3} + 0.32\,z^{-5} + 0.4\,z^{-7}] \\ &= [0.2 + 0.8\,z^{-2} + 0.6\,z^{-4} + 0.5\,z^{-6} + 0.9\,z^{-8}] \\ &\quad + z^{-1}[0.7 + 0.15\,z^{-2} + 0.32\,z^{-4} + 0.4\,z^{-6}] \\ &= [0.2(z^2)^0 + 0.8\,(z^2)^{-1} + 0.6\,(z^2)^{-2} + 0.5\,(z^2)^{-3} + 0.9\,(z^2)^{-4}] \\ &\quad + z^{-1}[0.7(z^2)^0 + 0.15\,(z^2)^{-1} + 0.32\,(z^2)^{-2} + 0.4\,(z^2)^{-3}] \\ &= E_0(z^2) + z^{-1}\,E_1(z^2)\end{aligned}$$

$$\text{where, } E_0(z^2) = 0.2 + 0.8z^{-2} + 0.6z^{-4} + 0.5z^{-6} + 0.9z^{-8}$$

$$E_1(z^2) = 0.7 + 0.15z^{-2} + 0.32z^{-4} + 0.4z^{-6}$$

Here, $E_0(z^2)$ and $E_1(z^2)$ are two sections (or two sub-filters) obtained by polyphase decomposition of H(z).

b) Polyphase decomposition into 3 sections

Given that,

$$H(z) = 0.2 + 0.7\,z^{-1} + 0.8\,z^{-2} + 0.15\,z^{-3} + 0.6\,z^{-4} + 0.32\,z^{-5} + 0.5\,z^{-6} + 0.4\,z^{-7} + 0.9\,z^{-8}$$

Let us express H(z) as 3 sections as shown below.

$$\begin{aligned}H(z) &= [0.2 + 0.15\,z^{-3} + 0.5\,z^{-6}] \\ &\quad + [0.7z^{-1} + 0.6\,z^{-4} + 0.4\,z^{-7}] \\ &\quad + [0.8z^{-2} + 0.32\,z^{-5} + 0.9\,z^{-8}] \\ &= [0.2 + 0.15\,z^{-3} + 0.5\,z^{-6}] \\ &\quad + z^{-1}[0.7 + 0.6\,z^{-3} + 0.4\,z^{-6}] \\ &\quad + z^{-2}[0.8 + 0.32\,z^{-3} + 0.9\,z^{-6}] \\ &= [0.2(z^3)^0 + 0.15\,(z^3)^{-1} + 0.5\,(z^3)^{-2}] \\ &\quad + z^{-1}[0.7(z^3)^0 + 0.6\,(z^3)^{-1} + 0.4\,(z^3)^{-2}] \\ &\quad + z^{-2}[0.8(z^3)^0 + 0.32\,(z^3)^{-1} + 0.9\,(z^3)^{-2}] \\ &= E_0(z^3) + z^{-1}\,E_1(z^3) + z^{-2}\,E_2(z^3)\end{aligned}$$

$$\text{where, } E_0(z^3) = 0.2 + 0.15z^{-3} + 0.5z^{-6}$$

$$E_1(z^3) = 0.7 + 0.6z^{-3} + 0.4z^{-6}$$

$$E_2(z^3) = 0.8 + 0.32z^{-3} + 0.9z^{-6}$$

Here, $E_0(z^3)$, $E_1(z^3)$ and $E_2(z^3)$ are three sections (or three sub-filters) obtained by polyphase decomposition of H(z).

c) Polyphase decomposition into 4 sections

Given that,

$$H(z) = 0.2 + 0.7\,z^{-1} + 0.8\,z^{-2} + 0.15\,z^{-3} + 0.6\,z^{-4} + 0.32\,z^{-5} + 0.5\,z^{-6} + 0.4\,z^{-7} + 0.9\,z^{-8}$$

Let us express H(z) as 4 sections as shown below.

$$\begin{aligned} H(z) &= [0.2 + 0.6\,z^{-4} + 0.9\,z^{-8}] && = [0.2 + 0.6\,z^{-4} + 0.9\,z^{-8}] \\ &\quad + [0.7z^{-1} + 0.32\,z^{-5}] && \quad + z^{-1}[0.7 + 0.32\,z^{-4}] \\ &\quad + [0.8z^{-2} + 0.5\,z^{-6}] && \quad + z^{-2}[0.8 + 0.5\,z^{-4}] \\ &\quad + [0.15z^{-3} + 0.4\,z^{-7}] && \quad + z^{-3}[0.15 + 0.4\,z^{-4}] \end{aligned}$$

$$\begin{aligned} &= [0.2\,(z^4)^0 + 0.6\,(z^4)^{-1} + 0.9\,(z^4)^{-2}] \\ &\quad + z^{-1}[0.7(z^4)^0 + 0.32\,(z^4)^{-1}] \\ &\quad + z^{-2}[0.8(z^4)^0 + 0.5\,(z^4)^{-1}] \\ &\quad + z^{-3}[0.15(z^4)^0 + 0.4\,(z^4)^{-1}] \\ &= E_0(z^4) + z^{-1}\,E_1(z^4) + z^{-2}\,E_2(z^4) + z^{-3}\,E_3(z^4) \end{aligned}$$

where, $E_0(z^4) = 0.2 + 0.6z^{-4} + 0.9z^{-8}$

$E_1(z^4) = 0.7 + 0.32z^{-4}$

$E_2(z^4) = 0.8 + 0.5z^{-4}$

$E_3(z^4) = 0.15 + 0.4z^{-4}$

Here, $E_0(z^4)$, $E_1(z^4)$, $E_2(z^4)$ and $E_3(z^4)$ are four sections (or four sub-filters) obtained by polyphase decomposition of H(z).

Example 9.14

The transfer function of an IIR filter is,

$$H(z) = \frac{1 + 0.7z^{-1}}{1 - 0.9z^{-1}}$$

Perform polyphase decomposition of H(z) to decompose into, **a)** 2 sections **b)** 4 sections.

Solution

a) Polyphase decomposition into 2 sections

Given that,

$$H(z) = \frac{1 + 0.7z^{-1}}{1 - 0.9z^{-1}}$$

Let us choose a polynomial, $P(z) = 1 + 0.9z^{-1}$

Let us multiply the numerator and denominator of H(z) by P(z).

$$\therefore\ H(z) = \frac{1 + 0.7z^{-1}}{1 - 0.9z^{-1}} \times \frac{P(z)}{P(z)} = \frac{1 + 0.7z^{-1}}{1 - 0.9z^{-1}} \times \frac{1 + 0.9z^{-1}}{1 + 0.9z^{-1}}$$

$$= \frac{1 + 0.9z^{-1} + 0.7z^{-1} + 0.63z^{-2}}{1 - 0.81z^{-2}} \qquad \boxed{(a - b)\,(a + b) = a^2 - b^2}$$

$$= \frac{1 + 1.6z^{-1} + 0.63z^{-2}}{1 - 0.81z^{-2}} \qquad .....(1)$$

Now, the transfer function of equation (1) can be decomposed into two sections as shown below.

$$H(z)=\frac{1+1.6z^{-1}+0.63z^{-2}}{1-0.81z^{-2}}=\frac{(1+0.63z^{-2})+z^{-1}(1.6)}{1-0.81z^{-2}}$$

$$=\frac{1+0.63z^{-2}}{1-0.81z^{-2}}+z^{-1}\frac{1.6}{1-0.81z^{-2}}=\frac{1+0.63(z^2)^{-1}}{1-0.81(z^2)^{-1}}+z^{-1}\frac{1.6}{1-0.81(z^2)^{-1}}$$

$$=E_0(z^2)+z^{-1}E_1(z^2)$$

$$\text{where, } E_0(z^2)=\frac{1+0.63z^{-2}}{1-0.81z^{-2}} \quad ; \quad E_1(z^2)=\frac{1.6}{1-0.81z^{-2}}$$

Here, $E_0(z^2)$ and $E_1(z^2)$ are two sections (or two sub-filters) obtained by decomposition of H(z).

b) Polyphase decomposition into 4 sections

Consider the transfer function of equation (1).

$$H(z)=\frac{1+1.6z^{-1}+0.63z^{-2}}{1-0.81z^{-2}}$$

Let us choose a polynomial, $P_1(z) = 1 + 0.81z^{-2}$

Let us multiply the numerator and denominator of above H(z) by $P_1(z)$.

$$\therefore\ H(z)=\frac{1+1.6z^{-1}+0.63z^{-2}}{1-0.81z^{-2}}\times\frac{P_1(z)}{P_1(z)}$$

$$=\frac{1+1.6z^{-1}+0.63z^{-2}}{1-0.81z^{-2}}\times\frac{1+0.81z^{-2}}{1+0.81z^{-2}}$$

$$=\frac{1+1.6z^{-1}+0.63z^{-2}+0.81z^{-2}+1.296z^{-3}+0.5103z^{-4}}{1-0.6561z^{-4}} \qquad \boxed{(a-b)(a+b)=a^2-b^2}$$

$$=\frac{1+1.6z^{-1}+1.44z^{-2}+1.296z^{-3}+0.5103z^{-4}}{1-0.6561z^{-4}}$$

Now, the above transfer function can be decomposed into four sections as shown below.

$$\therefore\ H(z)=\frac{1+1.6z^{-1}+1.44z^{-2}+1.296z^{-3}+0.5103z^{-4}}{1-0.6561z^{-4}}$$

$$=\frac{(1+0.5103z^{-4})+z^{-1}(1.6)+z^{-2}(1.44)+z^{-3}(1.296)}{1-0.6561z^{-4}}$$

$$=\frac{1+0.5103z^{-4}}{1-0.6561z^{-4}}+z^{-1}\frac{1.6}{1-0.6561z^{-4}}+z^{-2}\frac{1.44}{1-0.6561z^{-4}}+z^{-3}\frac{1.296}{1-0.6561z^{-4}}$$

$$=\frac{1+0.5103(z^4)^{-1}}{1-0.6561(z^4)^{-1}}+z^{-1}\frac{1.6}{1-0.6561(z^4)^{-1}}+z^{-2}\frac{1.44}{1-0.6561(z^4)^{-1}}+z^{-3}\frac{1.296}{1-0.6561(z^4)^{-1}}$$

$$=E_0(z^4)+z^{-1}E_1(z^4)+z^{-2}E_2(z^4)+z^{-3}E_3(z^4)$$

$$\text{where, } E_0(z^4)=\frac{1+0.5103z^{-4}}{1-0.6561z^{-4}} \quad ; \quad E_1(z^4)=\frac{1.6}{1-0.6561z^{-4}}$$

$$E_2(z^4)=\frac{1.44}{1-0.6561z^{-4}} \quad ; \quad E_3(z^4)=\frac{1.296}{1-0.6561z^{-4}}$$

Here, $E_0(z^4)$, $E_1(z^4)$, $E_2(z^4)$ and $E_3(z^4)$ are four sections (or four sub-filters) obtained by decomposition of H(z).

Example 9.15

The transfer function of an IIR filter is,

$$H(z) = \frac{1 + 0.5z^{-1} + 0.3z^{-2}}{1 + 0.9z^{-1} + 0.8z^{-2}}$$

Perform polyphase decomposition of H(z) to decompose into, **a)** 2 sections **b)** 4sections.

Solution

a) Polyphase decomposition into 2 sections

Given that,

$$H(z) = \frac{1 + 0.5z^{-1} + 0.3z^{-2}}{1 + 0.9z^{-1} + 0.8z^{-2}}$$

Let us choose a polynomial, $P(z) = 1 - 0.9z^{-1} + 0.8z^{-2}$

Let us multiply the numerator and denominator of H(z) by P(z).

$$\therefore\ H(z) = \frac{1 + 0.5z^{-1} + 0.3z^{-2}}{1 + 0.9z^{-1} + 0.8z^{-2}} \times \frac{P(z)}{P(z)}$$

$$= \frac{1 + 0.5z^{-1} + 0.3z^{-2}}{1 + 0.9z^{-1} + 0.8z^{-2}} \times \frac{1 - 0.9z^{-1} + 0.8z^{-2}}{1 - 0.9z^{-1} + 0.8z^{-2}}$$

$$= \frac{1 - 0.9z^{-1} + 0.8z^{-2} + 0.5z^{-1} - 0.45z^{-2} + 0.4z^{-3} + 0.3z^{-2} - 0.27z^{-3} + 0.24z^{-4}}{1 - \cancel{0.9z^{-1}} + 0.8z^{-2} + \cancel{0.9z^{-1}} - 0.81z^{-2} + \cancel{0.72z^{-3}} + 0.8z^{-2} - \cancel{0.72z^{-3}} + 0.64z^{-4}}$$

$$= \frac{1 - 0.4z^{-1} + 0.65z^{-2} + 0.13z^{-3} + 0.24z^{-4}}{1 + 0.79z^{-2} + 0.64z^{-4}} \qquad .....(1)$$

Now, the above transfer function can be decomposed into two sections as shown below.

$$H(z) = \frac{1 - 0.4z^{-1} + 0.65z^{-2} + +0.13z^{-3} + 0.24z^{-4}}{1 + 0.79z^{-2} + 0.64z^{-4}}$$

$$= \frac{(1 + 0.65z^{-2} + 0.24z^{-4}) + z^{-1}(-0.4 + 0.13z^{-2})}{1 + 0.79z^{-2} + 0.64z^{-4}}$$

$$= \frac{1 + 0.65z^{-2} + 0.24z^{-4}}{1 + 0.79z^{-2} + 0.64z^{-4}} + z^{-1}\,\frac{-0.4 + 0.13z^{-2}}{1 + 0.79z^{-2} + 0.64z^{-4}}$$

$$= \frac{1 + 0.65(z^2)^{-1} + 0.24(z^2)^{-2}}{1 + 0.79(z^2)^{-1} + 0.64(z^2)^{-2}} + z^{-1}\,\frac{-0.4 + 0.13(z^2)^{-1}}{1 + 0.79(z^2)^{-1} + 0.64(z^2)^{-2}}$$

$$= E_0(z^2) + z^{-1}\,E_1(z^2)$$

$$\text{where, } E_0(z^2) = \frac{1 + 0.65z^{-2} + 0.24z^{-4}}{1 + 0.79z^{-2} + 0.64z^{-4}} \quad ; \quad E_1(z^2) = \frac{-0.4 + 0.13z^{-2}}{1 + 0.79z^{-2} + 0.64z^{-4}}$$

Here, $E_0(z^2)$ and $E_1(z^2)$ are two sections (or two sub-filters) obtained by decomposition of H(z).

b) Polyphase decomposition into 4 sections

Consider the transfer function of equation(1).

$$H(z) = \frac{1 - 0.4z^{-1} + 0.65z^{-2} + 0.13z^{-3} + 0.24z^{-4}}{1 + 0.79z^{-2} + 0.64z^{-4}}$$

Let us choose a polynomial, $P_1(z) = 1 - 0.79z^{-2} + 0.64z^{-4}$

Let us multiply the numerator and denominator of above H(z) by $P_1(z)$.

$$\therefore\ H(z) = \frac{1-0.4z^{-1}+0.65z^{-2}+0.13z^{-3}+0.24z^{-4}}{1+0.79z^{-2}+0.64z^{-4}} \times \frac{P_1(z)}{P_1(z)}$$

$$= \frac{1-0.4z^{-1}+0.65z^{-2}+0.13z^{-3}+0.24z^{-4}}{1+0.79z^{-2}+0.64z^{-4}} \times \frac{1-0.79z^{-2}+0.64z^{-4}}{1-0.79z^{-2}+0.64z^{-4}}$$

$$= \frac{\begin{array}{c}1-0.79z^{-2}+0.64z^{-4}-0.4z^{-1}+0.316z^{-3}-0.256z^{-5}+0.65z^{-2}-0.5135z^{-4}\\ +0.416z^{-6}+0.13z^{-3}-0.1027z^{-5}+0.0832z^{-7}+0.24z^{-4}-0.1896z^{-6}+0.1536z^{-8}\end{array}}{1-0.79z^{-2}+0.64z^{-4}+0.79z^{-2}-0.6241z^{-4}+0.5056z^{-6}+0.64z^{-4}-0.5056z^{-6}+0.4096z^{-8}}$$

$$= \frac{\begin{array}{r}1-0.4z^{-1}-0.14z^{-2}+0.446z^{-3}+0.3665z^{-4}-0.3587z^{-5}+0.2264z^{-6}\\ +0.0832z^{-7}+0.1536z^{-8}\end{array}}{1+0.6559z^{-4}+0.4096z^{-8}}$$

Now, the above transfer function can be decomposed into four sections as shown below.

$$\therefore\ H(z) = \frac{\begin{array}{l}(1+0.3665z^{-4}+0.1536z^{-8})+z^{-1}(-0.4-0{\cdot}3587z^{-4})\\ +z^{-2}(-0.14+0.2264z^{-4})+z^{-3}(0.446+0.0832z^{-4})\end{array}}{1+0.6559z^{-4}+0.4096z^{-8}}$$

$$= \frac{1+0.3665z^{-4}+0.1536z^{-8}}{1+0.6559z^{-4}+0.4096z^{-8}} + z^{-1}\frac{-0.4-0.3587z^{-4}}{1+0.6559z^{-4}+0.4096z^{-8}}$$

$$+ z^{-2}\frac{-0.14+0.2264z^{-4}}{1+0.6559z^{-4}+0.4096z^{-8}} + z^{-3}\frac{0.446+0.0832z^{-4}}{1+0.6559z^{-4}+0.4096z^{-8}}$$

$$= \frac{1+0.3665(z^4)^{-1}+0.1536(z^4)^{-2}}{1+0.6559(z^4)^{-1}+0.4096(z^4)^{-2}} + z^{-1}\frac{-0.4-0{\cdot}3587(z^4)^{-1}}{1+0.6559(z^4)^{-1}+0.4096(z^4)^{-2}}$$

$$+ z^{-2}\frac{-0.14+0.2264(z^4)^{-1}}{1+0.6559(z^4)^{-1}+0.4096(z^4)^{-2}} + z^{-3}\frac{0.446+0.0832(z^4)^{-1}}{1+0.6559(z^4)^{-1}+0.4096(z^4)^{-2}}$$

$$= E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4)$$

$$\text{where, } E_0(z^4) = \frac{1+0.3665z^{-4}+0.1536z^{-8}}{1+0.6559z^{-4}+0.4096z^{-8}}$$

$$E_1(z^4) = \frac{-0.4-0{\cdot}3587z^{-4}}{1+0.6559z^{-4}+0.4096z^{-8}}$$

$$E_2(z^4) = \frac{-0.14+0.2264z^{-4}}{1+0.6559z^{-4}+0.4096z^{-8}}$$

$$E_3(z^4) = \frac{0.446+0.0832z^{-4}}{1+0.6559z^{-4}+0.4096z^{-8}}$$

Here, $E_0(z^4)$, $E_1(z^4)$, $E_2(z^4)$ and $E_3(z^4)$ are four sections (or four sub-filters) obtained by decomposition of H(z).

9.9 Applications of Multirate DSP

9.9.1 Digital Filter Banks

A ***digital filter bank*** is a set of bandpass filters. The digital filter banks can be classified into two types. They are,

i) Analysis filter banks

ii) Synthesis filter banks

Analysis Filter Banks

An ***analysis filter bank*** is a set of bandpass filters with common input as shown in fig 9.30. The analysis filter bank is used for spectrum analysis in which a signal is divided into a set of sub-band signals. The analysis filter bank shown in fig 9.30 consists of M numbers of sub-band filters so that the input signal x(n) is divided into M-numbers of sub-band signals $v_0(n)$, $v_1(n)$, $v_2(n)$,......$v_{M-1}(n)$. Here $H_0(z)$, $H_1(z)$, $H_2(z)$, ... $H_{M-1}(z)$ are transfer function of M-numbers of bandpass filters.

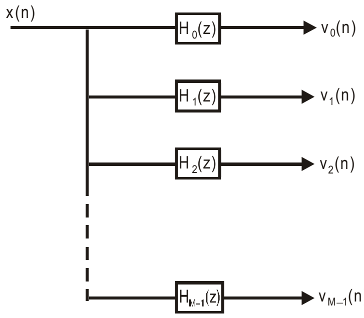


Fig 9.30 : Analysis filter bank.

Synthesis Filter Bank

A ***synthesis filter bank*** is a set of bandpass filters used to combine or synthesis a number of sub-band signals into a single composite signal as shown in fig 9.31. The synthesis filter bank shown in fig 9.31 accepts M-numbers of sub-band signals $w_0(n)$, $w_1(n)$, $w_2(n)$,......$w_{M-1}(n)$, combined to give a signal, y(n). In fact the synthesis filter bank perform the reverse process of analysis filter bank. Here $G_0(z)$, $G_1(z)$, $G_2(z)$, ... $G_{M-1}(z)$, are transfer function of M-numbers of bandpass filters.

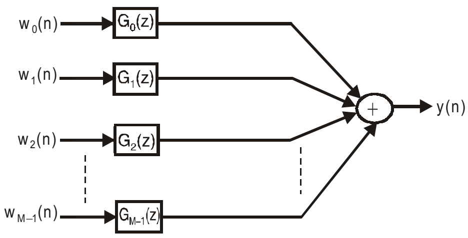


Fig 9.31 : Synthesis filter bank.

9.9.2 Sub-band Coding of Speech Signals

In ***sub-band coding*** of speech signals, the speech signal is divided into sub-bands, decimated, encoded and transmitted to the receiver system. On the receiver side the subband signals are decoded, interpolated and synthesized into the original speech signal. The fig 9.32 shows the subband coding of speech signal.

In the transmission side, the input signal is split into M-numbers of non-overlapping frequency bands using an analysis filter bank consisting of M-numbers of bandpass filters. The output of each bandpass filter is decimated by a factor of D. The output of decimators are encoded and transmitted.

On the reception side, the received sub-band signals are decoded and then interpolated to recover the missing samples. The output of interpolators are applied to a synthesis filter bank consisting of M-numbers of bandpass filters to recover the original signal.

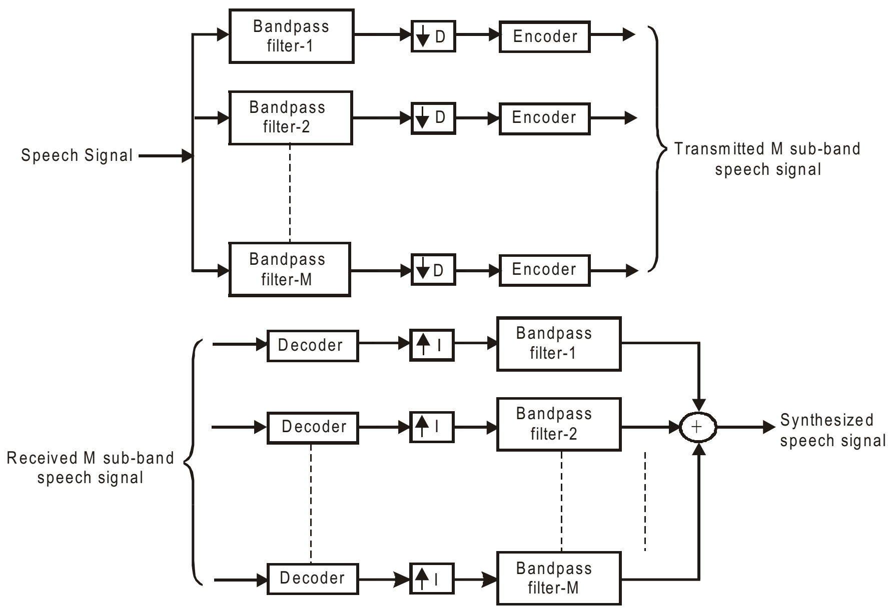


Fig 9.32 : Sub-band coding of speech signal.

9.9.3 Quadrature Mirror Filter (QMF) Bank

The ***QMF banks*** are filter banks with complementary frequency response. The basic building block of QMF bank is a two channel quadrature mirror filter (QMF). A two channel QMF consists of an analysis section and a synthesis section, as shown in fig 9.33.

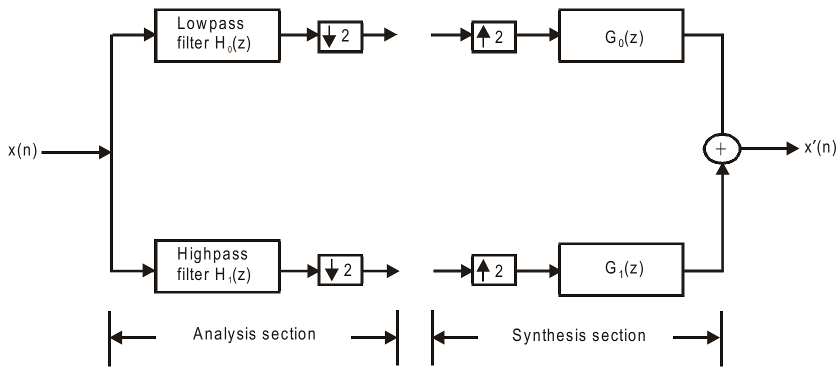


Fig 9.33 : Two-channel QMF bank.

The analysis section consists of a lowpass filter and highpass filter with symmetrical frequency response with centre of symmetry at p/2 as shown in fig 9.34. The output of the filters of analysis section are decimated by 2 and transmitted.

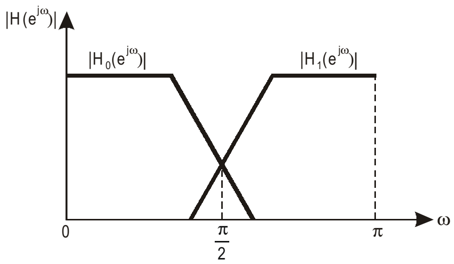


***Fig 9.34 :** Frequency response of analysis section QMF bank.*

In the synthesis section, the received signals are upsampled/interpolated by 2 and passed through two filters, whose frequency responses are selected to exactly cancel the effect of aliasing due to decimation by spectrum imaging due to interpolation.

The main advantage of QMF filter is that the aliasing resulting from decimation in the analysis section is exactly cancelled by the image signal spectrum that arises due to interpolation. Hence the two-channel QMF section behaves as a linear time invariant system.

9.10 Summary of Important Concepts

1. The discrete time systems that employ sampling rate conversion are called multirate DSP systems.
2. The process of converting a signal from one sampling rate to another sampling rate is called sampling rate conversion.
3. The two methods of sampling rate conversion in digital domain are decimation and interpolation.
4. Decimation is the process of reducing the sampling rate by an integer factor D.
5. Interpolation is the process of increasing the sampling rate by an integer factor I.
6. A decimator is a device which performs the process of downsampling.
7. An interpolator is a device which performs the process of upsampling.
8. The output spectrum of a decimator is the sum of scaled, stretched and shifted versions of the input spectrum.
9. For decimation by D, the input spectrum of the decimator should be bandlimited to p/D, to avoid aliasing in the output spectrum.
10. The filter used to limit the bandwidth of input signal to decimator is called anti-aliasing filter.
11. For interpolation by I, the spectrum of output signal of interpolator will have I images of input spectrum, in a period of 2p.
12. The filter used to eliminate the multiple images in the output spectrum of an interpolator is called anti-imaging filter.
13. The anti-aliasing and anti-imaging filters are lowpass filters.
14. The sampling rate conversion by a rational factor $\frac{I}{D}$ is employed when sample rate conversion is required by a noninteger factor.
15. In sampling rate conversion by a factor $\frac{I}{D}$, the interpolation is performed first and decimation is performed next in order to preserve the spectral characteristics of the input signal.

16. In sampling rate conversion by a factor $\frac{I}{D}$, a single lowpass filter with bandwidth minimum among $\frac{\pi}{I}$, $\frac{\pi}{D}$ is employed to prevent imaging due to interpolation and aliasing due to decimation.
17. When the sampling rate conversion factor D or I is very large then multistage sampling rate conversion is employed to achieve computational efficiency.
18. If D_1D_2 = D, then the cascade of two decimators with sampling rate reduction factor D_1 and $D_2$is equal to single decimator with sampling rate reduction factor D.
19. If I_1 I_2 = I, then the cascade of two interpolators with sampling rate multiplication factor I_1 and I_2 is equal to single interpolator with sampling rate multiplication factor I.
20. In FIR filter structures, in order to improve the computational efficiency, the decimators are employed before multiplier and interpolators are employed after multipliers.
21. Polyphase decomposition is dividing a filter into a number of sub-filters that can be realized in parallel.
22. In polyphase decomposition, the sub-filters will differ only in phase characteristics and so they are called polyphase filters.
23. The analysis filter bank is a set of bandpass filters that divide a signal into a set of sub-band signals.
24. The synthesis filter bank is a set of bandpass filters that combine a set of sub-band signals into a signle composite signal.
25. The QMF (Quadrature Mirror Filter) banks are filter banks with complementary frequency response.

9.11. Short Questions and Answers

Q9.1 What is multirate DSP?

The processing of discrete time signals of different sampling rate in different parts of a system is called multirate DSP.

Q9.2 What is a multirate DSP system?

The discrete time system that employs sampling rate conversion while processing the discrete time signal is called multirate DSP systems

Q9.3 What are the various basic methods of sampling rate conversion in digital domain?

The basic methods of sampling rate conversion are decimation (or downsampling) and interpolation (or upsampling).

Q9.4 What is decimation?

Decimation (or downsampling) is the process of reducing the sampling rate by an integer factor D.

Q9.5 What is interpolation?

Interpolation (or upsampling) is the process of increasing the sampling rate by an integer factor I.

Q9.6 Given any two applications of multirate DSP system.

1. Sub-band coding of speech signals and image compression.
2. Oversampling A/D and D/A converters for high quality digital audio systems and digital storage systems.

Q.9.7 Write some advantages of multirate processing.

1. The reduction in number of computations.
2. The reduction in memory requirement.
3. The reduction in finite word length effects.

Q9.8 What is a decimator? Draw the symbolic representation of a decimator.

The device which performs the process of decimation (or downsampling) is called decimator (or downsampler)

The symbolic representation of a decimator for decimation by an integer factor D is shown in fig Q9.8.

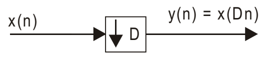


Fig Q9.8 : *Decimator.*

Q9.9 Show that the decimator is a time variant system.

Consider the decimator shown in fig Q9.9.

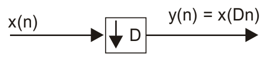


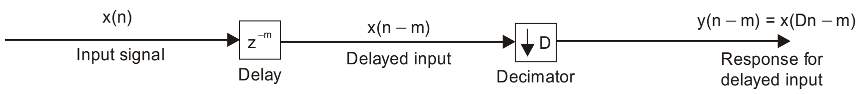
Fig Q9.9 : *Decimator.*

The input-output relation of a decimator is,

$$y(n) = x(Dn)$$

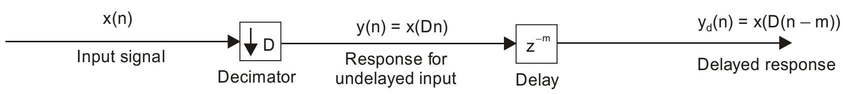
Test 1 : Response for delayed input

Let, $y(n - m)$ = Response for delayed input



Test 2 : Delayed response

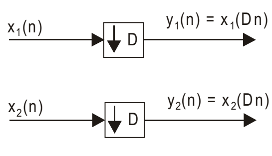
Let, $y_d(n)$ = Delayed response



Conclusion : Here, $y(n - m) \neq y_d(n)$, therefore the decimator is time variant system.

Q9.10 Show that the decimator is linear system.

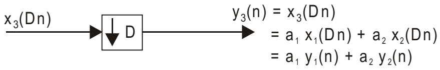
Let $x_1(n)$ and $x_2(n)$ be two different inputs to a decimator. Let $y_1(n)$ and $y_2(n)$ be the corresponding outputs.



Let, $x_3(n) = a_1\, x_1(n) + a_2\, x_2(n)$

A linear combination of inputs $x_1(n)$ and $x_2(n)$.

Let, $y_3(n)$ be output of decimator for the input $x_3(n)$.



Since, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$, the decimator is linear system.

Q9.11 ***Write the expression for output spectrum, $Y(e^{jw})$ of decimator in terms of input spectrum, $X(e^{jw})$.***

$$\text{Output spectrum, } Y(e^{j\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j(\omega-2\pi k)/D}\right)$$

where, D = Integer sampling rate reduction factor of decimator.

Q9.12 ***What is anti-aliasing filter?***

The lowpass filter used at the input of decimator is called anti-aliasing filter. It is used to limit the bandwidth of an input signal to $\frac{\pi}{D}$ in order to prevent the aliasing of output specturm of decimator for decimation by D.

Q9.13 ***What is interpolator? Draw the symbolic representation of an interpolator.***

The device which performs the process of interpolation (or upsampling) is called an interpolator (or upsampler).

The symbolic representation of interpolator for interpolation by an integer factor I is shown in fig Q9.13.

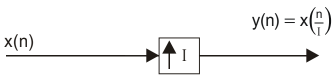


Fig Q9.13 : Interpolator.

Q9.14 ***Show that the interpolator is a time variant system.***

Consider the interpolator shown in fig Q9.14.

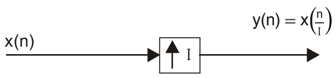


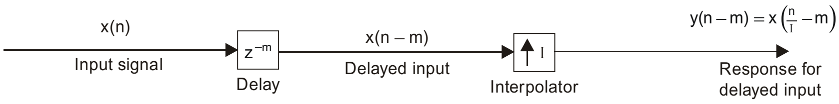
Fig Q9.14 : Interpolator.

The input-output relation of an interpolator is,

$$y(n) = x\left(\tfrac{n}{I}\right)$$

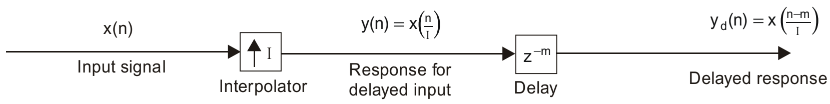
<u>Test 1 : Response for delayed input</u>

Let, y(n – m) = Response for delayed input



Test 2 : Delayed response

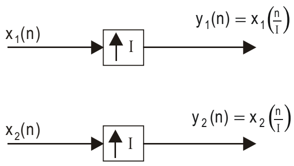
Let, $y_d(n)$ = Delayed response



Conclusion : **Here, $y(n-m) \neq y_d(n)$, therefore the interpolator is a time variant system.**

Q9.15 ***Show that the interpolator is a linear system.***

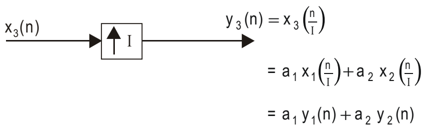
Let $x_1(n)$ and $x_2(n)$ be two different inputs to an interpolator. Let $y_1(n)$ and $y_2(n)$ be the corresponding outputs.



Let, $x_3(n) = a_1\, x_1(n) + a_2\, x_2(n)$

Let, $y_3(n)$ be output of interpolator for the input $x_3(n)$.

A linear combination of inputs $x_1(n)$ and $x_2(n)$.



Since, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$, the interpolator is linear system.

Q9.16 ***Write the expression for output spectrum, $Y(e^{jw})$ of an interpolator in terms of input spectrum, $X(e^{jw})$.***

Output spectrum, $Y(e^{j\omega}) = X(e^{j\omega I})$

where, I = Integer sampling rate multiplication factor of interpolator.

Q9.17 ***What is an anti-imaging filter?***

The lowpass filter used at the output of an interpolator is called anti-imaging filter. It is used to eliminate the multiple images in the output spectrum of the interpolator.

Q9.18 ***Write a short note on sampling rate conversion by a rational factor.***

When sampling rate conversion is required by a non-integer factor, then sampling rate conversion is performed by the rational factor $\frac{I}{D}$. In this method, the signal is first interpolated by an integer factor I, then passed through a lowpass filter with bandwidth minimum of $\left(\frac{\pi}{I}, \frac{\pi}{D}\right)$, and finally decimated by an integer factor, D.

Q9.19 Write a short note on multistage implementation of sampling rate conversion.

When the sampling rate conversion factor I or D is very large then the multistage sampling rate conversion will be computationally efficient realization.

In multistage interpolation, the interpolation by I is realized as cascade of interpolators with sampling rate multiplication factors I_1, I_2, ... I_L, where $I = I_1 \times I_2 \times \ldots\ldots \times I_L$.

In multistage decimation, the decimation by D is realized as a cascade of decimators with sampling rate reduction factors D_1, D_2, ... D_L, where $D = D_1 \times D_2 \times \ldots\ldots \times D_L$.

Q9.20 What is polyphase decomposition?

The process of dividing a filter into a number of sub-filters which differ only in phase characteristics is called polyphase decomposition.

9.12. MATLAB Programs

Program 9.1

Write a MATLAB program to downsample the signal x(n) by sampling rate reduction factor a) 2 b) 3.

x(n)={1,-1,1,-1,2,-2,2,-2,3,-3,3,-3}

```
%Program for downsampling a discrete time signal

clear all
clc

display(‘The given signal is,’);
xn=[1,-1,1,-1,2,-2,2,-2,3,-3,3,-3]
N=length(xn);
n=0:1:N-1;
stem(n,xn,’k’); xlim([0 12]); ylim([-3 3]);
xlabel(‘n’,’fontsize’,11,’fontweight’,’b’);
ylabel(‘x(n)’,’fontsize’,11,’fontweight’,’b’);
title(‘Signal x(n)’,’fontsize’,11,’fontweight’,’b’);

D=2;                        %Sampling rate reduction factor
display(‘The signal downsampled by reduction factor 2 is,’);
xD2n=xn(1:D:N)
n1=1:1:N/D;
figure, stem(n1-1,xD2n,’k’); xlim([0 12]); ylim([-3 3]);
xlabel(‘n’,’fontsize’,11,’fontweight’,’b’);
ylabel(‘x_D_2(n)’,’fontsize’,11,’fontweight’,’b’);
title(‘Downsampled Signal, D = 2’,’fontsize’,11,’fontweight’,’b’);

D=3;                        %Sampling rate reduction factor
display(‘The signal downsampled by reduction factor 3 is,’);
xD3n=xn(1:D:N)
n1=1:1:N/D;
figure,stem(n1-1,xD3n,’k’); xlim([0 12]); ylim([-3 3]);
xlabel(‘n’,’fontsize’,11,’fontweight’,’b’);
ylabel(‘x_D_3(n)’,’fontsize’,11,’fontweight’,’b’);
title(‘Downsampled Signal, D = 3’,’fontsize’,11,’fontweight’,’b’);
```

OUTPUT

```
     The given signal is,
     xn =
           1    -1     1    -1     2    -2     2    -2     3    -3     3    -3

     The signal downsampled by reduction factor 2 is,
     xD2n =
           1     1     2     2     3     3

     The signal downsampled by reduction factor 3 is,
     xD3n =
           1    -1     2    -3
```

The graphical representation of x(n) and the downsampled version of x(n) are shown in fig p9.1a, b, and c.

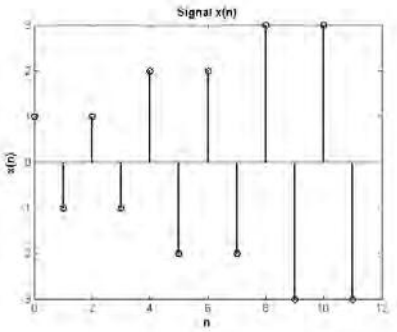


Fig P9.1a : Signal x(n).

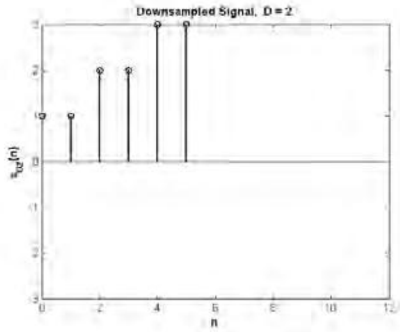


Fig P9.1b : Downsampled signal, D = 2.

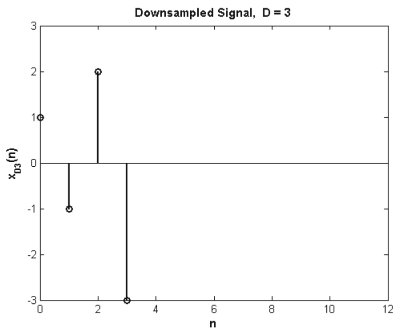


Fig P9.1c : Downsampled signal, D = 3.

Note : Verify the result with example 9.2.

Program 9.2

Consider the spectrum of the signal x(n) shown in fig 1. Determine the spectrum of downsampled version of the signal for sampling rate reduction factor, a) 2 b) 3 c) 4.

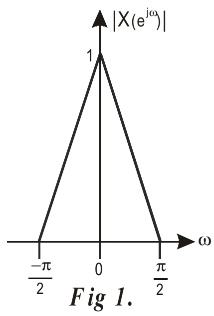


Fig 1.

```
%Program to determine the spectrum of downsampled signal

clear all
clc

F=.5;                              %Highest normalized frequency in spectrum
                                   %of x(n)
n=0:1:1024;
xn=F/2*sinc(F/2*(n-512)).^2;       %Generate the discrete time signal x(n)
f=-3:1/512:3;
Xjw=freqz(xn,1,pi*f);              %Determine the spectrum of x(n)
plot(f,abs(Xjw),'k')               %Plot the magnitude spectrum of x(n)
xlabel('Normalized frequency, \omega/\pi','fontsize', 11 fontweight', 'b');
ylabel('X(e^j^w)','fontsize',11,'fontweight','b');
title('Spectrum of x(n)','fontsize',11,'fontweight','b');

D=2;                               %Sampling rate reduction factor
y2n=F/2*sinc(F/2*(n-512)*D).^2;    %Generate the downsampled signal
Y2jw=freqz(y2n,1,pi*f*D);          %Determine the spectrum of downsampled signal
figure,plot(f,abs(Y2jw),'k')       %Plot the magnitude spectrum of downsampled
                                   %signal
xlabel('Normalized frequency, \omega/\pi','fontsize',11,'fontweight', 'b');
ylabel('Y(e^j^w)','fontsize',11,'fontweight','b');
title('Spectrum of Downsampled Signal, D = 2','fontsize',11,'fontweight','b');
axis([-3 3 0 1])
D=3;                               %Sampling rate reduction factor
y3n=F/2*sinc(F/2*(n-512)*D).^2;    %Generate the downsampled signal
Y3jw=freqz(y3n,1,pi*f*D);          %Determine the spectrum of downsampled signal
figure,plot(f,abs(Y3jw),'k')       %Plot the magnitude spectrum of downsampled
                                   %signal
xlabel('Normalized frequency, \omega/\pi','fontsize',11,'fontweight','b');
ylabel('Y(e^j^w)','fontsize',11,'fontweight','b');
title('Spectrum of Downsampled Signal, D = 3','fontsize',11,'fontweight','b');
axis([-3 3 0 1])

D=4;                               %Sampling rate reduction factor
y4n=F/2*sinc(F/2*(n-512)*D).^2;    %Generate the downsampled signal
Y4jw=freqz(y4n,1,pi*f*D);          %Determine the spectrum of downsampled signal
figure,plot(f,abs(Y4jw),'k')       %Plot the magnitude spectrum of downsampled
                                   %signal
xlabel('Normalized frequency, \omega/\pi','fontsize',11,'fontweight','b');
ylabel('Y(e^j^w)','fontsize',11,'fontweight','b');
title('Spectrum of Downsampled Signal, D = 4','fontsize',11,'fontweight','b');
axis([-3 3 0 1])
```

OUTPUT

The spectrum of x(n) and the spectrum of downsampled version of the signals are Shown in fig p9.2a, b, c and d.

Note : Verify the result with example 9.3.

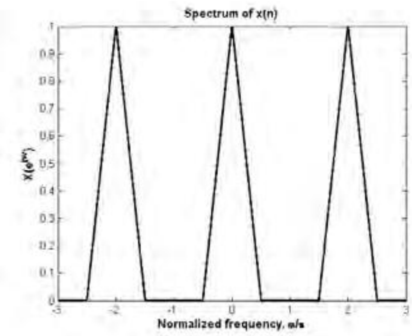


Fig P9.2a : *Spectrum of x(n).*

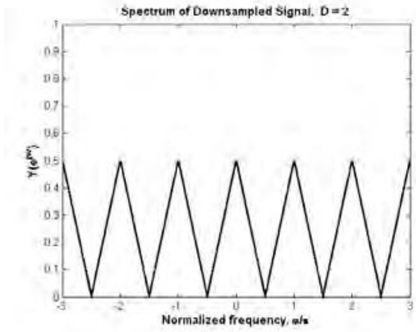


Fig P9.2b : *Spectrum of downsampled signal, D = 2.*

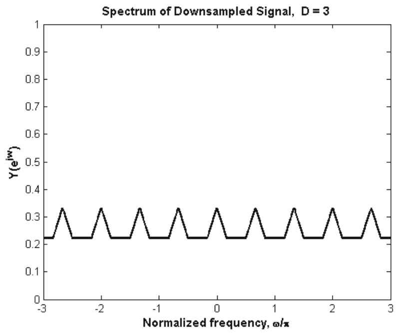


Fig P9.2c : *Spectrum of downsampled signal, D = 3.*

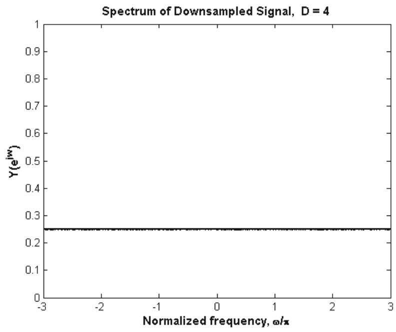


Fig P9.2d : *Spectrum of downsampled signal, D = 4.*

Program 9.3

Consider the spectrum of the signal x(n) shown in fig 1.Determine the spectrum of downsampled version of the signal for sampling rate reduction factor D=3.

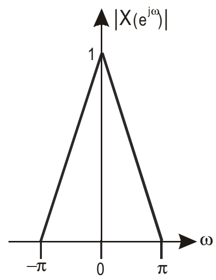


Fig 1.

```
%Program to determine the spectrum of downsampled signal

clear all
clc

F=1.0;                          %Highest normalized frequency in spectrum of
                                %x(n)
n=0:1:1024;
xn=F/2*sinc(F/2*(n-512)).^2;    %Generate the discrete time signal x(n)
f=-3:1/512:3;
Xjw=freqz(xn,1,pi*f);           %Determine the spectrum of x(n)
plot(f,abs(Xjw),'k')            %Plot the magnitude spectrum of x(n)
xlabel('Normalized Frequency','fontsize',11,'fontweight','b');
ylabel('X(e^j^w)','fontsize',11,'fontweight','b');
title('Spectrum of x(n)','fontsize',11,'fontweight','b');
axis([-3 3 0 1])

D=3;                            %Sampling rate reduction factor
yn=F/2*sinc(F/2*(n-512)*D).^2;  %Generate the downsampled signal
Yjw=freqz(yn,1,pi*f*D);         %Determine the spectrum of downsampled signal
figure,plot(f,abs(Yjw),'k')     %Plot the magnitude spectrum of downsampled signal
```

```
xlabel('Normalized Frequency','fontsize',11,'fontweight','b');
ylabel('Y(e^j^w)','fontsize',11,'fontweight','b');
title('Spectrum of Downsampled Signal, D = 3','fontsize',11,'fontweight','b');
axis([-3 3 0 1])
```

OUTPUT

The spectrum of x(n) and the spectrum of downsampled version of the signal are Shown in fig p9.3a, and b.

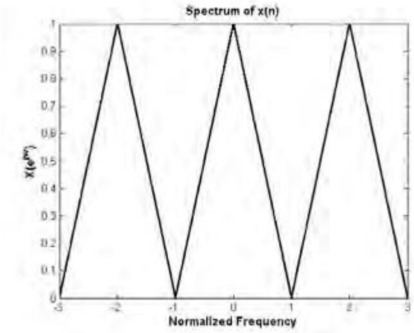


Fig P9.3a : *Spectrum of x(n).*

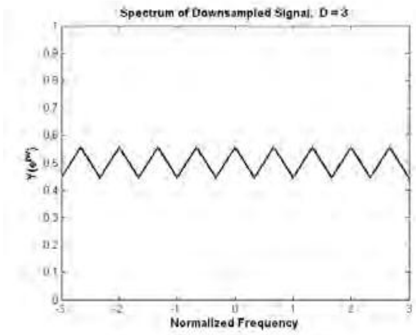


Fig P9.3b : *Spectrum of downsampled signal, D = 3.*

Note : Verify the result with example 9.6.

Program 9.4

Write a MATLAB program to upsample the signal x(n) by sampling rate multiplication factor a) 2 b) 3

x(n)={1,-1,2,-2}

```
%Program for upsampling a discrete time signal

clear all
clc

display('The given signal is,');
xn=[1,-1,2,-2]
N=length(xn);
n=0:1:N-1;
stem(n,xn,'k'); xlim([0 12]); ylim([-3 3]);
xlabel('n','fontsize',11,'fontweight','b');
ylabel('x(n)','fontsize',11,'fontweight','b');
title('Signal x(n)','fontsize',11,'fontweight','b');

I=2;                          %Sampling rate multiplication factor
xI2n=[zeros(1,I*N)];
n1=1:1:I*N; j=1:I:I*N;
display('The signal upsampled by multiplication factor 2 is,');
xI2n(j)=xn
figure,stem(n1,xI2n,'k'); xlim([0 12]); ylim([-3 3]);
xlabel('n','fontsize',11,'fontweight','b');
ylabel('x_I_2(n)','fontsize',11,'fontweight','b');
title('Upsampled Signal,  I = 2','fontsize',11,'fontweight','b');

I=3;                          %Sampling rate multiplication factor
xI3n=[zeros(1,I*N)];
n1=1:1:I*N; j=1:I:I*N;
display('The signal upsampled by multiplication factor 3 is,');
xI3n(j)=xn
```

```
figure,stem(n1,xI3n,'k'); xlim([0 12]); ylim([-3 3]);
xlabel('n','fontsize',11,'fontweight','b');
ylabel('x_I_3(n)','fontsize',11,'fontweight','b');
title('Upsampled Signal,  I = 3','fontsize',11,'fontweight','b');
```

OUTPUT

```
The given signal is,
xn =
   1    -1    2    -2
The signal upsampled by multiplication factor 2 is,
xI2n =
   1    0    -1    0    2    0    -2    0
The signal upsampled by multiplication factor 3 is,
xI3n =
   1    0    0    -1    0    0    2    0    0    -2    0    0
The graphical representation of x(n) and the upsampled version of
 x(n) are shown in fig p9.4a, b and c.
```

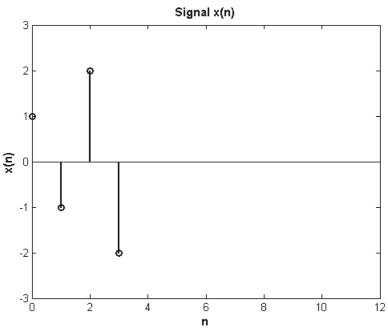


Fig P9.4a : *Signal x(n).*

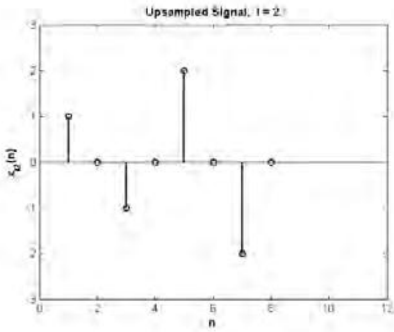


Fig P9.4b : *Upsampled signal, I = 2.*

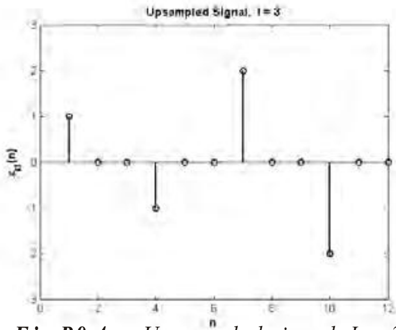


Fig P9.4c : *Upsampled signal, I = 3.*

Note : Verify the result with example 9.8.

Program 9.5

Consider the spectrum of the signal x(n) shown in fig 1. Determine the spectrum of upsampled version of the signal for sampling rate multiplication factor, a) 2 b) 3.

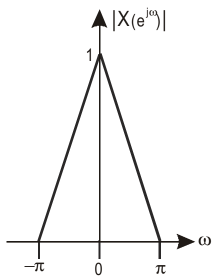


Fig 1.

```
%Program to determine the spectrum of upsampled signal
clear all
clc

F=1.0;                          %Highest normalized frequency in spectrum
                                %of x(n)
n=0:1:1024;
xn=F/2*sinc(F/2*(n-512)).^2;    %Generate the discrete time signal x(n)
f=-2:1/512:2;
Xjw=freqz(xn,1,2*pi*f);
plot(f,abs(Xjw),'k');
xlabel('Normalized frequency, \omega/\pi','fontsize',11,'fontweight','b');
ylabel('X(e^j^w)','fontsize',11,'fontweight','b');
title('Spectrum of x(n)','fontsize',11,'fontweight','b');

i=1:1025;
y2=[zeros(1,2048)];
I=2;                            %Sampling rate multiplication factor
y2(I*i)=xn;                     %Generate the upsampled signal
f=-2:1/512:2;
Y2jw=freqz(y2,1,2*pi*f);        %Determine the spectrum of upsampled signal
figure, plot(f,abs(Y2jw),'k');  %Plot the magnitude spectrum of upsampled
                                %signal
title('Spectrum of Upsampled Signal, I = 2','fontsize',12,'fontweight','b');
xlabel('Normalized frequency, \omega/\pi','fontsize',12,'fontweight','b');
ylabel('Y(e^j^w)','fontsize',12,'fontweight','b');

y3=[zeros(1,3072)];
I=3;                            %Sampling rate multiplication factor
y3(I*i)=xn;                     %Generate the upsampled signal
f=-2:1/512:2;
Y3jw=freqz(y3,1,2*pi*f);        %Determine the spectrum of upsampled signal
figure, plot(f,abs(Y3jw),'k');  %Plot the magnitude spectrum of upsampled
                                %signal
title('Spectrum of Upsampled Signal, I = 3','fontsize',12,'fontweight','b');
xlabel('Normalized frequency, \omega/\pi','fontsize',12,'fontweight','b');
ylabel('Y(e^j^w)','fontsize',12,'fontweight','b');
```

OUTPUT

The spectrum of x(n) and the spectrum of upsampled version of the signals are Shown in fig p9.5a, b, and c.

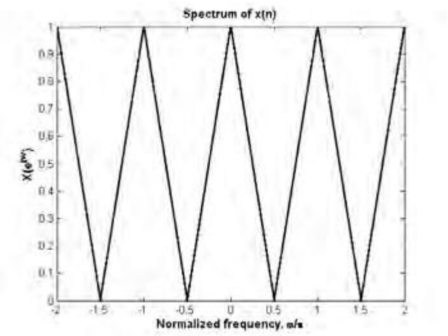


Fig P9.5a : Spectrum of x(n).

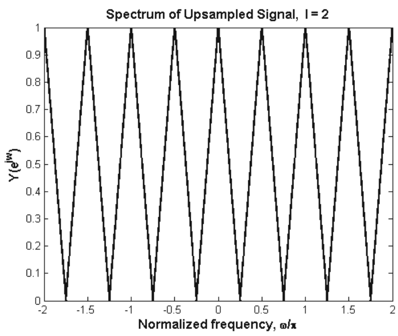


Fig P9.5b : Spectrum of upsampled signal, I = 2.

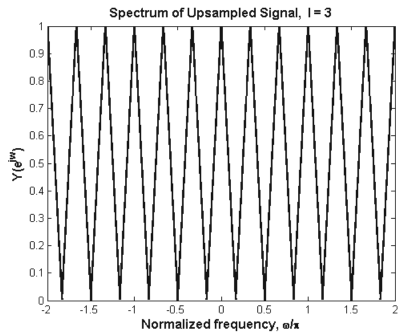


Fig P9.5c : *Spectrum of upsampled signal, I = 3.*

`Note : Verify the result with example 9.9.`

9.13 Exercises

I. Fill in the blanks with appropriate words

1. The processing of signal at different sampling rates is called_______ .
2. The _______ is the process of increasing the sampling rate.
3. The _______ is the process of decreasing the sampling rate.
4. To avoid aliasing of output spectrum of decimator for decmation by D, the input spectrum is _______ to p/D.
5. To eliminate multiple images in output spectrum of interpolator for interpolation by I, the output spectrum is bandlimited to _______.
6. In sampling rate conversion by rational factor, _______ is performed first.
7. When the sampling rate conversion factor is very large then _______ sampling rate conversion is preferred.
8. The process of dividing a filter into a number of sub-filters is called _______.
9. The digital filter bank is a set of _______ filters.
10. The _______ banks are filter banks with complementary frequency response.

Answers

1. multirate DSP	4. bandlimited	7. multistage	10. QMF
2. interpolation	5. p/I	8. polyphase decomposition	
3. decimation	6. interpolation	9. bandpass	

II. State whether the following statements are True/False

1. The multirate systems employ sampling rate conversion while processing signals.
2. In decimation, the sampling time is increased.
3. In interpolation, the sampling frequency is decreased.
4. The sampling rate conversion is possible only by integer factor.
5. The decimator employs lowpass filter for anti-imaging.
6. The interpolator employs lowpass filter for anti-aliasing.
7. In sampling rate conversion by rational factor, decimation is performed after interpolation to preserve spectral characteristics.

8. To improve computational efficiency of FIR filters, decimators are employed before multipliers.
9. To improve computational efficiency of FIR filters, interpolators are employed after multipliers.
10. The polyphase filters will have identical magnitude characteristics but different phase characteristics.

Answers

1. True	4. False	7. True	10. True
2. True	5. False	8. True	
3. False	6. False	9. True	

III. Choose the right answer for the following questions

1. If x(n) and y(n) are input and output of a decimator with sampling rate conversion factor A, then,

a) $y(n) = x(n-A)$ b) $y(n) = x\left(\frac{n}{A}\right)$

c) $y(n) = x(n+A)$ d) $y(n) = x(An)$

2. If $X(e^{jw})$ and $Y(e^{jw})$ are input and output spectrum of a decimator then,

a) $Y(e^{j\omega}) = \frac{1}{D}X(e^{j\omega/D})$ b) $Y(e^{j\omega}) = DX(e^{j\omega/D})$

c) $Y(e^{j\omega}) = \frac{1}{D}X(e^{j\omega D})$ d) $Y(e^{j\omega}) = DX(e^{j\omega D})$

3. To avoid aliasing at output during decimation by D, the input signal of a decimator should be bandlimited to,

a) $\frac{\pi}{2D}$ b) $\frac{2\pi}{D}$ c) $\frac{\pi}{D}$ d) $\frac{\pi}{D^2}$

4. If x(n) and y(n) are input and output of an interpolator with sampling rate conversion factor B, then,

a) $y(n) = x(Bn)$ b) $y(n) = x\left(\frac{n}{B}\right)$

c) $y(n) = \frac{x(n)}{B}$ d) $y(n) = B\,x(n)$

5. If $X(e^{jw})$ and $Y(e^{jw})$ are input and output spectrum of an interpolator then,

a) $Y(e^{j\omega}) = I\,X(e^{j\omega I})$ b) $Y(e^{j\omega}) = I\,X(e^{j\omega/I})$

c) $Y(e^{j\omega}) = X(e^{j\omega I})$ d) $Y(e^{j\omega}) = X(e^{j\omega/I})$

6. To eliminate multiple images at the output, during interpolation by I, the output is filtered to have a bandwidth of,

a) πI b) $\frac{\pi}{I}$ c) $\frac{I}{\pi}$ d) $\frac{\pi}{I^2}$

7. If A and B are integer sampling rate conversion factor for decimation and interpolation respectively, then sampling rate conversion factor for conversion by rational factor is,

a) $\frac{A}{B}$ b) $\frac{B}{A}$ c) $\frac{A^2}{B}$ d) $\frac{B}{A^2}$

8. In multistage decimation by D, where $D = D_1 D_2$, which of the following is correct implementation?

a) $x(n) \rightarrow \boxed{\downarrow D_1} \rightarrow \boxed{\uparrow D_2} \rightarrow y(n)$ **b)** $x(n) \rightarrow \boxed{\uparrow D_1} \rightarrow \boxed{\uparrow D_2} \rightarrow y(n)$

c) $x(n) \rightarrow \boxed{\downarrow D_1} \rightarrow \boxed{\downarrow D_2} \rightarrow y(n)$ **d)** $x(n) \rightarrow \boxed{\uparrow D_1} \rightarrow \boxed{\downarrow D_2} \rightarrow y(n)$

9. In multistage interpolation by I, where $I = I_1 I_2$ which of the following is correct implementatiion?

a) $x(n) \rightarrow \boxed{\uparrow I_1} \rightarrow \boxed{\uparrow I_2} \rightarrow y(n)$ **b)** $x(n) \rightarrow \boxed{\uparrow I_1} \rightarrow \boxed{\downarrow I_2} \rightarrow y(n)$

c) $x(n) \rightarrow \boxed{\downarrow I_1} \rightarrow \boxed{\uparrow I_2} \rightarrow y(n)$ **d)** $x(n) \rightarrow \boxed{\downarrow I_1} \rightarrow \boxed{\downarrow I_2} \rightarrow y(n)$

10. The polyphase decomposition of H(z) into L sections can be represented by the equation,

a) $H(z) = \sum_{m=1}^{L} z^{-m} E_m(z^L)$ **b)** $H(z) = \sum_{m=0}^{L-1} z^{-m} E_m(z^L)$

c) $H(z) = \sum_{m=1}^{L} z^{m} E_m(z^L)$ **d)** $H(z) = \sum_{m=1}^{L-1} z^{-m} E_m(z^L)$

Answers

1. d	4. b	7. b	10. b
2. a	5. c	8. c	
3. c	6. b	9. a	

IV. Answer the following questions

1. Write a detailed note on multirate DSP.
2. Explain the process of downsampling with an example.
3. Derive an expression for the spectrum of output signal of a decimator.
4. Discuss the concept of aliasing in the spectrum of output signal of a decimator with an example.
5. Explain the process of upsampling with an example.
6. Derive an expression for the spectrum of output signal of an interpolator.
7. Discuss the concept of imaging in the spectrum of output signal of an interpolator with an example.
8. Discuss the sampling rate conversion by a rational factor $\frac{I}{D}$.
9. Derive an expression for the spectrum of output signal of sampling rate convertor by a rational factor $\frac{I}{D}$.
10. Discuss the multistage implementation of sampling rate conversion.
11. Discuss the computationally efficient implementation of decimator in an FIR filter.
12. Discuss the computationally efficient implementation of interpolator in an FIR filter.
13. Write a detailed note on polyphase decomposition of filters.

14. Explain the process of polyphase decomposition of an FIR filter with an example.
15. Explain the process of polyphase decomposition of an IIR filter with an example.
16. Draw and explain the polyphase structure of a decimator.
17. Draw and explain the polyphase structure of an interpolator.
18. Explain the digital filter banks with suitable sketches.
19. Discuss the sub-band coding of speech signal with a suitable diagram.
20. Write a detailed note on QMF (Quadrature Mirror Filter) bank.

V. Solve the following problems

E9.1. ***Consider the discrete time signal, $x(n) = \{2, 4, 6, 8, 10, 12, 14, 16\}$***

Determine the downsampled version of the signals for the sampling rate reduction factor, a) $D = 2$ b) $D = 3$ c) $D = 4$.

E9.2. ***Consider the discrete time signal shown in fig E9.2. Sketch the downsampled version of the signals for the sampling rate reduction factor, a) $D = 3$ b) $D = 4$ c) $D = 5$.***

E9.3. ***Consider a spectrum of input signal $X(e^{jw})$ with a bandwidth of –p/5 to + p/5 as shown in fig E9.3. When the signal is downsampled by a factor D, sketch the spectrum of downsampled signal for sampling rate, reduction factor D = 3, 4 and 5.***

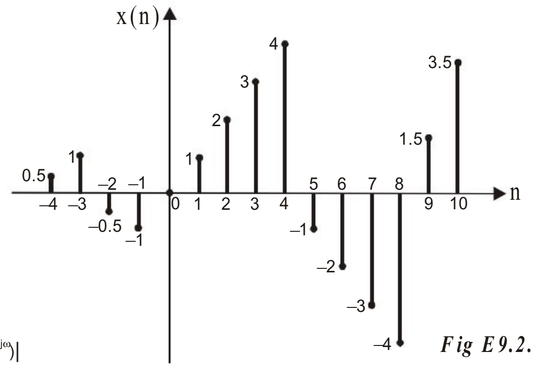


Fig E9.2.

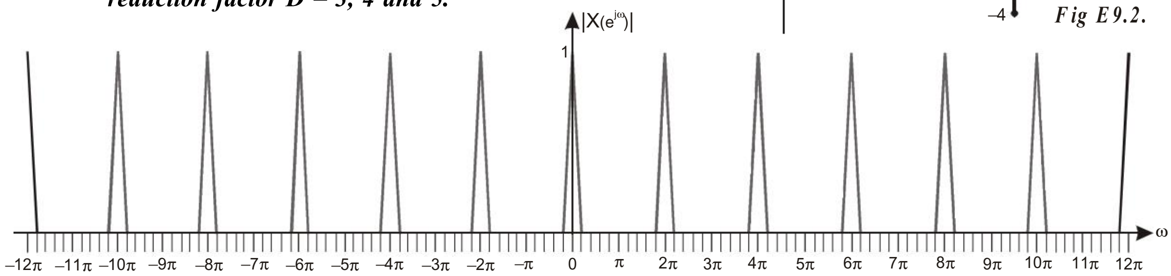


Fig E9.3.

E9.4. ***The spectrum of a discrete time signal is shown in fig E9.4. Draw the spectrum of the decimated signal, when the signal is decimated by D = 6.***

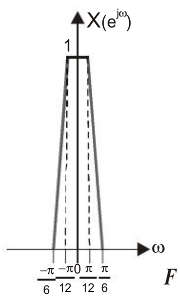


Fig E9.4.

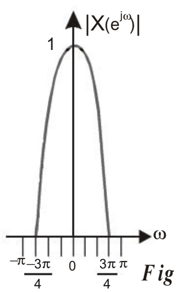


Fig E9.5.

E9.5. ***The spectrum of a discrete time signal is shown in fig E9.5. Draw the spectrum of the decimated signal, when the signal is decimated by D = 4.***

E9.6. ***The spectrum of a discrete time signal is shown in fig E9.6. Draw the spectrum of the decimated signal when the signal is decimated by 3.***

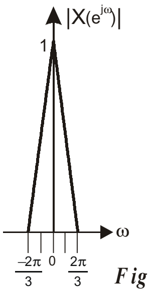


Fig E9.6.

E9.7. ***Consider the discrete time signal, x(n) = {1, 3, 5, 7, 9}***

Determine the upsampled version of the signals for the sampling rate multiplication factor, a) I = 2 b) I = 3 c) I = 4.

E9.8. ***Consider the discrete time signal shown in fig E9.8. Sketch the upsampled version of the signals for the sampling rate multiplication factor, a) I = 2 b) I = 3.***

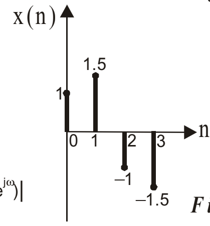


Fig E9.8.

E9.9 ***The spectrum of a discrete time signal is shown in fig E9.9. Draw the spectrum of the signal if it is upsampled by I = 2, 3 and 4.***

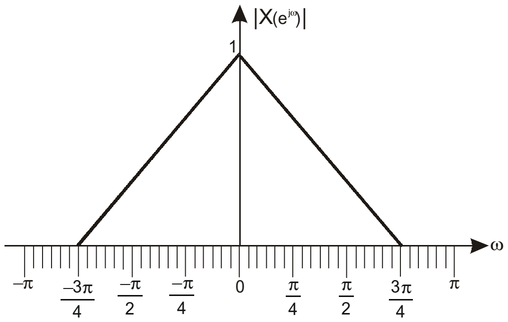


Fig E9.9.

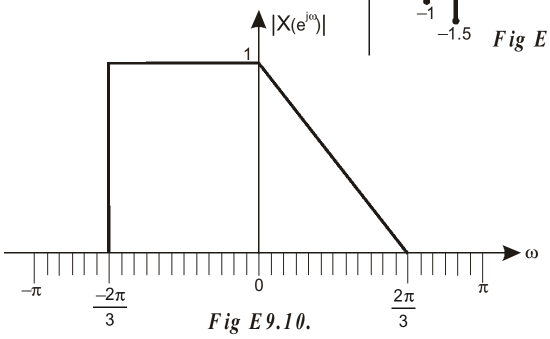


Fig E9.10.

E9.10 ***The spectrum of a discrete time signal is shown in fig E9.10. Draw the spectrum of the signal if it is upsampled by I = 2, 3 and 4.***

E9.11 ***For the multirate system shown in fig E9.11, determine y(n) as a function of x(n).***

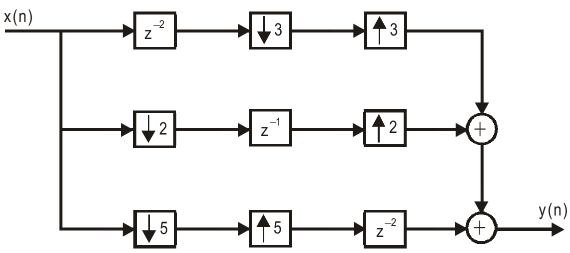


Fig E9.11.

E9.12 ***Determine the ouput y(n) in terms of input x(n) for the multirate system shown in fig E9.12.***

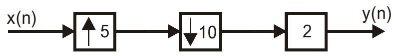


Fig E9.12.

E9.13 ***The transfer function of an FIR filter is,***

$$H(z) = 0.3 + 0.6\, z^{-1} + 0.7\, z^{-2} + 0.18\, z^{-3} + 0.85\, z^{-4} + 0.25\, z^{-5} + 0.28\, z^{-6} + 0.42\, z^{-7} + 0.89\, z^{-8}$$

Perform polyphase decomposition of H(z) to decompose into a) 2 sections b) 3 sections c) 4 sections.

E9.14 ***The transfer function of an IIR filter is,*** $H(z) = \dfrac{1 + 0.85z^{-1}}{1 - 0.65z^{-1}}$.

Perform polyphase decomposition of H(z) to decompose into a) 2 sections b) 4 sections.

E9.15 ***The transfer function of an IIR filter is,*** $H(z) = \dfrac{1 + 0.32z^{-1} + 0.58z^{-2}}{1 + 0.7z^{-1} + 0.4z^{-2}}$

Perform polyphase decomposition of H(z) to decompose into a) 2 sections b) 4 sections.

Answers

E9.1. ***a)*** $x(2n) = x_{D2}(n) = \{\underset{\uparrow}{2}, 6, 10, 14\}$

b) $x(3n) = x_{D3}(n) = \{\underset{\uparrow}{2}, 8, 14\}$

c) $x(4n) = x_{D4}(n) = \{\underset{\uparrow}{2}, 10\}$

E9.2. ***a)*** $x(3n) = x_{D3}(n) = \{1,\ \underset{\uparrow}{0},\ 3,\ -2,\ 1.5\}$

b) $x(4n) = \{0.5,\ \underset{\uparrow}{0},\ 4,\ -4\}$

c) $x(5n) = x_{D5}(n) = \{\underset{\uparrow}{0},\ -1,\ 3.5\}$

E9.3. ***Sampling rate reduction factor, D = 3***

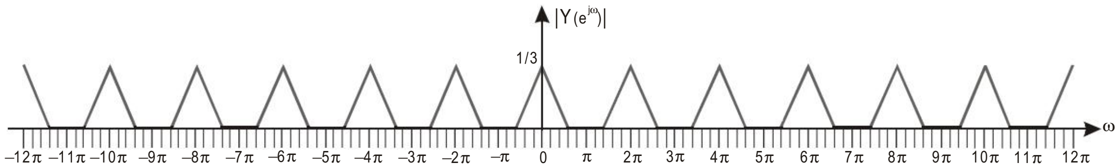


Fig E9.3.1.

Sampling rate reduction factor, D = 4

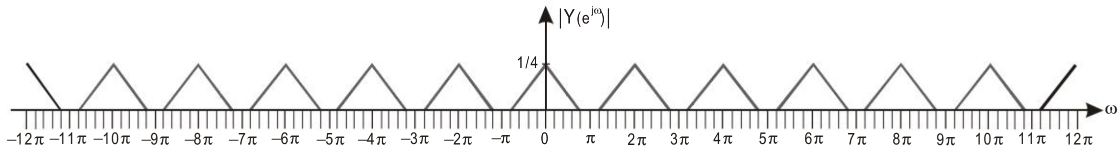


Fig E9.3.2.

Sampling rate reduction factor, D = 5

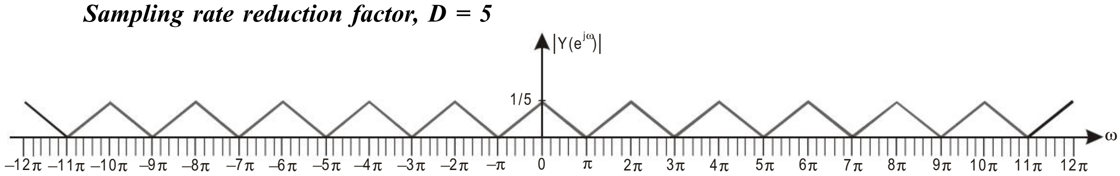


Fig E9.3.3.

E9.4.

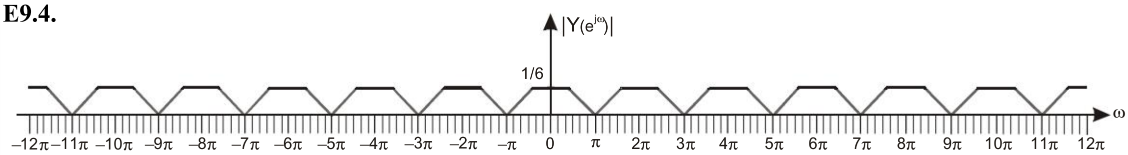


Fig E9.4.

E9.5.

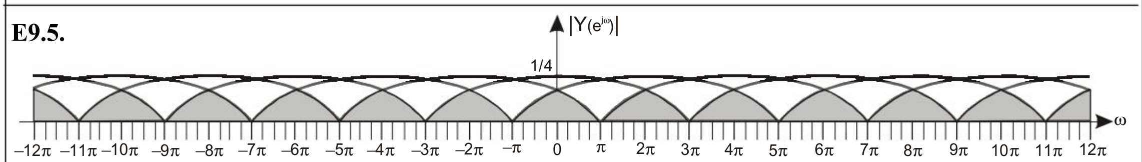


Fig E9.5.

E9.6.

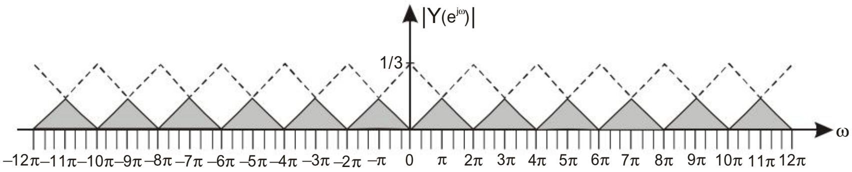


Fig E9.6.

E9.7. ***a)*** $x\left(\frac{n}{2}\right) = x_{I2}(n) = \{\underset{\uparrow}{1}, 0, 3, 0, 5, 0, 7, 0, 9, 0\}$

b) $x\left(\frac{n}{3}\right) = x_{I3}(n) = \{\underset{\uparrow}{1}, 0, 0, 3, 0, 0, 5, 0, 0, 7, 0, 0, 9, 0, 0\}$

c) $x\left(\frac{n}{4}\right) = x_{I4}(n) = \{\underset{\uparrow}{1}, 0, 0, 0, 3, 0, 0, 0, 5, 0, 0, 0, 7, 0, 0, 0, 9, 0, 0, 0\}$

E9.8. ***a)*** $x\left(\frac{n}{2}\right) = x_{I2}(n) = \{\underset{\uparrow}{1}, 0, 1.5, 0, -1, 0, -1.5, 0\}$

b) $x\left(\frac{n}{3}\right) = x_{I3}(n) = \{\underset{\uparrow}{1}, 0, 0, 1.5, 0, 0, -1, 0, 0, -1.5, 0, 0\}$

E9.9. ***Upsampling by I = 2***

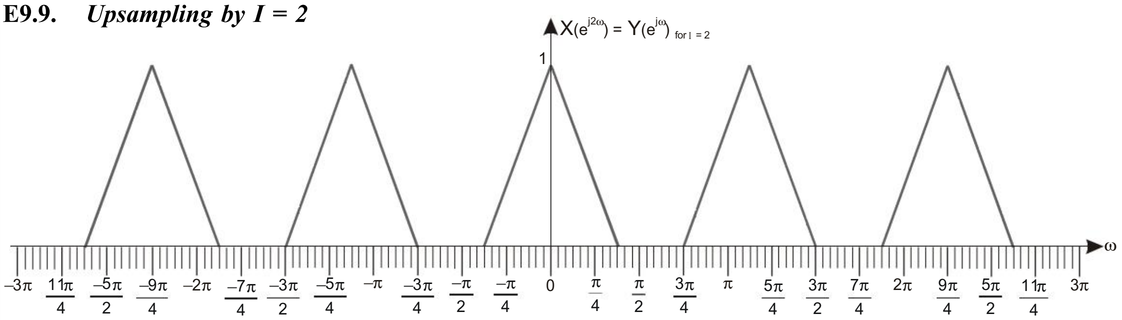


Fig E9.9.1.

Upsampling by I = 3

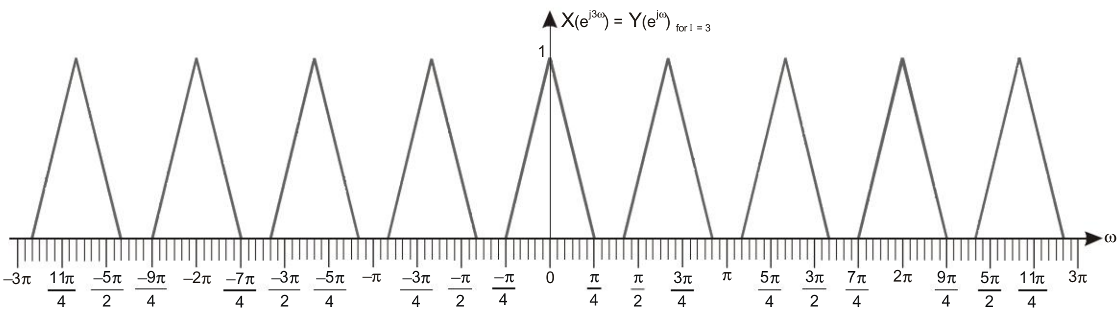


Fig E9.9.2.

Upsampling by I = 4

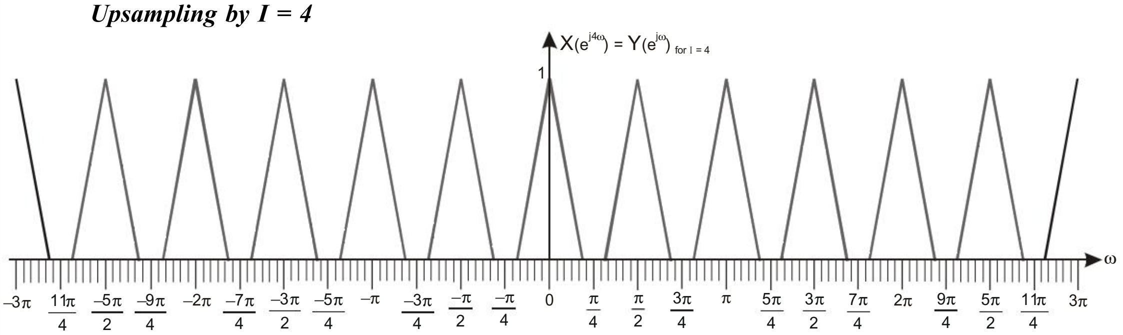


Fig E9.9.3.

E9.10. ***Upsampling by I = 2***

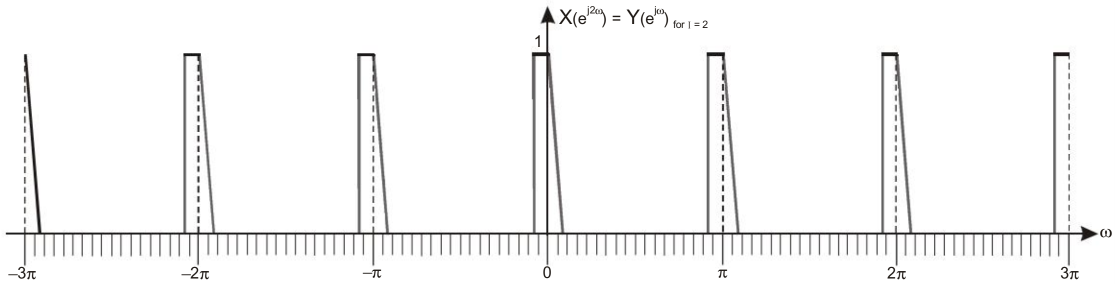


Fig E9.10.1.

Upsampling by I = 3

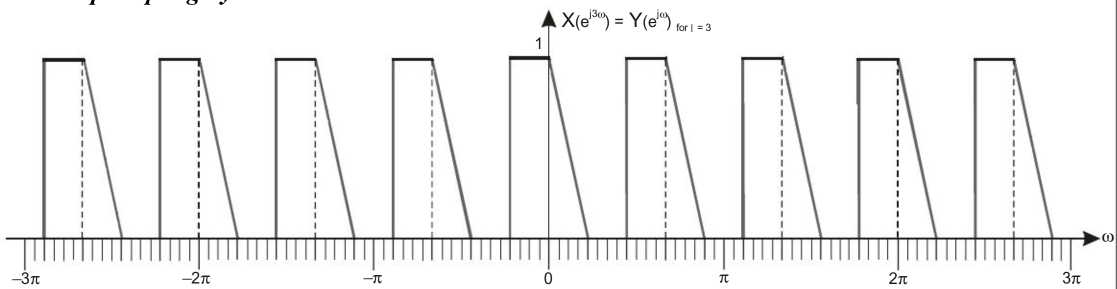


Fig E9.10.2.

Upsampling by I = 4

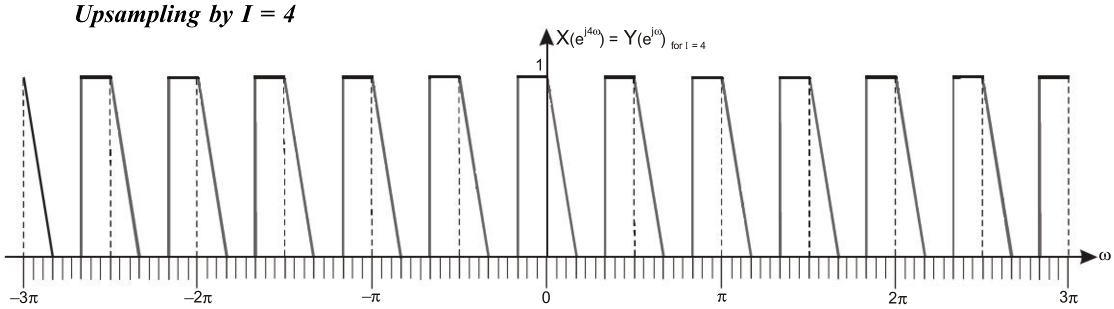


Fig E9.10.3.

E9.11. $y(n) = x(n-2)$

E9.12. $y(n) = \{....., \underset{\uparrow}{x(0)}, 0, x(2), 0, x(4), 0, x(6), ...\}$

E9.13. ***a)*** $H(z) = E_0(z^2) + z^{-1}\,E_1(z^2)$

where, $E_0(z^2) = 0.3 + 0.7z^{-2} + 0.85z^{-4} + 0.28z^{-6} + 0.89z^{-8}$

$E_1(z^2) = 0.6 + 0.18z^{-2} + 0.25z^{-4} + 0.42z^{-6}$

b) $H(z) = E_0(z^3) + z^{-1}\,E_1(z^3) + z^{-2}\,E_2(z^3)$

where, $E_0(z^3) = 0.3 + 0.18z^{-3} + 0.28z^{-6}$; $E_1(z^3) = 0.6 + 0.85z^{-3} + 0.42z^{-6}$;

$E_2(z^3) = 0.7 + 0.25z^{-3} + 0.89z^{-6}$

c) $H(z) = E_0(z^4) + z^{-1}\,E_1(z^4) + z^{-2}\,E_2(z^4) + z^{-3}\,E_3(z^4)$

where, $E_0(z^4) = 0.3 + 0.85z^{-4} + 0.89z^{-8}$; $E_1(z^4) = 0.6 + 0.25z^{-4}$

$E_2(z^4) = 0.7 + 0.28z^{-4}$; $E_3(z^4) = 0.18 + 0.42z^{-4}$

E9.14. ***a)*** $H(z) = \dfrac{1 + 1.5z^{-1} + 0.5525z^{-2}}{1 - 0.4225z^{-2}} = E_0(z^2) + z^{-1}\,E_1(z^2)$

where, $E_0(z^2) = \dfrac{1 + 0.5525z^{-2}}{1 - 0.4225z^{-2}}$; $E_1(z^2) = \dfrac{1.5}{1 - 0.4225z^{-2}}$

b) $H(z) = \dfrac{1 + 1.5z^{-1} + 0.975z^{-2} + 0.6338z^{-3} + 0.2334z^{-4}}{1 - 0.1785z^{-4}} = E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4)$

where, $E_0(z^4) = \dfrac{1 + 0.2334(z^4)^{-1}}{1 - 0.1785(z^4)^{-1}}$; $E_1(z^4) = \dfrac{1.5}{1 - 0.1785(z^4)^{-1}}$

$E_2(z^4) = \dfrac{0.975}{1 - 0.1785(z^4)^{-1}}$; $E_3(z^4) = \dfrac{0.6338}{1 - 0.1785(z^4)^{-1}}$

E9.15. ***a)*** $H(z) = \dfrac{1 - 0.38z^{-1} + 0.756z^{-2} - 0.2782z^{-3} + 0.232z^{-4}}{1 + 0.31z^{-2} + 0.16z^{-4}} = E_0(z^2) + z^{-1}\,E_1(z^2)$

where, $E_0(z^2) = \dfrac{1 + 0.756z^{-2} + 0.232z^{-4}}{1 + 0.31z^{-2} + 0.64z^{-4}}$; $E_1(z^2) = \dfrac{-0.38 - 0.278z^{-2}}{1 + 0.31z^{-2} + 0.16z^{-4}}$

b) $H(z) = \dfrac{1 - 0.38z^{-1} + 0.446z^{-2} - 0.1604z^{-3} + 0.1576z^{-4} + 0.0254z^{-5} + 0.0491z^{-6} - 0.0445z^{-7} + 0.0371z^{-8}}{1 + 0.2239z^{-4} + 0.0256z^{-8}}$

$= E_0(z^4) + z^{-1}\,E_1(z^4) + z^{-2}\,E_2(z^4) + z^{-3}\,E_3(z^4)$

where, $E_0(z^4) = \dfrac{1 + 0.1576(z^4)^{-1} + 0.0371(z^4)^{-2}}{1 + 0.2239(z^4)^{-1} + 0.0256(z^4)^{-2}}$; $E_1(z^4) = \dfrac{-0.38 + 0.0254(z^4)^{-1}}{1 + 0.2239(z^4)^{-1} + 0.0256(z^4)^{-2}}$

$E_2(z^4) = \dfrac{0.446 + 0.0491(z^4)^{-1}}{1 + 0.2239(z^4)^{-1} + 0.0256(z^4)^{-2}}$; $E_3(z^4) = \dfrac{-0.1604 - 0.0445(z^4)^{-1}}{1 + 0.2239(z^4)^{-1} + 0.0256(z^4)^{-2}}$

Solution for Exercise Problems

E9.1. ***Consider the discrete time signal, x(n) = {2, 4, 6, 8, 10, 12, 14, 16}***

Determine the downsampled version of the signals for the sampling rate reduction factor,

a) D = 2 b) D = 3 c) D = 4.

Solution

Given that,

$x(n) = \{2, 4, 6, 8, 10, 12, 14, 16\}$

$\therefore$ When n = 0, $x(n) = x(0) = 2$ | When n = 4, $x(n) = x(4) = 10$

When n = 1, $x(n) = x(1) = 4$ | When n = 5, $x(n) = x(5) = 12$

When n = 2, $x(n) = x(2) = 6$ | When n = 6, $x(n) = x(6) = 14$

When n = 3, $x(n) = x(3) = 8$ | When n = 7, $x(n) = x(7) = 16$

a) Sampling rate reduction factor, D = 2.

Now, $x(Dn) = x(2n)$ = Discrete time signal decimated by reduction factor 2.

Let, $x(2n) = x_{D2}(n)$

$\therefore$ When n = 0, $x_{D2}(n) = x_{D2}(0) = x(2 \times 0) = x(0) = 2$

When n = 1, $x_{D2}(n) = x_{D2}(1) = x(2 \times 1) = x(2) = 6$

When n = 2, $x_{D2}(n) = x_{D2}(2) = x(2 \times 2) = x(4) = 10$

When n = 3, $x_{D2}(n) = x_{D2}(3) = x(2 \times 3) = x(6) = 14$

$\therefore$ $x(2n) = x_{D2}(n) = \{2, 6, 10, 14\}$

b) Sampling rate reduction factor, D = 3.

Now, $x(Dn) = x(3n)$ = Discrete time signal decimated by reduction factor 3.

Let, $x(3n) = x_{D3}(n)$

$\therefore$ When n = 0, $x_{D3}(n) = x_{D3}(0) = x(3 \times 0) = x(0) = 2$

When n = 1, $x_{D3}(n) = x_{D3}(1) = x(3 \times 1) = x(3) = 8$

When n = 2, $x_{D3}(n) = x_{D3}(2) = x(3 \times 2) = x(6) = 14$

$\therefore$ $x(3n) = x_{D3}(n) = \{2, 8, 14\}$

c) Sampling rate reduction factor, D = 4.

Now, $x(Dn) = x(4n)$ = Discrete time signal decimated by reduction factor 4.

Let, $x(4n) = x_{D4}(n)$

$\therefore$ When n = 0, $x_{D4}(n) = x_{D4}(0) = x(4 \times 0) = x(0) = 2$

When n = 1, $x_{D4}(n) = x_{D4}(1) = x(4 \times 1) = x(4) = 10$

$\therefore$ $x(4n) = x_{D4}(n) = \{2, 10\}$

E9.2. ***Consider the discrete time signal shown in fig 1.***

Sketch the downsampled version of the signals for the sampling rate reduction factor, a) D = 3 b) D = 4 c) D = 5.

Solution

From fig 1, we can write the sample sequence

$x(n) = \{0.5, 1, -0.5, -1, 0, 1, 2, 3, 4, -1, -2, -3, -4, 1.5, 3.5\}$

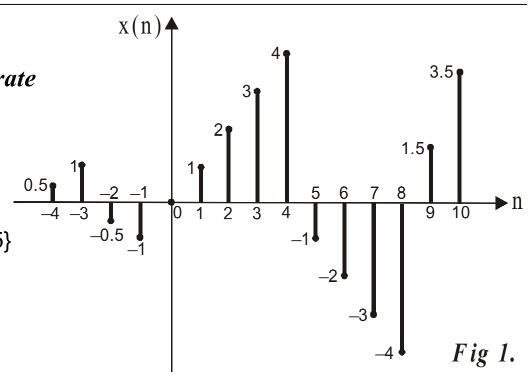


Fig 1.

$\therefore$ When n = –4, x(n) = x(–4) = 0.5	When n = 1, x(n) = x(1) = 1	When n = 6, x(n) = x(6) = –2
When n = –3, x(n) = x(–3) = 1	When n = 2, x(n) = x(2) = 2	When n = 7, x(n) = x(7) = –3
When n = –2, x(n) = x(–2) = –0.5	When n = 3, x(n) = x(3) = 3	When n = 8, x(n) = x(8) = –4
When n = –1, x(n) = x(–1) = –1	When n = 4, x(n) = x(4) = 4	When n = 9, x(n) = x(9) = 1.5
When n = 0, x(n) = x(0) = 0	When n = 5, x(n) = x(5) = –1	When n = 10, x(n) = x(10) = 3.5

a) Sampling rate reduction factor, D = 3

Let, $x_{D3}(n)$ = Discrete time signal decimated by reduction factor 3.

Now, $x_{D3}(n) = x(Dn) = x(3n)$

$\therefore$ When n = –1, $x_{D3}(n) = x_{D3}(-1) = x(3 \times -1) = x(-3) = 1$

When n = 0, $x_{D3}(n) = x_{D3}(0) = x(3 \times 0) = x(0) = 0$

When n = 1, $x_{D3}(n) = x_{D3}(1) = x(3 \times 1) = x(3) = 3$

When n = 2, $x_{D3}(n) = x_{D3}(2) = x(3 \times 2) = x(6) = -2$

When n = 3, $x_{D3}(n) = x_{D3}(3) = x(3 \times 3) = x(9) = 1.5$

$\therefore$ $x(3n) = x_{D3}(n) = \{1,\ \underset{\uparrow}{0},\ 3,\ -2,\ 1.5\}$ .....(1)

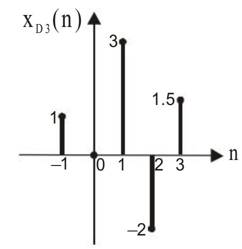


Fig 2 : x(n) decimated by 3.

Using equation (1), the decimated signal of x(n) by reduction factor 3, is drawn as shown in fig 2.

b) Sampling rate reduction factor, D = 4.

Let, $x_{D4}(n)$ = Discrete time signal decimated by reduction factor, 4.

Now, $x_{D4}(n) = x(Dn) = x(4n)$

$\therefore$ When n = –1, $x_{D4}(n) = x_{D4}(-1) = x(4 \times -1) = x(-4) = 0.5$

When n = 0, $x_{D4}(n) = x_{D4}(0) = x(4 \times 0) = x(0) = 0$

When n = 1, $x_{D4}(n) = x_{D4}(1) = x(4 \times 1) = x(4) = 4$

When n = 2, $x_{D4}(n) = x_{D4}(2) = x(4 \times 2) = x(8) = -4$

$\therefore$ $x(4n) = \{0.5,\ \underset{\uparrow}{0},\ 4,\ -4\}$ .....(2)

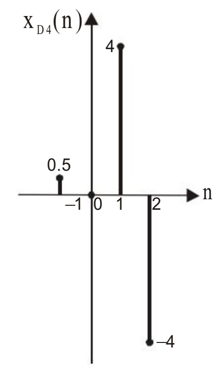


Fig 3 : x(n) decimated by 4.

Using equation (2), the decimated signal of x(n) by reduction factor 4, is drawn as shown in fig 3.

c) Sampling rate reduction factor, D = 5

Let, $x_{D5}(n)$ = Discrete time signal decimated by reduction factor, 5.

Now, $x_{D5}(n) = x(Dn) = x(5n)$

$\therefore$ When n = 0, $x_{D5}(n) = x_{D5}(0) = x(5 \times 0) = x(0) = 0$

When n = 1, $x_{D5}(n) = x_{D5}(1) = x(5 \times 1) = x(5) = -1$

When n = 2, $x_{D5}(n) = x_{D5}(2) = x(5 \times 2) = x(10) = 3.5$

$\therefore$ $x(5n) = x_{D5}(n) = \{\underset{\uparrow}{0},\ -1,\ 3.5\}$ .....(3)

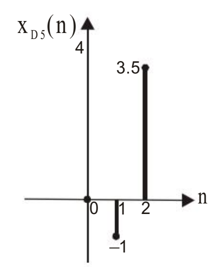


Fig 4 : x(n) decimated by 5.

Using equation (3), the decimated signal of x(n) by reduction factor 5, is drawn as shown in fig 4.

E9.3. Consider a spectrum of input signal $X(e^{j\omega})$ with a bandwidth of $-\pi/5$ to $+\pi/5$ as shown in fig 1. When the signal is downsampled by a factor D, sketch the spectrum of downsampled signal for sampling rate, reduction factor D = 3, 4 and 5.

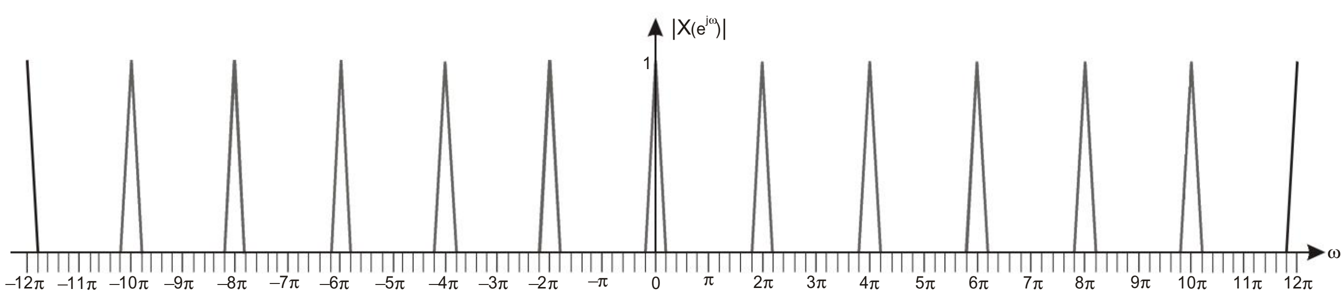


Fig 1.

Solution

Case (i) : Sampling rate reduction factor D = 3

Let, $Y(e^{jw})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 3 is given by,

$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j(\omega-2\pi k)/3}\right) \qquad \boxed{\text{Using equation (9.5).}}$$

$$= \frac{1}{3}X\left(e^{j\omega/3}\right) + \frac{1}{3}X\left(e^{j(\omega-2\pi)/3}\right) + \frac{1}{3}X\left(e^{j(\omega-4\pi)/3}\right)$$

From the above equation we can say that the spectrum of $Y(e^{jw})$ of decimated signal has three components.

The first component is frequency stretched version of input as shown in fig 2. The second component and third component is shifted version of first component, right shifted by 2π and 4π as shown in fig 3.

The frequency range of input spectrum is $-\pi/5$ to $+\pi/5$ and so its bandwidth is $2\pi/5$ [$\pi/5 - (-\pi/5) = 2\pi/5$]. It can be observed that this bandwidth $2\pi/5$ is stretched to $6\pi/5$ ($2\pi/5 \times 3 = 6\pi/5$) for decimation by 3 in each component of the spectrum of the decimated signal. Therefore, the frequency range of first component is stretched to $-3\pi/5$ to $+3\pi/5$. Also the magnitude of each component is scaled to 1/3 for decimation by 3.

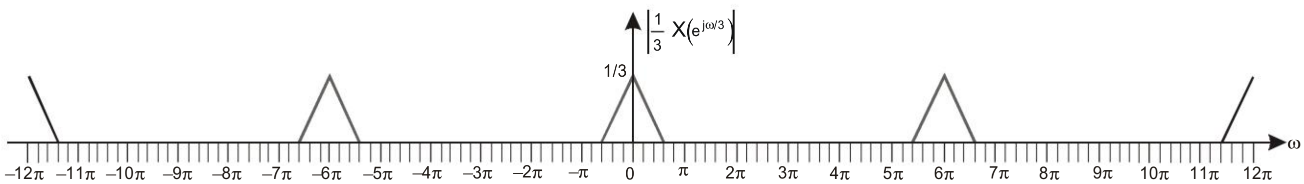


Fig 2.

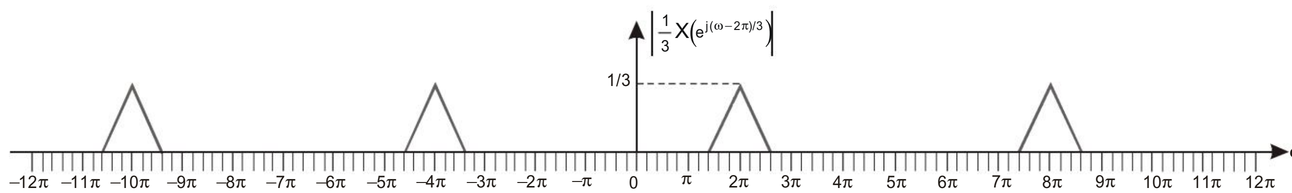


Fig 3.

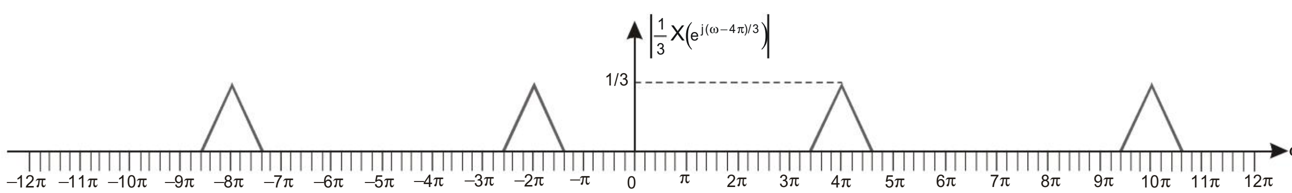


Fig 4.

The spectrum of decimated signal for decimation by 3 is shown in fig 5, which is obtained by adding the components of spectrum shown in fig 2, 3 and 4.

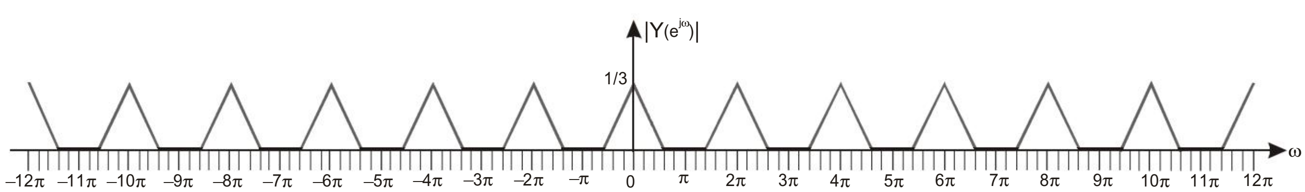


Fig 5.

Case (ii) : Sampling rate reduction factor, D = 4

Let, $Y(e^{jw})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 4 is given by,

$$Y(e^{j\omega}) = \frac{1}{4}\sum_{k=0}^{3} X\left(e^{j(\omega-2\pi k)/4}\right) \qquad \boxed{\text{Using equation (9.5).}}$$

$$= \frac{1}{4}X\left(e^{j\omega/4}\right) + \frac{1}{4}X\left(e^{j(\omega-2\pi)/4}\right) + \frac{1}{4}X\left(e^{j(\omega-4\pi)/4}\right) + \frac{1}{4}X\left(e^{j(\omega-6\pi)/4}\right)$$

From the above equation we can say that the spectrum $Y(e^{jw})$ of decimated signal has four components.

The first component is frequency stretched version of input as shown in fig 6. The second, third and fourth components are shifted version of first component, right shifted by 2π, 4π and 6π respectively as shown in fig 7, 8 and fig 9.

It can be observed that the bandwidth $2\pi/5$ of input spectrum is stretched to $8\pi/5$ ($2\pi/5 \times 4$) for decimation by 4 in each component of the spectrum of the decimated signal. Therefore, the frequency range of first component is stretched to $-4\pi/5$ to $+4\pi/5$. Also the magnitude of each component is scaled to 1/4 for decimation by 4.

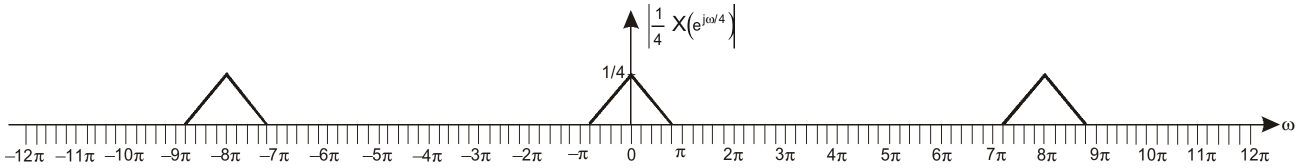


Fig 6.

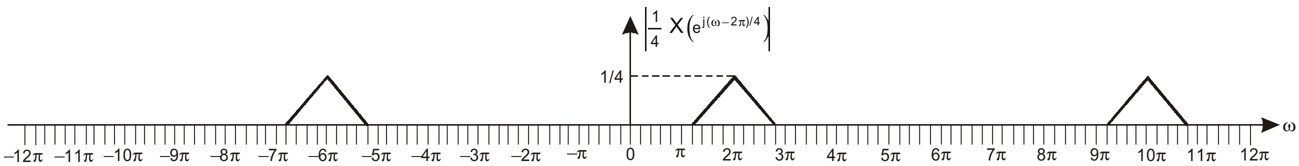


Fig 7.

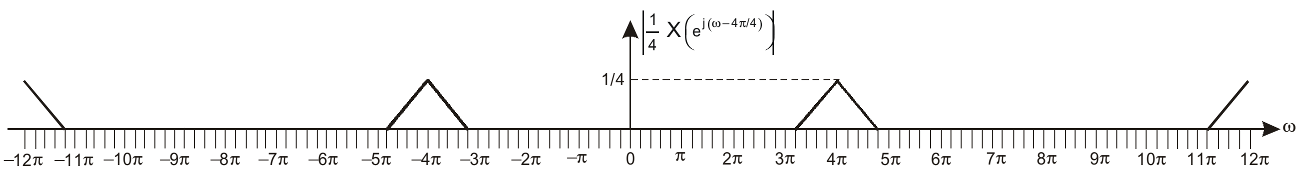


Fig 8.

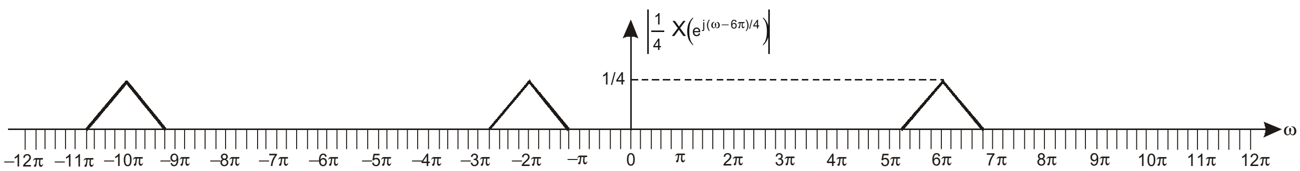


Fig 9.

The spectrum of decimated signal for decimation by 4, is shown in fig 9, which is obtained by adding the component of spectrum shown in fig 6 to 9.

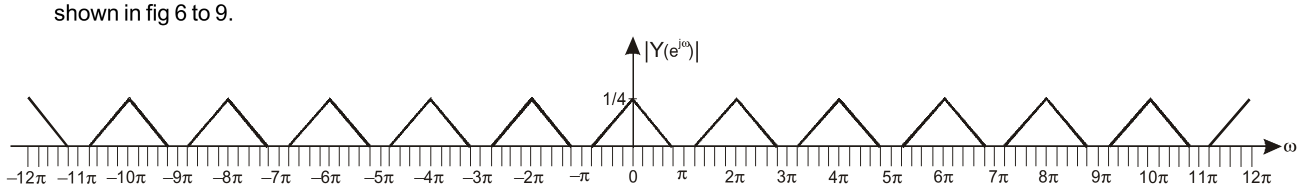


Fig 10.

***Case (iii) :* Sampling rate reduction factor, D = 5**

Let, $Y(e^{j\omega})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 5 is given by,

Using equation (9.5).

$$Y(e^{j\omega}) = \frac{1}{5}\sum_{k=0}^{4} X\left(e^{j(\omega-2\pi k)/5}\right)$$

$$= \frac{1}{5}X\left(e^{j\omega/5}\right) + \frac{1}{5}X\left(e^{j(\omega-2\pi)/5}\right) + \frac{1}{5}X\left(e^{j(\omega-4\pi)/5}\right) + \frac{1}{5}X\left(e^{j(\omega-6\pi)/5}\right) + \frac{1}{5}X\left(e^{j(\omega-8\pi/5)}\right)$$

From the above equation we can say that the spectrum $Y(e^{j\omega})$ of decimated signal has five components.

The first component is frequency stretched version of input as shown in fig 11. The second, third, fourth and fifth components are shifted version of first component, right shifted by 2π, 4π, 6π and 8π respectively as shown in fig 12, 13, 14 and 15.

It can be observed that the bandwidth $2\pi/5$ is stretched to 2π ($2\pi/5 \times 5 = 2\pi$) for decimation by 5 in each component of the spectrum of the decimated signal. Therefore, the frequency range of first component is stretched to $-\pi$ to π. Also, the magnitude of each component is scaled to 1/5 for decimation by 5.

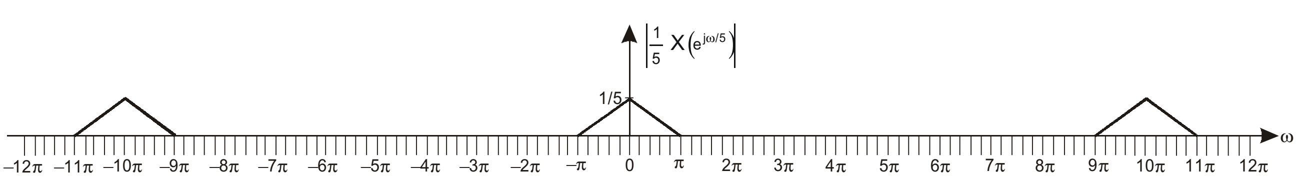


Fig 11.

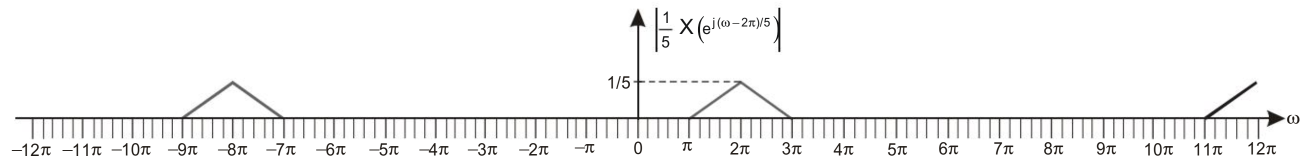


Fig 12.

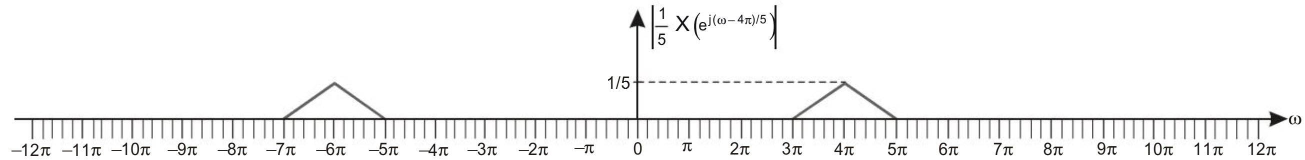


Fig 13.

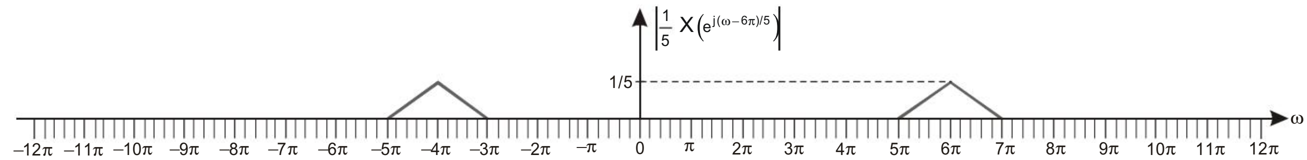


Fig 14.

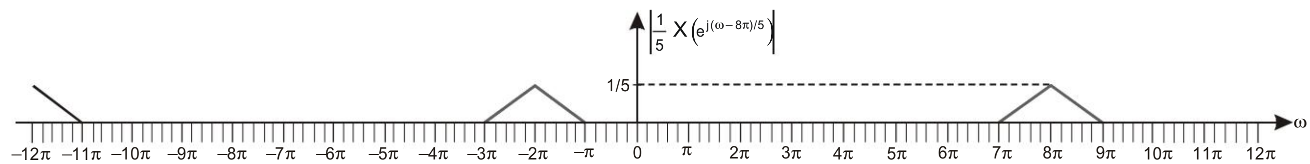


Fig 15.

The spectrum of decimated signal for decimation by 4, is shown in fig 16, which is obtained by adding the component of spectrum shown in fig 11 to 15.

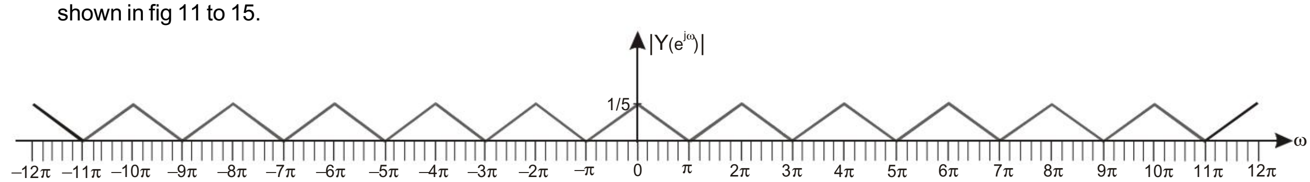


Fig 16.

Conclusion:

From the above three cases of decimation it is observed that, for decimation by a factor D,as long as the input spectrum is bandlimited to $\frac{\pi}{D}$ the spectrum of decimated signal does not overlaps. Hence we can say that, there is no aliasing in the spectrum of decimated signal if the spectrum of input signal is band limited to $\frac{\pi}{D}$.

E9.4. The spectrum of a discrete time signal is shown in fig 1. Draw the spectrum of the decimated signal, when the signal is decimated by D = 6.

Solution

The spectrum of a discrete time signal is periodic, with periodicity of 2p. Hence the spectrum of given signal can be drawn as shown in fig 2.

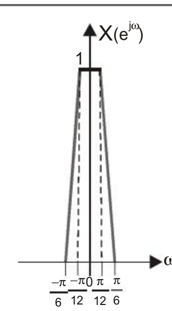


Fig 1.

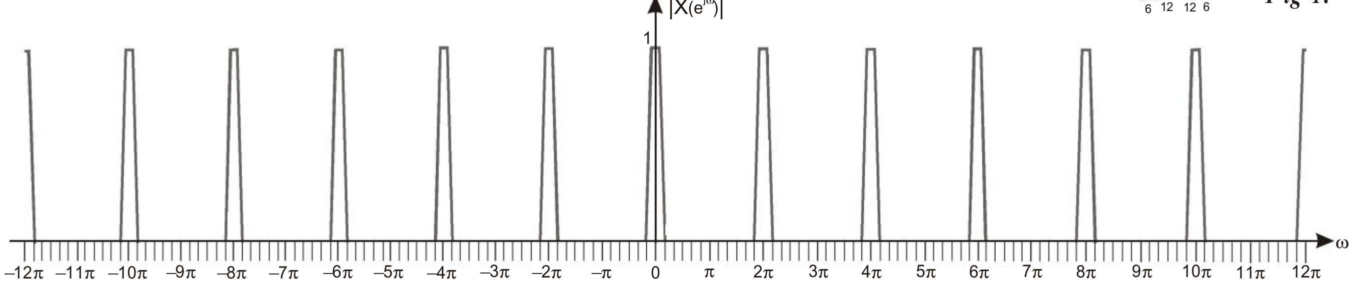


Fig 2.

Let, $Y(e^{jw})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 6 is given by,

$$Y(e^{j\omega}) = \frac{1}{6}\sum_{k=0}^{5} X\left(e^{j(\omega-2\pi k)/6}\right) \qquad \boxed{\text{Using equation (9.5).}}$$

$$= \frac{1}{6}X\left(e^{j\omega/6}\right) + \frac{1}{6}X\left(e^{j(\omega-2\pi)/6}\right) + \frac{1}{6}X\left(e^{j(\omega-4\pi)/6}\right) + \frac{1}{6}X\left(e^{j(\omega-6\pi)/6}\right) + \frac{1}{6}X\left(e^{j(\omega-8\pi)/6}\right) + \frac{1}{6}X\left(e^{j(\omega-10\pi)/6}\right)$$

From the above equation we can say that the spectrum $Y(e^{jw})$ of decimated signal has six components.

The first component is frequency stretched version of input as shown in fig 3. The second, third, fourth, fifth and sixth components are shifted version of first component, right shifted by 2p, 4p, 6p, 8p,and 10p respectively as shown in fig 4, 5, 6, 7 and 8.

The frequency range of input spectrum is –p/6 to +p/6 and so its bandwidth is 2p/6 [p/6 – (–p/6) = 2p/6]. It can be observed that this bandwidth 2p/6 is stretched to 2p (2p/6 ´ 6 = 2p) for decimation by 6 in each component of the spectrum of the decimated signal. Therefore, the frequency range of first component is stretched to –p to +p. Also the magnitude of each component is scaled to 1/6 for decimation by 6.

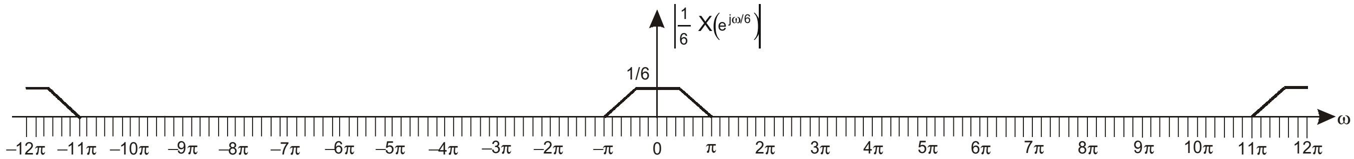


Fig 3.

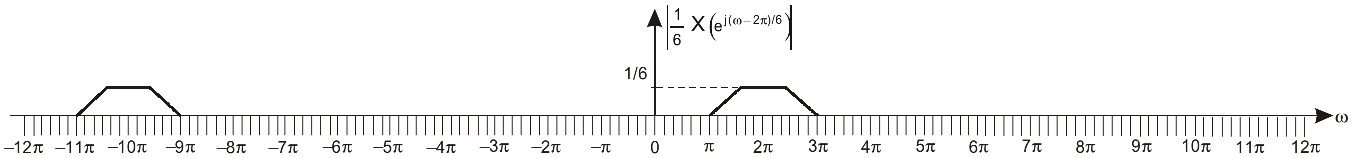


Fig 4.

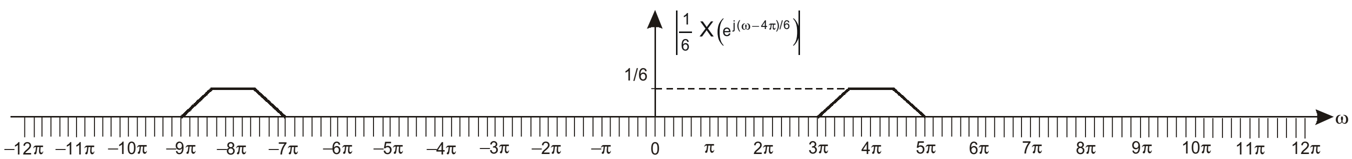


Fig 5.

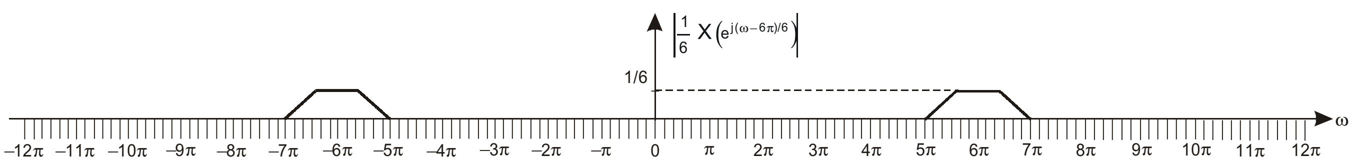


Fig 6.

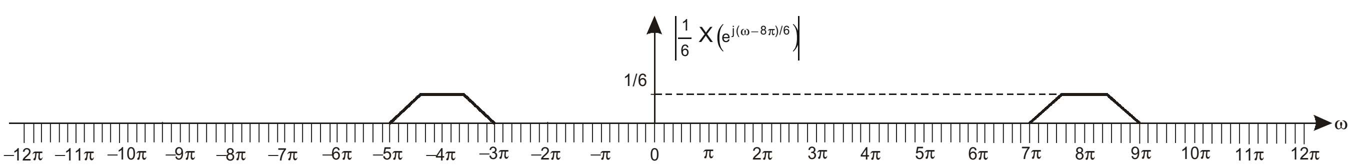


Fig 7.

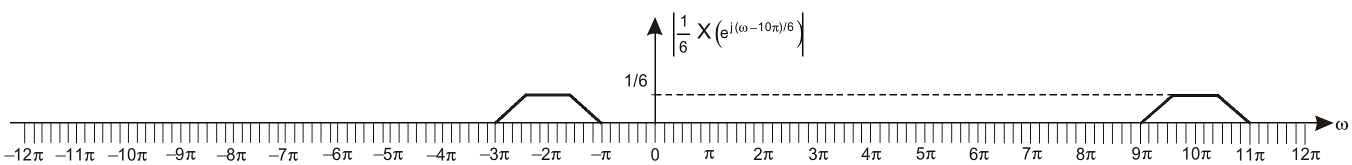


Fig 8.

The spectrum of decimated signal for decimation by 6 is shown in fig 9, which is obtained by adding the components of spectrum shown in fig 3 to 8.

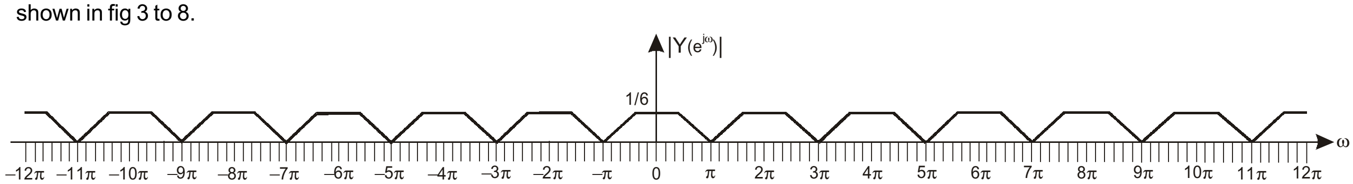


Fig 9.

E9.5 The spectrum of a discrete time signal is shown in fig 1. Draw the spectrum of the decimated signal, when the signal is decimated by D = 4.

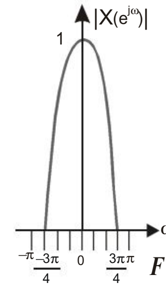


Fig 1.

Using equation (9.5).

Solution

The spectrum of a discrete time signal is periodic, with periodcity of 2p. Hence the spectrum of given signal can be drawn as shown in fig 2.

Let, $Y(e^{jw})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 4 is given by,

$$Y(e^{j\omega}) = \frac{1}{4}\sum_{k=0}^{3} X\left(e^{j(\omega-2\pi k)/4}\right)$$

$$= \frac{1}{4}X\left(e^{j\omega/4}\right) + \frac{1}{4}X\left(e^{j(\omega-2\pi)/4}\right) + \frac{1}{4}X\left(e^{j(\omega-4\pi)/4}\right) + \frac{1}{4}X\left(e^{j(\omega-6\pi)/4}\right)$$

From the above equation we can say that the spectrum $Y(e^{jw})$ of decimated signal has four components.

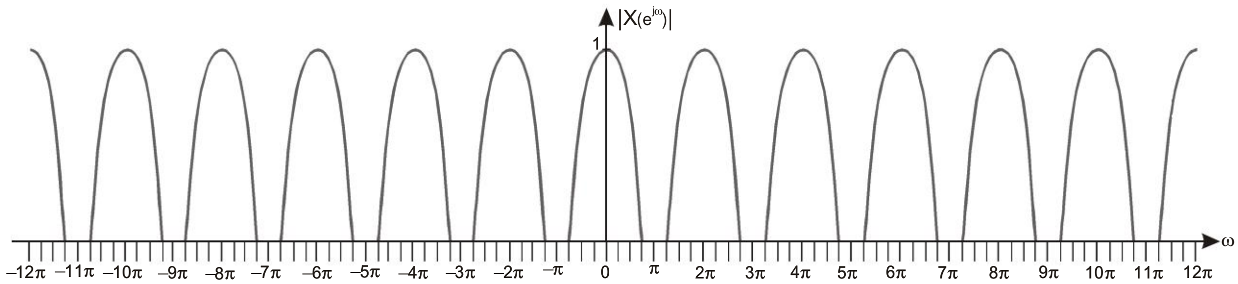


Fig 2.

The first component is frequency stretched version of input as shown in fig 3. The second, third and fourth components are frequency shifted version of first component, right shifted by 2p, 4p and 6p respectively as shown in fig 4, 5 and 6.

The frequency range of input spectrum is –3p/4 to +3p/4 and so its bandwidth is 3p/2 [3p/4 – (–3p/4) = 6p/4 = 3p/2]. It can be observed that this bandwidth 3p/2 is stretched to 6p (3p/2 ´ 4 = 6p) for decimation by 4 in each component of the spectrum of the decimated signal. Therefore, the frequency range of first component is stretched to –3p to +3p. Also, the magnitude of each component is scaled to 1/4 for decimation by 4.

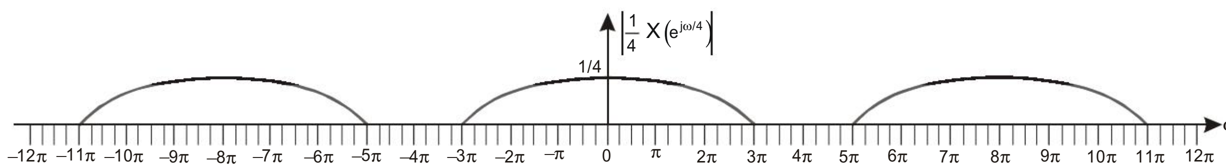


Fig 3.

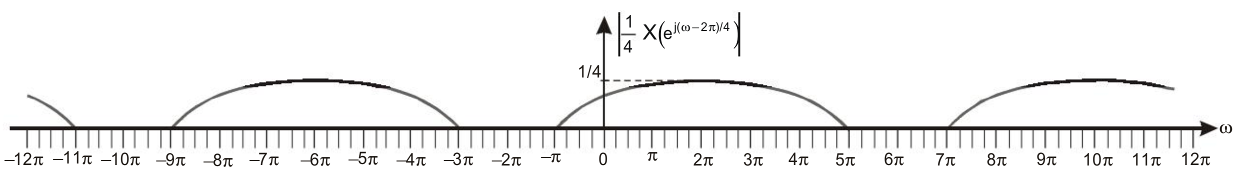


Fig 4.

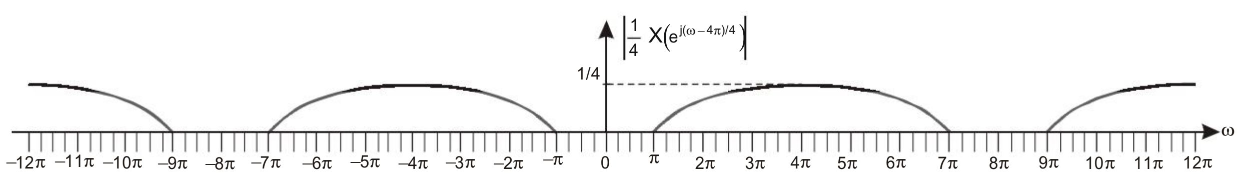


Fig 5.

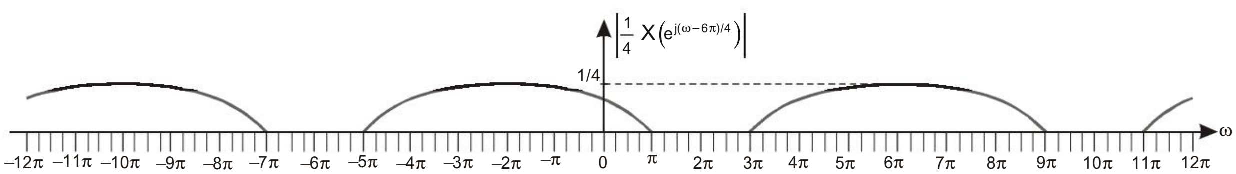


Fig 6.

The spectrum of decimated signal for decimation by 4 is shown in fig 7, which is obtained by adding the components of spectrum shown in fig 3 to 6.

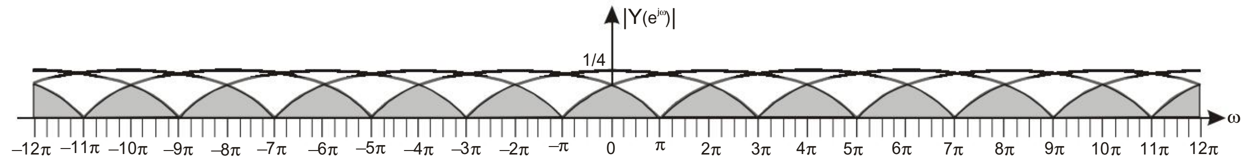


Fig 7.

E9.6 The spectrum of a discrete time signal is shown in fig 1. Draw the spectrum of the decimated signal when the signal is decimated by 3.

Fig 1.

Solution

The spectrum of a discrete time signal is periodic, with periodcity of 2π. Hence the spectrum of given signal can be drawn as shown in fig 2.

Let, $Y(e^{j\omega})$ = Spectrum of decimated signal.

The spectrum of decimated signal for D = 3 is given by,

$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j(\omega-2\pi k)/3}\right)$$

Using equation (9.5).

$$= \frac{1}{3}X\left(e^{j\omega/3}\right) + \frac{1}{3}X\left(e^{j(\omega-2\pi)/3}\right) + \frac{1}{3}X\left(e^{j(\omega-4\pi)/3}\right)$$

From the above equation we can say that the spectrum $Y(e^{j\omega})$ of decimated signal has four components.

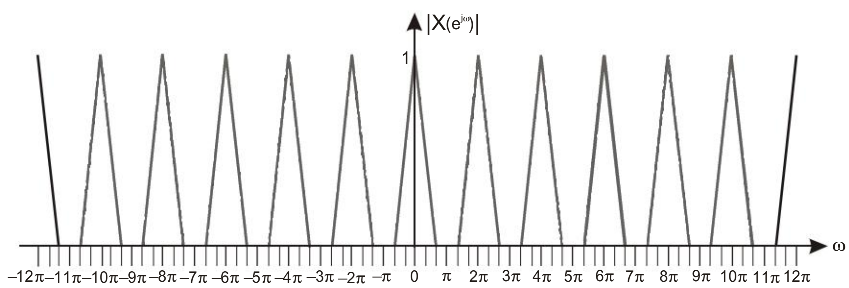


Fig 2.

The first component is frequency stretched version of input as shown in fig 3. The second and third components are shifted version of first component, right shifted by 2π, and 4π respectively as shown in fig 4 and 5.

The frequency range of input spectrum is $-2\pi/3$ to $+2\pi/3$ and so its bandwidth is $4\pi/3$ [$2\pi/3 - (-2\pi/3) = 4\pi/3$]. It can be observed that this bandwidth is stretched to 4π ($4\pi/3 \times 3 = 4\pi$) for decimation by 3 in each component of the spectrum of the decimated signal. Therefore, the frequency range of first component is stretched to -2π to $+2\pi$. Also, the magnitude of each component is scaled to 1/3 for decimation by 3.

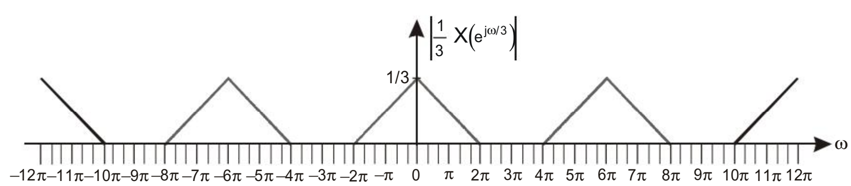


Fig 3.

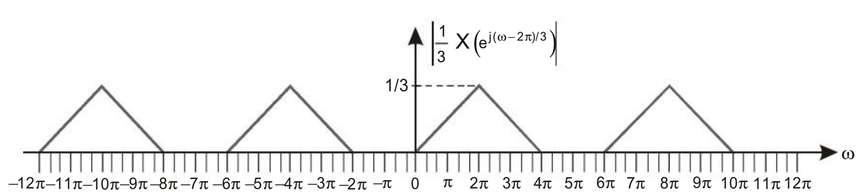


Fig 4.

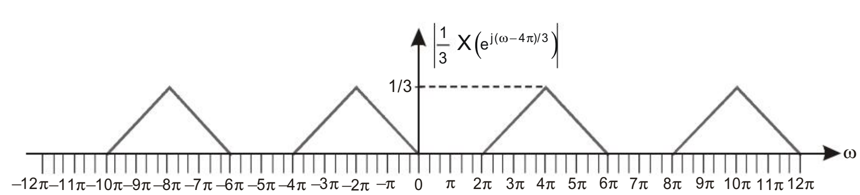


Fig 5.

The spectrum of decimated signal for decimation by 3 is shown in fig 6, which is obtained by adding the components of spectrum shown in fig 3, 4 and 5.

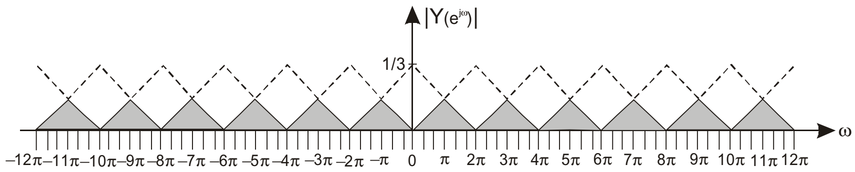


Fig 6.

E9.7 Consider the discrete time signal, x(n) = {1, 3, 5, 7, 9}

Determine the upsampled version of the signals for the sampling rate multiplication factor,

a) I = 2 b) I = 3 c) I = 4

Solution

Given that, $x(n) = \{\underset{\uparrow}{1}, 3, 5, 7, 9\}$

\ When n = 0, x(n) = x(0) = 1
When n = 1, x(n) = x(1) = 3
When n = 2, x(n) = x(2) = 5
When n = 3, x(n) = x(3) = 7
When n = 4, x(n) = x(4) = 9

a) Sampling rate multiplication factor, I = 2.

Now, $x\left(\frac{n}{I}\right) = x\left(\frac{n}{2}\right)$ = Discrete time signal interpolated by multiplication factor 2.

Let, $x\left(\frac{n}{2}\right) = x_{I2}(n)$

\ When n = 0, $x_{I2}(n) = x_{I2}(0) = x\left(\frac{0}{2}\right) = x(0) = 1$

When n = 1, $x_{I2}(n) = x_{I2}(1) = x\left(\frac{1}{2}\right) = x(0.5) = 0$

When n = 2, $x_{I2}(n) = x_{I2}(2) = x\left(\frac{2}{2}\right) = x(1) = 3$

When n = 3, $x_{I2}(n) = x_{I2}(3) = x\left(\frac{3}{2}\right) = x(1.5) = 0$

When n = 4, $x_{I2}(n) = x_{I2}(4) = x\left(\frac{4}{2}\right) = x(2) = 5$

When n = 5, $x_{I2}(n) = x_{I2}(5) = x\left(\frac{5}{2}\right) = x(2.5) = 0$

When n = 6, $x_{I2}(n) = x_{I2}(6) = x\left(\frac{6}{2}\right) = x(3) = 7$

When n = 7, $x_{I2}(n) = x_{I2}(7) = x\left(\frac{7}{2}\right) = x(3.5) = 0$

When n = 8, $x_{I2}(n) = x_{I2}(8) = x\left(\frac{8}{2}\right) = x(4) = 9$

When n = 9, $x_{I2}(n) = x_{I2}(9) = x\left(\frac{9}{2}\right) = x(4.5) = 0$

$\therefore\ x\left(\frac{n}{2}\right) = x_{I2}(n) = \{\underset{\uparrow}{1}, 0, 3, 0, 5, 0, 7, 0, 9, 0\}$

> **Note :** Discrete time signals are defined only for integer values of n. Therefore, the value of discrete time signal for noninteger value of n will be zero.

b) Sampling rate multiplication factor, I = 3.

Now, $x\left(\frac{n}{I}\right) = x\left(\frac{n}{3}\right)$ = Discrete time signal interpolated by multiplication factor 3.

Let, $x\left(\frac{n}{3}\right) = x_{I3}(n)$

\ When n = 0, $x_{I3}(n) = x_{I3}(0) = x\left(\frac{0}{3}\right) = x(0) = 1$

When n = 1, $x_{I3}(n) = x_{I3}(1) = x\left(\frac{1}{3}\right) = x(0.3) = 0$

When n = 2, $x_{I3}(n) = x_{I3}(2) = x\left(\frac{2}{3}\right) = x(0.7) = 0$

When n = 3, $x_{I3}(n) = x_{I3}(3) = x\left(\frac{3}{3}\right) = x(1) = 3$

When n = 4, $x_{I3}(n) = x_{I3}(4) = x\left(\frac{4}{3}\right) = x(1.3) = 0$

When n = 5, $x_{I3}(n) = x_{I3}(5) = x\left(\frac{5}{3}\right) = x(1.7) = 0$

When n = 6, $x_{I3}(n) = x_{I3}(6) = x\left(\frac{6}{3}\right) = x(2) = 5$

When n = 7, $x_{I3}(n) = x_{I3}(7) = x\left(\frac{7}{3}\right) = x(2.3) = 0$

When n = 8, $x_{I3}(n) = x_{I3}(8) = x\left(\frac{8}{3}\right) = x(2.7) = 0$

When n = 9, $x_{I3}(n) = x_{I3}(9) = x\left(\frac{9}{3}\right) = x(3) = 7$

When n = 10, $x_{I3}(n) = x_{I3}(10) = x\left(\frac{10}{3}\right) = x(3.3) = 0$

When n = 11, $x_{I3}(n) = x_{I3}(11) = x\left(\frac{11}{3}\right) = x(3.7) = 0$

When n = 12, $x_{I3}(n) = x_{I3}(12) = x\left(\frac{12}{3}\right) = x(4) = 9$

When n = 13, $x_{I3}(n) = x_{I3}(13) = x\left(\frac{13}{3}\right) = x(4.3) = 0$

When n = 14, $x_{I3}(n) = x_{I3}(14) = x\left(\frac{14}{3}\right) = x(4.7) = 0$

$\therefore\ x\left(\frac{n}{3}\right) = x_{I3}(n) = \{\underset{\uparrow}{1}, 0, 0, 3, 0, 0, 5, 0, 0, 7, 0, 0, 9, 0, 0\}$

c) Sampling rate multiplication factor, I = 4.

Now, $x\left(\frac{n}{I}\right) = x\left(\frac{n}{4}\right)$ = Discrete time signal interpolated by multiplication factor 4.

Let, $x\left(\frac{n}{4}\right) = x_{I4}(n)$

\ When n = 0, $x_{I4}(n) = x_{I4}(0) = x\left(\frac{0}{4}\right) = x(0) = 1$

When n = 1, $x_{I4}(n) = x_{I4}(1) = x\left(\frac{1}{4}\right) = x(0.25) = 0$

When n = 2, $x_{I4}(n) = x_{I4}(2) = x\left(\frac{2}{4}\right) = x(0.5) = 0$

When n = 3, $x_{I4}(n) = x_{I4}(3) = x\left(\frac{3}{4}\right) = x(0.75) = 0$

When n = 4, $x_{I4}(n) = x_{I4}(4) = x\left(\frac{4}{4}\right) = x(1) = 3$

When n = 5, $x_{I4}(n) = x_{I4}(5) = x\left(\frac{5}{4}\right) = x(1.25) = 0$

When n = 6, $x_{I4}(n) = x_{I4}(6) = x\left(\frac{6}{4}\right) = x(1.5) = 0$

When n = 7, $x_{I4}(n) = x_{I4}(7) = x\left(\frac{7}{4}\right) = x(1.75) = 0$

When n = 8, $x_{I4}(n) = x_{I4}(8) = x\left(\frac{8}{4}\right) = x(2) = 5$

When n = 9, $x_{I4}(n) = x_{I4}(9) = x\left(\frac{9}{4}\right) = x(2.25) = 0$

When n = 10, $x_{I4}(n) = x_{I4}(10) = x\left(\frac{10}{4}\right) = x(2.5) = 0$

When n = 11, $x_{I4}(n) = x_{I4}(11) = x\left(\frac{11}{4}\right) = x(2.75) = 0$

When n = 12, $x_{I4}(n) = x_{I4}(12) = x\left(\frac{12}{4}\right) = x(3) = 7$

When n = 13, $x_{I4}(n) = x_{I4}(13) = x\left(\frac{13}{4}\right) = x(3.25) = 0$

When n = 14, $x_{I4}(n) = x_{I4}(14) = x\left(\frac{14}{4}\right) = x(3.5) = 0$

When n = 15, $x_{I4}(n) = x_{I4}(15) = x\left(\frac{15}{4}\right) = x(3.75) = 0$

When n = 16, $x_{I4}(n) = x_{I4}(16) = x\left(\frac{16}{4}\right) = x(4) = 9$

When n = 17, $x_{I4}(n) = x_{I4}(17) = x\left(\frac{17}{4}\right) = x(4.75) = 0$

When n = 18, $x_{I4}(n) = x_{I4}(18) = x\left(\frac{18}{4}\right) = x(4.5) = 0$

When n = 19, $x_{I4}(n) = x_{I4}(19) = x\left(\frac{19}{4}\right) = x(4.75) = 0$

$$\therefore\ x\left(\frac{n}{4}\right) = x_{I4}(n) = \{\underset{\uparrow}{1},\ 0,\ 0,\ 0,\ 3,\ 0,\ 0,\ 0,\ 5,\ 0,\ 0,\ 0,\ 7,\ 0,\ 0,\ 0,\ 9,\ 0,\ 0,\ 0\}$$

E9.8 Consider the discrete time signal shown in fig 1. Sketch the upsampled version of the signals for the sampling rate multiplication factor, a) I = 2 b) I = 3.

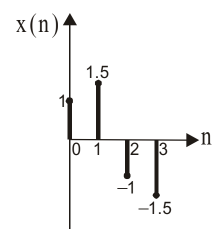


Fig 1.

Solution

From fig 1, we can write the above sample sequence as shown below.

$$x(n) = \{1,\ 1.5,\ -1,\ -1.5\}$$

\ When n = 0, $x(n) = x(0) = 1$

When n = 1, $x(n) = x(1) = 1.5$

When n = 2, $x(n) = x(2) = -1$

When n = 3, $x(n) = x(3) = -1.5$

a) Sampling rate multiplication factor, I = 2.

Let, $x_{I2}(n)$ = Discrete time signal interpolated by multiplication factor 2.

Now, $x_{I2}(n) = x\left(\frac{n}{I}\right) = x\left(\frac{n}{2}\right)$

\ When n = 0, $x_{I2}(n) = x_{I2}(0) = x\left(\frac{0}{2}\right) = x(0) = 1$

When n = 1, $x_{I2}(n) = x_{I2}(1) = x\left(\frac{1}{2}\right) = x(0.5) = 0$

When n = 2, $x_{I2}(n) = x_{I2}(2) = x\left(\frac{2}{2}\right) = x(1) = 1.5$

When n = 3, $x_{I2}(n) = x_{I2}(3) = x\left(\frac{3}{2}\right) = x(1.5) = 0$

When n = 4, $x_{I2}(n) = x_{I2}(4) = x\left(\frac{4}{2}\right) = x(2) = -1$

When n = 5, $x_{I2}(n) = x_{I2}(5) = x\left(\frac{5}{2}\right) = x(2.5) = 0$

When n = 6, $x_{I2}(n) = x_{I2}(6) = x\left(\frac{6}{2}\right) = x(3) = -1.5$

When n = 7, $x_{I2}(n) = x_{I2}(7) = x\left(\frac{7}{2}\right) = x(3.5) = 0$

$$\therefore\ x\left(\frac{n}{2}\right) = x_{I2}(n) = \{\underset{\uparrow}{1},\ 0,\ 1.5,\ 0,\ -1,\ 0,\ -1.5,\ 0\} \quad .....(1)$$

Note : Discrete time signals are defined only for integer values of n. Therefore, the value of discrete time signal for noninteger value of n will be zero.

Using equation (1), the interpolated signal of x(n) by multiplication factor 2, is drawn as shown in fig 2.

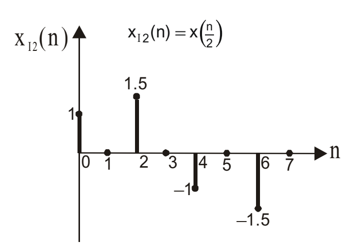


***Fig 2 :** x(n) interpolated by 2.*

b) Sampling rate multiplication factor, I = 3.

Let, $x_{I3}(n)$ = Discrete time signal interpolated by multiplication factor 3.

Now, $x_{I3}(n) = x\left(\frac{n}{I}\right) = x\left(\frac{n}{3}\right)$

\ When n = 0, $x_{I3}(n) = x_{I3}(0) = x\left(\frac{0}{3}\right) = x(0) = 1$

When n = 1, $x_{I3}(n) = x_{I3}(1) = x\left(\frac{1}{3}\right) = x(0.3) = 0$

When n = 2, $x_{I3}(n) = x_{I3}(2) = x\left(\frac{2}{3}\right) = x(0.7) = 0$

When n = 3, $x_{I3}(n) = x_{I3}(3) = x\left(\frac{3}{3}\right) = x(1) = 1.5$

When n = 4, $x_{I3}(n) = x_{I3}(4) = x\left(\frac{4}{3}\right) = x(1.3) = 0$

When n = 5, $x_{I3}(n) = x_{I3}(5) = x\left(\frac{5}{3}\right) = x(1.7) = 0$

When n = 6, $x_{I3}(n) = x_{I3}(6) = x\left(\frac{6}{3}\right) = x(2) = -1$

When n = 7, $x_{I3}(n) = x_{I3}(7) = x\left(\frac{7}{3}\right) = x(2.3) = 0$

When n = 8, $x_{I3}(n) = x_{I3}(8) = x\left(\frac{8}{3}\right) = x(2.7) = 0$

When n = 9, $x_{I3}(n) = x_{I3}(9) = x\left(\frac{9}{3}\right) = x(3) = -1.5$

When n = 10, $x_{I3}(n) = x_{I3}(10) = x\left(\frac{10}{3}\right) = x(3.3) = 0$

When n = 11, $x_{I3}(n) = x_{I3}(11) = x\left(\frac{11}{3}\right) = x(3.7) = 0$

$$\therefore\ x\left(\frac{n}{3}\right) = x_{I3}(n) = \{\underset{\uparrow}{1},\ 0,\ 0,\ 1.5,\ 0,\ 0,\ -1,\ 0,\ 0,\ -1.5,\ 0,\ 0\} \quad .....(2)$$

Using equation (2), the interpolated signal of x(n) by multiplication factor 3, is drawn as shown in fig 3.

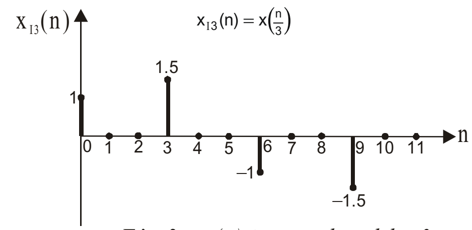


***Fig 3 :** x(n) interpolated by 3.*

E9.9 ***The spectrum of a discrete time signal is shown in fig 1. Draw the spectrum of the signal if it is upsampled by I = 2, 3 and 4.***

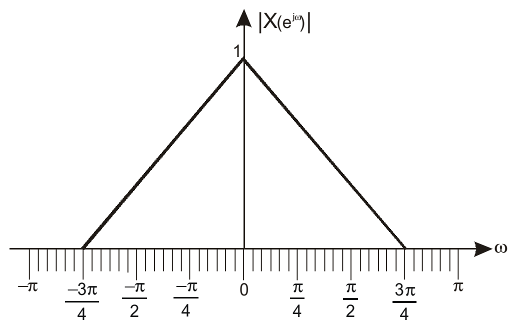


Fig 1.

Solution

Since, the frequency spectrum of a discrete time signal is periodic with periodicity of 2p, the spectrum of given signal can be drawn as shown in fig 2.

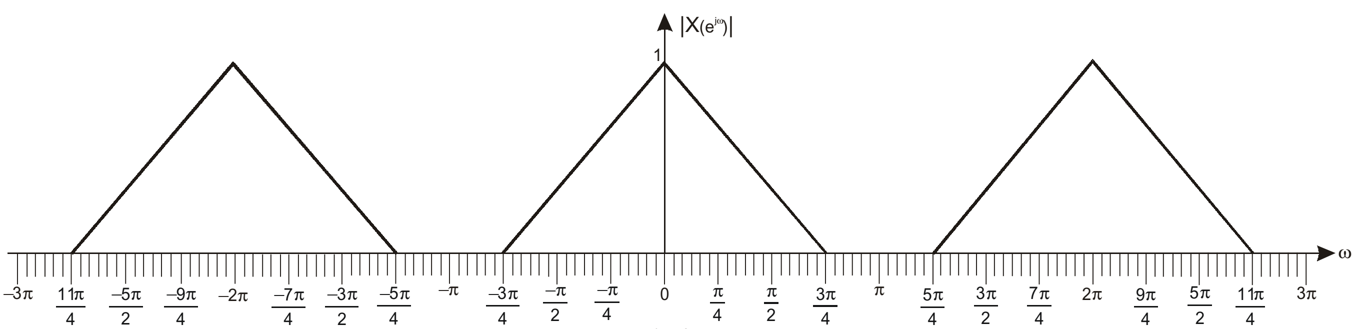


Fig 2.

Case (i) : Upsampling by I = 2

Let, $Y(e^{jw})$ = Spectrum of upsampled signal.

Now, the spectrum $X(e^{jw})$ of upsampled signal is given by,

$Y(e^{jw}) = X(e^{jwI}) = X(e^{j2w})$ $\boxed{\text{Using equation (9.11).}}$

From the above equation we can say that the spectrum of the signal is compressed by a factor of two and so the spectrum of upsampled version of the signal consists of two images of the compressed spectrum as shown in fig 3.

The frequency range of the spectrum of given signal is –3p/4 to +3p/4 and so its bandwidth is 3p/2 [3p/4 – (–3p/4) = 6p/4 = 3p/2]. This bandwidth of 3p/2 is compressed to 3p/4 (3p/2 ´ 1/2 = 3p/4) for interpolation by 2, in every image in the spectrum of upsampled signal.

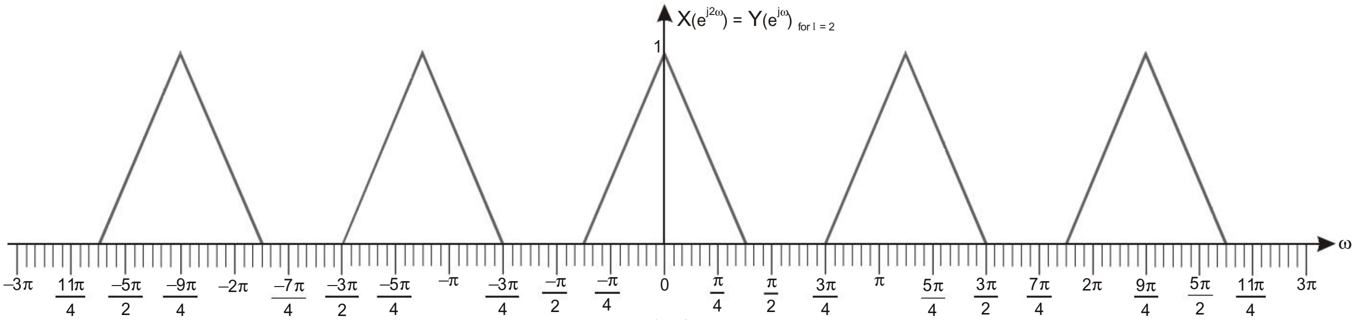


Fig 3.

Case (ii) : **Upsampling by I = 3**

Let, $Y(e^{jw})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{jw})$ of upsampled signal is given by,

$Y(e^{jw}) = X(e^{jwI}) = X(e^{j3w})$ Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of three and so the spectrum of upsampled version of the signal consists of three images of the compressed spectrum as shown in fig 4.

This bandwidth 3p/2 of given signal is compressed to p/2 (3p/2 ´ 1/3 = p/2) for interpolation by 3, in every image in the spectrum of upsampled signal.

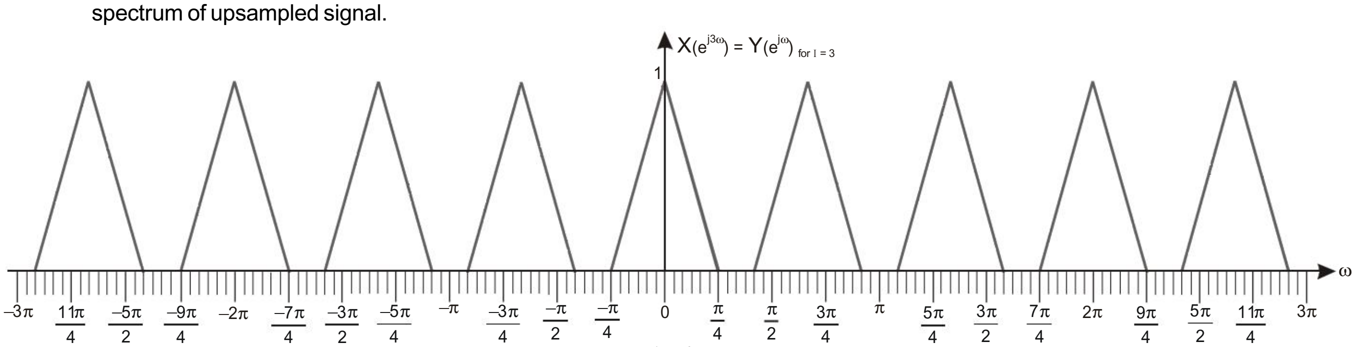


Fig 4.

Case (iii) : **Upsampling by I = 4**

Let, $Y(e^{jw})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{jw})$ of upsampled signal is given by,

$Y(e^{jw}) = X(e^{jwI}) = X(e^{j4w})$ Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of three and so the spectrum of upsampled version of the signal consists of three images of the compressed spectrum as shown in fig 4.

This bandwidth 3p/2 of given signal is compressed to 3p/8 (3p/2 ´ 1/4 = 3p/8) for interpolation by 3, in every image in the spectrum of upsampled signal.

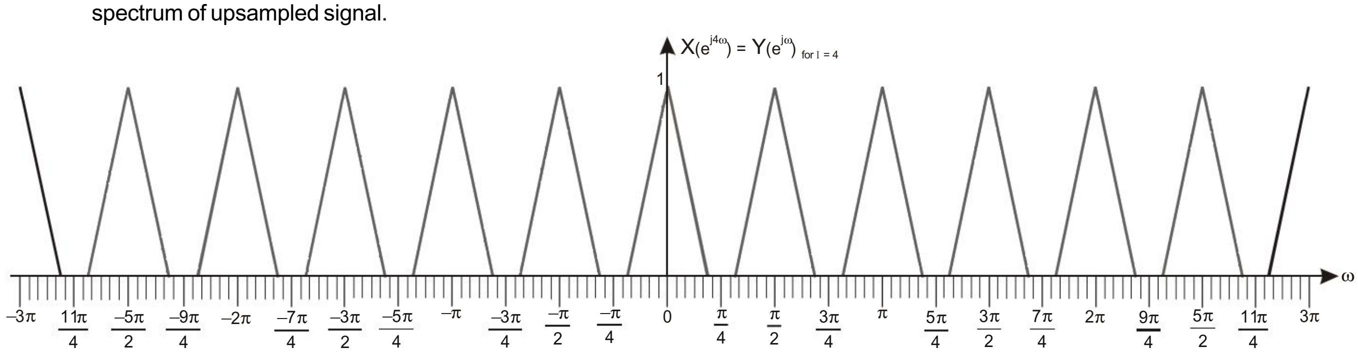


Fig 5.

E9.10 The spectrum of a discrete time signal is shown in fig 1. Draw the spectrum of the signal if it is upsampled by I = 2, 3 and 4.

Solution

Since, the frequency spectrum of a discrete time signal is periodic with periodicity of 2p, the spectrum of given signal can be drawn as shown in fig 2.

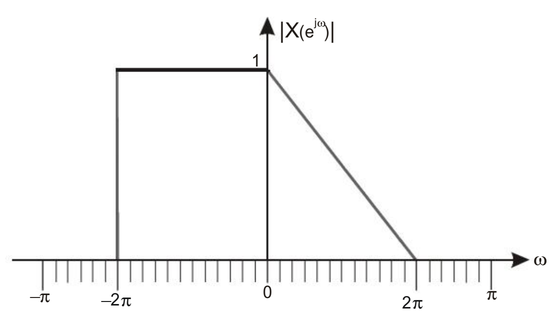


Fig 1.

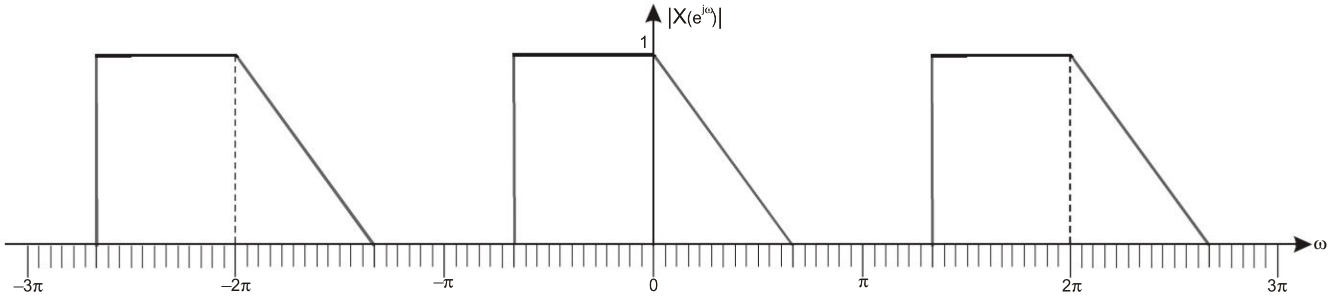


Fig 2.

Case (i) : Upsampling by I = 2

Let, $Y(e^{jw})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{jw})$ of upsampled signal is given by,

$Y(e^{jw}) = X(e^{jwI}) = X(e^{j2w})$ Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of two and so the spectrum of upsampled version of the signal consists of two images of the compressed spectrum in a period of 2p as shown in fig 3.

The frequency range of the spectrum of given signal is –2p/3 to +2p/3 and so its bandwidth is [2p/3 – (–2p/3) = 4p/3]. This bandwidth 4p/3 is compressed to p/6 (4p/3 ´ 1/2 = p/6) for interpolation by 2, in every image in the spectrum of upsampled signal.

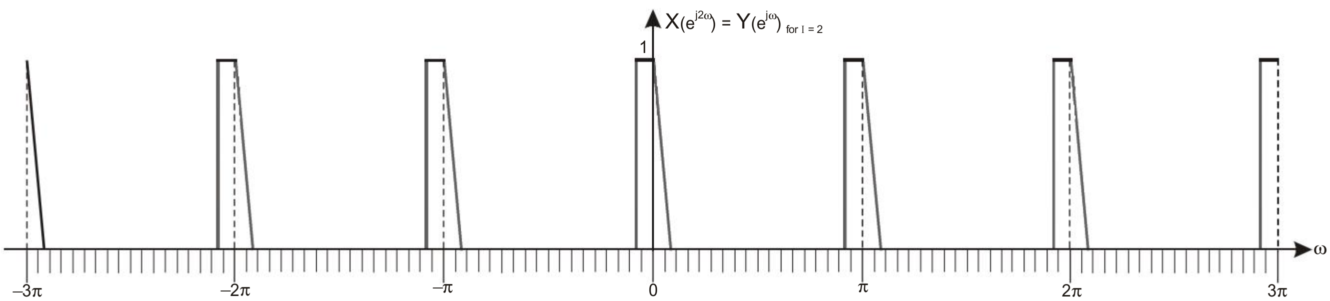


Fig 3.

Case (ii) : Upsampling by I = 3

Let, $Y(e^{jw})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{jw})$ of upsampled signal is given by,

$Y(e^{jw}) = X(e^{jwI}) = X(e^{j3w})$ Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of three and so the spectrum of upsampled version of the signal consists of three images of the compressed spectrum in a period of 2p as shown in fig 4.

The bandwidth 4p/3 of given signal is compressed to 4p/9 (4p/3 ´ 1/3 = 4p/9) for interpolation by 3, in every image in the spectrum of upsampled signal.

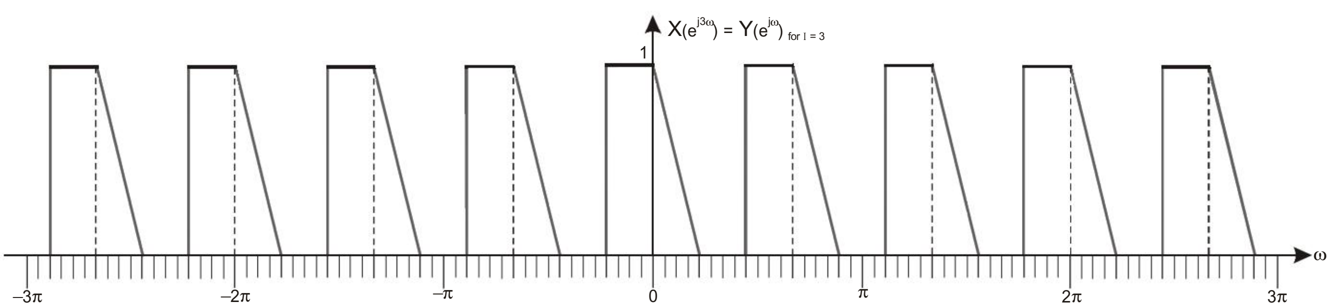


Fig 4.

Case (iii) : Upsampling by I = 4

Let, $Y(e^{jw})$ = Spectrum of upsampled signal.

Now, the spectrum $Y(e^{jw})$ of upsampled signal is given by,

$Y(e^{jw}) = X(e^{jwI}) = X(e^{j4w})$ Using equation (9.11).

From the above equation we can say that the spectrum of the signal is compressed by a factor of four and so the spectrum of upsampled version of the signal consists of four images of the compressed spectrum in a period of 2p as shown in fig 5.

The bandwidth 4p/3 of given signal is compressed to p/3 (4p/3 ´ 1/4 = p/3) for interpolation by 4, in every image in the spectrum of upsampled signal.

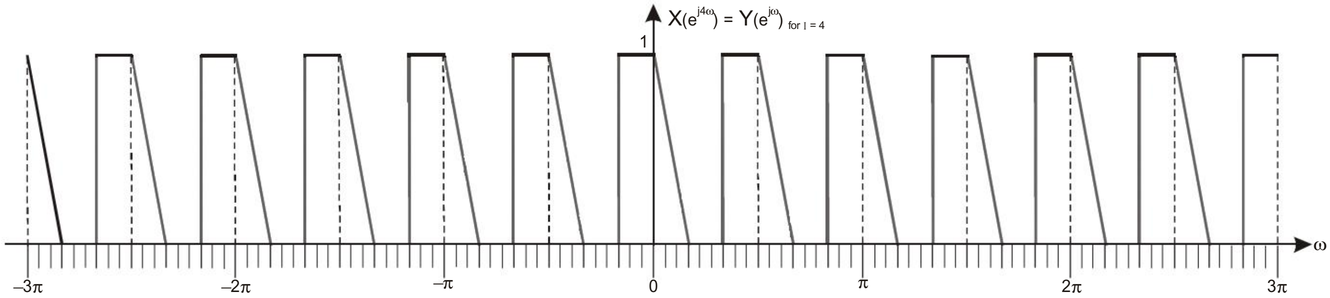


Fig 5.

E9.11 For the multirate system shown in fig 1, determine y(n) as a function of x(n).

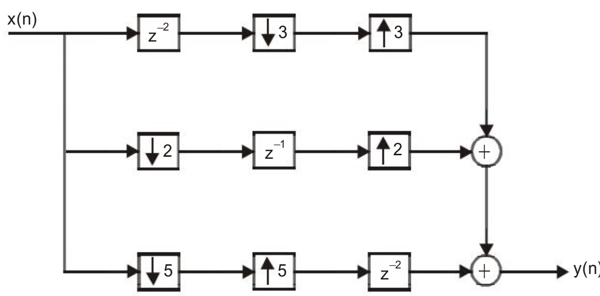


Fig 1.

Solution

Method 1

The z-domain representation of the given system with intermediate signals in z-domain is shown in fig 2.

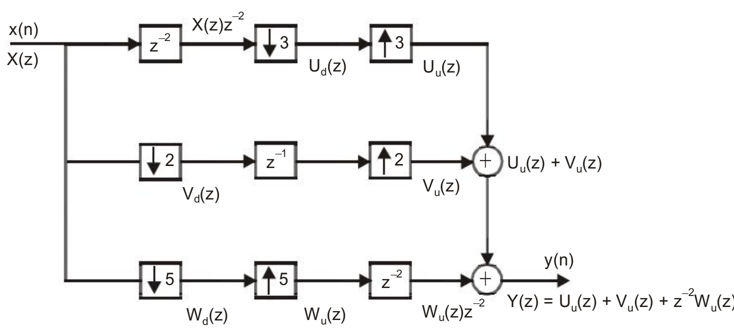


Fig 2.

Using equation (9.8), the z-domain output of decimators are obtained as shown below.

$$U_d(z) = \frac{1}{3}\ X\left(z^{1/3}\right)\left(z^{-2/3}\right) \qquad .....(1)$$

$$V_d(z) = \frac{1}{2}X\left(z^{1/2}\right) \qquad .....(2)$$

$$W_d(z) = \frac{1}{5}X\left(z^{1/5}\right) \qquad .....(3)$$

Using equation (9.9), the z-domain output of interpolators are obtained as shown below.

$$U_u(z) = U_d(z^3) = \frac{1}{3}\ X\left(z^3\right)^{1/3}\left(z^3\right)^{-2/3} \qquad \boxed{\text{Using equation (1).}}$$

$$\therefore\ U_u(z) = \frac{1}{3}\ z^{-2}\,X(z) \qquad .....(4)$$

$$V_u(z) = \left(z^{-1}\right)^2 V_d(z^2) = \left(z^{-1}\right)^2 \frac{1}{2}\ X\left(z^2\right)^{1/2} \qquad \boxed{\text{Using equation (2).}}$$

$$\therefore\ V_u(z) = \frac{1}{2}z^{-2}\ X(z) \qquad .....(5)$$

$$W_u(z) = W_u(z^5) = \frac{1}{5}X\left(z^5\right)^{1/5} \qquad \boxed{\text{Using equation (3).}}$$

$$\therefore\ W_u(z) = \frac{1}{5}\ X(z) \qquad .....(6)$$

Now, the z-domain output Y(z) of the system is,

$$Y(z) = U_u(z) + V_u(z) + z^{-2}\,W_u(z)$$

$$= \frac{1}{3}z^{-2}X(z) + \frac{1}{2}z^{-2}X(z) + \frac{1}{5}z^{-2}X(z)$$

$$= z^{-2}\,X(z)\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{5}\right)$$

$$= z^{-2}\,X(z) \qquad .....(7)$$

On taking inverse $\mathcal{Z}$-transform of equation (7) we get,

$$y(n) = x(n-2) \qquad .....(8)$$

The equation (8) is the output of system in time domain.

E9.12 Determine the ouput y(n) in terms of input x(n) for the multirate system shown in fig 1.

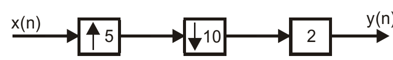


Fig 1.

Solution

Using identity 9, the downsampler with reduction factor 10 can be expressed as cascade of two down samplers with reduction factor 5 and 2 (5 ´ 2 = 10) as shown in fig 2.

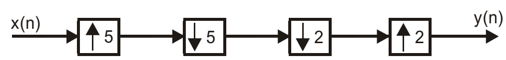


Fig 2.

Using identity 7, the cascade of upsample with multiplication factor 5 and the downsampler with reduction factor 5 can be replaced by unity gain branch as shown in fig 3.

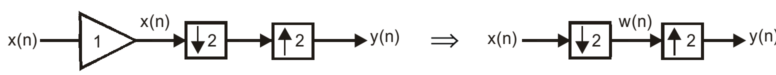


Fig 3.

Using indentity 8, the system shown in fig 3 can be replaced by an equivalent system as shown in fig 4.

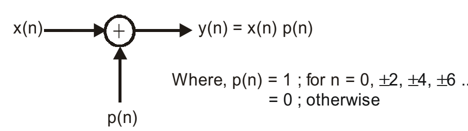


Fig 4.

$$\therefore\; y(n) = x(n)\,p(n) = x(n) \;\;;\;\; \text{for } n = 0,\ \pm 2,\ \pm 4,\ \pm 6,\ ......$$

$$= 0 \;\;;\;\; \text{for other } n$$

$$\therefore\; y(n) = \{.....,\ \underset{\uparrow}{x(0)},\ 0,\ x(2),\ 0,\ x(4),\ 0,\ x(6),\ .....\}$$

E9.13 The transfer function of an FIR filter is,

$$H(z) = 0.3 + 0.6\,z^{-1} + 0.7\,z^{-2} + 0.18\,z^{-3} + 0.85\,z^{-4} + 0.25\,z^{-5} + 0.28\,z^{-6} + 0.42\,z^{-7} + 0.89\,z^{-8}$$

Perform polyphase decomposition of H(z) to decompose into a) 2 sections b) 3 sections c) 4 sections.

Solution

a) Polyphase decomposition into 2 sections

Given that,

$$H(z) = 0.3 + 0.6\,z^{-1} + 0.7\,z^{-2} + 0.18\,z^{-3} + 0.85\,z^{-4} + 0.25\,z^{-5} + 0.28\,z^{-6} + 0.42\,z^{-7} + 0.89\,z^{-8}$$

Let us express H(z) as 2 sections as shown below.

$$H(z) = [0.3 + 0.7\,z^{-2} + 0.85\,z^{-4} + 0.28\,z^{-6} + 0.89\,z^{-8}] + [0.6z^{-1} + 0.18\,z^{-3} + 0.25\,z^{-5} + 0.42\,z^{-7}]$$

$$= [0.3 + 0.7\,z^{-2} + 0.85\,z^{-4} + 0.28\,z^{-6} + 0.89\,z^{-8}] + z^{-1}[0.6 + 0.18\,z^{-2} + 0.25\,z^{-4} + 0.42\,z^{-6}]$$

$$= [0.3(z^2)^0 + 0.7\,(z^2)^{-1} + 0.85\,(z^2)^{-2} + 0.28\,(z^2)^{-3} + 0.89\,(z^2)^{-4}] + z^{-1}[0.6(z^2)^0 + 0.18\,(z^2)^{-1} + 0.25\,(z^2)^{-2} + 0.42\,(z^2)^{-3}]$$

$$= E_0(z^2) + z^{-1}\,E_1(z^2)$$

where, $E_0(z^2) = 0.3 + 0.7z^{-2} + 0.85z^{-4} + 0.28z^{-6} + 0.89z^{-8}$

$E_1(z^2) = 0.6 + 0.18z^{-2} + 0.25z^{-4} + 0.42z^{-6}$

Here, $E_0(z^2)$ and $E_1(z^2)$ are two sections (or two sub-filters) obtained by polyphase decomposition of H(z).

b) Polyphase decomposition into 3 sections

Given that,

$$H(z) = 0.3 + 0.6\,z^{-1} + 0.7\,z^{-2} + 0.18\,z^{-3} + 0.85\,z^{-4} + 0.25\,z^{-5} + 0.28\,z^{-6} + 0.42\,z^{-7} + 0.89\,z^{-8}$$

Let us express H(z) as 3 sections as shown below.

$$H(z) = [0.3 + 0.18\,z^{-3} + 0.28\,z^{-6}] + [0.6z^{-1} + 0.85\,z^{-4} + 0.42\,z^{-7}] + [0.7z^{-2} + 0.25\,z^{-5} + 0.89\,z^{-8}]$$

$$= [0.3 + 0.18\,z^{-3} + 0.28\,z^{-6}] + z^{-1}[0.6 + 0.85\,z^{-3} + 0.42\,z^{-6}] + z^{-2}[0.7 + 0.25\,z^{-3} + 0.89\,z^{-6}]$$

$\therefore$ $H(z) = [0.3(z^3)^0 + 0.18\,(z^3)^{-1} + 0.28\,(z^3)^{-2}] + z^{-1}[0.6(z^3)^0 + 0.85\,(z^3)^{-1} + 0.42\,(z^3)^{-2}] + z^{-2}[0.7(z^3)^0 + 0.25\,(z^3)^{-1} + 0.89\,(z^3)^{-2}]$

$= E_0(z^3) + z^{-1}\,E_1(z^3) + z^{-2}\,E_2(z^3)$

where, $E_0(z^3) = 0.3 + 0.18z^{-3} + 0.28z^{-6}$; $E_1(z^3) = 0.6 + 0.85z^{-3} + 0.42z^{-6}$; $E_2(z^3) = 0.7 + 0.25z^{-3} + 0.89z^{-6}$

Here, $E_0(z^3)$, $E_1(z^3)$ and $E_2(z^3)$ are three sections (or three sub-filters) obtained by polyphase decomposition of H(z).

c) Polyphase decomposition into 4 sections

Given that,

$H(z) = 0.3 + 0.6\,z^{-1} + 0.7\,z^{-2} + 0.18\,z^{-3} + 0.85\,z^{-4} + 0.25\,z^{-5} + 0.28\,z^{-6} + 0.42\,z^{-7} + 0.89\,z^{-8}$

Let us express H(z) as 4 sections as shown below.

$H(z) = [0.3 + 0.85\,z^{-4} + 0.89\,z^{-8}] + [0.6z^{-1} + 0.25\,z^{-5}] + [0.7z^{-2} + 0.28\,z^{-6}] + [0.18z^{-3} + 0.42\,z^{-7}]$

$= [0.3 + 0.85\,z^{-4} + 0.89\,z^{-8}] + z^{-1}[0.6 + 0.25\,z^{-4}] + z^{-2}[0.7 + 0.28\,z^{-4}] + z^{-3}\,[0.18 + 0.42\,z^{-4}]$

$= [0.3\,(z^4)^0 + 0.85\,(z^4)^{-1} + 0.89\,(z^4)^{-2}] + z^{-1}[0.6(z^4)^0 + 0.25(z^4)^{-1}] + z^{-2}[0.7(z^4)^0 + 0.28(z^4)^{-1}] + z^{-3}[0.18(z^4)^0 + 0.42(z^4)^{-1}]$

$= E_0(z^4) + z^{-1}\,E_1(z^4) + z^{-2}\,E_2(z^4) + z^{-3}\,E_3(z^4)$

where, $E_0(z^4) = 0.3 + 0.85z^{-4} + 0.89z^{-8}$; $E_1(z^4) = 0.6 + 0.25z^{-4}$

$E_2(z^4) = 0.7 + 0.28z^{-4}$; $E_3(z^4) = 0.18 + 0.42z^{-4}$

Here, $E_0(z^4)$, $E_1(z^4)$, $E_2(z^4)$ and $E_3(z^4)$ are four sections (or four sub-filters) obtained by polyphase decomposition of H(z).

E9.14. The transfer function of an IIR filter is, $H(z) = \dfrac{1 + 0.85z^{-1}}{1 - 0.65z^{-1}}$.

Perform polyphase decomposition of H(z) to decompose into a) 2 sections b) 4 sections.

Solution

a) Polyphase decomposition into 2 sections

Given that,

$$H(z) = \frac{1 + 0.85z^{-1}}{1 - 0.65z^{-1}}$$

Let us choose a polynomial, $P(z) = 1 + 0.65z^{-1}$

Let us multiply the numerator and denominator of H(z) by P(z).

$$\therefore\ H(z) = \frac{1 + 0.85z^{-1}}{1 - 0.65z^{-1}} \times \frac{P(z)}{P(z)} = \frac{1 + 0.85z^{-1}}{1 - 0.65z^{-1}} \times \frac{1 + 0.65z^{-1}}{1 + 0.65z^{-1}}$$

$$= \frac{1 + 0.85z^{-1} + 0.65z^{-1} + 0.5525z^{-2}}{1 - 0.4225z^{-2}}$$

$\boxed{(a - b)\,(a + b) = a^2 - b^2}$

$$= \frac{1 + 1.5z^{-1} + 0.5525z^{-2}}{1 - 0.4225z^{-2}} \qquad .....(1)$$

Now, the above transfer function can be decomposed into two sections as shown below.

$$\therefore\ H(z) = \frac{1 + 1.5z^{-1} + 0.5525z^{-2}}{1 - 0.4225z^{-2}} = \frac{(1 + 0.5525z^{-2}) + z^{-1}(1.5)}{1 - 0.4225z^{-2}}$$

$$= \frac{1 + 0.5525z^{-2}}{1 - 0.4225z^{-2}} + z^{-1}\,\frac{1.5}{1 - 0.4225z^{-2}} = \frac{1 + 0.5525(z^2)^{-1}}{1 - 0.4225(z^2)^{-1}} + z^{-1}\,\frac{1.5}{1 - 0.4225(z^2)^{-1}}$$

$$= E_0(z^2) + z^{-1}\,E_1(z^2)$$

$$\text{where, } E_0(z^2) = \frac{1 + 0.5525z^{-2}}{1 - 0.4225z^{-2}}\ ;\ E_1(z^2) = \frac{1.5}{1 - 0.4225z^{-2}}$$

Here, $E_0(z^2)$ and $E_1(z^2)$ are two sections (or two sub-filters) obtained by decomposition of H(z).

b) Polyphase decomposition into 4 sections

Consider the transfer function of equation (1).

$$H(z) = \frac{1 + 1.5z^{-1} + 0.5525z^{-2}}{1 - 0.4225z^{-2}}$$

Let us choose a polynomial, $P_1(z) = 1 + 0.4225z^{-2}$

Let us multiply the numerator and denominator of above H(z) by $P_1(z)$.

$$\therefore\ H(z) = \frac{1 + 1.5z^{-1} + 0.5525z^{-2}}{1 - 0.4225z^{-2}} \times \frac{P_1(z)}{P_1(z)}$$

$$= \frac{1 + 1.5z^{-1} + 0.5525z^{-2}}{1 - 0.4225z^{-2}} \times \frac{1 + 0.4225z^{-2}}{1 + 0.4225z^{-2}}$$

$\boxed{(a-b)\,(a+b) = a^2 - b^2}$

$$\therefore\ H(z) = \frac{1+1.5z^{-1}+0.5525z^{-2}+0.4225z^{-2}+0.6338z^{-3}+0.2334z^{-4}}{1-0.1785z^{-4}}$$

$$= \frac{1+1.5z^{-1}+0.975z^{-2}+0.6338z^{-3}+0.2334z^{-4}}{1-0.1785z^{-4}}$$

Now, the above transfer function can be decomposed into four sections as shown below.

$$H(z) = \frac{1+1.5z^{-1}+0.975z^{-2}+0.6338z^{-3}+0.2334z^{-4}}{1-0.1785z^{-4}}$$

$$= \frac{(1+0.2334z^{-4})+z^{-1}(1.5)+z^{-2}(0.975)+z^{-3}(0.6338)}{1-0.1785z^{-4}}$$

$$= \frac{1+0.2334z^{-4}}{1-0.1785z^{-4}} + z^{-1}\frac{1.5}{1-0.1785z^{-4}} + z^{-2}\frac{0.975}{1-0.1785z^{-4}} + z^{-3}\frac{0.6338}{1-0.1785z^{-4}}$$

$$= \frac{1+0.2334(z^4)^{-1}}{1-0.1785(z^4)^{-1}} + z^{-1}\frac{1.5}{1-0.1785(z^4)^{-1}} + z^{-2}\frac{0.975}{1-0.1785(z^4)^{-1}} + z^{-3}\frac{0.6338}{1-0.1785(z^4)^{-1}}$$

$$= E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4)$$

$$\text{where, } E_0(z^4) = \frac{1+0.2334(z^4)^{-1}}{1-0.1785(z^4)^{-1}}\ ;\ E_1(z^4) = \frac{1.5}{1-0.1785(z^4)^{-1}}$$

$$E_2(z^4) = \frac{0.975}{1-0.1785(z^4)^{-1}}\ ;\ E_3(z^4) = \frac{0.6338}{1-0.1785(z^4)^{-1}}$$

Here, $E_0(z^4)$, $E_1(z^4)$, $E_2(z^4)$ and $E_3(z^4)$ are four sections (or four sub-filters) obtained by decomposition of H(z).

E9.15 The transfer function of an IIR filter is, $H(z) = \dfrac{1+0.32z^{-1}+0.58z^{-2}}{1+0.7z^{-1}+0.4z^{-2}}$

Perform polyphase decomposition of H(z) to decompose into a) 2 sections b) 4 sections.

Solution

a) Polyphase decomposition into 2 sections

Given that,

$$H(z) = \frac{1+0.32z^{-1}+0.58z^{-2}}{1+0.7z^{-1}+0.4z^{-2}}$$

Let us choose a polynomial, $P(z) = 1 - 0.7z^{-1} + 0.4z^{-2}$

Let us multiply the numerator and denominator of H(z) by P(z).

$$\therefore\ H(z) = \frac{1+0.32z^{-1}+0.58z^{-2}}{1+0.7z^{-1}+0.4z^{-2}} \times \frac{P(z)}{P(z)}$$

$$= \frac{1+0.32z^{-1}+0.58z^{-2}}{1+0.7z^{-1}+0.4z^{-2}} \times \frac{1-0.7z^{-1}+0.4z^{-2}}{1-0.7z^{-1}+0.4z^{-2}}$$

$$= \frac{1-0.7z^{-1}+0.4z^{-2}+0.32z^{-1}-0.224z^{-2}+0.128z^{-3}+0.58z^{-2}-0.406z^{-3}+0.232z^{-4}}{1\cancel{-0.7z^{-1}}+0.4z^{-2}\cancel{+0.7z^{-1}}-0.49z^{-2}\cancel{+0.28z^{-3}}+0.4z^{-2}\cancel{-0.28z^{-3}}+0.16z^{-4}}$$

$$= \frac{1-0.38z^{-1}+0.756z^{-2}-0.2782z^{-3}+0.232z^{-4}}{1+0.31z^{-2}+0.16z^{-4}} \qquad .....(1)$$

Now, the above transfer function can be decomposed into two sections as shown below.

$$H(z) = \frac{1-0.38z^{-1}+0.756z^{-2}-0.2782z^{-3}+0.232z^{-4}}{1+0.31z^{-2}+0.16z^{-4}}$$

$$= \frac{(1+0.756z^{-2}+0.232z^{-4})+z^{-1}(-0.38-0.278z^{-2})}{1+0.31z^{-2}+0.16z^{-4}}$$

$$= \frac{1+0.756z^{-2}+0.232z^{-4}}{1+0.31z^{-2}+0.16z^{-4}} + z^{-1}\frac{-0.38-0.278z^{-2}}{1+0.31z^{-2}+0.16z^{-4}}$$

$$= \frac{1+0.756(z^2)^{-1}+0.232(z^2)^{-2}}{1+0.31(z^2)^{-1}+0.16(z^2)^{-2}} + z^{-1}\frac{-0.38-0.278(z^2)^{-1}}{1+0.31(z^2)^{-1}+0.16(z^2)^{-2}}$$

$$= E_0(z^2) + z^{-1}E_1(z^2)$$

$$\text{where, } E_0(z^2) = \frac{1+0.756z^{-2}+0.232z^{-4}}{1+0.31z^{-2}+0.64z^{-4}}\ ;\ E_1(z^2) = \frac{-0.38-0.278z^{-2}}{1+0.31z^{-2}+0.16z^{-4}}$$

Here, $E_0(z^2)$ and $E_1(z^2)$ are two sections (or two sub-filters) obtained by decomposition of H(z).

b) Polyphase decomposition into 4 sections

Consider the transfer function of equation(1).

$$H(z)=\frac{1-0.38z^{-1}+0.756z^{-2}-0.2782z^{-3}+0.232z^{-4}}{1+0.31z^{-2}+0.16z^{-4}}$$

Let us choose a polynomial, $P_1(z) = 1 - 0.31z^{-2} + 0.16z^{-4}$

Let us multiply the numerator and denominator of above H(z) by $P_1(z)$.

$$\therefore\ H(z)=\frac{1-0.38z^{-1}+0.756z^{-2}-0.2782z^{-3}+0.232z^{-4}}{1+0.31z^{-2}+0.16z^{-4}}\times\frac{P_1(z)}{P_1(z)}$$

$$=\frac{1-0.38z^{-1}+0.756z^{-2}-0.2782z^{-3}+0.232z^{-4}}{1+0.31z^{-2}+0.16z^{-4}}\times\frac{1-0.31z^{-2}+0.16z^{-4}}{1-0.31z^{-2}+0.16z^{-4}}$$

$$=\frac{\begin{array}{c}1-0.31z^{-2}+0.16z^{-4}-0.38z^{-1}+0.1178z^{-3}-0.0608z^{-5}+0.756z^{-2}-0.2344z^{-4}\\ +0.121z^{-6}-0.2782z^{-3}+0.0862z^{-5}-0.0445z^{-7}+0.232z^{-4}-0.0719z^{-6}+0.0371z^{-8}\end{array}}{1-\cancel{0.31z^{-2}}+0.16z^{-4}+\cancel{0.31z^{-2}}-0.0961z^{-4}+\cancel{0.0496z^{-6}}+0.16z^{-4}-\cancel{0.0496z^{-6}}+0.0256z^{-8}}$$

$$=\frac{1-0.38z^{-1}+0.446z^{-2}-0.1604z^{-3}+0.1576z^{-4}+0.0254z^{-5}+0.0491z^{-6}-0.0445z^{-7}+0.0371z^{-8}}{1+0.2239z^{-4}+0.0256z^{-8}}$$

Now, the above transfer function can be decomposed into four sections as shown below.

$$\therefore\ H(z)=\frac{(1+0.1576z^{-4}+0.0371z^{-8})+z^{-1}(-0.38-0.0254z^{-4})+z^{-2}(0.446+0.0491z^{-4})+z^{-3}(-0.1604-0.0445z^{-4})}{1+0.2239z^{-4}+0.0256z^{-8}}$$

$$=\frac{1+0.1576z^{-4}+0.0371z^{-8}}{1+0.2239z^{-4}+0.0256z^{-8}}+z^{-1}\frac{-0.38-0.0254z^{-4}}{1+0.2239z^{-4}+0.0256z^{-8}}$$

$$+z^{-2}\frac{0.446+0.0491z^{-4}}{1+0.2239z^{-4}+0.0256z^{-8}}\ +\ z^{-3}\frac{-0.1604-0.0442z^{-4}}{1+0.2239z^{-4}+0.0256z^{-8}}$$

$$=\frac{1+0.1576(z^4)^{-1}+0.0371(z^4)^{-2}}{1+0.2239(z^4)^{-1}+0.0256(z^4)^{-2}}+z^{-1}\frac{-0.38+0.0254(z^4)^{-1}}{1+0.2239(z^4)^{-1}+0.0256(z^4)^{-2}}$$

$$+z^{-2}\frac{0.446+0.0491(z^4)^{-1}}{1+0.2239(z^4)^{-1}+0.0256(z^4)^{-2}}\ +\ z^{-3}\frac{-0.1604-0.0445(z^4)^{-1}}{1+0.2239(z^4)^{-1}+0.0256(z^4)^{-2}}$$

$$=E_0(z^4)+z^{-1}E_1(z^4)+z^{-2}E_2(z^4)+z^{-3}E_3(z^4)$$

where, $E_0(z^4)=\dfrac{1+0.1576(z^4)^{-1}+0.0371(z^4)^{-2}}{1+0.2239(z^4)^{-1}+0.0256(z^4)^{-2}}$; $E_1(z^4)=\dfrac{-0.38+0.0254(z^4)^{-1}}{1+0.2239(z^4)^{-1}+0.0256(z^4)^{-2}}$

$E_2(z^4)=\dfrac{0.446+0.0491(z^4)^{-1}}{1+0.2239(z^4)^{-1}+0.0256(z^4)^{-2}}$; $E_3(z^4)=\dfrac{-0.1604-0.0445(z^4)^{-1}}{1+0.2239(z^4)^{-1}+0.0256(z^4)^{-2}}$

Here, $E_0(z^4)$, $E_1(z^4)$, $E_2(z^4)$ and $E_3(z^4)$ are four sections (or four sub-filters) obtained by decomposition of H(z).

Chapter 10

Energy and Power Spectrum Estimation

10.1 Introduction

The power (or energy) spectrum describes the power (or energy) level of various frequency components of a signal. The main objective of a power (or energy) spectrum is to differentiate the signal from noise based on the power (or energy) level of the frequency components. Most of the practical signals observed or measured are corrupted by some type of noise. It is believed that the signal power (or energy) will be more than the noise power (or energy), and so estimating the power (or energy) spectrum will help to identify the signal from noise.

In one way of classification, the discrete time signals can be broadly classified into ***deterministic*** signals and ***random*** (or nondeterministic) signals. The deterministic signals can be reproduced exactly with repeated measurements, whereas the random signals are not repeatable in a predictable manner. The deterministic signals have finite energy and so estimation of ***energy spectrum*** is possible for deterministic signals. The random signals have finite power and so estimation of ***power spectrum*** is possible for random signals.

10.2 Energy Spectrum of Discrete Time Signal

Let, $x(n)$ = Discrete time energy signal.

Let, $X(e^{jw})$ = Fourier transform of $x(n)$.

By definition of Fourier transform, we can write,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(10.1)$$

Since, w = 2pf, the equation (10.1), can be written as shown in equation (10.2).

$$X(f) = \sum_{n=-\infty}^{+\infty} x(n)\ e^{-j2\pi fn} \quad .....(10.2)$$

The energy of x(n) is defined as,

$$\text{Energy} = \sum_{n=-\infty}^{+\infty} |x(n)|^2 \quad .....(10.3)$$

By Parseval's relation of Fourier transform, we can write,

$$\sum_{n=-\infty}^{+\infty} |x(n)|^2 = \int_{-\pi}^{\pi} |X(f)|^2 df \quad .....(10.4)$$

From equation (10.3), we can say that, $|x(n)|^2$ represents energy as a function of discrete time n, in the time domain.

From equation (10.4), we can say that, $|x(n)|^2$ in the time domain is equivalent to $|X(f)|^2$ in the frequency domain. Therefore, from equations (10.3) and (10.4), we can say that $|X(f)|^2$ represents energy as a function of discrete frequency f, in the frequency domain. Hence, $|X(f)|^2$ is called ***energy spectral density*** or ***energy spectrum.*** The energy spectrum is denoted as $S_{xx}(f)$.

Also, it can be proved that the Fourier transform of the autocorrelation function of x(n) gives the energy spectrum.

Therefore, the energy spectrum of a deterministic finite duration discrete time signal x(n) can be computed in two different methods. In one method, the Fourier transform of x(n) is determined to get X(f), and square of magnitude of X(f) gives the energy spectrum. In another method, the autocorrelation sequence of x(n) is determined first and then the Fourier transform of autocorrelation sequence is determined which is the desired energy spectrum.

Computing Energy Spectrum of Discrete Time Signal

Method -1

Let, x(n) = Discrete time energy signal.

Let, X(f) = Fourier transform of x(n).

Now, by definition of Fourier transform,

$$X(f) = \sum_{n=-\infty}^{+\infty} x(n)\ e^{-j2\pi fn}$$

Since x(n) is an N-point sequence,

$$X(f) = \sum_{n=0}^{N-1} x(n)\ e^{-j2\pi fn} \quad .....(10.5)$$

The energy spectrum $S_{xx}(f)$ is defined as,

$$S_{xx}(f) = |X(f)|^2 \quad .....(10.6)$$

On substituting for X(f) from equation (10.5) in equation (10.6), we get,

$$S_{xx}(f) = |X(f)|^2 = \left| \sum_{n=0}^{N-1} x(n)\ e^{-j2\pi fn} \right|^2 \qquad .....(10.7)$$

Method -2

Let, x(n) = Discrete time energy signal.

Let, $r_{xx}(m)$ = Autocorrelation of x(n).

The autocorrelation of x(n) is defined as,

$$r_{xx}(m) = \sum_{n=-\infty}^{+\infty} x^*(n)\ x(n+m) \qquad .....(10.8)$$

The energy spectrum $S_{xx}(f)$ is defined as,

$$S_{xx}(f) = \mathcal{F}\{r_{xx}(m)\} = \sum_{m=-\infty}^{+\infty} r_{xx}(m)\ e^{-j2\pi fm} \qquad .....(10.9)$$

Proof

Let, x(n) = Discrete time energy signal.

The autocorrelation $r_{xx}(m)$ of x(n) is defined as,

$$r_{xx}(m) = \sum_{n=-\infty}^{+\infty} x^*(n)\ x(n+m) \qquad .....(10.10)$$

The Fourier transform of autocorrelation sequence $r_{xx}(m)$ is,

$$\mathcal{F}\{r_{xx}(m)\} = \sum_{m=-\infty}^{+\infty} r_{xx}(m)\ e^{-j2\pi fm}$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x^*(n)\ x(n+m)\ e^{-j2\pi fm}$$

Using equation (10.10).

$$= \sum_{n=-\infty}^{+\infty} x^*(n) \left[\sum_{m=-\infty}^{+\infty} x(m+n)\ e^{-j2\pi fm} \right]$$

$$= \sum_{n=-\infty}^{+\infty} x^*(n) \left[X(f) e^{j2\pi fn} \right]$$

Using shifting property of Fourier transform.

If $\mathcal{F}\{x(m)\} = X(f)$ then $\mathcal{F}\{x(m+n)\} = X(f)\ e^{j2pfn}$

$$= X(f) \left[\sum_{n=-\infty}^{+\infty} x^*(n)\ e^{j2\pi fn} \right]$$

$$= X(f) \left[\sum_{n=-\infty}^{+\infty} x(n)\ e^{-j2\pi fn} \right]^*$$

By definition Fourier transform,

$$X(f) = F\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n) e^{-j2\pi fn}$$

$$= X(f)\ [X(f)]^*$$

$$= X(f)\ X^*(f) = |X(f)|^2 = S_{XX}(f)$$

Hence proved.

10.3 Random Signal and Random Process

A ***random signal*** or ***random process*** is a signal that is not repeatable in a predictable manner. Many phenomena that occur in nature are random processes.

Examples of random process:

1. Quantization noise produced by A/D converters.
2. Quantization noise produced by a fixed point digital filter.
3. Engine noise in speech transmission from the cockpit of an airplane.
4. Sonar signals corrupted by ambient ocean noise.
5. Fluctuations in atmospheric temperature and pressure.
6. Seismic data from earthquakes.
7. Thermal noise generated in resistors and electronic devices.

A discrete time random process or signal is a collection or ensemble of discrete time signals, which are obtained from multiple realizations of the process. The random process is denoted as X(n). When a random process or a signal is obtained from a single realization of the process, it is called ***ergodic random process*** and is denoted as x(n). The power spectrum of ergodic random processes are discussed in this book.

Some of the mathematical operations on ergodic random signals, which are useful for power spectrum estimation and comparison of power spectrum estimation by various methods, are autocorrelation, mean or expected value and variance.

The random processes are best characterized statistically in terms of average. The autocorrelation function of a random process is the appropriate statistical average for characterizing the random process or signal in time domain.

The ***autocorrelation*** $\gamma_{xx}(m)$ of an ergodic random process estimated using infinite data is defined as,

$$\text{Autocorrelation, } \gamma_{xx}(m) = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+m)\ x^*(n) \qquad .....(10.11)$$

The autocorrelation $r_{xx}(m)$ of an ergodic random process estimated using finite data measured in the interval, n = 0 to N – 1 is defined as,

$$\text{Autocorrelation, } r_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-1} x(n+m)\ x^*(n) \qquad .....(10.12)$$

The ***mean*** or expected value of an ergodic random process is defined as,

$$\text{Mean, } m_x = E\{x(n)\} = \frac{1}{2N+1} \sum_{n=-N}^{N} x(n) \qquad .....(10.13)$$

The ***variance*** of an ergodic random process is defined as,

$$\text{Variance, } \sigma_x^2 = E\left\{|x(n)|^2\right\} - \left[E\{x(n)\}\right]^2 = E\left\{|x(n)|^2\right\} - m_x^2 \qquad .....(10.14)$$

10.4 Power Spectrum of a Random Process

The Fourier transform of the autocorrelation sequence $\gamma_{xx}(m)$ of a random process gives ***power spectral density*** or ***power spectrum***. The power spectrum is denoted as $P_{xx}(f)$. Therefore, estimating the power spectrum is equivalent to estimating the autocorrelation. Therefore, for an ergodic random process if x(n) is known for all n, then estimating the power spectrum is (in theory) straightforward. However, there are two primary limitations making spectral estimation an extremely challenging problem.

1. The amount of data available for the analysis is limited and, in many situations, might be very small due to very short observation time.
2. The data is often corrupted by noise or contaminated with an interfering signal. Therefore, power spectrum estimation has to be performed using a finite number of noisy data samples.

There are two major classes of spectral estimation techniques:

1. Nonparametric (or classical) methods: In these methods, first the autocorrelation sequence is estimated from the given data. Then the power spectrum is estimated by taking Fourier transform of the estimated autocorrelation sequence.
2. Parametric (or nonclassical) methods: In these methods, first an appropriate system model is selected for the given random process and then the parameters of the model are computed from the available data of the random process. Finally, the power spectrum is estimated from the constructed model.

The various nonparametric methods of power spectrum estimation are discussed in this book.

10.5 Periodogram

Let x(n) be an ergodic random process with an unlimited amount of data and $\gamma_{xx}(m)$ be the autocorrelation sequence of x(n). Now, the Fourier transform of the autocorrelation sequence $\gamma_{xx}(m)$ gives the power spectral density or power spectrum, $P_{xx}(f)$.

$$\text{Power spectrum, } P_{xx}(f) = \mathcal{F}\{\gamma_{xx}(m)\} = \sum_{m=-\infty}^{+\infty} \gamma_{xx}(m)\, e^{-j2\pi fm} \qquad .....(10.15)$$

$$\text{where, } \gamma_{xx}(m) = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+m)\, x^*(n) \qquad .....(10.16)$$

When x(n) is measured only over a finite interval, i.e., for n = 0,1,2,.......,N – 1, then the autocorrelation has to be estimated only using these finite data. Let $r_{xx}(m)$ be the autocorrelation sequence estimated using N-sample sequence of x(n). The Fourier transform of autocorrelation sequence $r_{xx}(m)$ will be an estimate of power spectrum and this estimate of power spectrum is called ***periodogram***. The periodogram is denoted as $P_{xx}^{Per}(f)$.

The autocorrelation sequence $r_{xx}(m)$ obtained using N-samples of x(n) is defined as,

$$\text{Autocorrelation sequence, } r_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-1} x(n+m)\, x^*(n) \qquad .....(10.17)$$

The autocorrelation sequence $r_{xx}(m)$ can be extended to negative values of m, by using conjugate symmetry $r_{xx}(-m) = r_{xx}^*(m)$. Then the periodogram is computed by taking Fourier transform of autocorrelation sequence $r_{xx}(m)$ as shown below.

$$\text{Periodogram, } P_{xx}^{Per}(f) = \mathcal{F}\{r_{xx}(m)\} = \sum_{m=-(N-1)}^{N-1} r_{xx}(m)\, e^{-j2\pi fm} \qquad .....(10.18)$$

The periodogram can also be expressed in terms of random process x(n) as shown below.

$$\text{Periodogram, } P_{xx}^{Per}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right|^2 \qquad .....(10.19)$$

Proof :

The periodogram estimate of power spectrum, $P_{xx}^{Per}(f)$ is defined as Fourier transform of autocorrelation sequence, $r_{xx}(m)$.

Therefore, from the definition of Fourier transform we can write,

$$P_{xx}^{Per}(f) = \mathcal{F}\{r_{xx}(m)\} = \sum_{m=0}^{N-1} r_{xx}(m)\, e^{-j2\pi fm}$$

$$= \sum_{m=0}^{N-1}\left[\frac{1}{N}\sum_{n=0}^{N-1} x(n+m)\, x^*(n)\right] e^{-j2\pi fm}$$

Using equation (10.17).

$$= \frac{1}{N}\sum_{n=0}^{N-1} x^*(n)\left[\sum_{m=0}^{N-1} x(m+n)\, e^{-j2\pi fm}\right]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x^*(n)\, X(f) e^{j2\pi fn}$$

Using shifting property of Fourier transform.

If $\mathcal{F}\{x(m)\} = X(f)$ then $\mathcal{F}\{x(m+n)\} = X(f)\, e^{j2pfn}$

$$= \frac{1}{N} X(f)\left[\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right]^*$$

$$= \frac{1}{N} X(f)\, [X(f)]^* = \frac{1}{N}\, X(f)\, X^*(f)$$

$$= \frac{1}{N}|X(f)|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right|^2$$

By definition of Fourier transform,

$$X(f) = F\{x(n)\} = \sum_{n=0}^{N-1} x(n) e^{-j2\pi fn}$$

Hence proved.

10.6 Use of DFT/FFT in Power Spectrum Estimation

The computation of energy density spectrum and power density spectrum can be efficiently performed by using the techniques of DFT computation via FFT.

The energy density spectrum of a discrete time signal is [from equation 10.7)],

$$S_{xx}(f) = \left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right|^2$$

The power density spectrum of a discrete time random process is [from equation (10.19)],

$$P_{xx}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right|^2$$

The definition of DFT of x(n) is [from equation (5.2), of Chapter 5],

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi kn}{N}} \qquad .....(10.20)$$

On comparing, the above three equations we can say that the computation of the term $\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}$ in the energy / power spectrum is same as computation of the term $\sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi kn}{N}}$ in DFT, if 'f' is replaced by 'k/N'. Here, k/N for k = 0, 1, 2, ... N – 1, represents the N frequency intervals (or frequency points) at which the power/energy spectrum has to be computed. Therefore using equation (10.20), the equations of energy and power spectrum can be written as shown below.

$$S_{xx}\left(\frac{k}{N}\right) = \left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{k}{N}n}\right|^2 = |X(k)|^2$$

$$P_{xx}\left(\frac{k}{N}\right) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{k}{N}n}\right|^2 = \frac{1}{N}|X(k)|^2$$

Now, X(k) in the above two equations can be computed using any of the FFT algorithm (radix-2 DIT FFT or radix-2 DIF FFT) for computing DFT.

10.7 Nonparametric Methods of Power Spectrum Estimation

The ***nonparametric*** methods of power spectrum estimation methods are based on estimating the autocorrelation sequence of a random process from a set of measured data, and then taking Fourier transform of autocorrelation sequence to obtain the power spectrum estimate.

The various nonparametric methods of power spectrum estimation are,

1. Bartlett method (***Averaging periodograms***)
2. Welch method (***Averaging modified periodograms***)
3. Blackman-Tukey method (***Periodogram smoothing***)

10.7.1 Bartlett Method of Power Spectrum Estimation

Bartlett suggests the method of power spectrum estimation of a random process by periodogram averaging. The Bartlett method of power spectrum estimation involves the following three steps.

1. Divide the N-point sequence into L non-overlapping segments.
2. Compute the periodogram for each segment.
3. Compute the average of the L periodograms.

The average of L periodograms is the Bartlett estimate of power spectrum.

Let, x(n) be an ergodic random process, measured only over a finite interval, i.e., for n = 0,1,2,.......,N-1. In Bartlett method, the N-point sequence x(n) is divided into L numbers of non-overlapping segments each of length M.

Let, $x_0(n)$, $x_1(n)$, $x_2(n)$, ... $x_{L-1}(n)$, be the L number of segments of x(n).

The, i^{th} segment of x(n) is given by,

$$x_i(n) = x(n+iM) \quad ; i = 0, 1, 2, ..., L-1$$
$$; n = 0, 1, 2, ..., M-1$$

Example :

Let, x(n) be a sequence consisting of 12 samples. Let us divide this sequence into 3 segments each consisting of 4 samples.

$\therefore$ N = 12, L = 3, M = 4

Let, x(n) = { x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7), x(8), x(9), x(10), x(11) }

Let, $x_0(n)$, $x_1(n)$ and $x_2(n)$ be the 3 segments of x(n).

When i = 0; $x_0(n) = x(n+iM) = \{x(n+0\times4)\}|_{\text{for } n=0,1,2,3}$ = { x(0), x(1), x(2), x(3) }

When i = 1; $x_1(n) = x(n+iM) = \{x(n+1\times4)\}|_{\text{for } n=0,1,2,3}$ = { x(4), x(5), x(6), x(7) }

When i = 2; $x_2(n) = x(n+iM) = \{x(n+2\times4)\}|_{\text{for } n=0,1,2,3}$ = { x(8), x(9), x(10), x(11) }

Let, $P_{xx}^{0b}(f)$, $P_{xx}^{1b}(f)$, $P_{xx}^{2b}(f)$, ... $P_{xx}^{(L-1)b}(f)$, be periodogram estimate of $x_0(n)$, $x_1(n)$, $x_2(n)$, ... $x_{L-1}(n)$, respectively.

Now, the periodogram, $P_{xx}^{ib}(f)$ is given by,

$$\text{Periodogram, } P_{xx}^{ib}(f) = \frac{1}{M}\left|\sum_{n=0}^{M-1} x_i(n)\, e^{-j2\pi fn}\right|^2 \qquad .....(10.21)$$

The Bartlett power spectrum estimate is given by average of periodogram estimate of all the L segments of x(n).

Let, $P_{xx}^{B}(f)$ = Barlett power spectrum estimate.

$$\therefore\ P_{xx}^{B}(f) = \frac{P_{xx}^{0b}(f) + P_{xx}^{1b}(f) + P_{xx}^{2b}(f) + ... + P_{xx}^{(L-1)b}(f)}{L}$$

$$\therefore\ P_{xx}^{B}(f) = \frac{1}{L}\sum_{i=0}^{L-1} P_{xx}^{ib}(f) \qquad .....(10.22)$$

10.7.2 Welch Method of Power Spectrum Estimation

Welch proposed the following two changes in Bartlett method.

1. Allowing the segments to overlap.
2. Windowing the segments before computing periodogram.

The periodogram of windowed segments will be a modified periodogram, and averaging these modified periodograms will give the Welch estimate of power spectrum. The Welch method of power spectrum estimation involves the following three steps.

1. Divide the N-point sequence into L overlapping segments.
2. Window each segment, and compute the periodogram for each windowed segment.
3. Compute the average of the modified periodograms.

Let, x(n) be an ergodic random process, measured only over a finite interval, i.e., for n = 0,1,2,......., N – 1. In Welch method, the N-point sequence x(n) is divided into L numbers of overlapping segments each of length M. Usually, the overlap is expressed as percentage of M. The overlap factor is denoted as D, where D = %M.

Let, $x_0(n)$, $x_1(n)$, $x_2(n)$, ... $x_{L-1}(n)$, be the L number of segments of x(n).

The, i^{th} segment of x(n) is given by,

$$x_i(n) = x(n+iD) \quad ; \; i = 0, 1, 2, ..., L-1 \quad ; \; n = 0, 1, 2, ..., M-1 \quad ; \; D = \%M \qquad .....(10.23)$$

Example :

Let, x(n) be a sequence consisting of 12 samples. Let us divide this sequence into 3 segments each consisting of 6 samples with 50% overlap.

$\therefore$ N = 12, L = 3, M = 6, D = 50% of M = (50/100) x 6 = 3

Let, x(n) = { x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7), x(8), x(9), x(10), x(11) }

Let, $x_0(n)$, $x_1(n)$ and $x_2(n)$ be the 3 segments of x(n).

When i = 0; $x_0(n) = x(n+iD) = \{x(n+0\times3)\}|_{\text{for } n=0,1,2,3,4,5}$ = { x(0), x(1), x(2), x(3), x(4), x(5) }

When i = 1; $x_1(n) = x(n+iD) = \{x(n+1\times3)\}|_{\text{for } n=0,1,2,3,4,5}$ = { x(3), x(4), x(5), x(6), x(7), x(8) }

When i = 2; $x_2(n) = x(n+iD) = \{x(n+2\times3)\}|_{\text{for } n=0,1,2,3,4,5}$ = { x(6), x(7), x(8), x(9), x(10), x(11) }

Each segment is multiplied by a window sequence, w(n) and then the periodogram is computed for each product sequence. Let, $P_{xx}^{0w}(f)$, $P_{xx}^{1w}(f)$, $P_{xx}^{2w}(f)$, ... $P_{xx}^{(L-1)w}(f)$, be the periodogram estimate of product sequence $x_0(n)\,w(n)$, $x_1(n)\,w(n)$, $x_2(n)\,w(n)$, ... $x_{L-1}(n)\,w(n)$, respectively.

Now, the periodogram, $P_{xx}^{iw}(f)$ is given by,

$$\text{Periodogram, } P_{xx}^{iw}(f) = \frac{1}{MU}\left|\sum_{n=0}^{M-1} x_i(n)\, e^{-j2\pi fn}\right|^2 \qquad .....(10.24)$$

where, U is the normalization factor for power in the window function.

The normalization factor U is selected such that,

$$U = \frac{1}{M}\sum_{n=0}^{M-1} w^2(n) \qquad .....(10.25)$$

The Welch power spectrum estimate is given by the average of periodogram estimates of all the L segments of x(n).

Let, $P_{xx}^{W}(f)$ = Welch power spectrum estimate.

$$\therefore\ P_{xx}^{W}(f) = \frac{P_{xx}^{0w}(f) + P_{xx}^{1w}(f) + P_{xx}^{2w}(f) + ... + P_{xx}^{(L-1)w}(f)}{L}$$

$$\therefore\ P_{xx}^{W}(f) = \frac{1}{L}\sum_{i=0}^{L-1} P_{xx}^{iw}(f) \qquad .....(10.26)$$

10.7.3 Blackman and Tukey Method of Power Spectrum Estimation

In the ***Blackman*** and ***Tukey*** method, the autocorrelation sequence is windowed first, and then the Fourier transform of the windowed autocorrelation sequence is computed, which is the desired power spectrum estimate.

Let x(n) be an ergodic random process, measured only over a finite interval, i.e., for n = 0,1,2,....., N – 1. Let, $r_{xx}(m)$ be the autocorrelation sequence estimated using N-sample sequence of x(n).

The autocorrelation sequence $r_{xx}(m)$ is defined as,

$$\text{Autocorrelation sequence, } r_{xx}(m) = \frac{1}{N}\sum_{n=0}^{N-1} x(n+m)\, x^{*}(n) \qquad .....(10.27)$$

The autocorrelation sequence, $r_{xx}(m)$ can be extended to negative values of m, by using conjugate symmetry, $r_{xx}(-m) = r_{xx}^{*}(m)$. Then the autocorrelation sequence is multiplied by a window sequence, w(m) of length 2M – 1. The periodogram is computed by taking Fourier transform of product sequence.

Let, $P_{xx}^{BT}(f)$ = Blackman-Tukey power spectrum estimate.

$$P_{xx}^{BT}(f) = \sum_{m=-(M-1)}^{M-1} r_{xx}(m)w(m)e^{-j2\pi fm} \qquad .....(10.28)$$

Example 10.1

Compute the periodogram of the signal vector {1, 1, 1, 1, 0, 0, 0, 0} and sketch the periodogram.

Solution

Given that, x(n) = {1, 1, 1, 1, 0, 0, 0, 0}

The periodogram is an estimate of power density spectrum of x(n), and it is defined as,

$$\text{Periodogram, } P_{xx}^{Per}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right|^2$$

Since the given sequence is an 8-point sequence, let us compute the periodogram at 8 frequency intervals given by,

$$f = \frac{k}{8}\ ;\ \text{ for } k = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$$

$$\therefore\ P_{xx}^{Per}(f) = P_{xx}^{Per}\left(\frac{k}{8}\right) = \frac{1}{8}\left|\sum_{n=0}^{7} x(n)\, e^{-j\frac{2\pi kn}{8}}\right|^2 = \frac{1}{8}\left|X(k)\right|^2$$

$$\text{where, } X(k) = 8\text{ - point DFT of } x(n) = \sum_{n=0}^{7} x(n)\, e^{-j\frac{2\pi kn}{8}}$$

Let us compute 8-point DFT of x(n) by radix-2 DIT FFT method.

8-Point DFT by Radix-2 DIT-FFT

The given sequence is first arranged in the bit reversed order.

The sequence x(n) in normal order	The sequence x(n) in bit reversed order
x(0) = 1	x(0) = 1
x(1) = 1	x(4) = 0
x(2) = 1	x(2) = 1
x(3) = 1	x(6) = 0
x(4) = 0	x(1) = 1
x(5) = 0	x(5) = 0
x(6) = 0	x(3) = 1
x(7) = 0	x(7) = 0

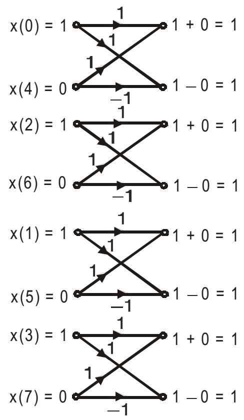


Fig 1 : Butterfly diagram for first-stage of radix-2 DIT FFT.

The 8-point DFT by radix-2 FFT involve 3 stages of computation with 4-butterfly computations in each stage. The sequence rearranged in the bit reversed order forms the input to the first-stage. For other stages of computation the output of previous stage will be the input for current stage.

First-Stage Computation

The input sequence of first-stage computation = { 1, 0, 1, 0, 1, 0, 1, 0}

The butterfly computations of first-stage are shown in fig 1.

> The phase factor involved in first-stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

The output sequence of first-stage of computation = { 1, 1, 1, 1, 1, 1, 1, 1}

Second-Stage Computation

The input sequence to second-stage computation = { 1, 1, 1, 1, 1, 1, 1, 1}

The phase factors involved in second-stage computation are W_4^0 and W_4^1.

The butterfly computations of second-stage are shown in fig 2.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

The output sequence of second-stage of computation = {2, 1−j, 0, 1+j, 2, 1 − j, 0,1+j}

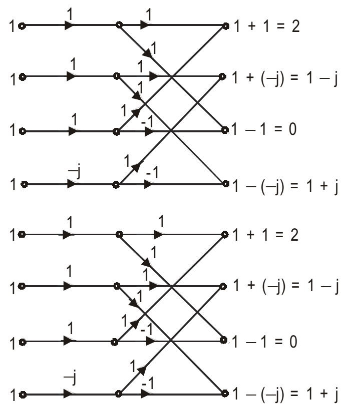


Fig 2 : Butterfly diagram for second-stage of radix-2 DIT FFT.

Third-Stage Computation

The input sequence to third-stage computation = {2, 1−j, 0, 1+j, 2, 1− j, 0, 1+ j}

The phase factors involved in third-stage computation are W_8^0, W_8^1, W_8^2 and W_8^3.

The butterfly computations of third-stage are shown in fig 3.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = 0.707 - j0.707$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -0.707 - j0.707$$

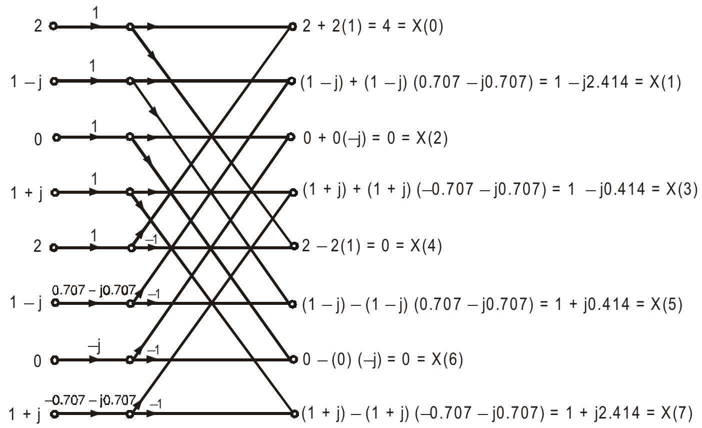


Fig 3 : Butterfly diagram for third-stage of radix-2 DIT FFT.

$$\left.\begin{array}{l}\text{The output sequence of third -}\\ \text{stage of computation}\end{array}\right\} = \{4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}$$

$$\therefore\ X(k) = \{4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}$$

$$|X(k)| = \{4,\ 2.613,\ 0,\ 1.082,\ 0,\ 1.082,\ 0,\ 2.613\}$$

$$|X(k)|^2 = \{16,\ 6.828,\ 0,\ 1.171,\ 0,\ 1.171,\ 0,\ 6.828\}$$

$$\therefore\ P_{xx}^{Per}(f) = P_{xx}^{Per}\left(\frac{k}{N}\right) = \frac{1}{8}|X(k)|^2 = \left\{\frac{16}{8},\ \frac{6.828}{8},\ 0,\ \frac{1.171}{8},\ 0,\ \frac{1.171}{8},\ 0,\ \frac{6.828}{8}\right\}$$

$$= \{2,\ 0.854,\ 0,\ 0.146,\ 0,\ 0.146,\ 0,\ 0.854\}$$

The sketch of the periodogram is shown in fig 4.

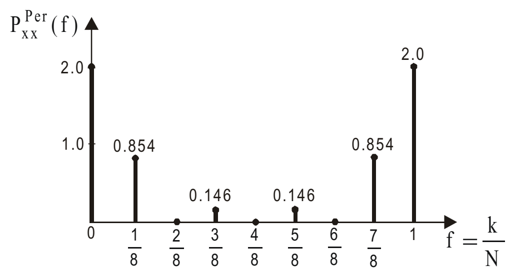


Fig 4 : Periodogram.

Example 10.2

Find the energy density spectrum of the discrete time signal, x(n) = {1, –1, 1, –1}.

Solution

Given that, x(n) = {1, –1, 1, –1}

The energy density spectrum of x(n) is defined as,

$$\text{Energy density spectrum, } S_{xx}(f) = \left|X(f)\right|^2 = \left|\sum_{n=-\infty}^{+\infty} x(n)\, e^{-j2\pi fn}\right|^2$$

The given sequence is a 4-point sequence.

Let us pad with zeros and convert the 4-point sequence to an 8-point sequence.

\ x(n) = {1, –1, 1, –1, 0, 0, 0, 0}

The energy density spectrum of this 8-point sequence is given by,

$$\text{Energy density spectrum, } S_{xx}(f) = \left|\sum_{n=0}^{7} x(n)\, e^{-j2\pi fn}\right|^2$$

Let us compute the energy density spectrum at 8 frequency points,

$$f = \frac{k}{8} \quad ; \quad \text{for } k = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$$

$$\therefore\ S_{xx}(f) = S_{xx}\left(\frac{k}{8}\right) = \left|\sum_{n=0}^{7} x(n)\, e^{-j\frac{2\pi kn}{8}}\right|^2 = \left|X(k)\right|^2$$

$$\text{where, } X(k) = \text{8-point DFT of } x(n) = \sum_{n=0}^{7} x(n)\, e^{-j\frac{2\pi kn}{8}}$$

The 8-point DFT of x(n) can be computed using radix-2 DIT FFT.

8-point DFT by Radix-2 DIT-FFT

The given sequence is first arranged in the bit reversed order.

The sequence x(n) in normal order	The sequence x(n) in bit reversed order
x(0) = 1	x(0) = 1
x(1) = –1	x(4) = 0
x(2) = 1	x(2) = 1
x(3) = –1	x(6) = 0
x(4) = 0	x(1) = –1
x(5) = 0	x(5) = 0
x(6) = 0	x(3) = –1
x(7) = 0	x(7) = 0

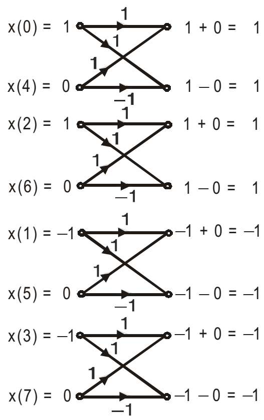


Fig 1 : Butterfly diagram for first-stage of radix-2 DIT FFT.

The 8-point DFT by radix-2 FFT involve 3 stages of computation with 4-butterfly computations in each stage. The sequence rearranged in the bit reversed order forms the input to the first-stage. For other stages of computation the output of previous stage will be the input for current stage.

First-Stage Computation

The input sequence of first-stage computation = { 1, 0, 1, 0, –1, 0, –1, 0}

The butterfly computations of first-stage are shown in fig 1.

> The phase factor involved in first-stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

The output sequence of first-stage of computation = { 1, 1, 1, 1, –1, –1, –1, –1}

Second-Stage Computation

The input sequence to second-stage computation = { 1, 1, 1, 1, –1, –1, –1, –1}

The phase factors involved in second-stage computation are W_4^0 and W_4^1.

The butterfly computations of second-stage are shown in fig 2.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

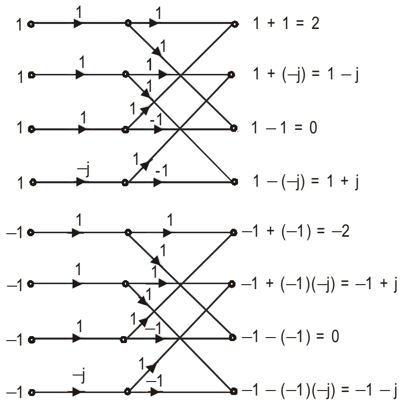


***Fig 2 :** Butterfly diagram for second-stage of radix-2 DIT FFT.*

$$\left.\begin{array}{l}\text{The output sequence of} \\ \text{second-stage of computation}\end{array}\right\} = \{2, 1-j, 0, 1+j, -2, -1+j, 0, -1-j\}$$

Third-Stage Computation

The input sequence to third-stage computation = {2, 1–j, 0, 1+j, –2, –1+ j, 0, –1– j}

The phase factors involved in third-stage computation are W_8^0, W_8^1, W_8^2 and W_8^3.

The butterfly computations of third-stage are shown in fig 3.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = 0.707 - j0.707$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -0.707 - j0.707$$

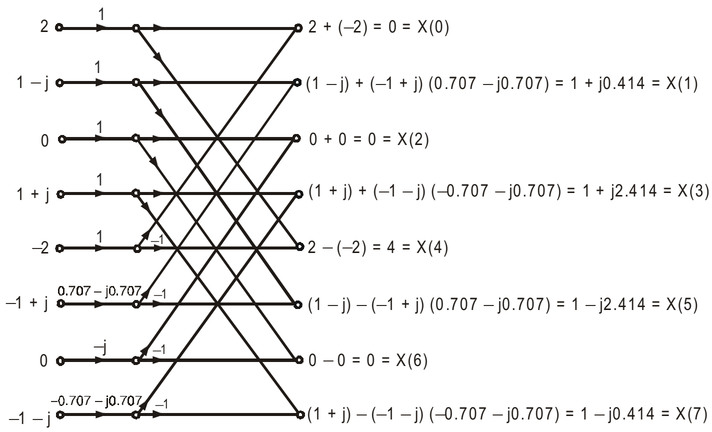


***Fig 3 :** Butterfly diagram for third-stage of radix-2 DIT FFT.*

$$\left.\begin{array}{l}\text{The output sequence of}\\ \text{third-stage computation}\end{array}\right\} = \{0,\ 1+j0.414,\ 0,\ 1+j2.414,\ 4,\ 1-j2.414,\ 0,\ 1-j0.414\}$$

$$X(k) = \{0,\ 1+j0.414,\ 0,\ 1+j2.414,\ 4,\ 1-j2.414,\ 0,\ 1-j0.414\}$$

$$|X(k)| = \{0,\ 1.082,\ 0,\ 2.613,\ 4,\ 2.613,\ 0,\ 1.082\}$$

$$\therefore\ S_{xx}(f) = S_{xx}\left(\tfrac{k}{N}\right) = |X(k)|^2 = \{0,\ 1.171,\ 0,\ 6.828,\ 16,\ 6.828,\ 0,\ 1.171\}$$

The sketch of energy density spectrum is shown in fig 4.

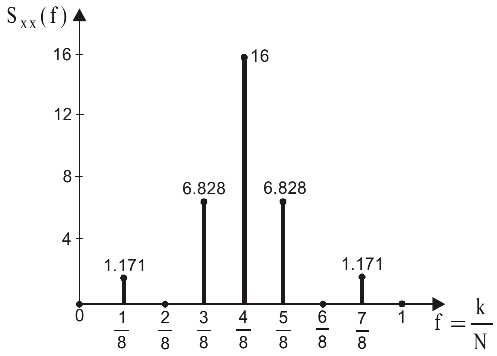


Fig 4 : Energy density spectrum.

Example 10.3

Compute the periodogram of the random signal by taking 8 samples of the signal.

$x(n) = \sin 2\pi f_1 n + \cos 2\pi(f_1 + \Delta f)n$, $f_1 = 0.2$, $\Delta f = 0.05$. Also sketch the periodogram.

Solution

$$x(n) = \sin 2\pi f_1 n + \cos 2\pi(f_1 + \Delta f)n = \sin(2\pi \times 0.2 \times n) + \cos(2\pi(0.2 + 0.05)n)$$

$$= \sin(0.4\pi n) + \cos(0.5\pi n)$$

Let us calculate x(n) for n = 0 to 7 as shown below.

Use calculator in radian mode.

When $n = 0$; $x(n) = x(0) = \sin(0.4\pi \times 0) + \cos(0.5\pi \times 0) = 1$

When $n = 1$; $x(n) = x(1) = \sin(0.4\pi \times 1) + \cos(0.5\pi \times 1) = 0.9511$

When $n = 2$; $x(n) = x(2) = \sin(0.4\pi \times 2) + \cos(0.5\pi \times 2) = -0.4122$

When $n = 3$; $x(n) = x(3) = \sin(0.4\pi \times 3) + \cos(0.5\pi \times 3) = -0.5878$

When $n = 4$; $x(n) = x(4) = \sin(0.4\pi \times 4) + \cos(0.5\pi \times 4) = 0.0489$

When $n = 5$; $x(n) = x(5) = \sin(0.4\pi \times 5) + \cos(0.5\pi \times 5) = 0$

When $n = 6$; $x(n) = x(6) = \sin(0.4\pi \times 6) + \cos(0.5\pi \times 6) = -0.0489$

When $n = 7$; $x(n) = x(7) = \sin(0.4\pi \times 7) + \cos(0.5\pi \times 7) = 0.5878$

$$\therefore\ x(n) = \{1, 0.9511, -0.4122, -0.5878, 0.0489, 0, -0.0489, 0.5878\}$$

The periodogram is an estimate of power density spectrum of x(n), and it is defined as,

$$\text{Periodogram},\ P_{xx}^{Per}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right|^2$$

Since x(n) is 8-point sequence, let us compute the periodogram at 8 frequency intervals given by,

$$f = \frac{k}{8} \quad ; \quad \text{for } k = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$$

$$\therefore\ P_{xx}^{Per}(f) = P_{xx}^{Per}\left(\frac{k}{8}\right) = \frac{1}{8}\left|\sum_{n=0}^{7} x(n)\, e^{-j2\pi kn/8}\right|^2 = \frac{1}{8}\,|X(k)|^2$$

$$\text{where, } X(k) = 8\text{ - point DFT of } x(n) = \sum_{n=0}^{7} x(n)\, e^{-j2\pi kn/8}$$

Let us compute 8-point DFT of x(n) by radix-2 DIT FFT method.

8-point DFT by Radix - 2 DIT FFT

The given sequence is first arranged in the bit reversed order.

The sequence x(n) in normal order	The sequence x(n) in bit reversed order
x(0) = 1	x(0) = 1
x(1) = 0.9511	x(4) = 0.0489
x(2) = –0.4122	x(2) = –0.4122
x(3) = –0.5878	x(6) = –0.0489
x(4) = 0.0489	x(1) = 0.9511
x(5) = 0	x(5) = 0
x(6) = –0.0489	x(3) = –0.5878
x(7) = 0.5878	x(7) = 0.5878

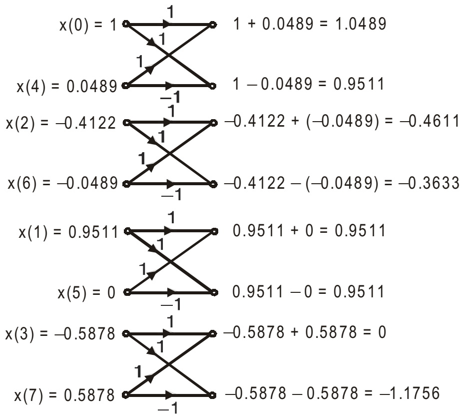


Fig 1 : Butterfly diagram for first-stage of radix-2 DIT FFT.

The 8-point DFT by radix-2 FFT involve 3 stages of computation with 4-butterfly computations in each stage. The sequence rearranged in the bit reversed order forms the input to the first-stage. For other stages of computation the output of previous stage will be the input for current stage.

First-stage Computation

$$\left.\begin{matrix}\text{The input sequence of}\\ \text{first - stage of computation}\end{matrix}\right\} = \left\{\begin{matrix}1,\ 0.0489,\ -0.4122,\ -0.0489,\\ 0.9511,\ 0,\ -0.5878,\ 0.5878\end{matrix}\right\}$$

The butterfly computations of first-stage are shown in fig 1.

$$\left.\begin{matrix}\text{The output sequence of}\\ \text{first - stage of computation}\end{matrix}\right\} = \left\{\begin{matrix}1.0489,\ 0.9511,\ -0.4611,\ -0.3633,\\ 0.9511,\ 0.9511,\ 0,\ -1.1756\end{matrix}\right\}$$

The phase factor involved in first-stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

Second-stage Computation

$$\left.\begin{matrix}\text{The input sequence of}\\ \text{second - stage of computation}\end{matrix}\right\} = \left\{\begin{matrix}1.0489,\ 0.9511,\ -0.4611,\ -0.3633,\ 0.9511,\\ 0.9511,\ 0,\ -1.1756\end{matrix}\right\}$$

The phase factors involved in second-stage computation are W_4^0 and W_4^1.

The butterfly computations of second-stage are shown in fig 2.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

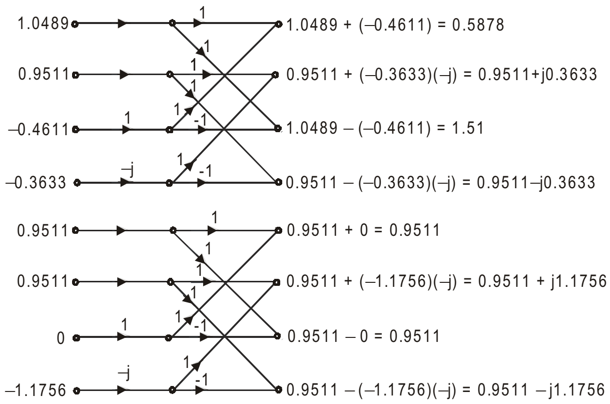


Fig 2 : Butterfly diagram for second-stage of radix-2 DIT FFT.

$$\left.\begin{array}{l}\text{The output sequence of}\\ \text{second-stage of computation}\end{array}\right\} = \left\{\begin{array}{l}0.5878,\ 0.9511+j0.3633,\ 1.51,\ 0.9511-j0.3633,\\ 0.9511,\ 0.9511+j1.1756,\ 0.9511, 0.9511-j1.1756\end{array}\right\}$$

Third-stage Computation

$$\left.\begin{array}{l}\text{The input sequence of}\\ \text{third-stage of computation}\end{array}\right\} = \left\{\begin{array}{l}0.5878,\ 0.9511+j0.3633,\ 1.51,\ 0.9511-j0.3633,\\ 0.9511,\ 0.9511+j1.1756,\ 0.9511, 0.9511-j1.1756\end{array}\right\}$$

The phase factors involved in third-stage computation are W_8^0, W_8^1, W_8^2 and W_8^3.

The butterfly computations of third-stage are shown in fig 3.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = 0.707 - j0.707$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -0.707 - j0.707$$

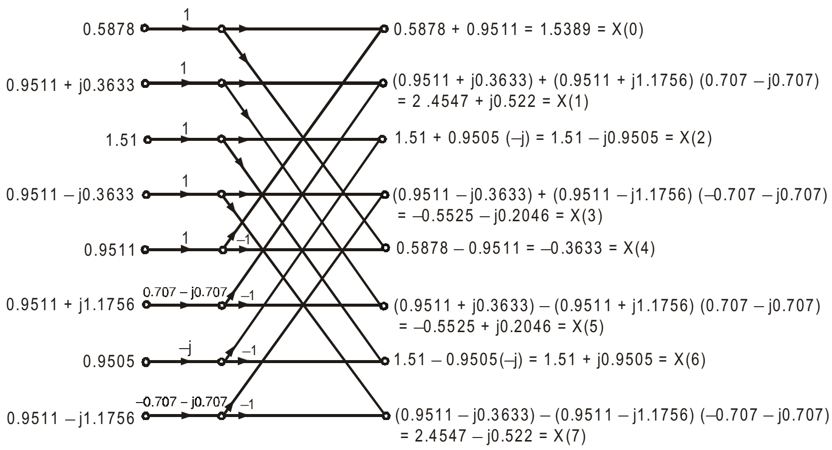


Fig 3 : Butterfly diagram for third-stage of radix-2 DIT FFT.

$$\left.\begin{array}{l}\text{The output sequence of}\\ \text{third-stage of computation}\end{array}\right\} = \left\{\begin{array}{l}1.5389,\ 2.4547 + j0.522,\ 1.51 - j0.9505,\ -0.5525 - j0.2046,\\ -0.3633,\ -0.5525 + j0.2046,\ 1.51 + j0.9505,\ 2.4547 - j0.522\end{array}\right\}$$

$$X(K) = \left\{\begin{array}{l}1.5389,\ 2.4547 + j0.522,\ 1.51 - j0.9505,\ -0.5525 - j0.2046,\\ -0.3633,\ -0.5525 + j0.2046,\ 1.51 + j0.9505,\ 2.4547 - j0.522\end{array}\right\}$$

$$|X(k)| = \{1.5389,\ 2.5096,\ 1.7843,\ 0.5892,\ 0.3633,\ 0.5892,\ 1.7843,\ 2.5096\}$$

$$|X(k)|^2 = \{2.3682,\ 6.2981,\ 3.1837,\ 0.3472,\ 0.132,\ 0.3472,\ 3.1837,\ 6.2981\}$$

$$\frac{1}{8}\ |X(k)|^2 = \left\{\frac{2.3682}{8},\ \frac{6.2981}{8},\ \frac{3.1837}{8},\ \frac{0.3472}{8},\ \frac{0.132}{8},\ \frac{0.3472}{8},\ \frac{3.1837}{8},\ \frac{6.2981}{8},\right\}$$

$$= \{0.2960,\ 0.7873,\ 0.3980,\ 0.0434,\ 0.0165,\ 0.0434,\ 0.3980,\ 0.7873\}$$

$$\therefore\ P_{xx}^{Per}(f) = P_{xx}^{Per}\left(\frac{k}{N}\right) = \frac{1}{8}|X(k)|^2 = \{0.2960,\ 0.7873,\ 0.3980,\ 0.0434,\ 0.0165,\ 0.0434,\ 0.3980,\ 0.7873\}$$

The sketch of periodogram is shown in fig 4.

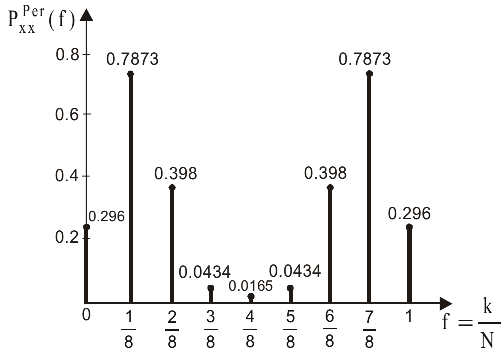


Fig 4 : Periodogram.

Example 10.4

Compute the periodogram of the random signal by taking 8 samples of the signal.

$x(n) = \cos 2\pi f_1 n + \cos 2\pi(f_1 + \Delta f)n$, $f_1 = 0.8$, $\Delta f = 0.1$. Also sketch the periodogram.

Solution

$$x(n) = \cos 2\pi f_1 n + \cos 2\pi(f_1 + \Delta f)n = \cos(2\pi \times 0.8 \times n) + \cos(2\pi(0.8 + 0.1)n)$$

$$= \cos(1.6\pi n) + \cos(1.8\pi n)$$

Let us calculate x(n) for n = 0 to 7 as shown below.

Use calculator in radian mode.

When n = 0 ; $x(n) = x(0) = \cos(1.6\pi \times 0) + \cos(1.8\pi \times 0) = 2$

When n = 1 ; $x(n) = x(1) = \cos(1.6\pi \times 1) + \cos(1.8\pi \times 1) = 1.118$

When n = 2 ; $x(n) = x(2) = \cos(1.6\pi \times 2) + \cos(1.8\pi \times 2) = -0.5$

When n = 3 ; $x(n) = x(3) = \cos(1.6\pi \times 3) + \cos(1.8\pi \times 3) = -1.118$

When n = 4 ; $x(n) = x(4) = \cos(1.6\pi \times 4) + \cos(1.8\pi \times 4) = -0.5$

When n = 5 ; $x(n) = x(5) = \cos(1.6\pi \times 5) + \cos(1.8\pi \times 5) = 0$

When n = 6 ; $x(n) = x(6) = \cos(1.6\pi \times 6) + \cos(1.8\pi \times 6) = -0.5$

When n = 7 ; $x(n) = x(7) = \cos(1.6\pi \times 7) + \cos(1.8\pi \times 7) = -1.118$

$$\therefore\ x(n) = \{2,\ 1.118,\ -0.5,\ -1.118,\ -0.5,\ 0,\ -0.5,\ -1.118\}$$

The periodogram is an estimate of power density spectrum of x(n), and it is defined as,

$$\text{Periodogram,}\ P_{xx}^{Per}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right|^2$$

Since x(n) is 8-point sequence, let us compute the periodogram at 8 frequency intervals given by,

$$f = \frac{k}{8}\ ;\ \text{for } k = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$$

$$\therefore\ P_{xx}^{Per}(f) = P_{xx}^{Per}\left(\frac{k}{8}\right) = \frac{1}{8}\left|\sum_{n=0}^{7} x(n)\, e^{-j2\pi kn/8}\right|^2$$

$$= \frac{1}{8}\left|X(k)\right|^2$$

where, X(k) = 8 - point DFT of x(n)

$$= \sum_{n=0}^{7} x(n)\, e^{-j2\pi kn/8}$$

Let us compute 8-point DFT of x(n) by radix-2 DIT FFT method.

8-Point DFT by Radix - 2 DIT FFT

The given sequence is first arranged in bit reversed order.

The sequence x(n) in normal order	The sequence x(n) in bit reversed order
x(0) = 2	x(0) = 2
x(1) = 1.118	x(4) = – 0.5
x(2) = –0.5	x(2) = –0.5
x(3) = –1.118	x(6) = –0.5
x(4) = – 0.5	x(1) = 1.118
x(5) = 0	x(5) = 0
x(6) = –0.5	x(3) = –1.118
x(7) = – 1.118	x(7) = – 1.118

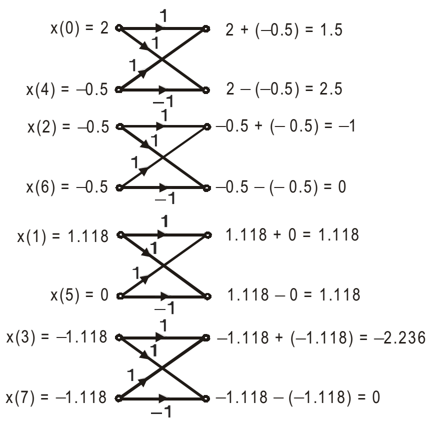


Fig 1 : Butterfly diagram for first stage of radix-2 DIT FFT.

The 8-point DFT by radix-2 FFT involve 3 stages of computation with 4-butterfly computations in each stage. The sequence rearranged in the bit reversed order forms the input to the first-stage. For other stages of computation the output of previous stage will be the input for current stage.

First-stage Computation

$$\left.\begin{array}{l}\text{The input sequence of}\\ \text{first - stage of computation}\end{array}\right\} = \{2,\ -0.5,\ -0.5,\ -0.5,\ 1.118,\ 0,\ -1.118,\ -1.118\}$$

The butterfly computations of first-stage are shown in fig 1.

$$\left.\begin{matrix}\text{The output sequence of} \\ \text{first - stage of computation}\end{matrix}\right\} = \{1.5,\ 2.5,\ -1,\ 0,\ 1.118,\ 1.118,\ -2.236,\ 0\}$$

The phase factor involved in first-stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

Second-stage Computation

$$\left.\begin{matrix}\text{The input sequence of} \\ \text{second - stage of computation}\end{matrix}\right\} = \{1.5,\ 2.5,\ -1,\ 0,\ 1.118,\ 1.118,\ -2.236,\ 0\}$$

The phase factors involved in second-stage computation are W_4^0 and W_4^1.

The butterfly computations of second-stage are shown in fig 2.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = e^0 = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

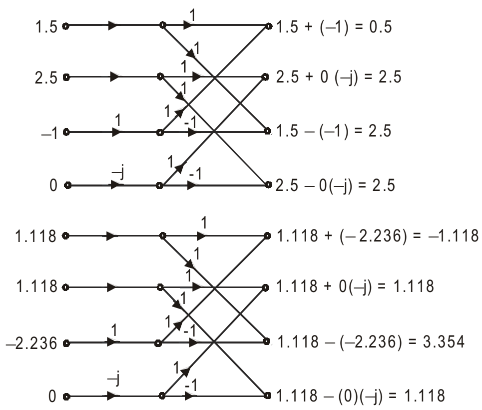


Fig 2 : *Butterfly diagram for second-stage of radix-2 DIT FFT.*

$$\left.\begin{matrix}\text{The output sequence of} \\ \text{second - stage of computation}\end{matrix}\right\} = \{0.5,\ 2.5,\ 2.5,\ 2.5,\ -1.118,\ 1.118,\ 3.354,\ 1.118\}$$

Third-stage Computation

$$\left.\begin{matrix}\text{The input sequence to} \\ \text{third - stage of computation}\end{matrix}\right\} = \{0.5,\ 2.5,\ 2.5,\ 2.5,\ -1.118,\ 1.118,\ 3.354,\ 1.118\}$$

The phase factors involved in third-stage computation are W_8^0, W_8^1, W_8^2 and W_8^3.

The butterfly computations of third-stage are shown in fig 3.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = e^0 = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = 0.707 - j0.707$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -0.707 - j0.707$$

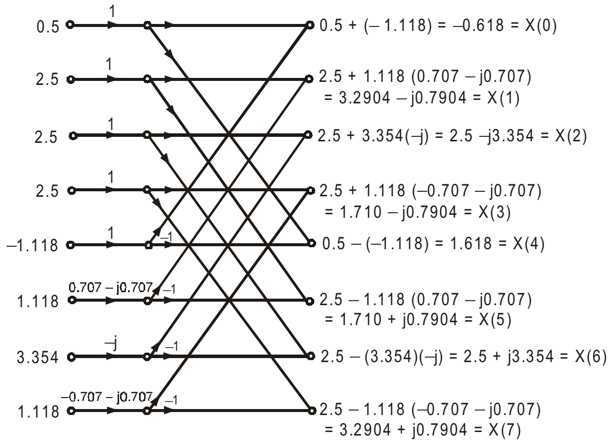


Fig 3 : Butterfly diagram for third-stage of radix-2 DIT FFT.

$$\left.\begin{array}{l}\text{The output sequence of}\\ \text{third-stage of computation}\end{array}\right\} = \left\{\begin{array}{l}-0.618,\ 3.2904 - j0.7904,\ 2.5 - j3.354,\ 1.710 - j0.7904,\\ 1.618,\ 1.710 + j0.7904,\ 2.5 + j3.354,\ 3.2904 + j0.7904\end{array}\right\}$$

$$\therefore\ X(K) = \left\{\begin{array}{l}-0.618,\ 3.2904 - j0.7904,\ 2.5 - j3.354,\ 1.710 - j0.7904,\\ 1.618,\ 1.710 + j0.7904,\ 2.5 + j3.354,\ 3.2904 + j0.7904\end{array}\right\}$$

$$|X(k)| = \{0.618,\ 3.384,\ 4.1832,\ 1.8838,\ 1.618,\ 1.8838,\ 4.1832,\ 3.384\}$$

$$|X(k)|^2 = \{0.3819,\ 11.4515,\ 17.4991,\ 3.5487,\ 2.6179,\ 3.5487,\ 17.4991,\ 11.4515\}$$

$$\frac{1}{8}\ |X(k)|^2 = \{0.0477,\ 1.4314,\ 2.1874,\ 0.4436,\ 0.3272,\ 0.4436,\ 2.1874,\ 1.4314\}$$

$$\therefore\ P_{xx}^{Per}(f) = P_{xx}\left(\frac{k}{N}\right)$$

$$= \frac{1}{8}|X(k)|^2 = \{0.0477,\ 1.4314,\ 2.1874,\ 0.4436,\ 0.3272,\ 0.4436,\ 2.1874,\ 1.4314\}$$

The sketch of periodogram is shown in fig 4.

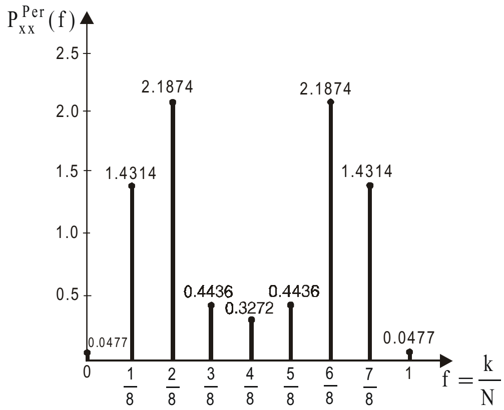


Fig 4 : Periodogram.

Example 10.5

White noise with power spectral density, $P_{xx}(e^{jw}) = s^2$ is passed through a filter with impulse response $h(n) = 0.5^n\, u(n)$. What is the output Power Spectral Density (PSD) ?

Solution

Let, x(n) = Input white noise

Given that PSD of input white noise, $P_{xx}(e^{jw}) = s^2$

We know that, $P_{xx}(e^{jw}) = |X(e^{jw})|^2$

$$\therefore \quad \left|X\left(e^{j\omega}\right)\right| = \sqrt{P_{xx}\left(e^{j\omega}\right)} = \sqrt{\sigma^2} = \sigma$$

Let, $X(e^{jw}) = s$ (Real signal) .....(1)

Let, y(n) = Output of the filter.

The output y(n) of the filter can be obtained by convolution of input x(n) with impulse response h(n).

$$\backslash \quad y(n) = x(n) * h(n)$$

On taking Fourier transform of above equation we get,

$$Y(e^{jw}) = X(e^{jw})\, H(e^{jw})$$

Using convolution property of Fourier transform.

$$\text{Here,} \quad H\left(e^{j\omega}\right) = \mathcal{F}\left\{h(n)\right\} = \mathcal{F}\left\{0.5^n\, u(n)\right\} = \frac{1}{1-0.5\,e^{-j\omega}} \qquad .....(2)$$

$$\therefore \quad Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right) = \sigma \times \frac{1}{1-0.5\,e^{-j\omega}}$$

Using equations (1) and (2).

$$Y\left(e^{j\omega}\right) = \frac{\sigma}{1-0.5\,e^{-j\omega}}$$

Let, $P_{yy}(e^{jw})$ = Power Spectral Density (PSD) of output signal y(n).

Now, $P_{yy}\left(e^{j\omega}\right) = \left|Y\left(e^{j\omega}\right)\right|^2 = Y\left(e^{j\omega}\right) Y^*\left(e^{j\omega}\right)$

$$= \frac{\sigma}{1-0.5e^{-j\omega}} \times \frac{\sigma}{1-0.5e^{j\omega}}$$

$$= \frac{\sigma^2}{1-0.5e^{j\omega}-0.5e^{-j\omega}+0.25}$$

$$= \frac{\sigma^2}{1.25-0.5\left(e^{j\omega}+e^{-j\omega}\right)}$$

$$= \frac{\sigma^2}{1.25-0.5\times 2\cos\omega} \qquad \boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

$$= \frac{\sigma^2}{1.25-\cos\omega}$$

$$\boxed{\therefore \quad \text{Output PSD} = \frac{\sigma^2}{1.25-\cos\omega}}$$

10.8 Performance Characteristics of Nonparametric Methods of Power Spectrum Estimation

The nonparametric methods use a finite set of data for estimating power spectrum of the random process. Hence, the estimates by various methods have to be compared to find its closeness to true value. A few common properties used to compare the estimates are bias, variance and consistency.

Bias:

The difference between the mean or expected value of an estimate and its true value is called the ***bias***. If the mean of an estimate is equal to the true value, then the bias is zero, and the estimate is considered to be unbiased. If the mean of an estimate is not equal to the true value, then the bias in nonzero, and the estimate is said to be biased. If the mean of an estimate is equal to the true value, as N tend to infinity, then the estimate is called ***asymptotically unbiased***.

Consistency:

If the bias and variance both tend to zero as the number of samples tends to infinity or the number of observations become large, the estimator is said to be consistent.

Frequency resolution:

It is the smallest frequency that can be identified by an estimator.

Quality factor:

It is defined as the ratio of square of the expected value of power spectrum estimate and the variance. The reciprocal of quality factor is called ***variability***.

Figure of merit:

It is defined as the product of variability and frequency resolution.

10.8.1 Performance Characteristics of Periodogram Power Spectrum Estimation

Expected Value of Autocorrelation Sequence $r_{xx}(m)$

$$\text{Expected value of } r_{xx}(m) = E\{r_{xx}(m)\}$$

$$= E\left\{\frac{1}{N}\sum_{n=0}^{N-1} x(n+m)\,x^*(n)\right\}$$ Using equation (10.12).

$$= E\left\{\frac{1}{N}\sum_{n=0}^{N-1-m} x(n+m)\,x^*(n)\right\}$$ To exclude the terms that lie outside the range of N – 1

$$= \frac{1}{N}\sum_{n=0}^{N-1-m} E\{x(n+m)x^*(n)\}$$ $E\{x(n+m)\,x^*(n)\} = \gamma_{xx}(m)$

$$= \frac{1}{N}\sum_{n=0}^{N-1-m} \gamma_{xx}(m)$$

$$= \frac{1}{N}\gamma_{xx}(m)\sum_{n=0}^{N-1-m} 1^n$$

$$= \frac{1}{N}\,\gamma_{xx}(m)\underbrace{[1+1+\;.....\;+1]}_{N-1-m+1 \text{ terms}}$$

$$= \frac{1}{N}\,\gamma_{xx}(m)\,[N-m]$$

$$= \gamma_{xx}(m)\,\frac{N-m}{N} \qquad .....(10.29)$$

$$= \gamma_{xx}(m)\,w_B(m) \qquad .....(10.30)$$

where, $w_B(m) = \dfrac{N-m}{N}$ = Bartlett window sequence

Expected Value of Periodogram $P_{xx}^{Per}(f)$

$$\text{Expected value of periodogram } = E\{P_{xx}^{Per}(f)\}$$

$$= E\left\{\sum_{m=-(N-1)}^{N-1} r_{xx}(m)\,e^{-j2\pi fm}\right\}$$ Using equation (10.18).

$$= \sum_{m=-(N-1)}^{N-1} E\{r_{xx}(m)\}\,e^{-j2\pi fm}$$

$$= \sum_{m=-(N-1)}^{N-1} \gamma_{xx}(m)\,w_B(m)e^{-j2\pi fm}$$ Using equation (10.30).

$$= P_{xx}(f) * W_B(f) \qquad .....(10.31)$$

where, $P_{xx}(f) = \mathcal{F}\{\gamma_{xx}(m)\}$

$$W_B(f) = \mathcal{F}\{w_B(m)\} = \frac{1}{N}\left(\frac{\sin \pi f N}{\sin \pi f}\right)^2$$

$*$ = Symbol for convolution

From the above equation we can say that the expected value of periodogram is the convolution of the power spectrum, $P_{xx}(f)$ with Fourier transform of Bartlett window sequence, $W_B(f)$. Therefore, the periodogram is biased estimate.

When, N tends to infinity, the $W_B(f)$ converges to impulse and so the expected value is equal to true value, $P_{xx}(f)$. Therefore, the periodogram is asymtotically unbiased.

$$\text{i.e., } \underset{N\to\infty}{\text{Lt}}\ E\left\{P_{xx}^{per}(f)\right\} = \underset{N\to\infty}{\text{Lt}}\ \left[P_{xx}(f) * W_B(f)\right] = P_{xx}(f) \qquad .....(10.32)$$

Therefore, for very large values of N,

$$E\left\{P_{xx}^{per}(f)\right\} = P_{xx}(f) \qquad .....(10.33)$$

Variance of Periodogram $P_{xx}^{Per}(f)$

The variance of the periodogram is given by equation (10.34). [The proof of this equation is beyond the scope of the book].

$$\text{Variance of periodogram } = \text{Var}\left\{P_{xx}^{per}(f)\right\} = \left[P_{xx}(f)\right]^2 \left[1 + \frac{\sin 2\pi f N}{N \sin 2\pi f}\right]^2 \qquad .....(10.34)$$

When, N tends to infinity, the term $\left[1 + \frac{\sin 2\pi f N}{N \sin 2\pi f}\right]$ becomes unity. Therefore, the variance can be expressed as shown in equation (10.35).

$$\therefore \text{ variance of periodogram } = \text{Var}\left\{P_{xx}^{per}(f)\right\} \approx \left[P_{xx}(f)\right]^2 \qquad .....(10.35)$$

From equations (10.34) and (10.35), we can say that the variance does not tend to zero as N tends to infinity, and so the periodogram is not a consistant estimate of power spectrum.

Frequency Resolution of Periodogram

The frequency resolution is the ability of the periodogram to resolve closely spaced frequency components. The periodogram is computed by considering only N sample of the random process and so it is equivalent to windowing the random process using rectangular window. Therefore, the frequency resolution can be defined as half power frequency width of main lobe of rectangular window. (Refer table 10.1 for half power width of various windows).

Let, Δf_{Per} = Frequency resolution of periodogram.

Now, $\Delta f_{Per} = \Delta f_{3dB}$ of N-point rectangular window.

$$\therefore\ \Delta f_{per} = \frac{0.89}{N} \qquad .....(10.36)$$

Quality Factor and Variability of Periodogram

The quality factor is defined as the ratio of square of the expected value of power spectrum estimate and the variance.

$$\therefore \text{ quality factor of periodogram, } Q_{per} = \frac{\left[E\left\{P_{xx}^{per}(f)\right\}\right]^2}{Var\left\{P_{xx}^{per}(f)\right\}}$$

$$\boxed{\text{Using equations (10.33) and (10.35).}}$$

$$= \frac{\left[P_{xx}(f)\right]^2}{\left[P_{xx}(f)\right]^2} = 1 \qquad .....(10.37)$$

The variability is a reciprocal of the quality factor.

$$\therefore \text{ variability of periodogram, } = \mathcal{V}_{per} = \frac{1}{Q_{per}} = 1 \qquad .....(10.38)$$

Figure of Merit of Periodogram

The figure of merit is defined as the product of variability and frequency resolution.

$$\therefore \text{ figure of merit of periodogram, } \mathcal{M}_{per} = \mathcal{V}_{per} \times \Delta f_{per} = 1 \times \frac{0.89}{N} = \frac{0.89}{N} \qquad .....(10.39)$$

10.8.2 Performance Characteristics of Bartlett Power Spectrum Estimation

Expected Value of Bartlett Estimate $P_{xx}^{B}(f)$

$$\boxed{\text{Using equation (10.22)}}$$

$$\text{Expected value } = E\left\{P_{xx}^{B}(f)\right\} = E\left\{\frac{1}{L}\sum_{i=0}^{L-1} P_{xx}^{ib}(f)\right\}$$

$$= \frac{1}{L}\sum_{i=0}^{L-1} E\left\{P_{xx}^{ib}(f)\right\} = E\left\{P_{xx}^{ib}(f)\right\}\frac{1}{L}\sum_{i=0}^{L-1} 1^i$$

$$= E\left\{P_{xx}^{ib}(f)\right\}\frac{1}{L}\underbrace{\left[1+1+......+1\right]}_{L-1+1 \text{ terms}} = E\left\{P_{xx}^{ib}(f)\right\}\frac{1}{L}\times L$$

$$\boxed{\text{Using equation (10.31)}}$$

$$= E\left\{P_{xx}^{ib}(f)\right\} = P_{xx}(f) * W_B(f) \qquad .....(10.40)$$

When N tends to infinity, the $W_B(f)$ converges to impulse and so the expected value is equal to true value, $P_{xx}(f)$. Therefore, the Bartlett estimate is asymptotically unbiased.

$$\text{i.e., } \underset{N\to\infty}{Lt} E\left\{P_{xx}^{B}(f)\right\} = \underset{N\to\infty}{Lt}\left[P_{xx}(f) * W_B(f)\right] = P_{xx}(f)$$

Therefore, for very large values of N,

$$E\left\{P_{xx}^{B}(f)\right\} = P_{xx}(f) \qquad .....(10.41)$$

Variance of Bartlett Estimate $P_{xx}^{B}(f)$

$$\text{Variance of Bartlett estimate } = \text{Var}\left\{P_{xx}^{B}(f)\right\}$$

$$= \frac{1}{L}\ \text{Var}\left\{P_{xx}^{ib}(f)\right\}$$

Using equation (10.35).

$$= \frac{1}{L}\left[P_{xx}(f)\right]^2 \qquad .....(10.42)$$

When L tends to infinity, the variance tends to zero.

$$\text{i.e., } \underset{N\to\infty}{\text{Lt}}\ \text{Var}\left\{P_{xx}^{B}(f)\right\} = \underset{L\to\infty}{\text{Lt}}\ \frac{1}{L}\left[P_{xx}(f)\right]^2 = \frac{1}{\infty}\times\left[P_{xx}(f)\right]^2 = 0$$

Since, the variance tend to zero as L tends to infinity, the Bartlett estimate is a consistant estimate of power spectrum.

Frequency Resolution of Bartlett Estimate

In the Bartlett method, each section of the periodogram is computed by considering only M sample of the random process and so it is equivalent to windowing the random process using rectangular window of length M. Therefore, the frequency resolution can be defined as half power frequency width of main-lobe of rectangular window.

Let, Δf_B = Frequency resolution of Bartlett estimate.

Now, $\Delta f_B = \Delta f_{3dB}$ of M-point rectangular window.

In Bartlett method, $M = \frac{N}{L}$

$$\therefore\ \Delta f_B = \frac{0.89}{M} = \frac{0.89}{N/L} = L\frac{0.89}{N} \qquad .....(10.43)$$

From equations (10.36) and (10.43), we can say that the frequency resolution of Bartlett estimate is L times more than that of periodogram and so the capability of Bartlett estimate to identify closely spaced spectra reduces.

Quality Factor and Variability of Bartlett Estimate

Let, Q_B = Quality factor of Bartlett estimate.

$$\text{Now, } Q_B = \frac{\left[E\left\{P_{xx}^{B}(f)\right\}\right]^2}{\text{Var}\left\{P_{xx}^{B}(f)\right\}} = \frac{\left[P_{xx}(f)\right]^2}{\frac{1}{L}\left[P_{xx}(f)\right]^2} = L \qquad .....(10.44)$$

$$\therefore\ \text{variability of Bartlett estimate, } \mathcal{V}_B = \frac{1}{Q_B} = \frac{1}{L} \qquad .....(10.45)$$

Alternatively, the quality factor, Q_B of Bartlett estimate can be expressed in terms of N and Δf_B as shown in equation (10.46).

$$\text{From equation (10.43), we get, } L = \frac{N}{0.89}\ \Delta f_B$$

$$\therefore\ Q_B = L = \frac{N}{0.89}\ \Delta f_B = 1.12N\ \Delta f_B \qquad .....(10.46)$$

Figure of Merit of Bartlett Estimate

Figure of merit of Bartlett estimate, $\mathcal{M}_B = \mathcal{V}_B \times \Delta f_B$

$$= \frac{1}{L} \times L \frac{0.89}{N} = \frac{0.89}{N} \qquad .....(10.47)$$

10.8.3 Performance Characteristics of Welch Power Spectrum Estimation

Expected Value of Welch Estimate $P_{xx}^{W}(f)$

The expected value of Welch estimate is given by equation (10.48). [The proof of this equation is beyond the scope of the book].

$$\text{Expected value of welch estimate } = E\left\{P_{xx}^{W}(f)\right\}$$

$$= \frac{1}{MU} P_{xx}(f) * |W(f)|^2 \qquad .....(10.48)$$

where, $W(f) = \mathcal{F}\{w(n)\}$

$w(n)$ = window sequence used in Welch estimate.

When N tends to infinity, the W(f) converges to impulse and so the expected value is equal to true value, $P_{xx}(f)$. Therefore, the Welch estimate is asymptotically unbiased.

$$\text{i.e., } \underset{N\to\infty}{Lt}\; E\left\{P_{xx}^{W}(f)\right\} = \underset{N\to\infty}{Lt}\left[\frac{1}{MU} P_{xx}(f) * [W(f)]^2\right] = P_{xx}(f)$$

Therefore, for very large values of N,

$$E\left\{P_{xx}^{W}(f)\right\} = P_{xx}(f) \qquad .....(10.49)$$

Variance of Welch Estimate $P_{xx}^{W}(f)$

The variance of Welch estimate, when Bartlett window is employed and with 50% overlap is given by equation (10.50).

$$\text{Variance of Welch estimate } = Var\left\{P_{xx}^{W}(f)\right\} = \frac{9}{8L}\left[P_{xx}(f)\right]^2 \qquad .....(10.50)$$

Frequency Resolution of Welch Estimate

In the Welch method, each section of the periodogram is computed by selecting only M samples of overlapped sections of the random process using an M-point window sequence. Therefore, the frequency resolution can be defined as half power frequency width of main lobe of the window used for selecting M samples. (Refer Table 10.1 for half power width of various windows).

The frequency resolution of the Welch estimate, when the Bartlett window is employed is given by equation (10.51).

Let, Δf_W = Frequency resolution of Welch estimate.

Now, $\Delta f_W = \Delta f_{3dB}$ of M-point Bartlett window.

$$\therefore\ \Delta f_W = \frac{1.28}{M} \qquad .....(10.51)$$

Quality Factor and Variability of Welch Estimate

Let, Q_W = Quality factor of Welch estimate.

Using equations (10.44) and (10.50).

$$\text{Now, } Q_W = \frac{\left[E\left\{P_{xx}^W(f)\right\}\right]^2}{Var\left\{P_{xx}^W(f)\right\}} = \frac{[P_{xx}(f)]^2}{\frac{9}{8L}[P_{xx}(f)]^2} = \frac{8L}{9} \qquad .....(10.52)$$

$$\therefore \text{ variability of Welch estimate, } \mathcal{V}_W = \frac{1}{Q_W} = \frac{9}{8}\frac{1}{L} \qquad .....(10.53)$$

Alternatively, the quality factor, Q_w of Welch estimate can be expressed in terms of N and Δf_W as shown in equation (10.56).

$$\text{From equation (10.50), we get, } M = \frac{1.28}{\Delta f_W} \qquad .....(10.54)$$

$$\text{For no overlap,} \quad L = \frac{N}{M}$$

$$\text{For 50\% overlap, } L = \frac{N}{\frac{50}{100}M} = \frac{2N}{M} \qquad .....(10.55)$$

Using equations (10.54) and (10.55), the Q_w can be expressed as shown in equation (10.56).

$$Q_W = \frac{8L}{9} = \frac{8}{9} \times \frac{2N}{M} = \frac{16}{9} \times \frac{N}{\frac{1.28}{\Delta f_W}} = 1.39N\,\Delta f_W \qquad .....(10.56)$$

Figure of Merit of Welch Estimate

From equation (10.55) LM = 2N.

$$\text{Figure of merit of Welch estimate, } \mathcal{M}_W = \mathcal{V}_W \times \Delta f_W$$

$$= \frac{9}{8}\frac{1}{L} \times \frac{1.28}{M} = \frac{1.44}{LM} = \frac{1.44}{2N} = \frac{0.72}{N} \qquad .....(10.57)$$

10.8.4 Performance Characteristics of Blackman-Tukey Power Spectrum Estimation

Expected Value of Blackman-Tukey Estimate $P_{xx}^{BT}(f)$

When a Bartlett window of length 2M is employed, the expected value of Blackman-Tukey estimate can be approximately expressed as shown in equation (10.58).

$$E\left\{P_{xx}^{BT}(f)\right\} = P_{xx}(f) * W_B(f) \qquad .....(10.58)$$

When N tends to infinity, the $W_B(f)$ converges to impulse and so the expected value is equal to true value, $P_{xx}(f)$. Therefore, the Blackman-Tukey estimate is asymtotically unbiased.

$$\text{i.e., } \underset{N\to\infty}{Lt}\ E\left\{P_{xx}^{BT}(f)\right\} = \underset{N\to\infty}{Lt}\ \left[P_{xx}(f) * W_B(f)\right] = P_{xx}(f)$$

Therefore, for very large values of N,

$$E\left\{P_{xx}^{BT}(f)\right\} = P_{xx}(f) \qquad .....(10.59)$$

Variance of Blackman-Tukey $P_{xx}^{BT}(f)$

When a Bartlett window of length 2M is employed, the variance of Blackman-Tukey estimate can be approximately expressed as shown in equation (10.60).

$$\text{Variance of Blackman-Tukey estimate } = \text{Var}\left\{P_{xx}^{BT}(f)\right\}$$

$$= \frac{2M}{3N}\left[P_{xx}(f)\right]^2 \qquad .....(10.60)$$

Frequency Resolution of Blackman-Tukey Estimate

In the Blackman-Tukey method, the periodogram is computed by selecting only M samples of the random process using a M-point window sequence. Therefore, the frequency resolution can be defined as half power frequency width of main lobe of the window used for selecting M samples. (Refer table 10.1 for half power width of various windows).

The frequency resolution of Blackman-Tukey estimate, when a Bartlett window of length 2M is employed, is given by equation (10.61).

Let, Df_{BT} = Frequency resolution of Blackman-Tukey estimate.

Now, $Df_{BT} = Df_{3dB}$ of 2M-point Bartlett window.

$$\therefore\ \Delta f_{BT} = \frac{1.28}{2M} = \frac{0.64}{M} \qquad .....(10.61)$$

Quality Factor and Variability of Blackman-Tukey Estimate

Let, Q_{BT} = Quality factor of Blackman-Tukey estimate.

$$\text{Now, } Q_{BT} = \frac{\left[E\left\{P_{xx}^{BT}(f)\right\}\right]^2}{\text{Var}\left\{P_{xx}^{BT}(f)\right\}} = \frac{\left[P_{xx}(f)\right]^2}{\frac{2M}{3N}\left[P_{xx}(f)\right]^2} = \frac{3N}{2M} \qquad .....(10.62)$$

$$\therefore\ \text{variability of Blackman-Tukey estimate, } \mathcal{V}_{BT} = \frac{1}{Q_{BT}} = \frac{2M}{3N} \qquad .....(10.63)$$

Alternatively, the quality factor, Q_{BT} of Blackman-Tukey estimate can be expressed in terms of N and Df_{BT} as shown in equation (10.64).

$$\text{From equation (10.61), we get, } M = \frac{0.64}{\Delta f_{BT}}$$

$$\therefore\ Q_{BT} = \frac{3N}{2M} = \frac{3N}{2\times\frac{0.64}{\Delta f_{BT}}} = 2.34N\,\Delta f_{BT} \qquad .....(10.64)$$

Figure of Merit of Blackman-Tukey Estimate

$$\text{Figure of merit of Blackman-Tukey estimate, } \mathcal{M}_{BT} = \mathcal{V}_{BT}\times\Delta f_{BT}$$

$$= \frac{2M}{3N}\times\frac{0.64}{M} = \frac{0.43}{N} \qquad .....(10.65)$$

10.8.5 Summary of Performance Characteristics of Power Spectrum Estimation

Table 10.1: Properties of several commonly used windows with length N

Window	Sidelobe level	3 dB Bandwidth	
		$\Delta\omega_{3dB}$	Δf_{3dB}
Rectangular	-13 dB	$0.89\,(2\pi/N)$	$0.89\,/N$
Bartlett	-27 dB	$1.28\,(2\pi/N)$	$1.28\,/N$
Hanning	-32 dB	$1.44\,(2\pi/N)$	$1.44\,/N$
Hamming	-43 dB	$1.30\,(2\pi/N)$	$1.30\,/N$
Blackman	-58 dB	$1.68\,(2\pi/N)$	$1.68\,/N$

N t o . h t d i w d n a b d n a l e v e l e b o l e c

Table 10.2: Summary of performance measures of various power spectrum estimator

Method	Quality Factor Q	Variability $\mathcal{V}$	Frequency Resolution Δf	Figure of merit $\mathcal{M}$
Periodogram	1	1	$\frac{0.89}{N}$	$\frac{0.89}{N}$
Bartlett	$1.12N\,\Delta f_B$ (or L)	$\frac{1}{L}$	$L\,\frac{0.89}{N}$	$\frac{0.89}{N}$
Welch (50% overlap, Bartlett window)	$1.39N\,\Delta f_W$ $\left(\text{or } \frac{8L}{9}\right)$	$\frac{9}{8L}$	$\frac{1.28}{M}$	$\frac{0.72}{N}$
Blackman-Tukey (Bartlett window)	$2.34N\,\Delta f_{BT}$ $\left(\text{or } \frac{3N}{2M}\right)$	$\frac{2M}{3N}$	$\frac{0.64}{M}$	$\frac{0.43}{N}$

The following observations can be made from table 10.2.

1. The figure of merit of all methods is approximately the same.
2. The figure of merit is inversely proportional to N.
3. The frequency resolution decreases with increasing N and so when N is very large then its possible to identify closely spaced spectra.
4. Although each method differs in its resolution and variance, the overall performance is fundamentally limited by the amount of data that is available.

Example 10.6

Determine the frequency resolution, variability and figure of merit of the Bartlett, Welch (50% overlap) and Blackman-Tukey method of power spectrum estimations when x(n) has 800 samples (i.e., N = 800) and quality factor is 16 (i.e., Q = 16).

Solution

Bartlett Method

L = Q, in Bartlett estimate.

Given that, $Q_B = 16$ and $N = 800$

$$\text{Frequency resolution, } \Delta f_B = L\frac{0.89}{N} = \frac{16 \times 0.89}{800} = 0.0178$$

$$\text{Variability, } \mathcal{V}_B = \frac{1}{Q_B} = \frac{1}{16} = 0.0625$$

$$\text{Figure of merit, } \mathcal{M}_B = \Delta f_B \times \mathcal{V}_B = 0.0178 \times 0.0625 = 1.1125 \times 10^{-3}$$

Welch 50% Overlap

$$\text{Quality factor, } Q_W = 16 = \frac{8L}{9} \quad \text{for Welch} \qquad .....(1)$$

$$\text{From equation (1), } L = \frac{9Q_W}{8} = \frac{9 \times 16}{8} = 18$$

M = N/L for no overlap

M = 2N/L for Welch 50% overlap

$$\text{Frequency resolution, } \Delta f_W = \frac{1.28}{M} = \frac{1.28}{2N/L}$$

$$= \frac{1.28}{\frac{2 \times 800}{18}} = 0.0144$$

$$\text{Variability, } \mathcal{V}_W = \frac{1}{Q_W} = \frac{1}{\frac{8L}{9}} = \frac{9}{8L} = \frac{9}{8 \times 18} = 0.0625$$

$$\text{Figure of merit, } \mathcal{M}_W = \Delta f_W \times \mathcal{V}_W = 0.0144 \times 0.0625 = 9 \times 10^{-4}$$

Blackman-Tukey Method

$$\text{Quality factor, } Q_{BT} = \frac{3N}{2M} \qquad .....(2)$$

$$\text{From equation (2), } M = \frac{3N}{2Q_{BT}} = \frac{3 \times 800}{2 \times 16} = 75$$

$$\text{Frequency resolution, } \Delta f_{BT} = \frac{0.64}{M} = \frac{0.64}{75} = 8.5333 \times 10^{-3}$$

$$\text{Variability, } \mathcal{V}_{BT} = \frac{1}{Q_{BT}} = \frac{1}{\frac{3N}{2M}} = \frac{2M}{3N} = \frac{2 \times 75}{3 \times 800} = 0.0625$$

$$\text{Figure of merit, } \mathcal{M}_{BT} = \Delta f_{BT} \times \mathcal{V}_{BT}$$

$$= 8.5333 \times 10^{-3} \times 0.0625$$

$$= 5.3333 \times 10^{-4}$$

10.9 Summary of Important Concepts

1. The energy spectrum describes the energy level of various frequency components of a signal.
2. The deterministic signals have finite energy and so estimation of energy spectrum is possible.
3. The power spectrum describes the power level of various frequency components of a signal.
4. The random signals have finite power and so estimation of power spectrum is possible.
5. If X(f) is Fourier transform of discrete time signal x(n) then $|X(f)|^2$ gives the energy spectrum or energy spectral density.
6. The energy spectrum of discrete time signal x(n) is also given by Fourier transform of autocorrelation sequence of x(n).
7. Random signal is a signal that is not repeatable in a predictable manner.
8. A discrete time random process or signal is a collection or ensemble of discrete time signals, obtained from multiple realization.
9. An ergodic random process or signal is obtained from single realization of the process.
10. The autocorrelation function is the best statistical average for characterizing the random process or signal in time domain.
11. The Fourier transform of autocorrelation sequence $\gamma_{xx}(m)$ of a random process gives power spectral density or power spectrum $P_{xx}(f)$.
12. The periodogram $P_{xx}^{per}(f)$ is an estimate of power spectrum of random process obtained by taking Fourier transform of the autocorrelation of a random process.
13. The various methods of power spectrum estimation of a random process can be broadly classified into nonparametric (or classical) and parametric (or nonclassical) methods of power spectrum estimation.
14. If 'f' is replaced by k/N in the equation of energy/power spectrum, then it is convenient to use any FFT algorithm (DIT or DIF) for computing energy/power spectrum.
15. Periodogram, Bartlett, Welch and Blackman-Tukey methods are the various nonparametric methods of power spectrum estimation.
16. In the Bartlett method, the power spectrum estimation is obtained by periodogram averaging using non-overlaping segments.
17. In the Welch method, the power spectrum estimation is obtained by periodogram averaging using overlapped and windowed segments.
18. In the Blackman-Tukey method, the power spectrum estimation is obtained by taking Fourier transform of windowed autocorrelation sequence.
19. The parameters used to compare the performance of nonparametric methods of power spectrum estimates are bias, consistency, variance, quality factor and figure of merit.
20. The difference between the mean or expected value of an estimate and its true value is called bias.
21. Quality factor is defined as the ratio of square of the expected value of power spectrum estimate and the variance.
22. The reciprocal of quality factor is called variability.
23. Frequency resolution is the smallest frequency that can be identified by an estimator.
24. Figure of merit is the product of variability and frequency resolution.

10.10. Short Questions and Answers

Q10.1 What is known as energy spectrum?

The energy spectrum describes the energy level of various frequency components of discrete time signal. If X(f) is Fourier transform of x(n) then $|X(f)|^2$ is the energy spectrum of x(n).

Q10.2 What are the two methods of computing the energy spectrum of deterministic discrete signal?

In one method, the Fourier transform of x(n) is determined to get X(f) and square of magnitude of X(f) is computed (i.e., $|X(f)|^2$ is computed), which is the desired energy spectrum. In another method, the autocorrelation sequence of x(n) is determined first and the Fourier transform of autocorrelation sequence is determined which is the desired energy spectrum.

Q.10.3 What is a random signal or random process? Write any two examples.

A random signal or random process is a signal that is not repeatable in a predictable manner. Many phenomena that occur in nature are random processes.

Examples

1. Seismic data from earthquake.
2. Quantization noise produced by A/D converters.

Q10.4 What is an ergodic random process?

When a random process or a signal is obtained from single realization of the process then it is called ergodic random process and it is denoted as x(n).

Q10.5 What is known as a power spectrum?

The power spectrum describes the power level of various frequency components of the random process. It is obtained by taking Fourier transform of the autocorrelation sequence of the random process.

Q10.6 What are the two major classes of power spectral estimation techniques of random process?

The various power spectral estimation techniques are grouped under two major classes:

parametric methods and nonparametric methods.

Q10.7 How is power spectrum estimated in nonparametric methods?

In nonparametric methods, the autocorrelation sequence is estimated from the given data. Then the power spectrum is estimated by taking Fourier transform of the estimated autocorrelation sequence.

Q10.8 How is power spectrum estimated in parametric methods?

In parametric methods, first an appropriate system model is selected for the given random process and then the parameters of the model are computed from the available data of the random process. Finally, the power spectrum is estimated from the constructed model.

Q10.9 What is known as a periodogram?

The Fourier transform of the autocorrelation sequence $\gamma_{xx}(m)$ of a random process will be an estimate of the power spectrum. This estimate of the power spectrum is called a periodogram and it is denoted as $P_{xx}^{Per}(f)$.

$$\therefore \text{ periodogram, } P_{xx}^{Per}(f) = \mathcal{F}\{\gamma_{xx}(m)\} = \sum_{m=-(N-1)}^{N-1} \gamma_{xx}(m)\, e^{-j2\pi fm}$$

Q10.10 Write a short note on use of DFT in power spectrum estimation.

The power density spectrum of a discrete time random process is,

$$P_{xx}(f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\ e^{-j2\pi fn}\right|^2 \qquad .....(1)$$

The definition of DFT of x(n) is,

$$X(k) = \sum_{n=0}^{N-1} x(n)\ e^{-j2\pi kn/N} \qquad .....(2)$$

On comparing equations (1) and (2), the term $\sum_{n=0}^{N-1} x(n)\ e^{-j2\pi fn}$ in equation (1) is same as the term $\sum_{n=0}^{N-1} x(n)\ e^{-j2\pi kn/N}$ in equation (2), if 'f' is replaced by k/N. Here, k/N for k = 0, 1, 2, ... N – 1 represents N frequency intervals at which the power spectrum can be estimated.

$$\therefore\ P_{xx}\left(\frac{k}{N}\right) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\ e^{-j2\pi \frac{k}{N} n}\right|^2 = \frac{1}{N}\left|X(k)\right|^2$$

Now, X(k) can be computed using any of the FFT algorithm (DIT or DIF) used for computing DFT.

The power spectrum is obtained by taking square of magnitude of X(k) and dividing by N.

Q10.11 List the various nonparametric methods of power spectrum estimation.

The various nonparametric methods of power spectrum estimation are,

1. Periodogram
2. Bartlett method (Averaging periodograms)
3. Welch method (Averaging modified periodograms)
4. Blackman-Tukey method (Periodogram smoothing)

Q10.12 Explain the Bartlett method of power spectrum estimation.

The Bartlett method of power spectrum estimation involves the following three steps.

1. Divide the N-point sequence into L non-overlapping segments.
2. Compute the periodogram for each segment.
3. Compute the average of the L periodograms.

The average of L periodograms is the Bartlett estimate of power spectrum.

Q10.13 What are the changes made in the Bartlett method to form the Welch method of power spectrum estimation?

Welch proposed the following two changes in the Bartlett method.

1. Allowing the segments to overlap.
2. Windowing the segments before computing periodogram.

Q10.14 Explain the Welch method of power spectrum estimation.

The Welch method of power spectrum estimation involves the following three steps.

1. Divide the N-point sequence into L overlapping segments.
2. Window each segment and compute the periodogram for each windowed segment.
3. Compute the average of the L modified periodograms.

The average of the L modified periodograms is the Welch power spectrum estimate.

Q10.15 Explain Blackman-Tukey method of power spectrum estimation.

In this method, the autocorrelation sequence is windowed first, and then the Fourier transform of the windowed atuocorrelation sequence is computed, which is the desired power spectrum estimate.

Q10.16 Define bias.

The difference between the mean or expected value of an estimate and its true value is called the bias.

Q10.17 What is quality factor?

It is the ratio of square of the expected value of power spectrum estimate and the variance.

Q10.18 When is a power spectrum estimate said to be consistent?

In a power spectrum estimate, if the bias and variance both tend to zero as the number of samples tends to infinity or very large then the estimate is said to be consistent.

Q10.19 What is frequency resolution?

The frequency resolution will provide the smallest frequency that can be identified by a power spectrum estimator.

Q10.20 Calculate the frequency resolution for the Bartlett estimate when the quality factor is 8 and 200 samples of the signal is available.

Solution

$Q_B = 8$ and $N = 200$ samples.

$$\Delta f_B = \frac{0.89}{M} = \frac{0.89}{N/L} = \frac{0.89L}{N}$$

$$\therefore \ \Delta f_B = \frac{0.89 \times 8}{200} = 0.0356$$

In Bartlett method, $M = \frac{N}{L}$ and $L = Q_B$.

10.11. MATLAB Programs

Program 10.1

Write a MATLAB program to compute the power spectrum of the signal,

$x(n)=\{1,1,1,1,0,0,0,0\}$.

Sketch the spectrum for various lengths of FFT.

```
%Power spectrum estimation using periodogram

clear all
clc

x=[1,1,1,1,0,0,0,0];                %Input sequence
Ns=length(x);                       %Length of input sequence
```

```
N=8;                                  %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns)           %Calculation of PSD using 8-point FFT
stem((0:(N-1))/N, Psd);xlim([0 1]); %Plot PSD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,......N-1','fontsize',11,'fontweight','b');
ylabel('Power','fontsize',11,'fontweight','b');
title('Power Spectrum,  FFT Length N = 8','fontsize',11,'fontweight','b');

N=16;                                 %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns);          %Calculation of PSD using 16-point FFT
figure, stem((0:(N-1))/N, Psd);      %Plot PSD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,......N-1','fontsize',11,'fontweight','b');
ylabel('Power','fontsize',11,'fontweight','b');
title('Power Spectrum,  FFT Length N = 16','fontsize',11,'fontweight','b');

N=32;                                 %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns);          %Calculation of PSD using 32-point FFT
figure, stem((0:(N-1))/N, Psd);      %Plot PSD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,......N-1','fontsize',11,'fontweight','b');
ylabel('Power','fontsize',11,'fontweight','b');
title('Power Spectrum,  FFT Length N = 32','fontsize',11,'fontweight','b');

N=64;                                 %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns);          %Calculation of PSD using 64-point FFT
figure, stem((0:(N-1))/N, Psd);      %Plot PSD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,......N-1','fontsize',11,'fontweight','b');
ylabel('Power','fontsize',11,'fontweight','b');
title('Power Spectrum,  FFT Length N = 64','fontsize',11,'fontweight','b');
```

<u>OUTPUT</u>

```
Psd =

     2.0000      0.8536      0      0.1464      0     0.1464      0     0.8536
```

The power spectrum for various length of FFT are shown in fig P10.1a to P10.1d

Note : Verify the result with example 10.1

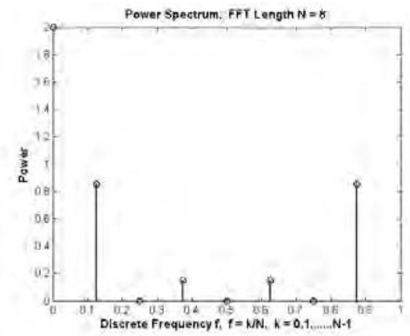


Fig P10.1a : *Power Spectrum, FFT Length N = 8.*

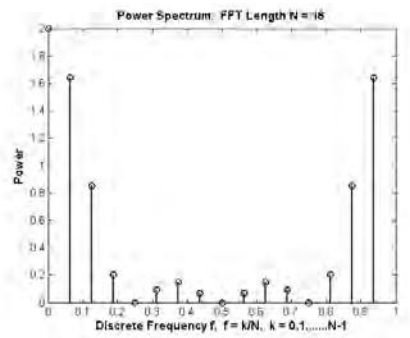


Fig P10.1b : *Power Spectrum, FFT Length N = 16.*

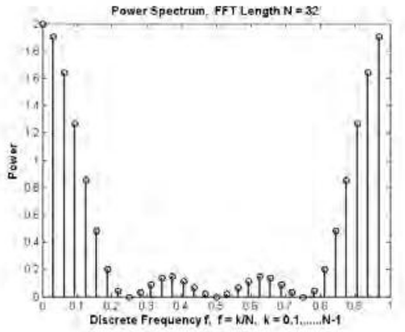


Fig P10.1c : *Power Spectrum, FFT Length N = 32.*

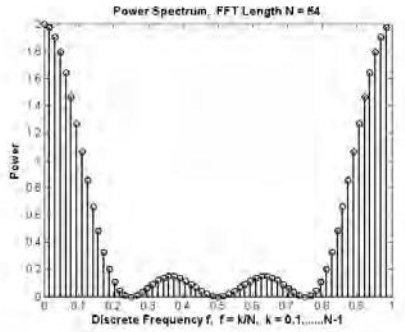


Fig P10.1d : *Power Spectrum, FFT Length N = 64.*

Program 10.2

Write a MATLAB program to compute the energy spectrum of the signal,
x(n)={1,-1,1,-1,0,0,0,0}.
Sketch the spectrum for various lengths of FFT.

```
% Energy spectrum estimation
clear all
clc

x=[1,-1,1,-1,0,0,0,0];            % Input sequence

N=8;                               %Length of FFT
Esd = abs(fft(x,N)).^2             %Calculation of ESD using 8-point FFT
stem((0:(N-1))/N, Esd);xlim([0 1]); %Plot ESD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,......N-1','fontsize',11,'fontweight','b');
ylabel('Energy','fontsize',11,'fontweight','b');
title('Energy Spectrum,  FFT Length N = 8','fontsize',11,'fontweight','b');

N=16;                              %Length of FFT
Esd = abs(fft(x,N)).^2;            %Calculation of ESD using 16-point FFT
figure, stem((0:(N-1))/N, Esd);    %Plot ESD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,......N-1','fontsize',11,'fontweight','b');
ylabel('Energy','fontsize',11,'fontweight','b');
title('Energy Spectrum,  FFT Length N = 16','fontsize',11,'fontweight','b');

N=32;                              %Length of FFT
Esd = abs(fft(x,N)).^2;            %Calculation of ESD using 32-point FFT
figure, stem((0:(N-1))/N, Esd);    %Plot ESD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,......N-1','fontsize',11,'fontweight','b');
ylabel('Energy','fontsize',11,'fontweight','b');
title('Energy Spectrum,  FFT Length N = 32','fontsize',11,'fontweight','b');

N=64;                              %Length of FFT
Esd = abs(fft(x,N)).^2;            %Calculation of ESD using 64-point FFT
figure, stem((0:(N-1))/N, Esd);    %Plot ESD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,......N-1','fontsize',11,'fontweight','b');
ylabel('Energy','fontsize',11,'fontweight','b');
title('Energy Spectrum,  FFT Length N = 64','fontsize',11,'fontweight','b');
```

OUTPUT

```
Esd =
     0    1.1716     0    6.8284    16.0000    6.8284     0    1.1716
```

The energy spectrum for various length of FFT are shown in fig P10.2a to P10.2d

Note : Verify the result with example 10.2

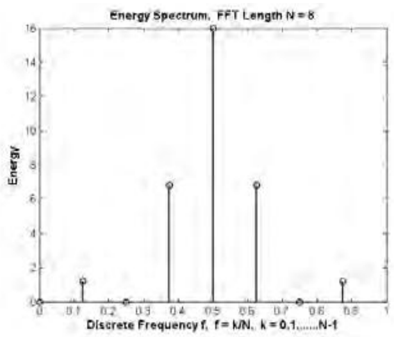


Fig P10.2a : *Energy Spectrum, FFT Length N = 8.*

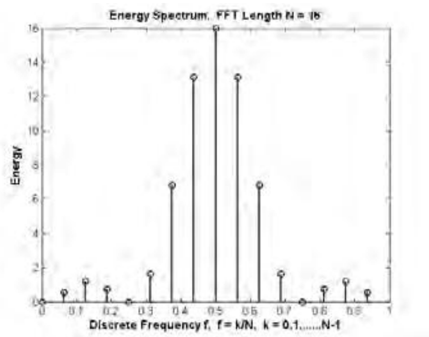


Fig P10.2b : *Energy Spectrum, FFT Length N = 16.*

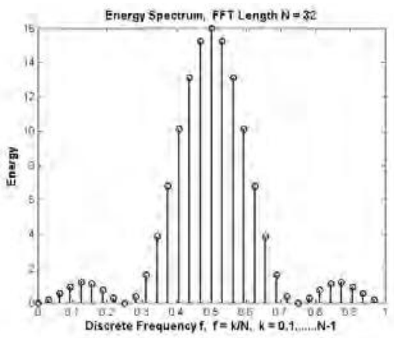


Fig P10.2c : *Energy Spectrum, FFT Length N = 32.*

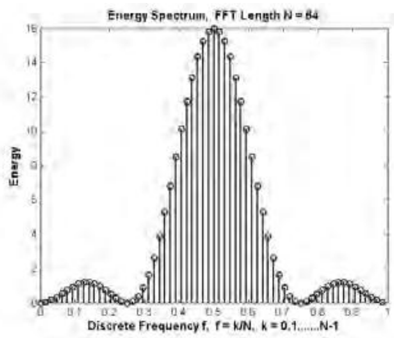


Fig P10.2d : *Energy Spectrum, FFT Length N = 64.*

Program 10.3

Write a MATLAB program to compute the periodogram power spectrum estimate of the random signal,

$$x(n) = \sin 2pf_1n + \cos 2p(f_1+Df)n, \quad f_1 = 0.2, \quad Df = 0.05$$

Sketch the periodogram for various lengths of FFT.

```
% Periodogram power spectrum estimate of sine and cosine signals with frequency
%separation of 0.05
clc
clear all

f1=0.2;                              %Frequency of 1st cosine signal;
f2=f1+0.05;                          %Frequency of 2nd cosine signal;
Ns=8;                                %Length of the signal
n=0:Ns-1;
x=sin(2*pi*f1*n)+ cos(2*pi*f2*n)     %Generation of Ns-point signal x(n)
```

```
N=8;                                   %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns)            %Calculation of PSD using 8-point FFT
stem((0:(N-1))/N, Psd); xlim([0 1]);%Plot PSD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,.....N-1','fontsize',11,'fontweight','b');
ylabel('Power','fontsize',11,'fontweight','b');
title('Power Spectrum,  FFT Length N = 8','fontsize',11,'fontweight','b');

N=16;                                  %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns);           %Calculation of PSD using 16-point FFT
figure, stem((0:(N-1))/N, Psd);        %Plot PSD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,.....N-1','fontsize',11,'fontweight','b');
ylabel('Power','fontsize',11,'fontweight','b');
title('Power Spectrum,  FFT Length N = 16','fontsize',11,'fontweight','b');

N=32;                                  %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns);           %Calculation of PSD using 32-point FFT
figure, stem((0:(N-1))/N, Psd);        %Plot PSD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,.....N-1','fontsize',11,'fontweight','b');
ylabel('Power','fontsize',11,'fontweight','b');
title('Power Spectrum,  FFT Length N = 32','fontsize',11,'fontweight','b');

N=64;                                  %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns);           % Calculation of PSD using 64-point FFT
figure, stem((0:(N-1))/N, Psd);        %Plot PSD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,.....N-1','fontsize',11,'fontweight','b');
ylabel('Power','fontsize',11,'fontweight','b');
title('Power Spectrum,  FFT Length N = 64','fontsize',11,'fontweight','b');
```

OUTPUT

```
x =
     1.0000    0.9511   -0.4122   -0.5878    0.0489    0.0000   -0.0489    0.5878
Psd =
     0.2960    0.7873    0.3981    0.0434    0.0165    0.0434    0.3981    0.7873
```

Note : Verify the result with example 10.3

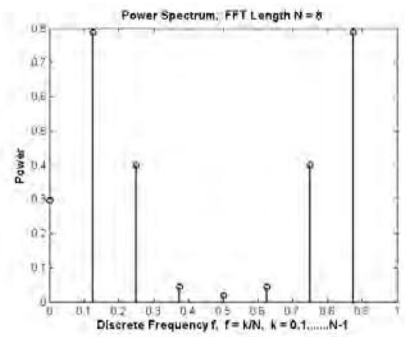


Fig P10.3a : *Power Spectrum, FFT Length N = 8.*

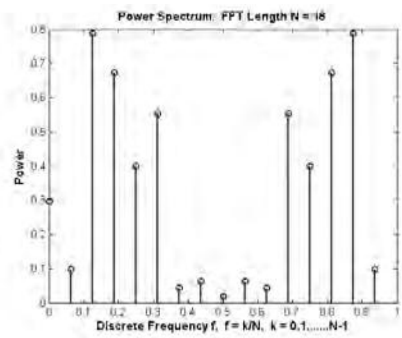


Fig P10.3b : *Power Spectrum, FFT Length N = 16.*

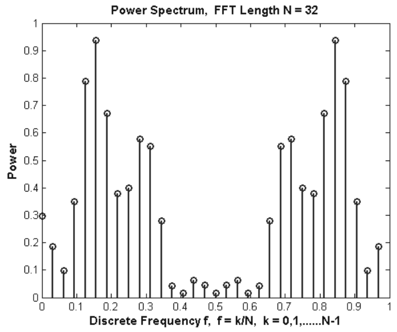


***Fig P10.3c :** Power Spectrum, FFT Length N = 32.*

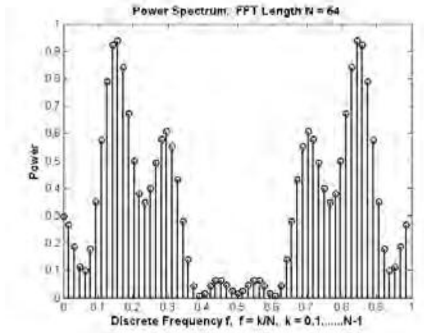


***Fig P10.3d :** Power Spectrum, FFT Length N = 64.*

Program 10.4

Write a MATLAB program to compute the periodogram power spectrum estimate of the random signal,

$$x(n) = \cos 2\pi f_1 n + \cos 2\pi(f_1+\Delta f)n, \quad f_1 = 0.8, \quad \Delta f = 0.1$$

Sketch the periodogram for various lengths of FFT.

```
% Power Spectrum of two cosine signals with frequency separation of 0.1
clc
clear all

f1=0.8;                                %Frequency of 1st cosine signal;
f2=f1+0.1;                             %Frequency of 2nd cosine signal;
Ns=8;                                  %Length of the signal
n=0:Ns-1;
x=cos(2*pi*f1*n)+ cos(2*pi*f2*n)       %Generation of Ns-point signal x(n)

N=8;                                   %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns)            %Calculation of PSD using 8-point FFT
stem((0:(N-1))/N, Psd);xlim([0 1]); %Plot PSD
xlabel(‘Discrete Frequency f,  f = k/N,  k = 0,1,......N-1’,’fontsize’,11,’fontweight’,’b’);
ylabel(‘Power’,’fontsize’,11,’fontweight’,’b’);
title(‘Power Spectrum,  FFT Length N = 8’,’fontsize’,11,’fontweight’,’b’);

N=16;                                  %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns);           %Calculation of PSD using 16-point FFT
figure, stem((0:(N-1))/N, Psd);        %Plot PSD
xlabel(‘Discrete Frequency f,  f = k/N,  k = 0,1,......N-1’,’fontsize’,11,’fontweight’,’b’);
ylabel(‘Power’,’fontsize’,11,’fontweight’,’b’);
title(‘Power Spectrum,  FFT Length N = 16’,’fontsize’,11,’fontweight’,’b’);

N=32;                                  %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns);           %Calculation of PSD using 32-point FFT
figure, stem((0:(N-1))/N, Psd);        %Plot PSD
xlabel(‘Discrete Frequency f,  f = k/N,  k = 0,1,......N-1’,’fontsize’,11,’fontweight’,’b’);
ylabel(‘Power’,’fontsize’,11,’fontweight’,’b’);
title(‘Power Spectrum,  FFT Length N = 32’,’fontsize’,11,’fontweight’,’b’);
```

```
N=64;                                  %Length of FFT
Psd = abs(fft(x,N)).^2/(Ns);           %Calculation of PSD using 64-point FFT
figure, stem((0:(N-1))/N, Psd);        %Plot PSD
xlabel('Discrete Frequency f,  f = k/N,  k = 0,1,.....N-1','fontsize',11,'fontweight','b');
ylabel('Power','fontsize',11,'fontweight','b');
title('Power Spectrum,  FFT Length N = 64','fontsize',11,'fontweight','b');
```

OUTPUT

```
x =

    2.0000  1.1180  -0.5000  -1.1180  -0.5000       0  -0.5000  -1.1180

Psd =

    0.0477  1.4316   2.1875   0.4434   0.3273  0.4434   2.1875   1.4316
```

Note : Verify the result with example 10.4

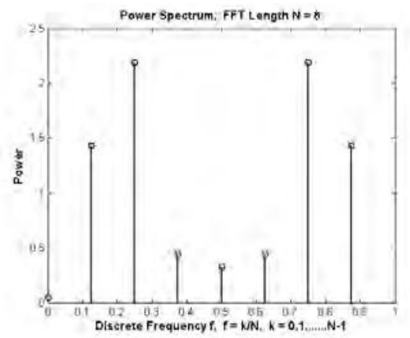


Fig P10.4a : *Power Spectrum, FFT Length N = 8.*

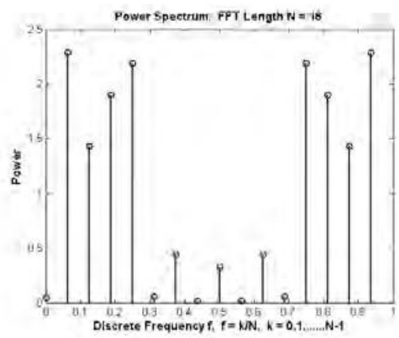


Fig P10.4b : *Power Spectrum, FFT Length N = 16.*

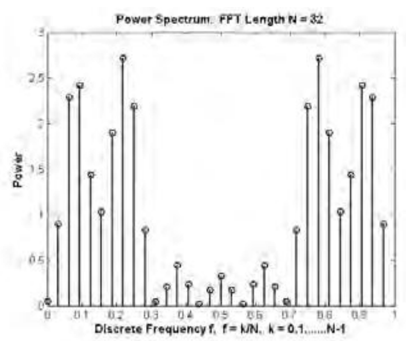


Fig P10.4c : *Power Spectrum, FFT Length N = 32.*

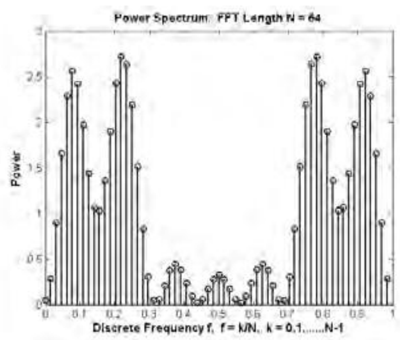


Fig P10.4d : *Power Spectrum, FFT Length N = 64.*

10.12. Exercises

I Fill in the blanks with appropriate words.

1. The ________ signals can be reproduced exactly with repeated measurements.
2. The ________ describes the power level of various frequency components of a signal.
3. The square of magnitude of X(f) gives the _______ spectrum.
4. The power spectrum estimate obtained via Fourier transform of autocorrelation sequence is called ________.
5. Overlapping of segments are allowed in _______ method of power spectrum estimation.
6. __________ methods uses a finite set of data for estimating power spectrum of the random process.
7. __________ is the product of variability and frequency resolution.
8. ______ is the difference between the mean of an estimate and its true value.
9. __________ is the ratio of square of the expected value of power spectrum estimate and the variance.
10. Expected value of the periodogram is equal to the _________ for large values of N.

Answers

1. deterministic	4. periodogram	7. Figure of merit	10. power spectrum
2. power spectrum	5. Welch	8. Bias	
3. energy	6. Nonparametric	9. Quality factor	

II State whether the following statements are True/False.

1. The energy spectrum describes the energy level of various frequency components of a signal.
2. A random process or a signal obtained from multiple realization of the process is called ergodic.
3. A random signal is a signal that is not repeatable in a predictable manner.
4. The autocorrelation of a random process is the statistical average in time domain.
5. The Bartlett method employs periodogram averaging for power spectrum estimate.
6. In Blackman-Tukey method, the autocorrelation sequence is windowed first and then the Fourier transform is computed.
7. Figure of merit($\mathcal{M}$) is directly proportional to the number of samples (N) used to estimate power spectrum.
8. The frequency resolution of Bartlett estimate is L times more than that of periodogram.
9. The estimate is said to be consistent if the variance tends to zero as the number of samples tends to infinity.
10. The frequency resolution is inversely proportional to the number of data N used to estimate power spectrum.

Answers

1. True	4. True	7. False	10. True
2. False	5. Ture	8. True	
3. True	6. True	9. True	

III Choose the right answer for the following questions.

1. The estimate of power spectrum of random process is called,

a) periodogram **b)** energy spectrum

c) autocorrelation **d)** expected value

2. The Fourier transform of autocorrelation sequence $\gamma_{xx}(m)$ gives the,

a) periodogram **b)** energy spectrum

c) power spectrum **d)** variance

3. The periodogram power spectrum estimate is,

a) $N\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right|^2$ **b)** $\frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi fn}\right|^2$

c) $N\left|\sum_{m=0}^{N-1} \gamma_{xx}(m)\, e^{-j2\pi fm}\right|^2$ **d)** $\frac{1}{N}\left|\sum_{m=0}^{N-1} \gamma_{xx}(m)\, e^{-j2\pi fm}\right|^2$

4. The Welch power spectrum estimate is,

a) $\frac{1}{MUL}\sum_{i=0}^{L-1}\left|\sum_{n=0}^{M-1} x_i(n)\, e^{-j2\pi fn}\right|^2$ **b)** $\frac{M}{UL}\sum_{i=0}^{L-1}\left|\sum_{n=0}^{M-1} x_i(n)\, e^{-j2\pi fn}\right|^2$

c) $\frac{LU}{M}\sum_{i=0}^{L-1}\left|\sum_{n=0}^{L-1} x_i(n)\, e^{-j2\pi fn}\right|^2$ **d)** $\frac{L}{MU}\sum_{i=0}^{M-1}\left|\sum_{n=0}^{L-1} x_i(n)\, e^{-j2\pi fn}\right|^2$

5. The Blackman-Tukey method of power spectrum estimation is also known as,

a) averaging periodogram **b)** periodogram smoothing

c) averaging modified periodogram **d)** none of these

6. The Blackman-Tukey power spectrum estimation is given by,

a) $\sum_{m=-\infty}^{+\infty} \gamma_{xx}(m)\, w(m)\, e^{-j2\pi fm}$ **b)** $\sum_{m=-\infty}^{+\infty} x(m)\, w(m)\, e^{-j2\pi fm}$

c) $\sum_{m=-(M-1)}^{M-1} x(m)\, w(m)\, e^{-j2\pi fm}$ **d)** $\sum_{m=-(M-1)}^{M-1} \gamma_{xx}(m)\, w(m)\, e^{-j2\pi fm}$

7. Reciprocal of a quality factor is called,

a) frequency resolution **b)** figure of merit

c) bias **d)** variability

8. The frequency resolutions of Welch (50% overlap) and Blackman-Tukey methods are respectively,

a) $\frac{0.72}{N}$ and $\frac{0.64}{M}$ **b)** $\frac{0.64}{M}$ and $\frac{0.89}{M}$

c) $\frac{1.28}{M}$ and $\frac{0.64}{M}$ **d)** $\frac{0.89}{M}$ and $\frac{0.64}{M}$

9. Figure of merit is inversely proportional to,

a) N **b)** L **c)** M **d)** variability

10. In the Welch method, if there is no overlap, then $L = \frac{N}{M}$ **_and if overlap is 50% then L = ,_**

a) $\frac{N}{M}$ **b)** $\frac{2M}{N}$ **c)** $\frac{N}{2M}$ **d)** $\frac{2N}{M}$

Answers

1. a	4. a	7. d	10. d
2. c	5. b	8. c	
3. b	6. d	9. a	

IV Answer the following questions.

1. Define energy spectrum of a discrete time signal. Prove that the energy density spectrum of a discrete time signal can be obtained from the autocorrelation sequence.
2. Define power spectrum of a random signal. Prove that the power spectrum of a random signal can be obtained from the autocorrelation sequence.
3. How can DFT/FFT be used to compute the energy/power spectrum estimate?
4. Explain the periodogram method of power spectrum estimation.
5. Write a detailed note on nonparametric methods of power spectrum estimation.
6. Explain the Bartlett method of power spectrum estimation.
7. Explain the Welch method of power spectrum estimation.
8. Explain the Blackman-Tukey method of power spectrum estimation.
9. What are the properties used to compare the power spectrum estimates of different methods? Explain the properties.
10. Discuss the performance characteristics of any one power spectrum estimator.

V Solve the following problems.

E10.1 Compute the periodogram of the signal vector {0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55} and sketch the periodogram.

E10.2. Find the energy density spectrum of the discrete time signal, $x(n) = \{0.5, -2, 0.75, -1\}$.

E10.3. Compute the periodogram of the random signal by taking 8 samples of the signal.

$x(n) = \sin 2\pi f_1 n + \cos 2\pi(f_1 + \Delta f)n,\ f_1 = 0.35,\ \Delta f = 0.075$. **_Also sketch the periodogram._**

E10.4. Compute the periodogram of the random signal by taking 8 samples of the signal.

$x(n) = \cos 2\pi f_1 n + \cos 2\pi(f_1 + \Delta f)n,\ f_1 = 0.65,\ \Delta f = 0.25$. **_Also sketch the periodogram._**

E10.5. White noise with power spectral density $P_{xx}(e^{j\omega}) = \sigma^4$ **_is passed through a filter with_** $h(n) = 0.8^n\, u(n)$. **_What is the output power spectral density (PSD)?_**

E10.6 Determine the frequency resolution, variability and figure of merit of the Bartlett, Welch (50% overlap) and Blackman-Tukey method of power spectrum estimations when x(n) has 1200 samples (i.e., N = 1200) and quality factor is 8 (i.e., Q = 8).

Answers

E10.1 $P_{xx}^{Per}(f) = \{1.125,\ 0.0341,\ 0.01,\ 0.0059,\ 0.005,\ 0.0059,\ 0.01,\ 0.0341\}$

E10.2 $S_{xx}(f) = \{3.0625,\ 1.9224,\ 1.0625,\ 9.6995,\ 18.0625,\ 9.6995,\ 1.0625,\ 1.9224\}$

E10.3 $x(n) = \{2,\ 0.2212,\ 0,\ 0.6420,\ -1.6180,\ -1,\ 0,\ -1.2601\}$

$P_{xx}^{Per}(f) = \{0.3308,\ 0.4069,\ 0.9187,\ 0.0615,\ 0.0340,\ 0.0615,\ 0.9187,\ 0.4069\}$

E10.4 $x(n) = \{1,\ -0.082,\ -0.3633,\ 0.1526,\ 0.2788,\ -0.2929,\ -0.3633,\ 1.2967\}$

$P_{xx}^{Per}(f) = \{0.1287,\ 1.8393,\ 0.0215,\ 2.7101,\ 0.3956,\ 2.7101,\ 0.0215,\ 1.8393\}$

E10.5 Output power spectral density (PSD) $= \dfrac{\sigma^4}{1.64 - 1.6\cos\omega}$

E10.6 ***Bartlett Method***

Frequency resolution, $\Delta f_B = 5.9333 \times 10^{-3}$

Variability, $\mathcal{V}_B = 0.125$

Figure of merit, $\mathcal{M}_B = 7.4166 \times 10^{-4}$

Welch (50% overlap)

Frequency resolution, $\Delta f_W = 4.8 \times 10^{-3}$

Variability, $\mathcal{V}_W = 0.125$

Figure of merit, $\mathcal{M}_W = 6 \times 10^{-4}$

Blackman-Tukey Method

Frequency resolution, $\Delta f_{BT} = 2.8444 \times 10^{-3}$

Variability, $\mathcal{V}_{BT} = 0.125$

Figure of merit, $\mathcal{M}_{BT} = 3.5555 \times 10^{-4}$

Chapter 11

Digital Signal Processors

11.1 Introduction

The ***digital signal processors*** are microprocessors specially designed for efficient implementation of digital signal processing systems. The pioneers in developing digital signal processors are Texas Instruments and Analog Devices of USA. With increase in digital signal processing applications, almost all microprocessor/ microcontroller manufacturers are including some or all features of digital signal processors in their devices.

Texas Instruments has released TMS320 series of digital signal processors. The TMS320 family of processors includes four basic types of processors. They are 16-bit fixed point processors, 32-bit floating point processors, VLIW (**V**ery **L**arge **I**nstruction **W**ord) architecture processors and multiprocessor DSPs (**D**igital **S**ignal **P**rocessors).

The various generations of TMS320 family of processors are, TMS320C1x, TMS320C2x, TMS320C3x TMS320C4x, TMS320C5x, 6x and 8x. Here, 1, 2, 3, ... in C1x, C2x, C3x, ... denotes the generations (i.e., 1^{st} generation, 2^{nd} generation, 3^{rd} generation, ...). The TMS320C1x, C2x, C5x are 16-bit fixed point processors and TMS320C54x is the advanced version of C5x. The TMS320C3x, 4x are 32-bit floating point processors. The TMS320C6x are VLIW architecture processors. The TMS320C8x are multiprocessor DSPs. The various applications of digital signal processors are listed in table 11.1.

The ***fixed point processors*** are low power and low cost devices and they can operate at high speeds due to simple architecture. The ***floating point processors*** offer large dynamic range, wider instruction word size and support more addressing modes.

The ***VLIW architecture*** processors employ instruction level parallelism in which many instructions are issued at the same time and are executed in parallel by multiple execution units. The ***multiprocessor architecture*** processors provide parallel processing capability by integrating multiple DSPs on a single piece of silicon. The VLIW architecture processors and multiple DSPs supports both fixed point and floating point computations.

Table 11.1 : Typical Applications for the TMS320 Digital Signal Processors

AUTOMOTIVE	CONSUMER	CONTROL
Adaptive ride control Antiskid brakes Cellular telephones Digital radios Engine control Navigation and global positioning Vibration analysis Voice commands Anticollision radar	Digital radios/TVs Educational toys Music synthesizers Pagers Power tools Radar detectors Solid-state answering machines	Disk drive control Engine control Laser printer control Motor control Robotics control Servo control
GENERAL-PURPOSE	**GRAPHICS/IMAGING**	**INDUSTRIAL**
Adaptive filtering Convolution Correlation Digital filtering Fast Fourier transforms Hilbert transforms Waveform generation Windowing	3-D rotation Animation/digital maps Homomorphic processing Image compression/transmission Image enhancement Pattern recognition Robot vision Workstations	Numeric control Power-line monitoring Robotics Security access
INSTRUMENTATION	**MEDICAL**	**MILITARY**
Digital filtering Function generation Pattern matching Phase-locked loops Seismic processing Spectrum analysis Transient analysis	Diagnostic equipment Fetal monitoring Hearing aids Patient monitoring Prosthetics Ultrasound equipment	Image processing Missile guidance Navigation Radar processing Radio frequency modems Secure communications Sonar processing
TELECOMMUNICATIONS		**VOICE/SPEECH**
1200 to 33600 bps modems Adaptive equalizers ADPCM transcoders Cellular telephones Channel multiplexing Data encryption Digital PBXs Digital Speech Interpolation (DSI) DTMF encoding/decoding Echo cancellation	Faxing Line repeaters Personal Communications Systems (PCS) Personal Digital Assistants (PDA) Speaker phones Spread spectrum communications Video conferencing X.25 packet switching	Speaker verification Speech enhancement Speech recognition Speech synthesis Speech vocoding Text-to-speech Voice mail

11.2 Special Features of Digital Signal Processors

In the digital domain, a signal is represented by an array of N-numbers and as the value of N is large, the accuracy of representation will be high. The digital processing of signals involve operations like convolution and correlation, which in-turn involve multiplication and addition. Therefore, the real time processing of digital signals require very fast accessing of large volumes of data and very fast computation. The hardware requirement of digital signal processors to perform the above task are listed in table 11.2.

Table 11.2 : Special Requirements of Digital Signal Processors

Processing requirement	Hardware implementation to satisfy the processing requirement
Fast data access	· High-bandwidth memory architecture · Specialized addressing mode · **D**irect **M**emory **A**ccess (DMA)
Fast computation	· Multiply and accumulate unit · Pipelining of instruction execution · VLIW (**V**ery **L**arge **I**nstruction **W**ord) architecture · Multiprocessor architecture
Numerical fidelity	· Wide accumulator registers · Guard bits
Fast execution control	· Hardware-assisted zero-overhead loop · Shadow registers

11.2.1 Fast Data Access

In digital signal processors, the fast data access is achieved by employing high-bandwidth memory architecture, specialized addressing modes and **D**irect **M**emory **A**ccess (DMA). The high-bandwidth memory architecture are modified versions of Harvard architecture for simultaneous access of one or more data along with instruction code in a single clock cycle. The specialized addressing modes are provided for easy implementation of signal processing algorithms like FFT, convolution and correlation. The DMA help to transfer data from/to the external and internal memory without involving CPU, so that CPU is relieved for other task to run in parallel.

High-bandwidth Memory Architectures

The general purpose microprocessors are based on the ***Von Neumann architecture*** shown in fig 11.1, which consists of a single memory block to store both program and data and a single bus to transfer data and instruction from/to the CPU. The disadvantage in this architecture is that only one memory access per instruction cycle is possible.

The digital signal processors are based on either Harvard architecture shown in fig 11.2 or modified Harvard architecture shown in fig11.3. In ***Harvard architecture,*** there are separate memory blocks for program and data, and separate buses for transfer data and instruction from/to the CPU. The Harvard architecture facilitates the simultaneous access of instruction and data in a single cycle, i.e., two memory accesses in one cycle.

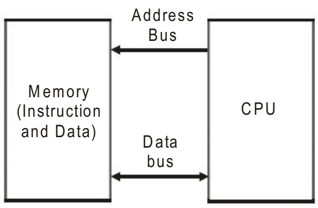


Fig 11.1 : Von Neumann architecture.

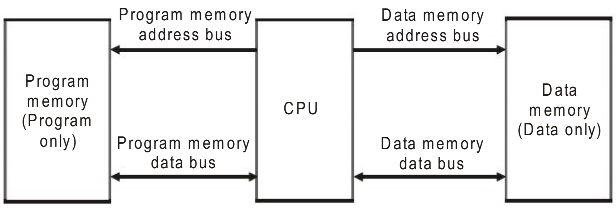


Fig 11.2 : Harvard architecture.

In ***modified Harvard architecture,*** one memory block is dedicated for storing data alone and another memory block for storing both instruction and data. This architecture will also have separate buses to access instruction and data simultaneously in one cycle.

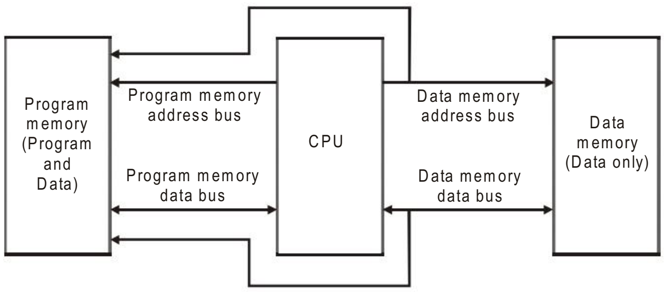


Fig 11.3 : Modified Harvard architecture.

The advanced digital signal processors employ two or more internal memory blocks connected to CPU via separate buses. It also has an external memory block common to program and data connected to CPU via a single external bus, as shown in fig 11.4.

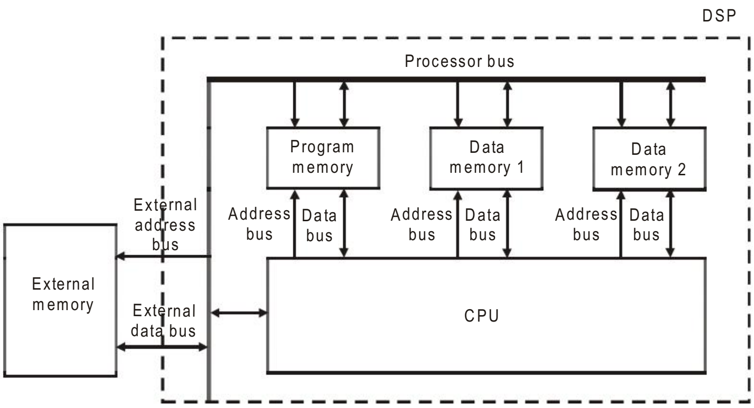


Fig 11.4 : Architecture of advanced digital signal processors.

In these processors, the program and data will be transferred from external memory to internal memory and execution starts from internal memory. Whenever the internal memory locations are free, the program and data will be copied from external memory to internal memory. In this way the execution speed is enhanced. The internal memory is also known as cache memory.

Specialized Addressing Modes

The digital signal processors support a number of specialized addressing modes, designed for efficient implementation of digital signal processing algorithms. Two of the special addressing modes are circular addressing and bit reversed addressing. The digital signal processors will also have address generator blocks to perform the task of address generation for specialized addressing modes.

The ***circular addressing*** mode can be used to access memory declared as circular buffer. The programmer can define a ***circular buffer*** by specifying a start address and end address. In this addressing, when the address pointer is incremented, the address will be checked with the end address of a circular buffer, and if the end address is reached then the pointer is loaded with the start address.

The ***bit reversed addressing*** mode can be used to access data in the bit reversed order for FFT computation. In this addressing mode, the address is incremented/decremented by the number represented in the bit reversed form.

Direct Memory Access (DMA)

The ***DMA*** is a technique of transferring information between two memory blocks/areas without involving the CPU, so that CPU is relieved for performing other tasks when DMA is performed.

The DMA is possible with the help of a DMA controller which is a dedicated processor for performing DMA. The DMA controller can simultaneously generate two addresses and control signals for data transfer between two memories or between memory and IO. Usually the digital signal processors employ DMA for bulk data transfer from the external memory to internal memory or vice versa.

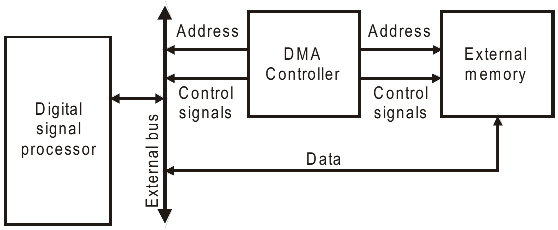


Fig 11.5 : *A typical DMA in DSP system.*

11.2.2 Fast Computation

The fast computation in digital signal processors is achieved by providing single cycle multiply/accumulate (MAC) unit, pipelining of instruction execution, VLIW architecture and multiprocessor architectures.

MAC (Multiply/Accumulate) Unit

The popular computations in digital signal processing are FFT, convolution and correlation. These operations involve multiplication and summation of lengthy numerical arrays. The ***MAC unit*** in the CPU of digital signal processors is capable of computing one multiplication and addition in a single clock cycle.

Typically a MAC unit will have a multiplier, a set of registers, a shifter and an ALU. The instruction "MACD pgm, dma", of the TMS320C5x processor will multiply the content of program memory (pgm) and data memory (dma) specified by the instruction and add to the sum of previous products in the accumulator with appropriate shift in a single clock cycle.

Pipelining of Instruction Execution

In processors without pipelining, the execution of instruction is performed one by one, i.e., after complete execution of an instruction the next instruction is fetched from memory. In processors with ***pipelining***, the instruction execution is divided into various phases/stages and execution of different phases of two or more instructions are performed in parallel. The number of instructions that can be executed in parallel is called ***depth*** or ***level of pipelining***.

Let us consider a processor in which the instruction execution is divided into the following four phases.

Phase 1 : Fetch the opcode (or instruction code) from program memory.

Phase 2 : Decode the instruction code.

Phase 3 : Read the operands (or data) from data/program memory.

Phase 4 : Execute the task specified by the instruction and store the result.

Let Inst1, Inst2, Inst3, ... be the instructions to be executed sequentially. The execution of the four phases of the instructions for subsequent clock cycles are listed in table 11.3.

In this pipelining when the phase 4 of 1st instruction is executed, the phase 3 of 2nd instruction, the phase 2 of 3rd instruction and the phase 1 of 4th instruction are also executed simultaneously.

Table 11.3 : Pipelining of Instruction Execution

Number of Clock Cycles	Phase 1	Phase 2	Phase 3	Phase 4
1	Inst1	–	–	–
2	Inst2	Inst1	–	–
3	Inst3	Inst2	Inst1	–
4	Inst4	Inst3	Inst2	Inst1
5	Inst5	Inst4	Inst3	Inst2
6	Inst6	Inst5	Inst4	Inst3
7	Inst7	Inst6	Inst5	Inst4
8	Inst8	Inst7	Inst6	Inst5
9	Inst9	Inst8	Inst7	Inst6
.	.	.	.	.
.	.	.	.	.

VLIW (Very Long Instruction Word) Architecture

The ***VLIW architecture*** has an enhanced parallelism in the instruction execution. In this processor, many instructions are fetched at the same time and issued to multiple execution units to be executed in parallel.

Consider an example of VLIW architecture shown in fig 11.6. In this example, the instructions are packed such that 8 numbers of 32-bit instructions are packet as a single 256-bit wide instruction and stored in on-chip program memory. When a packed instruction is fetched, it is simultaneously issued to 8 execution units to be executed in parallel. Here instruction packing is performed by the compiler during compile-time so as to ensure proper execution of the algorithm.

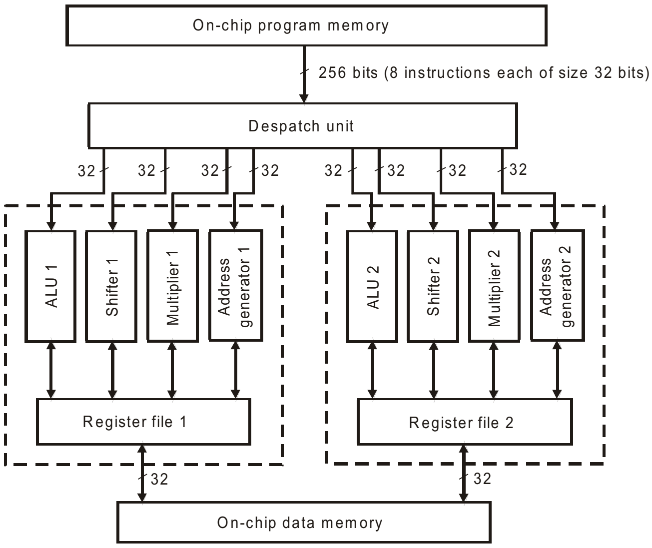


Fig 11.6 : *VLIW architecture.*

Multiprocessor Architecture

The ***multiprocessor architecture*** will consist of a number of digital signal processors running in parallel. For example, the TMS320C8x family of processors will have 4 parallel processors and one master processor. The master processor is a 32-bit RISC (**R**educed **I**nstruction **S**et **C**omputer) processor with an internal floating point unit. Each parallel processor is an advanced 32-bit digital signal processors. The communication between the master processor and parallel processors takes place via a high-speed crossbar network which provides simultaneous access to multiple on-chip memory banks.

11.2.3 Numerical Fidelity

The ***numerical fidelity*** refers to the faithfulness of the digital signal processor to perform any mathematical operation without errors like underflow and overflow. These errors occur when the size of the result of any computation exceeds the size of the register used to store the result. In order to avoid such errors in computations, the digital signal processors will have larger size accumulators and CPU registers. Most of the 16-bit digital signal processors will have 32-bit accumulators and CPU registers, to store the result of any computations.

The TMS320C54x family of processors have 40-bit accumulator and ALU, in which the upper 8 bits are called ***guard bits***. The guard bits are used as headmargin for computations like convolution/correlation to avoid overflow.

In fixed point processors, fractional numbers will be used for computations. The range of fractional numbers is -1.0_{10} to $+1.0_{10}$. In binary, the fixed point fraction numbers are represented in 1.X format where the uppermost bit is used to represent the sign and X is the number of bits used to represent the magnitude. The 32-bit fixed point binary fraction number is represented in 1.31 format, where 31 bits is used to represent the magnitude. Now the smallest value that can be represented in this format is $1/2^{31}$ and the largest value is 1. The ***dynamic range*** in dB for a 32-bit fixed point number is defined as,

$$\text{Dynamic range}_{\text{dB}} = 20\log\left(\frac{\text{Largest value}}{\text{Smallest value}}\right) = 20\log\left(\frac{1}{1/2^{31}}\right) = -186.6_{\text{dB}}$$

11.2.4 Fast Execution Control

Some of the features of digital signal processors that help to achieve fast execution control are zero-overhead hardware loop and very fast interrupt handling by employing shadow registers.

The zero-overhead hardware loop allows the programmers to initialize loops by setting a counter and defining the loop bounds, without spending any software overhead to update and test loop counters or branching back to the beginning of the loop.

The interrupts are usually given higher priority. When an interrupt occurs while executing a program, the processor will stop the current execution and execute an **I**nterrupt **S**ervice **R**outine (ISR) to service the interrupt. After executing the ISR, the processor will resume the program execution. When the program execution is stopped, the processor has to store the vital informations in the CPU registers and load them back to CPU registers when the program execution is resumed. In digital signal processors, a portion of internal/stack memory locations are dedicated for this purpose and they are called ***shadow registers***, which ensure very fast storage and retrial of information.

11.3 TMS320C5x Family of Digital Signal Processors

The TMS320C5x family of processors are fifth-generation digital signal processors from Texas Instruments, USA. They are 16-bit fixed point processors fabricated using high performance static CMOS technology. These processors have advanced Harvard architecture, with a variety of on-chip peripherals and memory and highly specialized instructions. They can execute 50 **M**illion **I**nstructions **P**er **S**econd(MIPS).

Some of the features of TMS320C5x family of digital signal processors are,

- 16-bit CPU
- 20 to 50 ns single cycle instruction execution time
- Single cycle 16 ´ 16-bit MAC (Multiply/Accumulate) unit
- 64k ´ 16-bit external program memory address space
- 64k ´ 16-bit external data memory address space
- 64k ´ 16-bit external IO address space

- 32k ´ 16-bit external global memory address space
- 2k to 32k ´ 16-bit single-access On-chip PROM
- 1k to 9k ´ 16-bit single-access On-chip program/data RAM
- 1k ´ 16-bit dual-access On-chip program/data RAM
- Synchronous, TDM and buffered serial ports
- Programmable timer and PLL (**P**hase **L**ocked **L**oops)
- IEEE standard JTAG ports
- 5 V/3 V operation with low power dissipation and power down modes
- DMA interface
- 100/128/132/144 pins in plastic QFP and TQFP

The various processors of TMS320C5x family and their characteristic features are listed in table 11.4.

Table 11.4 : Characteristics of TMS320C5x Family of Processors

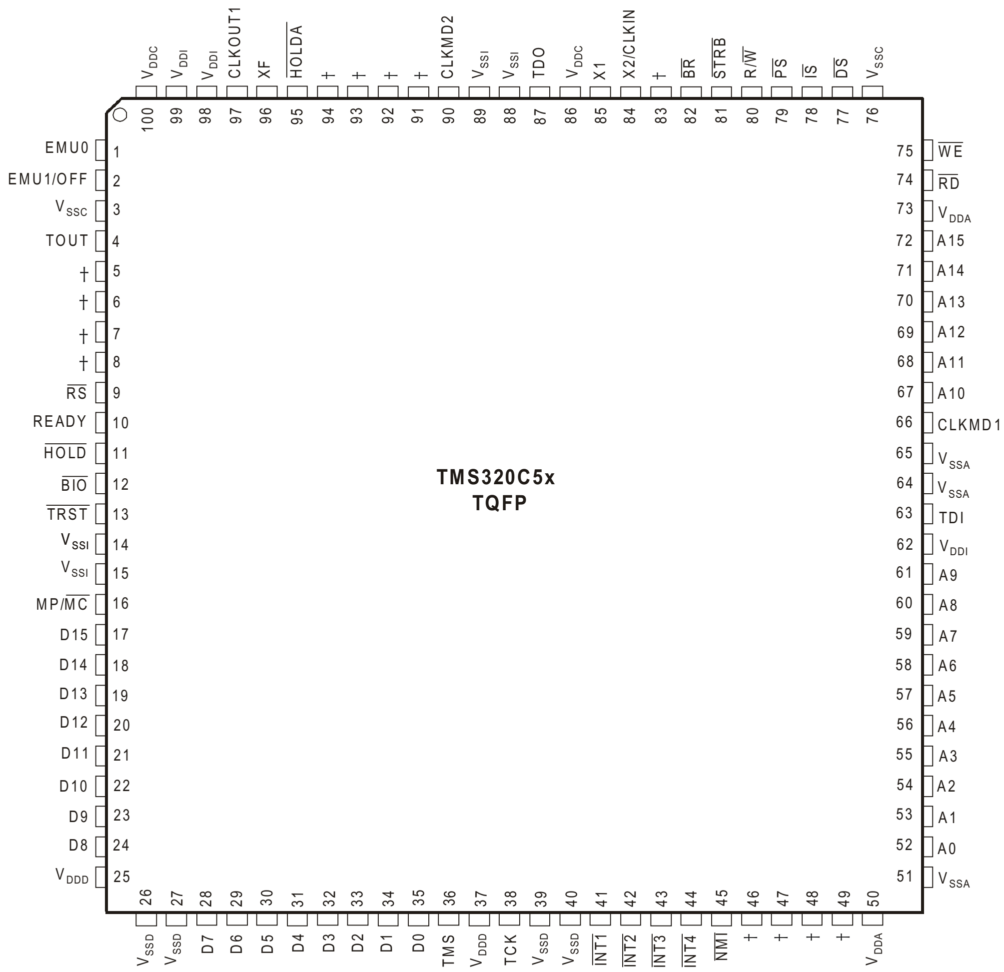
PROCESSOR	ON-CHIP MEMORY (16-BIT WORDS)				IO PORTS		POWER SUPPLY (V)	CYCLE TIME (ns)	NUMBER OF PINS
	DARAM		SARAM	ROM					
	DATA	DATA+ PROG	DATA+ PROG	PROG	SERIAL	PARALLEL			
TMS320C50	544	512	9k	2k	2	64k	5	50/35/25	132 pin
TMS320LC50	544	512	9k	2k	2	64k	3.3	50/40/25	132 pin
TMS320C51	544	512	1k	8k	2	64k	5	50/35/25/20	100/132 pin
TMS320LC51	544	512	1k	8k	2	64k	3.3	50/40/25	100/132 pin
TMS320C52	544	512	–	4k	1	64k	5	50/35/25/20	100 pin
TMS320LC52	544	512	–	4k	1	64k	3.3	50/40/25	100 pin
TMS320C53	544	512	3k	16k	2	64k	5	50/35/25	132 pin
TMS320LC53	544	512	3k	16k	2	64k	3.3	50/40/25	132 pin
TMS320C53S	544	512	3k	16k	2	64k	5	50/35/25	100 pin
TMS320LC53S	544	512	3k	16k	2	64k	3.3	50/40/25	100 pin
TMS320LC56	544	512	6k	32k	2	64k	3.3	35/25	100 pin
TMS320LC57	544	512	6k	32k	2	64k+HPI	3.3	35/25	128 pin
TMS320C57S	544	512	6k	2k	2	64k+HPI	5	50/35/25	144 pin
TMS320LC57S	544	512	6k	2k	2	64k+HPI	3.3	50/35	144 pin

11.3.1 Pin Diagram of TMS320C5x Processors

The TMS320C5x family of processors are available in the following plastic packages.

- 100/132 pins QFP (**Q**uad **F**lat **P**ackage)
- 100/128/144 pins TQFP (**T**hin **Q**uad **F**lat **P**ackage)

The pin configuration of TMS320C5x processors with 100 pins in TQFP is shown in fig 11.7. The functional groupings of the pins are shown in fig 11.8. The names or functions of various pins are listed in table 11.5.



Note : † *See Table 11.6 for device-specific pinouts.*

Fig 11.7 : *Pin diagram of TMS320C5x (TQFP).*

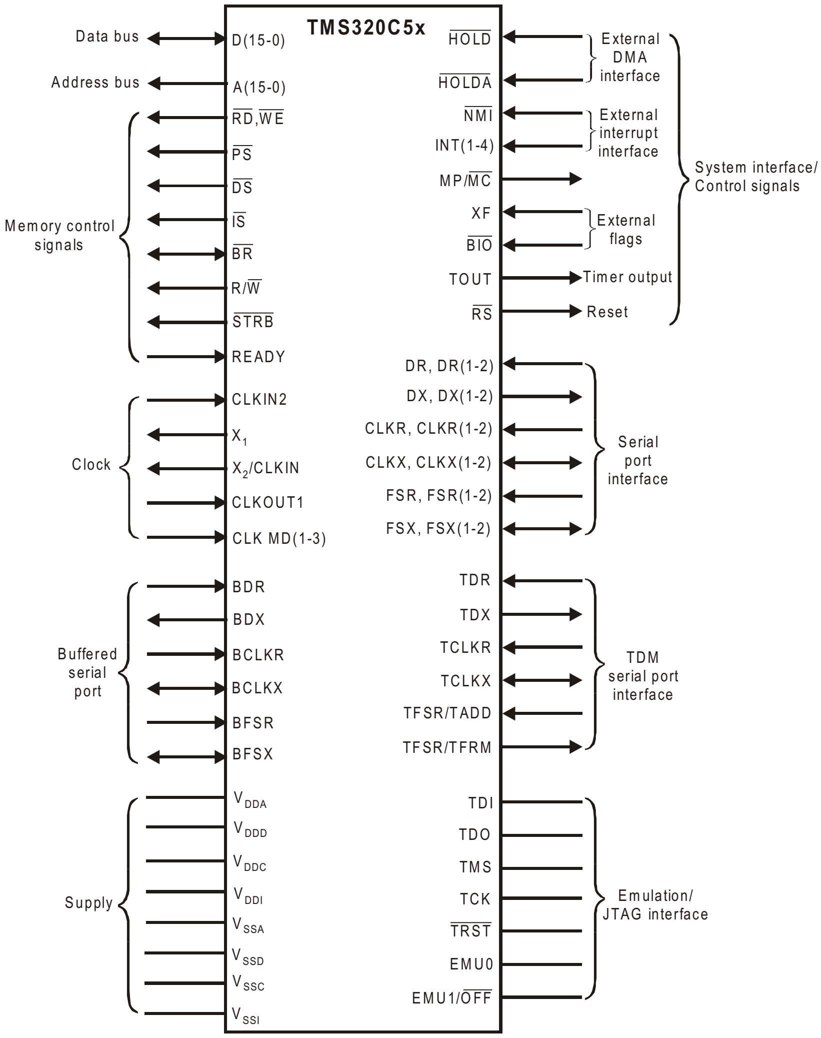


Fig 11.8 : Functional grouping of TMS320C5x pins (100 pin TQFP).

Table 11.5 : Pin Description of TMS320C5x (100 pin TQFP)

SIGNAL	TYPE	DESCRIPTION
		PARALLEL INTERFACE BUS
A0 – A15	IO, Tristate	16-bit external address bus
D0 – D15	IO, Tristate	16-bit external data bus
$\overline{PS}$, $\overline{DS}$, $\overline{IS}$	Output, Tristate	Program, data and IO space select outputs, respectively
$\overline{STRB}$	IO, Tristate	Timing strobe for external cycles and external DMA
R / $\overline{W}$	IO, Tristate	Read/write select for external cycles and external DMA
$\overline{RD}$, $\overline{WE}$	Output, Tristate	Read and write strobes, respectively, for external cycles

Table 11.5 : Continued...

SIGNAL	TYPE	DESCRIPTION
READY	Input	External bus ready/wait-state control input
$\overline{BR}$	IO, Tristate	Bus request
	SYSTEM INTERFACE/CONTROL SIGNALS	
$\overline{RS}$	Input	Reset
$MP/\overline{MC}$	Input	Microprocessor/microcomputer mode select
$\overline{HOLD}$	Input	Put parallel I/F bus in high-impedance state after current cycle
$\overline{HOLDA}$	Output, Tristate	Hold acknowledge
XF	Output, Tristate	External flag output
$\overline{BIO}$	Input	IO branch control input
TOUT	Output, Tristate	Timer output signal
$\overline{INT1}-\overline{INT4}$	Input	External interrupt inputs
$\overline{NMI}$	Input	Nonmaskable external interrupt
	SERIAL PORT INTERFACE	
DR, DR1, DR2	Input	Serial receive-data input
DX, DX1, DX2	Output, Tristate	Serial transmit-data output
CLKR, CLKR1, CLKR2	Input	Serial receive-data clock input
CLKX, CLKX1, CLKX2	IO, Tristate	Serial transmit-data clock
FSR, FSR1, FSR2	Input	Serial receive-frame-synchronization input
FSX, FSX1, FSX2	IO, Tristate	Serial transmit-frame-synchronization signal
	BUFFERED SERIAL PORT (BSP)	
BDR	Input	BSP receive-data input
BDX	Output, Tristate	BSP transmit-data output
BCLKR	Input	BSP receive-data clock input
BCLKX	IO, Tristate	BSP transmit-data clock
BFSR	Input	BSP receive frame-synchronization input
BFSX	IO, Tristate	BSP transmit frame-synchronization signal
	TDM SERIAL PORT INTERFACE	
TDR	Input	TDM serial receive-data input
TDX	Output, Tristate	TDM serial transmit-data output
TCLKR	Input	TDM serial receive-data clock input
TCLKX	IO, Tristate	TDM serial transmit-data clock
TFSR/TADD	IO, Tristate	TDM serial receive-frame-synchronization input
TFSX/TFRM	Input	TDM serial transmit-frame-synchronization signal

Table 11.5 : Continued...

SIGNAL	TYPE	DESCRIPTION
	EMULATION/JTAG INTERFACE	
TDI	Input	JTAG-test-port scan data input
TDO	Output, Tristate	JTAG-test-port scan data output
TMS	Input	JTAG-test-port mode select input
TCK	Input	JTAG-port clock input
$\overline{\text{TRST}}$	Input	JTAG-port reset (with pull-down resistor)
EMU0	IO, Tristate	Emulation control 0
EMU1 / $\overline{\text{OFF}}$	IO, Tristate	Emulation control 1
	CLOCK GENERATION AND CONTROL	
X1	Output	Oscillator output
X2/CLKIN, CLKIN2	Input	Clock/oscillator input
CLKMD1, CLKMD2	Input	Clock-mode select inputs
CLKOUT1	Output, Tristate	Device system-clock output
	POWER SUPPLY CONNECTIONS	
VDDA	Supply	Supply connection, address-bus output
VDDD	Supply	Supply connection, data-bus output
VDDC	Supply	Supply connection, control output
VDDI	Supply	Supply connection, internal logic
VSSA	Supply	Supply connection, address-bus output
VSSD	Supply	Supply connection, data-bus output
VSSC	Supply	Supply connection, control output
VSSI	Supply	Supply connection, internal logic

Table 11.6 : Device-Specific Pinouts for the 100 pin TQFP

PIN	TMS320C51 TMS320LC51	TMS320C52 TMS320LC52	TMS320C53S TMS320LC53S	TMS320LC56
5	TCLKX	V_{SSI}	CLKX2	BCLKX
6	CLKX	CLKX	CLKX1	CLKX
7	TFSR/TADD	V_{SSI}	FSR2	BFSR
8	TCLKR	V_{SSI}	CLKR2	BCLKR
46	DR	DR	DR1	DR
47	TDR	V_{SSI}	DR2	BDR
48	FSR	FSR	FSR1	FSR
49	CLKR	CLKR	CLKR1	CLKR

Table 11.6 : Continued...

PIN	TMS320C51 TMS320LC51	TMS320C52 TMS320LC52	TMS320C53S TMS320LC53S	TMS320LC56
83	CLKIN2	CLKIN2	CLKIN2	CLKMD3
91	FSX	FSX	FSX1	FSX
92	TFSX/TFRM	V_{SSI}	FSX2	BFSX
93	DX	DX	DX1	DX
94	TDX	NC	DX2	BDX

11.3.2 Architecture of TMS320C5x Processors

The TMS320C5x processors have an advanced version of Harvard architecture, with separate buses for program and data, which facilitate simultaneous access of program and data. The program bus has separate lines to transmit data and address. Similarly the data bus has separate lines to transmit data and address.

The internal architecture of TMS320C5x processor is shown in fig 11.9. The architecture of TMS320C5x processors can be broadly divided into three major areas. They are, CPU (**C**entral **P**rocessing **U**nit), memory and peripherals.

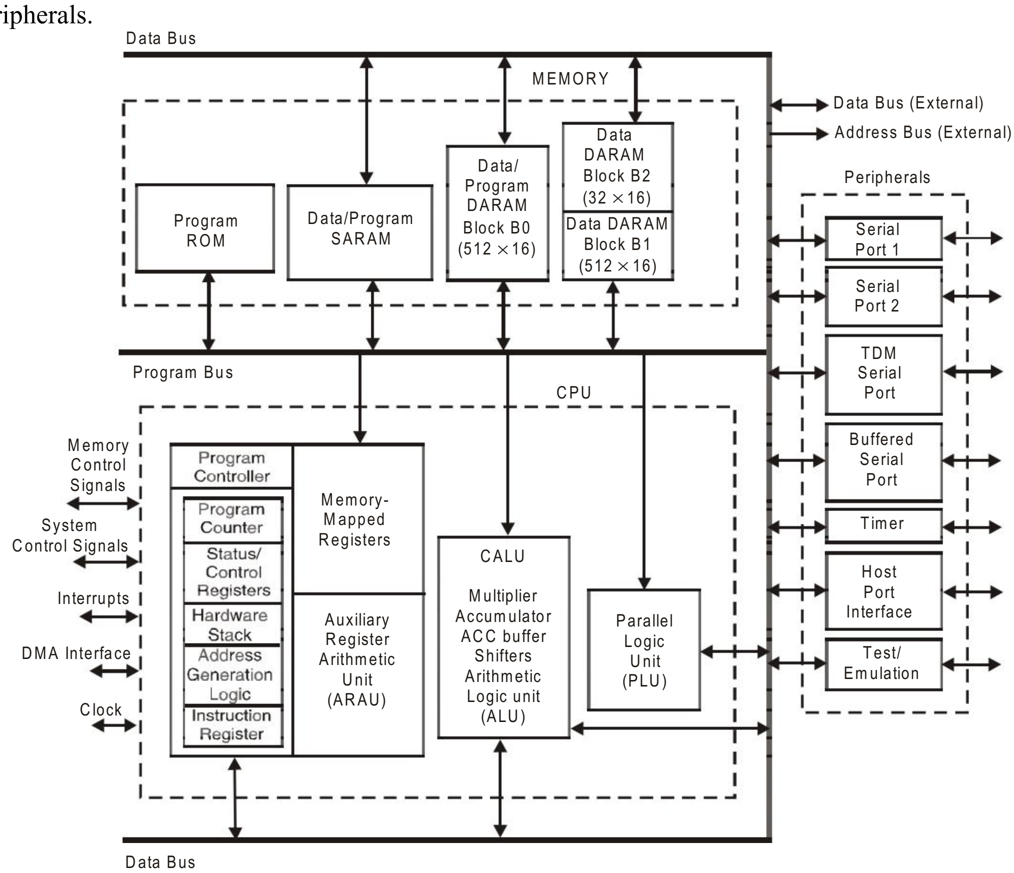


Fig 11.9 : *Internal architecture of TMS320C5x.*

The functional units of CPU are **P**arallel **L**ogic **U**nit (PLU), central ALU, memory mapped registers, **A**uxiliary **R**egister **A**rithmetic **U**nit (ARAU) and program controller.

The TMS320C5x processors has the following internal (or on-chip) memory.

- Program ROM (2k to 32k words)
- Data/Program **D**ual **A**ccess **RAM** (DARAM) (1024 + 32 = 1056 words)
- Data/Program **S**ingle **A**ccess **RAM**/(SARAM) (1k to 9k words)

The various on-chip or internal peripherals of TMS320C5x processors are clock generator, hardware timer, software programmable wait state generators, parallel IO ports, **H**ost **P**ort **I**nterface (HPI), serial port, **B**uffered **S**erial **P**ort (BSP), **T**ime **D**ivision **M**ultiplexed (TDM) serial port and user maskable interrupts.

The TMS320C5x processors have a total memory address space of 224k (including on-chip memory) with addressability of 16 bits. This address space is divided into four individually selectable address spaces as follows.

- 64k Program memory address space
- 64k Local data memory address space
- 32k Global data memory address space
- 64k IO ports address space

> *Note : The addressability refer to memory word size, which is the maximum size of binary that can be stored in one memory location.*

11.3.3 Functional Units of CPU of TMS320C5x Processors

Parallel Logic Unit (PLU)

The ***PLU*** is an additional logic unit, that permits logic operations without affecting accumulator or product register. It performs Boolean operations or bit manipulations. It can set, clear, test or toggle bits in the status register, control register and in any data memory location.

Central Arithmetic Logic Unit (CALU)

The ***CALU*** consists of 32-bit ALU, 32-bit accumulator (ACC), 32-bit accumulator buffer (ACCB), 16×16-bit multiplier, 32-bit product register (PREG), 0 to 16-bit left barrel shifter and 0 to 16-bit right barrel shifter.

The ***ALU*** is a general-purpose arithmetic/logic unit that performs usual arithmetic and logical operations. For ALU operations involving two operands, one of the operands is stored in the accumulator and the other operand can be from data memory/immediate data in the instruction/internal register. After ALU operation, the result is stored in the accumulator.

The 16×16-bit hardware ***multiplier*** is capable of multiplying two 16-bit two's complement numbers to generate a 32-bit product in a single machine cycle. For performing multiplication, one of the operands is stored in TREG0 (Temporary register 0) and the other operand can be from data memory/immediate operand in the instruction. After multiplication, the product is stored in the 32-bit ***product register*** (PREG).

The 0 to 16-bit left and right shifter permit the content of memory to be shifted before loading into ALU and vice versa. The content of accumulator (ACC) and product register (PREG) can also be shifted using these shifters.

Memory-Mapped Registers

The TMS320C5x has 96 numbers of 16-bit memory-mapped registers, and they are mapped into page-0 of data memory space. The memory-mapped registers includes various control and status registers for CPU, serial port, timer and software wait-state generators. Also they include 16 memory-mapped IO ports. The memory-mapped registers along with their memory address are listed in table 11.7.

Table 11.7 : Memory-Mapped Registers of TMS320C5x Processors

ADDRESS		NAME	DESCRIPTION
DEC	HEX		
0–3	0–3	–	Reserved
4	4	IMR	Interrupt mask register
5	5	GREG	Global memory allocation register
6	6	IFR	Interrupt flag register
7	7	PMST	Processor mode status register
8	8	RPTC	Repeat counter register
9	9	BRCR	Block repeat counter register
10	A	PASR	Block repeat program address start register
11	B	PAER	Block repeat program address end register
12	C	TREG0	Temporary register 0 (used for multiplicand)
13	D	TREG1	Temporary register 1 (used for dynamic shift count)
14	E	TREG2	Temporary register 2 (used as bit pointer in dynamic bit test)
15	F	DBMR	Dynamic bit manipulation register
16–23	10–17	AR0–AR7	Auxiliary register 0 - Auxiliary register 7
24	18	INDX	Index register
25	19	ARCR	Auxiliary register compare register
26	1A	CBSR1	Circular buffer 1 start register
27	1B	CBER1	Circular buffer 1 end register
28	1C	CBSR2	Circular buffer 2 start register
29	1D	CBER2	Circular buffer 2 end register
30	1E	CBCR	Circular buffer control register
31	1F	BMAR	Block move address register
32–35	20–23	–	Memory-mapped serial port registers

Table 11.7 : Continued...

ADDRESS		NAME	DESCRIPTION
DEC	HEX		
36–42	24–2A	–	Memory-mapped peripheral registers
43–47	2B–2F	–	Reserved for test/emulation
48–55	30–37	–	Memory-mapped serial port registers
56–79	38–4F	–	Reserved
80–95	50–5F	–	Memory-mapped IO ports

Auxiliary Register Arithmetic Unit (ARAU)

The ***ARAU*** contains eight 16-bit auxiliary registers AR0-AR7, a 3-bit **A**uxiliary **R**egister **P**ointer (ARP), a 16-bit index register (INDX) and a 16-bit **A**uxiliary **R**egister **C**ompare **R**egister (ARCR). An unsigned 16-bit arithmetic unit in the ARAU is used to calculate indirect addresses, using the contents of ARP, INDX and ARCR registers. Therefore, the CALU is relieved from task of address manipulation and so it is free for other operations in parallel.

The auxiliary registers can also be used as general-purpose registers for holding the operands for arithmetic and logical operation in CALU.

Program Controller

The ***program controller*** contains logic circuits that decodes the instructions, manages the CPU pipeline, stores the status of CPU operations and decodes the conditional operations. Due to parallelism in architecture, the program controller can perform three concurrent or simultaneous memory operations in any given machine cycle. They are fetch an instruction, read an operand and write an operand.

The program controller unit consists of a 16-bit **P**rogram **C**ounter (PC), 16-bit status registers ST0 and ST1, **P**rocessor **M**ode **St**atus register (PMST) and **C**ircular **B**uffer **C**ontrol **R**egister (CBCR), a 16 ´ 16-bit hardware stack, address generation logic, instruction register, interrupt flag register and interrupt mask register.

Status Registers (ST0 and ST1)

The TMS320C5x processors have two 16-bit ***status registers*** (ST0 and ST1) which holds the status of ALU result, pointers for indirect addressing and various bits for interrupt control, hold mode and product shift mode. The format of status registers are shown in fig 11.10 and 11.11. The functions of various bits of status registers are listed in Table 11.8 and 11.9.

The status registers can be stored into data memory and loaded from data memory, thereby allowing the processor status to be saved and restored for subroutines. The LST instruction writes to ST0 and ST1, and the SST instruction reads from them, except that the ARP bits and INTM bit are not affected by the LST #0 instruction.

The ST0 and ST1 each have an associated 1-level deep shadow register stack for automatic context-saving when an interrupt occurs. The INTM and OVM bits in ST0 and the C, CNF, HM, SXM, TC and XF bits in ST1 can be individually set using the SETC instruction or individually cleared using the CLRC instruction.

15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
ARP			OV	OVM	1	INTM	DP								

***Fig 11.10 :** Format of status register 0 (ST0) of TMS320C5x processors.*

Table 11.8 : Functions of Various Bits of ST0 of TMS320C5x Processors

BIT	NAME	RESET VALUE	FUNCTION
15-13	ARP	X	Auxiliary register pointer to select AR for indirect addressing
12	OV	0	Overflow flag bit. Indicates an overflow in ALU operation
11	OVM	X	Overflow mode bit. Enables/disables the saturation mode in ALU
10	–	1	Always 1
9	INTM	1	Interrupt mode bit. Globally masks or enables all interrupts except NMI
8-0	DP	X	Data memory page pointer to specify the current data memory page

15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
ARB			CNF	TC	SXM	C	1	1	HM	1	XF	1	1	PM	

***Fig 11.11 :** Format of status register 1 (ST1) of TMS320C5x processors.*

Table 11.9 : Function of Various Bits of ST1 of TMS320C5x Processors

BIT	NAME	RESET VALUE	FUNCTION
15-13	ARB	XXX	Auxiliary register buffer which holds the previous value in ARP of ST0
12	CNF	0	Configuration control bit to map Block 0 of DARAM either as data or program memory
11	TC	X	Test/control flag bit which stores the results of ALU or PLU test bit operations
10	SXM	1	Sign-extension mode bit. Enables/disables sign extension of an arithmetic operation
9	C	1	Carry bit which indicates a carry or borrow in ALU operation
8-7	–	11	Always 1
6	HM	1	Hold mode bit
5	–	1	Always 1
4	XF	1	Status of external flag pin
3-2	–	11	Always 1
1-0	PM	00	Product shift mode bits

11.3.4 On-Chip Memory in TMS320C5x Processors

The TMS320C5x family of processors consists of three different types of on-chip memory and they are mask-programmable ROM, **S**ingle-**A**ccess **RAM** (SARAM) and **D**ual-**A**ccess **RAM** (DARAM). The various members of TMS320C5x will have different capacity of on-chip memory which are listed in table 11.4.

Program ROM

The various models of TMS320C5x processors have internal maskable-**P**rogram **ROM** (PROM) of size 2k to 32k words. The processor has an option for including or excluding the on-chip PROM addresses in the processor program memory address space.

The main purpose of PROM is to permanently store the program code for a specific application during manufacturing of the chip itself. The processor has an option of boot loading the content of PROM to internal/external RAM during power-ON reset. The content of the PROM can be protected so that any external device cannot have access to the program code. This feature provides security for proprietary algorithms.

Data / Program Dual-Access RAM (DARAM)

The TMS320C5x processor has 1056 words [1056 ´ 16 bits] on-chip **D**ual-**A**ccess RAM (DARAM), which is divided into three blocks, B0, B1 and B2.

- The block B0 has 512 words [512 ´ 16 bits] data / program RAM.
- The block B1 has 512 words [512 ´ 16 bits] data RAM.
- The block B2 has 32 words [32 ´ 16 bits] data RAM.

Data / Program Single-Access RAM (SARAM)

The various models or TMS320C5x processor has 1k words to 9k words of SARAM.

The internal SARAM can be configured as data memory, program memory and combination of data and program memory

The SARAM can be divided into block of 1k/2k words with continuous address. The processor CPU can access one block for reading while writing in another block.

11.3.5 On-Chip Peripherals of TMS320C5x Processors

The various on-chip peripherals of TMS320C5x processors are clock generator, hardware timer, software-programmable wait-state generators, parallel IO ports, **H**ost **P**ort **I**nterface (**HPI**) and serial ports.

Clock Generator

The ***clock generator*** of the TMS320C5x processor consists of an internal oscillator and a **P**hase **L**ocked **L**oop (PLL) circuit. The clock generator can be driven by an external crystal resonator circuit or supplied by an external clock source. The ***PLL*** circuit can generate an internal CPU clock by multiplying the clock source by a specified factor, so that CPU is driven by high frequency clock and clock source can be used as source for other peripherals which runs at low frequency clock.

Hardware Timer

A 16-bit hardware timer with a 4-bit prescaler is available in TMS320C5x processor. This programmable ***timer*** generates clock at a rate that is between 1/2 and 1/32 of the machine cycle rate (CLKOUT1), depending upon the timer divide-down ratio. The timer can be stopped, restarted, reset or disabled by specific status bits. The processor has three registers to control and operate the timer and they are **T**imer **C**ontrol **R**egister (TCR), timer counter register (TIM) and timer period register (PRD). The timer counter register gives the current count of the timer. The timer period register defines the period for the timer. The 16-bit timer control register controls the operations of the timer.

Software Programmable Wait - State Generators

The TMS320C5x processor has software-programmable ***wait-state generators***, which can insert/ generate wait-states in external bus cycles for interfacing with slow speed external memory and IO devices. The processor consists of multiple wait-state generating circuits,and each circuit is user-programmable to insert different number of wait states for external memory accesses. These wait-state generators can extend the external bus cycles up to seven machine cycles.

Parallel IO Ports

The TMS320C5x processor has 64k ***IO address space*** which can be used as 64k IO ports and 16 of these ports are memory-mapped in data memory space. Each of the IO ports can be addressed by the IN or the OUT instruction. The memory-mapped IO ports can be accessed with any instruction that reads from or writes to data memory. TheTMS320C5x genearates a hardware signal $\overline{\text{IS}}$ during IO access to indicate a read or write operation through an IO port. The TMS320C5x can easily interface with external IO devices through the IO ports with minimal external address decoding circuits.

Host Port interface (HPI)

The HPI is available on the TMS320C57S and TMS320LC57 processors. The ***HPI*** is an 8-bit parallel IO port that provides an interface to a host processor for information exchange between the **D**igital **S**ignal **P**rocessor (DSP) and the host processor. The DSP has 2k word on-chip memory that is accessible to both the host processor and the DSP. The HPI is connected to this memory through a dedicated bus, so that the CPU can work uninterrupted while the host processor accesses the memory through host port.

Note : A host processor is an independent microprocessor/micorcontroller that is designed to carry out some specific tasks and deliver the results to digital signal processor.

Serial Ports

Three different kinds of serial ports are available in TMS320C5x processors and they are general-purpose serial port, **T**ime-**D**ivision **M**ultiplexed (TDM) serial port and **B**uffered **S**erial **P**ort (BSP). Every TMS320C5x processor contains at least one general-purpose, high-speed synchronous, full-duplexed serial port which can be used to provide direct communication with serial devices such as codec, serial analog-to digital (A/D) converters and other serial systems. The serial port is capable of operating at a clock rate up to one-fourth the machine cycle rate (CLKOUT1). The serial port transmitter and receiver are double-buffered and indiviually controlled by maskable external interrupt signals. For serial communication the data is framed either as bytes or as words.

The **B**uffered **S**erial **P**ort (BSP) consists of a full-duplex double-buffered **S**erial **P**ort **I**nterface (SPI) and an **A**uto-**B**uffering **U**nit (ABU). The ABU allows the SPI to read/write directly to processor internal memory using a dedicated bus which enhances the speed of serial communication.

The **T**ime-**D**ivision **M**ultiplexed (TDM) serial ports can be used for serial communication between multiple processors. A maximum of eight processors having TDM ports can be connected via a pair of data lines and a pair of address lines for serial communication.

11.3.6 Addressing Modes of TMS320C5x Processors

The addressing mode is the method of specifying the data to be operated by an instruction. The TMS320C5x family of processors supports the following six addressing modes.

1. Direct addressing
2. Memory-mapped register addressing
3. Indirect addressing
4. Immediate addressing
5. Dedicated-register addressing
6. Circular addressing

Direct Addressing

In direct addressing, the lower 7 bits of data memory address are specified directly in the instruction itself. The upper 9 bits of the address will be the content of data memory page pointer (DP) in status register-0 (ST0).

Example :

```
ADDC 2Ch   ; Add the content of data memory whose page offset address
           ; (2Ch) is specified in the instruction and the carry to accumulator
           ; with sign extension suppressed.
```

Note : The TMS320C5x data memory is organized as 512 (or 2^9) pages with each page having 128 (or 2^7) locations ($2^9 \times 2^7 = 2^{16} = 64k$). The DP holds the 9-bit current page address and the 7-bit address in the instruction is the page offset address.

Memory-Mapped Register Addressing

In memory-mapped register addressing, the address of the memory-mapped register can be specified as direct address in the instruction. The memory-mapped register addressing is a special case of direct addressing in which only page offset address is used to access the memory and the default page address is 000h. Therefore, the data pointer need not be loaded with page address for this addressing mode.

Example :

```
LAMM 16h   ; Load accumulator with the content of memory-mapped
           ; register mapped to address 0016h.
SAMM 16h   ; Store the content of accumulator in memory-mapped
           ; register mapped to address 0016h.
```

Note : The memory-mapped registers of TMS320C5x are mapped to page-0 of data memory address space.

Indirect Addressing

In indirect addressing mode, the data memory address is specified by the content of one of the eight auxiliary registers, AR0 - AR7. The AR (**A**uxiliary **R**egister) currently used for accessing data is denoted by ARP (**A**uxiliary **R**egister **P**ointer).

In indirect addressing mode, the content of AR can be updated automatically either after or before the operand is fetched. The syntax used in the operand field of instruction for modifying the content AR are listed in table 11.10.

Example :

```
LACC *,0      ; Load the content of data memory addressed by AR, (which in turn pointed by ARP) to
              ; accumulator without left shift. AR is not altered.
LACC *+, 0    ; Same as above, but AR is incremented by one.
LACC *-, 0    ; Same as above, but AR is decremented by one.
LACC *0+,0    ; Same as above, but AR is incremented by the value in index register.
LACC *0-,0    ; Same as above, but AR is decremented by the value in index register.
```

Table 11.10 : Syntax Used in Indirect Address for Modifying AR

Syntax	Modification of AR
*	AR unaltered
*+	AR incremented by 1
*–	AR decremented by 1
*0+	AR incremented by the content of index register
*0-	AR decremented by the content of index register
*BR0+	AR incremented for bit reversed addressing using the content of index register
*BR0-	AR decremented for bit reversed addressing using the content of index register

Bit Reversed Addressing

In bit reversed addressing, the data memory address is specified by AR like indirect addressing, but the content of AR is incremented/decremented in order to generate the data memory address in the bit reversed order, using the content of index register. (The bit reversed addressing is a special case of indirect addressing).

Example :

```
MAC 14F0h, *BR0+   ; The content of program memory is multiplied by the data memory and the product is
                   ; added to the accumulator. The address of the program memory is 14F0h (which is
                   ; specified in the instruction). The address of data memory is the content of AR currently
                   ; pointed by ARP. The AR is incremented to generate the bit reversed address of data
                   ; memory operand.
```

Immediate Addressing

In immediate addressing, the data is specified as a part of the instruction. In this addressing, the instruction will carry an 8-bit/9-bit/13-bit/16-bit constant, which is the data to be operated by the instruction. The immediate constant is specified with # symbol.

Example :

ADD # 4Ah	; Add the immediate, 8-bit data 4Ah given in the instruction to accumulator.
ADD # 8C4Ah	; Add the immediate 16-bit data 8C4Ah given in the instruction to accumulator.

Dedicated Register Addressing

In dedicated register addressing mode, the address of one of the operands is specified by a dedicated CPU register BMAR (**B**lock **M**ove **A**ddress **R**egister). In this addressing mode, the address of the memory block to be accessed can be changed during execution of the program.

In another case of dedicated register addressing, one of the operands is the content of a dedicated CPU register DBMR (**D**ynamic **B**it **M**anupulation **R**egister).

Example :

BLDD BMAR, 6Fh	; The source address is the content of BMAR. The lower 7 bits of destination address is ; 6Fh, (which is directly given in the instruction) and the upper 9 bits of destination ; address is the content of DP. This instruction will copy the content of source address to ; destination address.
OPL 6Fh	; The content of DBMR is logically 0Red with the content of data memory specified by the ; instruction and result is stored back in data memory. The lower 7 bits of data memory ; is 6Fh (which is directly given in the instruction) and the upper 9 bits is the ; content of DP.

Circular Addressing

The circular addressing is similar to indirect addressing. This addressing mode allows the specified memory buffer to be accessed sequentially with a pointer that automatically wraps around to the beginning of the buffer when the last location is accessed.

In circular addressing mode, when the address pointer is incremented, the address in AR will be checked with the end address of the circular buffer, and if it exceeds the end address then the address is made equal to start address of the circular buffer.

In order to hold the start and end addresses of the circular buffer, the TMS320C5x has four circular buffer registers, namely,

CBSR1 : **C**ircular **B**uffer-1 **S**tart address **R**egister

CBSR2 : **C**ircular **B**uffer-2 **S**tart address **R**egister

CBER1 : **C**ircular **B**uffer-1 **E**nd address **R**egister

CBER2 : **C**ircular **B**uffer-2 **E**nd address **R**egister

With the help of the above registers, at any one time, two circular buffers can be defined. A **C**ircular **B**uffer **C**ontrol **R**egister (CBCR) is used to enable/disable the circular buffers.

11.3.7 Instruction Pipelining in TMS320C5x Processors

The execution of TMS320C5x processor instructions involve four levels/phases of pipelining. The four phases of pipelining are fetch, decode, read and execute. The functions performed in the four phases are given below.

Fetch : The opcode (instruction code) is fetched from program memory and PC is updated.

Decode : The opcode is decoded, the data address is generated and data address generation registers are updated.

Read : The operand is read from data memory.

Execute : Perform the task specified by the instruction.

The TMS320C5x employs a four-phase clock so that one phase of four consecutive instructions can be processed simultaneously. When the first instruction is in execute phase, the second instruction will be in read phase, the third instruction will be in decode phase and the fourth instruction will be in fetch phase. Therefore, the average execution time of one word instruction will be one clock cycle. While executing two word instructions and branch instructions, some of the phases will be idle for some time and so average execution time will be more than one clock cycle.

11.3.8 Instructions of TMS320C5x Processors

The TMS320C5x processors instruction set consists of instructions that supports both numeric-intensive signal processing operations and general-purpose applications. The instructions can be classified into following groups.

1. Arithmetic instructions
2. Logical instructions
3. Branch/control instructions
4. Load/store instructions
5. Block move instructions

The instructions of TMS320C5x processors are classified into the above groups, arranged in alphabetical order and listed in table 11.11.

The size of TMS320C5x instructions is either 1 word or 2 words. When all the instructions and data reside in internal memory, most of the instructions are executed in one or two clock cycles. The execution time for some of the data transfer, branch and MAC instructions is 3 to 4 clock cycles.

Table 11.11 : Instruction Set Summary for TMS320C5x

ARITHMETIC INSTRUCTIONS	
INSTRUCTION	**DESCRIPTION**
ABS	Absolute value of ACC
ADCB	Add ACCB to ACC with carry
ADD *dma [,shift]*	Add memory to ACC with shift (Direct addressing)
ADD *{ind} [,shift [,next ARP]]*	Add memory to ACC with shift (Indirect addressing)
ADD # *k*	Add short immediate to low ACC
ADD # *lk [,shift2]*	Add long immediate to ACC with shift
ADDB	Add ACCB to ACC

Table 11.11 : Continued ....

ARITHMETIC INSTRUCTIONS	
INSTRUCTION	**DESCRIPTION**
ADDC *dma*	Add memory to ACC with carry (Direct addressing)
ADDC *{ind} [,next ARP]*	Add memory to ACC with carry (Indirect addressing)
ADDH *dma*	Add memory to higher 16 bits of accumulator (Direct addressing)
ADDH *{ind} [,next ARP]*	Add memory to higher 16 bits of accumulator (Indirect addressing)
ADDS *dma*	Add memory to ACC with sign extension suppressed (Direct addressing)
ADDS *{ind} [,next ARP]*	Add memory to ACC with sign extension suppressed (Indirect addressing)
ADDT *dma*	Add memory to ACC with shift specified by T-register (Direct addressing)
ADDT *{ind} [,next ARP]*	Add memory to ACC with shift specified by T-register (Indirect addressing)
ADRK *# k*	Add short immediate to AR
APAC	Add PREG to ACC
MAC *pma, dma*	Multiply and accumulate (Direct addressing)
MAC *pma, {ind} [,next ARP]*	Multiply and accumulate (Indirect addressing)
MACD *pma, dma*	Multiply and accumulate with data move (Direct addressing)
MACD *pma, {ind} [,next ARP]*	Multiply and accumulate with data move (Indirect addressing)
MADD *dma*	Multiply and ACC with data move, pma in BMAR (Direct addressing)
MADD *{ind} [, next ARP]*	Multiply and ACC with data move, pma in BMAR (Indirect addressing)
MADS *dma*	Multiply and ACC, pma in BMAR (Direct addressing)
MADS *{ind} [,next ARP]*	Multiply and ACC, pma in BMAR (Indirect addressing)
MPY *dma*	Multiply data in memory with TREG0 (Direct addressing)
MPY *{ind} [,next ARP]*	Multiply data in memory with TREG0 (Indirect addressing)
MPY *# k*	Multiply TREG0 by short immediate
MPY *# lk*	Multiply TREG0 by long immediate
MPYA *dma*	Multiply TREG0 by data, add previous product (Direct addressing)
MPYA *{ind} [,next ARP]*	Multiply TREG0 by data, add previous product (Indirect addressing)
MPYS *dma*	Multiply TREG0 by data, subtract previous product (Direct addressing)
MPYS *{ind} [,next ARP]*	Multiply TREG0 by data, subtract previous product (Indirect addressing)
MPYU *dma*	Unsigned multiplication of memory with TREG0 (Direct addressing)
MPYU *{ind} [,next ARP]*	Unsigned multiplication of memory with TREG0 (Indirect addressing)
NORM	Normalize ACC
SBB	Subtract ACCB from ACC

Table 11.11 : Continued...

INSTRUCTION	DESCRIPTION
SBBB	Subtract ACCB from ACC with borrow
SBRK *# k*	Subtract short immediate from AR
SPAC	Subtract P-register from ACC
SQRA *dma*	Square and accumulate previous product (Direct addressing)
SQRA *{ind} [,next ARP]*	Square and accumulate previous product (Indirect addressing)
SQRS *dma*	Square and subtract previous product (Direct addressing)
SQRS *{ind} [,next ARP]*	Square and subtract previous product (Indirect addressing)
SUB *dma [,shift]*	Subtract memory from ACC with shift (Direct addressing)
SUB *{ind} [,shift [,next ARP]]*	Subtract memory from ACC with shift (Indirect addressing)
SUB *# k*	Subtract short immediate from ACC
SUB *# lk [,shift 2]*	Subtract long immediate from ACC with shift
SUBB *dma*	Subtract memory from ACC with borrow (Direct addressing)
SUBB *{ind} [,next ARP]*	Subtract memory from ACC with borrow (Indirect addressing)
SUBC *dma*	Conditional subtract (Direct addressing)
SUBC *{ind} [,next ARP]*	Conditional subtract (Indirect addressing)
SUBH *dma*	Subtract memory from high ACC (Direct addressing)
SUBH *{ind} [,next ARP]*	Subtract memory from high ACC (Indirect addressing)
SUBS *dma*	Subtract memory from low ACC with sign extension suppressed (Direct addressing)
SUBS *{ind} [,next ARP]*	Subtract memory from low ACC with sign extension suppressed (Indirect addressing)
SUBT *dma*	Subtract memory from ACC with shift specified by TREG1 (Direct addressing)
SUBT *{ind} [,next ARP]*	Subtract memory from ACC with shift specified by TREG1 (Indirect addressing)
LOGICAL INSTRUCTIONS	
AND *dma*	AND memory with ACC (Direct addressing)
AND *{ind} [,next ARP]*	AND memory with ACC (Indirect addressing)
AND *# lk [,shift]*	AND long immediate with ACC with shift
ANDB	AND ACCB with ACC
APL *[# lk], dma*	AND memory with DBMR or long constant (Direct addressing)
APL *[# lk,] {ind} [,next ARP]*	AND memory with DBMR or long constant (Indirect addressing)
BSAR *[shift]*	Barrel shift ACC right
CMPL	Complement ACC
CMPR *CM*	Compare AR with AR0 as specified by CM, if true set TC else clear TC

Table 11.11 : Continued...

INSTRUCTION	DESCRIPTION
CPL *[, # lk] dma*	Compare DBMR or long immediate with memory (Direct addressing)
CPL *[, # lk] {ind} [,next ARP]*	Compare DBMR or long immediate with memory (Indirect addressing)
CRGT	Compare ACC and ACCB, store largest in both, if ACC > ACCB set carry
CRLT	Compare ACC and ACCB, store smallest in both, if ACC < ACCB clear carry
NEG	Negate ACC
OPL *[# lk,] dma*	OR memory with DBMR or long constant (Direct addressing)
OPL *[# lk,] {ind} [,next ARP]*	OR memory with DBMR or long constant (Indirect addressing)
OR *dma*	OR memory with ACC (Direct addressing)
OR *{ind} [,next ARP]*	OR memory with ACC (Indirect addressing)
OR # *lk [,shift]*	OR long immediate with ACC with shift
ORB	OR ACCB with ACC
ROL	Rotate ACC left by one bit
ROLB	Rotate ACCB and ACC left by one bit
ROR	Rotate ACC right by one bit
RORB	Rotate ACCB and ACC right by one bit
SATH	Shift ACC 16 bits right if TREG1[4] = 1
SATL	Shift low ACC right as specfied by TREG1 [3-0]
SFL	Shift ACC left by one bit
SFLB	Shift ACCB and ACC left by one bit
SFR	Shift ACC right by one bit
SFRB	Shift ACCB and ACC right by one bit
XOR *dma*	XOR memory with ACC (Direct addressing)
XOR *{ind} [,next ARP]*	XOR memory with ACC (Indirect addressing)
XOR # *lk [,shift]*	XOR long immediate with ACC with shift
XORB	XOR ACCB with ACC
XPL *[# lk] dma*	XOR memory with DBMR or long constant (Direct adressing)
XPL *[# lk,] {ind} [,next ARP]*	XOR memory with DBMR or long constant (Indirect adressing)
BRANCH / CONTROL INSTRUCTIONS	
B[D] *pma [,{ind} [,next ARP]]*	Branch unconditionally to specified pma with AR update and optional delay
BACC[D]	Branch to address specified by ACC with optional delay
BANZ[D] *pma [,{ind} [,next ARP]]*	Branch to pma if content of AR ¹ 0 with AR update and with optional delay

Table 11.11 : Continued...

INSTRUCTION	DESCRIPTION
BBNZ *pma [,{ind} [,next ARP]]*	Branch to specified pma if TC bit of ST1 is not zero with AR update
BBZ *pma*	Branch to specified pma if TC bit of ST1 is zero
BBZ *pma [,{ind} [,next ARP]]*	Branch to specified pma if TC bit of ST1 is zero with AR update
BC *pma*	Branch to specified pma if C bit of ST1 is one
BC *pma [,{ind} [,next ARP]]*	Branch to specified pma if C bit of ST1 is one with AR update
BCND[D] *pma [cond1] [,cond2] [...]*	Branch to pma if specified conditions are true with optional delay
BGEZ *pma*	Branch to specified pma if ACC $\geq$ 0
BGEZ *pma [, {ind} [, next ARP]]*	Branch to specified pma if ACC $\geq$ 0 with AR update
BGZ *pma*	Branch to specified pma if ACC > 0
BGZ *pma [, {ind} [, next ARP]]*	Branch to specified pma if ACC > 0 with AR update
BIOZ *pma*	Branch to specified pma if $\overline{BIO} = 0$
BIOZ *pma [, {ind} [, next ARP]]*	Branch to specified pma if $\overline{BIO} = 0$ with AR update
BIT *dma, bit code*	Copy the specified bit of memory to TC bit in ST1 (Direct addressing)
BIT *{ind}, bit code [,next ARP]*	Copy the specified bit of memory to TC bit in ST1 (Indirect addressing)
BITT *dma*	Copy the bit of memory specified by T-register to TC bit in ST1 (Direct addressing)
BITT *{ind} [,next ARP]*	Copy the bit of memory specified by T-register to TC bit in ST1 (Indirect addressing)
BLEZ *pma*	Branch to specified pma if ACC $\leq$ 0
BLEZ *pma [, {ind} [, next ARP]]*	Branch to specified pma if ACC $\leq$ 0 with AR update
BLZ *pma*	Branch to specified pma if ACC < 0
BLZ *pma [, {ind} [, next ARP]]*	Branch to specified pma if ACC < 0 with AR update
BNC *pma [, {ind} [, next ARP]]*	Branch to specified pma if C bit in ST1 is zero with AR update
BNV *pma [, {ind} [, next ARP]]*	Branch to specified pma if OV bit in ST0 is zero with AR update
BNZ *pma [, {ind} [, next ARP]]*	Branch to specified pma if ACC $\neq$ 0 with AR update
BV *pma [, {ind} [, next ARP]]*	Branch to specified pma if OV bit in ST0 is one with AR update
BZ *pma*	Branch to specified pma if ACC = 0
CALA[D]	Call subroutine addressed by ACC with optional delay
CALL[D] *pma [,{ind} [,next ARP]]*	Call subroutine unconditionally with AR update and with optional delay
CC[D] *pma [cond1] [,cond2] [...]*	Call subroutine if specified conditions are true and with optional delay
CLRC *control bit*	Clear the specified control bit
CNFB	Configure on-chip block0 RAM as program memory
CNFD	Configure on-chip block0 RAM as data memory

Table 11.11 : Continued...

INSTRUCTION	DESCRIPTION
DINT	Disable interrupts
EINT	Enable interrupts
IDLE	Idle until interrupt
IDLE2	Idle until interrupt and drive the processor to low-power mode
INTR *k*	Software interrupt with vector number k
NMI	Nonmaskable interrupt (Vector number 15) (Vector address 24h)
NOP	No operation
POP	Pop top of stack to low ACC
POPD *dma*	Pop top of stack to data memory (Direct addressing)
POPD *{ind} [,next ARP]*	Pop top of stack to data memory (Indirect addressing)
PSHD *dma*	Push the content of dma to stack (Direct addressing)
PSHD *{ind} [,next ARP]*	Push the content of dma to stack (Indirect addressing)
PUSH	Push low ACC to stack
RC	Reset carry
RET [D]	Return from subroutine with optional delay
RETC[D] *[cond1] [,cond2] [...]*	Return from subroutine if specified conditions are true with optional delay
RETE	Return from interrupt with interrupt enable
RETI	Return from interrupt
RHM	Reset hold mode
ROVM	Reset overflow mode
RPT *dma*	Repeat next instruction the number times specified by memory (Direct addressing)
RPT *{ind} [,next ARP]*	Repeat next instruction the number times specified by memory (Indirect addressing)
RPT *# k*	Repeat next instruction the number of times specified by short immediate
RPT *#lk*	Repeat next instruction the number of times specified by long immediate
RPTB *pma*	Repeat a block of instructions the number of times specified by BRCR
RPTZ *# lk*	Clear ACC and PREG, repeat next instruction, count specified by long immediate
RSXM	Reset sign-extension mode
RTC	Reset test/control flag
RXF	Reset external flag
SC	Set carry bit
SETC *control bit*	Set the specified control bit

Table 11.11 : Continued...

INSTRUCTION	DESCRIPTION
SHM	Set hold mode
SOVM	Set overflow mode
SPM *2-bit constant*	Set PREG shift mode by copying the specified 2-bit constant to pm field of ST1
SSXM	Set sign-extension mode
STC	Set test/control flag
SXF	Set external flag
TRAP	Software interrupt with vector number 14_{10} (Vector address 22h)
XC *n [,cond1] [,cond2] [...]*	Execute next one or two (n = 1 or 2) instructions if the specified conditions are true
ZAC	The content of ACC is made zero (i.e., cleared)
ZAP	The content of ACC and product register are made zero (i.e., cleared)
ZPR	The content of product register is made zero (i.e., cleared)
INPUT/OUTPUT INSTRUCTIONS	
IN *dma, PA*	Input data from port to the memory specified by dma
IN *{ind}, PA [, next ARP]*	Input data from port to memory specified by indirect addressing and update AR
OUT *dma, PA*	Output data from memory specified by dma to port
OUT *{ind}, PA [, next ARP]*	Output data from memory specified by indirect addressing to port and update AR
TBLR *dma*	Table read (Direct addressing)
TBLR *{ind} [, next ARP]*	Table read (Indirect addressing)
TBLW *dma*	Table write (Direct addressing)
TBLW *{ind} [, next ARP]*	Table write (Indirect addressing)
LOAD/STORE INSTRUCTIONS	
EXAR	Exchange ACCB with ACC
LACB	Load ACCB to ACC
LACC *dma [,shift]*	Load memory to ACC with shift (Direct addressing)
LACC *{ind} [,shift [,next ARP]]*	Load memory to ACC with shift and with AR update (Indirect addressing)
LACC # *lk [,shift2]*	Load long immediate with shift to ACC
LACK *8-bit constant*	Load immediate short constant to accumulator
LACL *dma*	Load from memory to low ACC and clear other bits (Direct addressing)
LACL *{ind} [,next ARP]*	Load from memory to low ACC and clear other bits (Indirect addressing)
LACL # *k*	Load short immediate to low ACC and clear other bits

Table 11.11 : Continued...

INSTRUCTION	DESCRIPTION
LACT *dma*	Load memory to ACC with shift specified by TREG1[3 - 0] (Direct addressing)
LACT {*ind*} *[,next ARP]*	Load memory to ACC with shift specified by TREG1[3 - 0] (Indirect addressing)
LAMM *dma*	Load the content of memory-mapped register to low ACC (Direct addressing)
LAMM {*ind*} *[,next ARP]*	Load the content of memory-mapped register to low ACC (Indirect addressing)
LAR *AR, dma*	Load memory to AR (Direct addressing)
LAR *AR, {ind} [,next ARP]*	Load memory to AR (Indirect addressing)
LAR *AR, # k*	Load specified short immediate to AR
LAR *AR, # lk*	Load specified long immediate to AR
LDP *dma*	Load memory to data page pointer register (Direct addressing)
LDP *{ind} [,next ARP]*	Load memory to data page pointer register (Indirect addressing)
LDP *# k*	Load specified short immediate to data page pointer register
LMMR *dma, # lk*	Load MMR addressed by dma to memory specified by long immediate
LMMR *{ind} [,next ARP]*	Load indirectly addressed MMR to memory specified by long immediate
LPH *dma*	Load data of directly addressed memory to high PREG
LPH *{ind} [,next ARP]*	Load data of indirectly addressed memory to high PREG
LST *#n dma*	Load memory to status register n (Direct addressing)
LST *#n, {ind} [,next ARP]*	Load memory to status register n (Indirect addressing)
LST1 *dma*	Load data of directly addressed memory to ST1
LST1 *{ind} [,next ARP]*	Load data of indirectly addressed memory to ST1
LT *dma*	Load data of directly addressed memory to TREG0
LT *{ind} [,next ARP]*	Load data of indirectly addressed memory to TREG0
LTA *dma*	Load memory to TREG0, accumulate previous product (Direct addressing)
LTA *{ind} [,next ARP]*	Load memory to TREG0, accumulate previous product (Indirect addressing)
LTD *dma*	Load TREG0, accumulate previous product and move data (Direct addressing)
LTD *{ind} [,next ARP]*	Load TREG0, accumulate previous product and move data (Indirect addressing)
LTP *dma*	Load memory to TREG0 and load PREG to ACC (Direct addressing)
LTP *{ind} [,next ARP]*	Load memory to TREG0 and load PREG to ACC (Indirect addressing)
LTS *dma*	Load memory to TREG0, subtract PREG from ACC (Direct addressing)

Table 11.11 : Continued...

INSTRUCTION	DESCRIPTION
LTS *{ind} [,next ARP]*	Load memory to TREG0, subtract PREG from ACC (Indirect addressing)
MAR *dma*	Modify auxiliary register (Direct addressing)
MAR *{ind} [,next ARP]*	Modify auxiliary register (Indirect addressing)
PAC	Load product register to ACC
SACB	Store ACC in ACCB
SACH *dma [,shift]*	Store high ACC with shift in memory (Direct addressing)
SACH *{ind} [,shift [,next ARP]]*	Store high ACC with shift in memory (Indirect addressing)
SACL *dma [,shift]*	Store low ACC with shift in memory (Direct addressing)
SACL *{ind} [,shift [,next ARP]]*	Store low ACC with shift in memory (Indirect addressing)
SAMM *dma*	Store ACCL in memory-mapped register (Direct addressing)
SAMM *{ind} [,next ARP]*	Store ACCL in memory-mapped register (Indirect addressing)
SAR *AR, dma*	Store AR in memory (Direct addressing)
SAR *AR, {ind} [,next ARP]*	Store AR in memory (Indirect addressing)
SMMR *dma, # lk*	Store MMR addressed by dma in memory specified by long immediate
SMMR *{ind}, # lk [,next ARP]*	Store indirectly addressed MMR in memory specified by long immediate
SPLK *# lk, dma*	Store long immediate in memory (Direct addressing)
SPLK *# lk, {ind} [,next ARP]*	Store long immediate in memory (Indirect addressing)
SPH *dma*	Store high product register in memory (Direct addressing)
SPH *{ind} [,next ARP]*	Store high product register in memory (Indirect addressing)
SPL *dma*	Store low product register in memory (Direct addressing)
SPL *{ind} [,next ARP]*	Store low product register in memory (Indirect addressing)
SST *# n dma*	Store status register n in memory (Direct addressing)
SST *# n {ind} [,next ARP]*	Store status register n in memory (Indirect addressing)
SST1 *dma*	Store status register 1 in memory (Direct addressing)
SST1 *{ind} [,next ARP]*	Store status register 1 in memory (Indirect addressing)
ZALH *dma*	Zero low ACC and load high ACC from memory (Direct addressing)
ZALH *{ind} [,next ARP]*	Zero low ACC and load high ACC from memory (Indirect addressing)
ZALR *dma*	Zero low ACC, load memory to high ACC with rounding (Direct addressing)
ZALR *{ind} [,next ARP]*	Zero low ACC, load memory to high ACC with rounding (Indirect addressing)

Table 11.11 : Continued...

INSTRUCTION	DESCRIPTION
ZALS *dma*	Zero ACC, load memory to low ACC without sign extension (Direct addressing)
ZALS *{ind} [,next ARP]*	Zero ACC, load memory to low ACC without sign extension (Indirect addressing)
	BLOCK MOVE INSTRUCTIONS
BLDD *# lk, dma*	Block move from directly addressed memory to immediate addressed memory
BLDD *# lk, {ind} [,next ARP]*	Block move from indirectly addressed memory to immediate addressed memory
BLDD *dma, # lk*	Block move from immediate addressed memory to directly addressed memory
BLDD *{ind}, # lk [,next ARP]*	Block move from immediate addressed memory to indirectly addressed memory
BLDD BMAR, DMA	Block move from directly addressed memory to memory addressed by BMAR
BLDD BMAR, *{ind} [,next ARP]*	Block move from indirectly addressed memory to memory addressed by BMAR
BLDD *dma BMAR*	Block move from memory addressed by BMAR to directly addressed memory
BLDD *{ind}, BMAR [,next ARP]*	Block move from memory addressed by BMAR to indirectly addressed memory
BLDP *dma*	Block move from data memory to program memory (Direct addressing)
BLDP *{ind} [,next ARP]*	Block move from data memory to program memory (Indirect addressing)
BLPD *# pma, dma*	Block move from program to data memory (Direct addressing)
BLPD *# pma, {ind} [,next ARP]*	Block move from program to data memory (Indirect addressing)
BLPD BMAR, *{ind} [,next ARP]*	Block move from program to data memory, pma in BMAR (Direct addressing)
BLPD BMAR, *dma*	Block move from program to data memory, pma in BMAR (Indirect addressing)
DMOV *dma*	Move the content of memory to next higher location (Direct addressing)
DMOV *{ind} [,next ARP]*	Move the content of memory to next higher location (Indirect addressing)

Table 11.12 : Symbols and Acronyms Used in the Instruction Set Summary

SYMBOL	DESCRIPTION
ACC	Accumulator
ACCB	Accumulator buffer
AR	Auxiliary register
ARCR	Auxiliary register compare
ARP	Auxiliary register pointer
BMAR	Block move address register
bit code	Bit to test
2-bit constant	2-bit short immediate value

SYMBOL	DESCRIPTION
8-bit constant	8-bit short immediate value
BRCR	Block repeat count register
C	Carry bit
CM	Compare mode (00/01/10/11)
control bit	Bit to be cleared/set
[D]	Optional delay
DBMR	Dynamic bit manipulation register
dma	Data memory address

Table 11.12 : Continued ....

SYMBOL	DESCRIPTION
DP	Data memory page pointer
{ind}	Indirect address * /*+ /*– /*0+ /*0– /*BR0+ /*BR0– (Refer table 11.8)
k	Interrupt vector number
# k	8-bit immediate value (0 to FFh)
# lk	16-bit immediate value (0 to FFFFh)
# n	Register number (0 or 1)
[,next ARP]	Update ARP for next instruction
OV	Overflow bit
PA	Port address

SYMBOL	DESCRIPTION
PC	Program counter
PC	Program counter
pma	Program memory address (o to FFFFh)
PREG	Product register
shift	Shift of 0-16 bits
shift 2	Shift of 0-15 bits
ST	Status register
T	Temporary register
TC	Test/control bit
TREGn	Temporary register (0-2)
TREGn	Temporary register (0-2)

Table 11.13 : Conditions for Branch, Call and Return Instructions

Code for Cond1 Cond2, etc.,	Condition	Description
BIO	$\overline{BIO} = 0$	$\overline{BIO}$ signal is low
C	C = 1	Carry bit set
EQ	ACC = 0	Accumulator equal to 0
GEQ	ACC ³ 0	Accumulator greater than or equal to 0
GT	ACC > 0	Accumulator greater than 0
LEQ	ACC £ 0	Accumulator less than or equal to 0
LT	ACC < 0	Accumulator less than 0
NC	C = 0	Carry bit cleared
NEQ	ACC ¹ 0	Accumulator not equal to 0
NOV	OV = 0	No accumulator overflow detected
NTC	TC = 0	Test/control flag cleared
OV	OV = 1	Accumulator overflow detected
TC	TC = 1	Test/control flag set
UNC	None	Unconditional operation

11.3.9 Assembly Language Programs in TMS320C5x

The assembly language programs for TMS320C5x processors are written using the mnemonics listed in table 11.11. The processor can execute only machine language programs and the conversion from assembly language to machine language is performed by a software tool called assembler.

Texas Instruments has released a number of assembly language program-development tools for their digital signal processors. Some of the tools are assembler, linker, absolute lister, hex-converter and loader.

The ***assembler*** is a software tool that can run on any standard PC (**P**ersonal **C**omputer) and permits to type, edit and convert the assembly language program to machine language object file called COFF (**C**ommon **O**bject **F**ile **F**ormat) file.The process of conversion from assembly language program to machine language program is called ***assembling***. The assembly language program can be developed in modules and each module can be assembled separately to generate COFF files and they can be combined to a single executable object file using a linker.

The ***absolute lister*** will map the executable object file of the program to a specific memory location of the system. The ***loader*** is used to download the executable object file into the processor RAM for execution by the processor. When the program has to be permanently stored in ROM/EPROM, the object files have to be converted to standard hex files. The ***hex-converter*** can be used to convert an executable object file to standard hex-file. Using any standard EPROM programmer, the hex-file can be loaded in EPROM for execution by the processor.

Assembly Language Program Statement Format

The assembly program for TMS320C5x processor consists of statements that contain labels, assembler directives, instructions, macro directives and comments. Each statement can have a maximum of 200 characters. A statement may have four fields, namely label, mnemonic, operands and comment. The general format of the statement is shown below.

[label] [:] mnemonic [operands] [;comment]

The guidelines for writing statements are given below.

- A statement must begin with a label, a blank, an asterisk or a semicolon.
- A label should begin in the first column and the label is optional.
- A comment should begin with a semicolon and the comment is optional.
- Each field should be separated by one or more blanks.

Constants

The decimal, binary or hexadecimal numbers used to represent the data or address in assembly language program statement are called ***constants*** or numerical constants. When constants are used to represent the address/data then their values are fixed and cannot be changed while running a program. The binary and hexadecimal constants can be differentiated by placing a specific alphabet at the end of the constant.

A valid binary constant/number is framed using the numeric characters 0 and 1, and the alphabet b is placed at the end.

A valid decimal (BCD) constant/number is framed using numeric characters 0 to 9, and no alphabet is placed at the end.

A valid hexadecimal constant/number is framed using numeric characters 0 to 9 and alphabets a to f, and the alphabet h is placed at the end. A zero should be placed/inserted at the beginning of a hexadecimal number if the first digit is an alphabet character from a to f, otherwise the assembler will consider the constant starting with a to f as a symbol.

Examples of valid constant

1011 - Decimal (BCD) constant

1101b - Binary constant

92ach - Hexadecimal constant

0e2h - Hexadecimal constant

Symbols

The ***symbols*** are variables (or terms) used in assembly language program statements in order to represent the variable data and address. While running a program, a value has to be attached to each symbol in the program. The advantage of using a symbol is that the value of the symbol can be dynamically varied while running the program.

Usually a symbol name is constructed such that it reflects the meaning of the value it holds. A variable name selected to represent the temperature of a device can be TEMP, a variable name selected to represent the speed of a motor can be M_SPEED, etc.

The guidelines for framing the symbols are given below:

- The symbol name can be constructed using A to Z, a to z, 0 to 9, $, _ (underscore).
- A number cannot be the first character in the symbol name.
- The maximum length of a symbol name is 32 characters.
- The symbol name is case sensitive.

Assembler Directives

The ***assembler directives*** are the instructions to the assembler regarding the program being assembled. They are also called ***pseudo instructions***. The assembler directives are used to specify start and end of a program, attach value to variables, allocate storage locations to input/output data, to define start and end of segments, procedures, macros, etc.

The assembler directives control the generation of machine code and organization of the program. But no machine codes are generated for assembler directives. Some of the assembler directives that can be used for TMS320C5x assembly language program development are listed in table 11.14.

Table 11.14 : Assembler Directives Summary

ASSEMBLER DIRECTIVE	DESCRIPTION
.align	Align the SPC (Section Program Counter) on a page boundary
.data [value]	Assemble into default or specified data memory
.ds xxxx	Assemble into data memory address xxxx
.end	End program
.endloop	End .loop code block
.entry	Initialize the starting of PC
.even	Align the SPC (Section Program Counter) on an even word boundary
.equ	Equate a value with a symbol
.int *value1 [, ... , value n]*	Initialize one or more 16-bit integers
.loop *[well-defined expression]*	Begin repeatable assembly of a code block
.mmregs	Enter memory-mapped registers into symbol table
.ps *xxxx*	Assemble into program memory address xxxx
.set	Equate a value with a symbol
.text [value]	Assemble into default or specified program memory
.word *value1 [, ... , value n]*	Initialize one or more 16-bit binary

Program 11.1

Write an assembly language program using instructions of TMS320C5x processors to add two numbers of 64-bit data. Assume that the two data are available in memory. Store the sum in memory.

Problem Analysis

The memory word size of the TMS320C5x processor is 16 bits and so each 64-bit data is stored as 4 words ($4 \times 16 = 64$). Let us use direct addressing. Let the address of 4 words of data-1 be named as AD1W1, AD1W2, AD1W3 and AD1W4. Let the address of 4 words of data-2 be named as AD2W1, AD2W2, AD2W3 and AD2W4. Let the address of 4 words of sum be named as ASW1, ASW2, ASW3 and ASW4.

Let us load the lower two words of data-1 in the 32-bit accumulator and then the word-1 of data-2 is added to the low accumulator, and the word-2 of data-2 is added to the high accumulator. The 32-bit sum in the accumulator is stored in the memory. The addition of the next two words are performed in a similar manner by considering the carry in the previous addition.

Assembly language program

```
;PROGRAM TO ADD TWO NUMBERS OF 64-BIT DATA
AD1W1   .set 00h        ;Offset address for data-1.
AD1W2   .set 01h
AD1W3   .set 02h
AD1W4   .set 03h
AD2W1   .set 04h        ;Offset address for data-2.
```

```
AD2W2   .set 05h
AD2W3   .set 06h
AD2W4   .set 07h
ASW1    .set 08h        ;Offset address for sum.
ASW2    .set 09h
ASW3    .set 0ah
ASW4    .set 0bh
        .mmregs         ;Include memory-mapped registers.
        .ps   0b00h     ;Origin of program address is 0b00h.
        .entry          ;Initialize program counter with starting address of program.
ADD64:   CLRC SXM       ;Clear sign extension mode bit.
         LACC AD1W2,16  ;Load word-2 of data-1 in high accumulator.
         ADDS AD1W1     ;Load word-1 of data-1 with sign extension suppressed in low accumulator.
         ADDS AD2W1     ;Add word-1 of data-2 to low accumulator.
         ADD  AD2W2,16  ;Add word-2 of data-2 to high accumulator.
         SACL ASW1      ;Store word-1 of sum in memory.
         SACH ASW2      ;Store word-2 of sum in memory.
         LACC AD1W4,16  ;Load word-4 of data-1 in high accumulator.
         ADDC AD1W3     ;Add word-3 of data-1 and carry to accumulator.
         ADDS AD2W3     ;Add word-3 of data-2 with sign extension suppressed to low accumulator.
         ADD  AD2W4,16  ;Add word-4 of data-2 to high accumulator.
         SACL ASW3      ;Store word-3 of sum in memory.
         SACH ASW4      ;Store word-4 of sum in memory.
         RET            ;Program end.
        .end            ;Assembly end.
```

Program 11.2

Write an assembly language program using instructions of TMS320C5x processors to subtract two numbers of 64-bit data. Assume that the two data are available in memory. Store the result in memory.

Problem Analysis

The memory word size of the TMS320C5x processor is 16 bits and so each 64-bit data is stored as 4 words (4 ´ 16 = 64). Let 4 words of data-1 be stored in memory at address 1101h to 1104h. Let 4 words of data-2 be stored in memory at address 1111h to 1114h. Let the 4 words of result be stored in memory at address 1121h to 1124h. Let us use indirect address using auxiliary registers.

Let us load the lower two words of data-1 in 32-bit accumulator and then the word-1 of data-2 is subtracted from the low accumulator, and the word-2 of data-2 is subtracted from the high accumulator. The 32-bit result in the accumulator is stored in memory. The subtraction of next two words are performed in a similar manner by considering the borrow in the previous subtraction.

Assembly language program

```
;PROGRAM TO SUBTRACT TWO NUMBERS OF 64-BIT DATA
         .mmregs            ;Include memory-mapped registers.
         .ps   0C00h        ;Origin of program address is 0C00h.
         .entry             ;Initialize program counter with starting address of program.
Ini_ARs:  LAR  AR1,#1101h   ;Load starting address of data-1 in AR1.
          LAR  AR2,#1111h   ;Load starting address of data-2 in AR2.
          LAR  AR3,#1121h   ;Load starting address of sum in AR3.
SUB64:    CLRC SXM          ;Clear sign extension mode bit.
          LACC *+,0,AR1     ;Load word-1 of data-1 in low accumulator.
          ADD  *+,16,AR1    ;Load word-2 of data-1 in high accumulator.
          SUBS *+,0,AR2     ;Subtract word-1 of data-2 with sign extension suppressed from low,
                            ;accumulator.
```

```
        SUB  *+,16,AR2   ;Subtract word-2 of data-2 from high accumulator.
        SACL *+,0,AR3    ;Store word-1 of result in memory.
        SACH *+,0,AR3    ;Store word-2 of result in memory.
        LACC *+,0,AR1    ;Load word-3 of data-1 in low accumulator.
        SUBB *+,0,AR2    ;Subtract word-3 of data-2 and previous borrow from low accumulator.
        ADD  *+,16,AR1   ;Load word-4 of data-1 in low accumulator.
        SUB  *+,16,AR2   ;Subtract word-4 of data-2 from high accumulator.
        SACL *+,0,AR3    ;Store word-3 of result in memory.
        SACH *+,0,AR3    ;Store word-4 of result in memory.
        RET              ;Program end.
       .end              ;Assembly end.
```

Program 11.3

Write an assembly language program using instructions of TMS320C5x processors to multiply two numbers of unsigned 32-bit data. Assume that the two data are available in memory. Save the 64-bit product in memory.

Problem Analysis

In the TMS320C5x processor, the 32-bit multiplication can be implemented in terms of 16-bit multiplication. The given 32-bit data can be divided into two words (16-bit data) as shown below.

Data-1 (D1) ® Data-1 word-2 (D1W2), Data-1 word-1 (D1W1)

Data-2 (D2) ® Data-2 word-2 (D2W2), Data-2 word-1 (D2W1)

Using the above four words (D1W1, D1W2, D2W1, D2W2), the following four products are computed. Each product will generate a 32-bit result and so they are divided into two words (16-bit) as shown below.

Product-1 (P1) : D1W1 ´ D2W1 = P1 ® P1W2, P1W1

Product-2 (P2) : D1W2 ´ D2W1 = P2 ® P2W2, P2W1

Product-3 (P3) : D1W1 ´ D2W2 = P3 ® P3W2, P3W1

Product-4 (P4) : D1W2 ´ D2W2 = P4 ® P4W2, P4W1

The results of the above four products can be added to get the final result as shown below. The maximum size of **F**inal **P**roduct (FP) will be 64 bits and so it is divided into four words, namely FPW1, FPW2, FPW3 and FPW4.

		D1W2	D1W1
		´ D2W2	D2W1
		P1W2	P1W1
	P2W2	P2W1	
	P3W2	P3W1	
P4W2	P4W1		
FPW4	FPW3	FPW2	FPW1

Let us use direct addressing for data and product. The address of data words and final product words are named as AD1W1, AD1W2, AD2W1, AD2W2, AFPW1, AFPW2, AFPW3, AFPW4.

The products are performed one by one by loading one of the data words in T-register-0 (TREG0) and multiplying with a memory data word to get the 32-bit product in P-register. The results of the product are added as shown below to get the final product words.

FPW1 = P1W1

FPW2 = P1W2 + P2W1 + P3W1

FPW3 = P2W2 + P3W2 + P4W1

FPW4 = P4W2

Assembly language program

```
;PROGRAM TO MULTIPLY TWO NUMBERS OF 32-BIT DATA
AD1W1   .set 10h      ;Offset address for data-1.
AD1W2   .set 11h
AD2W1   .set 12h      ;Offset address for data-2.
AD2W2   .set 13h
AFPW1   .set 14h      ;Offset address for product.
AFPW2   .set 15h
AFPW3   .set 16h
AFPW4   .set 17h
        .mmregs       ;Include memory-mapped registers.
        .ps   od00h   ;Origin of program address is 0d00h.
        .entry        ;Initialize program counter with starting address of program.
MUL32:   CLRC SXM     ;Clear sign extension mode bit.
         LT   AD2W1   ;Load D2W1 in TREG0.
         MPYU AD1W1   ;Multiply D1W1 and D2W1 to get product-1 (P1) in P-register.
         SPL  AFPW1   ;Store FPW1 in memory.
         SPH  AFPW2   ;Store partial FPW2 in memory.
         MPYU AD1W2   ;Multiply D1W2 and D2W1 to get product-2 (P2) in P-register.
         LTP  AD2W2   ;Save P2 in accumulator. Load D2W2 in TREG0.
         MPYU AD1W1   ;Multiply D1W1 and D2W2 to get product-3 (P3) in P-register.
         MPYA AD1W2   ;Get sum of P2 and P3 in accumulator. Multiply D1W2 and D2W2 to get,
                      ;product-4 (P4) in P-register.
         ADDS AFPW2   ;Add partial FPW2 in memory to low accumulator to get FPW2 in low,
                      ;accumulator.
         SACL AFPW2   ;Save FPW2 in memory.
         BSAR 16      ;Shift accumulator right by 16.
         APAC         ;Add P4 in P-register to accumulator to get FPW3 and FPW4 in low and,
                      ;high accumulator respectively.
         SACL AFPW3   ;Save FPW3 in memory.
         SACH AFPW4   ;Save FPW4 in memory.
         RET          ;Program end.
        .end          ;Assembly end.
```

Program 11.4

Write an assembly language program using instructions of TMS320C5x processors to divide a 16-bit data by an 8-bit data. Assume that the data are 2's complement positive integers available in memory. Store the quotient and remainder in memory.

Problem Analysis

The TMS320C5x processors have a special instruction "SUBC dma" to implement division algorithms. For 16-bit by 8-bit division of positive integers, the 16-bit dividend is stored in the low accumulator and the high accumulator is filled with zero. The 8-bit divisor is stored in data memory as lower 8 bits and upper 8 bits are filled with zero. Then the "SUBC dma" instruction is executed 16 times to generate the quotient in the low accumulator and remainder in the high accumulator.

Note : The SUBC instruction will left shift the divisor by 15 bits and subtract from dividend and store the
i nehT .rotalumucca eht ni tluser i rotalumucca eht ,evitisop s i 1 dna 2 yb deilpitlum s eht esle ,dedda s
i rotalumucca i 1 dna(2 yb deilpitlum ylpmis s .)dedda ton s

Let us use direct addressing. Let the address of dividend and divisor be ADIVD and ADIVS. Let the address of quotient and remainder be AQUO and AREM.

Assembly language program

```
;PROGRAM FOR 16-BIT BY 8-BIT DIVISION OF POSITIVE DATA
ADIVD    .set 30h       ;Offset address of dividend.
ADIVS    .set 31h       ;Offset address of divisor.
AQUO     .set 32h       ;Offset address of quotient.
AREM     .set 33h       ;Offset address of remainder.
         .mmregs        ;Include memory-mapped registers.
         .ps   0e00h    ;Origin of program address is 0e00h.
         .entry         ;Initialize program counter with starting address of program.
DIV16_8:  CLRC SXM      ;Clear sign extension mode bit.
          LACC ADIVD    ;Load dividend in low accumulator with high accumulator as zero.
          RPT  #15      ;Repeat the next instruction 16 times. Count is 1 less than number of,
                        ;repeatitions.
          SUBC ADIVS    ;Perform conditional subtraction of division algorithm.
          SACL AQUO     ;Store the quotient in memory.
          SACH AREM     ;Store the remainder in memory.
          RET           ;Program end.
         .end           ;Assembly end.
```

Program 11.5

Write an assembly language program using instructions of TMS320C5x processor to find the sum of an array stored in memory. Assume that the array has 10_{10} data each of size 16 bits and store the sum in memory.

Problem Analysis

The array of 10 data can be stored in memory with starting address named as "Arr_addr". The 32-bit accumulator is cleared to have an initial sum as zero. Then the data of the array can be added one by one to the accumulator. The final sum in the 32-bit accumulator is stored as two words in memory with address named as Asum_W1 and Asum_W2.

Assembly language program

```
;PROGRAM FOR SUM OF AN ARRAY
Arr_addr .set 1200h          ;Data array starting address is 1200h.
Cnt_val  .set 09h            ;Count value is one less than number of data in the array.
Asum_W1  .set 1300h          ;Address of sum.
Asum_W2  .set 1301h
         .mmregs             ;Include memory-mapped registers.
         .ps   0f00h         ;Origin of program address is 0f00h.
         .entry              ;Initialize program counter with starting address of program.
Sum_arr:  CLRC SXM           ;Clear sign extension mode bit.
          LAR  AR0,Cnt_val   ;Load count value in AR0.
          LAR  AR1,Arr_addr  ;Load starting address of array in AR1.
```

```
        ZAP                   ;Clear accumulator.
        MAR  *,AR1            ;Make ARP to point AR1 as current AR.
AGAIN:  ADDS *+,0,AR0         ;Add content of AR1 (without shift and sign extension suppressed),
                              ;to accumulator, increment AR1 and make ARP to point AR0 as next AR.
        BANZ AGAIN,AR1        ;If AR0 is not zero, then decrement AR0 by 1, branch to program,
                              ;memory address AGAIN, make ARP to point AR1 as next AR. If AR0 is,
                              ;zero then go to next instruction.
        SACL Asum_W1          ;Store word-1 of sum in memory.
        SCAH Asum_W2          ;Store word-2 of sum in memory.
        RET                   ;Program end.
       .end                   ;Assembly end.
```

Program 11.6

The input x(n) and impulse response h(n) of an LTI system is given by,

$$x(n) = \{2, 10, 13, 5\}$$

$$h(n) = \{7, 14, 4\}$$

The response y(n) of the LTI system is given by,

$y(n) = x(n) * h(n)$; where "*" symbol for convolution.

Write an assembly language program using instructions of TMS320C5x processors to determine the response of LTI system via convolution of input and impulse response.

Problem Analysis

The x(n) is a 4-point sequence and h(n) is a 3-point sequence. The convolution of x(n) and h(n) will produce a sequence of size 4 + 3 – 1 = 6-point sequence. Let us append zero to x(n) and h(n) to convert them to 6-point sequences as shown below and perform linear convolution via circular convolution.

$$\backslash\ \ x(n) = \{2, 10, 13, 5, 0, 0\}$$

$$h(n) = \{7, 14, 4, 0, 0, 0\}$$

The circular convolution of x(n) and h(n) is defined as,

$$y(n) = \sum_{m=0}^{N-1} x(m)\ h\big((n-m)\big)_N$$

When N = 6,

$$y(n) = \sum_{m=0}^{5} x(m)\ h\big((n-m)\big)_6$$

$$= x(0)\ h\big((n-0)\big)_6 + x(1)\ h\big((n-1)\big)_6 + x(2)\ h\big((n-2)\big)_6 + x(3)\ h\big((n-3)\big)_6 + x(4)\ h\big((n-4)\big)_6 + x(5)\ h\big((n-5)\big)_6$$

When n = 0 ; $y(0) = x(0)\,h(0) + x(1)\,h((-1))_6 + x(2)\,h((-2))_6 + x(3)\,h((-3))_6 + x(4)\,h((-4))_6 + x(5)\,h((-5))_6$

$$\backslash\ \ y(0) = x(0)\,h(0) + x(1)\,h(6-1) + x(2)\,h(6-2) + x(3)\,h(6-3) + x(4)\,h(6-4) + x(5)\,h(6-5)$$

$$\backslash\ \ y(0) = x(0)\,h(0) + x(1)\,h(5) + x(2)\,h(4) + x(3)\,h(3) + x(4)\,h(2) + x(5)\,h(1)$$

Similarly,

When n = 1 ; y(1) = x(0) h(1) + x(1) h(0) + x(2) h(5) + x(3) h(4) + x(4) h(3) + x(5) h(2)

When n = 2 ; y(2) = x(0) h(2) + x(1) h(1) + x(2) h(0) + x(3) h(5) + x(4) h(4) + x(5) h(3)

When n = 3 ; y(3) = x(0) h(3) + x(1) h(2) + x(2) h(1) + x(3) h(0) + x(4) h(5) + x(5) h(4)

When n = 4 ; y(4) = x(0) h(4) + x(1) h(3) + x(2) h(2) + x(3) h(1) + x(4) h(0) + x(5) h(5)

When n = 5 ; y(5) = x(0) h(5) + x(1) h(4) + x(2) h(3) + x(3) h(2) + x(4) h(1) + x(5) h(0)

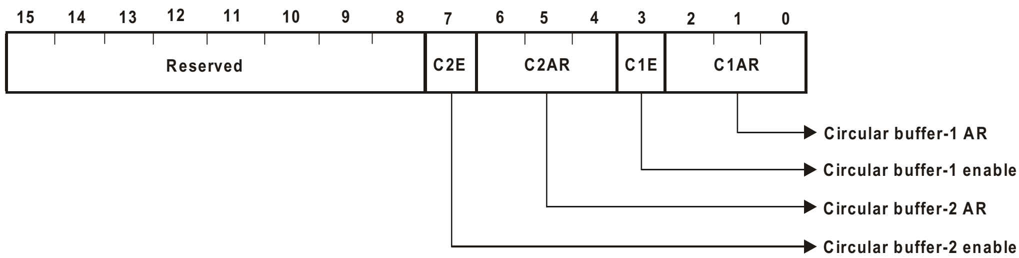
The computation of the above equations involves multiplication and addition, and so can be computed in TMS320C5x processors using RPT and MAC instructions. Let us store input array in program memory and impulse array in data memory. The data memory used to store impulse array can be declared as a circular memory.

The repeat count register is loaded with a count value of 5 for executing MAC instruction 6 times, to compute one data of the output sequence. The output data is stored in memory and the loading of count value and executing MAC instruction are repeated 5 more times to compute the next 5 data of output sequence.

In this computation process, the program memory address has to be incremented and circular data memory address has to be decremented.

In order to enable the circular memory and to specify the auxiliary register to be used for addressing circular memory, an appropriate word (called control word) should be loaded in CBCR (**C**ircular **B**uffer **C**ontrol **R**egister).

The format of CBCR is shown below.



Note : 1 = Enable, 0 = Disable.

Let us use circular buffer-1 and use AR0 to address buffer-1, and so the control word to be loaded in CBCR is 08h (0 000 1 000).

Assembly language program

```
;PROGRAM FOR CONVOLUTION
        .mmregs               ;Include memory-mapped registers.
        .ps    0c00h          ;Origin of program address is 0c00h.
        .word 02h, 0Ah, 0dh   ;Store the input array in program.
        .word 05h, 0h, 0h     ;Memory address starting from 0c00h.
        .ds   1100h           ;Origin of data address is 1100h.
```

```
        .word 07h, 0eh, 04h   ;Store the impulse array in data.
        .word 0h, 0h, 0h      ;memory address starting from 1100h.
        .entry                ;Initialize program counter with starting address of program.
CONV:    SPLK #1100h,CBSR1    ;Load start address of circular buffer-1.
         SPLK #1107h,CBER1    ;Load end address of circular buffer-1.
         SPLK #08h            ;Enable circular buffer-1 .set AR0 as pointer for circular,
                              ;buffer-1.
         LAR  AR0,#1100h      ;Initialize AR0 with start address of impulse array.
         LAR  AR1,#1200h      ;Initialize AR1 with start address of output array.
         LAR  AR2,#05h        ;Initialize AR2 as count for number of data in output array, count,
                              ;is 1 less than number of data.
LOOP:    ZAP                  ;Clear accumulator and P-register.
         MAR  *,AR0           ;Make ARP to point AR0 as current AR.
         RPT  #05h            ;Repeat next instruction 6 times. Count is 1 less than number of,
                              ;repeatitions.
         MAC  0c00h,*-,AR0    ;Add P-register to accumulator, load a word of input array,
                              ;addressed by program memory in T-registers, multiply,
                              ;T-register and a word of impulse array in data memory,
                              ;addressed by AR0, increment program memory address, decrement,
                              ;data memory address.
         MAR  *,AR1           ;Make ARP to point AR1 as current AR.
         SACL *+,AR2          ;Store one word of output array in memory, increment AR1, make ARP,
                              ;to point AR2 as current AR.
         BANZ LOOP            ;If AR2 is not zero, then decrement AR2 by 1, branch to program,
                              ;memory address LOOP. If AR2 is zero, then go to next instruction.
         RET                  ;Program end.
        .end                  ;Assembly end.
```

11.4 TMS320C54x Family of Digital Signal Processors

The TMS320C54x family of processors are advanced versions of TMS320C5x from Texas Instruments, USA. These processors are built with modified Harvard architecture with more internal buses and on-chip peripherals, larger size ALU and very rich instruction set than the TMS320C5x family. The TMS320C54x family of processors can execute 40 to 120 **M**illion **I**nstructions **P**er-**S**econd (MIPS).

Some of the features of the TMS320C54x family of digital signal processors are,

- 16-bit CPU
- 25 to 8.3 ns single cycle instruction execution time
- Single cycle 17 ´ 17-bit MAC (Multiply/Accumulate) unit
- 8M ´ 16-bit virtual program memory address space
- 64k ´ 16-bit physical program memory address space
- 64k ´ 16-bit external data memory address space
- 64k ´ 16-bit external IO address space
- 2k to 48k ´ 16-bit on-chip program/data ROM
- 5k to 32k ´ 16-bit on-chip program/data RAM

- Synchronous, TDM and buffered serial ports
- Programmable timer and PLL (**P**hase **L**ocked **L**oops)
- IEEE standard JTAG ports
- 5/3.3V operation with low power dissipation and power down modes
- DMA interface
- 100/128/144 pins in plastic TQFP and BGA package

The various processors of TMS320C54x family and their characteristic features are listed in table11.15. A comparision of various features of TMS320C5x and TMS320C54x family of digital signal processors are listed in table 11.16.

Table 11.15 : Characteristics of TMS320C54x Family of Processors

PROCESSOR	POWER SUPPLY (V)	ON-CHIP MEMORY		PERIPHERALS			CYCLE TIME(ns)	PACKAGE TYPE
		RAM (Word)	ROM (Word)	SERIAL PORT	TIMER	HPI		
TMS320C541	5.0	5k	28k	2	1	No	25	100-pin TQFP
TMS320LC541	3.3	5k	28k	2	1	No	20/25	100-pin TQFP
TMS320LC541B	3.3	5k	28k	2	1	No	20/25	100-pin TQFP
TMS320C542	5.0	10k	2k	2	1	Yes	25	144-pin TQFP
TMS320LC542	3.3	10k	2k	2	1	Yes	20/25	128/144-pin TQFP
TMS320LC543	3.3	10k	2k	2	1	No	20/25	100-pin TQFP
TMS320LC545	3.3	6k	48k	2	1	Yes	20/25	128-pin TQFP
TMS320LC545A	3.3	6k	48k	2	1	Yes	15/20/25	128-pin TQFP
TMS320LC545B	3.3	6k	48k	2	1	Yes	15/20/25	128-pin TQFP
TMS320LC546	3.3	6k	48k	2	1	No	20/25	100-pin TQFP
TMS320LC546A	3.3	6k	48k	2	1	No	15/20/25	100-pin TQFP
TMS320LC546B	3.3	6k	48k	2	1	No	15/20/25	100-pin TQFP
TMS320LC548	3.3	32k	2k	3	1	Yes	12.5/15/20	144-pin TQFP/144-pin BGA
TMS320LC549	3.3	32k	16k	3	1	Yes	12.5/15	144-pin TQFP/144-pin BGA
TMS320VC549	3.3(2.5core)	32k	16k	3	1	Yes	8.3/10/12.5	144-pin TQFP/144-pin BGA

Table 11.16 : Comparison of the features of TMS320C5x and TMS320C54x

FEATURE	TMS320C5x	TMS320C54x
Program bus	PB	PB
Data bus	DB	DB and CB (for Read), EB (for Write)
Address buses	PAB, DAB	PAB, CAB, DAB, EAB
Main ALU	32-bit ALU	40-bit ALU
Accumulators	32-bit ACC	40-bit ACCA and ACCB
Barrel shifter	0-16-bit left shift 0-16-bit right shift	40-bit: 0-31 left shift 0-15 right shift
Multiplier	16 ´ 16-bit	17 ´ 17-bit
Adder	32-bit	40-bit
Auxiliary register ALU	ARAU	ARAU0, ARAU1
Block repeat registers	16-bit BRCR, PASR, PAER	16-bit BRC, RSA, REA
Circular buffer register	Two 16-bit start and end register	16-bit BK
Wait state generator	PDWSR	SWWSR
Host port interface	8-bit standard HPI	8-bit standard HPI or enhanced 8-bit and 16-bit HPI
COMMON TO TMS320C5x AND TMS320C54x		
Auxiliary registers	AR0-AR7	
Status registers	16-bit PMST, ST0, ST1	
Program counter	16-bit PC	
Interrupt registers	16-bit IMR and IFR	
General purpose IO	BIO and XF	
Hardware timer	16-bit timer	
Clock generator	PLL based	
Synchronous serial port	full duplex and double buffered	
TDM serial ports	up to 7 devices using TDM can communicate serially	
Buffered serial port	standard 5x serial port with additional autobuffering unit	
AVAILABLE ONLY IN TMS320C54x		
Stack pointer (SP), 16-bit SP		
Extended prog memory, 7-bit XPC		
Multichannel buffered serial port		
Internal programmable clock		
On-chip ROM for look up table		

11.4.1 Pin Diagram of TMS320C54x Processors

The TMS320C54x family of processors are available in the following plastic packages.

- 100/128/144 pins TQFP (**T**hin **Q**uad **F**lat **P**ackage)
- 144-pin BGA (**B**all **G**rid **A**rray) package.

The pin configuration of TMS320C54x processor with 100 pins in TQFP is shown in fig 11.12. The names or functions of various pins are listed in table 11.17.

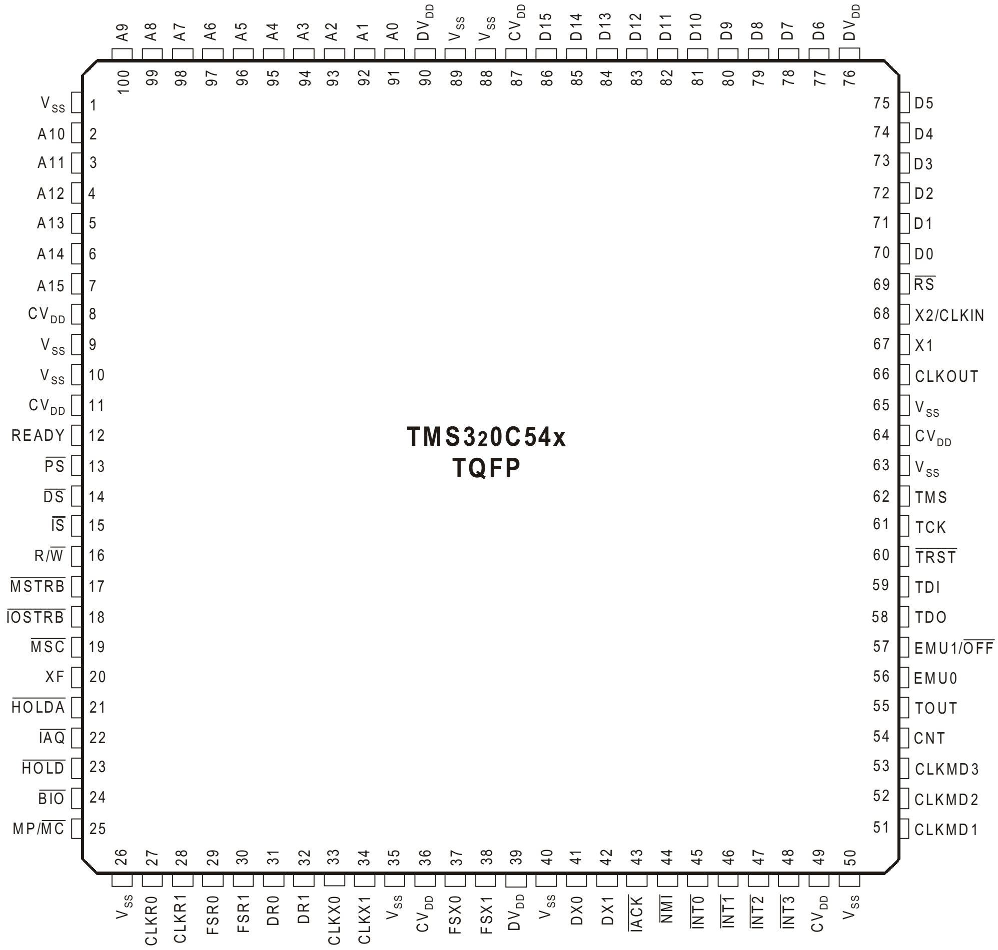


Fig 11.12 : *Pin diagram of TMS320C54x (TQFP).*

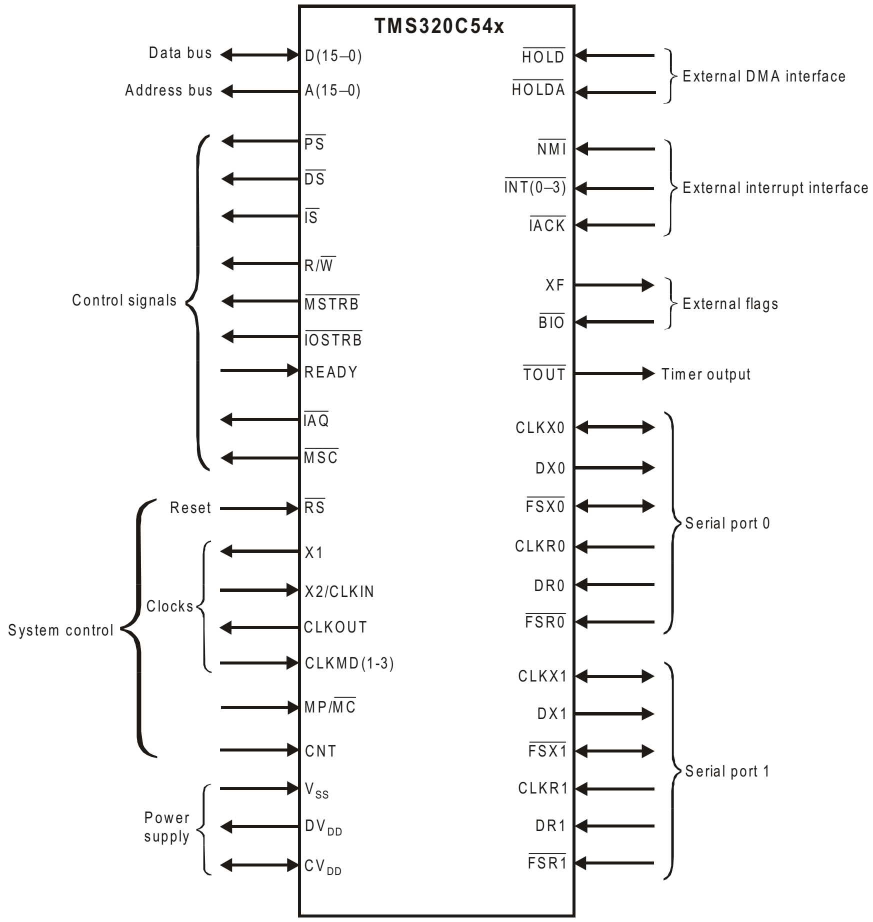


Fig 11.13 : Functional grouping of TMS320C54x pins.

Table 11.17 : Pin Description of TMS320C54x (100 Pin TQFP)

SIGNAL	TYPE	DESCRIPTION
	DATA SIGNALS	
A0– A15	Output, Tristate	Parallel port address bus/16-bit external address bus
D0–D15	IO, Tristate	Parallel port data bus/16-bit external data bus
	INITIALIZATION, INTERRUPT AND RESET OPERATIONS	
$\overline{\text{IACK}}$	Output, Tristate	Interrupt acknowledge signal
$\overline{\text{INT0}}-\overline{\text{INT3}}$	Input	External interrupt inputs
$\overline{\text{NMI}}$	Input	Nonmaskable interrupt
$\overline{\text{RS}}$	Input	Reset
MP / $\overline{\text{MC}}$	Input	Microprocessor/microcomputer mode select pin
CNT	Input	TTL/CMOS logic level select

Table 11.17 : Continued...

SIGNAL	TYPE	DESCRIPTION
	MULTIPROCESSING SIGNALS	
$\overline{BIO}$	Input	IO branch control input
XF	Output, Tristate	External flag output
	MEMORY CONTROL SIGNALS	
$\overline{DS}$, $\overline{PS}$, $\overline{IS}$	Output, Tristate	Data, program, and IO space select signals
$\overline{MSTRB}$	Output, Tristate	Memory strobe signal
READY	Input	External bus ready/wait-state control input
$R/\overline{W}$	Output, Tristate	Read/write signal
$\overline{IOSTRB}$	Output, Tristate	IO strobe signal
$\overline{HOLD}$	Input	Hold input
$\overline{HOLDA}$	Output, Tristate	Hold acknowledge signal
$\overline{MSC}$	Output, Tristate	Microstate complete signal
$\overline{IAQ}$	Output, Tristate	Instruction acquisition signal
	OSCILLATOR/TIMER SIGNALS	
CLKOUT	Output, Tristate	Master clock output signal
CLKMD 1-3	Input	Clock mode external/internal input signals
	OSCILLATOR/TIMER SIGNALS (CONTINUED)	
X2/CLKIN	Input	Input pin to internal oscillator from the crystal
X1	Output	Output pin from the internal oscillator for the crystal
TOUT	Output, Tristate	Timer output
	SERIAL PORT 0 AND SERIAL PORT 1 SIGNALS	
CLKR0, CLKR1	Input	Serial port receive clocks
CLKX0, CLKX1	IO, Tristate	Serial port transmit clocks
DR0, DR1	Input	Serial data receive
DX0, DX1	Output, Tristate	Serial data transmit
FSR0, FSR1	Input	Frame synchronization pulse for serial data receive
FSX0, FSX1	IO, Tristate	Frame synchronization pulse for serial data transmit
	SUPPLY PINS	
C_{VDD}	Supply	Dedicated power supply for the core CPU
DV_{DD}	Supply	Dedicated power supply for IO pins
V_{SS}	Supply	Dedicated power ground for the device

Table 11.17 : Continued...

SIGNAL	TYPE	DESCRIPTION
	IEEE1149.1 TEST PINS	
TCK	Input	IEEE standard 1149.1 test clock
TDI	Input	IEEE standard 1149.1 test data input
TDO	Output, Tristate	IEEE standard 1149.1 test data output
TMS	Input	IEEE standard 1149.1 test mode select
$\overline{\text{TRST}}$	Input	IEEE standard 1149.1 test reset
EMU0	IO, Tristate	Emulator interrupt 0 pin
EMU1 / $\overline{\text{OFF}}$	IO, Tristate	Emulator interrupt 1 pin/disable all outputs

11.4.2 Architecture of TMS320C54x Processors

The TMS320C54x processors employ an advanced, modified Harvard architecture that maximizes processing power by providing four pairs of separate bus structures, three pairs for data memory and one pair for program memory. The 4 pairs or 8 internal buses of TMS320C54x processors are,

PB : **P**rogram **B**us
PAB : **P**rogram **A**ddress **B**us
} Program memory bus to read opcode and immediate operand

CB : **C B**us
CAB : **C A**ddress **B**us
DB : **D B**us
DAB : **D A**ddress **B**us
} Two independent data memory buses to read two data simultaneously from memory

EB : **E B**us
EAB : **E A**ddress **B**us
} Data memory bus to write data in data memory

In TMS320C54x processors, the separate program and data memory spaces allow simultaneous access to program instructions and data, providing a high degree of parallelism. For example, two read and one write operations can be performed in a single cycle. Special instructions with parallel load/store and multiply/accumulate fully utilize this architecture. In addition, data can be transferred between data and program memory spaces. Such parallelism supports a powerful set of arithmetic, logic, and bit-manipulation operations that can be performed in a single machine cycle. In addition, the TMS320C54x processors include the control mechanisms to manage interrupts repeated operations and function calls.

The simplified internal architecture of TMS320C54x processor is shown in fig 11.14. The architecture can be broadly divided into three major areas. They are CPU (**C**entral **P**rocessing **U**nit), on-chip memory unit and on-chip peripherals.

The functional units of CPU are 40-bit ALU (**A**rithmetic **L**ogic **U**nit), two numbers of 40-bit accumulators (ACCA and ACCB), barrel shifter, 17×17-bit multiplier, 40-bit adder, CSSU (**C**ompare, **S**elect and **S**tore **U**nit), exponent encoder, status registers, data address generation unit, program address generation unit and system control interface.

The on-chip memory unit consists of 2k to 48k words (16-bits) program/data ROM, 5k to 32k words (16-bit) program/data RAM, DMA (**D**irect **M**emory **A**ccess) controller and external memory interface.

The on-chip peripherals of TMS320C54x processors are general purpose IO pins, software-programmable wait-state generator, programmable bank-switching logic, clock generator, timer, standard serial port, **B**uffered **S**erial **P**ort (BSP), **M**ulti**c**hannel **B**uffered **S**erial **P**ort (McBSP) and **H**ost **P**ort **I**nterface (HPI).

The TMS320C54x processors has a total physical memory address space of 192k (including on-chip memory), with addressability of 16-bits. This address space is divided into three individually selectable address spaces as follows.

- 64k Program memory address space
- 64k Data memory address space
- 64k IO ports address space

The 64k program memory address space can be extended to 8M virtual program memory address space in some versions of TMS320C54x processors.

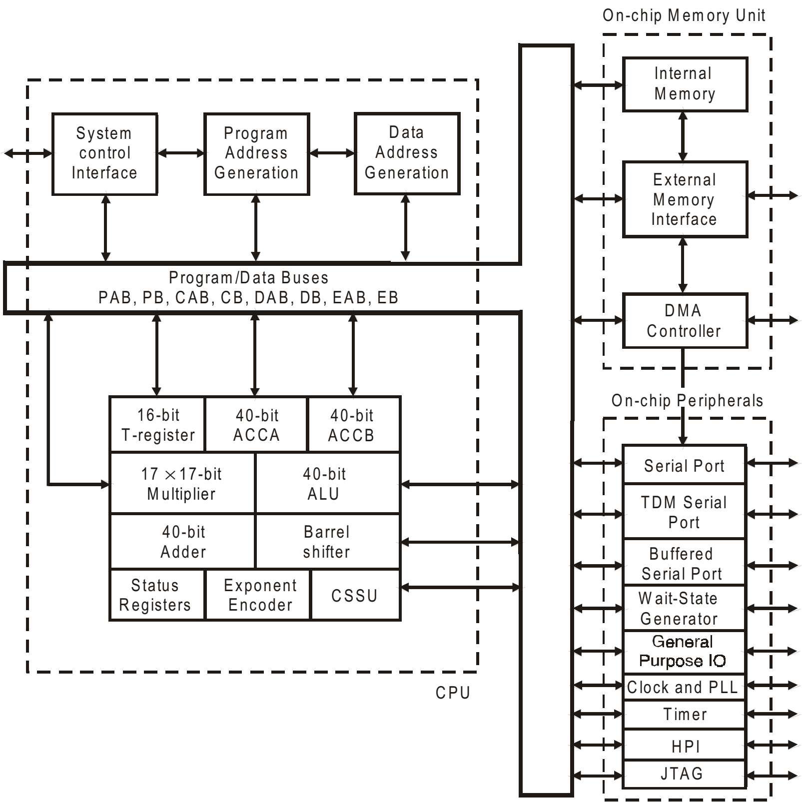


Fig 11.14 : *Simplified architecture of TMS320C54x.*

11.4.3 Functional Units of CPU of TMS320C54x Processors

The various functional units of TMS320C54x processors are,

- 40-bit **A**rithmetic **L**ogic **U**nit (ALU)
- Two numbers of 40-bit accumulators (ACCA and ACCB)

- Barrel shifter
- 17×17-bit multiplier
- 40-bit adder
- **C**ompare, **S**elect and **S**tore **U**nit (CSSU)
- **E**xponent encoder
- Data address generation unit
- Program address generation unit

Arithmetic Logic Unit (ALU)

The 40-bit ***ALU*** can perform a wide range of arithmetic and logical functions in a single clock cycle. After the ALU operation, the destination of the result is either accumulator or memory.

For ALU operations involving two data, one of the data is from barrel-shifter/memory and the other data is from accumulator/memory/T-register. The barrel-shifter and accumulator supply 40-bit data to ALU. When data is fed from memory to ALU, two 16-bit data are loaded to bits 0 to 15 and bits 16 to 31 with bits 32 to 39 either filled with zero or sign extended.

The ALU can function as two 16-bit ALUs and perform two 16-bit operations simultaneously when the C16 bit in status register 1 (ST1) is set.

Accumulators

The CPU has two 40-bit accumulators referred to as **acc**umulator **A** (ACCA) and **acc**umulator **B** (ACCB). The accumulators can act as source/destination for the ALU and the multiplier/adder. Also, any of the accumulators can be used as temporary storage for the other.

The accumulators are divided into three parts:

- Guard bits (bits 32-39)
- A high-order word (bits 16-31)
- A low-order word (bits 0-15)

The ***guard bits*** are used as a headmargin for computations, which prevent overflow in iterative computations like convolution/correlation. The instruction set of the TMS320C54x processor includes instructions for storing the guard bits, the high and the low-order accumulator words in data memory, and for manipulating 32-bit accumulator words in or out of data memory.

Barrel Shifter

The 40-bit ***barrel shifter*** can perform 0 to 31 bits left shift, 0 to 16 bits right shift and along with exponent encoder can normalize the accumulator content. The shift informations are specified in the shift count field of the instruction, the shift count field of status register 1 or in T-register. The shift and normalize operations of barrel shifter can be used to realize the following operations.

- Prescaling of the memory/accumulator operand before an ALU operation
- Logical or arithmetic shifting of accumulator value
- Normalizing the accumulator
- Postscaling the accumulator before storing in memory

The 40-bit shifter can handle 16-bit/32-bit/40-bit operands which are input from data buses (DB and CB buses) or from accumulators. The output of shifter can be loaded in ALU or EB bus.

Multiplier/Adder

The multiplier/adder unit consists of 17×17-bit multiplier, 40-bit adder, signed/unsigned input control logic, fractional control logic, zero detector, rounder, overflow/saturation logic and T-register. One of the inputs for the multiplier can be supplied from T-register/data-memory/accumulator, and the other input can be supplied from data-memory/program-memory/accumulator.

The multiplier/adder unit can perform 17×17-bit two's complement multiplication and 40-bit addition in parallel in a single instruction cycle. In addition, the multiplier and ALU together can perform MAC operation and an ALU operation in parallel in a single instruction cycle. These parallel operations can be used for efficient impelmentation of DSP computations like convolution, correlation and filtering.

Compare, Select and Store Unit (CSSU)

The ***CSSU*** is an application specific hardware unit dedicated to perform add/compare/select operations in order to support various viterbi butterfly algorithms used in equalizers and channel decoders.

The inputs to CSSU for comparision are from accumulator and the output is stored in data memory. The status of comparision is also stored in LSB of TRN register and TC bit of status register 0.

The instruction "CMPS *src, SMEM*", use the CSSU to compare the low and high word of specified source accumulator, to select the largest of the two words and store in specified data memory. If high accumulator is greater, then 0 is stored in LSB of TRN and TC, or if low accumulator is greater, then 1 is stored in LSB of TRN and TC.

Exponent Encoder

For implementation of floating point arithmetic in fixed point processors like TMS320C54x, require separation of exponent and mantissa of the floating point data.

The ***exponent encoder*** is an application-specific hardware device dedicated to extract the exponent value from floating point data in the accumulators and store in T-register.

The "EXP *src*" instruction is used to extract the exponent and save in T-register. The "NORM *src, dst*" instruction is used to normalize the accumulator using the exponent in T-register as count value.

Data Address Generation Unit

The data address generation units consist of two numbers of **A**uxiliary **R**egister **A**rithmetic **U**nits (ARAU0, ARAU1), eight numbers of **A**uxiliary **R**egisters (AR0-AR7), a 16-bit circular buffer size register (BK) and a 16-bit **S**tack **P**ointer (SP).

The auxiliary registers are used to hold the data-memory address in indirect addressing mode. The 3-bit ARP (**A**uxiliary **R**egister **P**ointer) field of status register 0 indicates the current AR used for indirect addressing. The auxiliary register-0 is also used as an index register for modifying the content of other auxiliary registers.

The ARAU perform arithmetic operations related to address generation for indirect addressing mode like increment, decrement, indexing, bit reversed address generation and circular address generation. The two independent ARAUs at any time can operate on two ARs to generate two data-memory address simultaneously.

The 9-bit DP (**D**ata-page **P**ointer) of status register-0 is used as upper 9 bits of data-memory address (page address) in direct addressing. The circular buffer register is loaded with circular buffer size which is used to generate the start and end address of circular memory along with AR specified in the instruction. The stack pointer is used to implement the **LIFO** stack for memory operands that uses stack addressing. The stack pointer always holds the address of top of stack.

Program Address Generation Unit

The program address generation unit consists of five registers, namely, **P**rogram **C**ounter (PC), **R**epeat **C**ounter (RC), **B**lock-**R**epeat **C**ounter (BRC), Block-**R**epeat **S**tart **A**ddress register (RSA) and Block-**R**epeat **E**nd **A**ddress register (REA). Some versions of TMS320C54x processors has an additional register called program counter extension register (XPC) to support addressing of virtual memory.

The ***program counter*** PC is a 16-bit register which holds the address of the program code. An instruction is fetched from program memory by loading the content of PC (address) on the program address bus (PAB) and then reading the code from program bus (PB). When the memory is read, the PC is incremented for the next fetch, so that when an instruction word is read, the PC holds the address of next word of same instruction or the next instruction. The XPC is a 7-bit register that selects the extended page of program memory in the processors that supports virtual addressing.

When the execution of a single instruction has to be repeated a number of times the 16-bit RC register is used to hold the count value and when a block of instruction has to be repeated the BRC is used to hold the count value. The registers RSA and REA are used to hold the start and end address of the block to be repeated respectively.

Status Registers (ST0 and ST1)

The TMS320C54x processors has two numbers of 16-bit ***status registers*** ST0 and ST1 which holds the status of ALU result, pointers for indirect addressing and various bits for interrupt control, hold mode, arithmetic mode and accumulator shift value. The format of status registers are shown in fig 11.12 and 11.13. The functions of various bits of status register are listed in tables 11.14 and 11.15.

The status registers can be stored into data memory and loaded from data memory, thereby allowing the processor status to be saved and restored for subroutines. The individual bit of the ST0 and ST1 registers can be set or cleared with the SSBX and RSBX instruction.

The ARP, DP and ASM bit fields can be loaded using the LD instruction with a short-immediate operand. The ASM and DP fields can be also loaded with data-memory values by using the LD instruction.

Table 11.18 : Functions of various bits of ST0 of TMS320C54x processors

BIT	NAME	RESET VALUE	FUNCTION
15-13	ARP	0	Auxiliary register pointer to select AR for indirect addressing
12	TC	1	Test/control flag bit which stores the results of ALU test bit operations
11	C	1	Carry bit which indicates a carry or borrow in ALU operation
10	OVA	0	Overflow flag for ACCA. Indicates an overflow in ALU operation with destination as ACCA
9	OVB	0	Overflow flag for ACCB. Indicates an overflow in ALU operation with destination as ACCB
8-0	DP	0	Data-memory page pointer to specify the current data memory page

15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
ARP			TC	C	OVA	OVB	DP								

Fig 11.15 : Format of status register 0 (ST0) of TMS320C54x processor.

Table 11.19 : Functions of various bits of ST1 of TMS320C54x processors

BIT	NAME	RESET VALUE	FUNCTION
15	BRAF	0	Block-repeat active flag. Indicates whether a block repeat is currently active
14	CPL	0	Compiler mode bit. Indicates which pointer is used in relative direct addressing
13	XF	1	Status of external flag pin
12	HM	0	Hold mode bit
11	INTM	1	Interrupt mode bit. Enables/disables all interrupts
10	–	0	Always 0
9	OVM	0	Overflow mode bit. Enables/disables the saturation mode in ALU.
8	SXM	1	Sign-extension mode bit. Enables/disables sign extension of an arithmetic operation
7	C16	0	Dual 16-Bit/double precision arithmetic mode selection bit
6	FRCT	0	Fractional mode bit
5	CMPT	0	Compatibility mode bit. Determines the compatibility mode for the ARP
4-0	ASM	0	Accumulator shift value in the range -16_{10} to 15_{10}

15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
BRAF	CPL	XF	HM	INTM	0	OVM	SXM	C16	FRCT	CMPT	ASM				

Fig 11.16 : Format of status register 1 (ST1) of TMS320C54x processor.

CPU Memory Mapped Registers

The TMS320C54x has 32 numbers of 16-bit CPU registers that are mapped into page-0 of data memory space. These memory-mapped registers includes registers for data and program memory address generation, various status and control registers for CPU and the accumulators. The memory-mapped registers along with their memory address are listed in table 11.20.

Table 11.20 : CPU Memory-Mapped Registers of TMS320C54x processors

Address		Name	Description
Dec	Hex		
0	0	IMR	Interrupt mask register
1	1	IFR	Interrupt flag register
2–5	2–5	–	Reserved for testing
6	6	ST0	Status register 0
7	7	ST1	Status register 1
8	8	AL	Accumulator A low word (bits 15–0)
9	9	AH	Accumulator A high word (bits 31–16)
10	A	AG	Accumulator A guard bits (bits 39–32)
11	B	BL	Accumulator B low word (bits 15–0)
12	C	BH	Accumulator B high word (bits 31–16)
13	D	BG	Accumulator B guard bits (bits 39–32)
14	E	T	Temporary register
15	F	TRN	Transition register
16–23	10–17	AR0–AR7	Auxiliary register 0 – Auxiliary register 7
24	18	SP	Stack pointer
25	19	BK	Circular-buffer size register
26	1A	BRC	Block-repeat counter
27	1B	RSA	Block-repeat start address
28	1C	REA	Block-repeat end address
29	1D	PMST	Processor mode status register
30	1E	XPC	Program counter extension register
31	1F	–	Reserved

11.4.4 On-Chip Memory in TMS320C54x Processors

The TMS320C54x family of processors consists of three different types of on-chip memory and they are mask-programmable ROM, **S**ingle-**A**ccess **RAM** (SARAM) and **D**ual-**A**ccess **RAM** (DARAM). The various members of TMS320C54x will have different capacity of on-chip memory which are listed in table 11.21.

Table 11.21 : On-chip Memory in TMS320C54x Processors

Memory Type		TMS320C54x Family of Processors						
		C541	C542	C543	C545	C546	C548	C549
ROM	Program ROM (PROM)	20k	2k	2k	32k	32k	2k	16k
	Program/Data ROM	8k	–	–	16k	16k	–	16k
RAM	DARAM	5k	10k	10k	6k	6k	8k	8k
	SARAM	–	–	–	–	–	24k	24k

On-chip ROM

The various models of TMS320C54x processors have internal maskable ROM of size 2k to 48k words. In majority of the processors, the on-chip ROM is mapped to program-memory space and in some processors a part of ROM can be mapped to data-memory space. The processor has an option for including or excluding the on-chip ROM addresses in the processor program memory address space.

The main purpose of internal ROM is to permanently store the program code and data for a specific application during manufacturing of the chip itself. The processor has an option of boot loading the content of on-chip ROM to internal/external RAM during power-ON reset. The content of the on-chip ROM can be protected so that any external device cannot have access to the program code. This feature provide security for proprietary algorithms.

On-chip DARAM

The TMS320C54x family of processors has 5k to 10k words of on-chip DARAM which are organized into blocks as shown below.

- TMS320C541 : 5k words organized as 5 blocks of 1k words each
- TMS320C542/543 : 10k words organized as 5 blocks of 2k words each
- TMS320C545/546 : 6k words organized as 3 blocks of 2k words each
- TMS320C548/549 : 8k words organized as 4 blocks of 2k words each

The DARAM blocks can be accessed twice per machine cycle. Upon reset, the DARAM is mapped to data memory address space and after reset the processor has provision to map the DARAM into program memory space.

On-chip SARAM

The TMS320C548/549 processors has 24k words of on-chip SARAM which are organized as three blocks of 8k words. Upon reset, the SARAM is mapped to data memory space and after reset the processor has provision to map the SARAM into program memory space.

11.4.5 On-Chip Peripherals of TMS320C54x Processors

The various on-chip peripherals available in TMS320C54x family of processors are,

- Software-programmable wait-state generator
- Programmable bank switching
- Parallel IO ports

- DMA controller
- Host Port Interface (HPI)
- Serial ports (Standard, TDM, BSP and McBSP)
- General purpose IO pins
- Timer
- Clock generator and Phase Locked Loop (PLL)

Software-programmable wait-state generator

The software-programmable ***wait-state generator*** can insert/generate wait-states in external bus cycles for interfacing with slow speed external memory and IO devices. The wait-state generator can extend the external bus cycles up to seven machine cycles. When all external accesses are configured to zero wait states, the internal clock to the wait-state generator is shut off to reduce power consumption.

Programmable Bank Switching

The programmable ***bank-switching logic*** can be used to insert one cycle automatically when the memory data access switches from data memory space to program memory space or vice versa. This extra cycle helps the memory to release the bus before the other memory starts driving the bus, thereby avoiding bus contention.

Parallel IO ports

The TMS320C54x family of processors has 64k ***IO address space*** which can be used as 64k IO ports. The IO ports can be addressed by the PORTR and PORTW instruction for data transfer between ports and data memory. The processor generates a signal $\overline{\text{IS}}$ during IO access to indicate a port read or port write operation. The processor can be easily interfaced to external IO devices through IO ports with minimal external address decoding circuits.

DMA (Direct Memory Access) Controller

The internal DMA controller in TMS320C54x processors can perform data transfer between various internal and external memory spaces without the intervention of CPU. The DMA has six independent programmable channels, allowing six different contexts for DMA operation. The DMA has higher priority than the CPU for both internal and external accesses. The DMA can perform single word or double word transfers. The DMA transfer from/to external to internal memory require 5 cycles.

Host Port Interface (HPI)

The ***HPI*** is an 8-bit parallel port that provides an interface to a host processor for information exchange between the Digital Signal Processor (DSP) and the host processor. The information exchange takes place via on-chip memory that is accessible to both DSP and host. The TMS320C54x family of processor has 2k words of internal DARAM mapped in data memory space 1000h to 17FFh as HPI memory.

Serial Ports

The TMS320C54x processors has the following four types of serial ports.

- Synchronous serial port
- **T**ime **D**ivision **M**ultiplexed (TDM) serial port

- Buffered serial port
- **M**ulti**c**hannel **B**uffered **S**erial **P**ort (McBSP)

The ***synchronous serial ports*** are high-speed, full-duplexed serial ports that provide direct communication with serial devices such as codecs, serial ADC, etc. These ports can operate up to one-fourth the machine cycle rate. The transmitter and receiver are double buffered and data is framed either as bytes or as words.

The ***TDM serial port*** employs the time-division multiplexing technique for serial communication to multiple devices having TDM ports. The time-division multiplexing is the process of dividing the time intervals into number of subintervals with each subinterval representing a communication channel. One TMS320C54x processor can communicate with up to seven devices/processors with TDM serial ports via a pair of data lines and a pair of address lines. Like synchronous serial port, the TDM port is also double-buffered for both transmit and receive data.

The ***buffered serial port*** consists of a full-duplex double-buffered serial-port interface and an auto-buffering unit. The processor internal memory is connected to an auto-buffering unit by a dedicated bus, so that the buffered serial port can directly read/write to processor internal memory without the intervention of CPU. This results in minimal overhead for serial port transactions and faster data rates.

The ***multichannel buffered serial port*** (McBSP) is an enhanced buffered serial port that can support multichannel transmit and receive up to 128 channels. The advanced features of McBSP are wide data sizes from 8-bit to 32-bit, μ-law and A-law companding and programmable internal clock and frame synchronization.

General-Purpose IO Pins

The TMS320C54x family of processors has two general-purpose ***IO pins*** and they are branch control input pin, $\overline{\text{BIO}}$ and external flag output pin, XF.

The $\overline{\text{BIO}}$ pin can be used to monitor the status of peripheral devices. A branch instruction can be conditionally executed depending upon the state of the $\overline{\text{BIO}}$ input. The $\overline{\text{BIO}}$ pin is an alternative to interrupt, when the interrupts are dedicated to time-critical applications.

The XF pin can be used to signal external devices. The XF pin is controlled using software. At reset the XF pin is set high. The SSBX instruction is used to set XF pin and RSBX instruction is used to reset XF pin.

Timer

The on-chip timer in TMS320C54x processors is a 16-bit timer with a 4-bit prescaler. The ***timer*** can be used to initiate any time-based event through interrupt. The timer has a count register, which is loaded with a count value and at every clock cycle the timer count is decremented by 1. At the end of the count an interrupt is generated. The timer has a control register to control its operations like start, stop, restart and disable.

Clock Generator and PLL (Phase Locked Loop)

There are two methods of clock generation in TMS320C54x processors. In one method, the internal oscillator connected to an external crystal is used to generate a clock at crystal frequency and then divided by 1, 2, or 4 and used for CPU.

In another method, a low-frequency external clock is supplied to an internal PLL circuit. The CPU clock is generated by a PLL circuit at multiple frequency of external clock. This method reduces system power consumption and clock-generated EMI and facilitate the use of low-cost crystal.

11.4.6 Addressing Modes of TMS320C54x Processors

The addressing mode refer to the method of specifying the operand or the data to be operated by the instruction. The TMS320C54x processors supports the following seven addressing modes.

1. Immediate addressing
2. Absolute addressing
3. Accumulator addressing
4. Direct addressing
5. Indirect addressing
6. Memory-mapped register addressing
7. Stack addressing

Immediate Addressing

In immediate addressing, the data is specified as a part of the instruction. In this addressing, the instruction will carry a 3-bit/5-bit/8-bit/9-bit/16-bit constant, which is the data to be operated by the instruction. The immediate constant is specified with # symbol. In the instructions listed in table 11.22, the syntax used for immediate addressing are *# k3, # k5, # k9, # K* and *# lk*.

Example :

```
LD # 1Ch, ASM      ;  Load the immediate 5-bit constant (1Ch) in ASM field of status register 1
LD # 12Ah, DP      ;  Load the immediate 9-bit constant (12Ah) in DP field of status register 0
LD # 37A5h, 16,A   ;  Shift the long immediate (16-bit) constant by 16-bit and load in accumulator A
```

Absolute Addressing

In absolute addressing, the 16-bit address of the operand is directly specified in the instruction. This addressing can be used to address an operand in all the three address spaces of the processor (i.e., address an operand in program memory, data memory and IO ports). In the instruction listed in table 11.22, the syntax used for absolute addrsssing are *pmad, dmad* and *PA*. In assembly language programs, the 16-bit address is specified as a 16-bit constant without # symbol.

Example :

```
MVKD 5F3Bh, *AR2     ;  Move the data from data memory addressed by the instruction (address = 5F3Bh)
                     ;  to another data memory location addressed by AR2
MVPD 3FCAh, *AR4     ;  Move the data from program memory addressed by the instruction
                     ;  (address = 3FCAh) to data memory location addressed by AR4
PORTR 7C20h, *AR1    ;  Move the data from the IO port addressed by the instruction (address = 7C20h)
                     ;  to data memory location addressed by AR1
```

Accumulator Addressing

In accumulator addressing, the content of accumulator is the address of the operand/data in program memory.

Example :

READA *AR3	;	Read the content of program memory addressed by accumulator A and store in data
	;	memory addressed by AR3
WRITA *AR4	;	Write the content of data memory addressed by AR4 in program memory addressed by
	;	accumulator A

Direct Addressing

In the direct addressing mode the lower 7 bits of data memory address are specified in the instruction itself. The 16-bit data memory address is formed by using either the 9 bits of DP (**D**ata **P**ointer) in status register-0 or the 16-bit of SP (**S**tack **P**ointer).

When DP is used, the 9 bits of DP is the upper 9 bits of the 16-bit address and the lower 7 bits are the address directly specified by the instruction.

When SP is used, the (16-bit) content of SP is added to 7 bits specified in the instruction to form 16-bit address.

In the instructions listed in table 11.22, the syntax used to represent direct addressing is *Smem*. In the assembly language programs, the 7-bit address is specified as a 7-bit constant without # symbol.

Example :

ADD 6Ch, A	;	Add the content of memory directly addressed by the instruction (address = 6Ch) to the
	;	accumulator A
SUB 57h, B	;	Subtract the content of memory directly addressed by the instruction (address = 57h) from the
	;	accumulator B

Indirect Addressing

In the indirect addressing mode, the data memory address is specified by the content of one of the eight auxiliary registers, AR0-AR7. The AR (**A**uxiliary **R**egister) currently used for accessing the data is denoted by 3-bit ARP (**A**uxiliary **R**egister **P**ointer) field of status register-0.

In this addressing mode, the content of AR can be updated automatically either after or before the operand is fetched. The syntax used for modifying the content of AR are listed in table 11.21.

In the instruction set listed in table 11.22, the syntax used to represent indirect addressing is *Smem/Xmem/Ymem*. In the assembly language programs, the syntax listed in table 11.21 are used.

Table 11.22 : Syntax Used in Indirect Address for Modifying AR

SYNTAX	MODIFICATION OF AR
*ARx	AR unaltered
*ARx−	AR decremented by 1 after data access
*ARx+	AR incremented by 1 after data access
*+ARx	AR incremented by 1 before data access
*ARx − 0	AR decremented by the content of index register (AR0)
*ARx + 0	AR incremented by the content of index register (AR0)

Table 11.22 : Continued ...

SYNTAX	MODIFICATION OF AR
*ARx – 0B	AR decremented for bit reversed addressing using index register (AR0)
*ARx + 0B	AR incremented for bit reversed addressing using index register (AR0)
*ARx – %	AR decremented for circular addressing
*ARx + %	AR incremented for circular addressing
*ARx – 0%	AR decremented for circular addressing using index register (AR0)
*ARx + 0%	AR incremented for circular addressing using index register (AR0)
*ARx(lk)	ARx = Base, *lk* = Offset, Data address = Base + Offset, ARx is not altered
*+ARx(lk)	Same as above, but ARx is modified by long immediate
*+ARx(lk)%	Same as above, but address modified for circular addressing
*(lk)	Absolute addressing

Example :

```
LD *AR3, A     ; Load the content of memory addressed by AR3 in accumulator A
LD *AR3–, A    ; Same as above, but after loading decrement AR3
LD *AR3+, A    ; Same as above, but after loading increment AR3
LD *AR3–0, A   ; Same as above, but after loading decrement AR3 using AR0
LD *AR3+0, A   ; Same as above, but after loading increment AR3 using AR0
```

Memory-Mapped Register Addressing

In memory-mapped register addressing, the address of the memory-mapped register is specified as direct or indirect address in the instruction.

The memory-mapped registers are mapped to page-0 of data memory address and so can be accessed by using only 7-bit address. In direct addressing, the 7 bits are directly specified in the instruction as a 7-bit constant without # symbol. In indirect addressing, the lower 7 bits of auxiliary register will be the address of memory-mapped register. In this addressing mode, the memory-mapped registers are accessed without affecting the content of DP (**D**ata **P**ointer) or SP (**S**tack **P**ointer).

Example :

```
LDM 06h, A     ; Load the content of MMR directly addressed by the instruction (address = 06h) in
               ; accumulator A
STLM A, 1Eh    ; Store the content of accumulator A in MMR directly addressed by the instruction
               ; (address = 1Eh)
```

Stack Addressing

In stack addressing mode, the data memory address is the content of **S**tack **P**ointer (SP).

The push and pop instructions access the stack memory using the stack addressing mode. The call interrupt and return instructions also use stack pointer address for automatic storage/retrieval of information to/from stack.

Note : Stack memory is a portion of data memory reserved by user/system designed for stack operations.

Example :

PSHM 1Ch ; Decrement SP by 2 and push the content of MMR addressed by the instruction
; (address = 1Ch) to stack memory addressed by SP

POPM 1Ch ; Pop the top of stack pointed by SP to MMR addressed by the instruction (address = 1Ch),
; then SP is incremented by 2.

11.4.7 Instruction Pipelining in TMS320C54x Processors

The execution of TMS320C54x processor instructions involve six level/phase of pipelining. The six phases of pipelining are program prefetch, program fetch, decode, access, read and execute. The functions performed in the six phases are given below.

Program prefetch : **P**rogram **A**ddress **B**us (PAB) is loaded with the address of the next instruction to be fetched.

Program fetch : The opcode (instruction word) is fetched from **P**rogram **B**us (PB) and loaded into the **I**nstruction **R**egister (IR).

Decode : The opcode is decoded to determine the type of memory access operation and the control sequence at the data address generation unit and the CPU.

Access : Operand address is loaded on the **D**ata **A**ddress **B**us (DAB). If a second operand is required, then another address loaded in CAB.

Read : The operands are read from buses DB and CB.

Execute : Perform the task specified by the instruction.

The six phases of pipeline are independent of each other, which permits overlapping of instruction execution. During any clock cycle, there is a possibility of execution of different phases of one to six instructions. Therefore, the average execution time of one word instruction is one clock cycle. While executing some of the instructions all the phases of pipeline are not fully utilized and so the average execution time will be 2 to 6 clock cycles.

11.4.8 Instruction of TMS320C54x Processors

The TMS320C54x processors instruction set consists of instructions for signal processing operations, high speed computations and general purpose applications. The instructions of TMS320C54x can be classified into the following groups.

1. Arithmetic instructions
2. Logical instructions
3. Branch/control instructions
4. Load/store instruction
5. Move instruction

The instructions of TMS320C54x processors are classified into the above groups, arranged in alphabetical order and listed in table 11.21.

The size of TMS320C54x instructions is 1 to 3 words. When all the instructions and data reside in internal memory, most of the one-word instructions are executed in one clock cycle. The execution time for 2/3 word instructions and some data transfer, branch and MAC instructions will be 2 to 6 clock cycles.

Table 11.23 : Instruction Set Summary for TMS320C54x

ARITHMETIC INSTRUCTIONS	
INSTRUCTION	**DESCRIPTION**
ABDST *Xmem, Ymem*	Absolute distance of two memory locations is determined and saved in ACCA
ABS *src[, dst]*	Absolute value of source ACC is determined and saved in destination ACC
ADD *Smem, src*	Add memory operand to source ACC. Sum in source ACC
ADD *Smem,* **TS**, *src*	Add memory with shift specified by T register to ACC
ADD *Smem,* **16**, *src [, dst]*	Add memory with 16-bit shift to source ACC. Sum in destination ACC
ADD *Smem [, SHIFT], src [, dst]*	Add memory with 5-bit signed shift to source ACC. Sum in destination ACC
ADD *Xmem, SHFT, src*	Add memory with 4-bit unsigned shift to source ACC. Sum in source ACC
ADD *Xmem, Ymem, dst*	Add two memory operands with 16-bit shift. Sum in destination ACC
ADD *#lk [, SHFT], src [, dst]*	Add long-immediate with 4-bit unsigned shift to src. Sum in dst
ADD *#lk,* **16**, *src [, dst]*	Add long-immediate with 16-bit shift to source ACC. Sum in destination ACC
ADD *src [, SHIFT], [, dst]*	Add src with 5-bit signed shift to dst
ADD *src,* **ASM** *[, dst]*	Add src with shift specified by ASM to dst
ADDC *Smem, src*	Add memory to accumulator with carry
ADDM *#lk, Smem*	Add long-immediate value to memory
ADDS *Smem, src*	Add memory to source ACC with sign-extension suppressed
DADD *Lmem, src [, dst]*	Add 32-bit memory data to source ACC. Sum in destination ACC
DADST *Lmem, dst*	Double/dual add/subtract of T, long operand
DELAY *Smem*	Delay the memory data
DRSUB *Lmem, src*	Double/dual 16-bit subtract from long word
DSADT *Lmem, dst*	Double/dual, subtract/add of T, long operand
DSUB *Lmem, src*	Double-precision/dual 16-bit subtract from ACC
DSUBT *Lmem, dst*	Double/dual, subtract/subtract of T, long operand
EXP *src*	Exponent of source ACC is determined and stored in T register
FIRS *Xmem, Ymem, pmad*	Execute symmetrical Finite Impulse Response (FIR) filter
LMS *Xmem, Ymem*	Execute Least Mean Square (LMS) algorithm
MAC[R] *Smem, src*	Multiply memory by TREG, add product to src with optional rounding
MAC[R] *Xmem, Ymem, src [, dst]*	Multiply two memory data, add to src, save in dst, optional rounding
MAC *#lk, src [, dst]*	Multiply TREG by long-immediate, add to src, save in dst
MAC *Smem, #lk, src [, dst]*	Multiply memory by long-immediate value, add to src, save in dst
MACA[R] *Smem [, B]*	Multiply memory by ACCA, add to ACCB with optional rounding

Table 11.23 : Continued...

INSTRUCTION	DESCRIPTION
MACA[R] **T**, *src [, dst]*	Multiply TREG by ACCA, add to src save in dst with optional rounding
MACD *Smem, pmad, src*	Multiply data and program memory, accumulate in src, delay data memory
MACP *Smem, pmad, src*	Multiply data and program memory, accumulate in src
MACSU *Xmem, Ymem, src*	Multiply signed and unsigned memory data, then accumulate in src
MAS[R] *Smem, src*	Multiply memory by T, subtract from src, save in src with optional rounding
MAS[R] *Xmem, Ymem, src [, dst]*	Multiply two memory data, subtract from src, save in dst with optional rounding
MASA *Smem [, B]*	Multiply memory by ACCA, subtract from ACCB, save in ACCB
MASA[R] **T**, *src [, dst]*	Multiply ACCA by T, subtract from src, save in dst with optional rounding
MAX *dst*	Maximum of ACCA and ACCB is determined and stored in dst
MIN *dst*	Minimum of ACCA and ACCB is determined and stored in dst
MPY[R] *Smem, dst*	Multiply TREG by memory, save in dst with optional rounding
MPY *Xmem, Ymem, dst*	Multiply two-memory operands, save product in destination ACC
MPY *Smem, #lk, dst*	Multiply memory by long-immediate, save product in destination ACC
MPY *#lk, dst*	Multiply TREG by long-immediate, save product in destination ACC
MPYA *Smem*	Multiply memory by ACCA, save product in ACCB
MPYA *dst*	Multiply TREG by ACCA, save product in destination ACC
MPYU *Smem, dst*	Multiply unsigned memory and TREG, save product in destination ACC
NEG *src [, dst]*	Negate source ACC and save in destination ACC
NORM *src [, dst]*	Normalize source ACC and save in destination ACC
POLY *Smem*	Evaluate polynomial
RND *src [, dst]*	Round source ACC and save in destination ACC
SAT *src*	Saturate source ACC
SQDST *Xmem, Ymem*	Square of distance between two memory data is determined and save in ACCA
SQUR *Smem, dst*	Square memory data, save in destination ACC
SQUR **A**, *dst*	Square ACCA high, save in destination ACC
SQURA *Smem, src*	Square memory data, add to src, save in src
SQURS *Smem, src*	Square memory data, subtract from src, store in src
SUB *Smem, src*	Subtract memory from source ACC, save in source ACC
SUB *Smem*, **TS**, *src*	Subtract memory with shift specified by T-register from src
SUB *Smem*, **16**, *src [, dst]*	Subtract memory with 16-bit shift from src, save result in dst
SUB *Smem [, SHIFT], src [, dst]*	Subtract memory with 5-bit signed shift from src, result in dst

Table 11.23 : Continued...

INSTRUCTION	DESCRIPTION
SUB *Xmem, SHFT, src*	Subtract memory with 4-bit unsigned shift from src
SUB *Xmem, Ymem, dst*	Shift memory, subtract Ymem from Xmem, store result in dst
SUB *#lk [, SHFT], src [, dst]*	Subtract long immediate with 4-bit unsigned shift from src, save result in dst
SUB *#lk,* **16**, *src [, dst]*	Subtract long immediate with 16-bit shift from src, save result in dst
SUB *src [, SHIFT], [, dst]*	Subtract src with 5-bit signed shift from dst
SUB *src,* **ASM** *[, dst]*	Subtract src with shift specifed by ASM from dst
SUBB *Smem, src*	Subtract memory from source accumulator with borrow
SUBC *Smem, src*	Subtract memory from source ACC conditionally
SUBS *Smem, src*	Subtract memory from source ACC with sign-extension suppressed
BRANCH/CONTROL INSTRUCTIONS	
B[D] *pmad*	Branch unconditionally to specified pmad with optional delay
BACC[D] *src*	Branch to address specified by source ACC with optional delay
BANZ[D] *pmad, Sind*	Branch to specified pmad if content of current AR ^{1} 0, optional delay
BC[D] *pmad, cond [, cond [, cond]]*	Branch to pmad if specified conditions are true, optional delay
CALA[D] *src*	Call subroutine addressed by source ACC, optional delay
CALL[D] *pmad*	Call subroutine unconditionally, optional delay
CC[D] *pmad, cond [, cond [, cond]]*	Call subroutine if specified conditions are true, optional delay
FB[D] *extpmad*	Far branch to specified extended pmad unconditionally, optional delay
FBACC[D] *src*	Far branch to address specified by source ACC, optional delay
FCALA[D] *src*	Far call subroutine addressed by source ACC, optional delay
FCALL[D] *extpmad*	Far call subroutine unconditionally, optional delay
FRAME *K*	Stack pointer immediate offset
FRET[D]	Far return, optional delay
FRETE[D]	Far return, enable interrupts, optional delay
IDLE *K*	Idle until interrupt
INTR *K*	Software interrupt with vector number k, mask other interrupts
MAR *Smem*	Modify auxiliary register specified by Smem
NOP	No operation
POPD *Smem*	Pop top of stack to data memory
POPM *MMR*	Pop top of stack to memory-mapped register
PSHD *Smem*	Push data-memory value onto stack

Table 11.23 : Continued...

INSTRUCTION	DESCRIPTION
PSHM *MMR*	Push memory-mapped register onto stack
RC[D] *cond [, cond [, cond]]*	Return from subroutine if specified conditions are true, optional delay
RESET	Software reset
RET[D]	Return from subroutine, optional delay
RETE[D]	Return from subroutine, enable interrupts, optional delay
RETF[D]	Return fast by loading return register to PC, enable interrupts, optional delay
RPT *Smem*	Repeat next instruction, count specified by memory
RPT *#K*	Repeat next instruction, count is short immediate
RPT *#lk*	Repeat next instruction, count is long immediate
RPTB[D] *pmad*	Block repeat, optional delay
RPTZ *dst, #lk*	Clear destination ACC, repeat next instruction, count is long immediate
RSBX *N, SBIT*	Clear/Reset the specified bit in status register
SSBX *N, SBIT*	Set the specified bit in status register
TRAP *K*	Software interrupt with vector number k
XC *n, cond [, cond [, cond]]*	Execute next one or two instructions if the specified conditions are true
IO INSTRUCTIONS	
PORTR *PA, Smem*	Read data from port and store in memory
PORTW *Smem, PA*	Write data from memory to port
LOAD/STORE INSTRUCTIONS	
CMPS *src, Smem*	Compare low and high of ACC, select maximum and store in memory
DLD *Lmem, dst*	Load long-word from memory to destination ACC
DST *src, Lmem*	Store long-word from source ACC to memory
LD *Smem, dst*	Load destination accumulator with memory operand
LD *Smem,* **TS***, dst*	Shift memory operand as specified by TREG, then load into destination ACC
LD *Smem,* **16***, dst*	Shift memory operand by 16 bits, then load into destination ACC
LD *Smem [, SHIFT], dst*	Shift memory operand, then load into ACC (5-bit signed shift)
LD *Xmem, SHFT, dst*	Shift memory operand, then load into ACC (4-bit unsigned shift)
LD *#K, dst*	Load destination ACC with short-immediate operand
LD *#lk [, SHFT], dst*	Shift long-immediate, then load into destination ACC (4-bit unsigned shift)
LD *#lk,* **16***, dst*	Shift long-immediate by 16 bits, then load into destination ACC
LD *src,* **ASM** *[, dst]*	Shift source ACC as specified by ASM and load in destination ACC

Table 11.23 : Continued...

INSTRUCTION	DESCRIPTION
LD *src [, SHIFT] [, dst]*	Shift source ACC and load in destination (5-bit signed shift)
LD *Smem,* **T**	Load TREG with single data-memory operand
LD *Smem,* **DP**	Load DP with single data-memory operand
LD *#k9,* **DP**	Load DP with 9-bit immediate operand
LD *#k5,* **ASM**	Load ACC shift-mode register with 5-bit immediate operand
LD *#k3,* **ARP**	Load ARP with 3-bit immediate operand
LD *Smem,* **ASM**	Load lower 5-bits of memory operand into ASM register
LD *Xmem, dst* || MAC[R] *Ymem [, dst_]*	Parallel load, multiply/accumulate [optional rounding]
LD *Xmem, dst* || MAS[R] *Ymem [, dst_]*	Parallel load, multiply/subtract [optional rounding]
LDM *MMR, dst*	Load memory-mapped register to destination ACC
LDR *Smem, dst*	Load memory value with rounding in destination ACC
LDU *Smem, dst*	Load unsigned memory value in destination ACC
LTD *Smem*	Load TREG and insert delay
SACCD *src, Xmem, cond*	Store source accumulator conditionally in memory
SRCCD *Xmem, cond*	Store block-repeat counter conditionally
ST T, *Smem*	Store TREG in memory
ST TRN, *Smem*	Store TRN in memory
ST *#lk, Smem*	Store long-immediate operand in memory
STH *src, Smem*	Store source accumulator high in memory
STH *src,* **ASM**, *Smem*	Shift source high ACC as specified by ASM, store in memory
STH *src, SHFT, Xmem*	Shift source ACC high, then store in memory (4-bit unsigned shift)
STH *src [, SHIFT], Smem*	Shift source ACC high, then store in memory (5-bit signed shift)
ST *src, Ymem* || ADD *Xmem, dst*	Store source ACC in memory with parallel memory add to destination high ACC
ST *src, Ymem* || LD *Xmem, dst*	Store source ACC in memory with parallel load to destination ACC from memory
ST *src, Ymem* || LD *Xmem, T*	Store source ACC in memory with parallel load from memory into TREG
ST *src, Ymem* || MAC[R] *Xmem, dst*	Store source ACC in memory and parallel multiply TREG with Xmem, and product to dst with optional rounding

Table 11.23 : Continued...

INSTRUCTION	DESCRIPTION
ST *src, Ymem* ‖ MAS[R] *Xmem, dst*	Store source ACC in memory and parallel multiply TREG with Xmem, subtract product from dst with optional rounding
ST *src, Ymem* ‖ MPY *Xmem, dst*	Store source ACC in memory and parallel multiply TREG with Xmem, store product in dst
ST *src, Ymem* ‖ SUB *Xmem, dst*	Store source ACC in memory and parallel subtract memory from destination high ACC
STL *src, Smem*	Store source ACC low to data memory
STL *src,* **ASM**, *Smem*	Shift source ACC low as specified by ASM, store in data memory
STL *src, SHFT, Xmem*	Shift source ACC low, then store in data memory (4-bit unsigned shift)
STL *src [, SHIFT], Smem*	Shift source ACC low, then store in data memory (5-bit signed shift)
STLM *src, MMR*	Store source ACC low to MMR
STM *#lk, MMR*	Store long-immediate to MMR
STRCD *Xmem, cond*	Store TREG to memory if specified condition is true
LOGICAL INSTRUCTIONS	
AND *Smem, src*	AND memory operand with source ACC
AND *#lk [, SHFT], src [, dst]*	Shift long-immediate, AND with src, save result in dst (4-bit unsigned shift)
AND *#lk,* **16**, *src [, dst]*	Shift long-immediate by 16 bits, AND with src, save result in dst
AND *src [, SHIFT], [, dst]*	Shift src, AND with dst, (5-bit signed shift)
ANDM *#lk, Smem*	AND memory with long-immediate operand
BIT *Xmem, BITC*	Copy the specified bit of memory to TC bit in ST0
BITF *Smem, #lk*	If logical AND of memory and long immediate is zero, then TC=0, else TC=1
BITT *Smem*	Copy the bit of memory specified by TREG to TC bit in ST0
CMPL *src [, dst]*	Complement source ACC and store in destination ACC
CMPM *Smem, #lk*	Compare memory with long-immediate, if equal then TC = 1, else TC=0
CMPR *CC, ARx*	Compare AR with AR0 as specified by CC, if true set TC else clear TC
OR *Smem, src*	OR memory operand with source ACC
OR *#lk [, SHFT], src [, dst]*	Shift long-immediate, OR with src, save result in dst (4-bit unsigned shift)
OR *#lk,* **16**, *src [, dst]*	Shift long-immediate by 16 bits, OR with src, save result in dst
OR *src [, SHIFT], [, dst]*	Shift src, AND with dst (5-bit signed shift)
ORM *#lk, Smem*	OR memory with long-immediate operand
ROL *src*	Rotate source accumulator left, previous C moved to LSB, MSB moved to C
ROLTC *src*	Rotate source accumulator left, TC moved to LSB, MSB moved to C

Table 11.23 : Continued...

INSTRUCTION	DESCRIPTION
ROR *src*	Rotate source accumulator right, previous C moved to MSB, LSB moved to C
SFTA *src, SHIFT [, dst]*	Shift source ACC arithmetically and save in destination ACC
SFTC *src*	Shift source accumulator conditionally
SFTL *src, SHIFT [, dst]*	Shift source ACC logically and store in destination ACC
XOR *Smem, src*	XOR memory operand with source ACC
XOR *#lk [, SHFT], src [, dst]*	Shift long-immediate, XOR with ACC (4-bit unsigned shift)
XOR *#lk,* **16**, *src [, dst]*	Shift long-immediate by 16 bits, XOR with src, save result in dst
XOR *src [, SHIFT] [, dst]*	Shift src, XOR with dst (5-bit signed shift)
XORM *#lk, Smem*	XOR memory with long-immediate operand
MOVE INSTRUCTIONS	
MVDD *Xmem, Ymem*	Move within data memory (Indirect addressing)
MVDK *Smem, dmad*	Move within data memory (Source: Indirect/direct addressing) (Destination: Absolute addressing)
MVDM *dmad, MMR*	Move memory data to memory-mapped register (Absolute addressing)
MVDP *Smem, pmad*	Move data to program memory (Source: Indirect/direct addressing)
MVKD *dmad, Smem*	Move within data memory (Source: Absolute addressing) (Destination: Indirect/direct addressing)
MVMD *MMR, dmad*	Move memory-mapped register to data memory (Absolute addressing)
MVMM *MMRx, MMRy*	Move between memory-mapped registers
MVPD *pmad, Smem*	Move program memory to data memory (Destination: Indirect/direct addressing)
READA *Smem*	Move ACCA to data memory (Indirect/direct addressing)
WRITA *Smem*	Move memory to ACCA (Indirect/direct addressing)

Table 11.24 : Meaning for the Operand Field Used in TMS320C54x

SYMBOL	DESCRIPTION
A	Accumulator A
ARx	Auxiliary register x (0 £ ´ £ 7)
ARP	Auxiliary register pointer field in ST0 that specifies current AR
ASM	5-bit accumulator shift mode field in ST1 (-16_{10} £ ASM £ 15_{10})
BITC	4-bit value that specify the bit of data memory to be tested (0 £ BITC £ 15_{10})
CC	2-bit condition code (0 £ CC £ 3)
cond	Condition to be tested by the instruction

Table 11.24 : Continued ...

SYMBOL	DESCRIPTION
[D]	Delay option
dmad	16-bit data-memory address (0 £ dmad £ FFFFh) (Absolute addressing)
DP	9-bit data-memory page pointer field in ST0 (0 £ DP £ 511_{10})
dst	Destination accumulator (A or B)
extpmad	Extended program-memory address (23-bit immediate address)
K	Short-immediate value (8-bit)
#k3/#k5/#k9	3-bit/5-bit/9-bit immediate constant
# lk	Long-immediate value (16-bit)
Lmem	32-bit data-memory address using long-word addressing
mmr, MMR	Memory-mapped register
MMRx, MMRy	Memory-mapped register, AR0-AR7 or SP
PA	16-bit port address (absolute addressing)
pmad	16-bit program-memory address (0 £ pmad £ FFFFh) (Absolute addressing)
SHFT	4-bit unsigned shift value (0 £ SHFT £ 15_{10})
SHIFT	5-bit signed shift value (-16_{10} £ SHIFT £ 15_{10})
Sind	Single data-memory operand using indirect addressing
Smem	Single data-memory operand using direct or indirect addressing
src	Source accumulator (A or B)
T	Temporary register
TS	5-bit shift value specified by T-register (-16_{10} £ TS £ 31_{10})
Xmem	Data-memory operand in dual-operand or single-operand instructions (Indirect addressing)
Ymem	Second data-memory operand in dual-operand instructions (Indirect addressing)

Table 11.25 : Conditions for Branch, Call and Return Instructions

Code for Cond	Condition	Description
AEQ	A = 0	Accumulator A equal to 0
BEQ	B = 0	Accumulator B equal to 0
ANEQ	A ¹ 0	Accumulator A not equal to 0
BNEQ	B ¹ 0	Accumulator B not equal to 0
ALT	A < 0	Accumulator A less than 0
BLT	B < 0	Accumulator B less than 0

Table 11.25 : Continued....

Code for Cond	Condition	Description
ALEQ	$A \leq 0$	Accumulator A less than or equal to 0
BLEQ	$B \leq 0$	Accumulator B less than or equal to 0
AGT	$A > 0$	Accumulator A greater than 0
BGT	$B > 0$	Accumulator B greater than 0
AGEQ	$A \geq 0$	Accumulator A greater than or equal to 0
BGEQ	$B \geq 0$	Accumulator B greater than or equal to 0
AOV	AOV = 1	Accumulator A overflow detected
BOV	BOV = 1	Accumulator B overflow detected
ANOV	AOV = 0	No accumulator A overflow detected
BNOV	BOV = 0	No accumulator B overflow detected
C	C = 1	ALU carry set to 1
NC	C = 0	ALU carry clear to 0
TC	TC = 1	Test/Control flag set to 1
NTC	TC = 0	Test/Control flag cleared to 0
BIO	$\overline{BIO}$ low	$\overline{BIO}$ signal is low
NBIO	$\overline{BIO}$ high	$\overline{BIO}$ signal is high
UNC	none	Unconditional operation

11.4.9 Assembly Language Programs in TMS320C54x Processors

The various concepts of assembly language program discussed in section 11.3.9 are applicable for assembly language programs in TMS320C54x processors. The assembly language programs for TMS320C54x processors are written using the mnemonics listed in table 11.22. Most of the assembler directives listed in table 11.14 can be used for assembly language programs of TMS320C54x processors.

Program 11.7

Write an assembly language program using instructions of TMS320C54x processors to add two numbers of 64-bit data. Assume that the two data are available in memory. Store the sum in memory.

Problem Analysis

The memory word size of TMS320C54x processor is 16 bits and so each 64-bit data is stored as 4 words ($4 \times 16 = 64$). Let 4 words of data-1 be stored in memory at address 1100h to 1103h. Let 4 words of data-2 be stored in memory at address 1104h to 1107h. Let the 4 words of result be stored in memory at address 1108h to 110bh. Let us use indirect address using auxiliary registers.

Let us load lower two words of data-1 in accumulator-A using double load instruction and then the lower two words of data-2 is added to accumulator-A using double add instruction. The lower two words (32-bit)of sum is stored in memory. Next the upper two words of data-1 is loaded in A and word-3 of data-2 and previous carry are added to low accumulator and then word-4 of data-2 is added to high accumulator. The upper two words (32-bit) of sum is stored in memory.

Assembly Language Program

```
;PROGRAM TO ADD TWO NUMBERS OF 64-BIT DATA
        .mmregs             ;Include memory-mapped registers.
        .text               ;Assemble the instructions into default program memory.
ADD64:   STM  #1100h,AR0    ;Load starting address of data-1 in AR0.
         STM  #1104h,AR1    ;Load starting address of data-2 in AR1.
         STM  #1108h,AR2    ;Load starting address of sum in AR2.
         DLD  *AR0+,A       ;Load the lower two words (32-bit) of data-1 from memory to,
                            ;accumulator-A. Increment AR0 by 2.
         DADD *AR1+,A       ;Add the lower two words (32-bit) of data-2 (in memory) to,
                            ;accumulator-A. Increment AR1 by 2.
         DST  A,*AR2+       ;Store the lower two words (32-bit)of sum in memory. Increment
                            ;AR2 by2.
         DLD  *AR0,A        ;Load the upper two words (32-bit) of data-1 from memory to,
                            ;accumulator-A.
         ADDC *AR1+,A       ;Add word-3 of data-2 and previous carry to low accumulator-A.
                            ;Increment AR1 by 1.
         ADD  *AR1,16,A     ;Add word-4 of data-2 to high accumulator-A.
         DST  A,*AR2        ;Store the upper two words (32-bit) of sum in memory
         RET                ;Program end
        .data 1100h         ;Store data, starting from data memory address 1100h
        .word 1205h, 3C57h  ;Store data-1 in memory
        .word 4273h, 1984h
        .word 2002h, 3A54h  ;Store data-2 in memory
        .word 7432h, 8AC9h
        .word 0, 0, 0, 0    ;Initialize sum as zero
        .end                ;Assembly end
```

Program 11.8

Write an assembly language program using instructions of TMS320C54x processors to subtract two numbers of 64-bit data. Assume that the two data are available in memory. Store the result in memory.

Problem Analysis

The memory word size of TMS320C54x processor is 16 bits and so each 64-bit data is stored as 4 words ($4 \times 16 = 64$). Let 4 words of data-1 be stored in memory at address 1200h to 1203h. Let 4 words of data-2 be stored in memory at address 1204h to 1207h. Let the 4 words of result be stored in memory at address 1208h to 120bh. Let us use indirect address using auxiliary registers.

Let us load lower two words of data-1 in accumulator-A using double load instruction and then the lower two words of data-2 is subtracted from accumulator-A, using double sub instruction. The lower two words (32-bit)of result is stored in memory. Next the upper two words of data-1 is loaded in A and word-3 of data-2 and previous borrow are subtracted from low accumulator-A, and then the word-4 of data-2 is subtracted from high accumulator-A. The upper two words (32-bit) of result is stored in memory.

Assembly Language Program

```
;PROGRAM TO SUBTRACT TWO NUMBERS OF 64-BIT DATA
        .mmregs                 ;Include memory-mapped registers.
        .text                   ;Assemble the instructions into default program memory.
SUB64:   STM  #1200h,AR3        ;Load starting address of data-1 in AR3.
         STM  #1204h,AR4        ;Load starting address of data-2 in AR4.
         STM  #1208h,AR5        ;Load starting address of result in AR5.
         DLD  *AR3+,A           ;Load the lower two words (32-bit) of data-1 from memory to,
                                ;accumulator-A. Increment AR3 by 2.
         DSUB *AR4+,A           ;Subtract the lower two words (32-bit) of data-2 (in memory) from,
                                ;accumulator-A. Increment AR4 by 2.
         DST  A,*AR5+           ;Store the lower two words (32-bit) of result in memory. Increment,
                                ;AR5 by 2.
         DLD  *AR3,A            ;Load the upper two words (32-bit) of data-1 from memory to,
                                ;accumulator-A.
         SUBB *AR4+,A           ;Subtract word-3 of data-2 and previous borrow from low,
                                ;accumulator-A. Increment AR4 by 1.
         SUB  *AR4,16,A         ;Subtract word-4 of data-2 from high accumulator-A.
         DST  A,*AR5            ;Store the upper two words (32-bit) of result in memory.
         RET                    ;Program end.
        .data 1200h             ;Store data, starting from data memory address 1200h.
        .word 72C4h, 23A5h      ;Store data-1 in memory.
        .word 3456h, 789Ah
        .word 5A64h, 9237h      ;Store data-2 in memory.
        .word 1A56h, 3478h
        .word 0, 0, 0, 0        ;Initialize result as zero.
        .end                    ;Assembly end.
```

Program 11.9

Write an assembly language program using instructions of TMS320C54x processors to multiply two numbers of unsigned 32-bit data. Assume that the two data are available in memory. Save the 64-bit product in memory.

Problem Analysis

Let the two words of data-1 be stored in memory at address 1300h and 1301h. Let the two words of data-2 be stored in memory at address 1302h and 1303h. Let the 4 words of the product be stored in memory at address 1304h to 1307h. Let us use indirect addressing.

N t o . n o i t a c i l p i t l eufme R

The four products are performed one by one by loading one of the data words in T-register and multiplying with a memory data word to get the 32-bit product in accumulators A or B. The results of the product are added appropriately to get the final product words.

Assembly Language Program

```
;PROGRAM TO MULTIPLY TWO NUMBERS OF 32-BIT DATA
        .mmregs              ;Include memory-mapped registers.
        .text                ;Assemble the instructions into default program memory.
MUL32:   STM  #1300h,AR1     ;Load starting address of data-1 in AR1.
         STM  #1302h,AR2     ;Load starting address of data-2 in AR2.
         STM  #1304h,AR3     ;Load starting address of product in AR3.
         LD   *AR2+,T        ;Load D2W1 in T-register. Increment AR2.
         MPYU *AR1+,A        ;Multiply D1W1 and D2W1 to get product-1 (P1) in A. Increment AR1.
```

```
        DST  A,*AR3          ;Store product-1(P1) in memory.
        MPYU *AR1,A          ;Multiply D1W2 and D2W1 to get product-2 (P2) in A.
        LD   *AR2,T          ;Load D2W2 in T-register.
        MAR  *AR1–           ;Decrement AR1 by 1.
        MPYU *AR1+,B         ;Multiply D1W1 and D2W2 to get product-3 (P3) in B.
        ADD  B,A             ;Get sum of P2 and P3 in A
        MPYU *AR1,B          ;Multiply D1W2 and D2W2 to get product-4 (P4) in B.
        MAR  *AR3+           ;Increment AR3 by 1.
        ADDS *AR3,A          ;Add upper word of P1 in memory to low A with sign extension,
                             ;suppressed to get FPW2 in low A.
        STL  A,*AR3+         ;save FPW2 in memory. Increment AR3.
        SFTL A,–16           ;Shift high A to low A.
        ADD  B,A             ;Add B to A to get FPW3 and FPW4 in A.
        DST  A,*AR3          ;Store FPW3 and FPW4 in memory.
        RET                  ;Program end.
       .data 1300h           ;Store data starting from data memory address 1300h.
       .word 1430h, 6270h    ;Store data-1 in memory.
       .word 1AC2h, 2A75h    ;Store data-2 in memory.
       .word 0, 0, 0, 0      ;Initialize product as zero.
       .end                  :Assembly end.
```

PROGRAM 11.10

Write an assembly language program using instructions of TMS320C54x processors to divide a 16-bit data by an 8-bit data. Assume that the data are 2's complement positive integers available in memory. Store the quotient and remainder in memory.

Problem Analysis

Let the 16-bit dividend and 8-bit divisor are stored in memory address 1100h and 1101h respectively. Let us store the quotient and remainder in memory address 1102h and 1103h. Let us use indirect addressing.

Note : Refer problem analysis of program 11.4 for logic of 16-bit by 8-bit division.

Assembly Language Program

```
;PROGRAM FOR 16-BIT BY 8-BIT DIVISION OF POSITIVE DATA

       .mmregs               ;Include memory-mapped registers.
       .text                 ;Assemble the instructions into default program memory.
DIV 16: STM  #1100H,AR0      ;Load starting address of data in AR0.
        RSBX SXM             ;Clear sign extension mode bit.
        LD   *AR0+,A         ;Load dividend in low A with high A as zero.
        RPT  #15             ;Repeat the next instruction 16 times.
                             ;Count is 1 less than number of repeatitions.
        SUBC *AR0,A          ;Perform conditional subtraction of division algorithm.
        MAR  *AR0+           ;Increment AR0 by 1.
        DST  A,*AR0          ;Store quotient and remainder in memory.
        RET                  ;Program end.
       .data 1100h           ;Store data starting from data memory address 1100h.
       .word 009Ah, 0007h    ;Store dividend and divisor in memory.
       .word 0, 0            ;Initialize quotient and remainder as zero.
       .end                  ;Assembly end.
```

11.5 Summary of Important Concepts

1. The digital signal processors (DSPs) are microprocessors specially designed for signal processing applications.
2. The various types of DSPs are fixed point processors, floating point processors, VLIW architecture processors and multiprocessor DSP.
3. The fixed point processors are low power, low cost and high speed processors.
4. The floating point processors offer large dynamic range, wider instruction word size and support large addressing modes.
5. The VLIW architecture processor can fetch and execute many instructions in parallel by multiple execution units.
6. The multiprocessor DSP provide parallel processing capability by integrating multiple DSPs on a single chip.
7. The special features of DSPs are fast data access, fast computation, numerical fidelity and fast execution control.
8. Fast data access is achieved by employing high-bandwidth memory architecture, specialized addressing modes and Direct Memory Access (DMA).
9. The fast computations are achieved by single cycle multiply/accumulate (MAC) unit, pipelining of instruction execution, VLIW architecture and multiprocessor architectures.
10. The numerical fidelity refer to faithfullness of the DSPs to perform mathematical operations without errors like underflow and overflow.
11. Some of the features that help fast execution control are zero overhead hardware loop and very fast interrupt handling.
12. The high bandwidth memory architecture are modified versions of Harvard architecture for simultaneous access of one or more data along with instruction code in a single clock cycle.
13. The specialized addressing modes are provided for easy implementation of signal processing algorithms like FFT, convolution and correlation.
14. The DMA helps to transfer data from/to the external and internal memory without involving CPU.
15. Von Neumann architecture consists of single memory block to store both program and data and a single bus to transfer data and instruction from/to the CPU.
16. Harvard architercture has separate memory blocks for program and data, and separate buses for transfer data and instruction from/to the CPU.
17. In modified Harvard architecture, one memory block is dedicated for storing data alone and another memory block for storing both instruction and data.
18. The advanced digital signal processors employ two or more internal memory blocks connected to CPU via separate buses.
19. The MAC unit in DSPs are capable of computing one multiplication and addition in a single clock cycle.
20. In pipelining, the instruction execution is divided into phases and different phases of two or more instructions are executed in parallel.
21. In pipelining, the number of instructions that can be executed in parallel is called depth or level of pipelining.
22. The TMS320C5x family of processors are fifth generation DSPs from Texas Instruments, USA.

23. The TMS320C5x are 16-bit fixed point processors fabricated using static CMOS technology.
24. The TMS320C5x processors has an advanced Harvard architecture, with separate buses for program and data.
25. The functional units of CPU of TMS320C5x processors are parallel logic unit, central ALU, memory-mapped registers, auxiliary register arithmetic unit and program controller.
26. The various types of on-chip memory in TMS320C5x processors are program ROM, data/program DARAM and SARAM.
27. The on-chip peripherals of TMS320C5x processors are clock generator, timer, programmable wait-state generators, parallel IO, HPI and standard/BSP/TDM serial ports.
28. The TMS320C5x processors has a total memory address space of 224k with addressability of 16 bits.
29. The 224k address space of TMS320C5x processors is divided into program, data and IO address space each of size 64k and global data memory address space of size 32k.
30. The TMS320C5x processors has 96 numbers of 16-bit memory-mapped registers, mapped into page-0 of data memory space.
31. The TMS320C5x processors has 8 numbers of auxiliary register AR0 to AR7 to hold address of data memory for indirect addressing.
32. The TMS320C5x processors has two numbers of 16-bit status registers ST0 and ST1 which holds the status of ALU result, pointer for indirect addressing and control bits for interrupt, hold mode and product shift.
33. The six addressing modes of TMS320C5x processors are direct, MMR, indirect, immediate, dedicated-register and circular addressing.
34. The TMS320C5x processor has four phase of pipelining and they are fetch, decode, read and execute.
35. The TMS320C54x family of processors are advanced version of TMS320C5x processors with more number of internal buses and 40-bit ALU.
36. The TMS320C54x porcessor has an advanced Harvard architecture with eight internal buses (or 4 pairs of buses).
37. The functional units of CPU of TMS320C54x processors are ALU, accumulators, barrel shifter, multiplier, adder and address generation units.
38. The two types of on-chip memory in TMS320C54x processors are program/data ROM and RAM.
39. The on-chip peripherals of TMS320C54x processors are clock generator, timer, programmable wait-satae generator and bank-switching logic, general-purpose IO pins, HPI and standard/BSP/McBSP/TDM serial ports.
40. The TMS320C54x processors has a total memory address space of 192k with addressability of 16-bits.
41. The 192k address space of TMS320C54x processors is divided into program, data and IO address space each of size 64k.
42. The 64k program memory address space can be extended to 8M virtual program memory address space in some versions of TMS320C54x processors.
43. In TMS320C54x processors, the upper 9 bits of 40-bit ALU are called guard bits and they serve as head margin in iterative computations to prevent overflow.
44. The TMS320C54x processor has 32 numbers of 16-bit CPU memory-mapped registers, mapped into page-0 of data memory space.

45. The seven addressing modes of TMS320C54x processors are immediate, absolute, accumulator, direct, indirect, MMR and stack addressing.
46. The TMS320C54x processor has six phases of pipleining and they are program prefetch, program fetch, decode, access, read and execute.
47. In the TMS320C54x processor, the bank-switching logic can be programmed to insert one dummy cycle when memory access switches from program to data address space or vice versa to avoid bus contention.
48. The internal DMA controller in TMS320C54x processors can perform data transfer between various internal and external memory spaces without the intervention of CPU.
49. The multichannel buffered serial port (McBSP) can support multichannel transmit and receive up to 128 channels.
50. The TMS320C54x processors has two general-purpose IO pins and they are $\overline{\text{BIO}}$ pin used to monitor the status of peripheral devices and XF pin used to signal external devices.

11.6 Short Questions and Answers

Q11.1 What are the different types of digital signal processors released by Texas Instruments?

Texas Instruments has released four basic types of digital signal processors and they are 16-bit fixed point processors, 32-bit floating point processors, VLIW architecture processors and multiprocessor digital signal processors.

Q11.2 What are the special features of digital signal processors?

The special features of digital signal processors are fast data access, fast computation, numerical fidelity and fast execution control.

Q11.3 How is fast data access achieved in digital signal processors?

In digital signal processors, the fast data access is achieved by high-band width memory architecture like modified Harvard architecture, specialized addressing modes like circular and bit reversed addressing and DMA.

Q11.4 What is the difference between Von Neumann and Harvard architecture?

The Von Neumann architecture has a single block of memory to store both code and data, and the memory is connected to CPU by a single bus, which permits the CPU to access memory for either code or data at any one time.

The Harvard architecture has two memory blocks to store code and data separately and the two memory blocks are connected to CPU by separate buses for simultaneous access of code and data.

Q11.5 What is the modified Harvard architecture employed in DSPs?

The modified Harvard architecture employed in DSPs will have two or more internal memory blocks connected to CPU by separate buses. One memory block is reserved for code and data, and the other blocks for only data. The CPU can access internal memory blocks for simultaneous code and two data at any one time.

Q11.6 Write a short note on any two special addressing modes in DSPs.

Two of the special addressing modes in DSPs are circular and bit reversed addressing. In circular addressing, a portion of memory is declared as circular buffer by specifying a start and end address, and an address pointer. When the address pointer is incremented for data access after the end address, the start address is automatically loaded in the pointer, so that the memory can be accessed in a circular fashion, for computing circular convolution. Similarly, when the address pointer is decremented, after the start address, the end address is automatically loaded in the pointer.

The bit reversed addressing mode can be used to access data in the bit reversed order for FFT computation. In this addressing mode, the address is incremented/decremented by the number represented in the bit reversed form.

Q11.7 How is fast computation achieved in DSPs?

The fast computation in DSPs are achieved by providing single cycle multiply/accumulate (MAC) unit, pipelining of instruction execution, VLIW architecture and mulitprocessor architectures.

Q11.8 Write a short note on MAC unit in DSPs.

The MAC unit in DSPs is capable of performing multiply-add operations involved in convolution and correlation. A typical MAC unit consists of a multiplier, a temporary register, a product register, an adder and an accumulator. Initially the product register and accumulator are cleared and then MAC instruction is executed, a number of times required to compute one data of convolution. The execution of MAC instruction will add the content of P-register to accumulator and multiply two memory data and store the product in P-register.

Q11.9 What is pipelining in DSPs?

The pipelining refers to overlapping of execution of various phases of different instructions so that a number of instructions can be executed in parallel. In DSPs, the execution of each instruction is divided in 4 or 6 phases. In 4 phase pipelining, when first instruction is in 4^{th} phase of execution, the second will be in 3^{rd} phase, the third will be in 2^{nd} phase and fourth will be in 1^{st} phase of execution.

Q11.10 What are the functional units of CPU of TMS320C5x processors?

The functional units of CPU of TMS320C5x processors are parallel logic unit, central ALU, memory-mapped registers, auxiliary register arithemtic unit and program controller.

Q11.11 What are the on-chip peripherals of TMS320C5x processors?

The on-chip peripherals of TMS320C5x processors are clock generator, hardware timer, programmable wait-state generaters, parallel IO ports, host port interface, standard serial port, buffered serial port and time division multiplexed serial port.

Q11.12 What is the total memory space in TMS320C5x processors and how it is divided between program, data and IO?

The TMS320C5x processors has a total memory space of 224k-words, and it is divided into following four sections.

- · 64k-words program memory address space
- · 64k-words local data memory address space
- · 32k-word global data memory address space
- · 64k-words IO ports address space

11.13 What are the addressing modes of TMS320C5x processing?

The TMS320C5x processors supports the following six addressing modes.

1. Direct addressing
2. Memory-mapped register addressing
3. Indirect addressing
4. Immediate addressing
5. Dedicated-register addressing
6. Circular addressing

11.14 *What is memory-mapped register addressing?*

In memory-mapped register addressing, the 7-bit address of memory-mapped register can be specified as direct address in the instruction. It is special case of direct addressing in which only offset address is used to access page-0 of memory.

11.15 *Write a short note on instruction pipelining in TMS320C5x processors.*

The execution of TMS320C5x processor instructions is divided into the following four phases of pipelining.

Fetch : The opcode (instruction code) is fetched from program memory and PC is updated.

Decode : The opcode is decoded, the data address is generated and data address generation registers are updated.

Read : The operand is read from data memory

Execute : Perform the task specified by the instruction

11.16 *What are the functional units of CPU of TMS320C54x processors?*

The functional units of CPU of TMS320C54x processors are 40-bit ALU, two numbers of 40-bit accumulators, barrel shifter, 17 ´ 17-bit multiplier, 40-bit adder, CSSU (**C**ompare, **S**elect and **S**tore **U**nit), exponent encoder, status registers, data address generation unit, program address generation unit and system control interface.

11.17 *What are the internal buses of TMS320C54x processors?*

The TMS320C54x processors has the following 4-pairs/8-buses.

PB : **P**rogram **B**us
PAB : **P**rogram **A**ddress **B**us
} Program memory bus to read opcode and immediate operand

CB : **C B**us
CAB : **C A**ddress **B**us
DB : **D B**us
DAB : **D A**ddress **B**us
} Two independent data memory buses to read two data simultaneously from memory

EB : **E B**us
EAB : **E A**ddress **B**us
} Data memory bus to write data in data memory

11.18 *What is the total memory space in TMS320C54x processors and how is it divided between program, data and IO?*

The TMS320C54x processors has a total memory space of 192k-words and it is equally divided into three sections, each of size 64k-words for program, data and IO address space.

11.19 *What are guard bits?*

In TMS320C54x processors, the upper 9 bits of the 40-bit accumulator are called guard bits. They serve as headmargin for iterative computation like convolution, correlation to prevent overflow.

11.20 *What are the components of data address generation unit of TMS320C54x processors?*

The components of data address generation unit of TMS320C54x processors are two numbers of auxiliary register arithmetic units, eight numbers of auxiliary registers, a 16-bit circular buffer size register and a 16-bit stack pointer.

11.21 ***What are the operations performed by auxiliary register arithmetic unit?***

The ARAU perform arithmetic operations related to address generation for indirect addressing mode like increment, decrement, indexing, bit-reversed address generation and circular address generation. The two independent ARAUs at any time can operate on two ARs to generate two data-memory address simultaneously.

11.22 ***What are the on-chip peripherals of TMS320C54x processors?***

The on-chip peripherals of TMS320C54x processors are general purpose IO pins, software-programmable wait-state generator, programmable bank-switching logic, clock generator, timer, standard serial port, **B**uffered **S**erial **P**ort (BSP), **M**ulti**c**hannel **B**uffered **S**erial **P**ort (McBSP) and **H**ost **P**ort **I**nterface (HPI).

11.23 ***What are the addressing modes of TMS320C54x processors?***

The TMS320C54x processors supports the following seven addressing modes.

1. Immediate addressing
2. Absolute addressing
3. Accumulator addressing
4. Direct addressing
5. Indirect addressing
6. Memory-mapped register addressing
7. Stack addressing

11.24 ***What is accumulator addressing? Give examples.***

In accumulator addressing, the content of accumulator is the address of the operand/data in program memory.

Example :

```
READA *AR3  ; Read the content of program memory addressed by accumulator A and store in data
            ; memory addressed by AR3

WRITA *AR4  ; Write the content of data memory addressed by AR4 in program memory addressed by
            ; accumulator A
```

11.25 ***Write a short note on instruction pipelining of TMS320C54x processors.***

The execution of TMS320C54x processor instructions involves the following six phases of pipelining.

Program prefetch : **P**rogram **A**ddress **B**us (PAB) is loaded with the address of the next instruction to be fetched

Program fetch : The opcode (instruction word) is fetched from **P**rogram **B**us (PB) and loaded into the **I**nstruction **R**egister (IR).

Decode : The opcode is decoded to determine the type of memory access operation and the control sequence at the data address generation unit and the CPU.

Access : Operand address is loaded on the **D**ata **A**ddress **B**us (DAB). If a second operand is required, then another address loaded in CAB.

Read : The operands are read from buses DB and CB.

Execute : Perform the task specified by the instruction.

11.7. Exercises

I Fill in the blanks with appropriate words.

1. The _______ are microprocessors specially designed for signal processing applications.
2. The _______ architecture employ separate buses for program and data.
3. The _______ unit in DSPs can perform multiplication and addition in a single cycle.
4. The number of instructions that can be executed in parallel is called _______ of pipelining.
5. The _______ addressing mode can be used to access memory declared as ______ buffer.
6. The TMS320C5x processors has a total memory space of ________.
7. In TMS320C5x processors the MMR are mapped into _______ of data memory space.
8. The auxiliary register arithmetic unit is a dedicated arithmetic unit to compute ________.
9. The _______ serial ports can be used for serial communication between multiple processors.
10. In _______ addressing the address of page-0 of memory can be directly specified in the instruction.
11. The execution of TMS320C5x processors involve _______ phase of pipelining.
12. The object file of TMS320C5x processors is called ________.
13. The upper 9-bits of the accumulators of TMS320C54x processors are called ________.
14. The _______ will extract the exponent of floating point data in accumulator and load in T-register.
15. The execution of TMS320C54x processors involve _______ phase of pipelining.

Answers

1. DSPs	5. circular, circular	9. TDM	13. guard bits
2. Harvard	6. 224k-words	10. MMR	14. exponent encode
3. MAC	7. page-0	11. four	15. six
4. depth/level	8. memory address	12. COFF	

II State whether the following statements are True/False.

1. In signal processing applications, the data are mostly represented as fixed point binary.
2. The VLIW architecture employ instruction level parallelism.
3. The Von Neumann architecture employ separate buses for program and data.
4. In TMS320C5x processors, after an ALU operation the result is stored in the accumulator or memory.
5. In TMS320C5x processors, after multiplication, the product is stored in the product register.
6. The ARP indicates the AR to be a used for address computation.
7. The TMS320C5x processor has a user programmable internal ROM.
8. The wait-state generator can be programmed to insert wait-state in internal memory cycles.
9. In indirect addressing, the content of AR can be updated automatically.
10. The TMS320C54x processors has 8 internal buses.

11. In TMS320C54x processors after an ALU operation the result can be stored in any of the two accumulators.
12. The ARAU performs arithmetic operations related to address generation.
13. In accumulator addressing, the content of the accumulator is the operand/data.
14. In direct addressing, the offset address is specified in the instruction and the memory page address is the content of DP.
15. In MMR addressing of TMS320C54x processors, the address of MMR can be specified either directly or indirectly using AR.

Answers

1. True	5. True	9. True	13. False
2. True	6. True	10. True	14. True
3. False	7. False	11. True	15. True
4. False	8. False	12. True	

III Choose the right answer for the following questions.

1. The architecture that employs instruction level parallelism is,

a) Von Neumann architecture **b)** Harvard architecture
c) Modified Harvard architecture **d)** VLIW architecture

2. The pipelining refers to,

a) Prefetching instructions and storing in a FIFO queue
b) Fetching instruction and data simultaneously
c) Executing different phases of two or more instructions in parallel
d) Executing different instruction in parallel using two or more computational units

3. The total memory space of TMS320C5x family of processors is,

a) 224k-words **b)** 224k-bytes
c) 192k-words **d)** 192k-bytes

4. The size of data bus, ALU and accumulator in TMS320C5x family of processors are respectively,

a) 16-bit, 32-bit, 40-bit **b)** 32-bit, 32-bit, 32-bit
c) 16-bit, 32-bit, 32-bit **d)** 16-bit, 40-bit, 32-bit

5. The size of mulitplier and product register in TMS320C5x processor is,

a) 17 ´ 17-bit, 32-bit **b)** 16 ´ 16-bit, 32-bit
c) 17 ´ 17-bit, 40-bit **d)** 16 ´ 16-bit, 40-bit

6. The MMRs of TMS320C5x processors can be directly addressed by,

a) 7-bit address **b)** 8-bit address
c) 9-bit address **d)** 11-bit address

7. ***Which of the following is true with respect to auxiliary register arithmetic unit?***
   a) It can be used for any arithmetic operation.
   b) It can be used for unsigned addition/subtraction.
   c) It can be used for signed addition/subtraction.
   d) It is used by the processor exclusively for address computations.

8. ***In TMS320C5x processors, the size of page address and page offset address are respectively,***
   a) 6-bit, 10-bit **b)** 7-bit, 9-bit
   c) 8-bit, 8-bit **d)** 9-bit, 7-bit

9. ***In TMS320C5x processors, the maximum number of address pointers for indirect addressing that can be employed in a program is,***
   a) 8 **b)** 5 **c)** 4 **d)** 2

10. ***In TMS320C5x processors, the maximum number of independent circular buffers that can be defined in a program is,***
    a) 1 **b)** 2 **c)** 3 **d)** 4

11. ***The depth of pipelining in TMS320C5x and TMS320C54x processors are respectively,***
    a) 4, 5 **b)** 6, 4 **c)** 4, 6 **d)** 6, 8

12. ***Which of the following is true with respect to "RPT #n" instruction?***
    a) Execute the next instruction n – 1 times.
    b) Execute the next instruction n times.
    c) Execute the next n – 1 instructions.
    d) Execute the next n instructions.

13. ***The total physical memory space of TMS320C54x family of processors is,***
    a) 64k-words **b)** 128k-words
    c) 192k-words **d)** 224k-words

14. ***The size of data bus, ALU and accumulator in TMS320C54x family of processors are respectively,***
    a) 16-bit, 40-bit, 40-bit **b)** 32-bit, 32-bit, 40-bit
    c) 40-bit, 32-bit, 40-bit **d)** 32-bit, 32-bit, 16-bit

15. ***The size of multiplier and adder in the MAC unit of TMS320C54x processors are respectively,***
    a) 17×17-bit, 32-bit **b)** 17×17-bit, 40-bit
    c) 16×16-bit, 32-bit **d)** 16×16-bit, 40-bit

16. ***The number of programs and data buses in TMS320C54x processors are,***
    a) Two pairs of program buses and two pairs of data buses
    b) One pair of program bus and three pairs of data buses
    c) Four pairs of data/program buses
    d) Eight pairs of data/program buses

17. The upper 8 bits of 40-bit accumulator of TMS320C54x processor is called,

a) overflow bits **b)** sign extension bits

c) guard bits **d)** carry bits

18. The function of exponent encoder in TMS320C54x processor is,

a) to extract the exponent from floating point data

b) to add the exponent to form a floating point data

c) to normalize the exponent of floating point data

d) to add/extract the exponent of floating point data

19. The number of independent addresses that can be generated at any one time by the address units of TMS320C54x processors are,

a) one program address, one data address

b) one program address, two data addresses

c) one program address, three data addresses

d) two program addresses, two data addresses

20. The function of a wait-state generator is,

a) to insert wait-states in internal and external bus cycles

b) to insert wait-states in data memory cycles

c) to insert wait-states in program memory cycles

d) to insert wait states in external bus cycles

Answers

1. d	6. a	11. c	16. b
2. c	7. d	12. a	17. c
3. a	8. d	13. c	18. a
4. c	9. a	14. a	19. b
5. b	10. b	15. a	20. d

IV Answer the following questions.

1. Explain the various types of digital signal processors released by Texas Instruments.
2. Discuss the various special hardware requirements of digital signal processors.
3. Explain Von Neumann and Harvard architectures with simple sketches.
4. Explain in detail the pipelining of instruction execution.
5. Write the salient features of TMS320C5x family of digital signal processors.
6. Draw the simplified architecture of TMS320C5x processor and explain.
7. Write a short note on various functional units of CPU of TMS320C5x processors.
8. Explain the various types of internal memory in TMS320C5x family of processors.
9. Write a short note on various on-chip peripherals of TMS320C5x processors.

10. Explain any four addressing modes of TMS320C5x processors with examples.
11. Write any four special instructions of TMS320C5x processors that are suitable for signal processing applications and explain.
12. Write the salient features of TMS320C54x family of digital signal processors.
13. Draw the simplified architecture of TMS320C54x processors and explain.
14. Write a short note on various functional units of CPU of TMS320C54x processors.
15. Explain the various types of on-chip memory in TMS320C54x family of processors.
16. Write a short note on various on-chip peripherals of TMS320C54x processors.
17. Compare the features of TMS320C5x and TMS320C54x processors.
18. Write a short note on data and program address generation units of TMS320C54x processors.
19. Explain any four addressing modes of TMS320C54x processor with examples.
20. Write any four special instructions of TMS320C54x processors that are suitable for signal processing applications and explain.

V Write assembly language programs using instructions of TMS320C5x processors.

1. Multiplication of two numbers of 32-bit signed fixed point data.
2. Division of 32-bit data by 16-bit data. Assume signed fixed point data.
3. To store an array of data in memory in the bit reversed order.
4. To store an array of data in memory in the reverse order.
5. Autocorrelation of 4-point sequence.

VI Write assembly language programs using instructions of TMS320C54x processors.

1. Addition of two numbers of 32-bit floating point data in IEEE standard format.
2. Multiplication of two numbers of 32-bit floating point data in IEEE standard format.
3. To search largest data in an array of N-data.
4. Circular convolution of two 8-point sequences.
5. Crosscorrelation of 5-point and 3-point sequences.

Table 11.3 : Characteristics of TMS320C5x Family of Processors

PROCESSOR	ON-CHIP MEMORY (16-BIT WORDS)				I/O PORTS		POWER SUPPLY (V)	CYCLE TIME (ns)	NUMBER OF PINS
	DARAM		SARAM	ROM					
	DATA	DATA+ PROG	DATA+ PROG	PROG	SERIAL	PARALLEL			
TMS320C50	544	512	9K	2K	2	64K	5	50/35/25	132 pin
TMS320LC50	544	512	9K	2K	2	64K	3.3	50/40/25	132 pin
TMS320C51	544	512	1K	8K	2	64K	5	50/35/25/20	100/132 pin
TMS320LC51	544	512	1K	8K	2	64K	3.3	50/40/25	100/132 pin
TMS320C52	544	512	–	4K	1	64K	5	50/35/25/20	100 pin
TMS320LC52	544	512	–	4K	1	64K	3.3	50/40/25	100 pin
TMS320C53	544	512	3K	16K	2	64K	5	50/35/25	132 pin
TMS320LC53	544	512	3K	16K	2	64K	3.3	50/40/25	132 pin
TMS320C53S	544	512	3K	16K	2	64K	5	50/35/25	100 pin
TMS320LC53S	544	512	3K	16K	2	64K	3.3	50/40/25	100 pin
TMS320LC56	544	512	6K	32K	2	64K	3.3	35/25	100 pin
TMS320LC57	544	512	6K	32K	2	64K+HPI	3.3	35/25	128 pin
TMS320C57S	544	512	6K	2K	2	64K+HPI	5	50/35/25	144 pin
TMS320LC57S	544	512	6K	2K	2	64K+HPI	3.3	50/35	144 pin

Table 11.6 : Characteristics of TMS320C54x Family of Processors

PROCESSOR	POWER SUPPLY (V)	ON-CHIP MEMORY		PERIPHERALS			CYCLE TIME(ns)	PACKAGE TYPE
		RAM (Word)	ROM (Word)	SERIAL PORT	TIMER	HPI		
TMS320C541	5.0	5K	28K	2	1	No	25	100-pin TQFP
TMS320LC541	3.3	5K	28K	2	1	No	20/25	100-pin TQFP
TMS320LC541B	3.3	5K	28K	2	1	No	20/25	100-pin TQFP
TMS320C542	5.0	10K	2K	2	1	Yes	25	144-pin TQFP
TMS320LC542	3.3	10K	2K	2	1	Yes	20/25	128/144-pin TQFP
TMS320LC543	3.3	10K	2K	2	1	No	20/25	100-pin TQFP
TMS320LC545	3.3	6K	48K	2	1	Yes	20/25	128-pin TQFP
TMS320LC545A	3.3	6K	48K	2	1	Yes	15/20/25	128-pin TQFP
TMS320LC545B	3.3	6K	48K	2	1	Yes	15/20/25	128-pin TQFP
TMS320LC546	3.3	6K	48K	2	1	No	20/25	100-pin TQFP
TMS320LC546A	3.3	6K	48K	2	1	No	15/20/25	100-pin TQFP
TMS320LC546B	3.3	6K	48K	2	1	No	15/20/25	100-pin TQFP
TMS320LC548	3.3	32K	2K	3	1	Yes	12.5/15/20	144-pin TQFP/144-pin BGA
TMS320LC549	3.3	32K	16K	3	1	Yes	12.5/15	144-pin TQFP/144-pin BGA
TMS320VC549	3.3(2.5core)	32K	16K	3	1	Yes	8.3/10/12.5	144-pin TQFP/144-pin BGA

Table 11. : Comparison of the features of TMS320C5x and TMS320C54x

FEATURE	TMS320C5x	TMS320C54x
Program bus	PB	PB
Data bus	DB	DB and CB (for Read), EB (for Write)
Address buses	PAB, DAB	PAB, CAB, DAB, EAB
Main ALU	32-bit ALU	40-bit ALU
Accumulators	32-bit ACC	40-bit ACCA and ACCB
Barrel shifter	0-16-bit left shift 0-16-bit right shift	40-bit: 0-31 left shift 0-15 right shift
Multiplier	16 ´ 16-bit	17 ´ 17-bit
Adder	32-bit	40-bit
Auxiliary register ALU	ARAU	ARAU0, ARAU1
Block repeat registers	16-bit BRCR, PASR, PAER	16-bit BRC, RSA, REA
Circular buffer register	Two 16-bit start and end register	16-bit BK
Wait state generator	PDWSR	SWWSR
Host port interface	8-bit standard HPI	8-bit standard HPI or enhanced 8-bit and 16-bit HPI
COMMON TO TMS320C5x AND TMS320C54x		
Auxiliary registers	AR0-AR7	
Status registers	16-bit PMST, ST0, ST1	
Program counter	16-bit PC	
Interrupt registers	16-bit IMR and IFR	
General purpose I/O	BIO and XF	
Hardware timer	16-bit timer	
Clock generator	PLL based	
Synchronous serial port	full duplex and double buffered	
TDM serial ports	upto 7 devices using TDM can communicate serially	
Buffered serial port	standard 5x serial port with additional autobuffering unit	
AVAILABLE ONLY IN TMS320C54x		
Stack pointer (SP), 16-bit SP		
Extended prog memory, 7-bit XPC		
Multichannel buffered serial port		
Internal programmable clock		
On-chip ROM for look up table		

Table 11.6 : Memory Mapped Registers

ADDRESS		NAME	DESCRIPTION
DEC	HEX		
0–3	0–3	–	Reserved
4	4	IMR	Interrupt mask register
5	5	GREG	Global memory allocation register
6	6	IFR	Interrupt flag register
7	7	PMST	Processor mode status register
8	8	RPTC	Repeat counter register
9	9	BRCR	Block repeat counter register
10	A	PASR	Block repeat program address start register
11	B	PAER	Block repeat program address end register
12	C	TREG0	Temporary register 0 (used for multiplicand)
13	D	TREG1	Temporary register 1 (used for dynamic shift count)
14	E	TREG2	Temporary register 2 (used as bit pointer in dynamic bit test)
15	F	DBMR	Dynamic bit manipulation register
16–23	10–17	AR0–AR7	Auxiliary register 0 to auxiliary register 7
24	18	INDX	Index register
25	19	ARCR	Auxiliary register compare register
26	1A	CBSR1	Circular buffer 1 start register
27	1B	CBER1	Circular buffer 1 end register
28	1C	CBSR2	Circular buffer 2 start register
29	1D	CBER2	Circular buffer 2 end register
30	1E	CBCR	Circular buffer control register
31	1F	BMAR	Block move address register
32–35	20–23	–	Memory-mapped serial port registers
36–42	24–2A	–	Memory-mapped peripheral registers
43–47	2B–2F	–	Reserved for test/emulation
48–55	30–37	–	Memory-mapped serial port registers
56–79	38–4F	–	Reserved
80–95	50–5F	–	Memory-mapped I/O ports

Table 11. : Conditions for Branch, Call and Return Instructions

Code for Cond	Condition	Description
AEQ	A = 0	Accumulator A equal to 0
BEQ	B = 0	Accumulator B equal to 0
ANEQ	A ¹ 0	Accumulator A not equal to 0
BNEQ	B ¹ 0	Accumulator B not equal to 0
ALT	A < 0	Accumulator A less than 0
BLT	B < 0	Accumulator B less than 0
ALEQ	A £ 0	Accumulator A less than or equal to 0
BLEQ	B £ 0	Accumulator B less than or equal to 0
AGT	A > 0	Accumulator A greater than 0
BGT	B > 0	Accumulator B greater than 0
AGEQ	A ³ 0	Accumulator A greater than or equal to 0
BGEQ	B ³ 0	Accumulator B greater than or equal to 0
AOV	AOV = 1	Accumulator A overflow detected
BOV	BOV = 1	Accumulator B overflow detected
ANOV	AOV = 0	No accumulator A overflow detected
BNOV	BOV = 0	No accumulator B overflow detected
C	C = 1	ALU carry set to 1
NC	C = 0	ALU carry clear to 0
TC	TC = 1	Test/Control flag set to 1
NTC	TC = 0	Test/Control flag cleared to 0
BIO	$\overline{\text{BIO}}$ low	$\overline{\text{BIO}}$ signal is low
NBIO	$\overline{\text{BIO}}$ high	$\overline{\text{BIO}}$ signal is high
UNC	none	Unconditional operation

Table 11. : CPU Memory-Mapped Registers of TMS32054x processors

Dec	Address Hex	Name	Description
0	0	IMR	Interrupt mask register
1	1	IFR	Interrupt flag register
2–5	2–5	–	Reserved for testing
6	6	ST0	Status register 0
7	7	ST1	Status register 1
8	8	AL	Accumulator A low word (bits 15–0)
9	9	AH	Accumulator A high word (bits 31–16)
10	A	AG	Accumulator A guard bits (bits 39–32)
11	B	BL	Accumulator B low word (bits 15–0)
12	C	BH	Accumulator B high word (bits 31–16)
13	D	BG	Accumulator B guard bits (bits 39–32)
14	E	T	Temporary register
15	F	TRN	Transition register
16–23	10–17	AR0	Auxiliary register 0 – Auxiliary register 7
24	18	SP	Stack pointer
25	19	BK	Circular-buffer size register
26	1A	BRC	Block-repeat counter
27	1B	RSA	Block-repeat start address
28	1C	REA	Block-repeat end address
29	1D	PMST	Processor mode status register
30	1E	XPC	Program counter extension register
31	1F	–	Reserved

Table 11. : Assembler Directives Summary

MNEMONIC AND SYNTAX	DESCRIPTION
.data	Assemble into the .data (initialized data) section
.text	Assemble into the .text (executable code) section
.int *value*$_1$ *[, ... , value*$_n$*]*	Initialize one or more 16-bit integers
.word *value*$_1$ *[, ... , value*$_n$*]*	Initialize one or more 16-bit integers
.align	Align the SPC on a page boundary
.even	Align the SPC on an even word boundary
.endloop	End .loop code block
.loop *[well-defined expression]*	Begin repeatable assembly of a code block
.equ	Equate a value with a symbol
.set	Equate a value with a symbol
.end	End program
.mmregs	Enter memory-mapped registers into symbol table
.ps *xxxx*	Assemble into program memory address xxxx
.entry	Initialize the starting of PC

Addressing Modes of TMS320C54x Processors

The addressing mode refer to the method of specifying the operand or the data to be operated by the instruction. The TMS320C54x processors supports the following seven addressing modes.

1. Immediate addressing
2. Absolute addressing
3. Accumulate addressing
4. Direct addressing
5. Indirect addressing
6. Memory-mapped register addressing
7. Stack addressing

Immediate Addressing

In immediate addressing the data is specified as a part of the instruction. In this addressing, the instruction will carry a 3-bit/5-bit/8-bit/9-bit/16-bit constant, which is the data to be operated by the instruction. The immediate constant is specified with # symbol. In the instructions listed in table 11.21 the syntax used for immediate addressing are # k3, # k5, # k9, # k and # *l*k.

Example :

LD # 1Ch, ASM ; Load the immediate 5-bit constant (1Ch) in ASM field of status register 1

Barrel Shifter

The 40-bit barrel shifter can perform 0 to 31 bits left shift, 0 to 16 bits right shift and along with exponent encoder can normalize the accumulator content. The shift informations are specified in the shift count field of the instruction, the shift count field of status register 1 or in T-register. The shift and normalize operations of barrel shifter can be used to realize the following operations.

- Prescaling of the memory/accumulator operand before an ALU operation.
- Logical or arithmetic shifting of accumulator value.
- Normalizing the accumulator.
- Postscaling the accumulator before storing in memory

The 40-bit shifter can handle 16-bit/32-bit/40-bit operands which are input from data buses (DB and CB buses) or from accumulators. The output of shifter can be loaded in ALU or EB bus.

Multiplier/Adder Unit

The multiplier/adder unit consists of 17 ´ 17-bit multiplier, 40-bit adder, signed/unsigned input control logic, fractional control logic, zero detector, rounder, overflow/saturation logic and T-register. One of the input for multiplier can be supplied from T-register/data-memory/accumulator, and the other input can be supplied from data-memory/program-memory/accumulator.

The multiplier/adder unit can perform 17 ´ 17-bit two's complement multiplication and 40-bit addition in parallel in a single instruction cycle. In addition, the multiplier and ALU together can perform MAC operation and an ALU operation in parallel in a single instruction cycle. These parallel operations can be used for efficient impelmentation of DSP computations like convolution, correlation and filtering.

Compare, Select and Store Unit (CSSU)

The CSSU is an application specific hardware unit dedicated to perform add/compare/select operations in order to support various viterbi butterfly algorithms used in equalizers and channel decoders.

The inputs to CSSU for comparision are from accumulator and the output is stored in data memory. The status of comparision is also stored in LSB of TRN register and TC bit of status register 0.

The instruction "CMPS *src, SMEM*", use the CSSU to compare the low and high word of specified source accumulator, to select the largest of the two words and store in specified data memory. If high accumulator is greater, then 0 is stored in LSB of TRN and TC, or if low accumulator is greater, then 1 is stored in LSB of TRN and TC.

Exponent Encoder

For implementation of floating point arithmetic in fixed point processors like TMS320C54x, require separation of exponent and mantissa of the floating point data.

The exponent encoder is an application-specific hardware device dedicated to extract the exponent value from floating point data in the accumulators and store in T-register.

The "EXP *src*" instruction is used to extract the exponent and save in T-register. The "NORM *src, dst*" instruction is used to normalize the accumulator using the exponent in T-register as count value.

Data Address Generation Unit

The data address generation units consists of two numbers of **A**uxiliary **R**egister **A**rithmetic **U**nits (ARAU0, ARAU1), eight numbers of **A**uxiliary **R**egisters (AR0-AR7), a 16-bit circular buffer size register (BK) and a 16-bit **S**tack **P**ointer (SP).

The auxiliary registers are used to hold the data-memory address in indirect addressing mode. The 3-bit ARP (**A**uxiliary **R**egister **P**ointer) field of status register 0 indicates the current AR used for indirect addressing. The auxiliary register-0 is also used as an index register for modifying the content of other auxiliary registers.

The ARAU perform arithmetic operations related to address generation for indirect addressing mode like increment, decrement, indexing, bit reversed address generation and circular address generation. The two independent ARAUs at any time can operate on two ARs to generate two data-memory address simultaneously.

The 9-bit DP (**D**ata-page **P**ointer) of status register-0 is used as upper 9-bits of data-memory address (page address) in direct addressing. The circular buffer register is loaded with circular buffer size which is used to generate the start and end address of circular memory along with AR specified in the instruction. The stack pointer is used to implement the **LIFO** stack for memory operands that uses stack addressing. The stack pointer always holds the address of top of stack.

Program Address Generation Unit

The program address generation unit consists of five registers namely, **P**rogram **C**ounter (PC), **R**epeat **C**ounter (RC), **B**lock-**R**epeat **C**ounter (BRC), Block-**R**epeat **S**tart **A**ddress register (RSA) and Block-**R**epeat **E**nd **A**ddress register (REA). Some versions of TMS320C54x processors has an additional register called program counter extension register (XPC) to support addressing of virtual memory.

The PC is a 16-bit register which holds the address of the program code. An instruction is fetched from program memory by loading the content of PC (address) on the program address bus (PAB) and then reading the code from program bus (PB). When the memory is read, the PC is incremented for the next fetch, so that when an instruction word is read, the PC holds the address of next word of same instruction or the next instruction. The XPC is a 7-bit register that selects the extended page of program memory in the processors that supports virtual addrsssing.

When the execution of a single instruction has to be repeated a number of times the 16-bit RC register is used to hold the count value and when a block of instruction has to be repeated the BRC is used to hold the count value. The registers RSA and REA are used to hold the start and end address of the block to be repeated respectively.

11.4.4 On-chip Memory in TMS320C54x Processors

The TMS320C54x family of processors consists of three different types of on-chip memory and they are mask-programmable ROM, **S**ingle-**A**ccess **RAM** (SARAM) and **D**ual-**A**ccess **RAM** (DARAM). The various members of TMS320C54x will have different capacity of on-chip memory which are listed in table 11.21.

Table 11.21 : On-chip Memory in TMS320C54x Processors

Memory Type		TMS320C54x Family of Processors						
		C541	C542	C543	C545	C546	C548	C549
ROM	Program ROM (PROM)	20k	2k	2k	32k	32k	2k	16k
	Program/Data ROM	8k	–	–	16k	16k	–	16k
RAM	DARAM	5k	10k	10k	6k	6k	8k	8k
	SARAM	–	–	–	–	–	24k	24k

On-chip ROM

The various models of TMS320C54x processors has internal maskable ROM of size 2k to 48k words. In majority of the processors the on-chip ROM is mapped to program memory space and in some processors a part of ROM can be mapped to data-memory space. The processor has an option for including or excluding the on-chip ROM addresses in the processor program memory address space.

The main purpose of internal ROM is to permanently store the program code and data for a specific application during manufacturing of the chip itself. The processor has an option of boot loading the content of on-chip ROM to internal/external RAM during power-ON reset. The content of the on-chip ROM can be protected so that any external device cannot have access to the program code. This feature provide security for proprietary algorithms.

On-chip DARAM

The TMS320C54x family of processors has 5k to 10k words of on-chip DARAM which are organized into blocks as shown below.

- TMS320C541 : 5k words organized as 5 blocks of 1k words each
- TMS320C542/543 : 10k words organized as 5 blocks of 2k words each
- TMS320C545/546 : 6k words organized as 3 blocks of 2k words each
- TMS320C548/549 : 8k words organized as 4 blocks of 2k words each

The DARAM blocks can be accessed twice per machine cycle. Upon reset the DARAM is mapped to data memory address space and after reset the processor has provision to map the DARAM into program memory space.

On-chip SARAM

The TMS320C548/549 processors has 24k words of on-chip SARAM which are organized as three blocks of 8k words. Upon reset the SARAM is mapped to data memory space and after reset the processor has provision to map the SARAM into program memory space.

11.4.5 On-chip Peripherals of TMS320C54x Processors

The various on-chip peripherals available in TMS320C54x family of processors are,

- Software-programmable wait-state generator
- Programmable bank switching
- Parallel IO ports

- DMA controller
- Host Port Interface (HPI)
- Serial ports (Standard, TDM, BSP and McBSP)
- General purpose IO pins
- Timer
- Clock generator and Phase Locked Loop (PLL)

Software-programmable wait-state generator

The software-programmable wait-state generator can insert/generate wait-states in external bus cycles for interfacing with slow speed external memory and IO devices. The wait-state generator can extend the external bus cycles up to seven machine cycles. When all external accesses are configured to zero wait states, the internal clock to the wait-state generator is shut off to reduce power consumption.

Programmable Bank Switching

The programmable bank-switching logic can be used to insert one cycle automatically when the memory data access switches from data memory space to program memory space or vice-versa. This extra cycle help the memory to release the bus before the other memory start driving the bus, thereby avoiding bus contention.

Parallel IO ports

The TMS320C54x family of processors has 64k IO address space which can be used as 64k IO ports. The IO ports can be addressed by the PORTR and PORTW instruction for data transfer between ports and data memory. The processor generates a signal $\overline{\text{IS}}$ during IO access to indicate a port read or port write operation. The processor can be easily interfaced to external IO devices through IO ports with minimal external address decoding circuits.

DMA (Direct Memory Access) Controller

The internal DMA controller in TMS320C54x processors can perform data transfer between various internal and external memory spaces without the intervention of CPU. The DMA has six independent programmable channels, allowing six different contexts for DMA operation. The DMA has higher priority than the CPU for both internal and external accesses. The DMA can perform single word or double word transfers. The DMA transfer from/to external to internal memory require 5 cycles.

Host Port Interface (HPI)

The HPI is an 8-bit parallel port that provides an interface to a host processor for information exchange between the Digital Signal Processor (DSP) and the host processor. The information exchange takes place via on-chip memory that is accessible to both DSP and host. The TMS320C54x family of processor has 2k words of internal DARAM mapped in data memory space 1000h to 17FFh as HPI memory.

Serial Ports

The TMS320C54x processors has the following four types of serial ports.

- Synchronous serial port
- **T**ime **D**ivision **M**ultiplexed (TDM) serial port

- Buffered serial port
- **M**ulti**c**hannel **B**uffered **S**erial **P**ort (McBSP)

The synchronous serial ports are high-speed, full-duplexed serial ports that provide direct communication with serial devices such as codecs, serial ADC, etc.,. These ports can operate up to one-fourth the machine cycle rate. The transmitter and receiver are double buffered and data is framed either as bytes or as words.

The TDM serial port employ the time-division multiplexing technique for serial communication to multiple devices having TDM ports. The time-division multiplexing is the process of dividing the time intervals into number of subintervals with each subinterval representing a communication channel. One TMS320C54x processor can communicate with up to seven devices/processors with TDM serial ports via a pair of data lines and a pair of address lines. Like synchronous serial port, the TDM port is also double-buffered for both transmit and receive data.

The buffered serial port consists of a full-duplex double-buffered serial-port interface and an auto-buffering unit. The processor internal memory is connected to auto-buffering unit by a dedicated bus, so that the buffered serial port can directly read/write to processor internal memory without the intervention of CPU. This results in minimal overhead for serial port transactions and faster data rates.

The multichannel buffered serial port (McBSP) is an enhanced buffered serial port that can support multichannel transmit and receive up to 128 channels. The advanced features of McBSP are wide data sizes from 8-bit to 32-bit, m-law and A-law companding and programmable internal clock and frame synchronization.

General Purpose IO Pins

The TMS320C54x family of processors has two general purpose IO pins and they are branch control input pin, $\overline{\text{BIO}}$ and external flag output pin, XF.

The $\overline{\text{BIO}}$ pin can be used to monitor the satus of peripheral devices. A branch instruction can be conditionally executed depending upon the state of the $\overline{\text{BIO}}$ input. The $\overline{\text{BIO}}$ pin is an alternative to interrupt, when the interrupts are dedicated to time-critical applications.

The XF pin can be used to signal external devices. The XF pin is controlled using software. At reset the XF pin is set high. The SSBX instruction is used to set XF pin and RSBX instruction is used to reset XF pin.

Timer

The on-chip timer in TMS320C54x processors is a 16-bit timer with a 4-bit prescaler. The timer can be used to initiate any time based event through interrupt. The timer has a count register, which is loaded with a count value and at every clock cycle the timer count is decremented by 1. At the end of the count an interrupt is generated. The timer has a control register to control its operations like start, stop, restart and disable.

Clock Generator and PLL (Phase Locked Loop)

There are two methods of clock generation in TMS320C54x processors. In one method the internal oscillator connected to an external crystal is used to generate a clock at crystal frequency and then divided by 1, 2, or 4 and used for CPU.

In another method, a low frequency external clock is supplied to an internal PLL circuit. The CPU clock is generated by PLL circuit at multiple frequency of external clock. This method reduces system power consumption and clock-generated EMI and facilitate the use of low cost crystal.

11.3.9 Assembly Language Programs in TMS320C5x

The assembly language programs for TMS320C5x processors are written using the mnemonics listed in table 11.11. The processor can execute only machine language programs and the conversion from assembly language to machine language is performed by a software tool called assembler.

The texas instruments has released a number of assembly language program development tools for their digital signal processors. Some of the tools are assembler, linker, absolute lister, hex-converter and loader.

The assembler is a software tool that can run on any standard PC (**P**ersonal **C**omputer) and permits to type, edit and convert the assembly language program to machine language object file called COFF (**C**ommon **O**bject **F**ile **F**ormat) file.The process of conversion from assembly language program to machine language program is called assembling. The assembly language program can be developed in modules and each module can be assembled separately to generate COFF files and they can be combined to single executable object file using linker.

The absolute lister will map the executable object file of the program to specific memory location of the system. The loader is used to download the executable object file into processor RAM for execution by the processor. When the program has to be permanently stored inROM/EPROM, the object files has to be converted to standard hex files. The hex-converter can be used to convert executable object file to standard hex-file. Using any standard EPROM programmer the hex-file can be loaded in EPROM for execution by the processor.

Assembly Language Program Statement Format

The assembly program for TMS320C5x processor consists of statements that contain labels, assembler directives, instructions, macro directives and comments. Each statement can have a maximum of 200 characters. A statement may have four fields namely label, mnemonic, operands and comment. The general format of the statement is shown below.

[label] [:] mnemonic [operands] [;comment]

The guidelines for writing statements are given below.

- A statement must begin with a label, a blank, an asterist or a semicolon.
- A label should begin in first column and the label is optional.
- Comment should begin with semicolon and the comment is optional.
- Each field should be separated by one or more blanks.

Constants

The decimal, binary or hexadecimal number used to represent the data or address in assembly language program statement are called constants or numerical constants. When constants are used to represent the address/data then their values are fixed and cannot be changed while running a program. The binary and hexadecimal constants can be differentiated by placing a specific alphabet at the end of the constant.

A valid binary constant/number is framed using numeric characters 0 and 1, and the alphabet b is placed at the end.

A valid decimal (BCD) constant/number is framed using numeric characters 0 to 9, and no alphabet is placed at the end.

A valid hexadecimal constant/number is framed using numeric characters 0 to 9 and alphabets a to f, and the alphabet h is placed at the end. A zero should be placed/inserted at the beginning of hexadecimal number if the first digit is an alphabet character from a to f, otherwise the assembler will consider the constant starting with a to f as a symbol.

Examples of valid constant

1011 - Decimal (BCD) constant

1101b - Binary constant

92ach - Hexadecimal constant

0e2h - Hexadecimal constant

Symbols

The symbols are variables (or terms) used in assembly language program statements in order to represent the variable data and address. While running a program, a value has to be attached to each symbol in the program. The advantage of using symbol is that the value of the symbol can be dynamically varied while running the program.

Usually a symbol name is constructed such that it reflect the meaning of the value it holds. A variable name selected to represent the temperature of a device can be TEMP, a variable name selected to represent the speed of a motor can be M_SPEED, etc.

The guidelines for framing the symbols are given below:

- The symbol name can be constructed using A to Z, a to z, 0 to 9, $, _ (underscore).
- A number cannot be the first character in the symbol name.
- The maximum length of a symbol name is 32 characters.
- The symbol name is case sensitive.

Assembler Directives

The assembler directives are the instructions to the assembler regarding the program being assembled. They are also called pseudo instructions. The assembler directives are used to specify start and end of a program, attach value to variables, allocate storage locations to input/output data, to define start and end of segments, procedures, macros, etc.

The assembler directives control the generation of machine code and organization of the program. But no machine codes are generated for assembler directives. Some of the assembler directives that can be used for TMS320C5x assembly language program development are listed in table-11.14.

PROGRAM 11.1

Write an assembly language program using instructions of TMS320C5x processors to add two numbers of 64-bit data. Assume that the two data are available in memory. Store the sum in memory.

Problem Analysis

The memory word size of TMS320C5x processor is 16-bit and so each 64-bit data is stored as 4 words ($4 \times 16 = 64$). Let us use direct addressing. Let the address of 4 words of data-1 be named as AD1W1, AD1W2, AD1W3 and AD1W4. Let the address of 4 words of data-2 be named as AD2W1, AD2W2, AD2W3 and AD2W4. Let the address of 4 words of sum be named as ASW1, ASW2, ASW3 and ASW4.

Let us load lower two words of data-1 in 32-bit accumulator and then the word-1 of data-2 is added to low accumulator and the word-2 of data-2 is added to high accumulator. The 32-bit sum in accumulator is stored in memory. The addition of the next two words are performed in similar manner by considering the carry in previous addition.

Assembly language program

```
;PROGRAM TO ADD TWO NUMBERS OF 64-BIT DATA
AD1W1  .set    00h      ;Offset address for data-1
AD1W2  .set    01h
AD1W3  .set    02h
AD1W4  .set    03h
AD2W1  .set    04h      ;Offset address for data-2
AD2W2  .set    05h
AD2W3  .set    06h
AD2W4  .set    07h
ASW1   .set    08h      ;Offset address for sum
ASW2   .set    09h
ASW3   .set    0ah
ASW4   .set    0bh
       .mmregs          ;Include memory-mapped registers
       .ps     0b00h    ;Origin of program address is 0b00h
       .entry           ;Initialize program counter with starting address of program
ADD64: CLRC    SXM      ;Clear sign extension mode bit
       LACC    AD1W2,16 ;Load word-2 of data-1 in high accumulator
       ADDS    AD1W1    ;Load word-1 of data-1 with sign extension suppressed in low accumulator
       ADDS    AD2W1    ;Add word-1 of data-2 to low accumulator
       ADD     AD2W2,16 ;Add word-2 of data-2 to high accumulator
       SACL    ASW1     ;Store word-1 of sum in memory
       SACH    ASW2     ;Store word-2 of sum in memory
       LACC    AD1W4,16 ;Load word-4 of data-1 in high accumulator
       ADDC    AD1W3    ;Add word-3 of data-1 and carry to accumulator
       ADDS    AD2W3    ;Add word-3 of data-2 with sign extension suppressed to low accumulator
       ADD     AD2W4,16 ;Add word-4 of data-2 to high accumulator
       SACL    ASW3     ;Store word-3 of sum in memory
       SACH    ASW4     ;Store word-4 of sum in memory
       RET              ;Program end
       .end             ;Assembly end
```

PROGRAM 11.2

Write an assembly language program using instructions of TMS320C5x processors to subtract two numbers of 64-bit data. Assume that the two data are available in memory. Store the result in memory.

Problem Analysis

The memory word size of TMS320C5x processor is 16-bit and so each 64-bit data is stored as 4 words ($4 \times 16 = 64$). Let 4 words of data-1 be stored in memory at address 1101h to 1104h. Let 4 words of data-2 be stored in memory at address 1111h to 1114h. Let the 4 words of result be stored in memory at address 1121h to 1124h. Let us use indirect address using auxiliary registers.

Let us load lower two words of data-1 in 32-bit accumulator and then the word-1 of data-2 is subtracted from low accumulator and the word-2 of data-2 is subtracted from high accumulator. The 32-bit result in accumulator is stored in memory. The subtraction of next two words are performed in similar manner by considering the borrow in previous subtraction.

Assembly language program

```
;PROGRAM TO SUBTRACT TWO NUMBERS OF 64-BIT DATA
        .mmregs            ;Include memory-mapped registers
        .ps         0C00h  ;Origin of program address is 0C00h
        .entry             ;Initialize program counter with starting address of program
Ini_ARs: LAR AR1,    #1101h ;Load starting address of data-1 in AR1
        LAR AR2,    #1111h ;Load starting address of data-2 in AR2
        LAR AR3,    #1121h ;Load starting address of sum in AR3
SUB64:  CLRC SXM           ;Clear sign extension mode bit
        LACC *+, 0, AR1    ;Load word-1 of data-1 in low accumulator
        ADD *+, 16, AR1    ;Load word-2 of data-1 in high accumulator
        SUBS *+, 0, AR2    ;Subtract word-1 of data-2 with sign extension suppressed from low
                           ;accumulator
        SUB *+, 16, AR2    ;Subtract word-2 of data-2 from high accumulator
        SACL *+, 0, AR3    ;Store word-1 of result in memory
        SACH *+, 0, AR3    ;Store word-2 of result in memory
        LACC *+, 0, AR1    ;Load word-3 of data-1 in low accumulator
        SUBB *+, 0, AR2    ;Subtract word-3 of data-2 and previous borrow from low accumulator
        ADD  *+,16, AR1    ;Load word-4 of data-1 in low accumulator
        SUB  *+,16, AR2    ;Subtract word-4 of data-2 from high accumulator
        SACL *+, 0, AR3    ;Store word-3 of result in memory
        SACH *+, 0, AR3    ;Store word-4 of result in memory
        RET                ;Program end
        .end               ;Assembly end
```

PROGRAM 11.3

Write an assembly language program using instructions of TMS320C5x processors to multiply two numbers of unsigned 32-bit data. Assume that the two data are available in memory. Save the 64-bit product in memory.

Problem Analysis

In TMS320C5x processor the 32-bit multiplication can be implemented in terms of 16-bit multiplication. The given 32-bit data can be divided into two words (16-bit data) as shown below.

Data-1 (D1) ® Data-1 word-2 (D1W2), Data-1 word-1 (D1W1)

Data-2 (D2) ® Data-2 word-2 (D2W2), Data-2 word-1 (D2W1)

Using the above four words (D1W1, D1W2, D2W1, D2W2) the following four products are computed. Each product will generate a 32-bit result and so they are divided into two words (16-bit) as shown below.

Product-1 (P1) : D1W1 ´ D2W1 = P1 ® P1W2, P1W1

Product-2 (P2) : D1W2 ´ D2W1 = P2 ® P2W2, P2W1

Product-3 (P3) : D1W1 ´ D2W2 = P3 ® P3W2, P3W1

Product-4 (P4) : D1W2 ´ D2W2 = P4 ® P4W2, P4W1

The results of the above four products can be added to get the final result as shown below. The maximum size of **F**inal **P**roduct (FP) will be 64-bit and so it is divided into four words namely FPW1, FPW2, FPW3 and FPW4.

			D1W2	D1W1
		×	D2W2	D2W1
			P1W2	P1W1
	P2W2		P2W1	
	P3W2		P3W1	
P4W2	P4W1			
FPW4	FPW3		FPW2	FPW1

Let us use direct addressing for data and product. The address of data words and final product words are named as AD1W1, AD1W2, AD2W1, AD2W2, AFPW1, AFPW2, AFPW3, AFPW4.

The products are performed one-by-one by loading one of the data word in T-register-0 (TREG0) and multiplying with a memory data word to get the 32-bit product in P-register. The results of the product are added as shown below to get the final product words.

FPW1 = P1W1

FPW2 = P1W2 + P2W1 + P3W1

FPW3 = P2W2 + P3W2 + P4W1

FPW4 = P4W2

Assembly language program

```
;PROGRAM TO MULTIPLY TWO NUMBERS OF 32-BIT DATA

AD1W1    .set     10h    ;Offset address for data-1
AD1W2    .set     11h

AD2W1    .set     12h    ;Offset address for data-2
AD2W2    .set     13h

AFPW1    .set     14h    ;Offset address for product
AFPW2    .set     15h
AFPW3    .set     16h
AFPW4    .set     17h
         .mmregs         ;Include memory-mapped registers
         .ps      od00h  ;Origin of program address is 0d00h
         .entry          ;Initialize program counter with starting address of program
MUL32:   CLRC     SXM    ;Clear sign extension mode bit
         LT       AD2W1  ;Load D2W1 in TREG0
         MPYU     AD1W1  ;Multiply D1W1 and D2W1 to get product-1 (P1) in P-register
         SPL      AFPW1  ;Store FPW1 in memory
         SPH      AFPW2  ;Store partial FPW2 in memory
         MPYU     AD1W2  ;Multiply D1W2 and D2W1 to get product-2 (P2) in P-register
         LTP      AD2W2  ;Save P2 in accumulator. Load D2W2 in TREG0
         MPYU     AD1W1  ;Multiply D1W1 and D2W2 to get product-3 (P3) in P-register
         MPYA     AD1W2  ;Get sum of P2 and P3 in accumulator. Multiply D1W2 and D2W2 to get
                         ;product-4 (P4) in P-register
         ADDS     AFPW2  ;Add partial FPW2 in memory to low accumulator to get FPW2 in low
                         ;accumulator
         SACL AFPW2      ;Save FPW2 in memory
         BSAR 16         ;Shift accumulator right by 16
         APAC            ;Add P4 in P-register to accumulator to get FPW3 and FPW4 in low and
                         ;high accumulator respectively
         SACL     AFPW3  ;Save FPW3 in memory
         SACH     AFPW4  ;Save FPW4 in memory
         RET             ;Program end
         .end            ;Assembly end
```

PROGRAM 11.4

Write an assembly language program using instructions of TMS320C5x processors to divide a 16-bit data by an 8-bit data. Assume that the data are 2's complement positive integers available in memory. Store the quotient and remainder in memory.

Problem Analysis

The TMS320C5x processors has a special instruction "SUBC dma" to implement division algorithms. For 16-bit by 8-bit division of positive integers, the 16-bit dividend is stored in low accumulator and the high accumulator is filled with zero. The 8-bit divisor is stored in data memory as lower 8-bits and upper eight bits are filled with zero. Then the "SUBC dma" instruction is executed 16 times to generate the quotient in low accumulator and remainder in high accumulator.

> *Note : The SUBC instruction will left shift the divisor by 15-bits and subtract from dividend and store the talumuccaotnilnuluacer revitisop si 1 dna 2 yb deilpitlum si rotalumucca esle ,dedda s s i 1 dd(2 ybdeilpitlumyls .)dd tons*

Let us use direct addressing. Let the address of dividend and divisor be ADIVD and ADIVS. Let the address of quotient and remainder be AQUO and AREM.

Assembly language program

```
;PROGRAM FOR 16-BIT BY 8-BIT DIVISION OF POSITIVE DATA

ADIVD    .set    30h      ;Offset address of dividend
ADIVS    .set    31h      ;Offset address of divisor

AQUO     .set    32h      ;Offset address of quotient
AREM     .set    33h      ;Offset address of remainder

         .mmregs          ;Include memory-mapped registers
         .ps     0e00h    ;Origin of program address is 0e00h
         .entry           ;Initialize program counter with starting address of program

DIV16_8: CLRC    SXM      ;Clear sign extension mode bit
         LACC    ADIVD    ;Load dividend in low accumulator with high accumulator as zero
         RPT     #15      ;Repeat the next instruction 16 times. Count is 1 less than number of
                          ;repeatitions.
         SUBC    ADIVS    ;Perform conditional subtraction of division algorithm
         SACL    AQUO     ;Store the quotient in memory
         SACH    AREM     ;Store the remainder in memory
         RET              ;Program end
         .end             ;Assembly end
```

PROGRAM 11.5

Write an assembly language program using instructions of TMS320C5x processor to find the sum of an array stored in memory. Assume that the array has 10_{10} data each of size 16 bits and store the sum in memory.

Problem Analysis

The array of 10 data can be stored in memory with starting address named as "Arr_addr". The 32-bit accumulator is cleared to have an initial sum as zero. Then the data of the array can be added one by one to accumulator. The final sum in the 32-bit accumulator is stored as two words in memory with address named as Asum_W1 and Asum_W2.

Assembly language program

```
;PROGRAM FOR SUM OF AN ARRAY

Arr_addr .set       1200h    ;Data array starting address is 1200h
Cnt_val  .set       09h      ;Count value is one less than number of data in the array
Asum_W1  .set       1300h    ;Address of sum
Asum_W2  .set       1301h
         .mmregs             ;Include memory mapped registers
         .ps        0f00h    ;Origin of program address is 0f00h
         .entry              ;Initialize program counter with starting address of program
Sum_arr  CLRC       SXM      ;Clear sign extension mode bit
         LAR AR0,   Cnt_val  ;Load count value in AR0
         LAR AR1,   Arr_addr ;Load starting address of array in AR1
         ZAP                 ;Clear accumulator
         MAR *, AR1          ;Make ARP to point AR1 as current AR
AGAIN:   ADDS *+, 0, AR0     ;Add content of AR1 (without shift and sign extension suppressed)
                             ;to accumulator, increment AR1 and make ARP to point AR0 as next AR
         BANZ AGAIN, AR1     ;If AR0 is not zero, then decrement AR0 by 1, branch to program
                             ;memory address AGAIN, make ARP to point AR1 as next AR. If AR0 is
                             ;zero then go to next instruction
         SACL       Asum_W1  ;Store word-1 of sum in memory
         SCAH       Asum_W2  ;Store word-2 of sum in memory
         RET                 ;Program end
         .end                ;Assembly end
```

PROGRAM 11.6

The input x(n) and impulse response h(n) of an LTI system is given by,

$x(n) = \{2, 10, 13, 5\}$

$h(n) = \{7, 14, 4\}$

The response y(n) of the LTI system is given by,

$y(n) = x(n) * h(n)$; where "*" symbol for convolution.

Write an assembly language program using instructions of TMS320C5x processors to determine the response of LTI system via convolution of input and impulse response.

Problem Analysis

The x(n) is 4-point sequence and h(n) is 3-point sequence. The convolution of x(n) and h(n) will produce a sequence of size 4 + 3 – 1 = 6-point sequence. Let us append zero to x(n) and h(n) to convert them to 6-point sequences as shown below and perform linear convolution via circular convolution.

$\therefore\ x(n) = \{2, 10, 13, 5, 0, 0\}$

$h(n) = \{7, 14, 4, 0, 0, 0\}$

The circular convolution of x(n) and h(n) is defined as,

$$y(n) = \sum_{m=0}^{N-1} x(m)\ h((n-m))_N$$

When N = 6,

$$y(n) = \sum_{m=0}^{5} x(m)\ h((n-m))_6$$

$$= x(0)\ h((n-0))_6 + x(1)\ h((n-1))_6 + x(2)\ h((n-2))_6 + x(3)\ h((n-3))_6$$
$$+ x(4)\ h((n-4))_6 + x(5)\ h((n-5))_6$$

When n = 0 ; $y(0) = x(0)\,h(0) + x(1)\,h((-1))_6 + x(2)\,h((-2))_6 + x(3)\,h((-3))_6 + x(4)\,h((-4))_6 + x(5)\,h((-5))_6$

$\backslash\ \ y(0) = x(0)\,h(0) + x(1)\,h(6-1) + x(2)\,h(6-2) + x(3)\,h(6-3) + x(4)\,h(6-4) + x(5)\,h(6-5)$

$\backslash\ \ y(0) = x(0)\,h(0) + x(1)\,h(5) + x(2)\,h(4) + x(3)\,h(3) + x(4)\,h(2) + x(5)\,h(1)$

Similarly,

When n = 1 ; $y(1) = x(0)\,h(1) + x(1)\,h(0) + x(2)\,h(5) + x(3)\,h(4) + x(4)\,h(3) + x(5)\,h(2)$

When n = 2 ; $y(2) = x(0)\,h(2) + x(1)\,h(1) + x(2)\,h(0) + x(3)\,h(5) + x(4)\,h(4) + x(5)\,h(3)$

When n = 3 ; $y(3) = x(0)\,h(3) + x(1)\,h(2) + x(2)\,h(1) + x(3)\,h(0) + x(4)\,h(5) + x(5)\,h(4)$

When n = 4 ; $y(4) = x(0)\,h(4) + x(1)\,h(3) + x(2)\,h(2) + x(3)\,h(1) + x(4)\,h(0) + x(5)\,h(5)$

When n = 5 ; $y(5) = x(0)\,h(5) + x(1)\,h(4) + x(2)\,h(3) + x(3)\,h(2) + x(4)\,h(1) + x(5)\,h(0)$

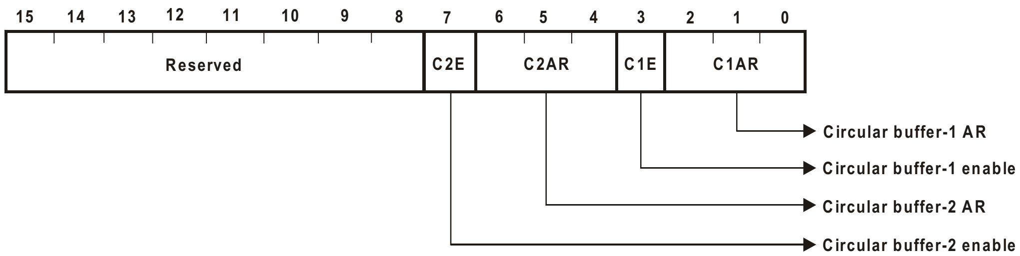
The computation of above equations involves multiplication and addition, and so can be computed in TMS320C5x processors using RPT and MAC instructions. Let us store input array in program memory and impulse array in data memory. The data memory used to store impulse array can be declared as a circular memory.

The repeat count register is loaded with a count value of 5 for executing MAC instruction 6 times, to compute one data of the output sequence. The output data is stored in memory and the loading of count value and executing MAC instruction are repeated 5 more times to compute the next 5 data of output sequence.

In this computation process the program memory address has to be incremented and circular data memory address has to be decremented.

In order to enable the circular memory and to specify the auxiliary register to be used for addressing circular memory, an appropriate word (called control word) should be loaded in CBCR (**C**ircular **B**uffer **C**ontrol **R**egister).

The format of CBCR is shown below.



Note : 1 = Enable, 0 = Disable.

Let us use circular buffer-1 and use AR0 to address buffer-1, and so the control word to be loaded in CBCR is 08h (0 000 1 000)

Assembly language program

```
;PROGRAM FOR CONVOLUTION
        .mmregs                       ;Include memory-mapped registers
        .ps             0c00h         ;Origin of program address is 0c00h
        .word 02h, 0Ah, 0dh           ;Store the input array in program
        .word 05h, 0h,  0h            ;Memory address starting from 0c00h
        .ds             1100h         ;Origin of data address is 1100h
        .word 07h, 0eh, 04h           ;Store the impulse array in data
        .word 0h,  0h,  0h            ;memory address starting from 1100h
        .entry                        ;Initialize program counter with starting address of program
CONV:   SPLK            #1100h, CBSR1 ;Load start address of circular buffer-1
        SPLK            #1107h, CBER1 ;Load end address of circular buffer-1
        SPLK            #08h          ;Enable circular buffer-1 .set AR0 as pointer for circular
                                      ;buffer-1
        LAR AR0,        #1100h        ;Initialize AR0 with start address of impulse array
        LAR AR1,        #1200h        ;Initialize AR1 with start address of output array
        LAR AR2,        #05h          ;Initialize AR2 as count for number of data in output
                                      ;array, count is 1 less than number of data
LOOP:   ZAP                           ;Clear accumulator and P-register
        MAR *,          AR0           ;Make ARP to point AR0 as current AR
        RPT             #05h          ;Repeat next instruction 6 times. Count is 1 less than
                                      ;number of repeatitions
        MAC 0c00h, *–,  AR0           ;Add P-register to accumulator, load a word of input array
                                      ;addressed by program memory in T-registers, multiply
                                      ;T-register and a word of impulse array in data memory
                                      ;addressed by AR0, increment program memory address, decrement
                                      ;data memory address
        MAR *,          AR1           ;Make ARP to point AR1 as current AR
        SACL *+,        AR2           ;Store one word of output array in memory, increment AR1,
                                      ;make ARP to point AR2 as current AR
        BANZ            LOOP          ;If AR2 is not zero, then decrement AR2 by 1, branch to
                                      ;program memory address LOOP. If AR2 is zero, then go to
                                      ;next instruction
        RET                           ;Program end
        .end                          ;Assembly end
```

Signal Processing

When we pass a signal through a system, as in filtering, for example, we say that we have processed the signal. In this case theprocessing of the signal involves filtering the noise and interference from the desired signal. In general, the system is characterized by the type of operation that it performs on the signal. For example, if the operation is linear, the system is called linear. If the operation on the signal is not-linear, the system is said to be nonlinear, and so forth. Such operations are usually referred to as signal processing.

Digital Signal Processing

Digital signal processing provides an alternative methos for processing the analog signal, as illustrated in fig 1.3. To perform the processing digitally, there is a need for an interface between the analog signal and the digital processor. This interface is called an analog-to-digital (A/D) converter. The output of the A/D converter is a digital signal that is appropriate as an input to the digital processor.

The digital signal processor may be a large programmable digital computer or a small microprocessor that is programmed to perform the desired operations on the input signal. It may also be a hardwired digital processor that is configured to perform a specified set fo operations on the input signal. Programmable machines provide the flexibility to change the signal processing operations through a change in the software, whereas hardwired machines are difficult to reconfigure. Consequently, programmable signal processors are very common in practice. On the other hand, when the signal processing operations are well defined, as in some applications, a hardwired implementation of the operations can be optimized so that it results in a cheaper signal processor and, usually, one that runs at a faster speed than its programmable counterpart. In applications where the digital output from the digital signal processor is to be given to the user in analog form, as in speech communications, for example, we must provide another interface from the digital domain to the analog domain. Such an interface si called a digital-to-analog (A/D) converter. Thus the signal is provided to the user in analog form.

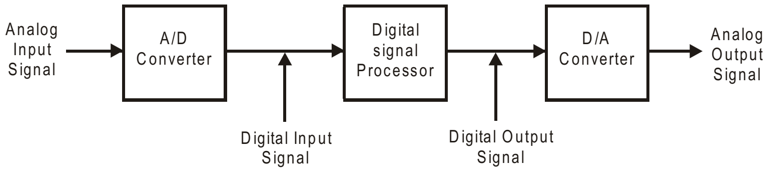


Fig 11. : Block diagram of a digital signal processing system.

Analog to Digital Conversion

A/D conversion as a three-step process. This process is illustrated in fig 11. .

1. Sampling: This is the conversion of a continuous-time signal into a discrete time signal obtained by taking "samples" of the continuous-time signal at discret-time instants. Thus, if x(t) is the input to the sampler, the output is x(nT) = x(n), where T is called the sampling interval.

2. Quantization: This is the conversion of a discrete-time continuous-valued signal into a discrete-time, discrete-valued (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample x(n) and the quantized output x(n) is called the quantization sample x(n) and the quantized output $x_q(n)$ is called the quantization error.

3. Coding: In the coding process, each discrete value x(n) is represented by a b-bit binary sequence.

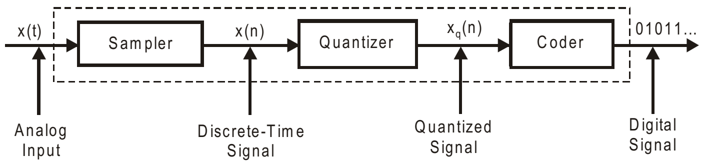


Fig 11. : Basic parts of an analog-to-digital (A/D) converter.

PROGRAM 11.10

Write an assembly language program using instructions of TMS320C54x processors to divide a 16-bit data by an 8-bit data. Assume that the data are 2's complement positive integers available in memory. Store the quotient and remainder in memory.

Problem Analysis

Let the 16-bit dividend and 8-bit divisor are stored in memory address 1100h and 1101h respectively. Let us store the quotient and remainder in memory address 1102h and 1103h. Let us are indirect addressing.

Note : Phase refer problem analysis of program 11.4 for logic of 16-bit by 8-bit division.

Assembly language program

```
;PROGRAM FOR 16-BIT BY 8-BIT DIVISION OF POSITIVE DATA
        .mmregs                 ;Include memory mapped registers.
        .text                   ;Assemble the instructions into default program memory.
DIV 16: STM     #1100H, AR0     ;Load starting address of data in AR0.
        RSBX    SXM             ;Clear sign extension mode bit.
        LD      *AR0+, A        ;Load dividend in low A with high A as zero.
        RPT     #15             ;Repeat the next instruction 16 times.
                                ;Count is 1 less than number of repeatitions.
        SUBC    *AR0, A         ;Perform conditional subtraction of division algorithm.
        MAR     *AR0+           ;Increment AR0 by 1.
        DST A,  *AR0            ;Store quotient and remainder in memory.
        RET                     ;Program end
        .data   1100h           ;Store data starting from data memory address 1100h.
        .word   009Ah, 0007h    ;Store dividend and divisor in memory.
        .word   0, 0            ;Initialize quotient and remainder as zero.
        .end                    ;Assembly end.
```

Chapter 12

Applications of DSP

12.1 Introduction

The advancement in Computer/Microprocessor hardware and software technology leads to complete dominance of digital signal processing in all fields of engineering. Some of the applications of DSP in voice processing, musical sound processing, audio/video signal processing, communication and biomedical signal processing are discussed in this chapter.

Some of the DSP applications in voice processing are,

- Speech coding and decoding
- Speech recognition
- Speech synthesis
- Digital vocoder

Some of the DSP applications in musical sound processing are,

- Digital music synthesis
- Musical sound processing for recording

Some of the DSP applications in audio/video processing are,

- Digital radio
- Digital television

Some of the DSP applications in communication are,

- DTMF generation and detection
- RADAR

12.2 Speech Processing

The speech signal is a slowly timed varying signal. The speech signal can be broadly classified into voiced and unvoiced signal. The voiced signals are periodic and unvoiced signals are random in nature. The voiced signals will have a fundamental frequency in a segment of 15 to 20 msec, representing a characteristic sound of the speech. The various frequency components of sounds in speech signal will lie within 4 kHz.

The various DSP based speech processing techniques can be classified into speech analysis and speech synthesis.

Speech analysis: In general, the process of extracting the features of speech and then coding or directly digitizing the speech and then reducing the bit rate are called speech analysis. It is used in speech recognition, speaker verification and speaker identification.

Speech synthesis: In general, the process of decoding the speech signal represented in the form of codes are called speech synthesis. It is used in conversion of text to speech.

12.2.1 Speech Coding and Decoding

The ***speech coding*** is digital representation of speech using minimum bit rate without affecting the voice quality. The speech decoding is conversion of digital speech data to analog speech.

The old method for quality transmission and reception of digital speech signal through telephone lines, employs a bit rate of 64 kbps (kilo bits per second). This digital representation is called **P**ulse **C**ode **M**odulation (PCM), in which the speech signal is sampled at 8 kHz and each sample is quantized to 13 bits and then compressed to 8 bits using m-law or A-law standards to achieve a transmission rate of 64 kbps (8000 samples per second ´ 8 bits per sample = 64000 bits per second) needed for transmission.

A number of digital speech coding techniques are developed to represent the speech at lower bit rates up to 1000 bits per second to effectively utilize the transmission channels and also to reduce memory requirements for storage and retrival of speech.

The speech coding techniques can be broadly classified into waveform coding techniques and parametric coding techniques.

Some of the popular waveform coding techniques are **A**daptive **P**ulse **C**ode **M**odulation (APCM), **D**ifferential **P**ulse **C**ode **M**odulation (DPCM) and **A**daptive **D**ifferential **P**ulse **C**ode **M**odulation (ADPCM).

Some of the parametric method of speech coding are **L**inear **P**rediction **C**oding (LPC), **M**el-**F**requency **C**epstrum **C**oefficients (MFCC), **C**ode **E**xcited **L**inear **P**redictive **C**oding (CELP) and **V**ector **S**um **E**xcited **L**inear **P**rediction (VSELP).

Adaptive Differential Pulse Code Modulation (ADPCM)

The DPCM (**D**ifferential **P**ulse **C**ode **M**odulation) method of speech analysis/coding is based on the assumption that a speech sample can be effectively represented by the difference between previous and current sample. In the ADPCM method, the difference signal is computed between an adaptively predicted sample and current sample. Since, the difference between two samples can be represented by fewer bits, a 2:1 compression can be achieved, so that a 64 kbps speech signal can be coded to 32 kbps signal.

The ADPCM method of speech coding is shown in fig 12.1. The analog speech signal is converted to 64 kbps digital speech signal by sampling at 8 kHz with 8-bit per sample using an audio codec. The ADPCM algorithm expand this 8-bit samples to 14-bit samples, and subtract each expanded sample with an adaptively predicted sample to generate a difference signal which is quantized to 4 bits. The output of quantizer is the coded speech signal. The adaptively predicted signal is a weighed average of some dequantized difference signals and some predicted samples.

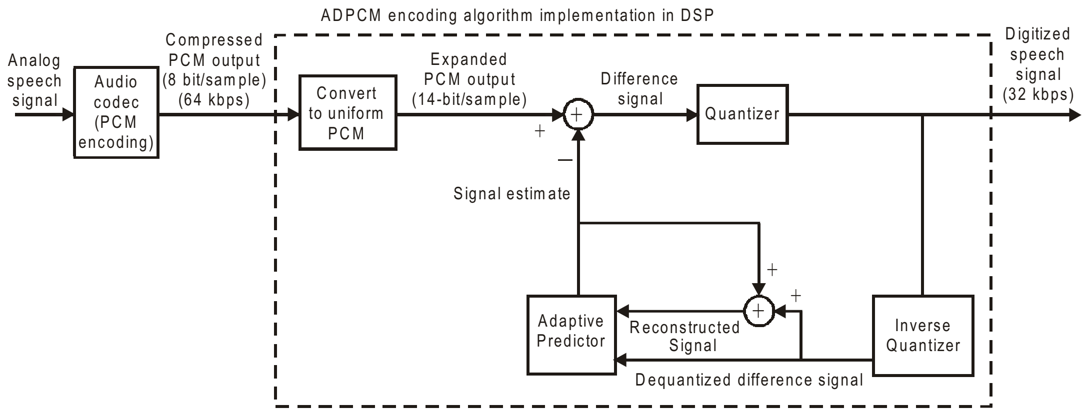


Fig 12.1 : *ADPCM speech encoder using digital signal processor (DSP).*

The ADPCM method of speech decoding is shown in fig 12.2. The ADPCM algorithm employs an inverse quantizer to generate the dequantized difference signal, from the coded speech sample. The ADPCM algorithm reconstruct the 14-bit sample of speech by adding the dequantized difference signal and an adaptively predicted signal estimate. Then the 14-bit speech samples are converted to 8-bit samples, which represents the decoded speech. The decoded speech can be converted to analog speech using an audio codec.

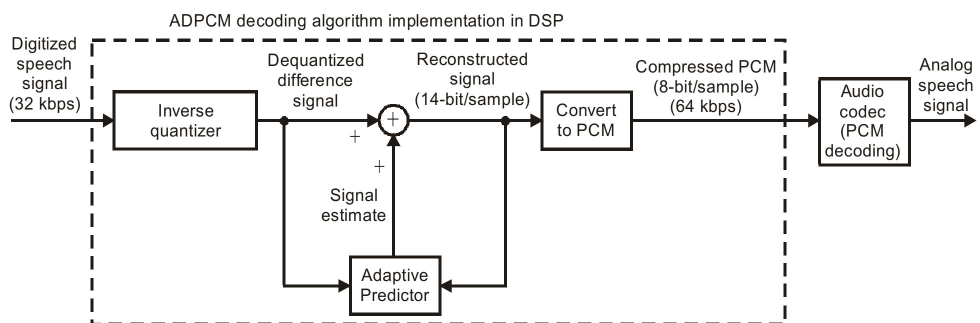


Fig 12.2 : *ADPCM speech decoder using digital signal processor (DSP).*

Linear Predictive Coding (LPC)

The LPC method of speech analysis/coding is based on the assumption that a speech sample can be approximated as a linear combination of previous speech samples. In LPC coding method, bit rates up to 24000 bits per second can be achieved.

For speech coding, first the speech signal is digitized using a coding system, in which the speech signal is segmented to 20 msec, sampled at 8 to 12 kHz, and using these samples a set of filter coefficients are determined. For each voiced speech segment a pitch is also calculated. The filter coefficients and the pitch represents the coded speech.

In the decoding process, a digital filter is constructed using the filter coefficients, and with input as a train of impulses at the pitch frequency for voiced segments and random noise sequence for unvoiced segments. The output of this filter is the synthesized/decoded speech signal.

Mel-Frequency Cepstrum Coefficients (MFCC)

MFCC method of speech coding/analysis is based on the known variation of the human ear's critical bandwidths with frequency. In this method, the phonetically important characteristics of speech are obtained using filters with linear frequency scale at low frequencies and logarithmic scale at high frequencies, and these characteristics are expressed in *mel-frequency* scale.

The various process in MFCC method of speech coding are shown in fig 12.3. The input 64 kbps speech data are converted to frames representing 20 msec of speech. Each frame data is overlapped by 5 msec on either side with adjacent frame and then windowed using Hamming window to minimize the effects of signal discontinuity at the beginning and end of each frame. Then FFT is computed to determine the frequency spectrum of the frame. For each tone frequency in the spectrum, a mel-spectrum coefficient is assigned using mel-scale, and this process is called mel-frequency wrapping. The mel-spectrum coefficients are converted to time domain coefficients called MFCC using discrete cosine transform. Thus, each overlapped 30 msec speech frame is coded into a set of **M**el-**F**requency **C**epstrum **C**oefficients (MFCC). This set of coefficients is also called an acoustic vector.

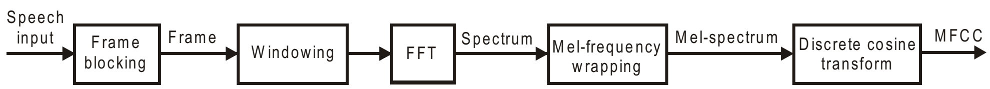


Fig 12.3 : *Various process in MFCC method of speech coding.*

Vector Sum Excited Linear Prediction (VSELP)

VSELP method of speech coding technique compress the 64 kbps digital speech signal to 7.95 kbps code. The 64 kbps speech data are converted to frames at a rate of 50 frames per second, so that each frame represent 20 msec speech and will consist of 160 samples. Then each frame is coded into 159 bits using code book search techniques.

Note : A code book is a collection of codewords representing vector quantized speech.

12.2.2 Speech Recognition

A speech recognition system can operate in many different conditions such as speaker-dependent/ independent and isolated/continuous speech recognition. A speaker-dependent system is a system that recognizes a specific speaker's speech, while speaker-independent systems can be used to recognize the speech of any unspecified speaker.

In an isolated word recognition system, each word or a simple utterance is assumed to be surrounded by silence or background noise. Connected speech or connected utterance recognition is similar to isolated word recognition. But it allows several words/digits to be spoken together with minimal pause between them.

The various processes in speech recognition systems are shown in fig 12.4. The front-end analysis, extracts the acoustic features of input speech. Some of the popular techniques used for extracting the acoustic

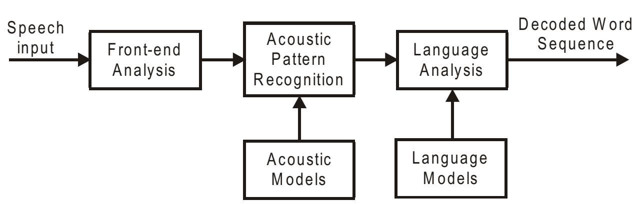


Fig 12.4 : *Various process in speech recognition.*

features of speech are **L**inear **P**redictive **C**oding (LPC) , **M**el-**F**requency **C**epstrum **C**oefficients (MFCC) and **P**erceptual **L**inear **P**rediction (PLP). The output of front-end analysis is a compact, efficient set of parameters that represent the acoustic properties observed from input speech signals, for subsequent utilization by acoustic modeling.

The acoustic models represent the acoustic properties, phonetic properties, microphone and environmental variability, as well as gender and dialectal differences among speakers. The language models contain the syntax, semantics, and pragmatics knowledge for the intended recognition task. These models can be dynamically modified according to the characteristics of the speech to be recognized during the training process.

Note : A speech recognition system has to be trained with known speech, before using the system for recognization.

Acoustic pattern recognition aims at measuring the similarity between an input speech and a reference model (obtained during training) and determines the best match for the input speech. Some popular methods of acoustic pattern matching are **D**ynamic **T**ime **W**arping (DTW), **H**idden **M**arkov **M**odeling (HMM), discrete HMM (DHMM), **C**ontinuous-**D**ensity **HMM** (CDHMM) and **V**ector **Q**uantization (VQ).

The language analysis is important in speech recognition, for **L**arge **V**ocabulary **C**ontinuous **S**peech **R**ecognition (LVCSR) tasks. The speech decoding process needs to invoke knowledge of pronunciation, lexicon, syntax, and pragmatics in order to produce a satisfactory output text sequence.

12.2.3 Speech Synthesis

Speech synthesis is either artificial production of human speech or decoding of coded speech parameters to recover the original speech. Some examples of speech synthesis are generation of speech signals from the speech parameters received through a transmission line and generation of speech signal from input text to a digital system like computer.

Text to Speech Conversion

The various process in speech synthesis from text is shown in Fig 12.5. The process of transforming text into speech contains two phases. The first phase consists of text analysis and phonetic analysis. The second phase is generation of speech signal, which can be divided into two sub-phases : the search of speech segments from a database or the creation of these segments and the implementation of the prosodic features.

Text analysis includes the task of text normalization and linguistic analysis. In text normalization, the numbers and symbols are converted to words and abbreviations are replaced by their corresponding whole words or phrases, so that the whole text is converted to human utterance like words.

The linguistic analysis aims at understanding the content of the text, exact meaning of utterance word and provide prosodic informations like position of pause, differentiate interrogative clause from statements, etc., for subsequent processing.

Phonetic analysis assigns phonetic transcription to each word, and this process is called grapheme-to-phoneme conversion.

Note : Grapheme is the smallest unit of written word, and phoneme is the smallest unit of speech.

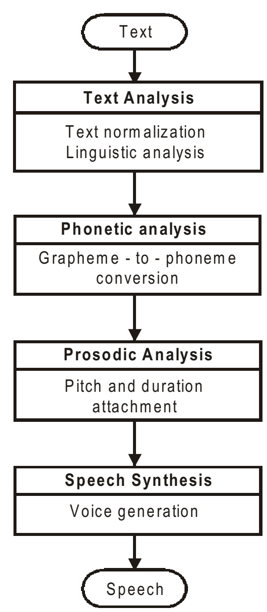


Fig 12.5 : *Various process in speech synthesis from text.*

Prosody refers to the rhythm of speech, stress patterns, pitch, duration, intonation, etc., and it plays a very important role in the understandability of speech. In prosodic analysis, some prosody features are added to synthetic speech so that it resembles natural speech. Moreover, some hierarchical rules have been developed to control the timing and fundamental frequency, which makes the flow of speech in synthesis systems to resemble natural sounding.

Speech synthesis block, finally generates the speech signal. This can be done by selecting speech unit for every phoneme from a database, using an appropriate search process. The resulting short units of speech are joined together to produce the final speech signal.

12.2.4 Digital Vocoder

The digital speech coding and decoding systems are generally called ***digital vocoders*** and they are analysis/synthesis systems, mostly used for secured transmission and reception of speech in telephones, cellphones and in wireless communication systems. The ***analysis vocoder*** is used for speech coding before transmission and the ***synthesis vocoder*** is used for decoding the received digital speech signal.

The general block diagram of speech analysis digital vocoder is shown in fig 12.6. An audio codec is used to sample the analog speech at 12 to 14 bits per sample and compressed to 8 bits using μ-law or A-law standard. This digital speech is read by digital signal processor for coding at a lower bit rate, using any of the techniques mentioned in section 12.2.1. The coded digital bitstream is passed through the error-coding block to protect the data against channel errors. Finally, the error-coded bitstream is modulated using modem and transmitted through transmission channels.

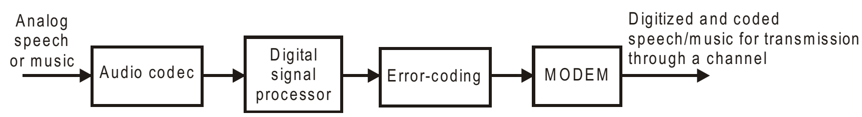


Fig 12.6 : *Block diagram of speech analysis digital vocoder.*

Note : The audio codec is an application specific IC for conversion of analog speech/music to digital or vice versa consisting of an ADC, DAC and μ-law/A-law compander.

The general block diagram of speech synthesis digital vocoder is shown in fig 12.7. A modem is used to demodulate the speech signal received from a transmission channel. The demodulated signal is error corrected by using error correction devices. Then the signal is read by digital signal processor for decoding using the same algorithm used for encoding. The decoded signal will have a size of 8-bit per sample, which is sent to codec for expansion to 12 to 14 bits using μ-law or A-law standard and then converted to analog speech.

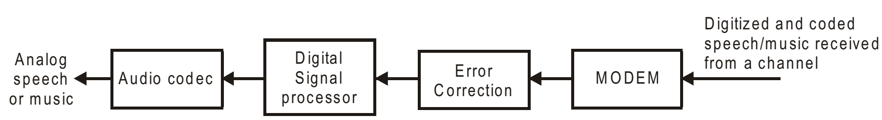


Fig 12.7 : *Block diagram of speech synthesis digital vocoder.*

12.3 Musical Sound Processing

The ***musical sound*** generated by a musical instrument is due to mechanical vibrations produced by a primary oscillator and then making other parts of the instrument to vibrate. For example, in a violin the primary oscillator is a stretched piece of string and it is vibrated by drawing a bow across it, which in turn vibrates the wooden body of the violin, and these vibrations make the surrounding air to vibrate, which produces the musical sound.

12.3.1 Digital Music Synthesis

Music synthesis plays an important role in multimedia applications, modern entertainment, and professional music systems.

The various music synthesis techniques used in the commercial systems are wavetable synthesis, spectral modeling synthesis, nonlinear synthesis (or FM synthesis) and physical modeling synthesis.

In wavetable synthesis method, the digital data of one period of the desired musical tone is stored in a table called wavetable. Then, using an IIR filter with no input and the stored data as initial condition, the musical signal is constructed whenever needed.

In spectral modeling synthesis, the mathematical equation representing the sound signal is used to generate the required music. The musical sound can be represented by an equation consisting of summation of sinusoidal signals. A musical tone consists of a fundamental tone frequency and its harmonics. Using suitable signal generation algorithm, the desired musical tone can be generated.

In nonlinear synthesis, the musical sound signal is represented as a nonlinear frequency modulated sinusoidal signal containing a fundamental frequency and harmonics of modulating signal. Using signal generation algorithm, various musical tones can be generated for various fundamental frequency. This method cannot be used to generate musics of natural instruments.

In physical modeling synthesis, a model of musical instrument like transfer function is constructed and the system model is implemented in a digital hardware, that can be used to generate the musics of an instrument.

12.3.2 Musical Sound Processing For Recording

The recording of musical programs are generally made in an acoustically inert studio. The sound of each instrument is separately recorded using microphones placed closed to it and then they are mixed using mixing system by a sound engineer. During mixing phase, various audio effects are artificially generated using signal processing circuits and devices. The modern trend is to use digital signal processing for these applications. Some of the special effects that can be implemented during mixing process are echo generation, reverberation, and chorus generation. Also, the musical sound signals can be passed through equalizers to provide amplification or attenuation of some of the tone frequencies.

Echo : The echoes can be generated by FIR filters with variable delay. The difference equation of the FIR filter used for echo generation will be in the form,

$$y(n) = x(n) + b\ x(n-M)$$

where, b represents signal loss in propagation and reflection

M is the delay parameter

The magnitude response of the filter represented by the above equation will have M peaks and M dips and look like a comb and so these filters are also called comb filters.

Reverberation : The musical sound in a concert hall will consists of direct sound, early reflections and reverberations. The early reflections are closely spaced echoes from a single source and reverberations are densely packed echoes from multiple sources. The reverberations can be introduced to musical sounds in order to create the concert hall effect by a set of parallel IIR comb filters with variable delays and cascaded with an all-pass filter.

Chorus Generation : The chorus is achieved by playing the same music by multiple musicians with different amplitudes. This effect can be created in a chorus generator consisting of a set of parallel IIR comb filters with slightly variable delay.

Equalizers : The equalizers are a set of bandpass IIR filters that are employed to amplify or attenuate the amplitude of the desired tone frequencies of the music.

12.4 Digital Radio

The digital radio has converted the old broadcast medium into the digital age with hundreds of new stations, crsytal-clear music, on-demand radio, surround sound, rewind radio, etc. The various digital signal processor manufacturers has released a number of special processors exclusively for digital radio.

The basic block diagram of digital radio is shown in fig.12.8. The system is built around a digital signal processor with analog front end consisting of antenna, low-noise high-speed amplifiers, local oscillator, mixer and high-speed ADC. The broadcasted signals from radio stations are received through antenna and amplified using low-noise amplifiers and then converted to IF (**I**ntermediate **F**requency) modulated signal by the local oscillator-mixer unit. These IF signals are digitized by high speed ADC, which is read by the digital signal processor for demodulation and extracting the audio signal. The digital audio signal is output to codec for conversion to analog, which is amplified and output to loudspeaker.

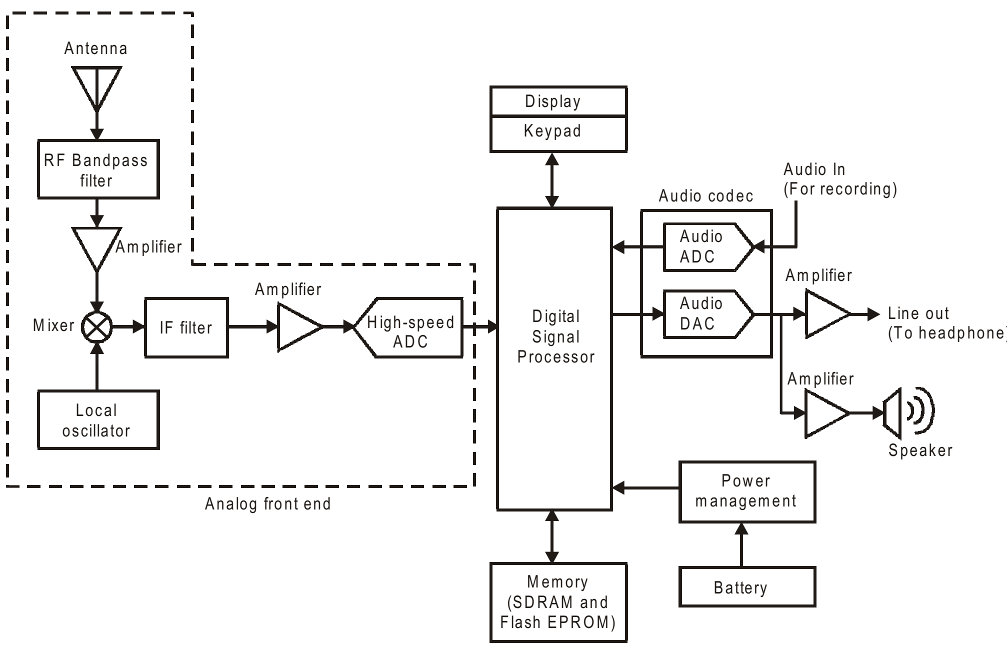


***Fig 12.8** : Block diagram of digital radio.*

The digital radio has memory for temporary and permanent storage, and also can read and play music/song stored in memory cards and CDs. Also, the digital radio has an audio-in input terminal which can be used for recording/storing any audio signal in memory. Besides, a keypad with menus for selecting stations, play/record mode, volume control, etc.

12.5 Digital Television

The digital television employs digital signal processing to enhance video and audio presentations and to reduce noise and ghosting. Also, a variety of features can be implemented, including frame store, picture-in-picture, improved sound quality and zoom. The general block diagram for digital processing of video signal is shown in fig 12.9. The reception of video signal from the antenna to IF (**I**ntermediate **F**requency) conversion is carried by analog devices. The **I**ntermediate **F**requency (IF) video signal is converted to digital by an 8-bit video ADC. The digital output can be processed in the digital domain using a digital signal processor to provide noise reduction, interpolation or averaging for digitally increased sharpness, and higher quality audio. The DSP digital output is converted back to analog by a video DAC, and displayed in the CRT.

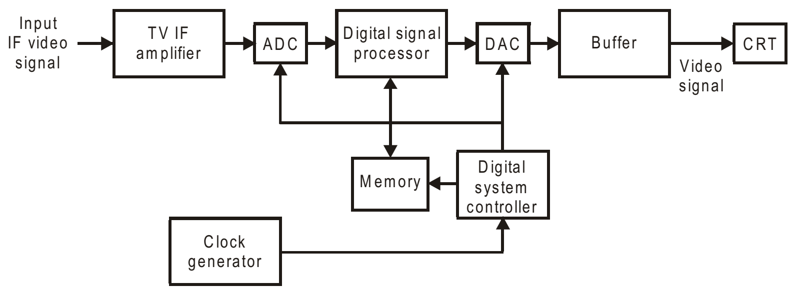


Fig 12.9 : *DSP based video signal processing system.*

12.6 DTMF in Telephone Dialing

The DTMF (**D**ual **T**one **M**ulti **F**requency) are standard tones proposed by CCITT (**C**onsultative **C**ommittee for **I**nternational **T**elephone and **T**elegraphy) for pushbutton telephones. The 16 standard DTMF signals are proposed using sinewaves of eight different frequencies (or tones), with each signal having sinewaves of two different frequencies (or tones). For each key/pushbutton of telephone, a standard DTMF signal is assigned, which consists of two tone frequencies chosen one from a row and one from a column as shown in fig 12.10.

	Column-1 1209 Hz	Column-2 1336 Hz	Column-3 1477 Hz	Column-4 1633 Hz
Row-1 697 Hz	1	2	3	A
Row-2 770 Hz	4	5	6	B
Row-3 852 Hz	7	8	9	C
Row-4 941 Hz	*	0	#	D

Fig 12.10 : *DTMF tone frequencies of telephone keypad.*

The digital signal processing plays a key role in generation and detection of DTMF signals. The DTMF signals can be easily generated using a suitable DSP algorithm running in a DSP system. The DSP system generates the digital samples of two sinewaves assigned to a digit and add together to get the samples of the corresponding DTMF signal. Then, these samples are send to a DAC (**D**igital to **A**nalog **C**onverter) or to an audio codec for construction of analog DTMF signal.

For detecting the DTMF signals, the signal is passed through an ADC (**A**nalog to **D**igital **C**onverter) or an audio codec to convert the analog DTMF signals to digital samples, and then FFT analysis of the samples is performed to detect the frequency components present in the received signal. The DSP system can also programmed to generate the hex / binary key-code corresponding to the received signal, which can be used to initiate a predefined activity.

The DTMF signals are used in all telephone based enquiry systems, banking systems, etc.

12.7 RADAR

Radar is an object-detection system which transmits electromagnetic waves in the radio/microwave frequency range and analyze the reflections to determine the range, altitude, direction, or speed of both moving and fixed objects such as aircraft, ships, spacecraft, guided missiles, motor vehicles, weather formations and terrain. Also, the radar can identify an object through its distinct reflection called "signature".

Note : The military applications of radar were developed in secret in nations across the world during World War II.The term RADAR was coined in 1940 by the U.S. Navy as an acronym for radio detection and ranging.

The major components of modern radar are digital signal processor interfaced to computer, RF (**R**adio **F**requency) transmitter and receiver system, ADC and DAC interfaced to digital signal processor and an antenna as shown in fig 12. 11.

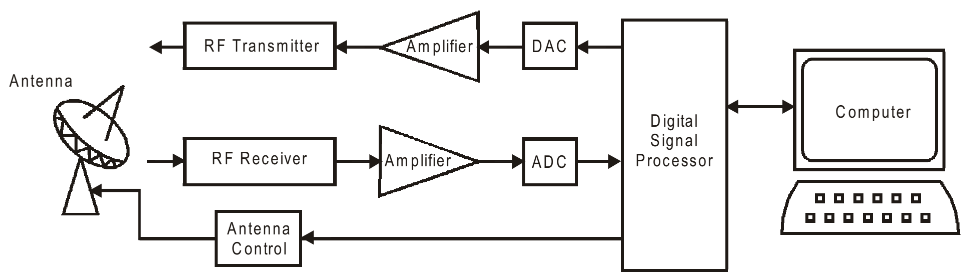


***Fig 12.11 :** Basic components of radar.*

The digital signal processor in the modern radar does all the functions like scheduling the appropriate antenna positions, generating and transmitting RF signals as a function of time, keeps track of targets, reading and analysis of reflected signals and feeding the processed information to computer for display.

The digital signal processor generates digital RF pulses, which are converted to analog RF pulses using fast acting DAC and then amplified and send to antenna for transmission. The transmitted RF pulses are reflected, when they encounter any object in their path. The reflected RF pulses are captured by antenna and digitized using fast acting ADC. The processor reads the digital information from ADC for analysis and detection of objects.

In radar, the decision-making is severely hampered by noise due to atmospheric noise entering into the system through the antenna. Also, all the electronics in the radar's signal path will add noise. In the modern radar systems, the digital signal processor is capable of extracting objects from very high noise levels.

The modern uses of radar are highly diverse . Some applications of radar are air traffic control, radar astronomy, air-defense systems, antimissile systems, aircraft anticollision systems, ocean-surveillance systems, outer-space surveillance and rendezvous systems, meteorological precipitation monitoring, altimetry and flight-control systems, guided-missile target-locating systems. The nautical radar is used to locate landmarks and other ships and ground-penetrating radar is used for geological observations.

12.8 Biomedical Signal Processing

All living things, from cells to organism, deliver signals of biological origin. Such signals can be electrical, mechanical or chemical. All such signals can be of interest for diagnosis, for patient monitoring and biomedical research. Most of the biomedical signals are acquired along with background noise, and so the main task of processing biomedical signals is to filter the signal of interest from the noisy background, and to extract the relevant information or parameters of interest.

Biomedical signal classification

The biomedical signals can be classified as shown below.

Bioelectric signals : Signals generated by nerve cells and muscle cells.

Biomagnetic signals: The brain, heart and lungs produce extremely weak magnetic fields, and this contains additional information to that obtained from bioelectric signals.

Bioimpedance signals: The tissue impedance reveals information about tissue composition, blood volume and distribution and more. Usually obtained as a ratio of voltage measured at the desired spot, and current injected using electrodes.

Bioacoustic signals: Sound or acoustic signals are created by flow of blood through the heart, its valves, or vessels and flow of air through upper and lower airways and lungs. Sound signals are also produced by digestive tract, joints and contraction of muscles. These signals can be recorded using microphones.

Biomechanical signals: Motion and displacement signals, pressure, tension and flow signals.

Biochemical signals: Chemical measurements from living tissue or samples analyzed in a laboratory.

Biooptical signals: Blood oxygenation obtained by measuring transmitted and backscattered light from a tissue, estimation of heart output by dye dilution.

Processing of biomedical signals

The processing of biomedical signals usually consists of at least four stages:

· Measurement or observation of signals, which is also called signal acquisition.

· Transformation and reduction of the signals.

· Computation of signal parameters that are diagnostically significant.

· Interpretation or classification of the signals.

The various stages of biosignal processing are shown in fig 12.12.

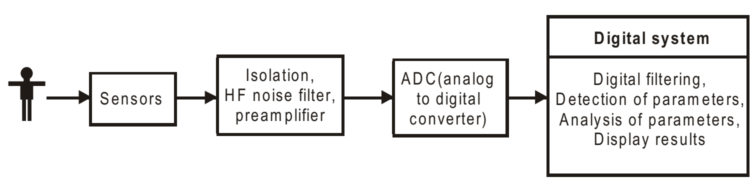


Fig 12.12 : *Biomedical signal processing.*

Sensors attached to a patient/human body convert biological signals into electrical signals. The signals from sensors will be corrupted by high frequency ambient noise, and so they are filtered by using an analog high frequency filter. Also, the signals from the sensors will be very weak signals and so they are amplified and if necessary, they are isolated to prevent the entry of high frequency noise into digital system.

The biosignals are analog signals and so they has to be digitized for processing by digital system. Therefore, the amplified signals are fed to an ADC for convertion to digital and the output of ADC is fed to a digital system for processing and analysis. A display device is usually attached to the digital system to display the result or to monitor the acquired signal.

Biomedical application domains

The various applications of biomedical signal processing can be classified as shown below.

Information gathering: The measurement of phenomena to understand the system.

Diagnosis: Detection of malfunction, pathology or abnormality.

Monitoring: To obtain continuous or periodic information about the system.

Therapy and control: Modify the behaviour of the system and ensure the result.

Evaluation: Proof of performance, quality control, effect of treatment.

APPENDIX 1

Important Mathematical Relations

Table - A1.1 : Trigonometric Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \text{cosec}\,\theta = \frac{1}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1, \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \text{cosec}^2\theta$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\,\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\,\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\,\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\,\sin\frac{A-B}{2}$$

$$\cos(\theta \pm 90^o) = \mp\sin\theta$$

$$\sin(\theta \pm 90^o) = \pm\cos\theta$$

$$\tan(\theta \pm 90^o) = -\cot\theta$$

$$\cos(\theta \pm 180^o) = -\cos\theta$$

$$\sin(\theta \pm 180^o) = -\sin\theta$$

$$\tan(\theta \pm 180^o) = \tan\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

Table - A1.2 : Complex Variables

A complex number 'z' may be represented as,

$$z = x + jy = r\angle\theta = re^{j\theta} = r(\cos\theta + j\sin\theta)$$

where, $x = \text{Re}(z) = r\cos q$, $\quad y = \text{Im}(z) = r\sin q$

$$r = |z| = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$j = \sqrt{-1}, \qquad \frac{1}{j} = -j, \qquad j^2 = -1$$

The conjugate of the complex number 'z' may be represented as,

$$z^* = x - jy = r\angle-\theta = re^{-j\theta} = r(\cos\theta - j\sin\theta)$$

Demovier's theorem : $\left(e^{j\theta}\right)^n = e^{jn\theta} = \cos n\theta + j\sin n\theta$

Let, z_1 and z_2 be two complex numbers defined as, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$.
Now, $z_1 = z_2$ only if $x_1 = x_2$ and $y_1 = y_2$.

$$z_1 \pm z_2 = (x_1 + x_2) \pm j(y_1 + y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \quad \text{or} \quad z_1 z_2 = r_1 r_2 \, e^{j(\theta_1 + \theta_2)} = r_1 r_2 \angle(\theta_1 + \theta_2)$$

$$\frac{z_1}{z_2} = \frac{(x_1 + jy_1)}{(x_2 + jy_2)} \times \frac{(x_2 - jy_2)}{(x_2 - jy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

$$\text{or} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} = \frac{r_1}{r_2} \angle(\theta_1 - \theta_2)$$

The following relations hold good for a complex number 'z'.

$$\sqrt{z} = \sqrt{x + jy} = \sqrt{re^{j\theta}} = \sqrt{r}\, e^{j\frac{\theta}{2}} = \sqrt{r} \angle \tfrac{\theta}{2}$$

$$z^n = (x + jy)^n = r^n e^{jn\theta} = r^n \angle n\theta \qquad \text{where, n is an integer.}$$

$$z^{1/n} = (x + jy)^{1/n} = r^{1/n} e^{j\theta/n} = r^{1/n} \angle\left(\tfrac{\theta}{n} + \tfrac{2\pi k}{n}\right) \quad \text{for, } k = 0, 1, 2, \ldots\ldots\ldots n - 1$$

$$\ell n z = \ell n(re^{j\theta}) = \ell n\, r + \ell n\, e^{j\theta} = \ell n\, r + j(\theta + 2k\pi) \quad \text{where, k is an integer.}$$

Table - A1.3 : Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}$$

$$\text{cosech } x = \frac{1}{\sinh x}, \qquad \text{sech } x = \frac{1}{\cosh x}$$

$$\sin jx = j \sinh x, \qquad \cos jx = \cosh x$$

$$\sinh jx = j \sin x, \qquad \cosh jx = \cos x$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\sinh(x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y$$

$$\cosh(x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y$$

$$\tanh(x \pm jy) = \frac{\sinh 2x}{\cosh 2x + \cos 2y} \pm j \frac{\sin 2y}{\cosh 2x + \cos 2y}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\text{sech}^2 x + \tanh^2 x = 1$$

$$\sin(x \pm iy) = \sin x \cosh y \pm j \cos x \sinh y$$

$$\cos(x \pm jy) = \cos x \cosh y \mp j \sin x \sinh y$$

Table - A1.4 : Derivatives

Let, $U = U(x)$, $V = V(x)$, and $a = \text{constant}$.

$$\frac{d}{dx}(a\,U) = a\frac{dU}{dx}$$

$$\frac{d}{dx}(U\,V) = U\frac{dV}{dx} + V\frac{dU}{dx}$$

$$\frac{d}{dx}\left[\frac{U}{V}\right] = \frac{V\frac{dU}{dx} - U\frac{dV}{dx}}{V^2}$$

$$\frac{d}{dx}\left(a\,U^n\right) = n\,a\,U^{n-1}$$

$$\frac{d}{dx}\log_a U = \frac{\log_a e}{U}\;\frac{dU}{dx}$$

$$\frac{d}{dx}\ell n\ U = \frac{1}{U}\;\frac{dU}{dx}$$

$$\frac{d}{dx}\ a^U = a^U\ \ell n\ a\frac{dU}{dx}$$

$$\frac{d}{dx}\ e^U = e^U\ \frac{dU}{dx}$$

$$\frac{d}{dx}\ U^V = V\,U^{V-1}\ \frac{dU}{dx} + U^V\ \text{In}\ U\frac{dV}{dx}$$

$$\frac{d}{dx}\ \sin U = \cos U\ \frac{dU}{dx}$$

$$\frac{d}{dx}\ \cos U = -\sin U\ \frac{dU}{dx}$$

$$\frac{d}{dx}\ \tan U = \sec^2 U\ \frac{dU}{dx}$$

Table - A1.5 : Indefinite Integrals

Let, $U = U(x)$, $V = V(x)$, and $a = \text{constant}$.

$$\int a\,dx = ax + C$$

$$\int UV = U\int V - \int\left[\int V\,dU\right] \quad \text{or} \quad \int U\,dV = UV - \int V\,dU$$

$$\int U^n\,dU = \frac{U^{n+1}}{n+1} + C, \qquad n \neq -1$$

$$\int \frac{1}{U}\,dU = \ell n\,U + C$$

$$\int a^U\,dU = \frac{a^U}{\ell n\,a} + C, \qquad a > 0 \text{ and } a \neq 1$$

$$\int e^U\,dU = e^U + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int x e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1) + C$$

$$\int x^2 e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2) + C$$

$$\int \ell n\,x\,dx = x\,\ell n\,x - x + C$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ell n(\sec ax) + C = -\frac{1}{a}\ell n(\cos ax) + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ell n(\sec ax + \tan ax) + C$$

APPENDIX 2

MATLAB Commands and Functions

	Operators and Special Characters
+	Plus; addition operator.
–	Minus; subtraction operator.
*	Scalar and matrix multiplication operator.
.*	Array multiplication operator.
^	Scalar and matrix exponentiation operator.
.^	Array exponentiation operator.
\	Left-division operator.
/	Right-division operator.
.\	Array left-division operator.
./	Array right-division operator.
:	Colon; generates regularly spaced elements and represents an entire row/column.
( )	Parentheses; encloses function arguments and array indices; overrides precedence.
[ ]	Brackets; enclosures array elements.
.	Decimal point.
…	Ellipsis; line-continuation operator.
,	Comma; separates statements and elements in a row.
;	Semicolon; separates columns and suppresses display.
%	Percent sign; designates a comment and specifies formatting.
_	Quote sign and transpose operator.
._	Nonconjugated transpose operator.
=	Assignment (replacement) operator.

	Logical and Relational Operators
==	Relational operator : equal to.
~=	Relational operator : not equal to.
<	Relational operator : less than.
<=	Relational operator : less than or equal to.
>	Relational operator : greater than.
>=	Relational operator : greater than or equal to.
&	Logical operator : AND.
|	Logical operator : OR.
~	Logical operator : NOT.
xor	Logical operator : EXCLUSIVE OR.

Special Variables and Constants	
`ans`	Most recent answer.
`eps`	Accuracy of floating-point precision.
`i,j`	The imaginary unit ; $\sqrt{-1}$.
`Inf`	Infinity.
`NaN`	Undefined numerical result (not a number).
`pi`	The number p .

Commands for Managing a Session	
`clc`	Clears Command window.
`clear`	Removes variables from memory.
`exist`	Checks for existence of file or variable.
`global`	Declares variables to be global.
`help`	Searches for a help topic.
`lookfor`	Searches help entries for a keyword.
`quit`	Stops MATLAB.
`who`	Lists current variables.
`whos`	Lists current variables (long display).

Input/Output Commands	
`disp`	Displays contents of an array or string.
`fscanf`	Read formatted data from a file.
`format`	Controls screen-display format.
`fprintf`	Performs formatted writes to screen or file.
`input`	Displays prompts and waits for input.
`;`	Suppresses screen printing.

Format Codes for fprintf and fscanf	
`%s`	Format as a string.
`%d`	Format as an integer.
`%f`	Format as a floating point value.
`%e`	Format as a floating point value in scientific notation.
`%g`	Format in the most compact form : %f or %e.
`\n`	Insert a new line in the output string.
`\t`	Insert a tab in the output string.

Array Commands	
`cat`	Concatenates arrays.
`find`	Finds indices of nonzero elements.
`length`	Computes number of elements.
`linspace`	Creates regularly spaced vector.
`logspace`	Creates logarithmically spaced vector.
`max`	Returns largest element.
`min`	Returns smallest element.
`prod`	Product of each column.
`reshape`	Change size
`size`	Computes array size.
`sort`	Sorts each column.
`sum`	Sums each column.

Special Matrices	
`eye`	Creates an identity matrix.
`ones`	Creates an array of ones.
`zeros`	Creates an array of zeros.

Program Flow Control	
`break`	Terminates execution of a loop.
`case`	Provides alternate execution paths within switch structure.
`else`	Delineates alternate block of statements.
`elseif`	Conditionally executes statements.
`end`	Terminates for, while, and if statements.
`error`	Displays error messages.
`for`	Repeats statements a specific number of times
`if`	Executes statements conditionally.
`otherwise`	Default part of switch statement.
`return`	Return to the invoking function.
`switch`	Directs program execution by comparing point with case expressions.
`warning`	Display a warning message.
`while`	Repeats statements an indefinite number of times.

Basic xy Plotting Commands	
`axis`	Sets axis limits.
`fplot`	Intelligent plotting of functions.
`grid`	Displays gridlines.
`plot`	Generates xy plot.
`print`	Prints plot or saves plot to a file
`title`	Puts text at top of plot.
`xlabel`	Adds text label to x-axis.
`ylabel`	Adds text label to y-axis.

Plot Enhancement Commands	
`axes`	Creates axes objects.
`close`	Closes the current plot.
`close`	Closes all plots.
`figure`	Opens a new figure window.
`gtext`	Enables label placement by mouse.
`hold`	Freezes current plot.
`legend`	Legend placement by mouse.
`refresh`	Redraws current figure window.
`set`	Specifies properties of objects such as axes.
`subplot`	Creates plots in subwindows.
`text`	Places string in figure.

Specialized Plot Commands	
`bar`	Creates bar chart.
`loglog`	Creates log-log plot.
`polar`	Creates polar plot.
`semilogx`	Creates semilog plot (logarithmic abscissa).
`semilogy`	Creates semilog plot (logarithmic ordinate).
`stairs`	Creates stairs plot.
`stem`	Creates stem plot.

Convolution Functions	
`conv(x,h)`	Returns convolution of x and h.
`deconv(y,x)`	Returns deconvolution of y and x.

Logical Functions	
`any`	True if any elements are nonzero.
`all`	True if all elements are nonzero.
`find`	Finds indices of nonzero elements.
`finite`	True if elements are finite.
`isnan`	True if elements are undefined.
`isinf`	True if elements are infinite.
`isempty`	True if matrix is empty.
`isreal`	True if all elements are real.

Exponential and Logarithmic Functions	
`exp(x)`	Exponential; e^x.
`log(x)`	Natural logarithm; ln(x).
`log10(x)`	Common (base 10) logarithm; $\log(x) = \log_{10}(x)$.
`sqrt(x)`	Square root of x; $\sqrt{x}$.

Trigonometric Functions	
`acos(x)`	Inverse cosine; $\cos^{-1}(x)$.
`acot(x)`	Inverse cotangent; $\cot^{-1}(x)$.
`acsc(x)`	Inverse cosecant; $\mathrm{cosec}^{-1}(x)$.
`asec(x)`	Inverse secant; $\sec^{-1}(x)$.
`asin(x)`	Inverse sine; $\sin^{-1}(x)$.
`atan(x)`	Inverse tangent; $\tan^{-1}(x)$.
`atan2(y,x)`	Four-quadrant inverse tangent.
`cos(x)`	Cosine; cos(x).
`cot(x)`	Cotangent; cot(x).
`csc(x)`	Cosecant; cosec(x).
`sec(x)`	Secant; sec(x).
`sin(x)`	Sine; sin(x).
`tan(x)`	Tangent; tan(x).

Hyperbolic Functions	
`acosh(x)`	Inverse hyperbolic cosine; $\cosh^{-1}(x)$.
`acoth(x)`	Inverse hyperbolic cotangent; $\coth^{-1}(x)$.
`acsch(x)`	Inverse hyperbolic cosecant; $\mathrm{cosech}^{-1}(x)$.
`asech(x)`	Inverse hyperbolic secant; $\mathrm{sech}^{-1}(x)$.
`asinh(x)`	Inverse hyperbolic sine; $\sinh^{-1}(x)$.
`atanh(x)`	Inverse hyperbolic tangent; $\tanh^{-1}(x)$.
`cosh(x)`	Hyperbolic cosine; cosh(x).
`coth(x)`	Hyperbolic cotangent; cosh(x)/sinh(x).
`csch(x)`	Hyperbolic cosecant; 1/sinh(x).
`sech(x)`	Hyperbolic secant; 1/cosh(x).
`sinh(x)`	Hyperbolic sine; sinh(x).
`tanh(x)`	Hyperbolic tangent; sinh(x)/cosh(x).

Complex Functions	
`abs(x)`	Absolute value; $\lvert x \rvert$.
`angle(x)`	Angle of a complex number x.
`conj(x)`	Complex conjugate of x.
`imag(x)`	Imaginary part of a complex number x.
`real(x)`	Real part of a complex number x.

State Space Functions	
`ss2tf`	Computes transfer function from state model.
`tf2ss`	Computes state model from transfer function.

Transform Functions	
`fft`	Computes DFT via FFT.
`fourier`	Returns the Fourier transform.
`ifft`	Computes inverse DFT via FFT.
`ifourier`	Returns the inverse Fourier transform.
`ilaplace`	Returns the inverse Laplace transform.
`iztrans`	Returns the inverse $\mathbb{Z}$-transform.
`laplace`	Returns the Laplace transform.
`ztrans`	Returns the $\mathbb{Z}$-transform.

APPENDIX 3

Summary of Various Standard Transform Pairs

Table - A3.1 : Standard Z-transform Pairs

x(t)	x(n)	X(z) With positive power of z	X(z) With negative power of z	ROC
	$\delta(n)$	1	1	Entire z-plane
	u(n) or 1	$\frac{z}{z-1}$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$
	$a^n\,u(n)$	$\frac{z}{z-a}$	$\frac{1}{1-az^{-1}}$	$\|z\|>\|a\|$
	$n\,a^n\,u(n)$	$\frac{az}{(z-a)^2}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|>\|a\|$
	$n^2\,a^n\,u(n)$	$\frac{az\,(z+a)}{(z-a)^3}$	$\frac{az^{-1}\,(1+az^{-1})}{(1-az^{-1})^3}$	$\|z\|>\|a\|$
	$-\,a^n u(-n-1)$	$\frac{z}{z-a}$	$\frac{1}{1-az^{-1}}$	$\|z\|<\|a\|$
	$-na^n\,u(-n-1)$	$\frac{az}{(z-a)^2}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|<\|a\|$
$t\,u(t)$	$nT\,u(nT)$	$\frac{Tz}{(z-1)^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$	$\|z\|>1$
$t^2\,u(t)$	$(nT)^2 u(nT)$	$\frac{T^2z\,(z+1)}{(z-1)^3}$	$\frac{T^2z^{-1}\,(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\|>1$
$e^{-at}u(t)$	$e^{-anT}\,u(nT)$	$\frac{z}{z-e^{-aT}}$	$\frac{1}{1-e^{-aT}\,z^{-1}}$	$\|z\|>\|e^{-aT}\|$
$te^{-at}u(t)$	$nTe^{-anT}u(nT)$	$\frac{z\,T\,e^{-aT}}{(z-e^{-aT})^2}$	$\frac{z^{-1}\,T\,e^{-aT}}{(1-e^{-aT}\,z^{-1})^2}$	$\|z\|>\|e^{-aT}\|$
$\sin\Omega_0 t\;u(t)$	$\sin\Omega_0 nT\;u(nT)$ $=\sin wn\;u(nT)$ where, $w=\Omega_0 T$	$\frac{z\sin\omega}{z^2-2z\cos\omega+1}$	$\frac{z^{-1}\sin\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\|z\|>1$
$\cos\Omega_0 t\;u(t)$	$\cos\Omega_0 nT\;u(nT)$ $=\cos wn\;u(nT)$	$\frac{z\,(z-\cos\omega)}{z^2-2z\cos\omega+1}$	$\frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\|z\|>1$

Table - A3.2 : Standard Discrete Time Fourier Transform Pairs

x(t)	x(n)	$X(e^{jw})$	
		with positive power of e^{jw}	**with negative power of e^{jw}**
	d(n)	1	1
	$d(n-n_0)$	$\dfrac{1}{e^{j\omega n_0}}$	$e^{-j\omega n_0}$
	u(n)	$\dfrac{e^{j\omega}}{e^{j\omega}-1}+\sum\limits_{m=-\infty}^{+\infty}\pi\,\delta(\omega-2\pi m)$	$\dfrac{1}{1-e^{-j\omega}}+\sum\limits_{m=-\infty}^{+\infty}\pi\,\delta(\omega-2\pi m)$
	$a^n\,u(n)$	$\dfrac{e^{j\omega}}{e^{j\omega}-a}$	$\dfrac{1}{1-ae^{-j\omega}}$
	$n\,a^n\,u(n)$	$\dfrac{ae^{j\omega}}{(e^{j\omega}-a)^2}$	$\dfrac{ae^{-j\omega}}{(1-a\,e^{-j\omega})^2}$
	$n^2\,a^n\,u(n)$	$\dfrac{ae^{j\omega}\,(e^{j\omega}+a)}{(e^{j\omega}-a\,)^3}$	$\dfrac{ae^{-j\omega}\,(1+ae^{-j\omega})}{(1-ae^{-j\omega})^3}$
$e^{-at}u(t)$	$e^{-anT}u(nT)$	$\dfrac{e^{j\omega}}{e^{j\omega}-e^{-aT}}$	$\dfrac{1}{1-e^{-j\omega}e^{-aT}}$
	1	$2\pi\sum\limits_{m=-\infty}^{+\infty}\delta(\omega-2\pi m)$	
	$a^{\lvert n\rvert}$	$\dfrac{1-a^2}{1-2a\cos\omega+a^2}$	
	$\sum\limits_{m=-\infty}^{+\infty}\delta(n-mN)$	$\dfrac{2\pi}{N}\sum\limits_{m=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi m}{N}\right)$	
$e^{j\Omega_0 t}$	$e^{j\Omega_0 nt}=e^{j\omega_0 n}$ where, $\omega_0=\Omega_0 T$	$2\pi\sum\limits_{m=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi m)$	
$\sin\Omega_0 t$	$\sin\Omega_0 nT$ $=\sin\omega_0 n$ where, $\omega_0=\Omega_0 T$	$\dfrac{\pi}{j}\sum\limits_{m=-\infty}^{+\infty}\left[\delta(\omega-\omega_0-2\pi m)-\delta(\omega+\omega_0-2\pi m)\right]$	
$\cos\Omega_0 t$	$\cos\Omega_0 nT$ $=\cos\omega_0 n$ where, $\omega_0=\Omega_0 T$	$\pi\sum\limits_{m=-\infty}^{+\infty}\left[\delta(\omega-\omega_0-2\pi m)+\delta(\omega+\omega_0-2\pi m)\right]$	

Table - A3.3 : Standard Discrete Time Fourier Transform Pairs of Causal Signals via Z-transform

x(t)	x(n)	X(z)	X(e^jw)
	$\delta(n)$	1	1
	$a^n\, u(n)$; $\lvert a \rvert < 1$	$\dfrac{z}{z-a}$	$\dfrac{e^{j\omega}}{e^{j\omega}-a}$
	$n\, a^n\, u(n)$; $\lvert a \rvert < 1$	$\dfrac{az}{(z-a)^2}$	$\dfrac{a\, e^{j\omega}}{(e^{j\omega}-a)^2}$
	$n^2\, a^n\, u(n)$; $\lvert a \rvert < 1$	$\dfrac{az\,(z+a)}{(z-a)^3}$	$\dfrac{a\, e^{j\omega}\,(e^{j\omega}+a)}{(e^{j\omega}-a)^3}$
$e^{-at} u(t)$	$e^{-anT} u(nT)$; $\lvert e^{-aT} \rvert < 1$	$\dfrac{z}{z-e^{-aT}}$	$\dfrac{e^{j\omega}}{e^{j\omega}-e^{-aT}}$
$te^{-at} u(t)$	$nTe^{-anT} u(nT)$; $\lvert e^{-aT} \rvert < 1$	$\dfrac{z\,T\,e^{-aT}}{(z-e^{-aT})^2}$	$\dfrac{e^{j\omega}\,T\,e^{-aT}}{(e^{j\omega}-e^{-aT})^2}$

APPENDIX 4

Summary of Properties of Various Transforms

Table - A4.1 : Properties of Ƶ-Transform

Note : $X(z) = Ƶ\{x(n)\}$; $X_1(z) = Ƶ\{x_1(n)\}$; $X_2(z) = Ƶ\{x_2(n)\}$; $Y(z) = Ƶ\{y(n)\}$

Property		Discrete time signal	Ƶ-transform
Linearity		$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$
Shifting (m ³ 0)	x(n); for n ³ 0	$x(n-m)$	$z^{-m} X(z) + \sum_{i=1}^{m} x(-i)\, z^{-(m-i)}$
		$x(n+m)$	$z^{m} X(z) - \sum_{i=0}^{m-1} x(i)\, z^{m-i}$
	x(n); for all n	$x(n-m)$	$z^{-m} X(z)$
		$x(n+m)$	$z^{m} X(z)$
Multiplication by n^m (or differentiation in z-domain)		$n^m x(n)$	$\left(-z \frac{d}{dz}\right)^m X(z)$
Scaling in z-domain (or multiplication by a^n)		$a^n x(n)$	$X(a^{-1} z)$
Time reversal		$x(-n)$	$X(z^{-1})$
Conjugation		$x^*(n)$	$X^*(z^*)$
Convolution		$x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)$	$X_1(z)\, X_2(z)$
Corrrelation		$r_{xy}(m) = \sum_{m=-\infty}^{+\infty} x(n)\, y(n-m)$	$X(z)\, Y(z^{-1})$
Initial value		$x(0) = \underset{z \to \infty}{\mathrm{Lt}}\ X(z)$	
Final value		$x(\infty) = \underset{z \to 1}{\mathrm{Lt}}\ (1 - z^{-1})\, X(z)$ $= \underset{z \to 1}{\mathrm{Lt}}\ \frac{(z-1)}{z} X(z)$ if X(z) is analytic for $\lvert z \rvert > 1$	
Complex convolution theorem		$x_1(n)\, x_2(n)$	$\frac{1}{2\pi j} \oint_C X_1(v)\, X_2\!\left(\frac{z}{v}\right) v^{-1}\, dv$

Table - A4.2 : Properties of Fourier Series Coefficients of Discrete Time Signals

Note : c_k are Fourier series coefficients of x(n) and d_k are Fourier series coefficients of y(n).

Property	Discrete time periodic signal	Fourier series coefficients
Linearity	$A\,x(n) + B\,y(n)$	$A\,c_k + B\,d_k$
Time shifting	$x(n-m)$	$c_k\,e^{\frac{j2\pi km}{N}}$
Frequency shifting	$e^{\frac{j2\pi nm}{N}}\,x(n)$	c_{k-m}
Conjugation	$x^*(n)$	c^*_{-k}
Time reversal	$x(-n)$	c_{-k}
Time scaling	$x(\frac{n}{m})$; for n multiple of m (periodic with period mN)	$\frac{1}{m}\,c_k$
Multiplication	$x(n)\,y(n)$	$\sum_{m=0}^{N-1} c_m\,d_{k-m}$
Circular convolution	$\sum_{m=0}^{N-1} x(m)\,y((n-m))_N$	$N\,c_K\,d_K$
Symmetry of real signals	x(n) is real	$c_k = c^*_{-k}$ $\lvert c_k\rvert = \lvert c_{-k}\rvert$ $\angle c_k = -\angle c_{-k}$ $Re\{c_k\} = Re\{c_{-k}\}$ $Im\{c_k\} = -Im\{c_{-k}\}$
Real and even	x(n) is real and even	c_k are real and even
Real and odd	x(n) is real and odd	c_k are imaginary and odd

Table - A4.3 : Properties of Discrete Time Fourier Transform

Note : $X(e^{jw}) = \mathcal{F}\{x(n)\}$; $X_1(e^{jw}) = \mathcal{F}\{x_1(n)\}$; $X_2(e^{jw}) = \mathcal{F}\{x_2(n)\}$; $Y(e^{jw}) = F\{y(n)\}$

Property	Discrete time signal	Fourier transform
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(e^{jw}) + a_2 X_2(e^{jw})$
Periodicity	$x(n)$	$X(e^{jw+2pm}) = X(e^{jw})$
Time shifting	$x(n-m)$	$e^{-jwm} X(e^{jw})$
Time reversal	$x(-n)$	$X(e^{-jw})$
Conjugation	$x^*(n)$	$X^*(e^{-jw})$
Frequency shifting	$e^{j\omega_0 n}\, x(n)$	$X(e^{j(\omega-\omega_0)})$
Multiplication	$x_1(n)\, x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega-\lambda)})\, d\lambda$
Differentiation in frequency domain	$n\,x(n)$	$j\frac{d}{d\omega}X(e^{j\omega})$
Convolution	$x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)$	$X_1(e^{jw})\, X_2(e^{jw})$
Correlation	$r_{xy}(m) = \sum_{m=-\infty}^{+\infty} x(n)\, y(n-m)$	$X(e^{jw})\, Y(e^{-jw})$
Symmetry of real signals	x(n) is real	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $\mathrm{Re}\{X(e^{j\omega})\} = \mathrm{Re}\{X(e^{-j\omega})\}$ $\mathrm{Im}\{X(e^{j\omega})\} = -\mathrm{Im}\{X(e^{-j\omega})\}$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$, $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
Symmetry of real and even signal	x(n) is real and even	$X(e^{jw})$ is real and even
Symmetry of real and odd signal	x(n) is real and odd	$X(e^{jw})$ is imaginary and odd

Table - A4.5 : Properties of Discrete Fourier Transform (DFT)

Note : $X(k) = \mathcal{DFT}\{x(n)\}$; $X_1(k) = \mathcal{DFT}\{x_1(n)\}$; $X_2(k) = \mathcal{DFT}\{x_2(n)\}$; $Y(k) = \mathcal{DFT}\{y(n)\}$

Property	Discrete time signal	Discrete Fourier Transform
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Periodicity	$x(n+N) = x(n)$	$X(k+N) = X(k)$
Circular time shift	$x((n-m))_N$	$X(k)\,e^{\frac{-j2\pi km}{N}}$
Time reversal	$x(N-n)$	$X(N-k)$
Conjugation	$x^*(n)$	$X^*(N-k)$
Circular frequency shift	$x(n)\,e^{\frac{j2\pi mn}{N}}$	$X((k-m))_N$
Multiplication	$x_1(n)\,x_2(n)$	$\frac{1}{N}\left[X_1(k) \circledast X_2(k)\right]$
Circular convolution	$x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)\,x_2((n-m))_N$	$X_1(k)\,X_2(k)$
Circular correlation	$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\,y^*((n-m))_N$	$X(k)\,Y^*(k)$
Symmetry of real signals	x(n) is real	$X(k) = X^*(N-k)$ $X_r(k) = X_r(N-k)$ $X_i(k) = -X_i(N-k)$ $\lvert X(k)\rvert = \lvert X(N-k)\rvert$ $\angle X(k) = -\angle X(N-k)$
Symmetry of real and even signal	x(n) is real and even $x(n) = x(N-n)$	$X(k) = X_r(k)$ and $X_i(k) = 0$
Symmetry of real and odd signal	x(n) is real and odd $x(n) = -x(N-n)$	$X(k) = jX_i(k)$ and $X_r(k) = 0$

Table - A4.5 : Parseval's Relation in Various Transforms

Parseval's relation in $\mathcal{Z}$-transform	$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(z)\, X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz$	
Parseval's relation in discrete time Fourier series	Average power P of x(n) is defined as, $P = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x(n)\rvert^2$	The Average power P in terms of Fourier series coefficients is, $P = \sum_{k=0}^{N-1} \lvert c_k\rvert^2$
	$\frac{1}{N} \sum_{n=0}^{N-1} \lvert x(n)\rvert^2 = \sum_{k=0}^{N-1} \lvert c_k\rvert^2$	
Parseval's relation in discrete time Fourier transform	Energy E in time domain, $E = \sum_{n=-\infty}^{+\infty} \lvert x(n)\rvert^2$	Energy E in frequency domain, $E = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \lvert X(e^{j\omega})\rvert^2\, d\omega$
	$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)$	$\frac{1}{2\pi} \int_{-\pi}^{+\pi} X_1(e^{j\omega}) X_2^*(e^{j\omega})\, d\omega$
	$\sum_{n=-\infty}^{+\infty} \lvert x(n)\rvert^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\lvert X(e^{j\omega})\right\rvert^2\, d\omega$	
Parseval's relation in Discrete Fourier Transform (DFT)	$\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n)$	$\frac{1}{N} \sum_{k=0}^{N-1} X_1(k)\, X_2^*(k)$

APPENDIX 5

Summary of Important Equations for FIR Filter Design

Table - A5.1 : Summary of Magnitude Function for Linear Phase FIR Filters

Case	h(n) [Impulse response]	N	Symmetry condition	Magnitude function, $\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
i	Symmetric	Odd	$h(N-1-n) = h(n)$	$\left\lvert h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2\,h\left(\frac{N-1}{2} - n\right)\cos\omega n \right\rvert$
ii	Symmetric	Even	$h(N-1-n) = h(n)$	$\left\lvert \sum_{n=1}^{\frac{N}{2}} 2\,h\left(\frac{N}{2} - n\right)\cos\left(\omega\left(n - \frac{1}{2}\right)\right) \right\rvert$
iii	Antisymmetric	Odd	$h(N-1-n) = -h(n)$	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} 2\,h\left(\frac{N-1}{2} - n\right)\sin\omega n \right\rvert$
iv	Antisymmetric	Even	$h(N-1-n) = -h(n)$	$\left\lvert \sum_{n=1}^{\frac{N}{2}} 2\,h\left(\frac{N}{2} - n\right)\sin\left(\omega\left(n - \frac{1}{2}\right)\right) \right\rvert$
v	Symmetric	Odd	$h(-n) = h(n)$	$\left\lvert h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2\,h(n)\cos\omega n \right\rvert$
vi	Antisymmetric	Odd	$h(-n) = -h(n)$	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} 2\,h(n)\sin\omega n \right\rvert$

Table - A5.2 : Specification and Desired Impulse Response for FIR Filter Design by Fourier Series Method

Type of filter	Specifications	Impulse response
Lowpass	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\omega_c \le \omega \le +\omega_c \\ 0 \ ; \ \text{for} & -\pi \le \omega < -\omega_c \\ 0 \ ; \ \text{for} & \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{+\omega_c} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_c \text{ and } +\omega_c < \omega \le \pi\right]$
Highpass	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\pi \le \omega \le -\omega_c \\ 1 \ ; \ \text{for} & \omega_c \le \omega \le \pi \\ 0 \ ; \ \text{for} & -\omega_c < \omega < +\omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_c < \omega < +\omega_c\right]$
Bandpass	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\omega_{c2} \le \omega \le -\omega_{c1} \\ 1 \ ; \ \text{for} & \omega_{c1} \le \omega \le \omega_{c2} \\ 0 \ ; \ \text{for} & -\pi \le \omega < -\omega_{c2} \\ 0 \ ; \ \text{for} & -\omega_{c1} < \omega < +\omega_{c1} \\ 0 \ ; \ \text{for} & \omega_{c2} < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_{c2} \ ; \ -\omega_{c1} < \omega < +\omega_{c1} \text{ and } +\omega_{c2} < \omega \le \pi\right]$
Bandstop	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\pi \le \omega \le -\omega_{c2} \\ 1 \ ; \ \text{for} & -\omega_{c1} \le \omega \le +\omega_{c1} \\ 1 \ ; \ \text{for} & \omega_{c2} \le \omega \le \pi \\ 0 \ ; \ \text{for} & -\omega_{c2} < \omega < -\omega_{c1} \\ 0 \ ; \ \text{for} & \omega_{c1} < \omega < \omega_{c2} \end{cases}$	$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{-\omega_{c2}} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{\omega_{c2}}^{+\pi} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_{c2} < \omega < -\omega_{c1} \text{ and } +\omega_{c1} < \omega < +\omega_{c2}\right]$

Table - A5.3 : Window Sequences for FIR Filter Design

Name of window	Window sequence
Rectangular	$w_R(n) = 1 \; ; \; \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \; \text{other } n$
	$w_R(n) = 1 \; ; \; \text{for } n = 0 \text{ to } N-1$ $= 0 \; ; \; \text{other } n$
Triangular	$w_T(n) = 1 - \frac{2\lvert n\rvert}{N-1} \; ; \; \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \; \text{other } n$
	$w_T(n) = 1 - \frac{2\lvert n-(N-1)/2\rvert}{N-1} \; ; \; \text{for } n = 0 \text{ to } N-1$ $= 0 \; ; \; \text{other } n$
Hanning	$w_C(n) = 0.5 + 0.5 \cos \frac{2\pi n}{N-1} \; ; \; \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \; \text{other } n$
	$w_C(n) = 0.5 - 0.5 \cos \frac{2\pi n}{N-1} \; ; \; \text{for } n = 0 \text{ to } N-1$ $= 0 \; ; \; \text{other } n$
Hamming	$w_H(n) = 0.54 + 0.46 \cos \frac{2\pi n}{N-1} \; ; \; \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \; \text{other } n$
	$w_H(n) = 0.54 - 0.46 \cos \frac{2\pi n}{N-1} \; ; \; \text{for } n = 0 \text{ to } N-1$ $= 0 \; ; \; \text{other } n$
Blackman	$w_B(n) = 0.42 + 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1} \; ; \; \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \; \text{other } n$
	$w_B(n) = 0.42 - 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1} \; ; \; \text{for } n = 0 \text{ to } N-1$ $= 0 \; ; \; \text{other } n$
Kaiser	$w_k(n) = \frac{I_0(\beta_1)}{I_0(a)} \; ; \; \text{for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \; \text{other } n$ where, $\beta_1 = a\left[1 - \left(\frac{2n}{N-1}\right)^2\right]^{0.5}$
	$w_K(n) = \frac{I_0(\beta_2)}{I_0(a_2)} \; ; \; \text{for } n = 0 \text{ to } N-1$ $= 0 \; ; \; \text{other } n$ where, $\beta_2 = a\left[\left(\frac{N-1}{2}\right)^2 - \left(n - \frac{N-1}{2}\right)^2\right]^{0.5}$ $a_2 = a\frac{N-1}{2}$

Table - A5.4 : The Normalized Ideal (Desired) Frequency Response and Impulse Response for FIR Filter Design Using Windows

Type of filter	Ideal (desired) frequency response	Ideal (desired) impulse response
Lowpass filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \ -\omega_c \le \omega \le +\omega_c \\ 0 & ; \ -\pi \le \omega < -\omega_c \\ 0 & ; \ \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_c \text{ and } +\omega_c < \omega \le +\pi\right]$
Highpass filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \ -\pi \le \omega \le -\omega_c \\ e^{-j\omega\alpha} & ; \ \omega_c \le \omega \le \pi \\ 0 & ; \ -\omega_c < \omega < +\omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_c < \omega < +\omega_c\right]$
Bandpass filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \ -\omega_{c2} \le \omega \le -\omega_{c1} \\ e^{-j\omega\alpha} & ; \ \omega_{c1} \le \omega \le \omega_{c2} \\ 0 & ; \ -\pi \le \omega < -\omega_{c2} \\ 0 & ; \ -\omega_{c1} < \omega < +\omega_{c1} \\ 0 & ; \ \omega_{c2} < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_{c2}\ ;\ -\omega_{c1} < \omega < +\omega_{c1} \text{ and } +\omega_{c2} < \omega \le +\pi\right]$
Bandstop filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \ -\pi \le \omega \le -\omega_{c2} \\ e^{-j\omega\alpha} & ; \ -\omega_{c1} \le \omega \le +\omega_{c1} \\ e^{-j\omega\alpha} & ; \ \omega_{c2} \le \omega \le \pi \\ 0 & ; \ -\omega_{c2} < \omega < -\omega_{c1} \\ 0 & ; \ \omega_{c1} < \omega < \omega_{c2} \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega}) e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{+\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_{c2} < \omega < -\omega_{c1} \text{ and } +\omega_{c1} < \omega < +\omega_{c2}\right]$

Table - A5.1 : Summary of A(w) for Linear Phase FIR Filters

Case	h(n) [Impulse response]	N	Symmetry condition	Magnitude function, $\lvert H(e^{jw})\rvert = \lvert A(w)\rvert$
i	Symmetric	Odd	$h(N-1-n) = h(n)$	$\left\lvert h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2\,h\left(\frac{N-1}{2} - n\right) \cos \omega n \right\rvert$
ii	Symmetric	Even	$h(N-1-n) = h(n)$	$\left\lvert \sum_{n=1}^{\frac{N}{2}} 2\,h\left(\frac{N}{2} - n\right) \cos\left(\omega\left(n - \frac{1}{2}\right)\right) \right\rvert$
iii	Antisymmetric	Odd	$h(N-1-n) = -h(n)$	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} 2\,h\left(\frac{N-1}{2} - n\right) \sin \omega n \right\rvert$
iv	Antisymmetric	Even	$h(N-1-n) = -h(n)$	$\left\lvert \sum_{n=1}^{\frac{N}{2}} 2\,h\left(\frac{N}{2} - n\right) \sin\left(\omega\left(n - \frac{1}{2}\right)\right) \right\rvert$
v	Symmetric	Odd	$h(-n) = h(n)$	$\left\lvert h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2\,h(n) \cos \omega n \right\rvert$
vi	Antisymmetric	Odd	$h(-n) = -h(n)$	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} 2\,h(n) \sin \omega n \right\rvert$

Table - A5.2 : Specification and Desired Impulse Response for FIR Filter Design by Fourier Series Method

Type of filter	Specifications	Impulse response
Lowpass	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\omega_c \le \omega \le +\omega_c \\ 0 \ ; \ \text{for} & -\pi \le \omega < -\omega_c \\ 0 \ ; \ \text{for} & \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \dfrac{1}{2\pi} \displaystyle\int_{-\omega_c}^{+\omega_c} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_c \text{ and } +\omega_c < \omega \le \pi\right]$
Highpass	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\pi \le \omega \le -\omega_c \\ 1 \ ; \ \text{for} & \omega_c \le \omega \le \pi \\ 0 \ ; \ \text{for} & -\omega_c < \omega < +\omega_c \end{cases}$	$h_d(n) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{-\omega_c} e^{j\omega n}\, d\omega + \dfrac{1}{2\pi} \displaystyle\int_{\omega_c}^{\pi} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_c < \omega < +\omega_c\right]$
Bandpass	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\omega_{c2} \le \omega \le -\omega_{c1} \\ 1 \ ; \ \text{for} & \omega_{c1} \le \omega \le \omega_{c2} \\ 0 \ ; \ \text{for} & -\pi \le \omega < -\omega_{c2} \\ 0 \ ; \ \text{for} & -\omega_{c1} < \omega < +\omega_{c1} \\ 0 \ ; \ \text{for} & \omega_{c2} < \omega \le \pi \end{cases}$	$h_d(n) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \dfrac{1}{2\pi} \displaystyle\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega n}\, d\omega + \dfrac{1}{2\pi} \displaystyle\int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_{c2} \ ; \ -\omega_{c1} < \omega < +\omega_{c1} \text{ and } +\omega_{c2} < \omega \le \pi\right]$
Bandstop	$H_d(e^{j\omega}) = \begin{cases} 1 \ ; \ \text{for} & -\pi \le \omega \le -\omega_{c2} \\ 1 \ ; \ \text{for} & -\omega_{c1} \le \omega \le +\omega_{c1} \\ 1 \ ; \ \text{for} & \omega_{c2} \le \omega \le \pi \\ 0 \ ; \ \text{for} & -\omega_{c2} < \omega < -\omega_{c1} \\ 0 \ ; \ \text{for} & \omega_{c1} < \omega < \omega_{c2} \end{cases}$	$h_d(n) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{-\omega_{c2}} e^{j\omega n}\, d\omega + \dfrac{1}{2\pi} \displaystyle\int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega n}\, d\omega + \dfrac{1}{2\pi} \displaystyle\int_{\omega_{c2}}^{+\pi} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_{c2} < \omega < -\omega_{c1} \text{ and } +\omega_{c1} < \omega < +\omega_{c2}\right]$

Table - A5.3 : Window Sequences for FIR Filter Design

Name of window	Window sequence
Rectangular	$w_R(n) = 1 \; ; \text{ for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \text{ other } n$
	$w_R(n) = 1 \; ; \text{ for } n = 0 \text{ to } N-1$ $= 0 \; ; \text{ other } n$
Triangular	$w_T(n) = 1 - \frac{2\|n\|}{N-1} \; ; \text{ for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \text{ other } n$
	$w_T(n) = 1 - \frac{2\|n-(N-1)/2\|}{N-1} \; ; \text{ for } n = 0 \text{ to } N-1$ $= 0 \; ; \text{ other } n$
Hanning	$w_C(n) = 0.5 + 0.5 \cos \frac{2\pi n}{N-1} \; ; \text{ for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \text{ other } n$
	$w_C(n) = 0.5 - 0.5 \cos \frac{2\pi n}{N-1} \; ; \text{ for } n = 0 \text{ to } N-1$ $= 0 \; ; \text{ other } n$
Hamming	$w_H(n) = 0.54 + 0.46 \cos \frac{2\pi n}{N-1} \; ; \text{ for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \text{ other } n$
	$w_H(n) = 0.54 - 0.46 \cos \frac{2\pi n}{N-1} \; ; \text{ for } n = 0 \text{ to } N-1$ $= 0 \; ; \text{ other } n$
Blackman	$w_B(n) = 0.42 + 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1} \; ; \text{ for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \text{ other } n$
	$w_B(n) = 0.42 - 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1} \; ; \text{ for } n = 0 \text{ to } N-1$ $= 0 \; ; \text{ other } n$
Kaiser	$w_k(n) = \frac{I_0(\beta_1)}{I_0(a)} \; ; \text{ for } n = -\frac{N-1}{2} \text{ to } +\frac{N-1}{2}$ $= 0 \; ; \text{ other } n$ where, $\beta_1 = a\left[1-\left(\frac{2n}{N-1}\right)^2\right]^{0.5}$
	$w_K(n) = \frac{I_0(\beta_2)}{I_0(a_2)} \; ; \text{ for } n = 0 \text{ to } N-1$ $= 0 \; ; \text{ other } n$ where, $\beta_2 = a\left[\left(\frac{N-1}{2}\right)^2 - \left(n-\frac{N-1}{2}\right)^2\right]^{0.5}$ $a_2 = a\frac{N-1}{2}$

Table - A5.4 : The Normalized Ideal (Desired) Frequency Response and Impulse Response for FIR Filter Design Using Windows

Type of filter	Ideal (desired) frequency response	Ideal (desired) impulse response
Lowpass filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \quad -\omega_c \le \omega \le +\omega_c \\ 0 & ; \quad -\pi \le \omega < -\omega_c \\ 0 & ; \quad \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_c \text{ and } +\omega_c < \omega \le +\pi\right]$
Highpass filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \quad -\pi \le \omega \le -\omega_c \\ e^{-j\omega\alpha} & ; \quad \omega_c \le \omega \le \pi \\ 0 & ; \quad -\omega_c < \omega < +\omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_c < \omega < +\omega_c\right]$
Bandpass filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \quad -\omega_{c2} \le \omega \le -\omega_{c1} \\ e^{-j\omega\alpha} & ; \quad \omega_{c1} \le \omega \le \omega_{c2} \\ 0 & ; \quad -\pi \le \omega < -\omega_{c2} \\ 0 & ; \quad -\omega_{c1} < \omega < +\omega_{c1} \\ 0 & ; \quad \omega_{c2} < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\pi \le \omega < -\omega_{c2}\ ;\ -\omega_{c1} < \omega < +\omega_{c1} \text{ and } +\omega_{c2} < \omega \le +\pi\right]$
Bandstop filter	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \quad -\pi \le \omega \le -\omega_{c2} \\ e^{-j\omega\alpha} & ; \quad -\omega_{c1} \le \omega \le +\omega_{c1} \\ e^{-j\omega\alpha} & ; \quad \omega_{c2} \le \omega \le \pi \\ 0 & ; \quad -\omega_{c2} < \omega < -\omega_{c1} \\ 0 & ; \quad \omega_{c1} < \omega < \omega_{c2} \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d(e^{j\omega}) e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{+\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega$ $\left[\because H_d(e^{j\omega}) = 0 \text{ in the range } -\omega_{c2} < \omega < -\omega_{c1} \text{ and } +\omega_{c1} < \omega < +\omega_{c2}\right]$

A5.5 : Transformations

Impulse Invariant

$$\frac{1}{(s + p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{dp_i^{m-1}} \frac{1}{1 - e^{-p_i T} z^{-1}}$$

$$\frac{(s + a)}{(s + a)^2 + b^2} \longrightarrow \frac{1 - e^{-aT} (\cos bT) z^{-1}}{1 - 2e^{-aT} (\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(s + a)^2 + b^2} \longrightarrow \frac{e^{-aT} (\sin bT) z^{-1}}{1 - 2e^{-aT} (\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

Bilinear Transformation

$$s\,Y(s) \xrightarrow[\text{(is transformed to)}]{} \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} Y(z)$$

Table - A5.6 : Summary of Butterworth Lowpass Filter Normalized Transfer Function

Order, N	Normalized tansfer function, $H(s_n)$
1	$\frac{1}{s_n + 1}$
2	$\frac{1}{s_n^2 + 1.414 s_n + 1}$
3	$\frac{1}{(s_n + 1)(s_n^2 + s_n + 1)}$
4	$\frac{1}{(s_n^2 + 0.765 s_n + 1)(s_n^2 + 1.848 s_n + 1)}$
5	$\frac{1}{(s_n + 1)(s_n^2 + 0.618 s_n + 1)(s_n^2 + 1.618 s_n + 1)}$
6	$\frac{1}{(s_n^2 + 1.932 s_n + 1)(s_n^2 + 1.414 s_n + 1)(s_n^2 + 0.518 s_n + 1)}$

Table - A5.7 : Summary of Transformation for Analog Filter

Filter Type	Transformation
Lowpass	$s_n \rightarrow \dfrac{s}{\Omega_c}$
Highpass	$s_n \rightarrow \dfrac{\Omega_c}{s}$
Bandpass	$s_n \rightarrow \dfrac{Q\,(s^2 + \Omega_0^2)}{\Omega_0 s}$
Bandstop	$s_n \rightarrow \dfrac{\Omega_0 s}{Q\,(s^2 + \Omega_0^2)}$

Table -A5.8 : Properties of several commonly used windows with length N

Window	Sidelobe level	3 dB Bandwidth	
		Dw_{3dB}	Df_{3dB}
Rectangular	-13 dB	0.89 (2p/N)	0.89 /N
Bartlett	-27 dB	1.28 (2p/N)	1.28 /N
Hanning	-32 dB	1.44 (2p/N)	1.44 /N
Hamming	-43 dB	1.30 (2p/N)	1.30 /N
Blackman	-58 dB	1.68 (2p/N)	1.68 /N

Table - A5.9 : Summary of performance measures of various power spectrum estimator

Method	Quality Factor Q	Variability $\mathcal{V}$	Frequency Resolution Df	Figure of merit $\mathcal{M}$
Periodogram	1	1	$\dfrac{0.89}{N}$	$\dfrac{0.89}{N}$
Bartlett	$1.12N\,Df_B$ (or L)	$\dfrac{1}{L}$	$L\,\dfrac{0.89}{N}$	$\dfrac{0.89}{N}$
Welch (50% overlap, Bartlett window)	$1.39N\;Df_W$ $\left(\text{or } \dfrac{8L}{9}\right)$	$\dfrac{9}{8L}$	$\dfrac{1.28}{M}$	$\dfrac{0.72}{N}$
Blackman-Tukey (Bartlett window)	$2.34N\;Df_{BT}$ $\left(\text{or } \dfrac{3N}{2M}\right)$	$\dfrac{2M}{3N}$	$\dfrac{0.64}{M}$	$\dfrac{0.43}{N}$

APPENDIX 6

Summary of Important Equations for IIR Filter Design

Table - A6.1 : Transformations

Impulse Invariant Transformation

$$\frac{1}{(s + p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{dp_i^{m-1}} \frac{1}{1 - e^{-p_i T} z^{-1}}$$

$$\frac{(s+a)}{(s+a)^2 + b^2} \longrightarrow \frac{1 - e^{-aT} (\cos bT) z^{-1}}{1 - 2e^{-aT} (\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(s+a)^2 + b^2} \longrightarrow \frac{e^{-aT} (\sin bT) z^{-1}}{1 - 2e^{-aT} (\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

Bilinear Transformation

$$s\,Y(s) \xrightarrow[\text{(is transformed to)}]{} \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} Y(z)$$

Table - A6.2 : Summary of Butterworth Lowpass Filter Normalized Transfer Function

Order, N	Normalized tansfer function, $H(s_n)$
1	$\frac{1}{s_n + 1}$
2	$\frac{1}{s_n^2 + 1.414 s_n + 1}$
3	$\frac{1}{(s_n + 1)(s_n^2 + s_n + 1)}$
4	$\frac{1}{(s_n^2 + 0.765 s_n + 1)(s_n^2 + 1.848 s_n + 1)}$
5	$\frac{1}{(s_n + 1)(s_n^2 + 0.618 s_n + 1)(s_n^2 + 1.618 s_n + 1)}$
6	$\frac{1}{(s_n^2 + 1.932 s_n + 1)(s_n^2 + 1.414 s_n + 1)(s_n^2 + 0.518 s_n + 1)}$

Table - A6.3 : Summary of Transformation for Analog Filter

Filter Type	Transformation	Filter Type	Transformation
Lowpass	$s_n \rightarrow \frac{s}{\Omega_c}$	Bandpass	$s_n \rightarrow \frac{Q(s^2 + \Omega_0^2)}{\Omega_0 s}$
Highpass	$s_n \rightarrow \frac{\Omega_c}{s}$	Bandstop	$s_n \rightarrow \frac{\Omega_0 s}{Q(s^2 + \Omega_0^2)}$

APPENDIX 7

Summary of Properties of Power Spectrum Estimator

Table -A7.1 : Properties of Several Commonly used Windows with Length N

Window	Sidelobe level	3 dB Bandwidth	
		$\Delta\omega_{3dB}$	Δf_{3dB}
Rectangular	−13 dB	0.89 (2π/N)	0.89 /N
Bartlett	−27 dB	1.28 (2π/N)	1.28 /N
Hanning	−32 dB	1.44 (2π/N)	1.44 /N
Hamming	−43 dB	1.30 (2π/N)	1.30 /N
Blackman	−58 dB	1.68 (2π/N)	1.68 /N

Table - A7.2 : Summary of Performance Measures of Various Power Spectrum Estimator

Method	Quality Factor Q	Variability $\mathcal{V}$	Frequency Resolution Δf	Figure of merit $\mathcal{M}$
Periodogram	1	1	$\frac{0.89}{N}$	$\frac{0.89}{N}$
Bartlett	1.12N Δf_B (or L)	$\frac{1}{L}$	$L\,\frac{0.89}{N}$	$\frac{0.89}{N}$
Welch (50% overlap, Bartlett window)	1.39N Δf_W $\left(\text{or } \frac{8L}{9}\right)$	$\frac{9}{8L}$	$\frac{1.28}{M}$	$\frac{0.72}{N}$
Blackman-Tukey (Bartlett window)	2.34N Δf_{BT} $\left(\text{or } \frac{3N}{2M}\right)$	$\frac{2M}{3N}$	$\frac{0.64}{M}$	$\frac{0.43}{N}$

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