

Unit :1 Application of Functions of One Variable

Function of One Variable

Function: A function is a rule for transforming a member of one set A to a unique member of another set B. A function $f : A \rightarrow B$ is a rule which associates with each member of A to a unique member of B.

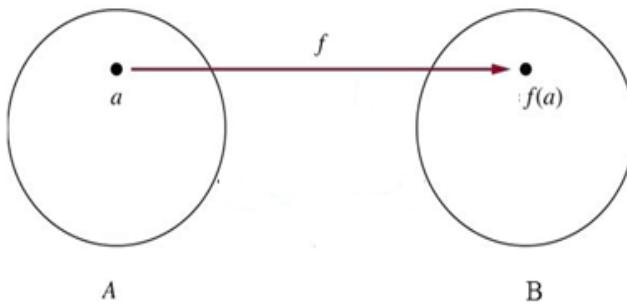


Figure 1: Function rule from A to B

“A” is called the **domain** of the function and “B” the **codomain**. A subset of the co-domain, which is collection of all the images of the elements of the domain is called the **Range**.

Examples of function of one variable

1. If $f(x) = 2x + 1$, then if $x = 0, 1, 2, 3$, $f(x) = 1, 3, 5$ respectively.
2. If $f(x) = x^2 + 2$, then if $x = 0, 1, 5$, $f(x) = 2, 3, 27$ respectively.

Example: Determine the domain and range of the following functions

1) $f(x) = x^2$

Solution: For any real number, its square is uniquely defined. Therefore, the domain of f is the set \mathbb{R} . The square of any number is never negative and the square root of any positive real number exists. Therefore, the range is the set of positive real numbers.

Say for $x = -1, -2, \sqrt{2}, \frac{1}{4}$ etc we have $f(x) = 1, 4, 2, \frac{1}{16}$.

$$2) f(x) = \sqrt{x+2}$$

Solution: Domain - There can only square roots of non-negative numbers.

$$\text{So, } x+2 \geq 0 \Rightarrow x \geq -2$$

$$\therefore \text{Domain} = [-2, \infty).$$

Range –The square root function always gives non-negative output. The smallest value of $f(x)$ exist and is zero at $x = -2$ and as x increases, $f(x)$ also increases.

$$\therefore \text{Range} = [0, \infty).$$

$$3) f(x) = \frac{1}{x-3}$$

Solution: Domain- We cannot divide by zero, so $x-3 \neq 0 \Rightarrow x \neq 3$. So, Domain has all real numbers except 3.

$$\therefore \text{Domain} = (-\infty, 3) \cup (3, \infty).$$

Range- This function can never be zero , because numerator is always 1.

$$\therefore \text{Range} = (-\infty, 0) \cup (0, \infty).$$

Types and Properties of Functions

1) Periodic function: A function which repeats itself at a regular interval of x is called periodic. i.e., if $f(x+T) = f(x)$ for all x , then $f(x)$ is said to be a periodic function of period T.

a) Sine Function

$$\text{Function : } f(x) = \sin x$$

$$\text{Period : } 2\pi$$

Reason: Repeats its value after every 2π .

$$\sin(x) = \sin(x + 2\pi)$$

b) Cosine Function

$$\text{Function : } f(x) = \cos x$$

$$\text{Period : } 2\pi$$

Reason: Repeats its value after every 2π .

$$\cos(x) = \cos(x + 2\pi)$$

c) Tangent Function

$$\text{Function : } f(x) = \tan x$$

$$\text{Period : } \pi$$

Reason: Tangent has shorter period than sine /cosine

$$\tan x = \tan(x + \pi)$$

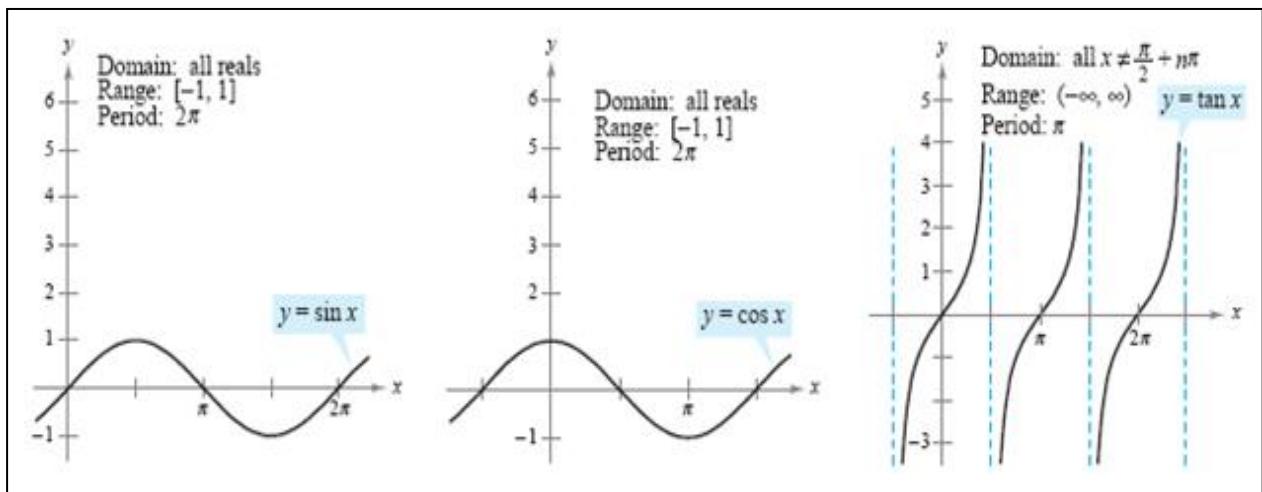


Figure 2: Graphs of periodic function (a) $\sin x$ (b) $\cos x$ (c) $\tan x$

2) Even and Odd function: A function is said to be even if $f(x) = f(-x)$ for all x .

For Example: $f(x) = x^2, f(-x) = (-x)^2 = x^2$

$$f(x) = x^2 + 1, f(-x) = (-x)^2 + 1 = x^2 + 1$$

$$f(x) = \cos x, f(-x) = \cos(-x) = \cos x$$

A function is said to be odd if $f(-x) = -f(x)$ for all x .

For Example: $f(x) = x^3, f(-x) = (-x)^3 = -x^3$

$$f(x) = \sin x, f(-x) = \sin(-x) = -\sin x$$

3) Some basic functions

Type	Form	Example
Constant	$f(x) = c$	$f(x) = 100$
Identity	$f(x) = x$	$f(t) = t$
Linear	$f(x) = ax + b$	$f(x) = 7t - 4$
Quadratic	$f(x) = ax^2 + bx + c$	$f(x) = -x^2 + 10x + 12$
Cubic	$f(x) = ax^3 + bx^2 + cx + d$	$f(x) = 3x^3 + 2x - 10$
Polynomial	$f(x) = \sum a_n x^n$	$f(x) = 6x^5 + 2x + 1$

Rational	$f(x) = \frac{P(x)}{Q(x)}$	$f(x) = \frac{1}{x+2}$
Radical	$f(x) = \sqrt[n]{x}$	$f(x) = \sqrt[3]{x}, \sqrt{x+2}$
Exponential	$f(x) = a^x$	$f(x) = e^x, 2^x$
Logarithmic	$f(x) = \log_a(x)$	$f(x) = \log_{10}(x), \ln(x)$

4) Monotonic Functions:

→ $f(x_1)$ is monotonic increasing if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

For example: $f(x) = x + 1, e^x$, increase with increase in value of x .

→ $f(x_1)$ is monotonic decreasing if $f(x_1) > f(x_2)$ whenever $x_1 > x_2$.

For example: $f(x) = \frac{1}{x+1}, e^{-x}$, decrease with decrease in value of x .

Limit of A Function of One Variable:

Definition: The limit of a function $f(x)$ as x approaches a value a is the value that $f(x)$ gets closer to as x gets closer to a .

Mathematically, $\lim_{x \rightarrow a} f(x) = L$ means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

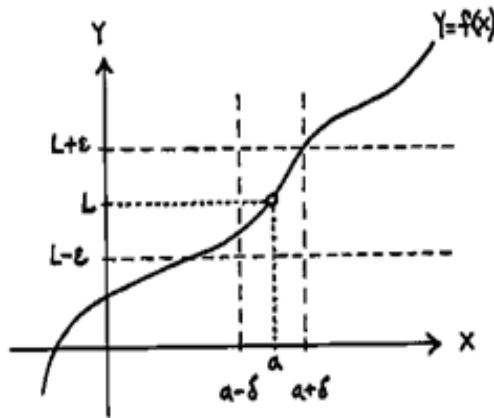


Figure 3: Limit of a function $f(x)$ as x approaches a

Left -Hand Limit (LHL) and Right-Hand Limit (RHL):

Let $f(x)$ be the function defined around a point a .

- Left-Hand Limit (LHL): $\lim_{x \rightarrow a^-} f(x)$

This means the value $f(x)$ approaches as x comes from the left (less than a)

- Right-Hand Limit (RHL): $\lim_{x \rightarrow a^+} f(x)$

This means the value $f(x)$ approaches as x comes from the right (greater than a)

If both LHL and RHL exist and are equal, then the limit exists at that point i.e.

If $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$, then $\lim_{x \rightarrow a} f(x) = L$.

Evaluation of Limits: Limits can be evaluated by either of the following techniques.

- (1) By definition
- (2) Direct Substitution
- (3) Factorization of numerator or denominator
- (4) In case of $\frac{0}{0}$ or $\frac{\infty}{\infty}$, by L'Hospital's Rule

Example 1: Show that $\lim_{x \rightarrow 1} 5x - 4 = 1$ exists.

Solution: Here, $f(x) = 5x - 4$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} 5(1-h) - 4 = 1$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} 5(1+h) - 4 = 1$$

$$\text{As } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 1$$

Therefore, $\lim_{x \rightarrow 1} 5x - 4 = 1$,

Example 2: Evaluate $\lim_{x \rightarrow 2} x + 3$

Solution: By Direct Substitution, $\lim_{x \rightarrow 2} x + 3 = 2 + 3 = 5$

Example 3: Evaluate $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

Solution: Here direct substitution gives 0/0. Factorising the numerator, we have

$$\lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \rightarrow 2} x + 2 = 2 + 2 = 4.$$

Example 4: Evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

Solution: It is an indeterminate 0/0 form. Here we apply L'Hospitals Rule

$$\lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$$

Example 5: Evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$

Solution: It is an indeterminate 0/0 form. Here we apply L'Hospitals Rule

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{e^x}{1} = e^0 = 1$$

Example 6: Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$

Solution: $\rightarrow \frac{0}{0}$ form

Differentiate: $\frac{\sin x}{2x}$

Still $\frac{0}{0}$, \rightarrow Apply again $\lim_{x \rightarrow 0} \frac{\cos x}{2} = \frac{1}{2}$

Exercise: Evaluate the following

(1) $\lim_{x \rightarrow 0} \frac{\ln(1+2x)}{x}$ (Ans :2)

(2) $\lim_{x \rightarrow 0} \frac{1}{x}$ (Ans : ∞)

(3) $\lim_{x \rightarrow \infty} \frac{3x^2 + 5}{2x^2 + 7x}$ (Ans : $\frac{3}{2}$)

(4) $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$ (Ans :3)

(5) $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$ (Ans : $\frac{1}{4}$)

Continuity: A function $f(x)$ is continuous at $x = a$ if the following three conditions are satisfied:

(i) $f(a)$ is defined

(ii) $\lim_{x \rightarrow a} f(x)$ exists (that is finite)

(iii) $\lim_{x \rightarrow a} f(x) = f(a)$

Example 1: Check the continuity of $f(x) = x^2$ at $x = 2$.

Solution: First note that the function is defined at $x = 2$ and its value is 4.

Then, find the limit of the function at $x = 2$. Clearly $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} x^2 = 4$ exists.

Thus, $\lim_{x \rightarrow 2} f(x) = 4 = f(2)$.

Hence, f is continuous at $x = 2$.

Example 2: Discuss the continuity of the function f given by

$$f(x) = \begin{cases} x^3 + 3, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases} \text{ at } x = 0.$$

Solution: The function is defined at $x = 0$ and its value is 1.

When $x \neq 0$, the function is given by a polynomial.

Hence, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^3 + 3 = 0^3 + 3 = 3$.

Since the limit of f at $x = 0$ does not coincide with $f(0)$, ie., $\lim_{x \rightarrow 0} f(x) \neq f(0)$.

Therefore, the function is not continuous at $x = 0$.

Example 3: Discuss the continuity of the function f given by $f(x) = \begin{cases} x^2 + 1, & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$

at $x = 1$.

Solution: The function is defined at $x = 1$ and its value is 2.

The Limit when x approaches 1 is $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} x^2 + 1 = 1^2 + 1 = 2$

Also, $\lim_{x \rightarrow 1} f(x) = f(1)$

Since, all three conditions are satisfied. Given function is continuous at $x = 1$.

Note: Polynomials are continuous everywhere.

Example 4: Discuss the continuity of the function f given by $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ at $x = 0$.

Solution: Since $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. (Proved earlier).

Also, $f(0) = 1$.

Therefore, the given function is continuous at $x = 0$.

Example 5: Discuss the continuity of the function f given by $f(x) = |x|$ at $x = 0$.

Solution: By definition

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly the function is defined at $x = 0, f(0) = 0$

Left hand limit of f at 0 is $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-x) = 0$

Right hand limit of f at 0 is $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$

Thus, the LHL,RHL and value of function at 0 coincide .

Hence, f is continuous at $x = 0$.

Example 6: Discuss the continuity of the function f given by $f(x) = \begin{cases} x + 2, & \text{if } x \leq 1 \\ x - 2, & \text{if } x > 1 \end{cases}$

at $x = 1$.

Solution: For $x < 1$: $f(x) = x + 2$ is a polynomial, which is continuous.

For $x \leq 1$: $f(x) = x - 2$ is a polynomial, which is continuous.

At $x = 1$: We check :

- Left Hand limit (LHL): $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x + 2 = 1 + 2 = 3$.
- Right Hand limit (RHL): $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x - 2 = 1 - 2 = -1$.
- Function value at $x = 1$: Since $x \leq 1$, we use $x + 2, f(1) = 3$

Since LHL \neq RHL, function is not continuous at $x = 1$.

Exercise: Discuss the continuity of the function f given by following functions

(1) $f(x) = \tan x$, at $x = \frac{\pi}{2}$ (Ans: No)

(2) $f(x) = \begin{cases} x^2 - 1, & \text{if } x < 1 \\ 2, & \text{if } x \geq 1 \end{cases}$ (Ans: No)

(3) $f(x) = \begin{cases} x + 1, & \text{if } x \leq 0 \\ x^2, & \text{if } x > 0 \end{cases}$ (Ans: Yes)

Differentiability:

Derivative of a Function is the rate of change in the given function with respect to an independent variable. The derivative of a given function in Calculus is found using the First Principle of differentiation. The process of finding the derivative of a function is also called the Differentiation of function.

Derivative of a Function gives the slope of the particular function at the point of differentiation and is used to get the extreme value of the function. It is also defined as the rate of change of the function with respect to any point lying in the domain of the function. A function $f(x)$ is differentiable at a point $x = a$ if it is continuous at that point. For any function $f(x)$ is said to be differentiable if the differentiation of the function exists and it is denoted by $f'(x)$ or $\frac{df(x)}{dx}$.

First Principle of Differentiation

A function f is said to be differentiable at $x = a$ if the derivative $f'(a)$ exists at every point in its domain. Mathematically, $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$ is the derivative of $f(x)$ at $x = a$ provided limit exists.

Example 1: Find the derivative of $f(x) = x^2$ at $x = 1$ by the first principle of derivative.

$$\begin{aligned}\text{Solution: } f'(1) &= \frac{f(1+h)-f(1)}{h} = \lim_{h \rightarrow 0} \frac{(1+h)^2 - (1)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{1+2h+h^2 - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h+h^2}{h} \\ &= \lim_{h \rightarrow 0} 2 + h \\ &= 2\end{aligned}$$

Since, the limit exists therefore the derivative of $f(x) = x^2$ at $x = 1$ is 2.

Example 2. Find the derivative of $f(x) = \sqrt{x}$. Also, show that $f'(x)$ at $x = 0$ doesn't exist.

$$\begin{aligned}\text{Solution: } f'(x) &= \frac{f(x+h)-f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Since, the limit exists therefore the derivative of $f(x) = \sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

At $x = 0$, $f'(0)$ becomes infinite. Hence, $f'(x)$ at $x = 0$ doesn't exist.

Example 3: Find $\frac{df}{dx}$ if $f(x) = \sin x$ by definition.

Solution: Here, $f(x) = \sin x$

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \rightarrow 0} \frac{2\cos\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{h}{2}\right)}{h} \\
&= \lim_{h \rightarrow 0} \cos\left(x + \frac{h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \\
&= \lim_{h \rightarrow 0} \cos\left(x + \frac{h}{2}\right) \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \\
&= \cos x
\end{aligned}$$

Exercise

- 1) Find the derivative of $f(x) = 3x + 4$ using the first principle of derivative. [Ans. 3]
- 2) Find the derivative of $f(x) = \frac{1}{x^2}$ using the first principle of derivative. [Ans. $-\frac{2}{x^3}$]

Geometrical interpretation of derivative:

Let $y = f(x)$ be the given function. Let $P(a, f(a))$ be a point on the graph of the function.

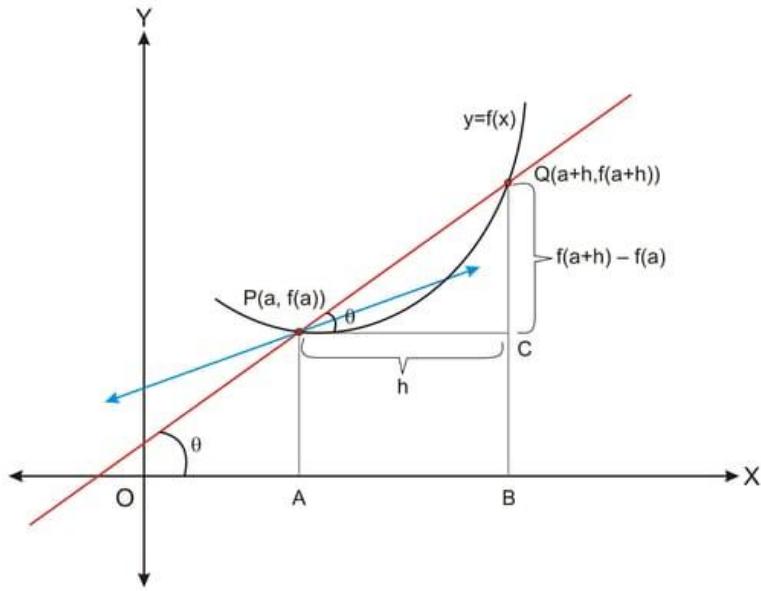


Figure 4: Geometrical interpretation of derivative

Let $Q(a + h, f(a + h))$ be a point close to P . Joining P and Q by a chord.

Then, $\frac{f(a+h)-f(a)}{h} = \frac{QC}{PC}$ which is the slope of the chord PQ .

Now, as $h \rightarrow 0$, Q will tend to P .

Then, the chord PQ will tend to the tangent at P .

Thus, $f'(a)$ represents the slope of the tangent to the curve $y = f(x)$ at the point $x = a$.

Differentiation standard formulas

General Formulas

1. Constant Rule: $\frac{d}{dx}[c] = 0$

2. Power Rule: $\frac{d}{dx}[x^n] = nx^{n-1}$

3. Scalar Multiple of a Function: $\frac{d}{dx}[c \cdot f(x)] = c f'(x)$

4. Sum and Difference of Functions: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

5. Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + g'(x) \cdot f(x)$

6. Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{[g(x)]^2}$

7. Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Exponential and Logarithmic Derivatives

$$8. \frac{d}{dx}[e^x] = e^x$$

$$9. \frac{d}{dx}[b^x] = b^x \cdot \ln(b)$$

$$10. \frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$11. \frac{d}{dx}[\log_b(x)] = \frac{1}{x \ln(b)}$$

Trigonometric Derivatives

$$12. \frac{d}{dx}[\sin x] = \cos x$$

$$13. \frac{d}{dx}[\cos x] = -\sin x$$

$$14. \frac{d}{dx}[\tan x] = \sec^2 x$$

$$15. \frac{d}{dx}[\cosec x] = -\cosec x \cot x$$

$$16. \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$17. \frac{d}{dx}[\cot x] = -\cosec^2 x$$

Inverse Trigonometric Derivatives

$$18. \frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$$

$$19. \frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$$

$$20. \frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$$

$$21. \frac{d}{dx}[\cosec^{-1} x] = -\frac{1}{x\sqrt{x^2-1}}$$

$$22. \frac{d}{dx}[\sec^{-1} x] = \frac{1}{x\sqrt{x^2-1}}$$

$$23. \frac{d}{dx}[\cot^{-1} x] = -\frac{1}{1+x^2}$$

Hyperbolic Derivatives

$$24. \frac{d}{dx}[\sinh x] = \cosh x$$

$$25. \frac{d}{dx}[\cosh x] = \sinh x$$

$$26. \frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x$$

$$27. \frac{d}{dx} [\operatorname{cosech} x] = -\operatorname{cosech} x \coth x$$

$$28. \frac{d}{dx} [\operatorname{sech} x] = -\operatorname{sech} x \tanh x$$

$$29. \frac{d}{dx} [\coth x] = -\operatorname{cosech}^2 x$$

Inverse Hyperbolic Derivatives

$$30. \frac{d}{dx} [\sinh^{-1} x] = \frac{1}{\sqrt{1+x^2}}$$

$$31. \frac{d}{dx} [\cosh^{-1} x] = \frac{1}{\sqrt{x^2-1}}$$

$$32. \frac{d}{dx} [\tanh^{-1} x] = \frac{1}{1-x^2}$$

$$33. \frac{d}{dx} [\operatorname{cosech}^{-1} x] = -\frac{1}{|x|\sqrt{x^2+1}}$$

$$34. \frac{d}{dx} [\operatorname{sech}^{-1} x] = -\frac{1}{x\sqrt{1-x^2}}$$

$$35. \frac{d}{dx} [\coth^{-1} x] = \frac{1}{1-x^2},$$

Example: Evaluate $\frac{dy}{dx}$ for

$$1) y = x^5 - \log x + 7$$

$$\text{Solution: } \frac{dy}{dx} = \frac{d}{dx} (x^5 - \log x + 7)$$

$$= \frac{d}{dx} (x^5) - \frac{d}{dx} (\log x) + \frac{d}{dx} (7)$$

$$= 5x^4 - \frac{1}{x} + 0$$

$$= 5x^4 - \frac{1}{x}$$

$$2) y = x^2 \cos x$$

$$\text{Solution: } \frac{dy}{dx} = \frac{d}{dx} (x^2 \cos x)$$

$$= x^2 \frac{d}{dx} (\cos x) + \cos x \frac{d}{dx} (x^2)$$

$$= x^2(-\sin x) + \cos x (2x)$$

$$= -x^2 \sin x + 2x \cos x$$

$$3) y = \frac{x^2+1}{x-3}$$

Solution: $\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^2+1}{x-3} \right)$

$$= \frac{(x-3)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x-3)}{(x-3)^2}$$

$$= \frac{(x-3)(2x) - (x^2+1)(1)}{(x-3)^2}$$

$$= \frac{2x^2 - 6x - x^2 - 1}{(x-3)^2}$$

$$= \frac{x^2 - 6x - 1}{(x-3)^2}$$

Exercise

Find the derivative of the following functions

- 1) $y = 3x^2 + \sin x + e^x$ [Ans. $\frac{dy}{dx} = 6x + \cos x + e^x$]
- 2) $y = x^2 \ln x$ [Ans. $\frac{dy}{dx} = 2x \ln x + x$]
- 3) $y = \frac{\sin x}{x^2}$ [Ans. $\frac{dy}{dx} = \frac{x^2 \cos x - 2x \sin x}{x^4}$]

Chain Rule: If $y = f(t)$, $t = g(x)$ be two differentiable functions then, their composite function is $y = f(g(x))$ is also, differentiable w.r.t x

Applying chain rule $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

Example: Find the derivative of the following functions

1) $y = \sqrt{1+x^2}$

Solution: Consider $y = f(t)$ and $t = g(x)$,

Here, $y = \sqrt{t}$ and $t = 1+x^2$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{1}{2\sqrt{t}} \cdot 2x \\ &= \frac{x}{\sqrt{1+x^2}}\end{aligned}$$

2) $y = \sin(x^2 + x)$

Solution: Consider $y = f(t)$ and $t = g(x)$,

Here, $y = \sin t$ and $t = x^2 + x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \cos t \cdot (2x + 1) \\ &= (2x + 1) \cos(x^2 + x)\end{aligned}$$

3) $y = \cos^3 2x$

Solution: Consider $y = f(t)$ and $t = g(x)$,

Here, $y = \cos^3 t$ and $t = 2x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= 3 \cos^2 t (-\sin t) \cdot 2 \\ &= -6 \cos^2 2x \sin 2x.\end{aligned}$$

Exercise

Find the derivative of the following functions

1) $y = (5x^3 - x^4)^7$ [Ans. $\frac{dy}{dx} = 7(5x^3 - x^4)^6(15x^2 - 4x^3)$]

2) $y = \log(2x + 3)$ [Ans. $\frac{dy}{dx} = \frac{2}{(2x+3)}$]

3) $y = e^{\tan x}$ [Ans. $\frac{dy}{dx} = e^{\tan x} \sec^2 x$]

Differentiation of Implicit Functions:

So far we have discussed derivatives of functions of the form $y = f(x)$ called the explicit form of the function. If the variables x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible or convenient to express y as a function of x , then y is said to be an implicit function of x .

Example:1 If $x^3 + y^3 = 3axy$, find $\frac{dy}{dx}$.

Solution: $x^3 + y^3 = 3axy$ -----(1)

Differentiating equation (1) both sides w.r.t x , we get

$$3x^2 + 3x^2 \frac{dy}{dx} = 3a \left(x \frac{dy}{dx} + y \right)$$

$$\Rightarrow x^2 + x^2 \frac{dy}{dx} = a \left(x \frac{dy}{dx} + y \right)$$

$$\Rightarrow (x^2 - ax) \frac{dy}{dx} = (ay - x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(ay - x^2)}{(x^2 - ax)}$$

Example:2 If $y + \sin y = x^2$, find $\frac{dy}{dx}$.

Solution: $y + \sin y = x^2$ ----- (2)

Differentiating equation (2) both sides w.r.t x , we get

$$\frac{dy}{dx} + \cos y \frac{dy}{dx} = 2x$$

$$\Rightarrow (1 + \cos y) \frac{dy}{dx} = 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{(1 + \cos y)}$$

Exercise

1) If $x^3 + y^3 = 3axy$, find $\frac{dy}{dx}$. [Ans. $-\frac{x}{y}$]

2) If $xy + \sin y = x^2$, find $\frac{dy}{dx}$. [Ans. $\frac{2x-y}{x+\cos y}$]

3) If $x^2 + xy + y^2 = 7$, find $\frac{dy}{dx}$. [Ans. $\frac{-2x-y}{x+2y}$]

Derivative of parametric equations:

If x and y are given functions of the form $x = x(t)$ and $y = y(t)$ then,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \text{ (provided } \frac{dx}{dt} \neq 0\text{)}$$

Example:1 If $x = t + \frac{1}{t}$ and $y = t - \frac{1}{t}$ then, find $\frac{dy}{dx}$.

$$\text{Solution: } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{t^2}}{\frac{1-1/t^2}{t^2}} = \frac{t^2+1}{t^2-1}.$$

Example:2 If $x = a \cos \theta$ and $y = a \sin \theta$ then, find $\frac{dy}{dx}$.

$$\text{Solution: } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a \cos \theta}{-a \sin \theta} = -\cot \theta.$$

Example:3 If $x = a(\theta + \sin \theta)$ and $y = a(1 - \cos \theta)$ then, find $\frac{dy}{d\theta}$.

$$\text{Solution: } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a \sin \theta}{a(1 + \cos \theta)} = \frac{\sin \theta}{(1 + \cos \theta)}.$$

Exercise

- 1) If $x = t^2, y = t^3$ find $\frac{dy}{dx}$. [Ans. $\frac{3t}{2}$]
- 2) If $x = \sin t, y = \cos t$ find $\frac{dy}{dx}$. [Ans. $-\tan t$]
- 3) If $x = 2t + 3, y = t^2 - 4$ find $\frac{dy}{dx}$. [Ans.]

Applications of derivative:

- **Maximum and minimum value of the function**

In calculus, one of the central problems is finding the **maximum** and **minimum** values that a function can attain. These values are known as the **extrema** of the function. Understanding these concepts helps in solving real-world problems like maximizing profit, minimizing cost, or finding optimal paths.

Maximum Value: A function $f(x)$ has a maximum value at $x = a$ if $f(a) \geq f(x)$ for all x in the domain of f .

Minimum Value: A function $f(x)$ has a minimum value at $x = b$ if $f(b) \leq f(x)$ for all x in the domain of f .

Global (Absolute) Maximum/Minimum: The highest or lowest value a function attains over its entire domain.

Local (Relative) Maximum/Minimum: The highest or lowest value the function attains within a small neighborhood around a point.

Consider the graph given in following figure of a continuous function defined on a closed interval $[a, d]$.

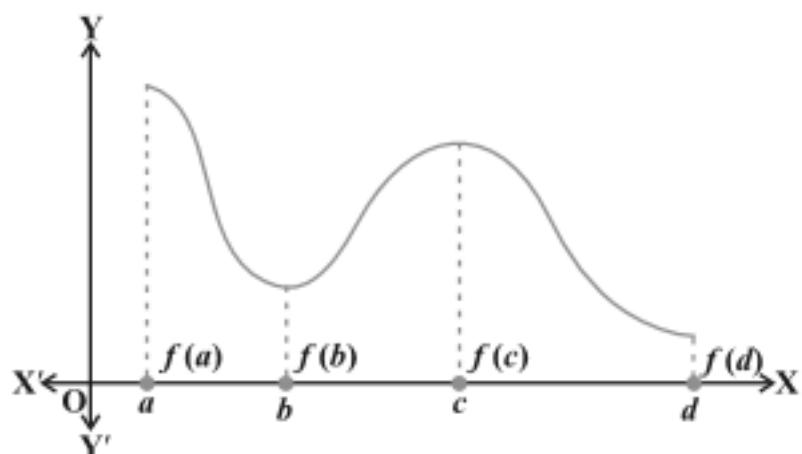


Figure 5: Geometrical interpretation of Maxima and Minima

Observe that the function f has a local minima at $x = b$ and local minimum value is $f(b)$. The function also has a local maxima at $x = c$ and local maximum value is $f(c)$. Also from the graph, it is evident that f has absolute maximum value $f(a)$ and absolute minimum value $f(d)$. Further note that the absolute maximum (minimum) value of f is different from local maximum (minimum) value of f .

Working Rule

Step 1: Find all critical points of f in the interval, i.e., find points x where $f'(x) = 0$.

Step 2: Apply second derivative test : Let $f(x)$ be a function differentiable at $x = a$. Then,

(a) $x = a$ is a point of local maximum of $f(x)$ if

- (i) $f'(a) = 0$ (necessary condition)
- (ii) $f''(a) < 0$ (sufficient condition)

(b) $x = a$ is a point of local minimum of $f(x)$ if

- (i) $f'(a) = 0$ (necessary condition)
- (ii) $f''(a) > 0$ (sufficient condition)

Step 3: Obtain the extreme value of the function $f(a)$.

Example 1. Find the maximum and minimum values for $f(x) = x^4 - 4x^3$.

Solution: $f(x) = x^4 - 4x^3$

$$f'(x) = 4x^3 - 12x^2$$

$$\text{Now, } f'(x) = 0 \Rightarrow 4x^3 - 12x^2 = 0 \Rightarrow 4x^2(x - 3) = 0 \Rightarrow x = 0, 3$$

Therefore, $x = 0$ and $x = 3$ are critical points.

$$\text{Also, } f''(x) = 12x^2 - 24x$$

Now, $f''(0) = 0$. Therefore, $f(x)$ has neither maximum nor minimum value at $x = 0$.

$f''(3) = 36 > 0$. Therefore, $f(x)$ has minimum value at $x = 0$. Also, $f(3) = -27$.

Example 2: Find the local maximum and local minimum values of the function f given by $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$.

Solution: We have $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x - 1)(x + 2)$$

$$\text{Now, } f'(x) = 0 \Rightarrow x = 0, 1, -2$$

$$f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$$

$$f''(0) = -24 < 0$$

$$f''(1) = 36 > 0$$

$$f''(-2) = 72 > 0$$

Therefore, by second derivative test, $x = 0$ is a point of local maxima and local maximum value of f at $x = 0$ is $f(0) = 12$ while $x = 1$ and $x = -2$ are the points of local minima and local minimum values of f at $x = 1$ and $x = -2$ are $f(1) = 7$ and $f(-2) = -20$ respectively.

- **Rolle's Theorem:** If a function f is (i) continuous in a closed interval $[a, b]$, (ii) differentiable in the open interval (a, b) and (iii) $f(a) = f(b)$, then there exists atleast one value $c \in (a, b)$ such that $f'(c) = 0$.

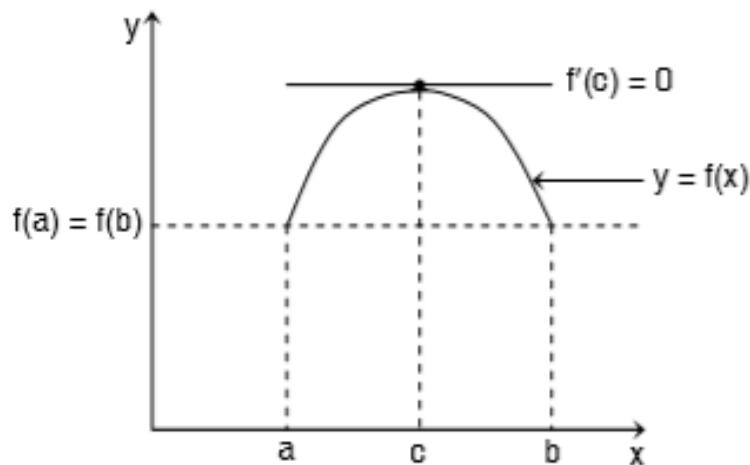


Figure 6: Geometrical interpretation of Rolle's theorem

Example: 1 If $f(x) = x^2 - 2x + 1$. Show that $f'(x) = 0$ has atleast one root in the interval $[0, 2]$ by using Rolle's theorem.

Solution: Here, $f(x) = x^2 - 2x + 1$ is continuous on $[0, 2]$ and differentiable in $(0, 2)$.

The function value of $f(x)$ at $x = 0$ and 2 are

$$f(0) = 1, \quad f(2) = 2^2 - 2 * 2 + 1 = 1,$$

$$\text{Therefore, } f(0) = f(2) = 1$$

$$\text{For } f'(c) = 0$$

$$\Rightarrow 2c - 2 = 0$$

$$\Rightarrow c = 1$$

Therefore, there exist $c = 1$ in the interval $[0, 2]$ such that $f'(c) = 0$.

Hence, Rolle's theorem is applicable.

Example: 2 Given a function $f(x) = x^2 - 5x + 4$. Check the applicability of Rolle's theorem for $f(x)$ in $[0, 4]$.

Solution: Here, $f(x) = x^2 - 5x + 4$ is continuous on $[0, 4]$ and differentiable in $(0, 4)$.

The function value of $f(x)$ at $x = 0$ and 4 are

$$f(0) = 4, \quad f(4) = 4^2 - 5 * 4 + 4 = 0,$$

$$\text{As } f(0) \neq f(4)$$

Hence, Rolle's theorem is not applicable.

Example: 3 For $f(x) = (x^2 - 2x)e^x$. Show that $f'(x) = 0$ has atleast one root in the interval $[0, 2]$ by using Rolle's theorem.

Solution: Given function $f(x) = (x^2 - 2x)e^x$ is continuous in $[0, 2]$ and differentiable in $(0, 2)$.

The function value of $f(x)$ at $x = 0$ and 2 are

$$f(0) = 0,$$

$$f(2) = (2^2 - 2 * 2)e^2 = 0,$$

$$\text{Therefore, } f(0) = f(2) = 0$$

From (1) and (2) it is confirmed that according to Rolle's theorem there exist a point where $f'(x) = 0$ in the interval $[0, 2]$

$$f'(x) = (x^2 - 2x)e^x + e^x(2x - 2)$$

$$\text{Now, } f'(x) = 0$$

$$\Rightarrow (x^2 - 2x + 2x - 2)e^x = 0$$

$$\Rightarrow e^x(x^2 - 2) = 0$$

$$\Rightarrow x^2 - 2 = 0 \quad (\because e^x \neq 0)$$

$$\Rightarrow x^2 = 2$$

$$\Rightarrow x = \pm\sqrt{2}$$

$$\text{As } c = \sqrt{2} \in [0, 2]$$

Therefore, if $f'(x) = 0$ then, there has atleast one root in the interval $[0, 2]$.

Hence, Rolle's theorem is verified for the given function.

Exercise:

- 1) Examine if Rolle's theorem is applicable for $f(x) = x^{3/2}$ in $[0, 4]$. [Ans. Not applicable]
- 2) Discuss the applicability of Rolle's theorem to the function

$$f(x) = \begin{cases} x^2 + 1 & \text{when } 0 \leq x < 1 \\ 3 - x & \text{when } 1 < x \leq 2 \end{cases}$$
 [Ans. Not applicable]
- 3) Verify Rolle's theorem for the function $f(x) = x(x - 4)^2$ on the interval $[0, 4]$. [Ans. show applicability]

- **Lagrange's Mean Value Theorem:**

If a function f is (i) continuous in a closed interval $[a, b]$, (ii) differentiable in the open interval (a, b) and then there exists atleast one value $c \in (a, b)$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$.

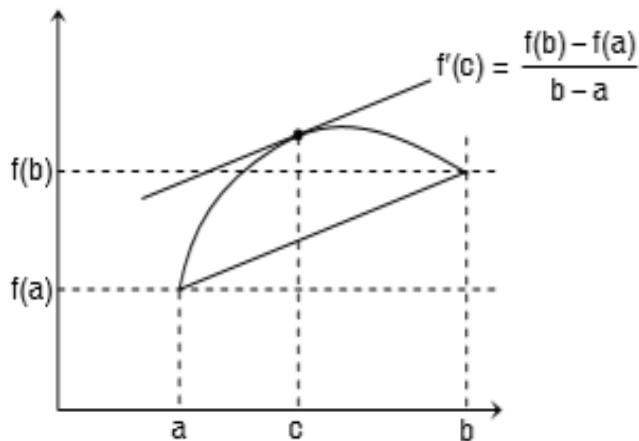


Figure 7: Geometrical interpretation of Lagrange's mean value theorem

Example: 1 For the function $f(x) = 1 - x^2$ in $0 \leq x \leq 2$, find the value of c in $0 \leq c \leq 2$ using LMVT.

Solution: We know that f is continuous on $[0, 2]$ and differentiable on $(0, 2)$. Therefore, on applying Lagrange's mean value theorem to f

$$\therefore f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$\therefore f'(c) = \frac{-3 - 1}{2}$$

$$\therefore f'(c) = -2 \quad \dots \dots \dots (1)$$

$$f'(x) = -2x$$

$$f'(c) = -2c$$

Substituting in equation (1)

$$-2c = -2$$

$$\therefore c = 1$$

Example: 2 If $f(x) = x^3 - x$; $x \in [0,2]$. Find c from Lagrange's mean value theorem.

Solution: We know that f is continuous on $[0,2]$ and differentiable on $(0,2)$. Therefore, on applying Lagrange's mean value theorem to f

$$\therefore f'(c) = \frac{f(2)-f(0)}{2-0}$$

$$\therefore f'(c) = \frac{6-0}{2}$$

$$\therefore f'(c) = 3 \text{ ----- (1)}$$

$$f'(x) = 3x^2 - 1$$

$$f'(c) = 3c^2 - 1$$

Substituting in equation (1)

$$3c^2 - 1 = 3$$

$$\therefore 3c^2 - 4 = 0$$

$$\therefore c^2 = \frac{4}{3}$$

$$\therefore c_1 = \frac{2}{\sqrt{3}}, \quad c_2 = \frac{-2}{\sqrt{3}}$$

Example: 3 If $f(x) = (x-1)(x-2)(x-3)$; $x \in [0,4]$. Find c from Lagrange's mean value theorem.

Solution: We know that, $f(x)$ is continuous on $[0,4]$ and differentiable on $(0,4)$. Therefore,

By Lagrange's mean value theorem

$$\begin{aligned} f'(c) &= \frac{f(b)-f(a)}{b-a} \\ \Rightarrow f'(c) &= \frac{f(4)-f(0)}{4-0} \\ \Rightarrow f'(c) &= \frac{6-(-6)}{4} \end{aligned}$$

$$\Rightarrow f'(c) = 3 \text{ ----- (1)}$$

$$f'(x) = (x-1)(x-2) + (x-1)(x-3) + (x-2)(x-3)$$

$$= x^2 - 3x + 2 + x^2 - 4x + 3 + x^2 - 5x + 6$$

$$= 3x^2 - 12x + 11$$

$$f'(c) = 3c^2 - 12c + 11$$

Substituting in equation (1)

$$3 = 3c^2 - 12c + 11$$

$$\therefore 3c^2 - 12c + 8 = 0$$

$$a = 3, b = -12, c = 8$$

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 4(3)(8)}}{6}$$

$$= \frac{12 \pm \sqrt{48}}{6}$$

$$= \frac{12 \pm 4\sqrt{3}}{6}$$

$$\therefore c_1 = \frac{12+4\sqrt{3}}{6}, c_2 = \frac{12-4\sqrt{3}}{6}.$$

Exercise

- 1) Find the value of c using LMVT, if the function $f(x) = x - 2 \sin x$ in the interval $[-\pi, \pi]$. [Ans. $c = \pm \frac{\pi}{2}$]
- 2) Verify the Lagrange's mean value theorem for $f(x) = e^x$ in $[0,1]$. [Ans. show applicability]
- 3) Find the value of c using LMVT, if the function $f(x) = x(x-1)(x-2)$ for $a = 0, b = \frac{1}{2}$. [Ans. $c = 0.236$]

Riemann Integral:

Introduction: The Riemann integral is one of the most fundamental concepts in real analysis and calculus. It provides a rigorous mathematical framework for defining and computing the area under a curve. Named after the German mathematician Bernhard Riemann, this integral formalizes the intuitive idea of summing up infinitely small quantities to find the total accumulation, such as area, displacement, or mass.

In simple terms, the Riemann integral evaluates the total value of a function over an interval by dividing that interval into small subintervals, calculating the value of the function at certain sample points within each subinterval, and then summing the products of these values

with the widths of the subintervals. As the subintervals become finer (i.e., as their width approaches zero), this sum approaches a limiting value — the Riemann integral. The Riemann integral is widely used in mathematics, physics, engineering, and economics for solving problems involving continuous change and accumulation.

Area of a Plane Region

In this section, we develop techniques to approximate the area between a curve (defined by a function $f(x)$), and the x -axis on a closed interval $[a, b]$. The main idea is to first approximate the area under the curve using shapes of known area (namely, rectangles). Then by using smaller and smaller rectangles, we get closer and closer approximations to the area. Finally, taking a limit allows us to calculate the exact area under the curve.

Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x -axis below, the line $x = a$ on the left, and the line $x = b$ on the right (Figure 8).

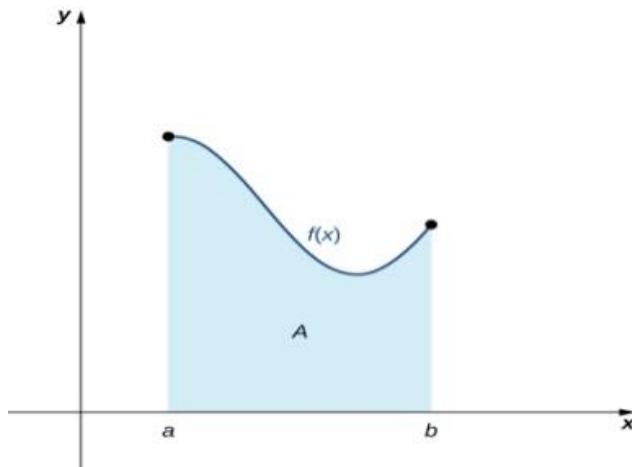


Figure 8: The area A bounded by $f(x)$ above, the x -axis below

We now take a geometric approach to approximate A . We divide A into many small shapes that have known area formulas; we can sum these areas and obtain an estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \dots, x_n$ with $x_0 = a$, $x_n = b$, and $x_i - x_{i-1} = \frac{b-a}{n}$ for $i = 1, 2, 3, \dots, n$.

We denote the width of each subinterval by Δx , so $\Delta x = \frac{b-a}{n}$ and $x_i = x_0 + i \Delta x$, for $i = 1, 2, 3, \dots, n$. The above division is called a **partition** of $[a, b]$. Since all the subintervals have the same width, the set of points forms a **regular partition** (or uniform partition) of the interval $[a, b]$. We next examine two methods: the **left-endpoint approximation** and the **right-endpoint approximation**.

Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \dots, n$), construct a rectangle with width Δx & height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1}) \Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure 9). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x = \sum_{i=1}^n f(x_{i-1})\Delta x$$

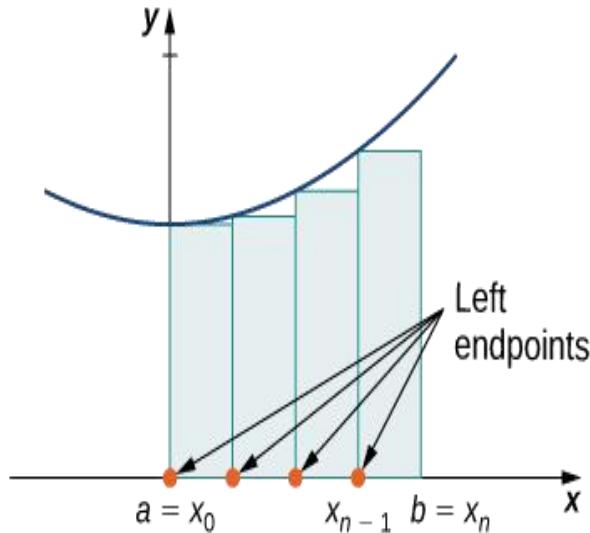


Figure 9: In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

Right-Endpoint Approximation

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i) \Delta x$ and the approximation for A is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x = \sum_{i=1}^n f(x_i)\Delta x.$$

The notation R_n indicates this is a **right-endpoint approximation** for A (Figure 10).

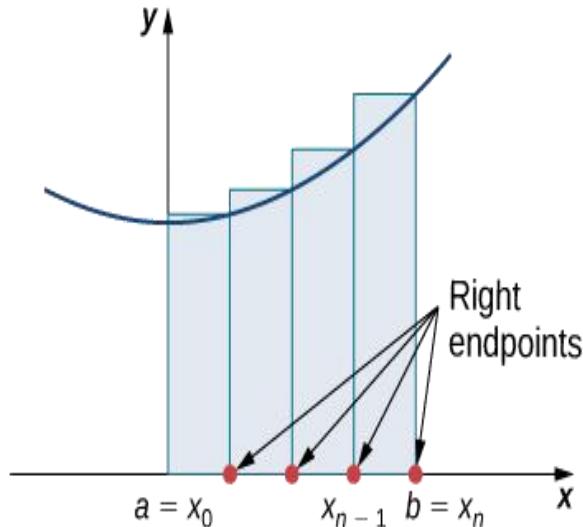


Figure 10: In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval.

Note that the right-endpoint approximation differs from the left-endpoint approximation.

Example: 1 Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution: First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \frac{2-0}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$ are shown in Figure 11. Using a left-endpoint approximation, the heights are $(0) = 0$, $(0.5) = 0.25$, $(1) = 1$, and $(1.5) = 2.25$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 1.75 \text{ units}^2 \end{aligned}$$

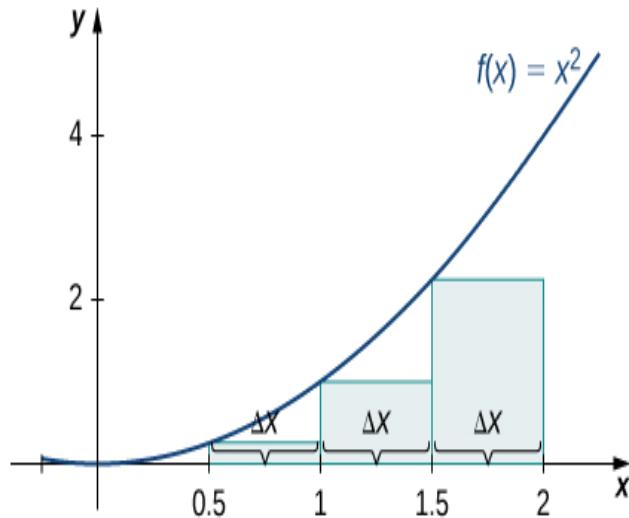


Figure 11: The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure 12. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned}
 R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\
 &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\
 &= 3.75 \text{ units}^2
 \end{aligned}$$

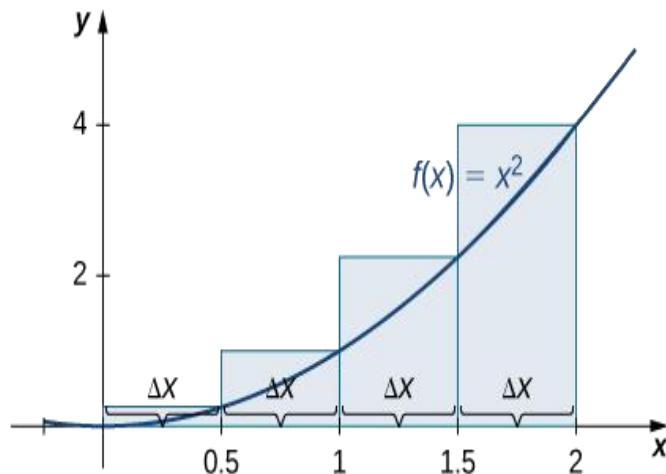


Figure 12: The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is 1.75 units²; the right-endpoint approximation is 3.75 units².

Observe that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, if we increase the number of points in our partition, our

estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

Practice Example: Use the five rectangles to find the approximation of the area of the region lying between the graph of $f(x) = -x^2 + 5$ and the x -axis between $x = 0$ and $x = 2$.

[Ans. $6.48 < \text{Area of region} < 8.08$]

Definition: Riemann Integral:

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^n f(c_i) \Delta x$ be a Riemann sum for $f(x)$ with a regular partition P . Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$\int_a^b f(x) dx = A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x. \quad x_{i-1} \leq c_i \leq x_i.$$

Example 1: Evaluate the definite integral as a limit of sum $\int_a^b x dx$.

Solution: The function $f(x) = x$ is integrable on the interval $[a, b]$ because it is continuous on $[a, b]$. Define Δ by subdividing $[a, b]$ into n subintervals of equal width:

$$\Delta x_i = \Delta x = \frac{b - a}{n}.$$

Choosing c_i as the right endpoint of each subinterval produces: $c_i = a + i(\Delta x) = a + i \frac{b-a}{n}$.

So, the definite integral is:

$$\begin{aligned} \int_a^b x dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(a + i \frac{b-a}{n} \right) \left(\frac{b-a}{n} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(a \frac{b-a}{n} + i \frac{(b-a)^2}{n^2} \right) \\ &= \lim_{n \rightarrow \infty} \left[\left(a \frac{b-a}{n} \sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2} \sum_{i=1}^n i \right) \right] \\ &= \lim_{n \rightarrow \infty} \left[\left(a \frac{b-a}{n} n + \left(\frac{(b-a)^2}{n^2} \right) \frac{n(n+1)}{2} \right) \right] \\ &= \lim_{n \rightarrow \infty} \left[a(b-a) + \frac{(b-a)^2}{2} \left(1 + \frac{1}{n} \right) \right] \\ &= a(b-a) + \frac{(b-a)^2}{2} \end{aligned}$$

$$= \frac{b^2 - a^2}{2}.$$

Example 2: Evaluate the definite integral as a limit of sum $\int_{-2}^1 2x \, dx$.

Solution: The function $f(x) = 2x$ is integrable on the interval $[-2, 1]$ because it is continuous on $[-2, 1]$. Define Δ by subdividing $[-2, 1]$ into n subintervals of equal width:

$$\Delta x_i = \Delta x = \frac{b - a}{n} = \frac{3}{n}.$$

Choosing c_i as the right endpoint of each subinterval produces: $c_i = a + i(\Delta x) = -2 + \frac{3i}{n}$.

So, the definite integral is:

$$\begin{aligned}\int_{-2}^1 2x \, dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 2 \left(-2 + \frac{3i}{n} \right) \left(\frac{3}{n} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(-\frac{12}{n} + \frac{18i}{n^2} \right) \\ &= \lim_{n \rightarrow \infty} \left[\left(-\frac{12}{n} \right) \sum_{i=1}^n 1 + \left(\frac{18}{n^2} \right) \sum_{i=1}^n i \right] \\ &= \lim_{n \rightarrow \infty} \left[\left(-\frac{12}{n} \right) n + \left(\frac{18}{n^2} \right) \frac{n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} \left[-12 + 9 \left(1 + \frac{1}{n} \right) \right] \\ &= -3.\end{aligned}$$

Exercise:

- 1) Evaluate the definite integral as a limit of sum $\int_a^b x^2 \, dx$. [Ans. $\frac{b^3 - a^3}{3}$]
- 2) Evaluate the definite integral as a limit of sum $\int_0^1 x^2 \, dx$. [Ans. 1/3]
- 3) Evaluate the definite integral as a limit of sum $\int_2^6 8 \, dx$. [Ans. 32]
- 4) Evaluate the definite integral as a limit of sum $\int_1^2 (x^2 + 1) \, dx$. [Ans. 11/3]
- 5) Evaluate the definite integral as a limit of sum $\int_{-2}^1 (2x + 3) \, dx$. [Ans. 6]

The Fundamental Theorem of Calculus:

The two major branches of calculus: differential and integral calculus might seem unrelated—but there is a very close connection. The connection was discovered independently by Isaac Newton and Gottfried Leibnitz and is stated in the Fundamental Theorem of Calculus. The theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations.

Theorem (Mean Value Theorem for Integrals): If $f(x)$ is continuous on the closed interval $[a, b]$ then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x) dx = f(c)(b - a).$$

First Fundamental Theorem of Calculus: If $f(x)$ is continuous on $[a, b]$, then $g(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is:

$$g'(x) = \frac{d}{dx} \left\{ \int_a^x f(t)dt \right\} = f(x).$$

Proof: Let $g(x) = \int_a^x f(t)dt$, then by definition of derivative

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_x^{x+h} f(t)dt \right] [\text{Use property of definite integral}] \end{aligned}$$

Since f is continuous at x , for small h , $f(t)$ is close to $f(x)$. By the Mean Value Theorem for integrals, there exists $c \in [x, x + h]$ such that

$$\int_x^{x+h} f(t)dt = f(c) \cdot h$$

As $h \rightarrow 0$, $x + h$ approaches to x and also c approaches to $(x \leq c \leq x + h)$. Since f continuous at x , $f(c)$ approaches to $f(x)$. Thus

$$g'(x) = \lim_{h \rightarrow 0} \frac{1}{h} f(c)h = \lim_{h \rightarrow 0} f(c) = f(x).$$

Hence the proof.

Second Fundamental Theorem of Calculus: If $f(x)$ is continuous on $[a, b]$ and $g(x)$ is antiderivative of $f(x)$ on $[a, b]$ then

$$\int_a^b f(x)dx = g(b) - g(a).$$

Proof: Let $F(x) = \int_a^x f(t)dt$, then by first theorem $F'(x) = f(x)$, i.e. $F(x)$ is antiderivative of $f(x)$. If $g(x)$ is any other antiderivative on $[a, b]$ then F and g differ by constant i.e.,

$$F(x) = g(x) + c$$

At $x = a$,

$$\begin{aligned} F(a) &= \int_a^a f(t)dt = 0 = g(a) + c \\ \Rightarrow c &= -g(a) \end{aligned}$$

$$\text{Therefore } F(x) = g(x) - g(a)$$

$$\text{Now, } \int_a^b f(x)dx = F(b) = g(b) - g(a).$$

This completes the proof.

1.4 Applications of Integral

1. Area of a Region between Two Curves:

Consider two functions f and g that are continuous on the interval $[a, b]$. Also, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f . This can be geometrically interpreted as the area of the region between the graphs when the area of the region under the graph of g subtracted from the area of the region under the graph of f as shown in Figure 13

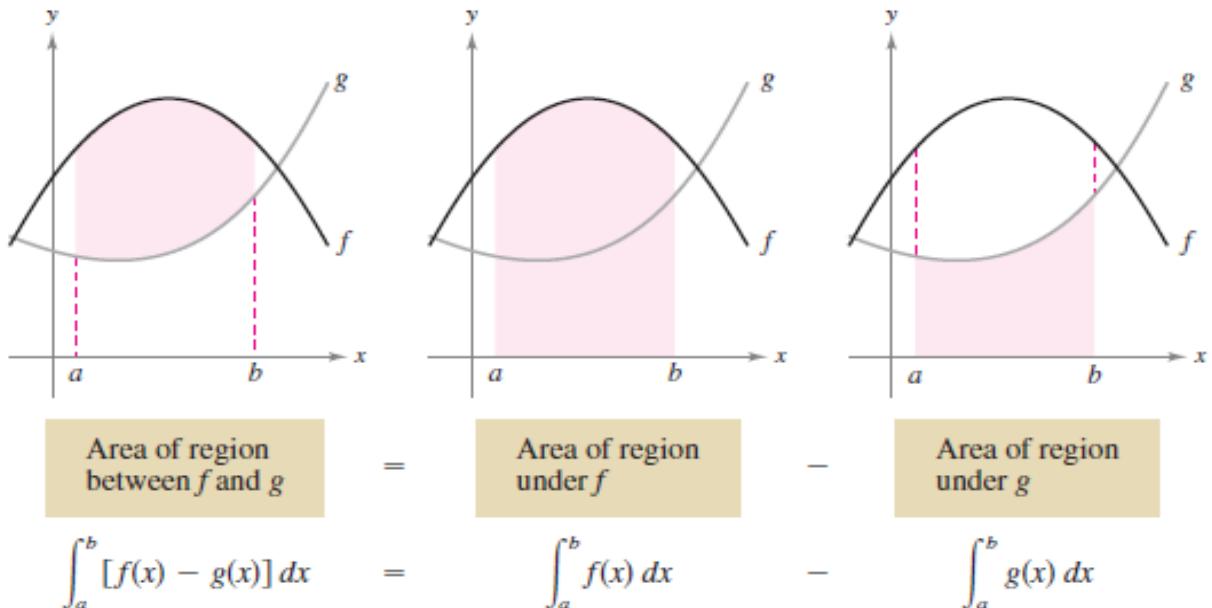


Figure 13: Area of the region between f and g

If functions f and g are continuous on the interval $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b [f(x) - g(x)] dx.$$

Example1: Find the area of the region bounded by the curves $y = x^2 + 2$ and $y = -x$, $x = 0$ and $x = 1$.

Solution: Let $f(x) = x^2 + 2$ and $g(x) = -x$ then $f(x) \geq g(x)$ for all x in $[0, 1]$ as shown in the Figure 14

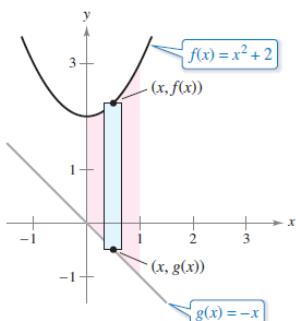


Figure 14

$$\begin{aligned} \text{Now, } A &= \int_a^b [f(x) - g(x)] dx \\ &= \int_0^1 [(x^2 + 2) - (-x)] dx \\ &= \int_0^1 [x^2 + x + 2] dx \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_0^1 \\ &= \frac{1}{3} + \frac{1}{2} + 2 = \frac{17}{6}. \end{aligned}$$

Example2: Find the area bounded by $f(x) = 2 - x^2$ and $g(x) = x$.

Solution: We notice that f and g intersect each other at two points $x = -2$ and $x = 1$, shown in the Figure 15. Let $a = -2$ and $b = 1$, then as $f(x) \geq g(x)$ for all x in $[-2, 1]$, the area of the region is

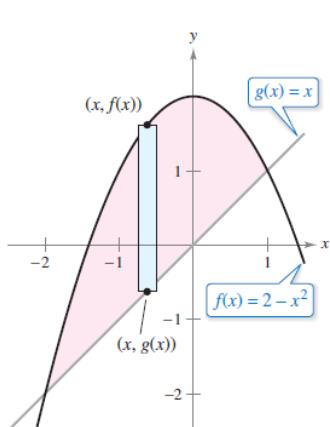


Figure 15

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)] dx \\ &= \int_{-2}^1 [(2 - x^2) - (x)] dx \\ &= \int_{-2}^1 [-x^2 - x + 2] dx \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 \\ &= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + \frac{4}{2} + 4 = \frac{9}{2}. \end{aligned}$$

Exercise:

- 1) Find the area bounded by $f(x) = x^2 + 2x + 1$ and $g(x) = 2x + 5$. [Ans. 32/3]
- 2) Find the area bounded by $f(x) = x^2$ and $g(x) = 6 - x$. [Ans. 125/6]
- 3) Find the area bounded by $f(x) = 2 - x^2$ and $g(x) = x$. [Ans. 9/2]
- 4) Find the area bounded by $f(x) = x^2/8$ and $g(x) = \frac{x+8}{2}$. [Ans. 52/3]
- 5) Find the area bounded by $f(x) = x^2 - 1$ and $g(x) = -x + 2$ between $x = 0$ and $x = 1$. [Ans. 13/6]

2. Arc length of the curve:

What do we mean by the length of a curve? One way to imagine it is by fitting a piece of string along the curve and then measuring the string with a ruler. However, this can be difficult to do accurately, especially for complicated curves. Therefore, we need a precise definition for the length of an arc of a curve, and for this, we use the **definite integral**. This is another important application of the definite integral.

Definition: Let the function $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The arc length of between a and b is defined by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Similarly, for the function $x = g(y)$ the arc length of g between c and d is defined by

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

Example 1: Find the length of the arc of the parabola $x^2 = 4ay$ measured from the vertex to one extremity of the latus rectum.

Solution: Let A be the vertex and L an extremity of the latus rectum, so that at A , $x = 0$ and at L , $x = 2a$ shown in the Figure 16

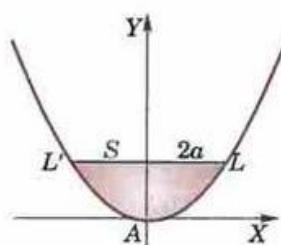


Figure 16

Now, $y = \frac{x^2}{4a}$, so that $\frac{dy}{dx} = \frac{1}{4a} \cdot 2x = \frac{x}{2a}$

Therefore

$$\begin{aligned} L &= \int_0^{2a} \sqrt{1 + \left[\frac{dy}{dx} \right]^2} dx \\ &= \int_0^{2a} \sqrt{1 + \left[\frac{x}{2a} \right]^2} dx \\ &= \frac{1}{2a} \int_0^{2a} \sqrt{(2a)^2 + x^2} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2a} \left[\frac{x}{2} \sqrt{(2a)^2 + x^2} + \frac{(2a)^2}{2} \sinh^{-1} \frac{x}{2a} \right]_0^{2a} \\
&= \frac{1}{2a} \left[\frac{2a}{2} \sqrt{(2a)^2 + (2a)^2} + \frac{(2a)^2}{2} \sinh^{-1} 1 \right] \\
&= a[\sqrt{2} + \sinh^{-1} 1] \\
&= a[\sqrt{2} + \log(1 + \sqrt{2})]
\end{aligned}$$

$$\because \sinh^{-1} x = \log [x + \sqrt{(1+x^2)}]$$

Example 2: Find the arc length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\left[\frac{1}{2}, 2\right]$.

Solution: Given curve is $y = \frac{x^3}{6} + \frac{1}{2x}$ so

$$\frac{dy}{dx} = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2} \left(x^2 - \frac{1}{x^2} \right)$$

$$\text{Therefore, } L = \int_{\frac{1}{2}}^2 \sqrt{1 + \left[\frac{dy}{dx} \right]^2} dx$$

$$\begin{aligned}
&= \int_{\frac{1}{2}}^2 \sqrt{1 + \left[\frac{1}{2} \left(x^2 - \frac{1}{x^2} \right) \right]^2} dx \\
&= \int_{\frac{1}{2}}^2 \sqrt{\frac{1}{4} \left(x^4 + 2 + \frac{1}{x^4} \right)} dx
\end{aligned}$$

$$\begin{aligned}
&= \int_{\frac{1}{2}}^2 \frac{1}{2} \left(x^2 + \frac{1}{x^2} \right) dx \\
&= \frac{1}{2} \left[\frac{x^3}{3} - \frac{1}{x} \right]_{\frac{1}{2}}^2 \\
&= \frac{1}{2} \left(\frac{13}{6} + \frac{47}{24} \right) = \frac{33}{16}
\end{aligned}$$

Exercise:

- 1) Find the length of the arc of the parabola $x^2 = 4ay$ cut off by the line $3y = 8x$.
[Ans. 32a/3]
- 2) Find the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = 2$.
[Ans. $\frac{1}{2}(e^2 - e^{-2})$]
- 3) Find the length of the curve $(y - 1)^3 = x^2$ on the x - interval $[0, 8]$.
[Ans. $\frac{1}{27}(40^{\frac{3}{2}} - 4^{\frac{3}{2}})$]

3. Surface Area of revolution:

When the graph of a continuous function is revolved about a line, the resulting surface is called a surface of revolution, which resembles a cylindrical object, as shown in the Figure 17. In this section, we will learn how to find the surface area of such a cylindrical object.

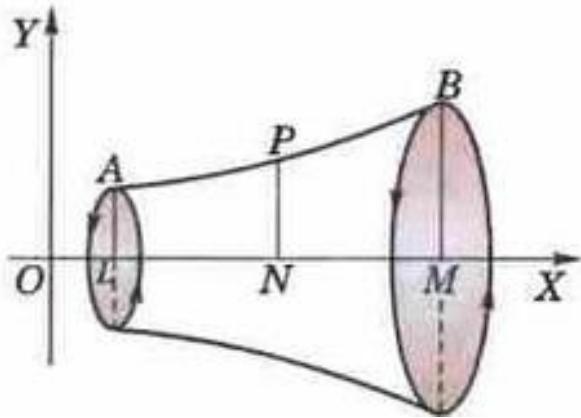


Figure 17

1. Revolution about x-axis

The surface area of the solid generated by the revolution about x – axis, of the arc of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$S = \int_a^b 2\pi y \sqrt{1 + [y']^2} dx$$

2. Revolution about y-axis

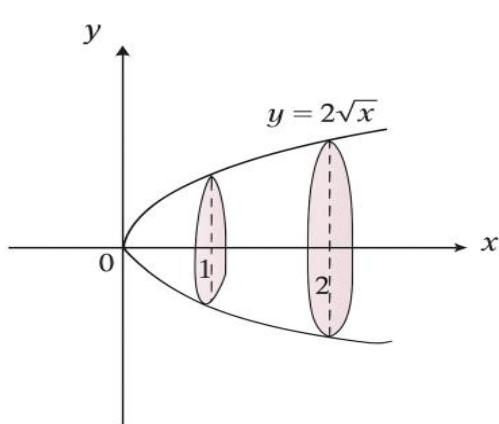
The surface area of the solid generated by the revolution about y – axis, of the arc of the curve $x = f(y)$ from $y = c$ to $y = d$ is

$$S = \int_c^d 2\pi x \sqrt{1 + [x']^2} dy$$

Example 1: Find the surface area of the solid generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x – axis.

Solution: Given curve is $y = 2\sqrt{x}$, so $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$

The surface area is

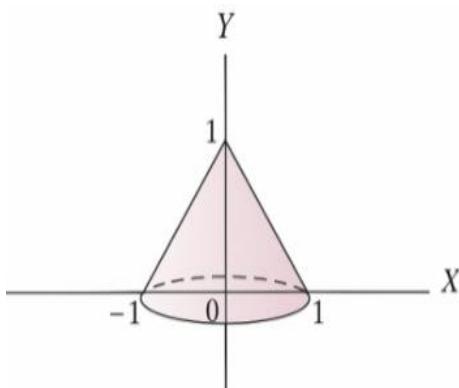


$$\begin{aligned}
 S &= \int_a^b 2\pi y \sqrt{1 + [y']^2} dx \\
 &= \int_1^2 2\pi 2\sqrt{x} \sqrt{1 + \left[\frac{1}{\sqrt{x}}\right]^2} dx \\
 &= \int_1^2 2\pi 2\sqrt{x} \sqrt{1 + \frac{1}{x}} dx \\
 &= 4\pi \int_1^2 \sqrt{1+x} dx \\
 &= 4\pi \cdot \frac{2}{3} (x+1)^{\frac{3}{2}} \Big|_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2})
 \end{aligned}$$

Figure 18

Example 2: The line segment $x = 1 - y$, $0 \leq y \leq 1$ is revolved about the y – axis to generate the cone. Find its lateral surface area.

Solution: Given curve is $x = 1 - y$, so $\frac{dx}{dy} = -1$



The surface area is

$$\begin{aligned}
 S &= \int_c^d 2\pi x \sqrt{1 + [x']^2} dy \\
 &= \int_0^1 2\pi(1-y) \sqrt{1 + [-1]^2} dy \\
 &= 2\pi\sqrt{2} \int_0^1 (1-y) dy \\
 &= 2\pi\sqrt{2} \left(y - \frac{y^2}{2}\right) \Big|_1^0 = 2\pi\sqrt{2} \left(1 - \frac{1}{2}\right) = \pi\sqrt{2}.
 \end{aligned}$$

Figure 19

Exercise:

- 1) Find the area of the surface formed by the revolution of $y^2 = 4ax$ about x – axis by the arc from vertex to one end of the latus-rectum. [Ans. $\frac{8\pi a^2}{3}(2\sqrt{2} - 1)$]

- 2) Obtain surface area of the sphere generated by the revolution of semi-circle about x – axis. [Ans. $4\pi a^2$]
- 3) Find the area of the surface formed by the revolution of $y = x^2$ about y – axis which lies between $y = 0$ and $y = 4$. [Ans. $\frac{\pi}{6}(17\sqrt{17} - 1)$]
- 4) Find the area of the surface formed by the revolution of $x = y^3/3$ about y – axis which lies between $y = 0$ and $y = 1$. [Ans. $\frac{\pi}{9}[\sqrt{8} - 1]$]
- 5) Find the surface area of the solid generated by revolving the curve $f(x) = x^3$ on the interval $[0, 1]$ about the x – axis. [Ans. $\frac{\pi}{27}(10^{\frac{3}{2}} - 1)$]

4. Volume of revolution:

In this section, we will learn how to find the volume of the cylindrical object generated by the graph of a continuous function about a line

1. Revolution about x-axis

The volume of the solid generated by the revolution about x – axis, of the arc of the curve $y = f(x)$ from $x = a$ to $x = b$ is

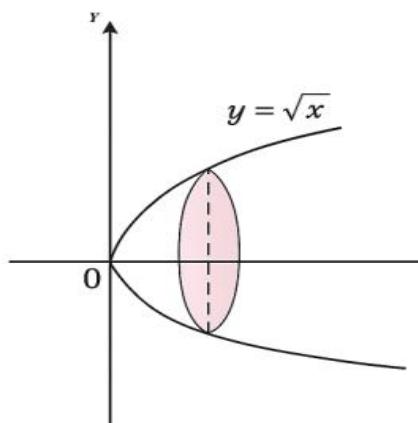
$$V = \int_a^b \pi y^2 dx$$

2. Revolution about y-axis

The volume of the solid generated by the revolution about y – axis, of the arc of the curve $x = f(y)$ from $y = c$ to $y = d$ is

$$V = \int_c^d \pi x^2 dy$$

Example 1: Find the volume of the solid generated by the curve $y = \sqrt{x}$ on the interval $[0, 1]$, about x – axis.



Solution: Given curve is $y = \sqrt{x}$, the volume of the revolution is

$$\begin{aligned} V &= \int_a^b \pi y^2 dx \\ &= \int_0^1 \pi(\sqrt{x})^2 dx \\ &= \int_0^1 \pi x dx = \frac{\pi}{2} \end{aligned}$$

Figure 20

Example 2: Find the volume of the reel-shaped solid formed by the revolution about the y – axis, of the part of the parabola $y^2 = 4ax$ cut off by the latus rectum.

Solution: Given parabola is $y^2 = 4ax$. From vertex to the one extremity of the latus rectum y varies from 0 to $2a$ shown in the Figure 18. Therefore, the volume of the revolution is

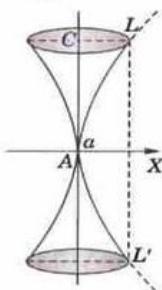


Figure 21

$$\begin{aligned}
 \text{Required volume} &= 2(\text{volume generated by the revolution about the } y\text{-axis of the area ALC}) \\
 &= 2 \int_0^{2a} \pi x^2 dy \\
 &= 2 \int_0^{2a} \pi \left(\frac{y^2}{4a}\right)^2 dy \\
 &= \frac{\pi}{8a^2} \int_0^{2a} y^4 dy \\
 &= \frac{\pi}{8a^2} \left|\frac{y^5}{5}\right|_0^{2a} \\
 &= \frac{4\pi a^3}{5}
 \end{aligned}$$

Exercise:

- 1) Find the volume of sphere of radius a . [Ans. $\frac{4}{3}\pi a^3$]
- 2) Find the volume generated by revolving the portion of the parabola $y^2 = 4ax$ cut off by one of the latus rectum about the axis of the parabola. [Ans. $2\pi a^3$]
- 3) Find the volume generated by revolving the portion of the parabola $x^2 = 4ay$ cut off by one of the latus rectum about x – axis. [Ans. $\frac{4}{5}\pi a^3$]
- 4) Find the volume of cone generated by revolving the triangle in the first quadrant bounded by $x + y = 2$ about y – axis. [Ans. $\frac{8}{3}\pi$]
- 5) Find the volume of cylinder generated by the square bounded by the line segments $x = 0, x = 2, y = 0$ and $y = 2$, about x – axis. [Ans. 8π]