Parul® University
Vadodara, Gujarat

NAAC GRADE A++

Applied Sciences and Humanities

Unit-3
Linear Transformation

Study Guide

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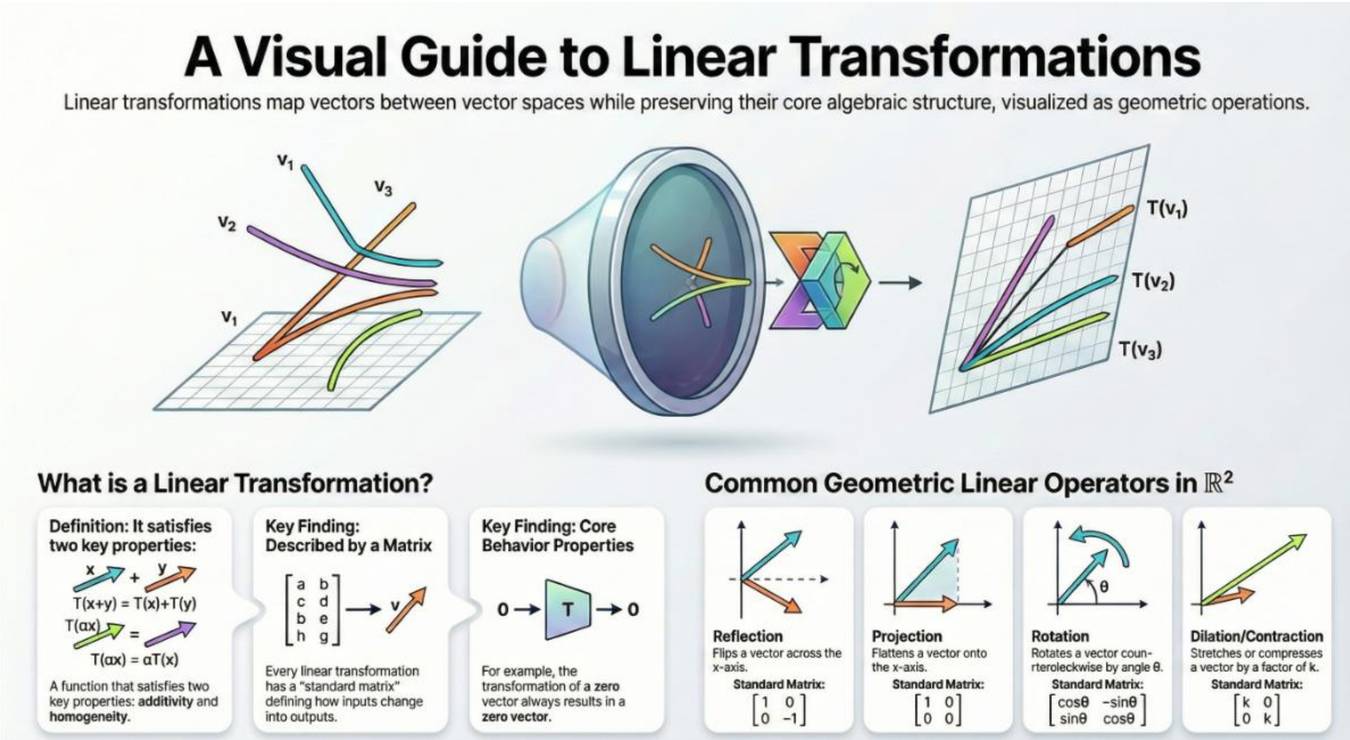
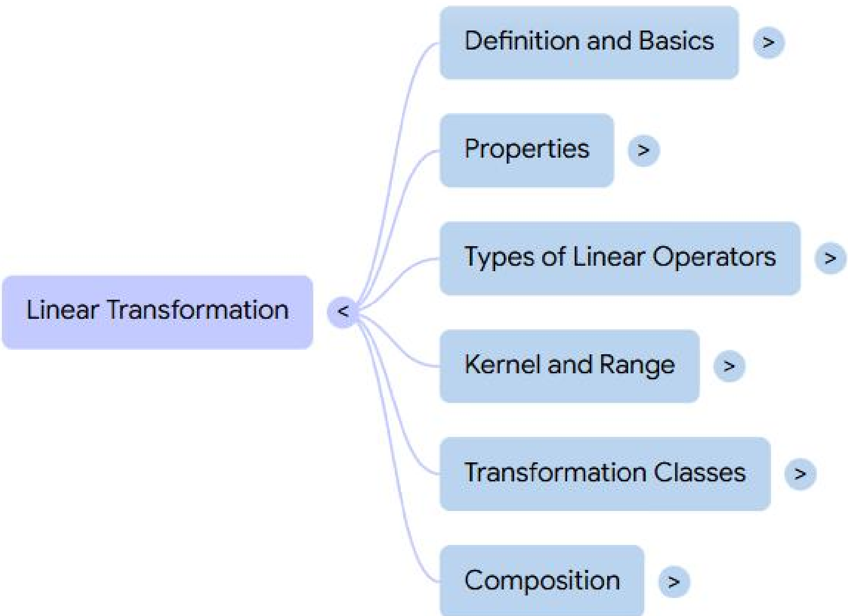


3.0 Transformation

By a transformation from R^n in to R^m, we mean a function of the type $T: R^n \to R^m$, with domain R^n and codomain R^m. For every vector $x \in R^n$, the vector $T(x) \in R^m$ is called the image of x under the transformation T, and the set

$$R(T) = \{T(x): x \in R^n\}$$

of all images under T, is called the range of the transformation T.



3.1 Euclidean Linear Transformations:

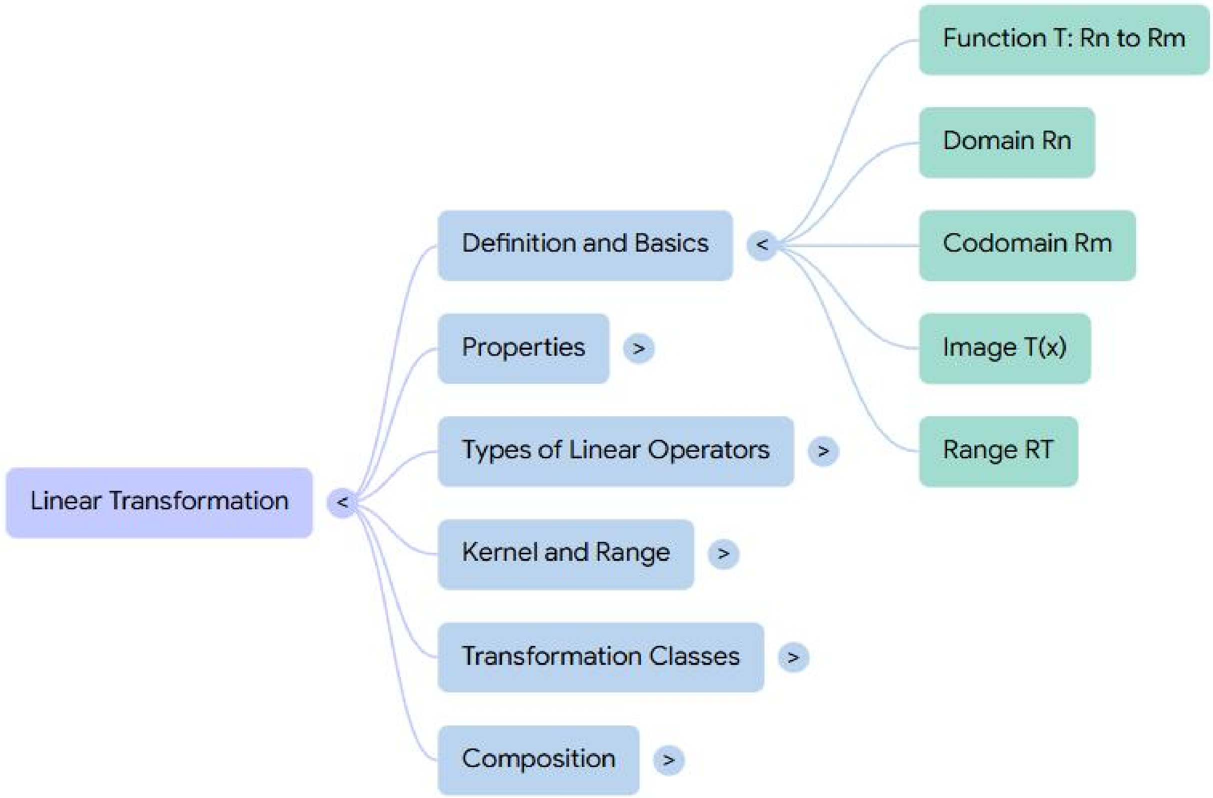
A function $T: R^n \rightarrow R^m$ is called a Euclidean transformation from R^n in to R^m. if

for any $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in R^n$ and $\alpha \in R$ the following two properties are satisfied:

(i) $T(x + y) = T(x) + T(y)$ (i.e. T preserves addition)

(ii) $T(\alpha x) = \alpha T(x)$ (i.e. T preserves scalar multiplication)

The linear transformation $T: R^n \rightarrow R^n$ is called a ***linear operator*** on R^n.



Some Standard Linear transformation:

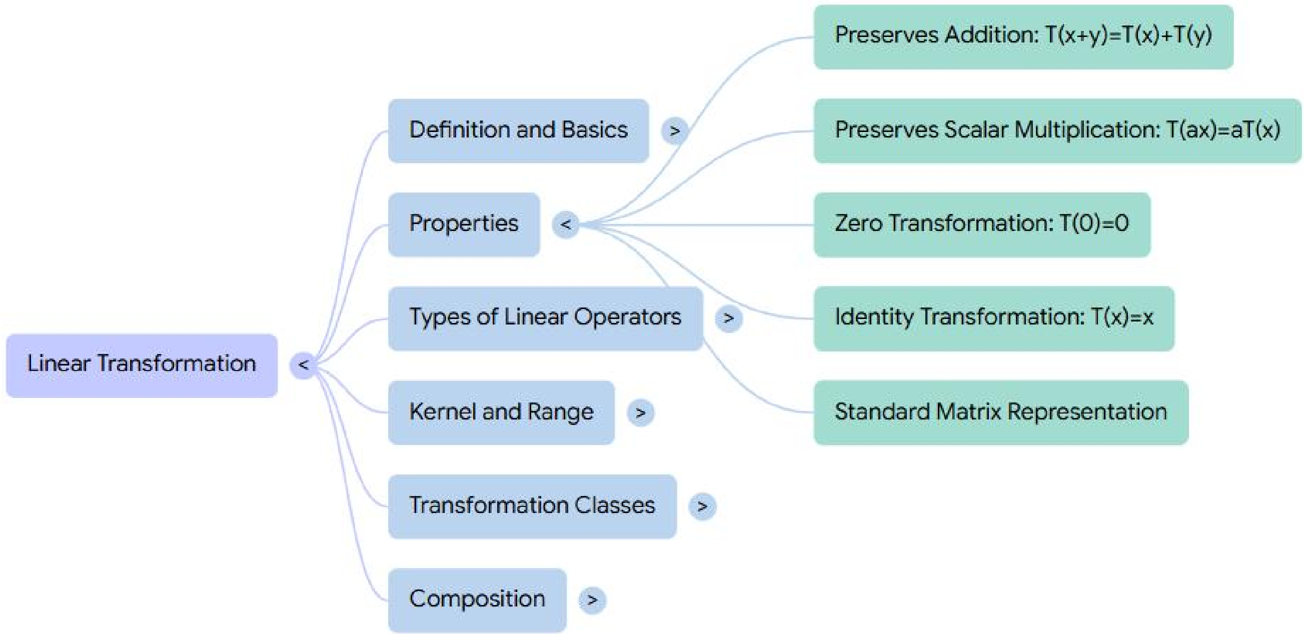
- A linear Transformation T is called the zero transformation if $T(x) = 0$, for every x.
- A linear Transformation T is called the identity transformation if $T(x) = x$, for every x.

Properties of Linear Transformation:

- $T(0) = 0$
- $T(-x) = -T(x)$
- $T(x - y) = T(x) - T(y)$

- $T(c_1x_1 + c_2x_2 + \dots + c_nx_n) = c_1T(x_1) + c_2T(x_2) + \dots + c_nT(x_n)$,

where $c_1, c_2, \dots, c_n$ are constants.



Example: Using definition of a linear transformation check whether the following functions are linear transformation or not:

1. $T: R^3 \to R^2$ is given by the formula $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$.

Solution: Let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in R^3 and \alpha \in R.$

$$\begin{aligned}
1)\, T(x+y) &= T((x_1,x_2,x_3)+(y_1,y_2,y_3)) \\
&= T(x_1+y_1, x_2+y_2, x_3+y_3) \\
&= ((x_1+y_1)-(x_2+y_2), (x_2+y_2)+(x_3+y_3)) \\
&= (x_1-x_2, x_2+x_3)+(y_1-y_2, y_2+y_3) \\
&= T(x)+T(y).
\end{aligned}$$

$$\begin{aligned}
2)\, T(\alpha x) &= T(\alpha(x_1,x_2,x_3)) \\
&= (\alpha x_1 - \alpha x_2, \alpha x_2 + \alpha x_3) \\
&= (\alpha(x_1-x_2), \alpha(x_2+x_3)) \\
&= \alpha(x_1-x_2, x_2+x_3) \\
&= \alpha T(x).
\end{aligned}$$

$\therefore T \text{ is a linear transformation.}$

2. $T: R^2 \to R \text{ is given by } T(x_1, x_2) = x_1x_2.$

3. $T: R \to R^2$ *is given by* $T(x) = (x, x)$.

4. $T: R^2 \to R^2$ *is given by* $T(x_1, x_2) = (x_1 + 2, -x_2)$.

Remark:

A transformation $T: R^n \to R^m$ is called a linear transformation if there exists a real matrix

$$A = (a_{11} \dots a_{1n} : \ : a_{m1} \dots a_{mn})$$

Such that for every $x = (x_1, x_2, \dots, x_n) \in R^n$, we have $T(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_m)$, where

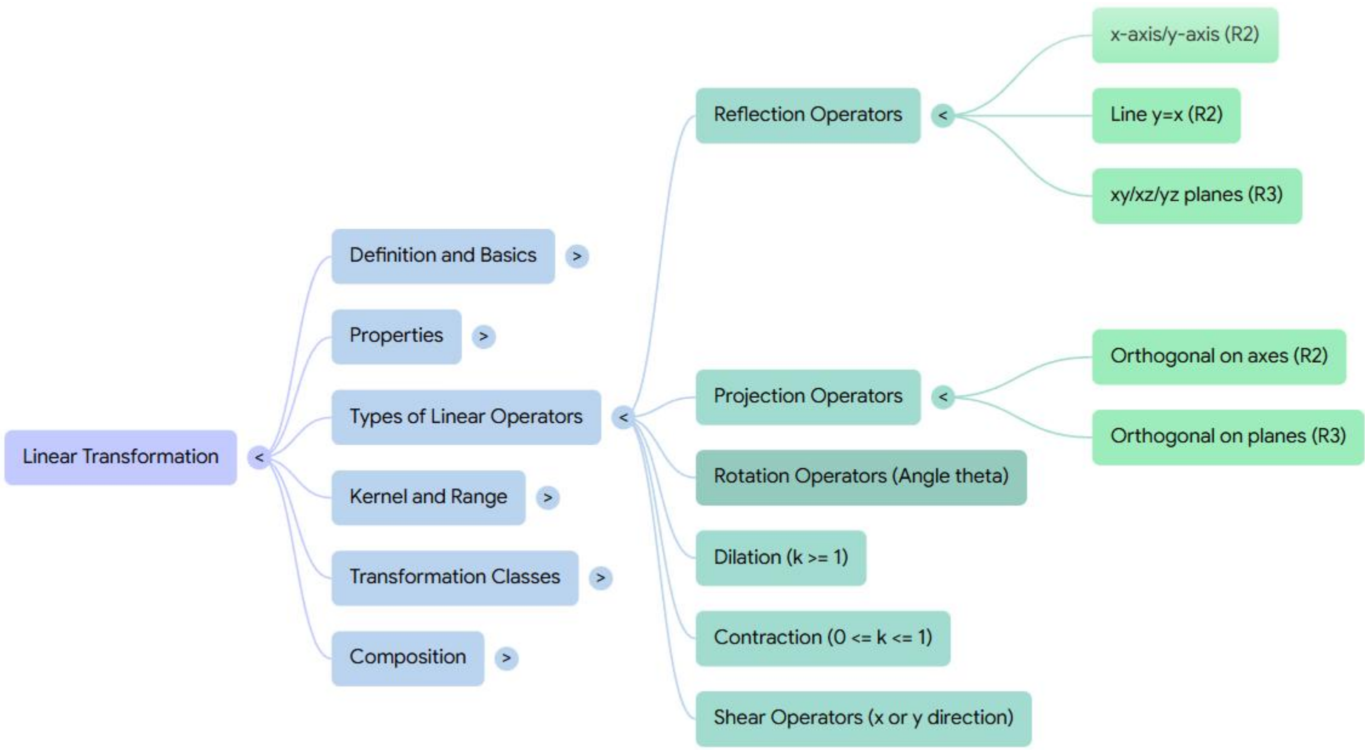
$$y_1 = a_{11}x_1 + \dots + a_{1n}x_n : y_m = a_{m1}x_1 + \dots + a_{mn}$$

In matrix notation,

$$(y_1 : y_m) = (a_{11} \dots a_{1n} : \ : a_{m1} \dots a_{mn})(x_1 : x_n)$$

In this case, the matrix A is called the standard matrix for the linear transformation T.

3.2 TYPES OF LINEAR OPERATORS



3.2.1 Reflection Operators:

An operator on R^2 *or* R^3 that maps each vector into its symmetric image about some line or plane is called a reflection operator. Let $T: R^2 \to R^2$ be a reflection operator define by

$$T(x, y) = (x, -y)$$

That maps each vector into its symmetric image about the x-axis.

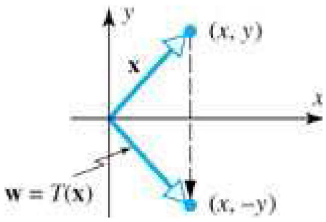
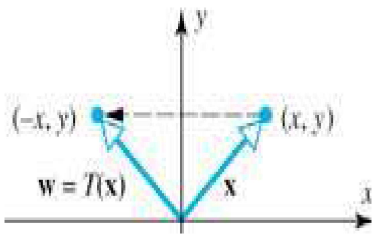
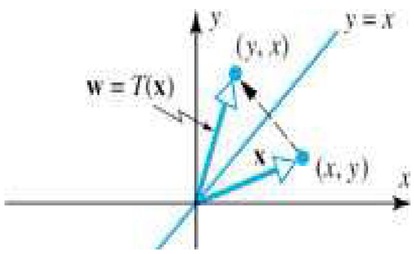
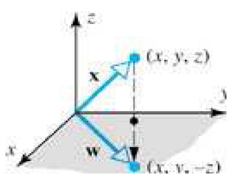
In matrix form ,

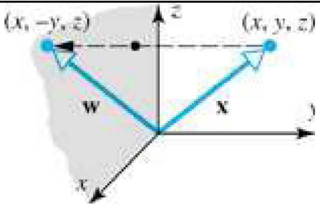
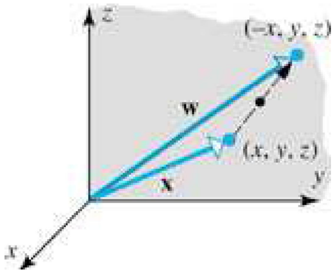
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

The standard matrix of T is

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

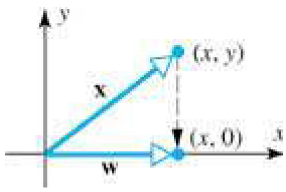
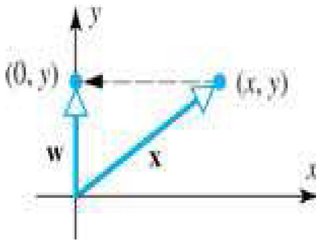
Some of the basic reflection operators are given below:

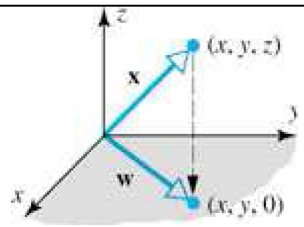
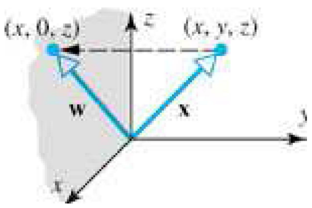
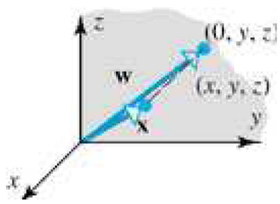
Operator	**Equation**	**Standard Matrix**
Reflection about the $x - axis$ on R^2 	$T(x, y) = (x, -y)$	$[1\ 0\ 0\ \ -1\ ]$
Reflection about the $y - axis$ on R^2 	$T(x, y) = (-x, y)$	$[-1\ 0\ 0\ 1\ ]$
Reflection about the line $y = x$ on R^2 	$T(x, y) = (y, x)$	$[0\ 1\ 1\ 0\ ]$
Reflection about the $xy - plane$ on R^3 	$T(x, y, z) = (x, y, -z)$	$[1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \ -1\ ]$
Reflection about the $xz - plane$ on R^3	$T(x, y, z) = (x, -y, z)$	$[1\ 0\ 0\ 0\ \ -1\ 0\ 0\ 0\ 1\ ]$

		
Reflection about the $yz-plane$ on R^3 	$T(x,y,z) = (-x,y,z)$	$[-1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ ]$

3.2.2 Projection Operators:

An operator on R^2 or R^3 that maps each vector into its orthogonal projection on a line or plane through the origin is called a projection operator.

Operator	Equation	Standard Matrix
Orthogonal projection on the $x-axis$ on R^2 	$T(x,y) = (x,0)$	$[1\ 0\ 0\ 0\ ]$
Orthogonal projection on the $y-axis$ on R^2 	$T(x,y) = (0,y)$	$[0\ 0\ 0\ 1\ ]$
Orthogonal projection on the $xy-plane$ on R^3	$T(x,y,z) = (x,y,0)$	$[1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ ]$

		
Orthogonal projection on the $xz-plane$ on R^3 	$T(x,y,z)=(x,0,z)$	$[1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ ]$
Orthogonal projection on the $yz-plane$ on R^3 	$T(x,y,z)=(0,y,z)$	$0\quad[0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ ]$

3.2.3 Rotation Operators:

An operator on R^2 that rotates each vector counterclockwise through a fixed angle θ is called a rotation operator.

Operator	**Equation**	**Standard Matrix**
Rotation through an angle θ on R^2	$T(x,y)=$ $(x\ cos\ cos\ \theta\ -y\ sin\ sin\ \theta\ , x\ sin\ sin\ \theta\ +y\ cos\ cos\ \theta\ ))$	$[cos\ cos\ \theta\quad -sin\ sin\ \theta\ \ sin\ sin\ \theta\quad cos\ cos\ \theta\ \ ]$
Counterclockwise rotation about the positive $x-axis$ through an angle θ on R^3	$T(x,y,z)=$ $(x, y\ cos\ cos\ \theta\ -z\ sin\ sin\ \theta\ , y\ sin\ sin\ \theta\ +z\ cos\ cos\ \theta\ ))$	$[1\ 0\ 0\ 0\ \ cos\ cos\ \theta\ \ -sin\ sin\ \theta\ \ 0\ \ sin\ sin\ \theta\ \ cos\ cos\ \theta\ \ ]$

Counterclockwise rotation about the positive $y - axis$ through an angle θ on R^3	$T(x, y, z) = (x\, cos\, cos\, \theta\ + z\, sin\, sin\, \theta\ , y,\ - x\, sin\, sin\, \theta\ + z\, cos\, cos\, \theta\ )\ )$	$[cos\, cos\, \theta\ \ 0\ \ sin\, sin\, \theta\ \ 0\, 1\, 0\ -\ sin\, sin\, \theta\ \ 0\ \ cos\, cos\, \theta\ \ ]$
Counterclockwise rotation about the positive $z - axis$ through an angle θ on R^3	$T(x, y, z) = (x\, cos\, cos\, \theta\ - y\, sin\, sin\, \theta\ , x\, sin\, sin\, \theta\ + y\, cos\, cos\, \theta\ , z)$	$[cos\, cos\, \theta\ \ -\ sin\, sin\, \theta\ \ 0\ \ sin\, sin\, \theta\ \ cos\, cos\, \theta\ \ 0\, 0\, 0\, 1\,]$

3.2.4 Dilation Operators:

An operator on $R^2\ or\ R^3$ that stretches each vector uniformly from the origin in all directions is called a dilation operator.

Operator	Equation	Standard Matrix
Dilation with factor k on R^2 $(k \geq 1)$	$T(x, y) = (kx, ky)$	$[k\, 0\, 0\, k\,]$
Dilation with factor k on R^3 $(k \geq 1)$	$T(x, y, z) = (kx, ky, kz)$	$[k\, 0\, 0\, 0\, k\, 0\, 0\, 0\, k\,]$

3.2.5 Contraction operators:

An operator on $R^2\ or\ R^3$ that compresses each vector uniformly towards the origin from all directions is called a contraction operator.

Operator	Equation	Standard Matrix
Contraction with factor k on R^2 $(0 \leq k \leq 1)$	$T(x, y) = (kx, ky)$	$[k\, 0\, 0\, k\,]$
Contraction with factor k on R^3 $(0 \leq k \leq 1)$	$T(x, y, z) = (kx, ky, kz)$	$[k\, 0\, 0\, 0\, k\, 0\, 0\, 0\, k\,]$

3.2.6 Shear Operators:

An operator on $R^2\ or\ R^3$ that moves each point parallel to the $x - axis$ by the amount ky is called a shear in the $x - direction$. Similarly, an operator on $R^2\ or\ R^3$ that moves each point parallel to the

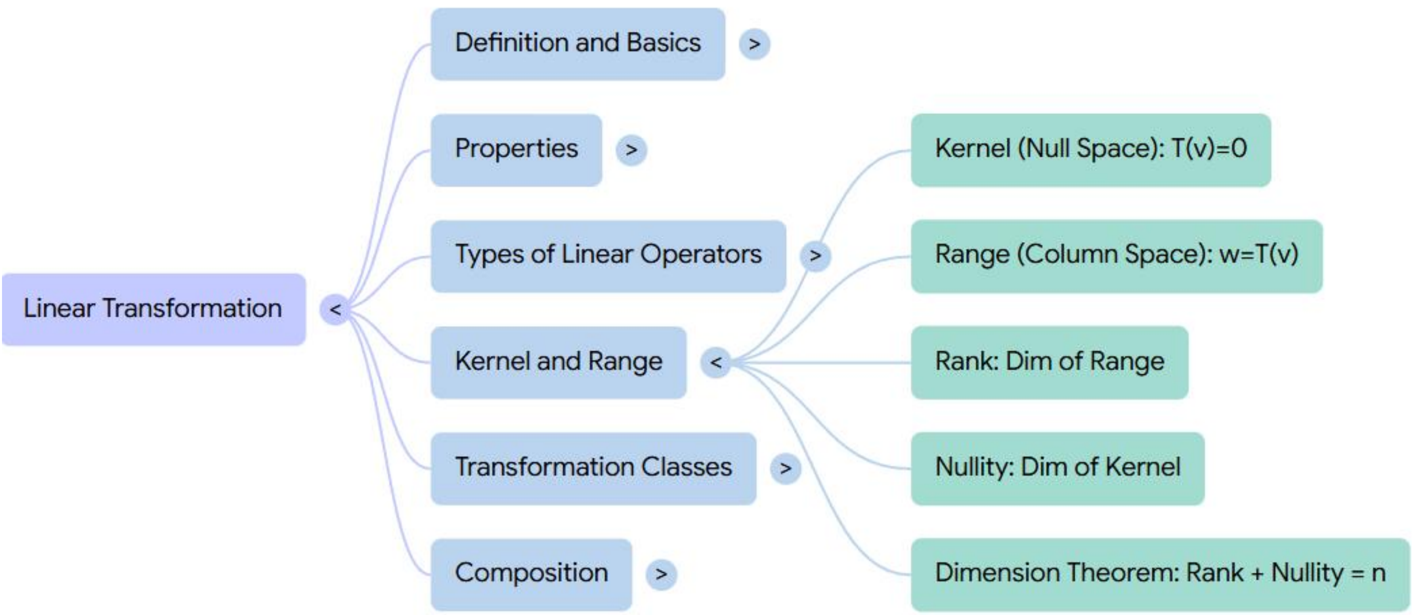
$y - axis$ by the amount kx is called a shear in the $y - direction$.

Operator	Equation	Standard Matrix
Shear in the $x - direction$ on $\mathbb{R}^2$	$T(x, y) = (x + ky, y)$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear in the $y - direction$ on $\mathbb{R}^2$	$T(x, y, z) = (x, y + kx)$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

3.3 <u>Composition of Linear Transformation</u>:

Let $T_1: U \to V$ and $T_2: V \to W$ be linear transformations. The application of T_1 followed by T_2 produces a transformation from U to V. This transformation is called the composition of T_2 with T_1 and is denoted by $T_2 \circ T_1$ and $T_2 \circ T_1(u) = T_2(T_1(u))$, where $u \in U$.

Theorem: If $T_1: U \to V$ and $T_2: V \to W$ are linear transformations then $(T_2 \circ T_1): U \to W$ is also a linear transformation.



Example 1: Find domain and codomain of $T^\circ T_1$ and find $T_2 \circ T_1(x, y)$.

(i) $T_1(x, y) = (2x, 3y); T_2(x, y) = (x - y, x + y)$.

Solution:

$Here\, T_1 : R^2 \to R^2$ *is given by* $T(x, y) = (2x, 3y)$ *and* $T_2 : R^2 \to R^2$ *is given by* $T(x, y) = (x - y, x + y)$.

$Therefore, domain\ of\ T_2 \circ T_1\ is\ R^2\ and\ codomain\ of\ T_2 \circ T_1\ is\ R^2\ and\ hence\ T_2 \circ T_1 : R^2 \to R^2\ is\ given\ by,$

$$\begin{aligned} T_2 \circ T_1(x, y) &= T_2(T_1(x, y)) \\ &= T_2(2x, 3y) \\ &= (2x - 3y, 2x + 3y). \qquad \# \end{aligned}$$

(ii) $T_1(x, y) = (x - y, y + z, x - z); T_2(x, y, z) = (0, x + y + z)$.

Example 2: Find the standard matrix of the following composition of linear operators on R^3.

(i) A rotation of $45°$ about $y - axis$ followed by a dilation with the factor $k = \sqrt{2}$.

Solution:

$Let\ T_1 : R^3 \to R^3\ is\ given\ by\ T_1(x, y, z) = (x\cos 45^\circ + z\sin 45^\circ, y, -x\sin 45^\circ + z\cos 45^\circ) = \left(\dfrac{x+y}{\sqrt{2}}, y, \dfrac{-x+z}{\sqrt{2}}\right)$

$and\ T_2 : R^3 \to R^3\ is\ given\ by\ T_2(x, y, z) = (\sqrt{2}\,x, \sqrt{2}\,y, \sqrt{2}\,z).$

$\therefore T_2 \circ T_1 : R^3 \to R^3\ is\ given\ by,$

$$\begin{aligned} T_2 \circ T_1(x, y, z) &= T_2(T_1(x, y, z)) \\ &= T_2\left(\left(\frac{x+y}{\sqrt{2}}, y, \frac{-x+z}{\sqrt{2}}\right)\right) \\ &= (x + y, \sqrt{2}\,y, -x + z). \qquad \# \end{aligned}$$

(ii) A rotation of $30°$ about $x - axis$ followed by A rotation of $30°$ about $z - axis$ followed by a contraction with factor $k = \frac{1}{4}$.

3.4 <u>Rank and Nullity of a linear transformation:</u>

Let $T: V \to W$ be a linear transformation.

- Then ***kernel of T,*** denoted by $ker\ (T)$ or $N(T)$, is the set of all vectors $v \in V$ such that $T(v) = 0$.

  In notation, $ker\ \ (T)\ = 0$.

- The ***range of*** T, denoted by $R(T)$, is the set of all images of vectors in W under that are images of at least one vector in V under T.

  In notation, $R(T) = \{T(v), v \in V\}$.

  Clearly, R $(T) \subseteq W$ and $N(T) \subseteq V$.

Note: 1) $Basis\ of\ \ ker\ \ (T) = Basis\ of\ Nullspace\ of\ A.\ \ i.e\ [T]$.

2) $Basis\ of\ \ R\ \ (T) = Basis\ of\ column\ space\ of\ A.\ \ i.e\ \ [T]$.

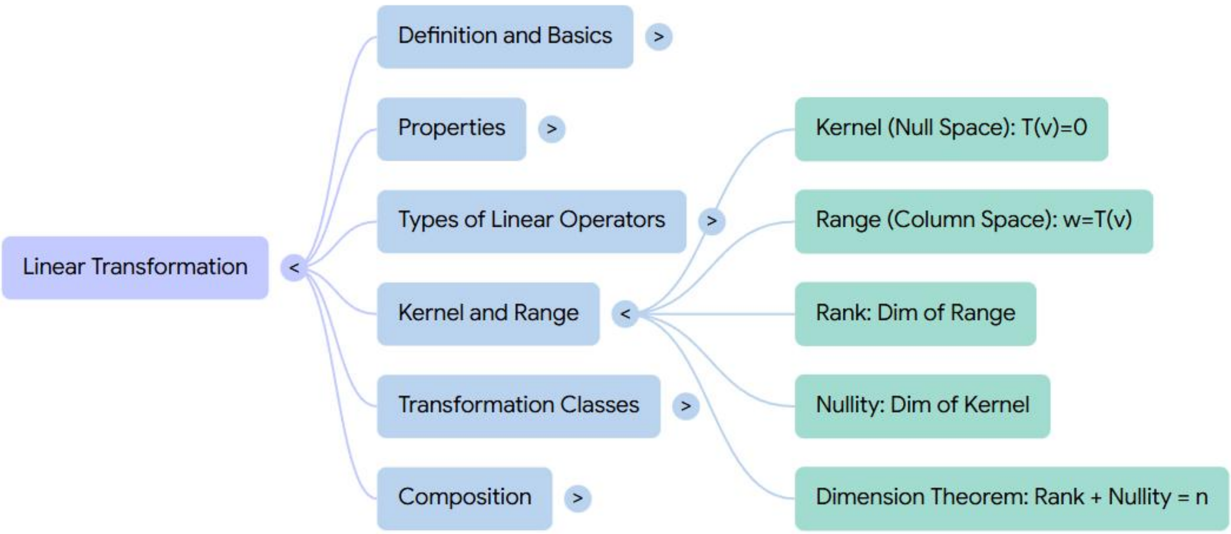
3) Dimension of range of T is called ***rank of T*** and is denoted by $rank(T)$**.**

4) Dimension of the kernel of T is called ***nullity of T*** and is denoted by $Nullity(T)$.

Dimension Theorem for Linear Transformation:

Let $T: V \to W$ be a linear transformation then ,

$$rank(T) + nullity(T) = dim\ (V)\ .$$



Example 1: $Let\, T : R^3 \to R^2\ is\ given\ by\ T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3). Find\ R(T)\ and\ N(T).$

Solution:

$R(T)\ consist\ of\ vectors\ of\ the\ form\ (x_1 - x_2, x_1 + x_3).$

$We\ want\ to\ determine\ the\ vectors\ of\ this\ form. For\ this, take\ a\ vector\ (a, b) \in R^2\ such\ that$

$$(x_1 - x_2, x_1 + x_3) = (a, b)$$
$$\Rightarrow x_1 - x_2 = a\ and\ x_1 + x_3 = b$$
$$\Rightarrow x_1 - a = x_2\ and\ b - x_1 = x_3$$

$Hence\ T(x_1, x_1 - a, b - x_1) = (a, b).$

$This\ shows\ that\ every\ vector\ (a, b) \in R^2 is\ in\ R(T). Therefore\ R(T) = R^2.$

$For\ N(T), T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3) = (0, 0).$

$$\Rightarrow x_1 = x_2 = -x_3, i.e. all\ vectors\ of\ the\ form\ (x_1, x_1, -x_1)\ will\ be\ mapped\ into\ zero.$$

$Therefore, N(T) = \{x_1(1,1,-1) \mid x_1\ any\ scalar\}.$ #

Example 2:

$A\ function\ T : R^3 \to R^4\ is\ defined\ by\ T(x, y, z) = (x, x, 2y, 3z). Find\ Kernel\ and\ Range\ of\ T.$ **Solution:**

$$\ker(T) = \{(x,y,z) \in R^3 \mid T(x,y,z) = 0\}$$
$$= \{(x,y,z) \in R^3 \mid (x,x,2y,3z) = (0,0,0,0)\}$$
$$= \{(0,0,0)\}$$

Now

$$(x,x,2y,3z) = x(1,1,0,0) + y(0,0,2,0) + z(0,0,0,3).$$

$$\textit{Therefore } Range(T) = [\{(1,1,0,0),(0,0,2,0),(0,0,0,3)\}]. \qquad \#$$

Exercises:

Que.1

A mapping $T : R^2 \to R^3$ is given by $T(x,y) = (x, x+y, y)$. Find rank and nullity of T. Verify rank − nullity theorem.

Que.2 $T: R^3 \to R^3$ be rotation about Y-axis through an angle 45° .

(i) Find a basis for R(T).

(ii) Find a basis for $ker\ (T)$.

(iii) Verify the dimension theorem.

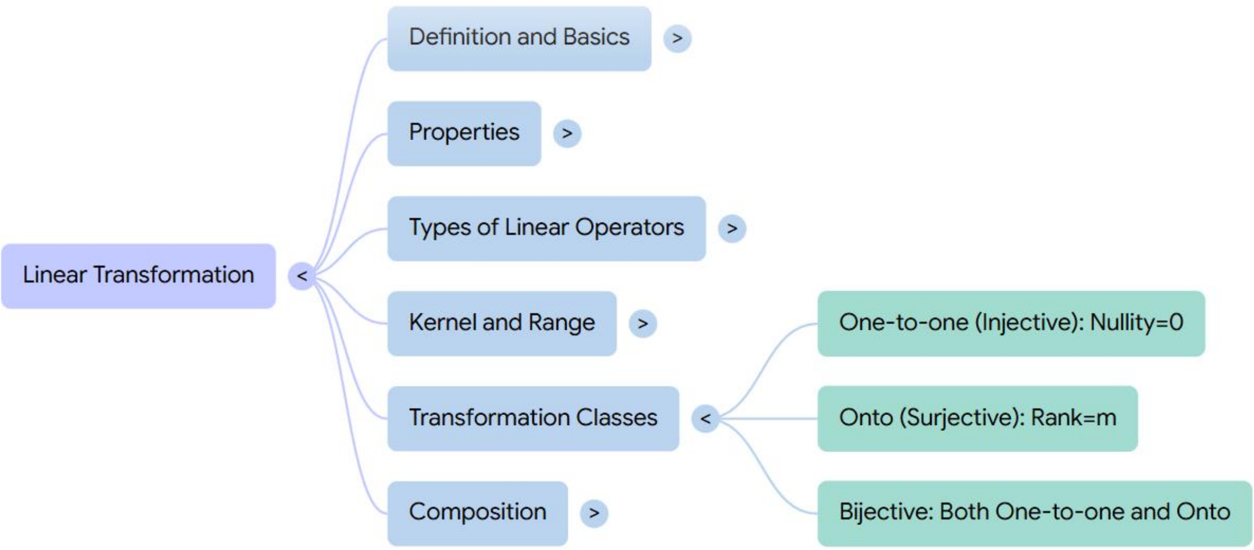
Que.3 Let $T: R^3 \to R^3$ be the linear transformation defined by

$T(x,y,z) = (x + 2y - z, x + y, x + y - 2z)$

(a)Find a basis and the dimension for the range of T.

(b)Find a basis and dimension for the kernel of T.

(c)Verify the dimension theorem.



3.5 <u>One-to-one Transformation</u>:

A linear transformation $T: V \to W$ is one-to-one if T maps distinct vectors in V to distinct vectors in W.

A one-to-one transformation is also called injective transformation.

Theorem:1: A linear transformation $T: V \to W$ is one-to-one if and only if $ker\ (T) = \{0\}$.

Theorem:2: A linear transformation $T: V \to W$ is one-to-one if and only if $(ker\ (T)) = 0$.i.e., $nullity(T) = 0$.

Theorem:3: A linear transformation $T: V \to W$ is one-to-one if and only if $rank(T) = dim\ (V)$.

Theorem:4: If A is an $m \times n$ matrix and $T_A: R^n \to R^m$ is multiplication by A then T_A is one-to-one if and only if $rank(A) = n$.

Theorem:5: If A is an $m \times n$ matrix and $T_A: R^n \to R^n$ is multiplication by A then T_A is one-to-one if and only if A is an invertible matrix.

3.6 Onto Transformation:

A linear transformation $T: V \to W$ is onto if for every $w \in W$, there is a $v \in V$ such that $T(v) = w$.

i.e. T is onto if and only if range of T is W.

An onto transformation is also called surjective transformation.

Theorem:1: $T: V \to W$ is onto if and only if $rank(T) = dim\ W$.

Theorem:2: If A is an $m \times n$ matrix and $T_A: R^n \to R^m$ is multiplication by A then T_A is onto if and only if $rank(A) = m$.

Theorem:3: Let $T: V \to W$ be a linear transformation then

(i) If T is one-to-one, then it is onto.

(ii) If T is onto, then it is one-to-one.

3.7 Bijective Transformation:

If a transformation $T: V \to W$ is both one-to-one and onto then it is called bijective transformation.

Example 1: $Let\ T: R^2 \to R\ define\ by\ T(x, y) = x + y. Is\ T\ one\ to\ one\ or\ onto?$

Solution:

$$Let\ z_1 = (1,2)\ and\ z_2 = (0,3), members\ of\ R^2.$$
$$Now, T(z_1) = T(1,2) = 3 = T(0,3) = T(z_2)\ but\ (1,2) \neq (0,3).$$
$$\therefore\ T\ is\ not\ one\ to\ one.$$

$$Let\ x\ be\ any\ member\ of\ R. Then\ there\ exist\ a\ member\ (x,0) \in R\ such\ that\ T(x,0) = x + 0 = x.$$

$$Hence\ T\ is\ onto.$$

Example 2: T is rotating counterclockwise about $positive\ z - axis$ through an angle θ on R^3.

$Is\ T\ one\,to\,one?$

Solution:

Here T is rotating counterclockwise about $positive\ z-axis$ through an angle θ on R^3,

$T(x, y, z) = (x\ cos\ cos\ \theta\ - y\ sin\ sin\ \theta\ , x\ sin\ sin\ \theta\ + y\ cos\ cos\ \theta\ , z)$

$$Let\ (x_1, y_1, z_1), (x_2, y_2, z_2) \in R^3\ and\ T(x_1, y_1, z_1) = T(x_2, y_2, z_2)$$
$$\Rightarrow (x_1 \cos\theta - y_1 \sin\theta, x_1 \sin\theta + y_1 \cos\theta, z_1) = (x_2 \cos\theta - y_2 \sin\theta, x_2 \sin\theta + y_2 \cos\theta, z_2)$$
$$\Rightarrow x_1 \cos\theta - y_1 \sin\theta = x_2 \cos\theta - y_2 \sin\theta, x_1 \sin\theta + y_1 \cos\theta = x_2 \sin\theta + y_2 \cos\theta\, and\ z_1 = z_2$$
$$\Rightarrow x_1 = x_2, y_1 = y_2\ and\ z_1 = z_2$$
$$\Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2)$$
$$\therefore T\ is\ one\ one.$$

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