Parul® University
Vadodara, Gujarat

NAAC GRADE A++

Linear Algebra

Applied Sciences and Humanities

# Unit 1 Matrices

## Study Guide

Dr. Monika Naik || Ms. Monali Darji
ASH, PIET
Parul University

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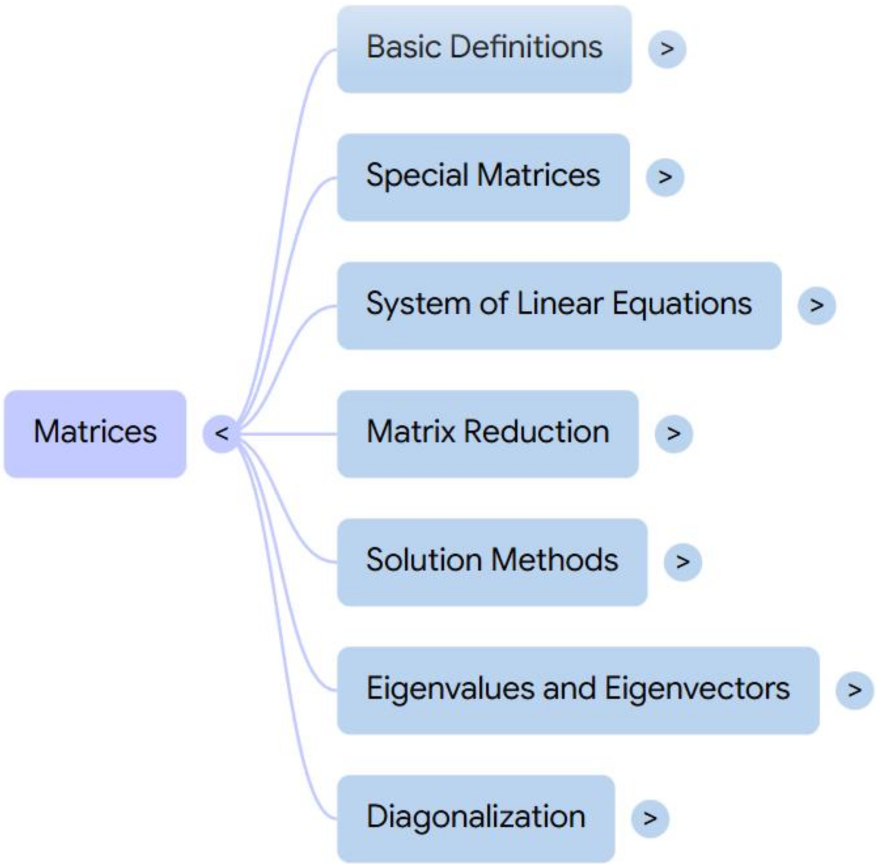



*Road map of matrices*

**1.0 Matrix**:

A Matrix is a rectangular array of numbers (or functions) enclosed in brackets. These number or functions are called entries or elements of the matrix.

*Example-1*

$\begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 5 \end{bmatrix}, \begin{bmatrix} \sin x & \cos x \\ -\cos x & \sin x \end{bmatrix}$

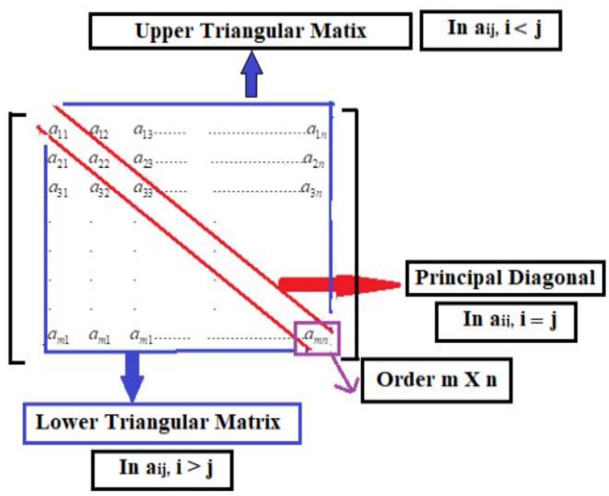
**1.1 Basic of Matrix and its types**

**Trace of a matrix:**

If $A$ is a square matrix, the trace of $A$, denoted by $tr(A)$ and is defined to be the sum of entries on the main diagonal of $A$. The trace of $A$ is undefined if$A$ is not a square matrix.

*Example-2*

If $A = \begin{bmatrix} 4 & 5 \\ 10 & 6 \end{bmatrix}$, then $tr(A) = 4 + 6 = 10$.



**Symmetric matrix: -** For any square matrix $A$, if $A = A^T$ then, it is known as symmetric matrix.

**Skew-symmetric matrix: -** For any square matrix $A$, if $A = -A^T$ then it is known as Skew symmetric matrix.

**Singular and non-singular matrix: -**

For any square matrix $A$, if $|A| \neq 0$ , then it is known as non-singular matrix and if $|A| = 0$ then it is known as singular matrix.

*Example-3*

$A = \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} \Rightarrow |A| = 8 - 8 = 0 \Rightarrow$
Singular Matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 4 - 6 = -2 \neq 0$$
$$\Rightarrow \text{Non} - \text{Singular Matrix}$$

**Orthogonal Matrix:-** The matrix is said to be an orthogonal matrix if the product of a matrix and its transpose give an identity value. i.e. $AA^T = I$

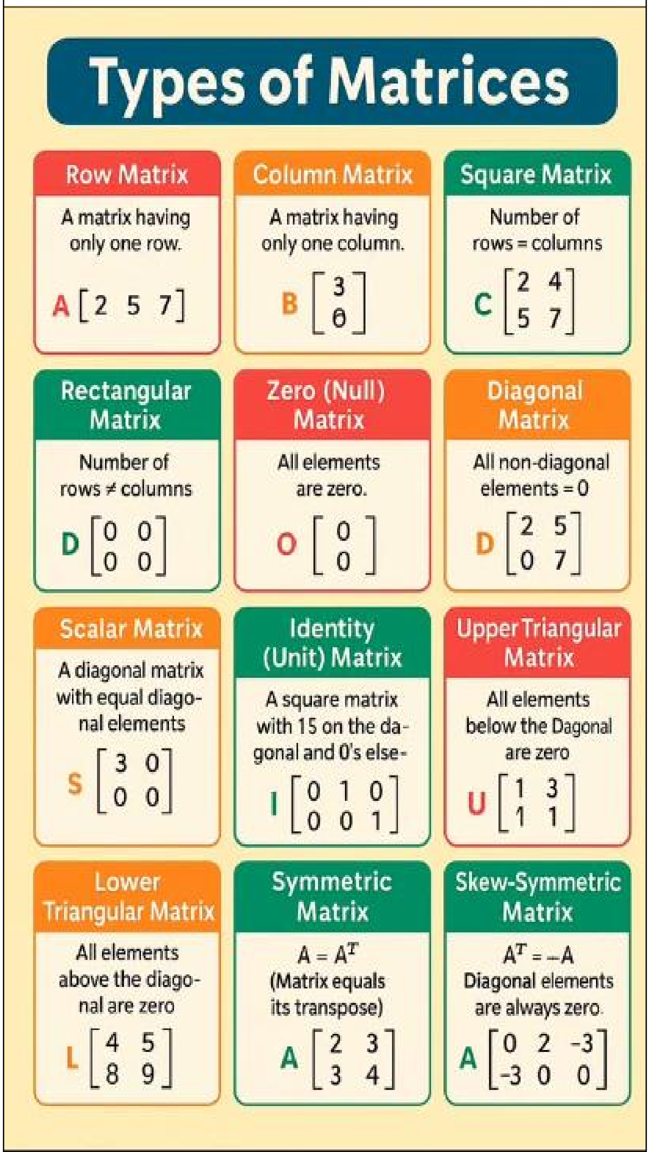
*Example-4*

Given $A$ is an orthogonal matrix because

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\therefore \ A^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

And

$$AA^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$




## 1.2 System of Linear Equations:

**1.2.1 Linear Equations:** Any straight line in the $xy$-plane can be represented algebraically by equation of the form $ax + by = c$, where $a\&b$ are real numbers.

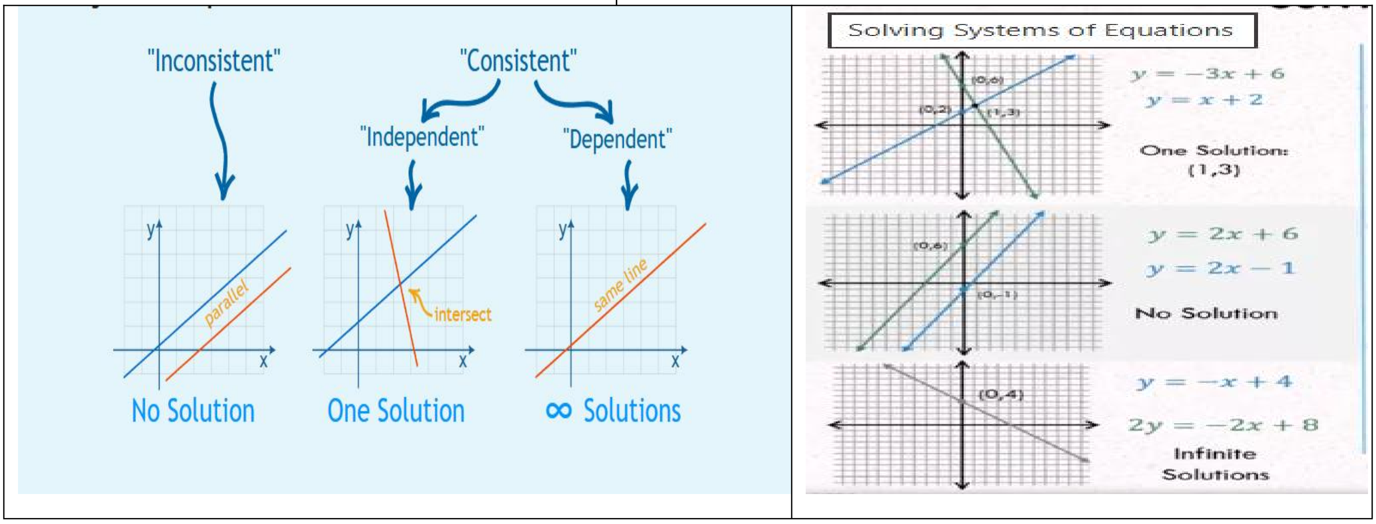
### 1.2.2 System of Linear Equations:

A **system of linear equation** is a collection of one or more linear equations involving the same variables. A linear system of m linear equations in n variables: An arbitrary system of m linear equations in n variables $x_1, x_2, x_3, \ldots. x_n$ is a set of equations of the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \ (i = 1,2,3, \ldots, m, j = 1,2,3, \ldots, n)$$

**A system of linear equations has either**

1. No solutions, or
2. Exactly one solution, or
3. Infinitely many solutions

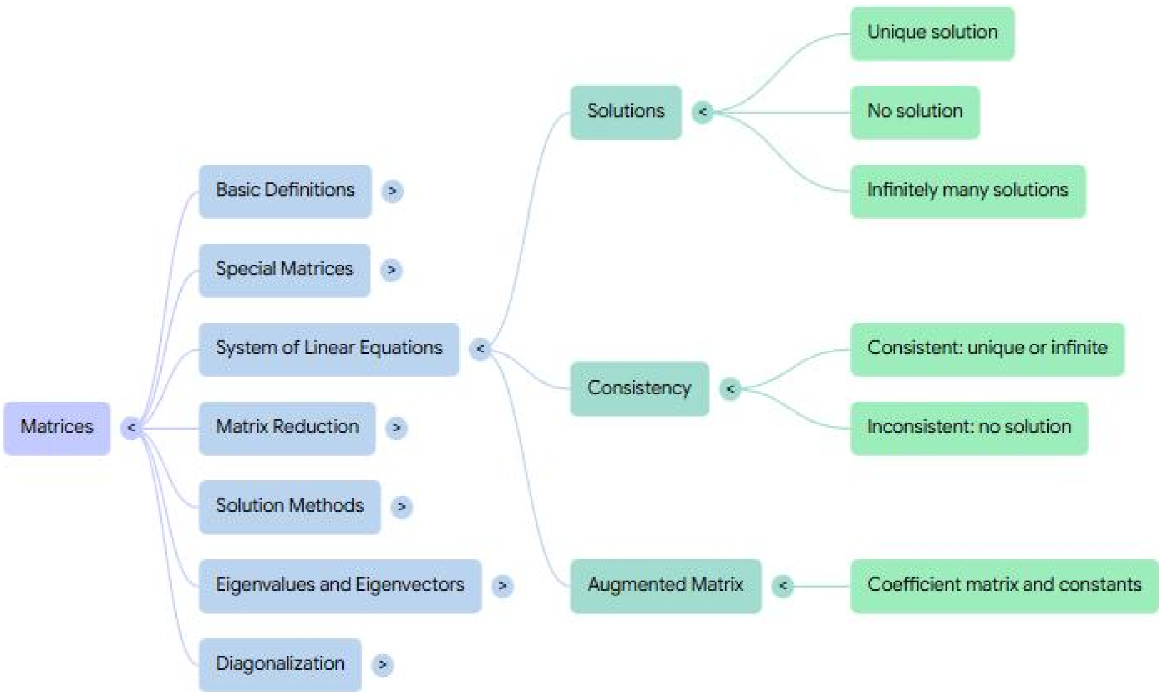



*Geometrical interpretation of system of Linear equations*

**Remarks:**

(i) The system is said to be consistent if we get infinitely many solutions or unique solution.

(ii) The system is said to be inconsistent if we get No solution.




*Schematic diagram for system of Linear equations*

## 1.2.3 Augmented matrix:

A system of $m$ equations in $n$ unknowns can be abbreviated by writing only the rectangular array of numbers.

$$\begin{bmatrix} a_{11} a_{12} \cdots a_{1n} | b_1 \\ a_{21} a_{22} \cdots a_{2n} | b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{m1} a_{m2} \ldots a_{mn} | b_m \end{bmatrix}$$

This is known as augmented matrix.

*Example-5*

Find the augmented matrix for each of the following system of linear equations:

$$2x_1 + \quad + 2x_3 = 1$$
$$3x_1 - x_2 + 4x_3 = 7$$
$$6x_1 + x_2 - x_3 = 0$$

Then, augmented matrix is given by $\begin{bmatrix} 2 & 0 & 2 & |1 \\ 3 & -1 & 4 & |7 \\ 6 & 1 & -1 & |0 \end{bmatrix}$.

**<u>1.2.4 Condition of Consistency for non-homogeneous system:</u>**

(1) If there is a zero row to left of the augmentation bar but the last entry of this row is non-zero then the system has **no solution**.

Example-6: $\begin{bmatrix} 1 & 0 & 2|1 \\ 0 & 1 & 4|7 \\ \mathbf{0} & \mathbf{0} & \mathbf{0|4} \end{bmatrix}$

(2) If at least one of the columns on the left of the augmentation bar has zero element pivot entry, then the system has **infinitely many solutions**.

Example-7: $\begin{bmatrix} 1 & 0 & 2|1 \\ 0 & 1 & 4|7 \\ \mathbf{0} & \mathbf{0} & \mathbf{0|0} \end{bmatrix}$

(3) If all the rows having the leading entry 1 then the system has **unique solution.**

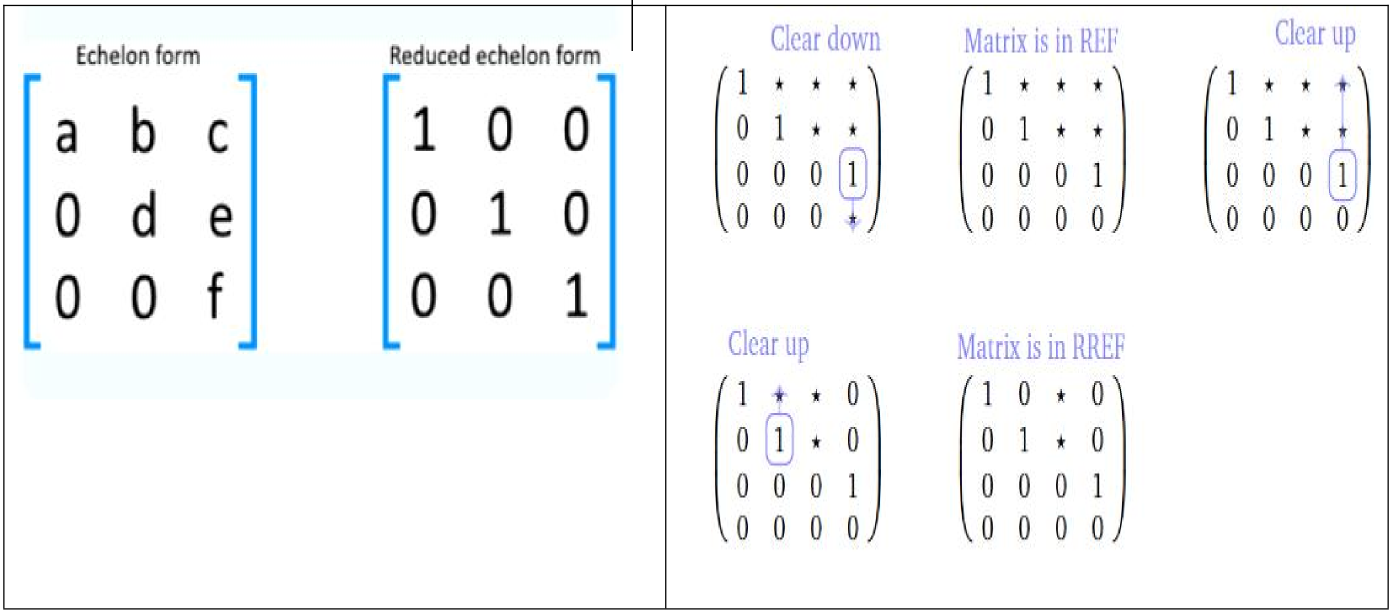
Example-8: $\begin{bmatrix} 1 & 0 & 2|1 \\ 0 & 1 & 4|7 \\ \mathbf{0} & \mathbf{0} & \mathbf{1|4} \end{bmatrix}$

**<u>1.3 Row-Echelon (RE)form and Row-Reduced Echelon (RRE) form of a matrix</u>**

**Definition:** A rectangular matrix is in row-echelon form (or echelon form) if it has the following three properties:

1. The **first** element in each row must be **non-zero** and equals to **1**, that is called **leading entry 1**.
2. All the **leading 1's** must be on the **right-hand** side of the matrix.
3. If any **zero row** is available, then it must be **below** to the **all-leading 1**.

If the matrix satisfies the **4$^{th}$ property** (i.e., In each column except leading 1 if all entries are zero) then **row-echelon form (RE form)** becomes **reduced row echelon form (RRE form)**.




*Illustrations of REF and RREF*

***Excercise-1***

1. Which of the following matrices are in row-echelon or echelon form?

(a) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ (b) $\begin{bmatrix}1&2&0\\0&1&0\\0&0&0\end{bmatrix}$ (c) $\begin{bmatrix}1&0&0\\0&1&0\\0&2&0\end{bmatrix}$ (d) $\begin{bmatrix}1&1&0&1&1\\0&2&0&2&2\\0&0&0&3&3\\0&0&0&0&4\end{bmatrix}$ (e) $\begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$

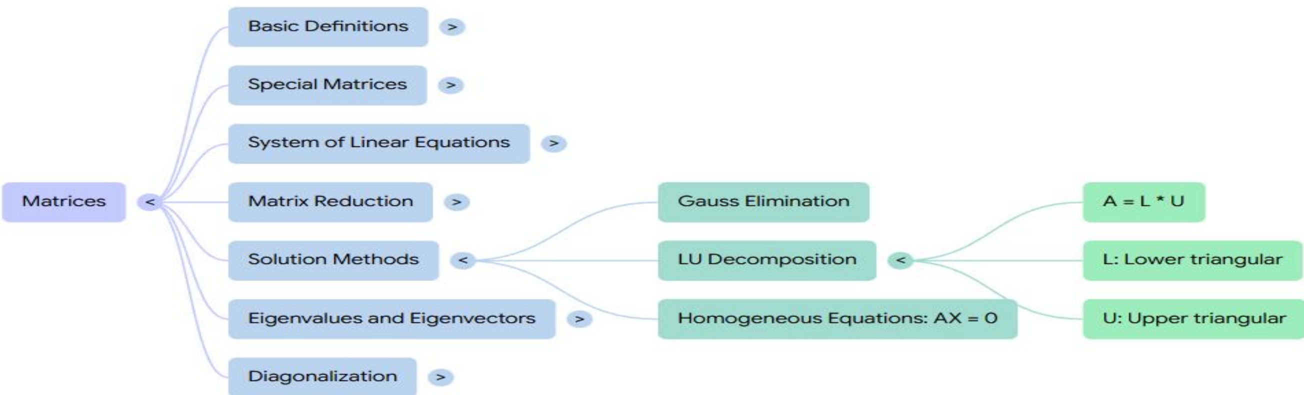
2. Which of the following matrices are in reduced row-echelon or reduced echelon form?

(a) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ (b) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ (c) $\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ (d) $\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$ (e) $\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$ (f) $\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$ (g) $\begin{bmatrix}0&0\\0&0\end{bmatrix}$

## <u>1.4 Methods of solving system of linear equations:</u>

1.4.1 Gauss – Elimination Method: (Using Row Echelon Form)

1.4.2 LU Decomposition Method




*Schematic diagram for Methods of Solution of system of Linear Equations*

**Case-1: Unique solution**

**Example 9:** Solve the following system by gauss-Elimination method

$$x+y+z=6$$
$$x+2y+3z=14$$
$$2x+4y+7z=30$$

Solution:

The matrix form of the given system is

$$\begin{bmatrix}1&1&1\\1&2&3\\2&4&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\14\\30\end{bmatrix}$$

The augmented matrix is

$$[A|B]=\left[\begin{array}{ccc|c}1&1&1&6\\1&2&3&14\\2&4&7&30\end{array}\right]$$

Now, to convert the given augmented matrix in row-echelon form we apply elementary operations as following.

$$R_2\to R_2-R_1, R_3\to R_3-2R_1$$
$$\left[\begin{array}{ccc|c}1&1&1&6\\0&1&2&8\\0&2&5&18\end{array}\right]$$
$$R_3\to R_3-2R_2$$
$$\left[\begin{array}{ccc|c}1&1&1&6\\0&1&2&8\\0&0&1&2\end{array}\right]$$

The corresponding system of equation is

$$x+y+z=6$$
$$y+2z=8$$
$$z=2$$

By using back substitution of $z=2$ in $y+2z=8$, we get $y=4$ and $z=2$ & $y=4$ in $x+y+z=6$ we get $x=0$.

$x=0, y=4, z=2$ is **unique solution** of given system.

**Case-3: Infinitely many solutions**

**Example 11:** Solve the following system by Gauss elimination method.

$$4x-2y+6z=8$$
$$x+y-3z=-1$$
$$15x-3y+9z=21$$

Solution:

The matrix form of the given system is

$$\begin{bmatrix}4&-2&6\\1&1&-3\\15&-3&9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}8\\-1\\21\end{bmatrix}$$

**Case-2: No solution**

**Example 10:** Solve the following system of equation by Gauss elimination.

$$-2b+3c=1$$
$$3a+6b-3c=-2$$

$$6a+6b+3c=5$$

Solution:

The matrix form of the given system is

$$\begin{bmatrix}0&-2&3\\3&6&-3\\6&6&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-2\\5\end{bmatrix}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c}0&-2&3&1\\3&6&-3&-2\\6&6&3&5\end{array}\right]$$
$$R_1\leftrightarrow R_2$$
$$\left[\begin{array}{ccc|c}3&6&-3&-2\\0&-2&3&1\\6&6&3&5\end{array}\right]$$

$$R_1\to\left({}^1\!/_3\right)R_1$$
$$\left[\begin{array}{ccc|c}1&2&-1&-2/3\\0&-2&3&1\\6&6&3&5\end{array}\right]$$
$$R_3\to R_3-6R_1$$
$$\left[\begin{array}{ccc|c}1&2&-1&-2/3\\0&-2&3&1\\0&-6&9&9\end{array}\right]$$

$$R_3\to R_3-3R_2$$

$$\left[\begin{array}{ccc|c}1&2&-1&-2/3\\0&-2&3&1\\0&0&0&6\end{array}\right]$$

The system of linear equation is

$a+2b-c=-2/3$

$-2b+3c=1$

$0a+0b+0c=6$ is not possible.

This shows that the system has **no solution**.

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**Example 12:**

Solve the following system by gauss elimination method.

$$\frac{-1}{x}+\frac{3}{y}+\frac{4}{z}=30$$
$$\frac{3}{x}+\frac{2}{y}-\frac{1}{z}=9$$

The augmented matrix is

$$[A|B]=\begin{bmatrix}4 & -2 & 6\,|\,8\\ 1 & 1 & -3|-1\\ 15 & -3 & 9\,|21\end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$
$$\begin{bmatrix}1 & 1 & -3|-1\\ 4 & -2 & 6\,|\,8\\ 15 & -3 & 9\,|21\end{bmatrix}$$

$$R_2 \to R_2-4R_1, R_3 \to R_3-15R_1$$
$$\begin{bmatrix}1 & 1 & -3|-1\\ 0 & -6 & 18|12\\ 0 & -18 & 54|36\end{bmatrix}$$

$$R_2 \to (-1/6)R_1, R_3 \to (-1/6)R_3$$
$$\begin{bmatrix}1 & 1 & -3|-1\\ 0 & 1 & -3|-2\\ 0 & 1 & -3|-2\end{bmatrix}$$
$$R_3 \to R_3-R_2$$
$$\begin{bmatrix}1 & 1 & -3|-1\\ 0 & 1 & -3|-2\\ 0 & 0 & 0\,|\,0\end{bmatrix}$$

The corresponding system of equations is

$$x+y-3z=-1$$
$$y-3z=-2$$

Assigning the free variable $z$ an arbitrary value $t$,
$y=3t-2$,
$$x=-1-3t+2+3t=1$$

Hence, $x=1, y=3t-2, z=t$ is solution of the given system of equations.

Since $t$ is arbitrary real number, The system has **infinitely many solutions.**

**Example 13:** Consider the following system
$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$
For what values of $\lambda$ and $\mu$ the system has (i) infinitely many solutions (ii) unique solution and (iii) no solution.

Solution: The Augmented matrix is

$$\frac{2}{x}-\frac{1}{y}+\frac{2}{z}=10$$

Solution:

Let $u=\frac{1}{x}, v=\frac{1}{y}, w=\frac{1}{z}$

Then the system of equations
$$-u+3v+4w=30$$
$$3u+2v-w=9$$
$$2u-v+2w=10$$
The matrix form of the system is
$$\begin{bmatrix}-1 & 3 & 4\\ 3 & 2 & -1\\ 2 & -1 & 2\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\begin{bmatrix}30\\ 9\\ 10\end{bmatrix}$$
The augmented matrix is
$$\begin{bmatrix}-1 & 3 & 4\,|30\\ 3 & 2 & -1|\,9\\ 2 & -1 & 2\,|10\end{bmatrix}$$

$$R_1 \to (-1)R_1$$
$$\begin{bmatrix}1 & -3 & -4|-30\\ 3 & 2 & -1|\ 9\\ 2 & -1 & 2\,|\ 10\end{bmatrix}$$

$$R_2 \to R_2-3R_1, R_3 \to R_3-2R_1$$
$$\begin{bmatrix}1 & -3 & -4|-30\\ 0 & 11 & 11|\ 99\\ 0 & 5 & 10|\ 70\end{bmatrix}$$

$$R_2 \to \left(\frac{1}{11}\right)R_2, R_3 \to \left(\frac{1}{5}\right)R_3$$
$$\begin{bmatrix}1 & -3 & -4|-30\\ 0 & 1 & 1\,|\ 9\\ 0 & 1 & 2\,|\ 14\end{bmatrix}$$

$$R_3 \to R_3-2R_2$$
$$\begin{bmatrix}1 & -3 & -4|-30\\ 0 & 1 & 1\,|\ 9\\ 0 & 0 & 1\,|\ 5\end{bmatrix}$$
The corresponding system of equations is
$$u-3v-4w=-30$$
$$v+w=9$$
$$w=5$$
By doing back substitution we get
$$v+5=9 \Rightarrow v=4 \Rightarrow y=\frac{1}{4}$$
$$u-12-20=-30 \Rightarrow u=2 \Rightarrow x=\frac{1}{2}$$
Hence, $x=\frac{1}{2}, y=\frac{1}{4}, z=\frac{1}{5}$ is required **unique solution** of the system.

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$$\begin{bmatrix}1 & 1 & 1 & | & 6\\ 1 & 2 & 3 & | & 10\\ 1 & 2 & \lambda & | & \mu\end{bmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\begin{bmatrix}1 & 1 & 1 & | & 6\\ 0 & 1 & 2 & | & 4\\ 0 & 1 & \lambda-1 & | & \mu-6\end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\begin{bmatrix}1 & 1 & 1 & | & 6\\ 0 & 1 & 2 & | & 4\\ 0 & 0 & \lambda-3 & | & \mu-10\end{bmatrix}$$

(i) If $\lambda - 3 = 0$ and $\mu - 10 = 0$, that is if $\lambda = 3$ and $\mu = 10$ then the system has infinitely many solutions.

(ii)If $\lambda - 3 = 0$ then the system has a unique solution. That is $\lambda \neq 3$ and $\mu$ can possess any real value.

(iii)If $\lambda - 3 = 0$ and $\mu - 10 \neq 0$, that is if $\lambda = 3$ and $\mu \neq 10$ then the system does not have any solution.

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**Exercise-2:**

Solve the following system of equations by using Gauss elimination method.

(1) $x + y + 2z = 9$<br>$2x + 4y - 3z = 1$<br>$3x + 6y - 5z = 0$

Ans:<br>$x = 1, y = 2, z = 3$

(2)$3x + y - 3z = 13$<br>$2x - 3y + 7z = 5$<br>$2x + 19y - 47z = 32$

Ans: No solution as the augmented matrix in row-echelon form is

$$\begin{bmatrix}1 & 1/3 & -1 & | & 13/\\ 0 & 1 & -27/11 & | & 1\\ 0 & 0 & 0 & | & 5\end{bmatrix}$$

(3) $2x + 2y + 2z = 0$<br>$-2x + 5y + 2z = 1$<br>$8x + y + 4z = -1$

Ans: Infinitely many solutions. The solution set is $\{(\frac{-3k-1}{7}, \frac{1-4k}{7}, k)/$ k $\in$R$\}$.

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**Example-14:**

**Solve the system of linear equations $x - 2y + z = 1$; $-x + y - z = 0$; $2x - y + z = -1$ using Gauss Elimination method.**

**Solution:** The Matrix form of the system of equation

$$\begin{bmatrix}1 & -2 & 1\\ -1 & 1 & -1\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$$

$$\text{Let A=}\begin{bmatrix}1 & -2 & 1\\ -1 & 1 & -1\\ 2 & -1 & 1\end{bmatrix}X + \begin{bmatrix}x\\ y\\ z\end{bmatrix}\ B = \begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$$

$$[A|B] = \begin{bmatrix}1 & -2 & 1 & 1\\ -1 & 1 & -1 & 0\\ 2 & -1 & 1 & -1\end{bmatrix}$$

$$R_2 \to R_1 + R_2 R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix}1 & -2 & 1 & 1\\ 0 & -1 & 0 & 1\\ 0 & 3 & -1 & -3\end{bmatrix}$$

$$R_2 \to -R_2$$

$$\sim \begin{bmatrix}1 & -2 & 1 & 1\\ 0 & 1 & 0 & -1\\ 0 & 3 & -1 & -3\end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix}1 & -2 & 1 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & -1 & 0\end{bmatrix}$$

$-z = 0 => z = 0$

$$y = -1$$
$$x - 2y + z = 1$$
$$x = -1$$

Hence, the required solution is $x = -1, y = -1$ $and$ $z = 0$

**Example-15: Solve the system of linear equation using Gauss Elimination method**

$$x + y + z = 6; x + 2y + 3z = 14; 2x + 4y + 7z = 30$$

$$[A|B] = \begin{bmatrix} 1 & 1 & 1\ 6 \\ 1 & 2 & 314 \\ 2 & 4 & 730 \end{bmatrix}$$
$$R_2 \to R_2 - R_1 R_3 \to R_3 - 2R_1$$
$$\sim \begin{bmatrix} 1 & 1 & 1\ 6 \\ 0 & 1 & 2\ 8 \\ 0 & 2 & 518 \end{bmatrix}$$
$$R_3 \to R_3 - 2R_2$$
$$\sim \begin{bmatrix} 1 & 1 & 16 \\ 0 & 1 & 28 \\ 0 & 0 & 12 \end{bmatrix}$$
$$z = 2$$
$$y + 2z = 8$$
$$\therefore y = 4$$
$$x = 0$$

Therefore, required solution is $x = 0; y = 4; z = 8$

**Exercise-3:**

1. Solve the system of linear equations using Gauss Elimination method

$$x + y + 2z = 9;\ \ 2x + 4y - 3z = 1;\ \ 3x + 6y - 5z = 0$$

Answer: $x = 1; y = 2;\ \ z = 3$

2. Solve the system of linear equation using Gauss Jordan method:

$$-x + 3y + 4z = 30;\ \ 3x + 2y - z = 9;\ \ 2x - y + 2z = 10$$

Answer: $x = 2; y = 4; z = 5$

3. Solve the following system of equation using Gauss Elimination Method.

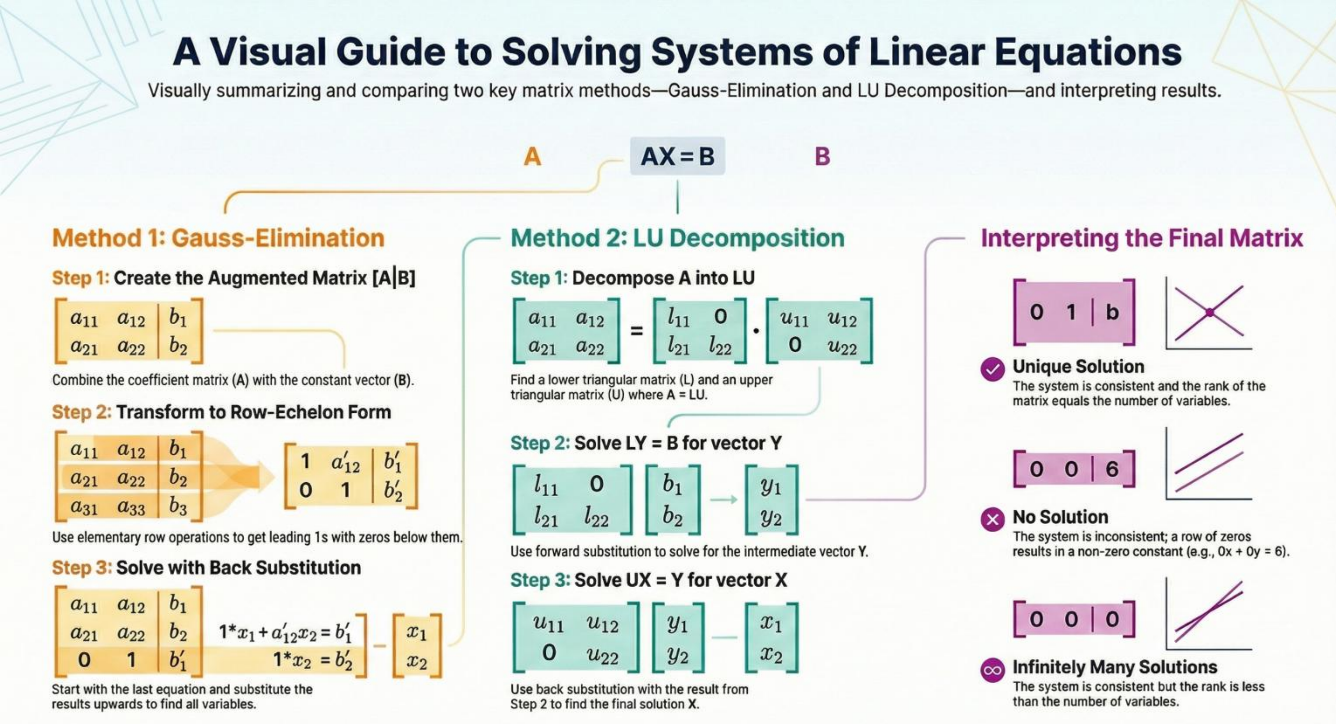
$$x + 2y + z = 5;\quad 3x - y + z = 6;\quad x + y + 4z = 7.$$

Answer:$x = 2, y = 1, z = 1$

4. Solve the following system of equations by using Gauss elimination method

$$x + y + z = 6;\ x + 2y + 3z = 14;\ 2x + 4y + 7z = 30.$$

Answer: $x = 0, y = 4, z = 2$




## LU Decomposition in Linear Algebra:

LU decomposition is a method of decomposing a square matrix A into the product of two matrices: A lower triangular matrix L and an upper triangular matrix U.
This is particularly useful for solving linear systems of equations, computing determinants, and performing matrix inversion efficiently.

If **A** is a matrix, the LU decomposition expresses it as:
Where:

(i)**L** is a lower triangular matrix (all elements above the diagonal are zero).

(ii) **U** is an upper triangular matrix (all elements below the diagonal are zero).

In some cases, a permutation matrix **P** may also be needed to reorder the rows of **A** to ensure stability, giving:

## Procedure for LU Decomposition:

Consider the system of linear equations in three variables:

$$a_{11x} + a_{12}y + a_{13}z = b_1; a_{21x} + a_{22}y + a_{23}z = b_2; a_{31x} + a_{32}y + a_{33}z = b_3$$

These can be written in the form of AX =B as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Here A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, X= $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, B = $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Now follow the steps given below to solve the above system of linear equations by LU Decomposition method.

**Step-1**: Generate a matrix A = LU such that L is the lower triangular matrix with principal diagonal elements being equal to 1 and U is the upper triangular matrix.

That means, L = $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and U = $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

**Step 2:** Now, we can write AX = B as: LUX = B….(1)

**Step 3:** Let us assume UX = Y….(2)

Where Y = $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

**Step 4:** From equations (1) and (2), we have; LY = B

On solving this equation, we get y1, y2, y3.

**Step 5:** Substituting Y in equation (2), we get UX = Y

By solving equation, we get X= (x , y, z)

The above process is also called the Method of Triangularisation.

## <u>Conditions for LU Decomposition</u>

- Let A must be a square matrix.
- LU decomposition may fail if **A** is singular or if pivot elements are zero.

**Example-16: Solve the following system using LU Decomposition Method:**

$$x + 5y + z = 14, 2x + y + 3z = 13, 3x + y + 4z = 17$$

**Solution:** We can solve the system using LU Decomposition
Let A = LU and Substitute into AX=B
So LUX =B for X to solve the system…..(1)
Let UX=Y ∴ LY=B and UX=Y
First we solve LY = B and then solve UX=Y for X
We need to find L and U such that A =LU.

$$\begin{bmatrix}1&5&1\\2&1&3\\3&1&4\end{bmatrix}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$$

$$=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$

$u_{11}$=1 , $u_{12}$=5, $u_{13}$= 1

$l_{21}u_{11}$ =2 ⇒ $l_{21}=2$

And $l_{31}u_{11}=3 \Rightarrow l_{31}=3$

$l_{21}u_{12}+u_{22}$ =1 ⇒ $u_{22}=-9$ and

$l_{21}u_{13}+u_{23}=3 \Rightarrow u_{23}=1$

$l_{31}u_{12}+l_{32}u_{22}$=1 ⇒ $l_{32}=14/9$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}$=4 ⇒ $u_{33}=-5/9$

$$A=LU=\begin{bmatrix}1&0&0\\2&1&0\\3&14/9&1\end{bmatrix}\begin{bmatrix}1&5&1\\0&-9&1\\0&0&-5/9\end{bmatrix}$$

From (i) L<u>UX</u>=B ⇒ LY=B

$$\begin{bmatrix}1&0&0\\2&1&0\\3&14/9&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}14\\13\\17\end{bmatrix}$$

$y_1$= 14 , $2y_2+y_3$=13 , $3y_1+\frac{14}{9}y_2+y_3=17$

$\Rightarrow y_1$= 14 , $y_2=-15$ , $y_3=-5/3$

Now UX=Y

$$\begin{bmatrix}1&5&1\\0&-9&1\\0&0&-5/9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}14\\-15\\-5/3\end{bmatrix}$$

$\therefore x+5y+z=14$ ,$-9y+z=-15$, z = 3

$\therefore x=1$ ,$y=2$, z = 3

**Example-17: Solve the following system using LU Decomposition Method:**

$$x+y-z=4, x-2y+3z=-6, 2x+3y+z=7$$

**Solution:** $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$

AX=B

We can solve the system using LU Decomposition

Let A = LU and Substitute into AX=B

So LUX =B for X to solve the system…..(1)

Let UX=Y ∴ LY=B and UX=Y

First we solve LY = B and then solve UX=Y for X

We need to find L and U such that A =LU.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$u_{11}$=1 , $u_{12}$=1, $u_{13}$= -1

$l_{21}u_{11}$ =1 ⇒ $l_{21} = 1$ and $l_{31}u_{11} = 2 \Rightarrow l_{31} = 2$

$l_{21}u_{12} + u_{22}$ =-2 ⇒ $u_{22} =- 1$ and $l_{21}u_{13} + u_{23} = 3 \Rightarrow u_{23} = 2$

$l_{31}u_{12} + l_{32}u_{22}$= 3 ⇒ $l_{32} =- 1/3$

$l_{31}u_{13} + l_{32}u_{23} + u_{33}$=1 ⇒ $u_{33} = 11/3$

A =LU = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 11/3 \end{bmatrix}$

Now we know that AX=B

For finding solution of equation, we are using

L$\underline{UX}$=B ⇒ LY=B

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1/3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

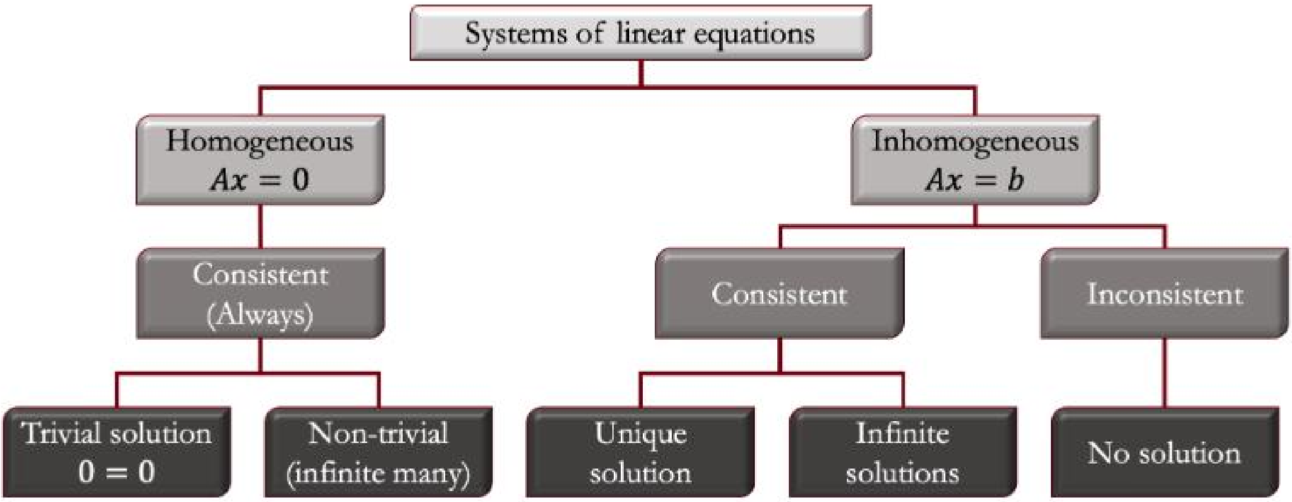
⇒ $y_1$= 4 , $y_2 =- 10$ , $y_3 =- 13/3$

Now UX=Y implies $\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ -13/3 \end{bmatrix}$

∴ $x = 1$ ,$y = 2$, z = -1

**Exercise-4:**

1. Solve the system of equations x + y + z = 1, 3x + y – 3z = 5 and x – 2y – 5z = 10 by LU decomposition method.(Ans: (6,-7,2))
2. Solve the below given system of equations by LU decomposition.
   x + y + z = 1, 4x + 3y – z = 6, 3x + 5y + 3z = 4
3. Find the solution of the system of equations by LU decomposition.
   x + 2y + 3z = 9, 4x + 5y + 6z = 24, 3x + y – 2z = 4




*Classification of types of Solution of Systems of Linear equations*

## 1.5 HOMOGENEOUS EQUATIONS

A system of linear equations in terms of $x_1, x_2, x_3, \ldots x_n$ having the matrix form AX= O, where A is $m \times n$ coefficient matrix, X is $n \times 1$ column matrix, O is a $m \times 1$ zero column matrix is called a system of homogeneous equations.

Example-18: $x + y + z = 0$ ; $x + 2y - z = 0$; $x + 3y + 2z = 0$

Example-19: $x + y = 0$ ; $x + 2y = 0$

Homogeneous equations are never inconsistent. They always have the solution "all variables = 0". The solution (0, 0, …, 0) is often called the **trivial solution**. Any other solution is called **nontrivial solution.**

**Example-20: Solve the following system:**

$$4x + 3y - z = 0$$
$$3x + 4y + z = 0$$
$$5x + y - 4z = 0$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 4 & 3 & -1 \\ 3 & 4 & 1 \\ 5 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is

$$[A|B]$$

**Example-21: Solve the following system**

$$-2x + 2y - 3z = 0$$
$$2x + y - 6z = 0$$
$$-x - 2y + 2z = 0$$
$$3x + y + 4z = 0$$

Solution:

$$\left[\begin{array}{ccc|c} -2 & 2 & -3 & 0 \\ 2 & 1 & -6 & 0 \\ -1 & -2 & 2 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right] R_1 \to -\frac{1}{2} R_1$$

$$\begin{bmatrix}4&3&-1&|0\\3&4&1&|0\\5&1&-4&|0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}4&3&-1&0\\3&4&1&0\\5&1&-4&0\end{array}\right] R_1\to\frac{1}{4}R_1$$

$$\left[\begin{array}{ccc|c}1&3/4&-1/4&0\\3&4&1&0\\5&1&-4&0\end{array}\right] \begin{matrix}R_2\to R_2-3R_1,\\R_3\to R_3-5R_1\end{matrix}$$

$$\left[\begin{array}{ccc|c}1&3/4&-1/4&0\\0&7/4&7/4&0\\0&-11/4&-11/4&0\end{array}\right] R_2\to\frac{4}{7}R_2$$

=

$$\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&1&0\\0&0&0&0\end{array}\right]$$

$x-z=0,\ y+z=0,\ 0=0.$

The last equation does not give any information about the equations**.**

Let. $z=k\Rightarrow y=-k\ and\ x=k$

$\therefore the\ solution\ set\ is\{(k,-k,k)/k\in R\}$

---

**Exercise-5:** Solve the following system of equations.

(1)

$x+y-z+w=0$

$x-y+2z-w=0$

$3x+y\quad +w=0$

Ans: Infinitely many solutions. The solution set is $\{(t/4\ ,-7t/4,t\ )/\ t\in R\}$.

(2) $2x+y+3z=0$

$x+2y\quad =0$

$y+z=0$

Ans: Trivial solution $x=0,y=0,z=0$

=

$$\left[\begin{array}{ccc|c}1&-1&3/2&0\\2&1&-6&0\\-1&-2&2&0\\3&1&4&0\end{array}\right] \begin{matrix}R_2\to R_2-2R_1,\\R_3\to R_3+R_1\\R_4\to R_4-3R_1\end{matrix}$$

=

$$\left[\begin{array}{ccc|c}1&-1&3/2&0\\0&3&-9&0\\0&-3&7/2&0\\0&4&-1/2&0\end{array}\right] R_2\to\frac{1}{3}R_2$$

=

$$\left[\begin{array}{ccc|c}1&-1&3/2&0\\0&1&-3&0\\0&-3&7/2&0\\0&4&-1/2&0\end{array}\right] \begin{matrix}R_1\to R_1+R_2,\\R_3\to R_3+3R_2\\R_4\to R_4-4R_2\end{matrix}$$

=

$$\left[\begin{array}{ccc|c}1&0&-3/2&0\\0&1&-3&0\\0&0&-11/2&0\\0&0&23/2&0\end{array}\right] R_3\to-\frac{2}{11}R_3$$

=

$$\left[\begin{array}{ccc|c}1&0&-3/2&0\\0&1&-3&0\\0&0&1&0\\0&0&23/2&0\end{array}\right] \begin{matrix}R_1\to R_1+3/2R_3,\\R_2\to R_2+3R_3\\R_4\to R_4-23/2R_3\end{matrix}$$

=

$$\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]$$

The required solution is$x=0,y=0,z=0$ which is trivial solution.

## 1.6 Rank of a Matrix

The positive integer $r$ is said to be a rank of a matrix $A$ if it possesses the following properties:

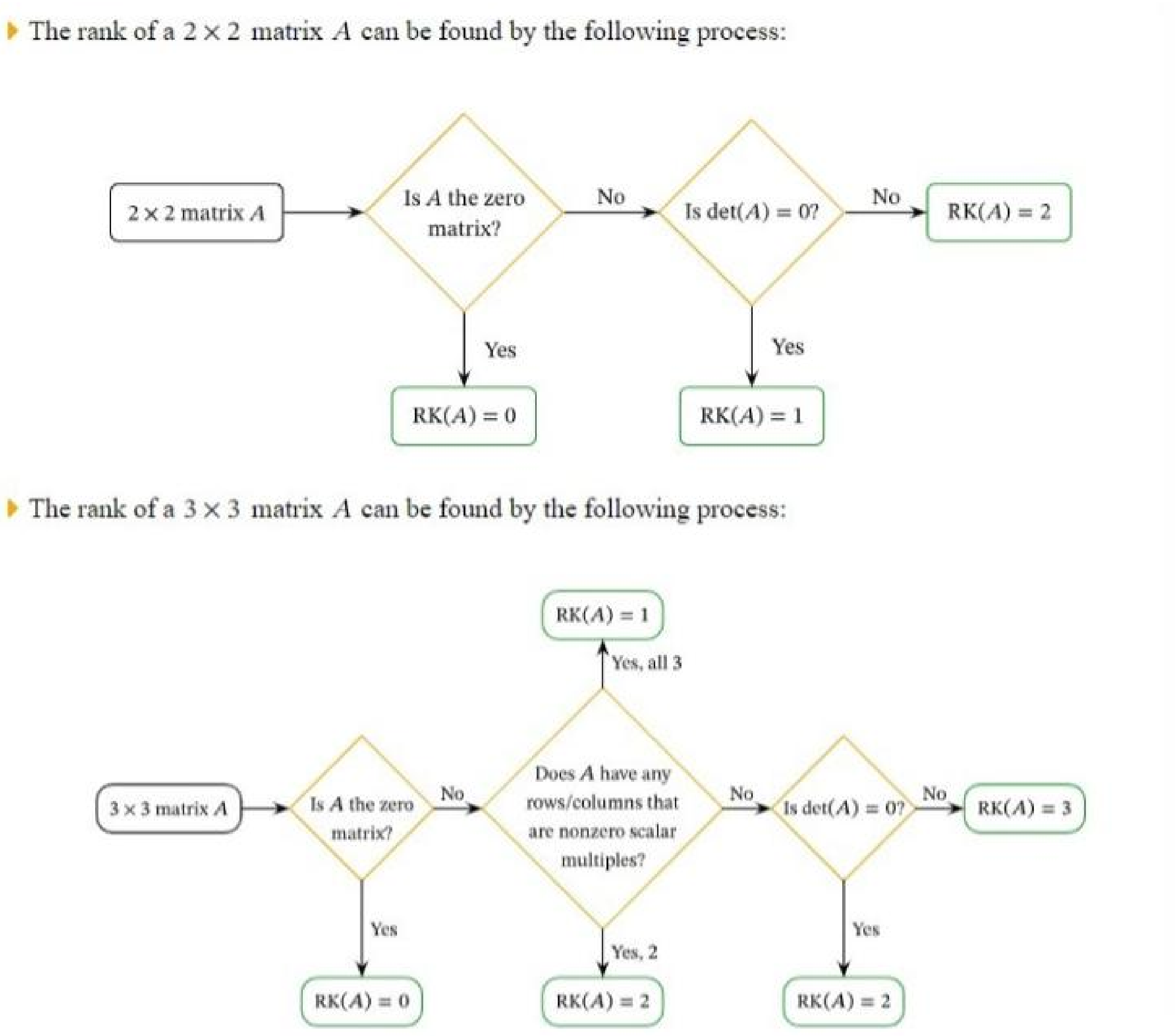
(i)There is at least one minor of order $r$ which is non-zero.

(ii) Every minor of order greater than $r$ is zero.

**Remarks:**

1. Rank of matrix $A$ is denoted by $\rho(A)$

2. The rank of matrix remains unchanged by elementary transformation
3. $\rho(A^T) = \rho(A)$
4. The rank of the product of two matrices always less than or equal to the rank of either matrix (i.e., $\rho(AB) \leq \rho(A)$ or $\rho(AB) \leq \rho(B)$).




*Flowcharts to find rank for 2 X2 and 3 X3 matrices*

## 1.6.1 Methods for finding Rank of a Matrix:

❖ **Method-1: Rank of a Matrix by Determinant Matrix**

Consider a square matrix $A$ of order $r$.

- **Step-1:** Find the determinant of $A$. If $det(A) \neq 0$ then $\rho(A) = r$. Otherwise $\rho(A) < r$.
- **Step-2:** Find the all-possible minors of order $r - 1$. If any one of them is non-zero then order is $r - 1$, otherwise $\rho(A) < r - 1$.
- **Step-3:** By continuing this process upto the non-zero determinant.

**Example 22:** Find the rank the following matrices by determinant method:

a. $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{4} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} \end{bmatrix}$

Solution: Given, $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ then $det(A) \neq 0$. Hence, the $\rho(A) = 3$

b. $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{3} & \mathbf{5} & \mathbf{7} \end{bmatrix}$

Solution: Given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ then $det(A) = 0$. Hence, the rank of $A$ is less than 3.

Now, minor of $1 = \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 21 - 20 = 1 \neq 0$. Hence, $\rho(A) = 2$.

c. $\boldsymbol{A} = \begin{bmatrix} \mathbf{4} & \mathbf{2} & \mathbf{3} \\ \mathbf{8} & \mathbf{4} & \mathbf{6} \\ \mathbf{-2} & \mathbf{-1} & -\frac{\mathbf{3}}{\mathbf{2}} \end{bmatrix}$

Solution: Given, $A = \begin{bmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ -2 & -1 & -\frac{3}{2} \end{bmatrix}$ then $det(A) = 0$. Hence $\rho(A) < 3$.

Consider all the minors of order 2, i.e.,

$$\begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix} = 0, \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0, \begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix} = 0, \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix} = 0, \begin{vmatrix} 2 & 3 \\ -1 & -\frac{3}{2} \end{vmatrix} = 0, \begin{vmatrix} 4 & 3 \\ -2 & -\frac{3}{2} \end{vmatrix} = 0$$

Here, all the minors of order 2 are zero. There rank is less than 2. Hence, $\rho(A) = 1$.

d. $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{-4} \\ \mathbf{2} & \mathbf{4} & \mathbf{3} & \mathbf{5} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{6} & \mathbf{-7} \end{bmatrix}$

Solution: Here, the order of matrix $A$ is $3 \times 4$. Hence the rank of $A$ is maximum 3 as we can find the square matrix of order 3. Therefore, consider all the minors of order 3, i.e.,

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & 3 \\ -1 & -2 & 6 \end{vmatrix} = 0, \begin{vmatrix} 2 & -1 & -4 \\ 4 & 3 & 5 \\ -2 & 6 & -7 \end{vmatrix} = 0, \begin{vmatrix} 1 & 2 & -4 \\ 2 & 4 & 5 \\ -1 & -2 & -7 \end{vmatrix} = 0, \begin{vmatrix} 1 & -1 & -4 \\ 2 & 3 & 5 \\ -1 & 6 & -7 \end{vmatrix} =- 120$$

Here, one minor of rank 3 is not equal to zero. Hence, $\rho(A) = 3$.

❖ **<u>Method-2: Rank of a Matrix by Row Echelon Form</u>**

The Rank of a Matrix in Row Echelon Form is equal to the number of non-zero rows of the matrix.

Example-23: A= $\begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{vmatrix}$ ; the matrix $A$ is in Row Echelon form with two non-zero rows. Hence, rank of matrix $A$ is 2.

**Example 24:** Find the rank the following matrices by reducing to Row Echelon Form:

<table>
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**(a)** $A=\begin{bmatrix} \mathbf{5} & \mathbf{3} & \mathbf{14} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{0} \end{bmatrix}$

Solution: Given

$$A=\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

By applying row-operations

$R_{13}:\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}$

$R_3-5R_1:\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix}$

$R_3-8R_2:\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix}$

$\left(-\frac{1}{12}\right)R_3:\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$

The equivalent matrix is in Row-Echelon Form.

Number of non-zero rows $=3$ .Hence, $\rho(A)=3$

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**(b)** $A=\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{-1} & \mathbf{-3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{-1} \end{bmatrix}$

Solution: Given

$$A=\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

By applying row-operations

$R_2+2R_1, R_3-R_1:\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$R_{24}:\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 3 & 3 & -3 \end{bmatrix}$

$R_3+2R_2, R_4-3R_2:\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The equivalent matrix is in Row-Echelon Form.

Number of non-zero rows $=2$. Hence, $\rho(A)=2$

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**Exercise-6:**

(1) Find the ranks of $A, B, AB$ and verify $\rho(AB)\le\rho(A)$ or $\rho(AB)\le\rho(B)$) where $A=\begin{vmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 4 & 8 & 3 \end{vmatrix}$,

$B=\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{vmatrix}$

(2) Find the rank the following matrices by reducing to Row Echelon Form:

(i) $A=\begin{bmatrix}1&1&-1&1\\1&-1&2&-1\\3&1&0&1\end{bmatrix}$, (ii) $A=\begin{bmatrix}3&-2&0&-1&-7\\0&2&2&1&-5\\1&-2&-3&-2&1\\0&1&2&1&6\end{bmatrix}$, (iii) $\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$, (iv) $\begin{bmatrix}1&1&-1&1\\1&-1&2&-1\\3&1&0&1\end{bmatrix}$

(v) $\begin{bmatrix}1&2&3\\2&3&4\\3&4&5\end{bmatrix}$

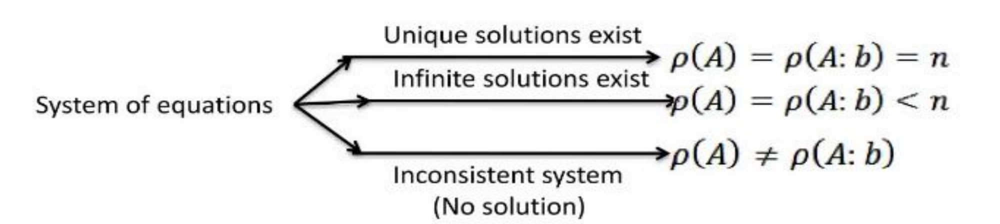
**Remarks:**

(1) If $\rho(A) \neq \rho(A|B)$ then the system is inconsistent.

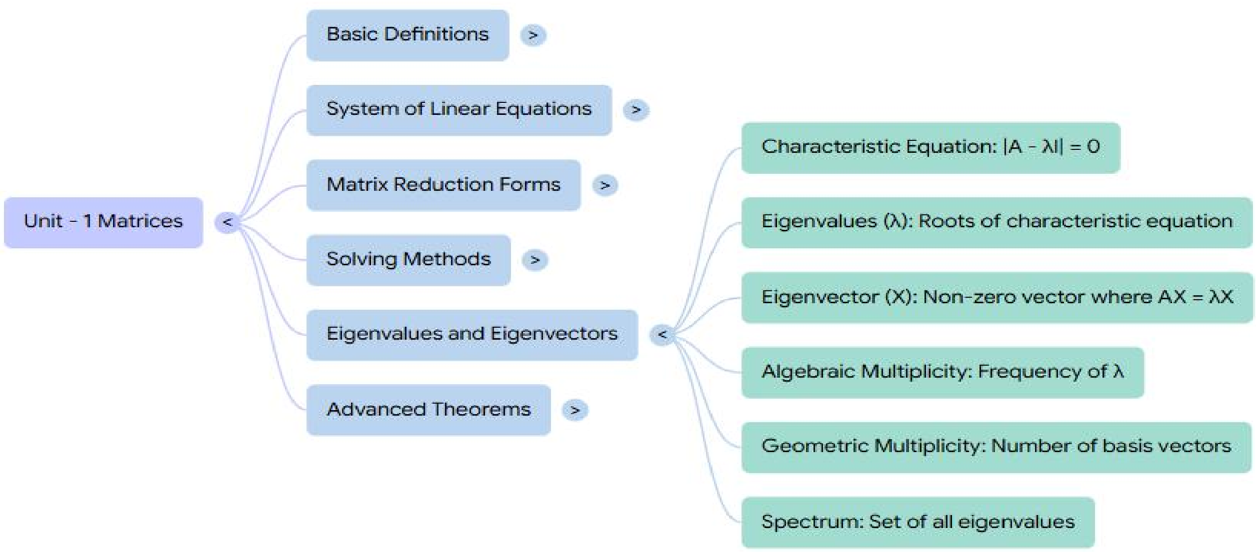
(2) If $\rho(A) = \rho(A|B)$ then the system is consistent.

(3) If $\rho(A) < n$ then there are infinitely many solutions ($n$ is the number of unknowns)

(4) If $\rho(A) = n$ then there is a unique solution.




*Schematic diagram for types of solutions*




*Schematic diagram for Eigen values and Eigen vectors concepts*

**1.7 Eigen values and Eigen vectors:**

Let $A$ be $n \times n$ matrix, then there exists a real number $\lambda$ and a nonzero vector $X$ such that

$$AX = \lambda X$$

then, $\lambda$ is called as the eigen value or characteristic value or proper roots of the matrix $A$, and $X$ is called as eigen vector or characteristic vector or real vector corresponding to eigen value $\lambda$ of the matrix $A$.

***Remarks:***

1. An Eigen vector is never the zero vector.
2. The matrix $[A - \lambda I_n]$ is known as the **characteristic matrix** of A.
3. The determinant of $(A - \lambda I_n)$ after expansion gives the polynomial in $\lambda$, it is known as the **characteristic polynomial** of the matrix A of order $n \times n$ and is of degree n.
4. $|A - \lambda I_n|=0$ is called the **characteristic equation** of matrix A.
5. The root of the characteristic equation is known as **characteristic value** or **eigenvalues** of the matrix.
6. The set of all characteristic roots (Eigen values) of the matrix A is called the **spectrum of A.**
7. Let A be $n \times n$ matrix and $\lambda$ be an eigen value for A. Then the set $E_\lambda = \{X / AX = \lambda X\}$ is called the **Eigen space of $\lambda$.**

***Results:***

1. The Eigen values of a diagonal matrix are its diagonal elements.
2. The sum of Eigen values of an $n \times n$ matrix is its trace and their product is $|A|$.
3. For the upper triangular (lower triangular) $n \times n$ matrix A, the Eigen values are its diagonal elements.

**Example-25:**

If $A = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}$, find the Eigen values for the given matrices:

(i) $A$, (ii) $A^T$, (iii) $A^{-1}$, (iv) $4A^{-1}$, (v) $A^2$, (vi) $A^2 - 2A + I$,
(vii) $A^3 + 2I$

Solution: Given, $A = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}$

The characteristic equation of matrix $A$ is

$$|A - \lambda I_2| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 12 = 0 \Rightarrow \lambda^2 - 3\lambda - 10 = 0 \Rightarrow (\lambda - 5)(\lambda + 2) = 0$$

$$\therefore \lambda = 5 \text{ or } \lambda = -2$$

| | |
|---|---|
| Eigenvalues of $A = \lambda$ | $5, -2$ |

| | |
|---|---|
| Eigenvalues of $A^T = \lambda^T$ | $5, -2$ |
| Eigenvalues of $A^{-1} = \lambda^{-1}$ | $\frac{1}{5}, -\frac{1}{2}$ |
| Eigenvalues of $4A^{-1} = 4\lambda^{-1}$ | $\frac{4}{5}, -2$ |
| Eigenvalues of $A^2 = \lambda^2$ | $25, 4$ |
| Eigenvalues of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$ | $16, 9$ |
| Eigenvalues of $A^3 + 2I = \lambda^3 + 2$ | $127, -6$ |

**Example 26:** Find the Eigen values of $A = \begin{vmatrix} 3 & 2 \\ 3 & 8 \end{vmatrix}$

Solution: Given $A = \begin{vmatrix} 3 & 2 \\ 3 & 8 \end{vmatrix}$, then the characteristic equation of matrix $A$ is

$|A - \lambda I_2| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 3 & 8-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(8-\lambda) - 6 = 0 \Rightarrow \lambda^2 - 11\lambda + 18 = 0 \Rightarrow (\lambda - 9)(\lambda - 2) = 0$$

$\therefore \lambda_1 = 9$ or $\lambda_2 = 2$

**Types of Eigen Values**

Eigenvalues are non-repeated, whether matrix is symmetric or non-symmetric

Eigenvalues are repeated and the matrix is non-symmetric

Eigenvalues are repeated and the matrix is symmetric

**Example 27**: Find the eigen values and eigen vector of the matrix A= $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

The characteristic equation is $|A - \lambda I_n|$=0

$$\begin{vmatrix} -2-\lambda & -8 & -13 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$S_1 = tr(A) = -2 + 4 + 1 = 3$

$S_2$ =Sum of minors of diagonal entries

$$= \begin{vmatrix} 4 & 4 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -2 & -12 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix} = 4 - 2 + 0 = 2$$

$|A| = -2(4) + 8(1) - 12(0) = -8 + 8 = 0$

$\therefore$ characteristic equation is

$$= \left[\begin{array}{ccc|c} 1 & 8/3 & 4 & 0 \\ 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 8/3 & 4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 \to 3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 8/3 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \to R_1 - 8/3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We suppose $z = k, y = 0, x + 4z = 0$

$\therefore z = k, y = 0, x = -4z = -4k$

$\lambda^3-3\lambda^2+2\lambda=0$

$\Rightarrow \lambda\left(\lambda^2-3\lambda+2\right)=0$

$\Rightarrow \lambda=0$ or $\lambda^2-3\lambda+2=0$

$\Rightarrow \lambda=0$ or $(\lambda-2)(\lambda-1)=0$

$\Rightarrow \lambda=0$ or $\lambda=1$ or $\lambda=2$

Here, one can observe that all eigenvalues are non-repeated and matrix is non-symmetric.

**When** $\lambda_1=0$

$$[A-\lambda I \mid O]=\left[\begin{array}{ccc|c}-2 & -8 & -12 & 0\\ 1 & 4 & 4 & 0\\ 0 & 0 & 1 & 0\end{array}\right] R_1\to -1/2R_1$$

$$=\left[\begin{array}{ccc|c}1 & 4 & 6 & 0\\ 1 & 4 & 4 & 0\\ 0 & 0 & 1 & 0\end{array}\right] R_2\to R_2-R_1$$

$$=\left[\begin{array}{ccc|c}1 & 4 & 6 & 0\\ 0 & 0 & -2 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

Therefore, we suppose

$x+4y+6z=0, -2z=0, y=k$

$\therefore z=0, y=k, x=-4k$

Therefore, eigen vector space is

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-4k\\k\\0\end{bmatrix}=k\begin{bmatrix}-4\\1\\0\end{bmatrix}$$

Therefore, eigen vector space for $\lambda_1=0$ is

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-4\\1\\0\end{bmatrix}$$

**When** $\lambda_2=1$

$$[A-\lambda I \mid O]=\left[\begin{array}{ccc|c}-2-1 & -8 & -12 & 0\\ 1 & 4-1 & 4 & 0\\ 0 & 0 & 1-1 & 0\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}-3 & -8 & -12 & 0\\ 1 & 3 & 4 & 0\\ 0 & 0 & 0 & 0\end{array}\right] R_1\to -1/3R_1$$

( Continue on previous page …)

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-4k\\0\\k\end{bmatrix}=k\begin{bmatrix}-4\\0\\1\end{bmatrix}$$

Therefore, eigen vector space for $\lambda_1=0$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-4\\0\\1\end{bmatrix}$$

**When** $\lambda_3=2$

$$[A-\lambda I \mid O]=\left[\begin{array}{ccc|c}-2-2 & -8 & -12 & 0\\ 1 & 4-2 & 4 & 0\\ 0 & 0 & 1-2 & 0\end{array}\right]$$

$$=\left[\begin{array}{ccc|c}-4 & -8 & -12 & 0\\ 1 & 2 & 4 & 0\\ 0 & 0 & -1 & 0\end{array}\right] R_1\to -1/4R_1$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 1 & 2 & 4 & 0\\ 0 & 0 & -1 & 0\end{array}\right] R_2\to R_2-R_1$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 0\end{array}\right] R_3\to -R_3$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0\end{array}\right] \begin{array}{l}R_1\to R_1-3R_3\\ R_2\to R_2-R_3\end{array}$$

$$=\left[\begin{array}{ccc|c}1 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

We suppose $z=0, y=k, x+2y=0$

$\therefore z=0, y=k, x=-2z=-2k$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2k\\k\\0\end{bmatrix}=k\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

Therefore, Eigen vector space for $\lambda_3=2$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

## 1.8 Algebraic multiplicity and Geometric multiplicity

Let $A$ be $n \times n$ matrix and $\lambda$ be an eigen value for $A$. If $\lambda$ occurs $(k \geq 1)$ times then $k$ is called the **Algebraic multiplicity** of $\lambda$, and the number of basis vectors is called **Geometric multiplicity**.

**Example-28:** Find Eigen values and Eigen vectors of the matrix. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Also determine algebraic and geometric multiplicity.

Solution: The characteristic equation is $|A - \lambda I_n| = 0$.

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix}$$

$$= (-2-\lambda)[(1-\lambda)(-\lambda)-(-2)(-6)] - 2[2(-\lambda)-(-1)(-6)] - 3[2(-2)-(-1)(1-\lambda)]$$

$$= (-2-\lambda)[-\lambda+\lambda^2-2] - 2[-2\lambda-6] - 3[-4+1-\lambda]$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45$$

$$= -(\lambda^3 + \lambda^2 - 21\lambda - 45)$$

$$\therefore -(\lambda^3 + \lambda^2 - 21\lambda - 45) = 0$$

$$\therefore \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\therefore (\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$$

$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0$$

$$\Rightarrow \lambda_1 = 5, \lambda_2 = -3, \lambda_3 = -3$$

Algebraic Multiplicity of $\lambda =- 3$ is 2 and of $\lambda = 5$ is 1.
We solve the following homogeneous system:

$$\therefore [A - \lambda I]X = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**Case I:** When $\lambda_1 = 5$

$$\therefore [A - \lambda I | O] = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & 2 & -5 & 0 \end{array}\right] R_1 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc|c} -1 & -2 & -5 & 0 \\ 2 & -4 & -6 & 0 \\ -7 & 2 & -3 & 0 \end{array}\right] R_1 \rightarrow -R_1$$

$$=$$

**Case II:** When $\lambda_2 =- 3, \lambda_3 =- 3$

$$\therefore [A - \lambda I | O] = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is in Row-Echelon from.
We suppose

$$\begin{bmatrix} 1 & 2 & 5 & | & 0 \\ 2 & -4 & -6 & | & 0 \\ -7 & 2 & -3 & | & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 7R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 5 & | & 0 \\ 0 & -8 & -16 & | & 0 \\ 0 & 16 & 32 & | & 0 \end{bmatrix} R_2 \to -1/8R_2$$

$$= \begin{bmatrix} 1 & 2 & 5 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 16 & 32 & | & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 - 16R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

which is in Row-Echelon from.

We suppose $x_3 = k, x_2 + 2x_3 = 0 \Rightarrow x_2 = -2k,$

$x_1 + x_3 = 0 \Rightarrow x_1 = -k$

Therefore, Eigen space is for $\lambda_1 = 5$ is $\{k(-1,-2,1) / k \in R\}$

$x_2 = k_1, x_3 = k_2, x_1 + 2x_2 - 3x_3 = 0 \Rightarrow x_1 = -2k_1 + 3k_2$

Therefore, Eigen space is for$\lambda_2 =-3, \lambda_3 =-3$ is $\{k_1(-2,1,0) + k_2(3,0,1) / k_1, k_2 \in R\}$

Hence, Geometric multiplicity of $\lambda_2 =-3$ is 2 and of$\lambda = 5$ is 1.

**Example-29:** Find Eigen values and Eigen vectors of the matrix. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Also determine algebraic and geometric multiplicity.

Solution: The characteristic equation is$|A - \lambda I_n| = 0$.

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$$

$= -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda)$

$= -\lambda^3 + 3\lambda + 2 = -(\lambda^3 - 3\lambda - 2)$

$\therefore -(\lambda^3 - 3\lambda - 2) = 0$

$\therefore \lambda^3 - 3\lambda - 2 = 0$

$\Rightarrow (\lambda - 2)(\lambda + 1)^2 = 0$

$\Rightarrow \lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$

Algebraic Multiplicity of $\lambda =-1$ is 2 and of $\lambda = 2$ is 1.

| | |
|---|---|
| Case-1:$\lambda_1 = 2$ | Case-2: $\lambda_2 =-1, \lambda_3 =-1$ |

$$\therefore [A-\lambda I|O] = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \quad R_2 \to -1/3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \quad \begin{matrix} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 - 3R_2 \end{matrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let

$x_3 = k, x_2 - x_3 = 0, \Rightarrow x_2 = k, x_1 - x_3 = 0, x_1$

.Therefore, eigen space is for $\lambda_1 = 2$ is

$\{k(1,1,1) / k \in R\}$

$$\therefore [A-\lambda I|O] = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let$x_3 = k_1, x_2 = k_2,$

$\Rightarrow x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -k_1 - k_2.$

Therefore, eigen space is for$\lambda_2 = -1, \lambda_3 = -1$

is $\{k_1(-1,0,1) + k_2(-1,1,0) / k_1, k_2 \in R\}$

Hence, Geometric Multiplicity of$\lambda_2 = -1$ is 2 and $\lambda_1 = 2$ of is 1.

**Example 30:** Determine algebraic and geometric multiplicity of matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$.

Answer: $\lambda = 1,2,2$ therefore algebraic multiplicity of $\lambda = 2$ is 2 and geometric multiplicity is 1. For $\lambda = 1$, A.M. is 1 and G.M. is 1.

**Example 31:** Find Eigen value and Eigen vector for following matrix $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: $|A-\lambda I|=0$. Also, find A.M.

$$\begin{vmatrix} -2-\lambda & -8 & -12 \\ 1 & 4-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore (-2-\lambda)((4-\lambda)\times(1-\lambda) - 4\times 0) - (-8)(1\times(1-\lambda) - 4\times 0) + (-12)(1\times 0 - (4-\lambda) \times 0) = 0$$

$$\therefore (-2-\lambda)((4-5\lambda+\lambda^2) - 0) + 8((1-\lambda) - 0) - 12(0-0) = 0$$

$$\therefore (-2-\lambda)(4-5\lambda+\lambda^2) + 8(1-\lambda) - 12(0) = 0$$

$\therefore (-8+6\lambda+3\lambda^2-\lambda^3)+(8-8\lambda)-0=0$

$\therefore (-\lambda^3+3\lambda^2-2\lambda)=0$
$\therefore -\lambda(\lambda-1)(\lambda-2)=0$

$\therefore \lambda = 0 or (\lambda-1) = 0 or (\lambda-2) = 0$

$\therefore$ *The eigenvalues of the matrix A are given by* $\lambda = 0,1,2$

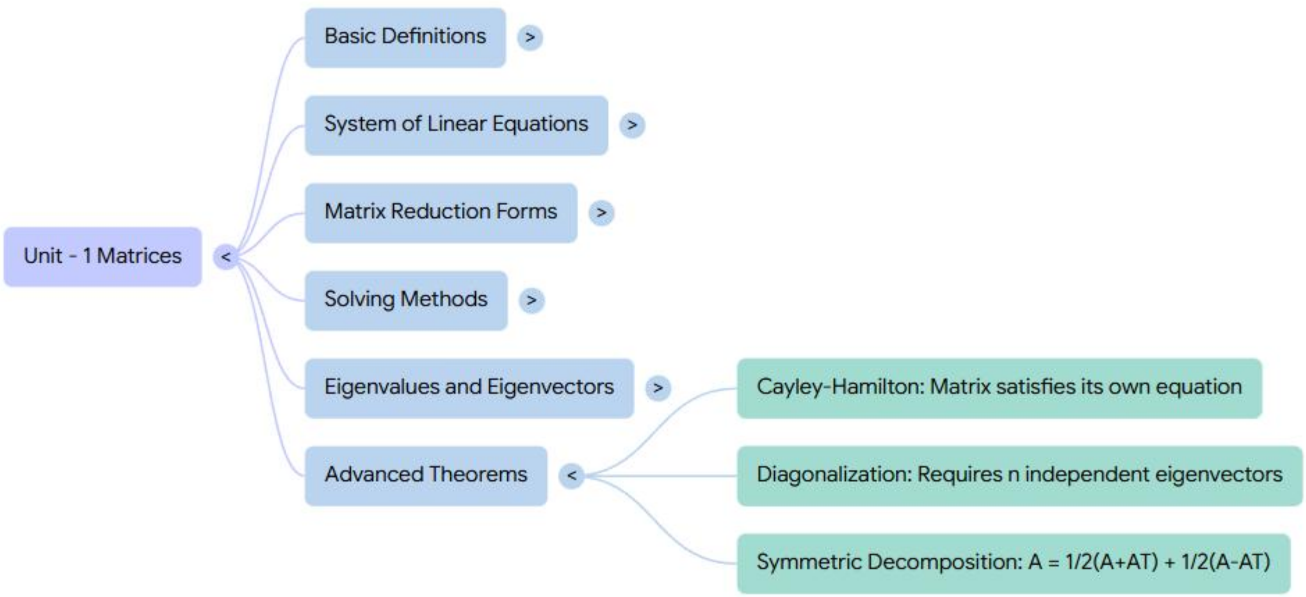
| 1. Eigenvectors for $\lambda$=0 | 2. Eigenvectors for $\lambda$=1 | 3. Eigenvector for $\lambda$=2 |
|---|---|---|
| $v_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$ | $v_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$ | $v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ |

A.M. for all the three eigen values are 1.

**Exercise-7**

Find the Eigen value and Eigenvector of the matrices

a. $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, b. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$




*Schematic diagram of matrix related results*

**Theorem:** Every square matrix can be decomposed as a sum of symmetric and skew-symmetric matrices.

**Proof:** Let $A$ be $m \times n$ matrix.

Let $B = \frac{1}{2}(A + A^T)$ *and* $C = \; = \frac{1}{2}(A - A^T)$ be two matrices.

Obviously, $A = B + C$

Now,$B^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}[(A + A^T)]^T = \frac{1}{2}[A^T + (A^T)^T] = \frac{1}{2}(A^T + A) = B$

As $B^T = B$ , $B$ is symmetric.

$$C^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}[(A - A^T)]^T = \frac{1}{2}[A^T - (A^T)^T] = \frac{1}{2}(A^T - A) = -C$$

Therefore, $C^T = -C$ , $C$ is skew- symmetric.

Therefore, $A$ is a sum of symmetric and skew-symmetric matrices.

**1.9 Cayley –Hamilton Theorem:**

Every square matrix satisfies its own characteristic equation i.e. The theorem states that, for a square matrix $A$ of order $n$, if $|A - \lambda I_n| = 0$.

**Example-32:** Verify Cayley-Hamilton theorem and hence find the inverse of $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and $A^4$.

Solution: The characteristic equation for given matrix is

$$|A - \lambda I_2| = 0.$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

Now, by putting $\lambda = A$, we have

$$A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence, Cayley-Hamilton theorem verified.

Now, by using Cayley-Hamilton theorem, we have

$A^2 - 4A - 5I = 0$, by applying $A^{-1}$ on both the sides

$$A^{-1}\left(A^2 - 4A - 5I\right) = A^{-1}(0)$$

$$\Rightarrow A - 4I - 5A^{-1} = 0$$

$$\Rightarrow 5A^{-1} = A - 4I$$

$$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

And for $A^4$, applying $A^2$ both the sides

$$A^2\left(A^2 - 4A - 5I\right) = A^2(0)$$

$$\Rightarrow A^4 - 4A^3 - 5A^2 = 0$$

$$\Rightarrow A^4 = 4A^3 + 5A^2$$

$$\Rightarrow A^4 = 4\begin{bmatrix} 41 & 84 \\ 42 & 83 \end{bmatrix} + 5\begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} \Rightarrow A^4 = \begin{bmatrix} 209 & 416 \\ 208 & 417 \end{bmatrix}$$

**Example-33:** Find the characteristics equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence prove that

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I=\begin{bmatrix}8&5&5\\0&3&0\\5&5&8\end{bmatrix}.$$

Solution: The characteristics equation is

$$|A-\lambda I|=\begin{vmatrix}2-\lambda&1&1\\0&1-\lambda&0\\1&1&2-\lambda\end{vmatrix}=0$$

$\therefore \lambda^3-5\lambda^2+7\lambda-3=0$

By Cayley-Hamilton Theorem

$\therefore A^3-5A^2+7A-3I=0$ ..............(1)

Now,

$$A^8-5A^7+7A^6-3A^4-5A^3+8A^2-2A+I$$

$$=A^5(A^3-5A^2+7A-3I)+A(A^3-5A^2+7A-3I)+A^2+A+I$$

$$=A^2+A+I \qquad using\ (1)$$

$$\therefore A^2+A+I=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}+\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}8&5&5\\0&3&0\\5&5&8\end{bmatrix}$$

**Exercise:-8**

(1)Verify Cayley-Hamilton theorem and hence find the inverse of $A=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$ and $A^4$.

(2) Compute $A^9-6A^8+10A^7-3A^6+A+I$, where $A=\begin{bmatrix}1&2&3\\-1&3&1\\1&0&2\end{bmatrix}$

(3) Using Cayley-Hamilton theorem find $A^{-1}$ for the following matrix $\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$.

(4) Using Cayley-Hamilton theorem find $A^{-1}$ for A=$\begin{bmatrix}1&-1\\2&3\end{bmatrix}$

**2.0 Diagonalization of a matrix:**

An $n\times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

**OR**

If $n \times n$ matrix $A$ has a basis of eigenvectors, then $D = P^{-1}AP$ is diagonal, with the eigenvalues of $A$ as the entries on the main diagonal. Here, $P$ is the matrix with these eigenvectors as column vectors.
Also, $D^n = P^{-1}A^nP$ and $A^n = PD^nP^{-1}$ 3

**Example-34**: Find a matrix $P$ that diagonalizes matrix $A$ and determine $P^{-1}AP$

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

Solution :

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

The characteristic equation is $|A - \lambda I_n| = 0$

$$\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\therefore (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$$\therefore \lambda = 1,2,3$$

For $\lambda = 1$

$$\therefore [A - \lambda I|O] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} -2 & 4 & -2 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \quad R_1 \to -1/2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \quad \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \quad R_2 \to -1/3R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \quad \begin{matrix} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 + 5R_2 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \quad \begin{matrix} R_1 \to R_1 + 4/3R_2 \\ R_3 \to R_3 + 3R_2 \end{matrix}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore z = k, y - z = 0 \ \& \ x - 2/3z = 0$$

$$\Rightarrow z = k, y = k, x = 2/3k$$

$$\therefore (x,y,z) = k(\frac{2}{3},1,1); k \in R$$

$$\therefore (x,y,z) = 3k(2,3,3); k \in R \quad (\because 3k = k')$$

$$E_1 = \{k'(2,3,3) / k' \in R\}$$

For $\lambda = 3$

$$\therefore [A - \lambda I|O] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} -4 & 4 & -2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right] \quad R_1 \to -1/4R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right] \quad \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & -2 & 3/2 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right] \quad R_2 \to -1/2R_2$$

$$=$$

=
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore z = k, y - z = 0 \;\&\; x - z = 0$

$\Rightarrow z = k, y = k, x = k$

$\therefore (x, y, z) = k(1,1,1); k \in R$

$E_1 = \{k(1,1,1) / k \in R\}$

For $\lambda = 2$

$$\therefore [A - \lambda I | O] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

=
$$\left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right] R_1 \to -1/3R_1$$

=
$$\left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

=
$$\left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] R_2 \to -1/2R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 2R_2 \end{array}$$

=
$$\left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore z = k, y - 3/4z = 0 \;\&\; x - 1/4z = 0$

$\Rightarrow z = k, y = 3/4k, x = 1/4k$

$\therefore (x, y, z) = k(\frac{1}{4}, \frac{3}{4}, 1); k \in R$

$\therefore (x, y, z) = 4k(1,3,4); k \in R$ (□ $4k = k'$)

$E_1 = \{k'(1,3,4) / k' \in R\}$

$$\therefore P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\therefore P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

**Example-35**: Find a matrix P that diagonalizes A and determine $P^{-1}AP$where

$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

Also find $A^{10}$ and find eigenvalues of $A^2$.

Solution : $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I_n| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)(-1-\lambda) - 0 = 0$

$\therefore \lambda = 1, -1$

$$\therefore [A - \lambda I | O] = \left[\begin{array}{cc|c} 1-\lambda & 0 & 0 \\ 6 & -1-\lambda & 0 \end{array}\right]$$

=
$$\left[\begin{array}{cc|c} 2 & 0 & 0 \\ 6 & 0 & 0 \end{array}\right]$$

$y = k, 6x = 0 => x = 0$

$\therefore (x, y) = \{k(0,1)/k \in R\}$

Now,

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For $\lambda = 1$

$\therefore [A-\lambda I|O] = \left[\begin{array}{cc|c} 1-\lambda & 0 & 0 \\ 6 & -1-\lambda & 0 \end{array}\right]$

$= \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 6 & -2 & 0 \end{array}\right]$

Suppose $x = k, 6x - 2y = 0$

$x = k, y = 3k$

$\therefore (x, y) = \{k(1,3)/k \in R\}$

For $\lambda = -1$

( Continue on previous page …)

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$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = D$

$P^{-1}AP = D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow A = PDP^{-1}$

$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1^{10} & 0 \\ 0 & (-1)^{10} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$A^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Eigenvalues of $A^2$ are: $1^2 = 1$ and $(-1)^2 = 1$.

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### 1.11 Application of Matrices:

#### 1.11.1 Coding and Encoding

**Example- 36**

Consider the message to be sent: BEST WISHES

We take the standard codes as follows:

$A = 1$ ; $B = 2$ ; ........; $Z = 26$ and Space $= 0$

1. We convert the above message in to a stream of numerical values as follows:

   BEST WISHES 2 5 19 20 0 23 9 19 8 5 19

2. We construct the message matrix M with this stream of numerals as

$$M = \begin{bmatrix} 2 & 5 & 19 \\ 20 & 0 & 23 \\ 9 & 19 & 8 \\ 5 & 19 & 0 \end{bmatrix}$$

which is of order $4 X 3$

3. Then we perform the product $MA$, where $A$ is an arbitrary nonsingular matrix given by

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \text{ whose inverse is given by } A^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{-1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

4. Then we get matrix

$$X = MA = \begin{bmatrix} 2 & 5 & 19 \\ 20 & 0 & 23 \\ 9 & 19 & 8 \\ 5 & 19 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 10 & 65 \\ 20 & 0 & 49 \\ 9 & 38 & 53 \\ 5 & 38 & 33 \end{bmatrix}$$

5. $$M = XA^{-1} = \begin{bmatrix} 2 & 10 & 65 \\ 20 & 0 & 49 \\ 9 & 38 & 53 \\ 5 & 38 & 33 \end{bmatrix}\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 2 & 5 & 19 \\ 20 & 0 & 23 \\ 9 & 19 & 8 \\ 5 & 19 & 0 \end{bmatrix}$$

This matrix $M$ is converted into numerical message as

2 5 19 20 0 23 9 19 8 5 19

This stream of numerical is converted into the text message as

2 5 19 20 0 23 9 19 8 5 19: *BEST WISHES*

**Example-37**

The Encoding Process

1. Convert the Message to Numbers

   The word "MATH" corresponds to the numbers: 13, 1, 20, 8.

2. Create the Message Matrix (M)

   Group the numbers into a matrix. Since there are 4 letters, we can form a 2x2 matrix.

$$M = \begin{bmatrix} 13 & 20 \\ 1 & 8 \end{bmatrix}$$

3. Choose an Invertible Encoding Matrix (A)

   Select a square encoding matrix that has an inverse (its determinant must be non-zero).

   We will use a simple 2x2 matrix as an example.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

   The determinant of A is det(A) = (2 * 2) - (1 * 3) = 4 - 3 = 1.

   Since the determinant is 1 (not zero), the matrix is invertible.

4. Encode the Message

   Multiply the message matrix (M) by the encoding matrix (A) to get the coded matrix (C).

$$C = AXM = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 13 & 20 \\ 1 & 8 \end{bmatrix}$$

$$C=\begin{bmatrix}(2\times13)+(1\times1) & (3\times13)+(2\times1)\\(2\times20)+(1\times8) & (3\times20)+(2\times8)\end{bmatrix}=\begin{bmatrix}26+1 & 39+2\\40+8 & 60+16\end{bmatrix}=\begin{bmatrix}27 & 41\\48 & 76\end{bmatrix}$$ ]

The encoded message is the sequence of numbers from matrix C: 27, 41, 48, This is what is transmitted.

The Decoding Process

The receiver gets the sequence 27, 41, 48, 76 and knows the encoding matrix A.

1. Form the Coded Matrix (C)

The receiver arranges the numbers back into the 2x2 matrix form: $C=\begin{bmatrix}27 & 48\\41 & 76\end{bmatrix}$

2. Find the Inverse of the Encoding Matrix

To decode, the receiver must find the inverse of the encoding matrix A.

For a 2 x 2 matrix

$A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$, the inverse is $A^{-1}=\dfrac{1}{\det(A)}[a*d-b*c]$

$\det(A)=1$. The inverse is: $A^{-1}=\dfrac{1}{1}\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}=\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}$

3. Decode the Message

Multiply the coded matrix (C) by the inverse matrix $(A^{-1})$ to recover the original message matrix (M).

$$M=A^{-1}X\,C=\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}X\begin{bmatrix}27 & 48\\41 & 76\end{bmatrix}$$

$$M=\begin{bmatrix}(2X27)+(-1X41) & (2X48)+(-1X76)\\(-3X27)+(2X41) & (-3X48)+(2X76)\end{bmatrix}$$

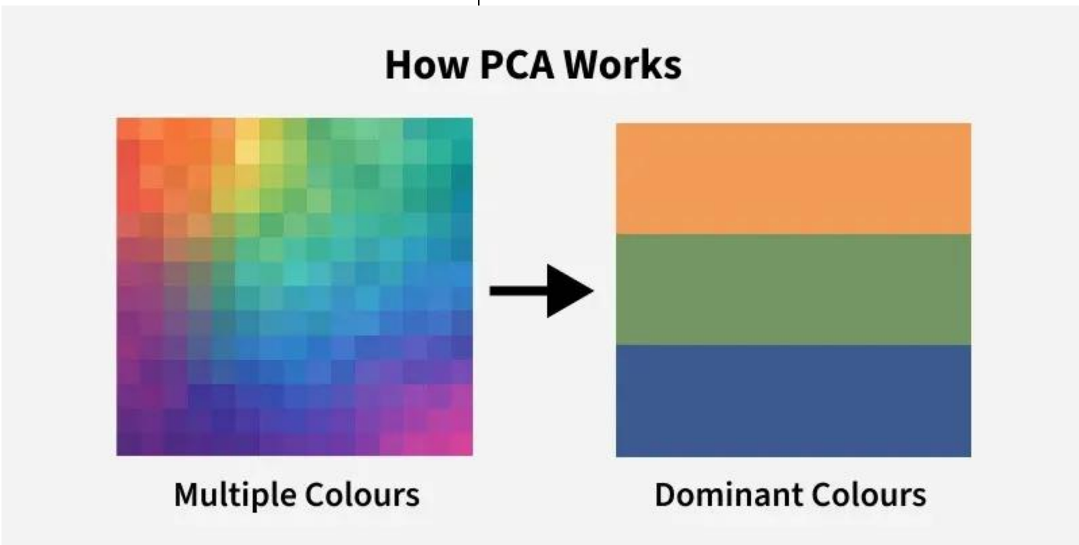
$$M=\begin{bmatrix}54-41 & 96-76\\-81+82 & -144+152\end{bmatrix}=\begin{bmatrix}13 & 20\\1 & 8\end{bmatrix}$$

4. Convert Numbers back to the Message

The resulting numbers are 13, 1, 20, 8, which, using the correspondence table, translate back to M, A, T, H. The original message is recovered.

### 1.11.2 Principal Component Analysis (PCA)

- PCA (Principal Component Analysis) is a dimensionality reduction technique and helps us to reduce the number of features in a dataset while keeping the most important information.
- It changes complex datasets by transforming correlated features into a smaller set of uncorrelated components.
- It helps us to remove redundancy, improve computational efficiency and make data easier to visualize and analyze




*Principal Component Analysis (PCA)*

- Principal Component Analysis (PCA) is a linear algebra–based technique used to reduce the number of variables (dimensions) in a dataset while keeping maximum information (variance).
- In simple words, PCA converts correlated variables into a new set of uncorrelated variables called principal components using eigenvalues and eigenvectors.
- Linear Algebra behind PCA is it is completely based on matrices, eigenvalues, and eigenvectors.
- Main concepts used in PCA are Mean, Covariance matrix, eigenvalues, eigen-vectors and Matrix Transformation.
- Principal components are orthogonal
- Eigenvectors = directions
- Eigenvalues = importance
- PCA is a dimension reduction technique

**Need for PCA**

- To reduce dimensionality of large datasets
- To remove redundancy in data
- To visualize high-dimensional data in 2D or 3D
- To speed up calculations in machine learning
- To remove noise from data

**Example-38: Find PCA for the matrix**

$$X = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 6 & 8 \end{bmatrix}$$

**Step 1: Arrange Data in Matrix Form**

Suppose we have a dataset with two variables x and y:

$$X = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 6 & 8 \end{bmatrix}$$

Each row = observation, each column = variable.

**Step 2: Mean Centering the Data**

Subtract mean of each column from the data:

Mean of column 1 = 4
Mean of column 2 = 6

$$X_{centered} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$

**Step 3: Find Covariance Matrix**

Formula: $cov = \frac{1}{n-1} X_c^T X_c$

For above data: $\text{cov} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$

**Step 4: Find Eigenvalues and Eigenvectors**

Solve: $|Cov - \lambda I| = 0$

Eigenvalues: $\lambda_1 = 8, \quad \lambda_2 = 0$

Corresponding eigenvectors:

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

**Interpretation of Eigenvalues and Eigenvectors**

**Eigenvector** → direction of new axis (principal component)

**Eigenvalue** → amount of variance in that direction

Largest eigenvalue = **most important principal component**

Here, eigenvalue 8 is largest → keep eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

**Step 5: Form Principal Components**

Principal Component 1 ( $PC_1$ ) is the eigenvector with highest eigenvalue: $PC_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This becomes the **new axis**.

**Step 6: Project Data onto Principal Component**

$Z = X_{centered}\ PC_1$

This reduces **2D data → 1D data**.

**Example-39:**

Given covariance matrix: $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Characteristic equation: $|A - \lambda I| = 0 \Rightarrow (3 - \lambda)^2 - 1 = 0$

Eigenvalues: $\lambda_1 = 4, \quad \lambda_2 = 2$

So, **first principal component** corresponds to eigenvalue 4.

**Example-40:** Find the First Principal Component of Data Matrix: $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$

**Step 1**: Mean of column 1 $= 2$

Mean of column 2 $= 3$

Mean of columns = (2, 3)

**Step 2: Mean Centered Matrix:**

Subtract column mean from each element: $\begin{bmatrix} 1-2 & 2-3 \\ 2-2 & 3-3 \\ 3-2 & 4-3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

**Step 3: Covariance Matrix**: $cov = \frac{1}{n-1} X_c^T X_c$

$$X_c^T = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$X_c^T X_c = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Since n = 3, $cov = \frac{1}{3-1} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$$cov = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

**Step 4: Find Eigenvalues**
Solve:

$$|Cov-\lambda I|=0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 2\lambda = 0$$
$$\Lambda_1 = 2, \lambda_2 = 0$$

Eigenvalues: $\lambda_1 = 2, \lambda_2 = 0$
First Principal Component corresponds to $\lambda_1$.:

**Example-41:** Find PCA Using Given Covariance Matrix $\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$

Given Covariance Matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{4} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$
Solution:

$$|A - \lambda I| = 0$$
$$\begin{bmatrix} 4-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)(3-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 8 = 0$$

$$(\lambda - 4)(\lambda - 3) = 0$$

$$\lambda_1 = 4, \lambda_2 = 3$$

First principal component corresponds to λ = 4
It explains maximum variance in the data

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