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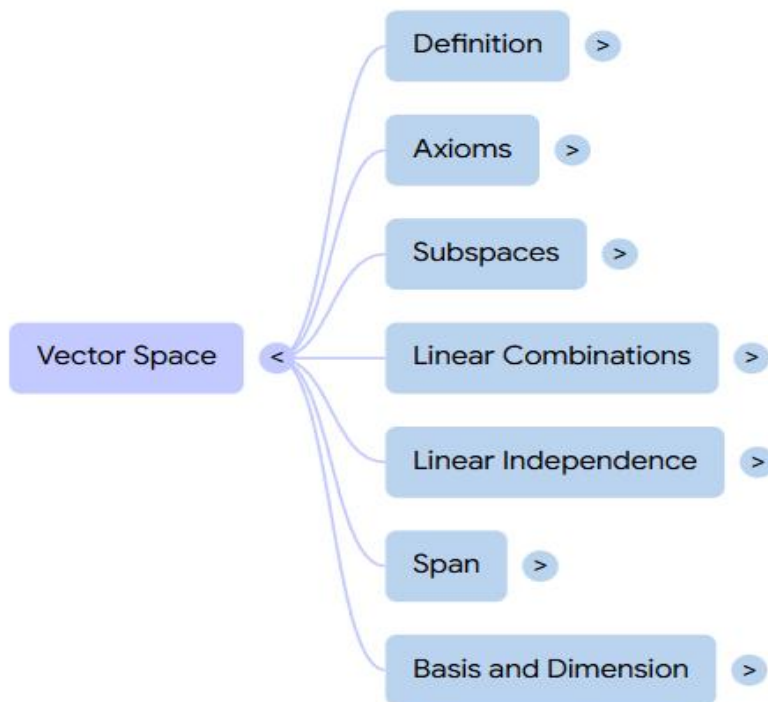
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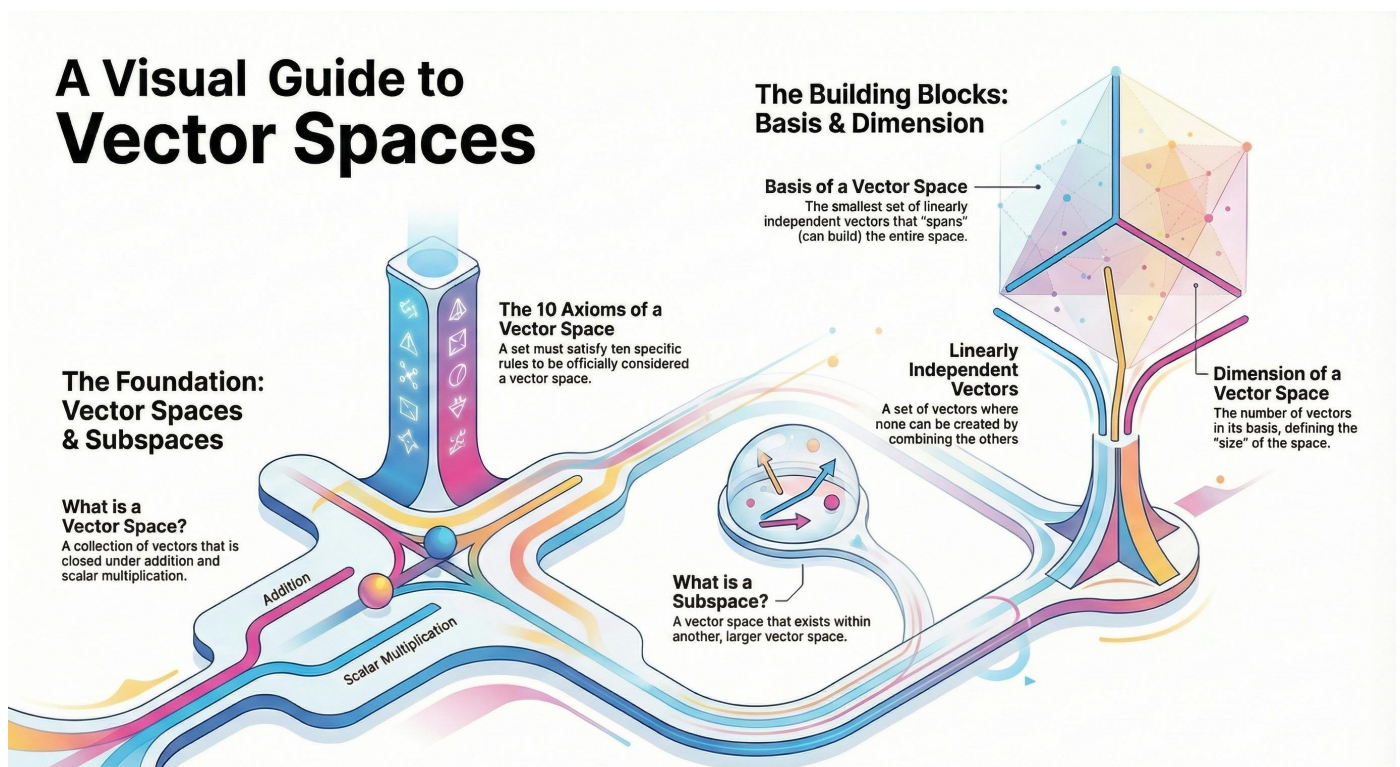**Applied Sciences and Humanities**

Unit 2 Vector Space

Study Guide
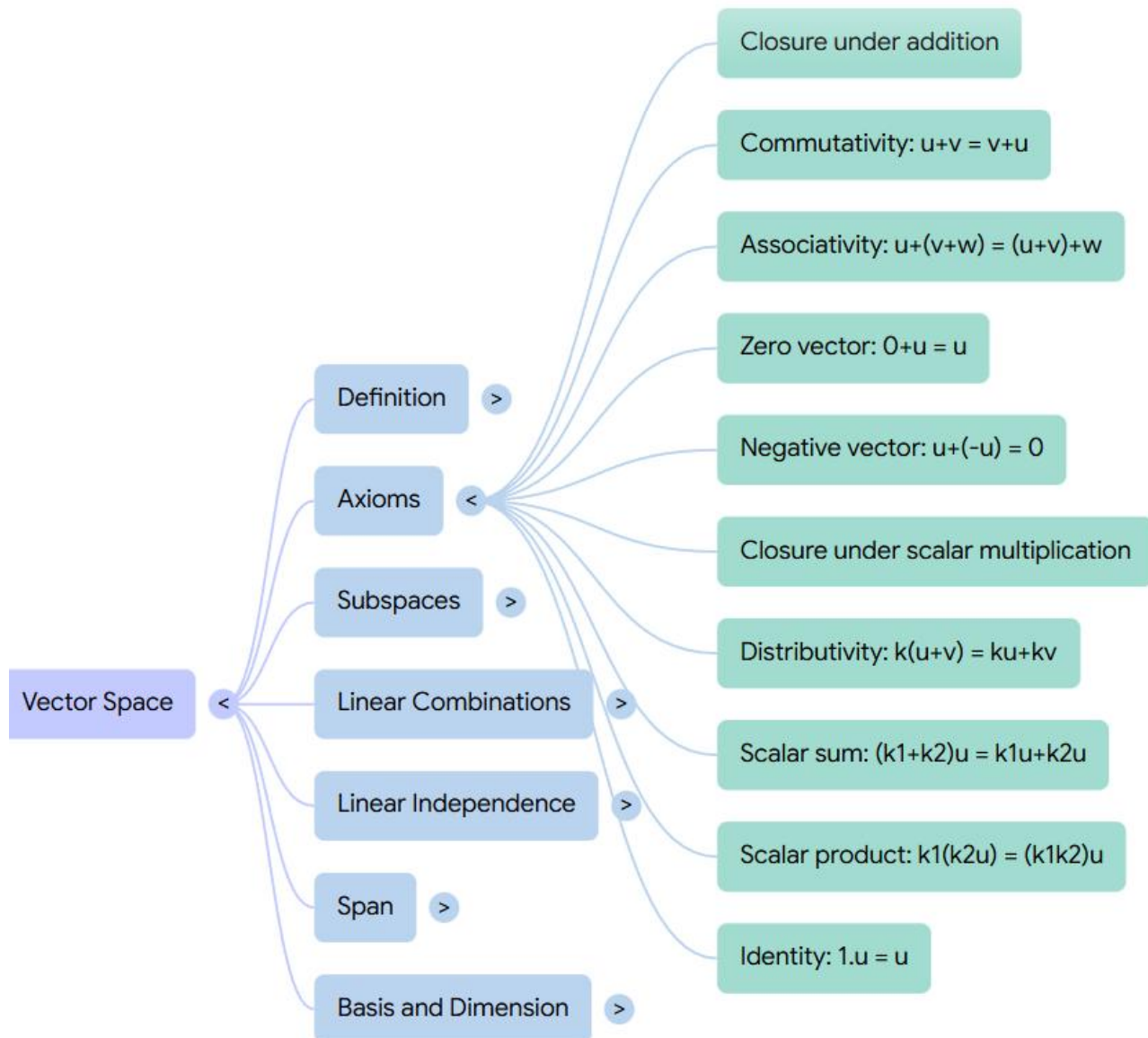
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A Visual Guide to **Vector Spaces**

The Building Blocks: Basis & Dimension

Basis of a Vector Space
The smallest set of linearly independent vectors that "spans" (can build) the entire space.

The Foundation: Vector Spaces & Subspaces

The 10 Axioms of a Vector Space
A set must satisfy ten specific rules to be officially considered a vector space.

Linearly Independent Vectors
A set of vectors where none can be created by combining the others

Dimension of a Vector Space
The number of vectors in its basis, defining the "size" of the space.

What is a Vector Space?
A collection of vectors that is closed under addition and scalar multiplication.

Addition

Scalar Multiplication

What is a Subspace?
A vector space that exists within another, larger vector space.

2. VECTOR SPACE:

Let V be a non-empty set of vectors on which the operations of addition and multiplication by scalars are defined.
If the following axioms are satisfied by all vectors u,v,w in V and all scalars k_1, k_2 then V is called a vector space and elements in V are called vectors
1. If u and v are vectors in V then u+v is in V.
2. u+v=v+u
3. u+(v+w)=(u+v)+w
4. There is an vector 0 in V, called Zero vector , such that 0+u = u+0=u for all u in V.

5. For each vector u in V, there exist an vector −u in V is called a negative of u, such that
u+ (-u) = (-u)+u=0.
6. If k_1 is any scalar and u is an vector in V,
then k_1u is in V.
7. $k_1(u+v)=k_1u+k_1v$
8. If k_1,k_2 are scalars and u is an vector in V,
$$then\ (k_1+k_2)u=k_1u+k_2u.$$
9. $k_1(k_2u)=(k_1k_2)u$
10. 1.u=u

Example1: Determine whether the set V of all pairs of real numbers (x,y)with the operations
$(x_1,y_1)+(x_2,y_2)=(x_1+x_2+1, y_1+y_2+1)$ and k(x,y)=(kx,ky)is a vector space.
Solution:-Let u=(x_1,y_1), v=$(x_2,y_2)$and w=(x_3,y_3) are vectors in V and k_1, k_2 are some scalars.
1.$u+v=(x_1,y_1)+(x_2,y_2)=(x_1+x_2+1, y_1+y_2+1) \in V$
2.$u+v=(x_1+x_2+1, y_1+y_2+1)=(x_2+x_1+1, y_2+y_1+1)=v+u$
3.$u+(v+w)=(x_1,y_1)+[(x_2,y_2)+(x_3,y_3)]$
$=(x_1,y_1)+[(x_2+x_3+1, y_2+y_3+1)]$
$=[x_1+(x_2+x_3+1)+1,y_1+(y_2+y_3+1)+1]$
$=[(x_1+x_2+1)+x_3+1,(y_1+y_2+1)+y_3+1]$
$=(x_1+x_2+1, y_1+y_2+1)+(x_3,y_3)=(u+v)+w$
4.Let (a,b)be an vector in V such that (a,b)+u=u
(a,b)+(x_1,y_1)=(x_1,y_1)
$(a+x_1+1, b+y_1+1)=(x_1,y_1)$=$a+x_1+1=x_1$, $b+y_1+1=y_1$
a=−1 and b=−1, Hence (−1,−1)is zero vector
5. Let (a,b)be an vector in V such that u+(a,b)=(−1,−1)
(x_1,y_1)+(a,b)=(−1,−1)
$(x_1+a+1, y_1+b+1)=(−1,−1)$=$x_1+a+1=−1$, $y_1+b+1=−1$
$a=−x_1−2$ and $b=−y_1−2$,
Hence $(−x_1−2 ,−y_1−2)$ is the negative of u in V.
6. $k_1u=k_1(x_1,y_1)=(k_1x_1,k_1y_1) \in V$
7. $k_1(u+v)=k_1(x_1+x_2+1, y_1+y_2+1)$
$=(k_1x_1+k_1x_2+k_1,k_1 y_1+k_1y_2+k_1)$
$\neq k_1u+k_1v$
∴V is not distributive under scalar muiltiplication, hence V is not a vector space.

Example 2:
Determine whether the set R$^+$of all positive real number with operations
x+y=xy and kx=x^k isa vector space.
Solution:-
Let x,y and z be positive real numbers in R$^+$ and k_1,k_2 are some scalars
1. x+y=xy, is also a positive real number.
2. x+y=xy=yx=y+x
3. x+(y+z)=x(y+z)=x(yz)=(xy)z=(x+y)z=(x+y)+z
4. Let a be a object in R$^+$ such that a+x=x, ax=x, a=1, so 1in zero vector in V
5. x+a=1, xa=1, a=$\frac{1}{x}$, so$\frac{1}{x}$ is the negative of x in R$^+$
6. If $k_1$is real then k_1x=x$^{k_1} \in$R$^+$
7. $k_1(x+y)=k_1(xy)=(xy)^{k_1}=x^{k_1}y^{k_1}=(k_1x)(k_1y)=k_1x+k_1y$

8. $(k_1+k_2)x=x^{k_1+k_2}=x^{k_1}x^{k_2}=k_1xk_2x=k_1x+k_1y$

9. $k_1(k_2x)=k_1x^{k_2}=\left(x^{k_2}\right)^{k_1}=x^{k_2k_1}=x^{(k_1k_2)}=(k_1k_2)x$

10. $1x=x^1=x$

   All axioms are satisfied by R^+ under operations.

   Hence , $R^+$is a vector space under given operations.

Exercise:

1. Check whether following are vector space or not?

a. The set of all pairs of real number (x,y) with the function

   $(x_1,y_1)+(x_2,y_2)=(x_1+x_2, y_1+y_2)$and $k(x,y)=(2kx, 2ky)$.

b. $(x_1, y_1,z_1)+(x_2,y_2,z_2)=(z_1+z_2,y_1+y_2, x_1+x_2)$

2. Check whether $V=R^2$ is a vector space with respect to

$(x_1,y_1)+(x_2,y_2)=(x_1+x_2-2,y_1+y_2-3)$ and $k(x,y)=(kx+2k-2, ky-3k+3)$, k is a real number.

3. The set of all 2*2 matrices of the form

$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with standard matrix addition and scalar multiplication.

4. The set of all $2*2$ matrices of the form $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}$ with $\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix}+\begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix}=\begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$ and

$$k\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}=\begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix}$$

5. The set off all ordered triples of real numbers of the form (0,0,z)with the operations

$$(0,0,z_1)+(0,0,z_2)=(0,0,z_1+z_2) \text{ and } k(0,0,z)=(0,0,kz)$$

2.1.SUBSPACES:

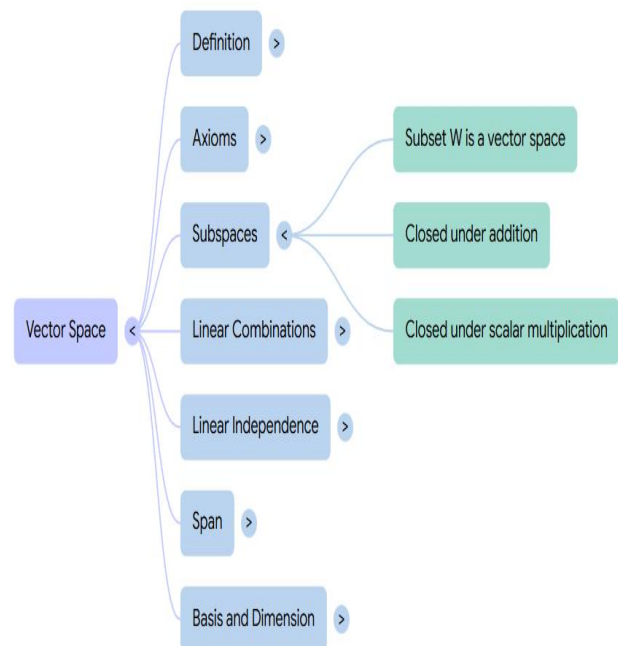|	
A non-empty subset W of a vector space is called subspace of V if W is itself a vector space under the operation defined on V. If W is a non-empty of vector space V, then W is a subspace of V if and only if following axioms hold. **Axiom 1:-** If u and v are vectors in W then u+v is in W. **Axiom 2 :-** If k is a scalar and u is a vector in W, then ku is in W	

Example 3: Show that $W=\{(x,y)=x=3y\}$ is a subspace of R^2.
Solution: Let $u=\{(x_1,y_1)\ /x_1=3y_1\}$ and $v=\{(x_2 y_2)|x_2=3y_2\}$ are in W and k is any scalar.
$$u+v=(x_1,y_1)+(x_2 y_2)= (x_1+x_2, y_1+y_2)x_1=3y_1$$
$$\text{and } x_2=3y_2 , x_1+x_2=3(y_1+y_2),$$
$$u+v=\{(x_1+x_2, y_1+y_2)|x_1+x_2=3(y_1+y_2)\} \qquad \text{thus u+v is in W}$$
$$ku=k(x_1,y_1)=(kx_1,ky_1), x_1=3y_1, kx_1=3(ky_1),$$
$$ku=\{(kx_1,ky_1),| kx_1=3(ky_1),$$

Thus ku is in W, so its a subspace.

Example 4:- $W=\{(x,y,z)|x^2+y^2+z^2\leq1\}$
Solution:- *Let $u=(1,0,0)$ and $v=(0,0,1)$ be two vectors of W satidfying $x^2+y^2+z^2\leq1$*
$$u+v=(1,0,0)+(0,0,1)=(1,0,1),$$

Here $x^2 + y^2 + z^2 = 2 > 1$. Thus $u+v$ is not in W. So W is not closed under addition and hence is not a subspace of R^3.

Exercise:

(i)Show that the following sets are the subspaces of the respective real vector space V
under standard operation (i)$W=\{a_0+a_1x+a_2x^2+a_3x^3\}|a_0=0\}$

(ii)$W=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\middle| a+b+c+d=0\right\}$

(iii)$W=\{(x,y)|x^2=y^2\}$, (iv)$W=\{A_{nn}|AB=BA$ for a fixed $B_{nn}\}$
(iv)$W=\{f|f(x)=a_1+a_2\sin x,$ where a_1 and a_2 are real numbers$\}$,
(v)$W=\{(x,y)|xy\geq0\}$

2.2.LINEAR COMBINATION:
If w is a vector in vector space V then w is said to be linear combination of the vectors
$v_1 , v_2 , \ldots , v_r$ if w can be expressed in the form,
$$w = k_1v_1 + k_2v_2 +\ldots\ldots+ k_r.v_r$$
where $k_1,k_2,\ldots,k_r$ are scalars and known as coefficient of linear combination.
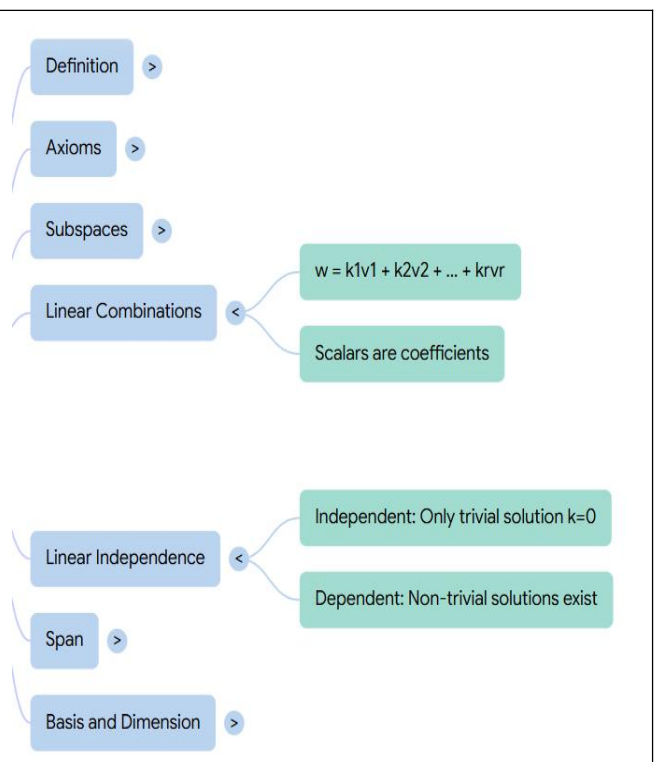
2.2.1.LINEAR DEPENDENCE AND INDEPENDENCE:
A
Let $S=\{v_1, v_2, \ldots,v_r\}$ is a nonempty set of vectors such that
$$k_1v_1+k_2v_2+\ldots+k_rv_r=0$$
If following homogeneous system obtained has trivi
a linearly independent.

the system has a non−trivial solution then S is called

Definition >

Axioms >

Subspaces >

Linear Combinations <

w = k1v1 + k2v2 + ... + krvr

Scalars are coefficients

Linear Independence <

Independent: Only trivial solution k=0

Dependent: Non-trivial solutions exist

Span >

Basis and Dimension >

Example 5:- Check following set in R^3 is linearly dependent or independent?

(i) $(4,-1,2), (-4,10,2), (4,0,1)$

Solution:

Let $v_1=(4,-1,2),\ v_2=(-4,10,2),\ v_3=(4,0,1)$ $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$ $k_1(4,-1,2)+k_2(-4,10,2)+k_3(4,0,1)=(0,0,0)$ $\begin{bmatrix} 4 & -4 & 4 \mid 0 \\ -1 & 10 & 0 \mid 0 \\ 2 & 2 & 1 \mid 0 \end{bmatrix}$ $\left(\frac{1}{4}\right)R_1$ $\begin{bmatrix} - & -1 & 1 \mid 0 \\ -1 & 10 & 0 \mid 0 \\ 2 & 2 & 1 \mid 0 \end{bmatrix}$ $R_2+R_1,\ \ R_3-2R_1$ Solution is $v_1=0, v_2=0, v_3=0$ They system has trivial solution so v_1, v_2, v_3 are linearly independent	$\begin{bmatrix} - & -1 & 1 \mid 0 \\ 0 & 9 & 1 \mid 0 \\ 0 & 4 & -1 \mid 0 \end{bmatrix}$ $\left(\frac{1}{9}\right)R_2$ $\begin{bmatrix} 1 & -1 & 1 \mid 0 \\ 0 & 1 & \frac{1}{9} \mid 0 \\ 0 & 4 & -1 \mid 0 \end{bmatrix}$ R_3-4R_2 $\begin{bmatrix} 1 & -1 & 1 & \mid 0 \\ 0 & 1 & \frac{1}{9} & \mid 0 \\ 0 & 0 & -\frac{13}{9} & \mid 0 \end{bmatrix}$

Example 6:-Let $v_1=(-2,0,1), v_2=(3,2,5),\ v_3=(6,-1,1),\ v_4=(7,0,-2)$

$k_1 v_1 + k_2 v_2 + k_3 v_3 + k_4 v_4 = 0$

$k_1(-2,0,1)+k_2(3,2,5)+k_3(6,-1,1)+k_4(7,0,-2)=(0,0,0)$

Equating corresponding components

$-2k_1+3k_2+6k_3+7k_4=0$

$2k_2-k_3=0$

$k_1+5k_2+k_3-2k_4=0$

The number of unknowns, r=4 and The number of equations ,n=3,

Hence v_1, v_2, v_3, v_4 are linearly dependent

Exercise: **Check L.D. OR L.I.**

(i)$2-x+4x^2,\ 3+6x+2x^2,\ 2+10x-4x^2$

(ii) $A_1=\begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $A_2=\begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}

(iii)Show that vectors $v_1=(0,3,1,-1),\ v_2=(6,0,5,1)$ and $v_3=(4,-7,1,3)$ form linearly dependent in R^4.

(iv)For what value of λ are vectors $v_1=\left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right),\ v_2=\left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right),\ v_3=\left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$

is linearly dependent in R^3

2.3. SPAN:

If S = $\{w_1, w_2, ..., w_r\}$ is a non-empty set of vectors in a vector space V , then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V generated by S, and we say that the vectors $w_1, w_2, ..., w_r$ span W.

We denote this subspace as W = span$\{w_1, w_2, ..., w_r\}$ or W = span(S).

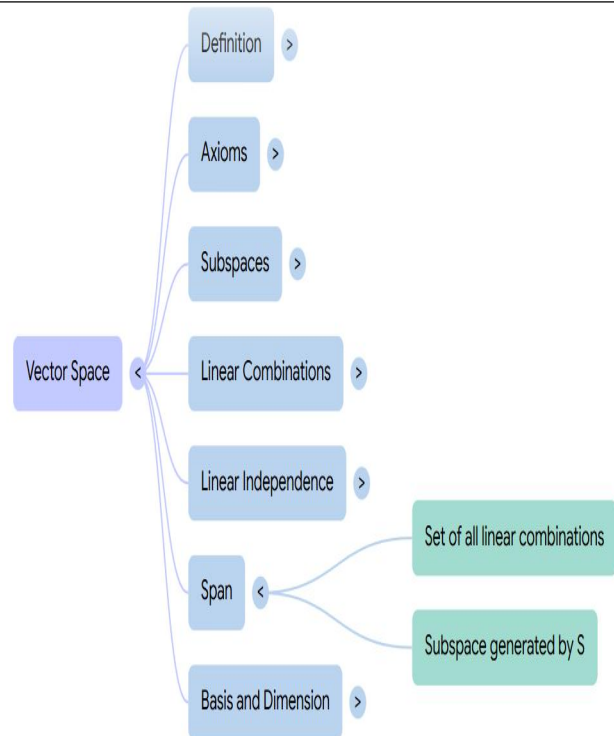$$Span(S) = c_1 v_1 + c_2 v_2 + ... + c_i v_i | \ \forall \ c_i \in R$$

(the set of all linear combinations of vectors in S)

2.4.BASIS:

The set of vectors S={v_1, v_2, v_3,....v_n}in a vector spa
(i)S is linearly independent
(ii)S spans V

- Definition
- Axioms
- Subspaces
- Vector Space
- Linear Combinations
- Linear Independence
- Span
  - Set of all linear combinations
  - Subspace generated by S
- Basis and Dimension

2.5.A SPANNING SET OF VECTOR

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called a spanning set of the vector space.
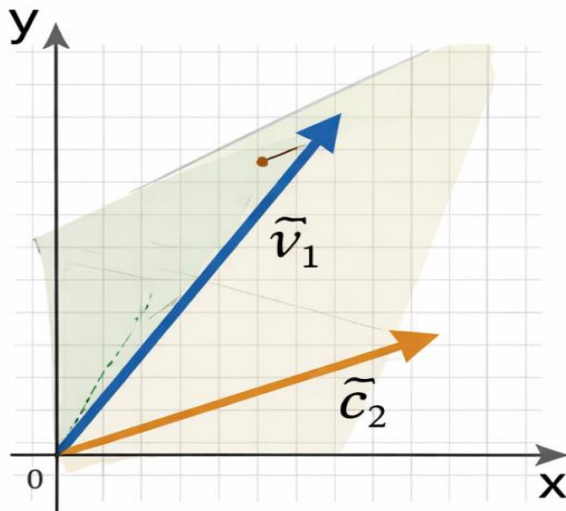OR
A vector **v** in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if **v** can be written in the form
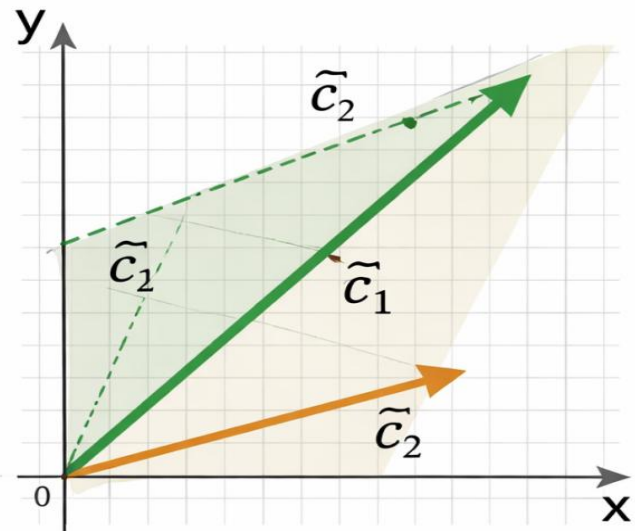
$$Span(S) = c_1 v_1 + c_2 v_2 + ... + c_i v_i | \ \forall \ c_i \in R$$

(the set of all linear combinations of vectors in S)

Spanning Set

$$\tilde{u} = c_1\tilde{v}_1 + c_2\tilde{v}_2$$

Example7:- Show that vectors $e_1=i=(1,0,0)$, $e_2=j=(0,1,0)$ and $e_3=k=(0,0,1)$ form a basis for R^3

Solution:- Let $b = (b_1, b_2, b_3)$ be an arbitrary vectors in R^3 and

can be expressed as a linear combination of $b=k_1e_1+k_2e_2+k_3e_3$
$(b_1, b_2, b_3)=k_1(1,0,0),+k_2(0,1,0)+k_3(0,0,1)=(k_1,k_2,k_3)$
Equating we get $k_1=b_1$, $k_2=b_2$, $k_3=b_3$

Since for each choice of (b_1, b_2, b_3) some scalars $k_1, k_2 k_3$ exist the given vectors span R^3.

Now consider , $k_1e_1+k_2e_2+k_3e_3=0$
$k_1(1,0,0),+k_2(0,1,0)+k_3(0,0,1)=(0,0,0)$, Equating we get $k_1=k_2=k_3=0$

Thus e_1, e_2 and e_3 are linearly independent. Hence e_1, e_2 and e_3 form a basis for R^3.

Example 8: Let $v_1=(1,2,1)$, $v_2=(2,9,0)$ and $v_3=(3,3,4)$ .Show that the set
$S=\{v_1,v_2,v_3\}$ is a basis for R^3.

Solution:- Let $b = (b_1, b_2, b_3)$ be an arbitrary vectors in R^3 and can be expressed as a linear combination of

$$b=k_1v_1+k_2v_2+k_3v_3$$
$$(b_1, b_2, b_3)=k_1(1,2,1),+k_2(2,9,0)+k_3(3,3,4)$$

Or $(b_1, b_2, b_3)=(k_1+2k_2+3k_3, 2k_1+9k_2+3k_3, k_1+4k_3)$

Or $k_1+2k_2+3k_3=b_1$, $2k_1+9k_2+3k_3=b_2$, $k_1+4k_3=b_3$

We form the matrix with v_1, v_2 , v_3 as a columns $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$

To check for linear independence, we determine if the determinant of A is nonzero.

Here $\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} \neq 0$,there fore the vectors are **linearly independent**.

Since S consists of three linearly independent vectors in R3, they must also span R3 . Therefore, S is a **basis** for R3.

Exercise:-

(i) Show that the set $S=\{1, x, x^2, ..., x^n\}$ is a basis for the vector space P_n.

(ii) Show that the set $S=\{v_1, v_2, v_3, v_4\}$ is a basis for the vector space M_{22} where
$$v_1=\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}, v_2=\begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}, v_3=\begin{bmatrix}0 & 0 \\ 1 & 0\end{bmatrix}, v_4=\begin{bmatrix}0 & 0 \\ 0 & 1\end{bmatrix}$$

(iii) Sow that the set $S=\{v_1, v_2, v_3\}$ is a basis for R^3, where $v_1(1,0,0)$, $v_2(2,2,0)$ and $v_3(3,3,3)$

(iv) Check which of following is basis for P_2, $-4+x+3x^2$, $6+5x+2x^2$, $8+4x+x^2$

(v) Let V be the space spanned by $v_1=\cos^2 x$, $v_2=\sin^2 x$, $v_3=\cos 2x$. Show that $S=\{v_1, v_2, v_3\}$ is not a basis
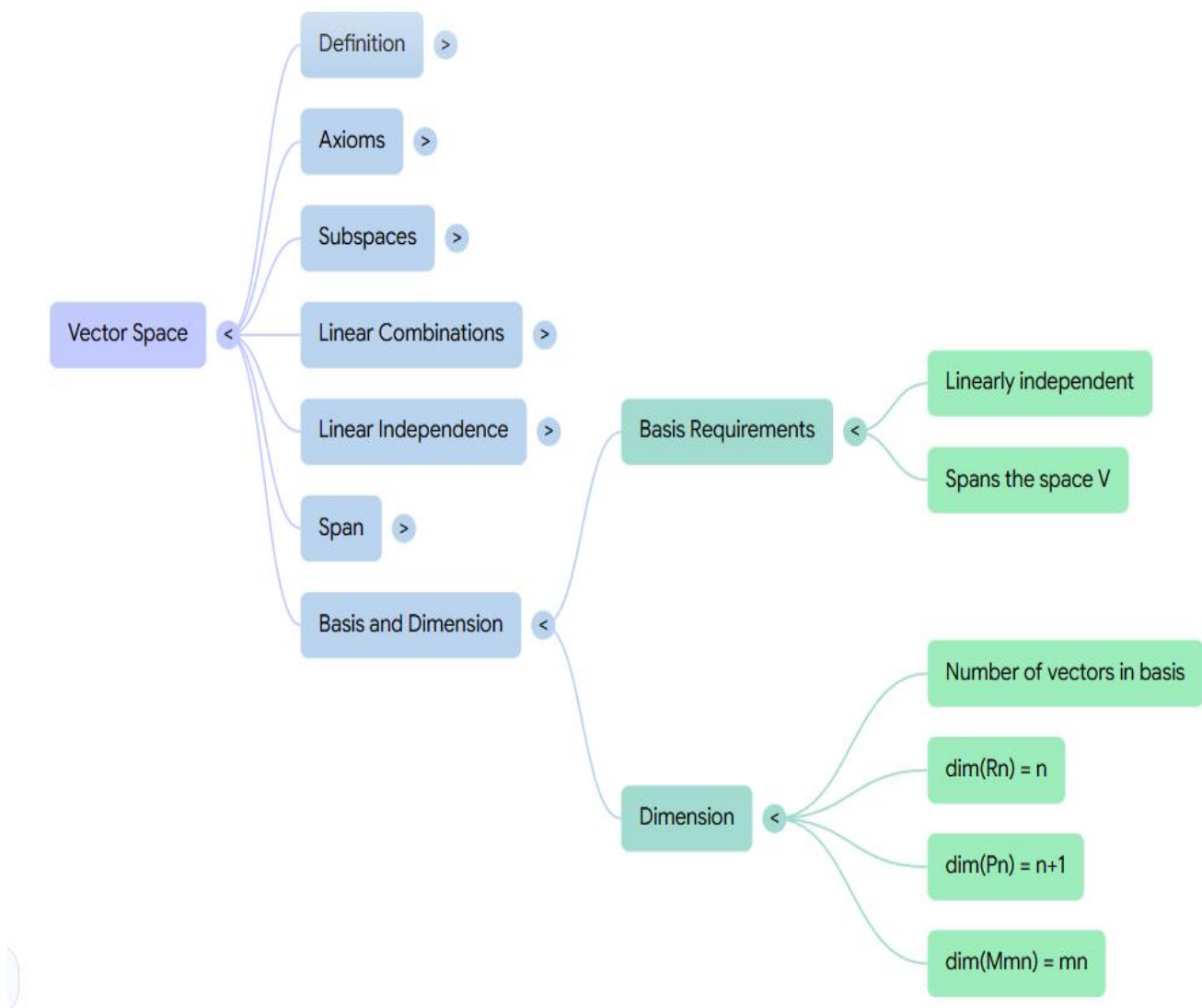
2.6.DIMENSIONS:

The number of vectors in basis of a non –zero finite dimensional vector space V is known space V is known as the dimension of V and is denoted by dim(V).

Remark :

(i) dim $(R^n) = n,$

(ii) dim $(P_n) = n+1,$

(iii) dim $(M_{mn}) = mn,$ (iv) dim $\{0\} = 0$

Example9: Determine the dimension and a basis for the solution space of the system

$$x_1 + x_2 - 2x_3 = 0, \ -2x_1 - 2x_2 + 4x_3 = 0, \ -x_1 - x_2 + 2x_3 = 0$$

Solution:-

The matrix form of the system is, $\begin{bmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The augmented matrix is $\begin{bmatrix} 1 & 1 & -2 & 0 \\ -2 & -2 & 4 & | & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix}$

$R_2 + 2R_1, \ R_3 + R_1$

$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The corresponding system of equation is $x_1 + x_2 - 2x_3 = 0$, solving for leading variables,

$x_1 = -x_2 2x_3$

Assigning the free variables x_2 and x_3 arbitrary values t_1 and t_2 respectively,

$x_1 = -t_1 + 2t_2, \ x_2 = t_1, \ x_3 = t_2$ is the solution of the system

The solution vector is x= $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t_1 + 2t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \ = t_1 x_1 + t_2 x_2$

Basis= $\{x_1, x_2\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$, Dimension=2

Exercise:-

(i)Find dimension and a basis for the solution space of the system

$3x_1 + x_2 + x_3 + x_4 = 0, \ 5x_1 - x_2 + x_3 - x_4 = 0$

(ii)Find a basis and dimension of $W = \{(a_1, a_2, a_3, a_4) \in R^4 | a_1 + a_2 = 0, \ a_3 + a_4 = 0\}$

(iii) Find the dimension and a basis for the following subspaces of P_2 and P_3,

all polynomials of the form $ax^3 + bx^2 + cx + d$, where $b = 3a - 5d$ and $c = d + 4a$

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