UNIT-3 MULTIPLE INTEGRATION

Introduction

Integration of function of normally two or more variables is normally called as multiple integration.

The particular case of integration of functions of two variables is called double integration and that of three variables is called triple integration.

Multiple integrals are useful in evaluating plane area, mass of lamina, mass and volume of a solid regions, etc.

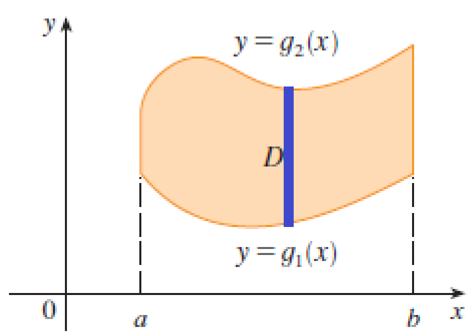
Evaluation of Double integration:

Double integration of a function $F(x,y)$ over the region R can be evaluated successive integration. There are two different methods to evaluate a double integral which are known as Fubini's theorem.

Method-I

Consider the region R, bounded by the curve $y = g_1(x)$ to $y = g_2(x)$ and the lines $x = a$ to $x = b$.

For $\int \int f(x,y)\,dy\,dx$

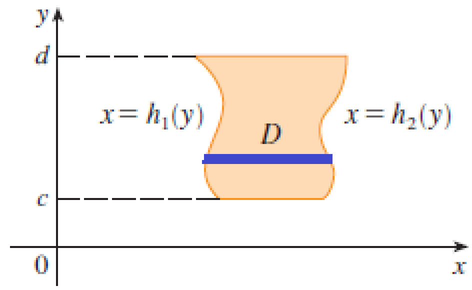


Draw vertical strip in region, lower end of strip touches curve $y = g_1(x)$ and upper end of strip touches curve $y = g_2(x)$, hence limit of y is $y = g_1(x)$ to $y = g_2(x)$.
This strip moves from $x = a$ to $x = b$.
Hence, limit of x is $x = a$ to $x = b$.

$$\int \int f(x,y)\,dy\,dx = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$$

Method-II

Consider the region R, bounded by the curve $x = h_1(y)$ to $x = h_2(x)$ and the lines $y = c$ to $y = d$



For $\int\int f(x,y)\,dx\,dy$

Draw horizontal strip in region, strip touches curve $x = h_1(y)$ and $x = h_2(y)$, hence limit of y : From $x = h_1(y)$ and $x = h_2(y)$.

This strip moves from $y = c$ to $y = d$.

Hence, limit of x: From $y = c$ to $y = d$.

Hence,

$$\int\int f(x,y)\,dx\,dy = \int_{y=c}^{y=d}\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy$$

Note:

i. If all the four limits are constant then the function $f(x,y)$ can be integrated with respect to any variable first. But if $f(x,y)$ is implicit and is discontinuous within or on the boundary of the region of integration then the change of order of integration will affect the result.

ii. If all the four limits are constant and $f(x,y)$ is explicit then the double integral can be written as the product of two single integrals.

iii. If inner limits depends on x then the function $f(x,y)$ first integrate with respect to y and vice-versa.

- **When limits of integration are given:**

Example 1: Evaluate $\int_0^3 \int_0^1 (x^2 + 3y^2)\,dy\,dx$

Solution: $\int_0^3 \int_0^1 (x^2 + 3y^2)\,dy\,dx = \int_0^3 \left(x^2 y + \frac{y^3}{3}\right)_0^1 dx$

$$= \int_0^3 2x^2 + \frac{8}{3}\,dx$$

$$= \left(\frac{2x^3}{3} + \frac{8x}{3}\right)_0^3 = \frac{78}{3}$$

Example 2: Evaluate $\int_{-1}^{1}\int_{0}^{2}(1-6x^2y)\,dx\,dy$

Solution: $\int_{-1}^{1}\int_{0}^{2}(1-6x^2y)\,dx\,dy$

$$= \int_{-1}^{1}\left(x+\frac{6x^3y}{3}\right)_0^2 dy$$

$$= \int_{-1}^{1}(2-16\mathrm{y})\mathrm{dy}$$

$$= \left(2\mathrm{y}-\frac{16y^2}{2}\right)_{-1}^{1} = 4$$

Example 3 : Evaluate $\int_0^1\int_0^{x^2}(x+y)dydx$.

Solution:: $\int_0^1\int_0^{x^2}(x+y)dydx$

$$= \int_0^1\left(xy+\frac{y^2}{2}\right)_0^{x^2} dx$$

$$= \int_0^1\left(x^3+\frac{x^4}{2}\right)-0\,dx$$

$$= \left(\frac{x^4}{4}+\frac{x^5}{10}\right)_0^1$$

$$= \frac{1}{4}+\frac{1}{10} = \frac{3}{20}$$

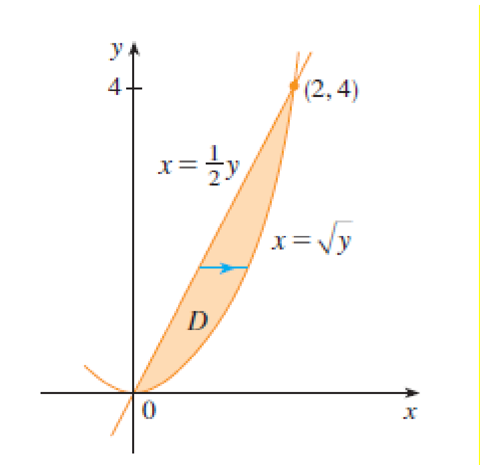
Exercise:

1) Evaluate $\int_o^2\int_1^3(x+2y)dxdy$. [Ans :16]
2) Evaluate $\int_o^2\int_0^1(x^2+2xy)dxdy$. [Ans :14/3]
3) Evaluate $\int_1^4\int_0^2(x^2y+y)dxdy$. [Ans : 35]
4) Evaluate $\int_0^1\int_0^y(x+y)dxdy$. [Ans : ½]
5) Evaluate $\int_0^1\int_x^1 y\,dydx$. [Ans ; 1/3]

- **When limits of integration are not given:**

Example 1: Evaluate the $\iint_R (x^2+y^2)\,dA$, where R is the region bounded by the line $y=2x$ and the parabola $y=x^2$.

Ans:



$$I=\iint_R (x^2+y^2)\,dA$$

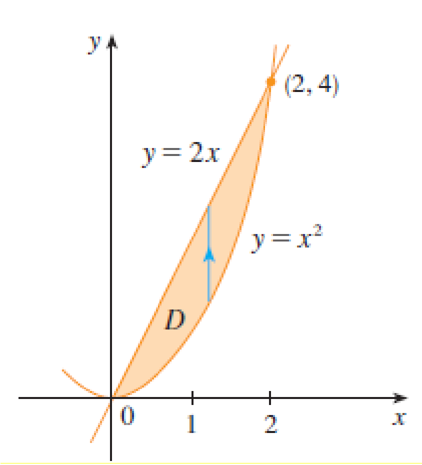
$$=\int_0^4\int_{\frac{y}{2}}^{\sqrt{y}}(x^2+y^2)\,dx\,dy$$

$$=\int_0^4\left(\frac{x^3}{3}+xy^2\right)_{\frac{y}{2}}^{\sqrt{y}}dy$$

$$=\int_0^4\left(\frac{y^{\frac{3}{2}}}{3}+y^{\frac{5}{2}}-\frac{13y^3}{24}\right)dy$$

$$=\left(\frac{2y^{\frac{5}{2}}}{15}+\frac{2y^{\frac{7}{2}}}{7}-\frac{13y^4}{96}\right)_0^4=\frac{216}{35}$$

OR



$$\iint_R (x^2+y^2)\,dA$$

$$=\int_0^2\int_{2x}^{x^2}(x^2+y^2)\,dy\,dx$$

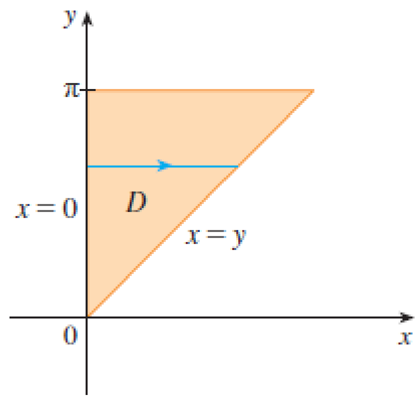
$$=\int_0^2\left(x^2y+\frac{y^3}{3}\right)_{2x}^{x^2}dx$$

$$=\int_0^2\left(x^4+\frac{x^6}{3}-\frac{14x^3}{3}\right)dx$$

$$=\left(\frac{x^5}{5}+\frac{x^7}{21}-\frac{7x^4}{6}\right)_0^2=\frac{216}{35}$$

Example 2: Evaluate $\iint_R \sin(y^2)\,dA$, where R is the region bounded by the lines

$$y = x, y = \pi, x = 0.$$



$$\int_0^{\pi}\int_0^{y} \sin(y^2)\,dx\,dy$$

$$= \int_0^{\pi} y\sin(y^2)\,dy$$

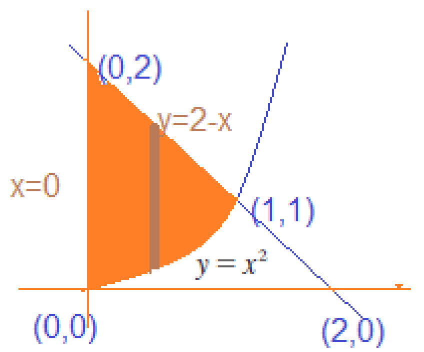
Suppose $y^2 = t$

$$\therefore\ y\,dy = \frac{dt}{2}$$

$$= \int_0^{\pi} \sin t \frac{dt}{2}$$

$$= \left(-\frac{\cos t}{2}\right)_0^{\pi} = 1$$

Example 3: Compute the double integral of the function $f(x, y) = 6 - x + 2y$ over the region bounded by the curves $x = y^2$ and $y = 2 - x$ in the x-y plane.



$$\int_0^{1}\int_{x^2}^{2-x} (6 - x + 2y)\,dy\,dx$$

$$= \int_0^{1} \left(6y - xy + y^2\right)_{x^2}^{2-x} dx$$

$$= \int_0^{1} \left(16 - 10x - 4x^2 + x^3 - x^4\right)dx$$

$$= \frac{-187}{660}$$

Example 4: Evaluate $I = \iint_R \left(6x + 2y^2\right)dA$, where R is the region enclosed by the parabola $x = y^2$ and the line $x + y = 2$.

$$I = \int_{-2}^{1}\int_{y^2}^{2-y} \left(6x + 2y^2\right)dx\,dy$$

$$I = \int_{-2}^{1} \left[3x^2 + 2xy^2\right]_{x=y^2}^{x=2-y} dy$$

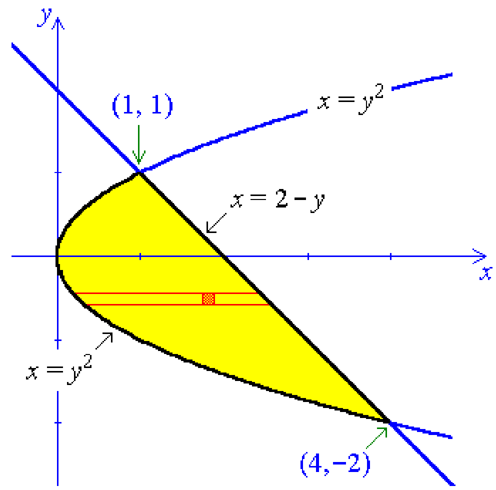
$$= \int_{-2}^{1} \left(\left(3(2-y)^2 + 2(2-y)y^2 \right) - \left(3y^4 + 2y^4 \right) \right) dy$$

$$= \int_{-2}^{1} \left(\left(12 - 12y + 3y^2 \right) + \left(4y^2 - 2y^3 \right) - 5y^4 \right) dy$$

$$= \int_{-2}^{1} \left(12 - 12y + 7y^2 - 2y^3 - 5y^4 \right) dy$$

$$= \left[12y - 6y^2 + \frac{7}{3}y^3 - \frac{1}{2}y^4 - y^5 \right]_{-2}^{+1}$$

$$= \left(12 - 6 + \frac{7}{3} - \frac{1}{2} - 1 \right) - \left(-24 - 24 - \frac{56}{3} - 8 + 32 \right) = \frac{99}{2}$$



Exercise:

1) Evaluate the $\iint_R (x^2 + y)\,dA$, where R is bounded by $y = 0$ and $y = \sqrt{x}$ for ,$0 \le x \le 1$.

[Ans : $\frac{15}{28}$]

2) Evaluate the $\iint_R (3x - y)\,dA$. where R is bounded by $x = y^2$ and the vertical line $x = 4$.

[Ans : $\frac{384}{5}$]

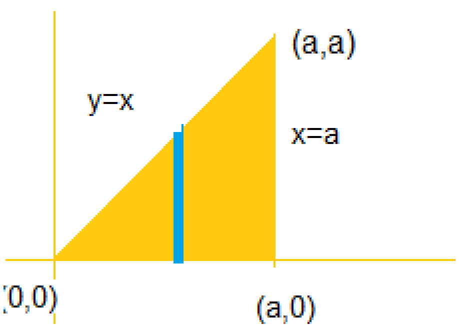
Change of Order:

The change of order often makes the evaluation of double integration easier e.g. in example 2 in previous section, $\int_0^{\pi}\int_x^{\pi}\sin(y^2)\,dy\,dx$ will be evaluated on reversing the order of integration.

In double integral with constant limits, the order of integration is immaterial provided the limits of integration are changed accordingly. But in case of double integral with variable limits, the limits of the integration changes with the change of order of integration. The new limits are obtained by sketching the region of integration. Sometime in changing the order of integration, it is required to split up the region of integration, and the given integral is expressed as the sum of number of double integrals with the changed limits.

Example 1

Change the order of integration in $\int_0^{a}\int_y^{a}\frac{x\,dx\,dy}{x^2+y^2}$, and evaluate the same



From the limits of integration, it is clear that the region of integration is bounded by y=x, x=a and y= a. Thus, the region of integration as shown in figure. Draw vertical strip. So new limits of integration are
Limits of x: from y=0 to y=x
Limits of y: from x=0 to x=a

$$\int_0^{a}\int_0^{x}\frac{x\,dy\,dx}{x^2+y^2}$$

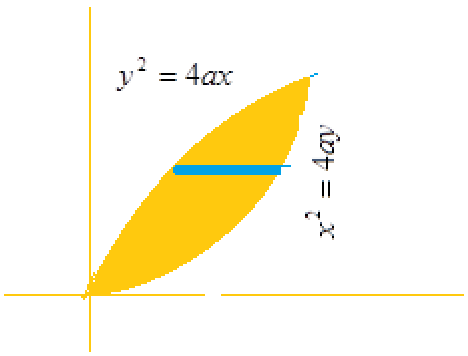
$$=\int_0^{a}x\left(\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right)_0^{x}dx$$

$$=\int_0^{a}\frac{\pi}{4}dx$$

$$=\frac{\pi}{4}(x)_0^{a}$$

$$=\frac{\pi a}{4}$$

Example 2: Change the order of integration in the following integral and evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{az}} dy\,dx$



After changing the order, new limits are

From $x = \frac{y^2}{4a}$ to $x = 2\sqrt{ax}$

From $y = 0$ to $y = 4a$

$$\int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy$$

$$= \int_0^{4a} (x)_{\frac{y^2}{4a}}^{2\sqrt{ay}}\, dy$$

$$= \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= \left(\frac{4\sqrt{a}\,y^{\frac{3}{2}}}{3} - \frac{y^3}{12a}\right)_0^{4a}$$

$$= \frac{16a^2}{3}$$

Example 3: Change the order of integration $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$

$y^2 = x$ y Region 1 : $x = 0$ to $x = \sqrt{y}$

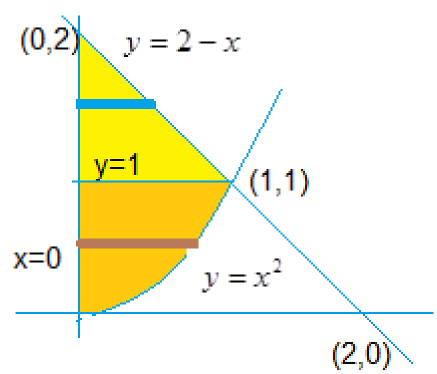
y=0 to y= 1

Region 2: $x = 0$ to $x = 2 - y$

y=1 to y=2

$$I - I_1 = \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} y\,dy = \int_0^1 \frac{y^2}{2}dy = \left(\frac{y^3}{6}\right)_0^1 = \frac{1}{6}$$



$$I_1 = \int_1^2 \int_0^{2-y} xy\,dx\,dy = \int_1^2 \left(\frac{x^2}{2}\right)_0^{2-y} y\,dy = \int_0^1 \frac{(2-y)^2 y}{2}\,dy$$

$$= \int_1^2 \frac{\left(4y - 4y^2 + y^3\right)}{2}\,dy = \left(y^2 - \frac{2}{3}y^3 + \frac{y^4}{8}\right)_1^2 = \frac{5}{24}$$

$$I = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

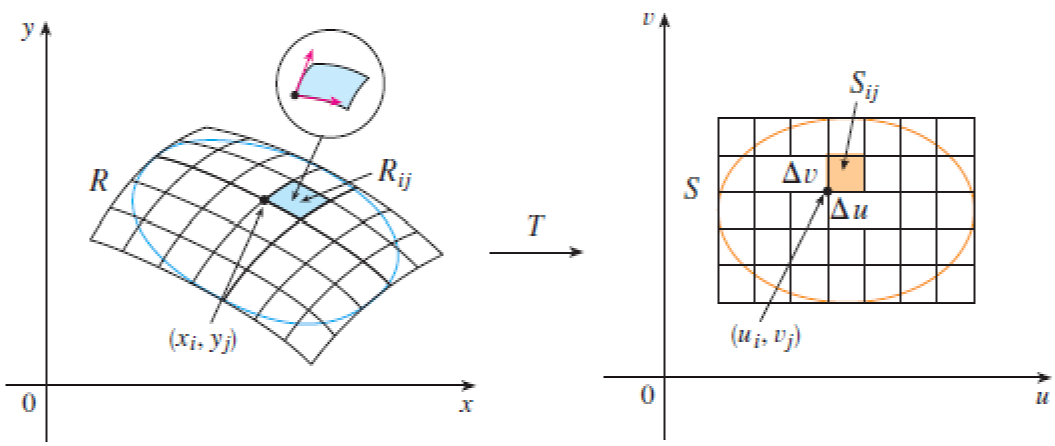
Exercise:

1) Change the order of integration and evaluate $\int_0^1 \int_x^{2x} (x+y)\,dydx$. [Ans: $\frac{5}{6}$]
2) Change the order of integration and evaluate $\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\,dxdy$. [Ans : $e-1$]
3) Change the order of integration and evaluate $\int_0^1 \int_x^1 \cos y^2\,dydx$. [Ans : $\frac{1}{2}\sin 1$]

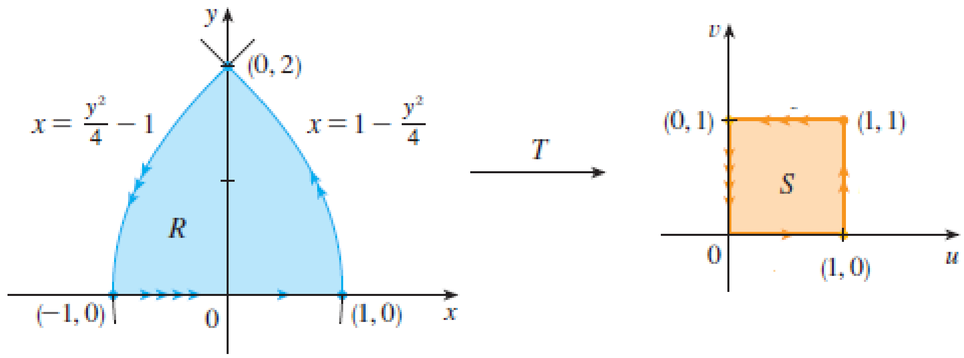
Change of Variables:

Suppose that T is transformation whose Jacobian is non-zero and that maps a region S in the uv – plane onto a region R in xy – plane. Suppose that f is continuous on R.

$$\iint_R f(x,y)\,dA = \iint_S f(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$



Example 1: Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\,dA$, where R is the region bounded by the $x-$axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.



Put $y = 0$ in equation $y = 2uv$, we get $u = 0,\ v = 0$

Put $x = u^2 - v^2$, $y = 2uv$ in $y^2 = 4 - 4x$,

we get $u^2v^2 = 1 - u^2 + v^2$

$\therefore u^2 = 1$

$\therefore u = 1$

Put $x = u^2 - v^2$, $y = 2uv$ in $y^2 = 4 + 4x$,

we get $u^2v^2 = 1 + u^2 - v^2$

$\therefore v^2 = 1$

$\therefore v = 1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4\left(u^2 + v^2\right)$$

$$\iint_R y\,dA = \int_0^1\int_0^1 8uv\left(u^2 + v^2\right)du\,dv = \int_0^1 \left(2u^4v + 4u^2v^3\right)_0^1 dv = \int_0^1 \left(2v + 4v^3\right)dv = \left[v^2 + v^4\right]_0^1 = 2$$

Example 2: Evaluate the integration $\iint_R e^{(x-y)/(x+y)}\,dy\,dx$ where R is the trapezoidal region with vertices $(1,0), (2,0), (0,-2)$ and $(0,-1)$.

Line joining (1,0) and (2,0) is $y = 0$.

Line joining (0,-2) and (0,-1) is $x = 0$.

Line joining (1,0) and (0,-1) is $x - y = 1$.

Line joining (2,0) and (0,-2) is $x - y = 2$.

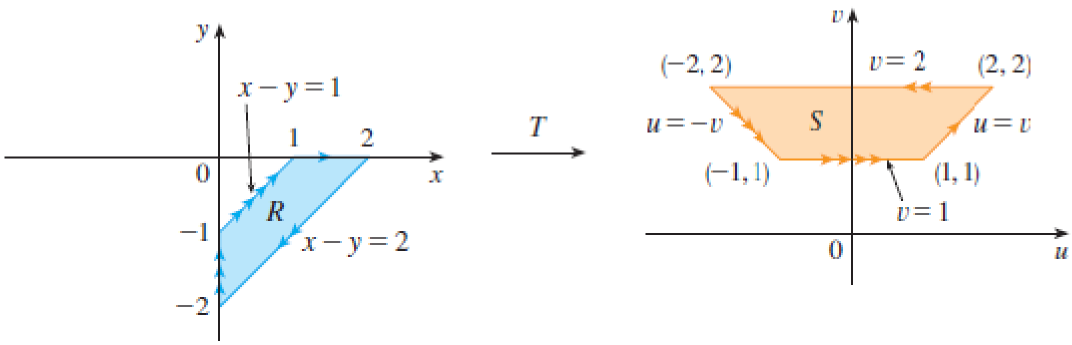
Since it is not easy to integrate $e^{(x-y)/\,(x+y)}$, we make a change of variables suggested by the form of the function

$u = x + y$ and $v = x - y$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left\|\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix}\right\| = \left\|\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\right\| = 2.$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$$

Put $y = 0$ in $u = x + y$ and $v = x - y$, we get $u = v$.
Put $x = 0$ in $u = x + y$ and $v = x - y$, we get $u = -v$.
$x - y = 1 \Rightarrow v = 1.$
$x - y = 2 \Rightarrow v = 2.$



$$\iint_R e^{(x-y)/(x+y)} dydx = \int_1^2 \int_{-v}^{v} e^{u/v} \frac{1}{2} dudv = \int_1^2 \left(v e^{u/v}\right)_{-v}^{v} dv = \int_1^2 \frac{\left(e - e^{-1}\right)}{2} v\, dv = \left(\frac{v^2}{2}\right)_1^2 \sinh 1 = \frac{3}{2}\sinh 1$$

Example 3: $\iint_R (x + y) e^{x^2 - y^2}\, dA$, where R is the rectangle enclosed by the lines $x - y = 0, x - y = 2,\ x + y = 0$ and $x + y = 3$.

We take transformation $u = x - y$, $v = x + y$

$x - y = 0, \Rightarrow u = 0 \qquad x - y = 2 \Rightarrow u = 2$

$x + y = 0 \Rightarrow v = 0 \quad x + y = 3 \Rightarrow v = 3.$

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left\|\begin{matrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{matrix}\right\| = \left\|\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}\right\| = 2.$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$$

$$\iint_R (x+y)e^{x^2-y^2}\,dA = \frac{1}{2}\int_0^3\int_0^2 ve^{uv}\,du\,dv = \frac{1}{2}\int_0^3 \left(v\frac{e^{uv}}{v}\right)_0^2 dv$$

$$= \frac{1}{2}\int_0^3 \left(e^{2v}-1\right)dv = \frac{1}{2}\left(\frac{e^{2v}}{2}-v\right)_0^3 = \frac{e^6}{4}-\frac{7}{4}.$$

Evaluation of Double integral in Polar Co-ordinates

To evaluate the double integral $\iint f(r,\theta)dr\,d\theta$ over the region R bounded by the curves $r=a$, $r=b$ and the straight lines $\theta=\alpha$ and $\theta=\beta$. First integrate with respect to r and θ.

Example: Evaluate $\int_0^{\frac{\pi}{4}}\int_0^1 r\ dr\ d\theta$

Solution:

$$\int_0^{\frac{\pi}{4}}\int_0^1 r\ dr\ d\theta = \int_0^{\frac{\pi}{4}}\left[\int_0^1 r\ dr\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_0^1 d\theta$$

$$= \int_0^{\frac{\pi}{4}}\frac{1}{2}\ d\theta$$

$$= \frac{1}{2}[\theta]_0^{\pi\backslash 4}$$

$$= \frac{\pi}{8}$$

Exercise

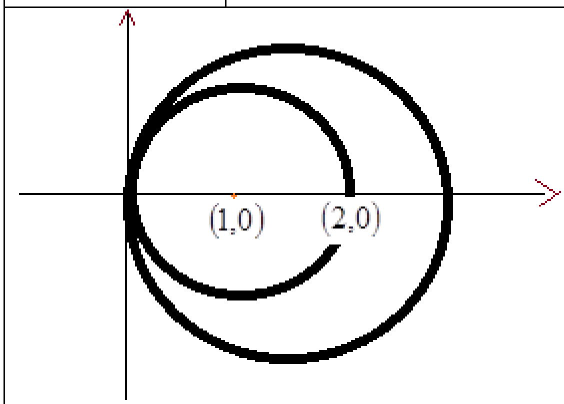
Example1: Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r\ dr\ d\theta$

Answer: $\frac{\pi}{4}$

Example 2: Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{1-\sin\theta} r^2\cos\theta\ dr\ d\theta$

Answer: $-\frac{1}{12}$

Example 1:	Evaluate $\iint_R r^3 dr\, d\theta$, over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r^3 dr d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r^3\, dr\, d\theta$$

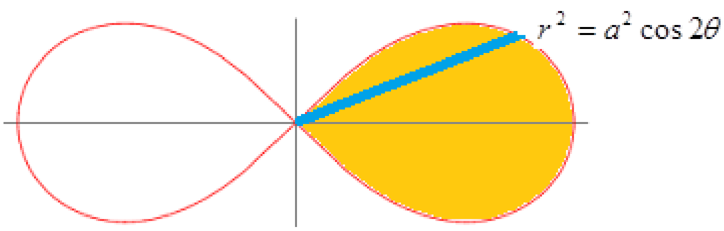
$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{r^4}{4}\right)_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= 120\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 120 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{45\pi}{2}$$

Example 2:	Evaluate $\iint_R \frac{r dr\, d\theta}{\sqrt{a^2 + r^2}}$ over one loop of lemniscate $r^2 = a^2 \cos 2\theta$.



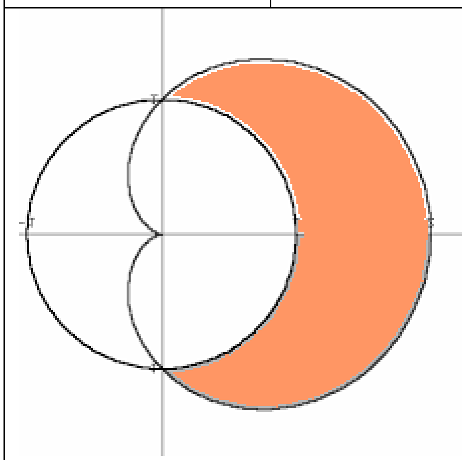
$$\iint_R \frac{r dr\, d\theta}{\sqrt{a^2 + r^2}}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} \frac{r dr\, d\theta}{\sqrt{a^2 + r^2}}$$

$$= 2\int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} \frac{r dr\, d\theta}{\sqrt{a^2 + r^2}}$$

$$= 2\int_0^{\frac{\pi}{4}} \left(2\sqrt{a^2 + r^2}\right)_0^{a\sqrt{\cos 2\theta}}$$

$$= 4\int_0^{\frac{\pi}{4}} \left(\sqrt{a^2 + a^2\cos 2\theta} - a\right) d\theta$$

		$= 4a\int_0^{\frac{\pi}{4}}(2\cos\theta - 1)d\theta$ $= 4a(2\sin\theta - \theta)_0^{\frac{\pi}{4}}$ $= 4a\left(\sqrt{2} - \frac{\pi}{4}\right)$
Example 3:	Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and the circle $r = 1$.	
		$\iint_R r dr d\theta$ $= 2\int_0^{\frac{\pi}{2}}\int_1^{1+\cos\theta} r dr d\theta$ $= 2\int_0^{\frac{\pi}{2}}\left(\frac{r^2}{2}\right)_1^{1+\cos\theta} d\theta$ $= \int_0^{\frac{\pi}{2}}\left[(1+\cos\theta)^2 - 1\right]d\theta$ $= \int_0^{\frac{\pi}{2}}(1 + 2\cos\theta + \cos^2\theta - 1)d\theta$ $= \int_0^{\frac{\pi}{2}}(2\cos\theta + \cos^2\theta)d\theta$ $= \int_0^{\frac{\pi}{2}}\left(2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$ $= \left(2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)_0^{\frac{\pi}{2}}$ $= \left(2 + \frac{\pi}{4}\right)$

Change into Polar Form:

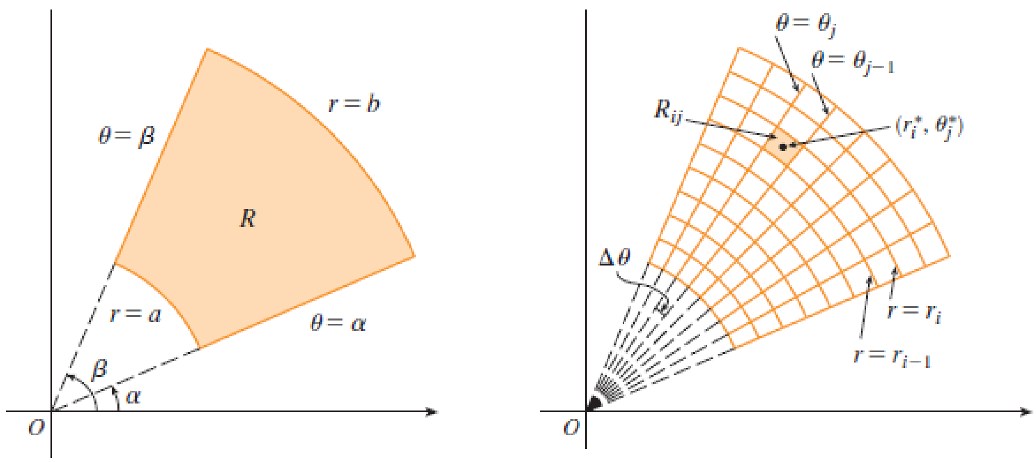
Integration is often transformed into polar form to make easier.

Suppose that we want to evaluate a double integral $\iint_R f(x,y)dA$, where R is either circle or semicircle.

In either case the description of in terms of rectangular coordinates is rather complicated but is easily

described using polar coordinates.

In order to evaluate the double integral $\iint_R f(x,y)dA$, where $R = \{(r,\theta): a \le r \le b;\ \alpha \le \theta \le \beta\}$, we divide the interval $[a,b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = \dfrac{b-a}{m}$ and we divide the interval $[\alpha,\beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$of the equal width $\Delta\theta = \dfrac{\beta-\alpha}{n}$.Then the circle will be divided into small polar rectangles as shown in figure



(a) (b)

The center of the Shaded polar sub rectangle $\{(r,\theta): r_{i-1} \le r \le r_i; \theta_{j-1} \le \theta \le \theta_j\}$ has polar coordinates

$$r_i^* = \frac{r_i + r_{i-1}}{2},\ \theta_j^* = \frac{\theta_j + \theta_{j-1}}{2}$$

$\therefore$ Area of polar rectangle $R_{i,j} = \dfrac{1}{2} r_i^2 \Delta\theta_j - \dfrac{1}{2} r_{i-1}^2 \Delta\theta_j$

$$= \frac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta_j$$

$$= \frac{1}{2}(r_i - r_{i-1})(r_i + r_{i-1})\Delta\theta_j \ = r_i^* \Delta r_i \Delta\theta_j$$

Although we have defined the double integral $\iint_R f(x,y)dA$ in terms of ordinary rectangle, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles, The rectangular coordinates of the centre of $R_{i,j}$ are $\left(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*,\right)$, so a typical Riemann sum is

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f\left(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*\right)\Delta A_{i,j} = \sum_{j=1}^{n}\sum_{i=1}^{m} f\left(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*\right)\Delta r_i \Delta\theta_j$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) dr\, d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r,\theta)\, r\, dr\, d\theta$$

Example 1: Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\ dx\ dy$ over the first quadrant of the circle $x^2+y^2=1$

Solution: Take $x = r\cos\theta\ , y = r\sin\theta$

$$\therefore r = 1\ \ (\because x^2+y^2=1)$$

As the region is the first quadrant of the circle therefore the limits of the integration are:

Limits of r: $r = 0$ to $r = 1$

Limits of θ: $\theta = 0$ to $\theta = \frac{\pi}{2}$

Hence,

$$\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\ dx\ dy = \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\ dr\ d\theta$$

Put $r^2 = \cos 2t\ , 2r\ dr = -2\sin 2t\ \ dt$
When $r = 0, t = \frac{\pi}{4}$
When $r = 1, t = 0$

$$\therefore \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\ dr\ d\theta = \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^0 \sqrt{\frac{1-\cos 2t}{1+\cos 2t}} (-\sin 2t\ \ dt)\ d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \sqrt{\frac{2\sin^2 t}{2\cos^2 t}} (2\sin t \cos t)\, dt\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t} (2\sin t \cos t)\, dt\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{4}} (1 - \cos 2t)\ dt$$

$$= [\theta]_0^{\frac{\pi}{4}} \left(t - \frac{\sin 2t}{2}\right)_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8}(\pi - 2)$$

Example 2: Evaluate $\iint \frac{4xy}{x^2+y^2} e^{-x^2-y^2}\, dx\, dy$ over the region bounded by the circle $x^2 + y^2 - x = 0$ in the first quadrant.

Answer: $\frac{1}{e}$

Application:

Area enclosed by a plane curves using double integrals:

Area: $A = \iint dxdy$

Example: Find the area of the plate in the form of a quadrant of a circle $x^2 + y^2 = a^2$

Solution: The required area of the region by using the vertical strip is

Area

$$A = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} dy\, dx$$

$$= \int_0^a (y)_0^{\sqrt{a^2-x^2}}\, dx$$

$$= \int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)_0^a$$

$$= \frac{\pi a^2}{4}$$

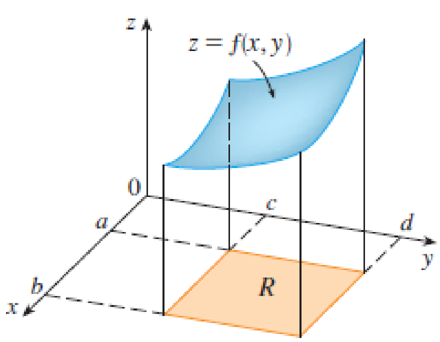
Example: Find the area bounded by $y = 2 - x$ and $y^2 = 2(x + 2)$.

Answer: 18

Example: Find the area common to the circles $r = a$ and $r = 2a\cos\theta\, ;a > 0$.

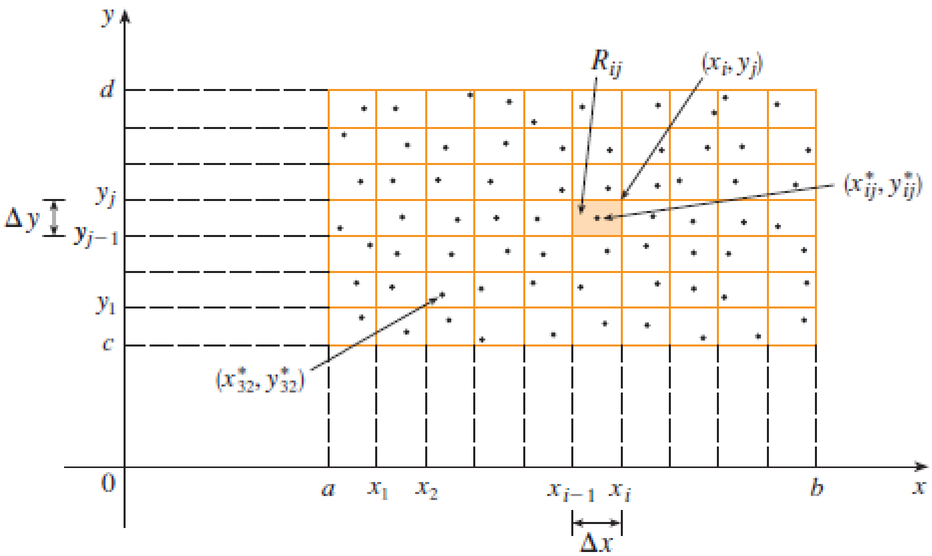
Answer: $a^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

Volume of Solid:



Consider, $z = f(x, y)$ defined on a closed rectangle $R = \{(x, y) / a \le x \le b, c \le y \le d\}$ and the graph of f
is shown in the figure.

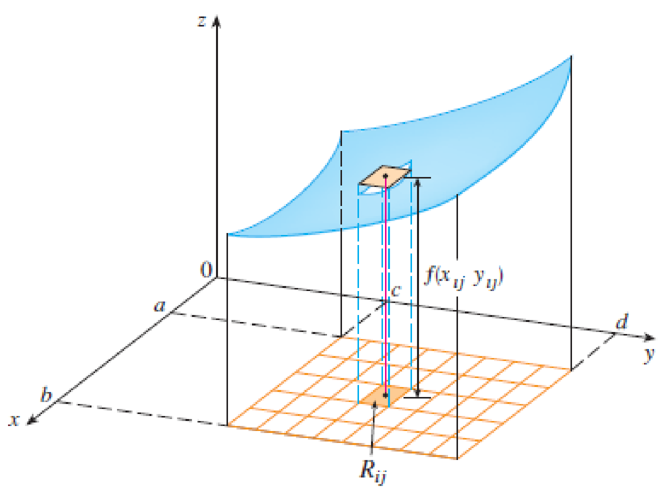
The first step is to divide the rectangle into sub rectangles. We accomplish this by dividing the rectangle by drawing lines parallel to co-ordinate axis. Number each rectangle 1,2,………
Let the area of i^{th} rectangle R_{ij} is $dx_{ij} \times dy_{ij}$.

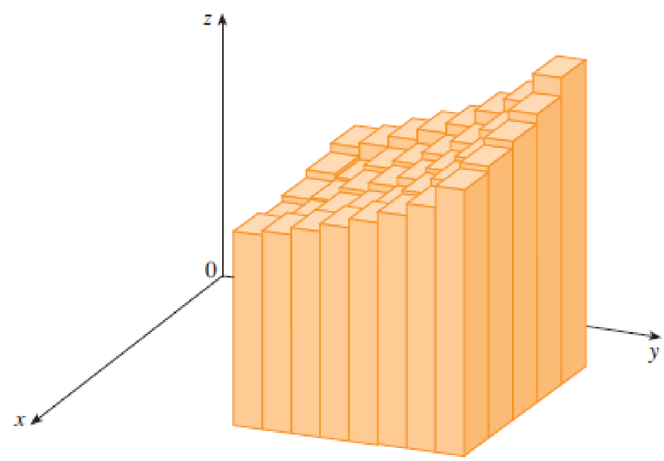


then we can approximate the part of S that lies above each R_{ij} by thin rectangular box (column) with base R_{ij} and height $f(x_{ij}, y_{ij})$ .The volume this thin box is

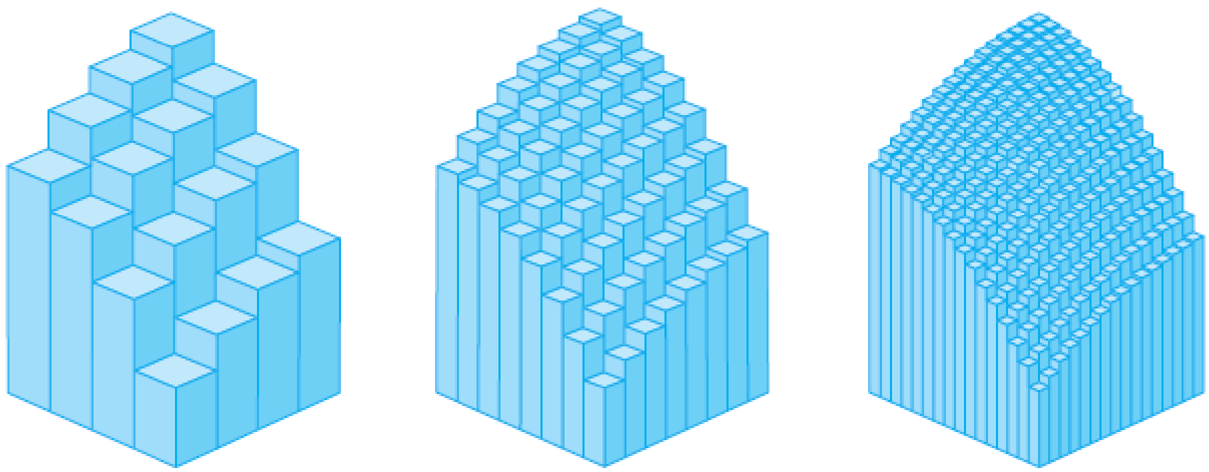
$$f(x_{ij}, y_{ij}) dx_{ij}\, dy_{ij} .$$

If we follow this procedure for all rectangles and add volumes of corresponding boxes. We get an approximate volume of S.





as number of rectangular boxes increases, accuracy of volume increases (as shown in figure).



Hence, Volume is

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{i\,j}, y_{i\,j}\right) dx_{i\,j}\ \ dy_{i\,j} = \iint_R f\left(x, y\right) dx\, dy$$

Example: Find the volume of the region bounded by the surface $x = 0, y = 0, z = 0$ and $2x + 3y + z = 6$.

Solution: Here $2x + 3y + z = 6$

$$\Rightarrow z = 6 - 2x - 3y$$

The required volume V is given by

$$V = \iint z\, dxdy = \int_{y=0}^{2} \int_{x=0}^{\frac{6-2y}{3}} (6 - 2x - 3y)\, dxdy$$

$$= \int_{y=0}^{2} \left(6x - x^2 - 3yx\right)_0^{\frac{6-2y}{3}} dy$$

$$= \int_0^2 \left[6\left(\frac{6-3y}{2}\right) - \left(\frac{6-3y}{2}\right)^2 - 3y\left(\frac{6-3y}{2}\right)\right] dy$$

$$= \frac{9}{4}\int_0^2 (y^2 - 4y + 4)\, dy$$

$$= \frac{9}{4}\left(\frac{y^3}{3} - \frac{4y^2}{2} + 4y\right)_0^2 = 6.$$

Example: Find the volume to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Answer: $\frac{16a^3}{3}$

Example: Find the volume of the solid bounded by the sphere $x^2 + y^2 + z^2 = 6$ and the paraboloid $z = x^2 + y^2$

Answer: $\left(2\sqrt{6} - \frac{11}{3}\right) 2\pi$

Triple Integration

A single integration is applicable when we have function of one variable only and double integration when we have function of two variables, similarly triple integration is applicable when we deal with the function of three variables.

Triple integration in cartesian coordinates:

$$\iiint_E f(x,y,z)dv = \int_a^b \int_{g1(x)}^{g2(x)} \int_{u2(x,y)}^{u1(x,y)} f(x,y,z)dxdydz$$

Note: The order of variables in $dxdydz$ indicates the order of integration. In some cases this order is not maintained. Therefore we have to identify the order of integration with the help of limits.

Triple Integration in cylinderical coordinates

Cylinderical coordinates are r, θ, z are used to evaluate the integral in the region which are bounded by cylinders along $z - axis$, plane through $z - axis$, plane perpendicular to $z - axis$.

Relation between cartesian coordinates (x, y, z) and cylinderical coordinates (r, θ, z) are given by $x = r\cos\theta\,, y = r\sin\theta\,, z = z$. Then

$$\iiint f(x,y,z) = \iiint f(r\cos\theta\,, r\sin\theta, z)|J|\, dzdrd\theta$$

Where $J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

Hence,

$$\iiint f(x,y,z) = \iiint f(r\cos\theta\,, r\sin\theta, z)\, rdzdrd\theta$$

Example 1: $\int_0^1 \int_0^2 \int_0^e dydxdz$

Solution: : $\int_0^1 \int_0^2 [\int_0^e dy]\, dxdz \ = \int_0^1 \int_0^2 [y]_0^e\, dxdz$

$$= \int_0^1 \int_0^2 e\, dxdz$$

$$= \int_0^1 [ex]_0^2\, dz$$

$$= 2e[z]_0^1 = 2e$$

Example 2: $\int_0^2 \int_1^3 \int_1^2 xy^2zdzdydx = \int_0^2 x\, dx \int_1^3 y^2dy \int_1^2 z\, dz$

$$= \left[\frac{x^2}{2}\right]_0^2 \left[\frac{y^3}{3}\right]_1^3 \left[\frac{z^2}{2}\right]_1^2 = 2\frac{26}{3}\frac{3}{2} = 26$$

Example 3: $\int_0^1 \int_0^{2-x} \int_0^{x-y} dzdydx$

$$= \int_0^1 \int_1^{2-x} [z]_0^{x-y} dydx \quad = \int_0^1 \int_1^{2-x} [x-y] dydx \quad = \int_0^1 \left[xy - \frac{y^2}{2} \right]_0^{2-x} dx \quad = \int_0^1 \left[x(2-x) - \frac{(2-x)^2}{2} \right] dx$$

$$= -\frac{1}{2} \int_0^1 [3x^2 - 8x + 4] dx = -\frac{1}{2} [x^3 - 4x^2 + 4x]_0^1 = -\frac{1}{2}$$

Example 4: $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2 \theta + z^2) r d\theta drdz$

Solution: $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2 \theta + z^2) r d\theta drdz = \int_0^1 \int_0^{\sqrt{z}} \left[\int_0^{2\pi} (r^2 \cos^2 \theta + z^2) \right] r d\theta drdz$

$$= \int_0^1 \int_0^{\sqrt{z}} \left[\int_0^{2\pi} \left(r^2 \frac{1 + cos2\theta}{2} + z^2 \right) \right] r d\theta drdz$$

$$= \int_0^1 \int_0^{\sqrt{z}} \left[\frac{r^2}{2} \left(\theta + \frac{\sin 2\theta}{2} + z^2 \theta \right) \right]_0^{2\pi} rdrdz$$

$$= 2\pi \int_0^1 \int_0^{\sqrt{z}} \left[\frac{1}{2} r^3 + z^2 r \right] drdz$$

$$= 2\pi \int_0^1 \left[\frac{1}{8} r^4 + \frac{z^2 r^2}{2} \right]_0^{\sqrt{z}} dz$$

$$= 2\pi \int_0^1 \left[\frac{z^2}{8} + \frac{z^3}{2} \right] dz$$

$$= 2\pi \left[\frac{z^3}{24} + \frac{z^4}{8} \right]_0^1$$

$$= 2\pi \left[\frac{1}{24} + \frac{1}{8} \right] = \frac{\pi}{3}$$

Example: Evaluate $\iiint xyz\, dxdydz$ over the region bounded by the planes $x = 0, y = 0, z = 0, z = 1$ and the cylinder $x^2 + y^2 = 1$

Solution: take

$$x = r\cos\theta\,, y = r\sin\theta\,, z = z$$

Equation of cylinder reduces to $r^2 = 1 \Longrightarrow r = 1$

$$r: r = 0\ to\ r = a$$
$$\theta: \theta = 0\ to\ \theta = \frac{\pi}{2}$$

Hence,

$$\iiint xyz\, dxdydz = \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cos\theta \sin\theta\, zr\, dzdrd\theta$$

$$= \int_0^1 zdz \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2} \int_0^1 r^3 dr = \frac{1}{16}$$

Exercise:

1) $\int_1^2 \int_2^3 \int_0^1 xyzdxdydz$ Answer: $\frac{15}{8}$

2) Evaluate $\int_0^1 \int_0^{2-x} \int_0^{x-y} dzdydx$ $-\frac{1}{2}$

Application of triple integration

$V(E)=\iiint f(x,y,z)dV=\iiint dV$

Let's begin with the special case where $f\ (x,y,z)\ =\ 1$ for all points in E. Then the triple integral does represent the volume of E.

$$\iiint 1dV=\iint_D\left[\int_{u_1(x,y)}^{u_2(x,y)}1\,dZ\right]dA=\iint_D u_2(x,y)-u_1(x,y)\ dA$$

Evaluation of triple integrals yields a volume instead of an area.

Example 1:

$$\int_1^2\int_x^{2x}\int_0^{y-x}\mathrm{dz\,dy\,dx}=\int_1^2\int_x^{2x}\mathrm{z}\Big|_0^{y-x}\mathrm{dy\,dx}=\int_1^2\int_x^{2x}(\mathrm{y-x})\mathrm{dy\,dx}=\int_1^2\left[\frac{\mathrm{y}^2}{2}-\mathrm{xy}\right]\Big|_x^{2x}\mathrm{dx}$$

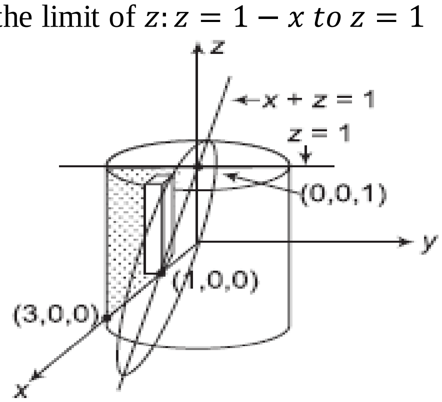
$$=\int_1^2\left[\left(\frac{(2\mathrm{x})^2}{2}-\mathrm{x}\cdot 2\mathrm{x}\right)-\left(\frac{\mathrm{x}^2}{2}-\mathrm{x}\cdot\mathrm{x}\right)\right]\mathrm{dx}=\int_1^2\frac{\mathrm{x}^2}{2}\mathrm{dx}$$

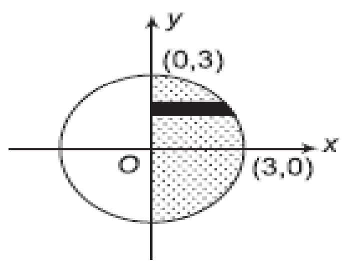
$$=\frac{\mathrm{x}^3}{6}\Big|_1^2=\frac{8}{6}-\frac{1}{6}=\frac{7}{6}$$

Example 2: Use triple integration to find the volume of the solid within the cylinder $x^2+y^2=9$ between the planes $z=1$ and $x+z=1$.

Solution: The region is bounded by the plane $x+z=1$ and $z=1$

Then the limit of z: $z=1-x\ to\ z=1$





The projection of the region in $xy - plane$ is the right half of the circle $x^2 + y^2 = 9$

From the above figure

$$x: x = 0 \; to \; x = \sqrt{9 - y^2}$$

$$y: y = 0 \; to \; y = 3$$

The volume

$$V = 2\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{1-x}^{1} dzdxdy$$

$$= 2\int_0^3 \int_0^{\sqrt{9-y^2}} [z]_{1-x}^{1} \, dxdy$$

$$= 2\int_0^3 \int_0^{\sqrt{9-y^2}} x \, dxdy$$

$$= 2\int_0^3 \left[\frac{x^2}{2}\right]_0^{\sqrt{9-y^2}} dy$$

$$= \int_0^3 (9 - y^2) \, dy$$

$$= \left[9y - \frac{y^3}{3}\right]_0^3 = 18$$

Exercise

Example: Find the volume in the first octant bounded by the circular cylinder $x^2 + y^2 = 2$ and planes $z = x + y, y = x, z = 0, x = 0.$

Answer: $\frac{2\sqrt{2}}{3}$

Use triple integration to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ between the planes $z = 1$ and $x + z = 1$.

Answer 18