

Applied Sciences and Humanities

Unit-4

Inner Product Space

Study Guide

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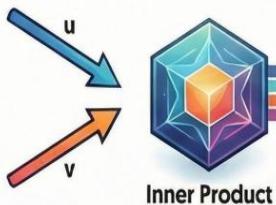
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A Visual Guide to Inner Product Spaces

CORE CONCEPTS OF INNER PRODUCT SPACES

What is an Inner Product Space?

A vector space with an "inner product" $\langle u, v \rangle$ that satisfies four key rules (axioms).



4 DEFINING AXIOMS

Symmetry | $\langle u, v \rangle = \langle v, u \rangle$

The order of vectors doesn't matter.

Additivity | $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$

It distributes over vector addition.

Homogeneity | $\langle cu, v \rangle = c\langle u, v \rangle$

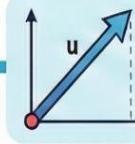
Scalar multiples can be factored out.

Positivity | $\langle u, v \rangle \geq 0$, and $= 0$ only if $v = 0$

The inner product of a vector with itself is non-negative.

Measuring Geometry with Inner Products

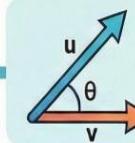
The inner product defines key geometric properties in any vector space.



LENGTH (NORM):

$$\|u\| = \sqrt{\langle u, u \rangle}$$

The square root of the inner product of a vector with itself.



ANGLE:

$$\cos(\theta) = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

Defines the angle θ between two vectors.

ORTHOGONAL VECTORS: The Inner Product is Zero

$$\langle u, v \rangle = 0$$

Two vectors u and v are orthogonal (perpendicular) if $\langle u, v \rangle = 0$.

THE GRAM-SCHMIDT ORTHONORMALIZATION PROCESS

Goal: Create an Orthonormal Basis

A method to convert any set of basis vectors into an "orthonormal" set.

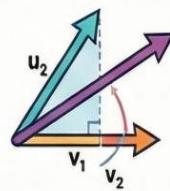


WHAT IS AN ORTHONORMAL SET?

A set where every vector is orthogonal to every other vector, and each vector has a length of 1.

STEP 1: SET THE FOUNDATION
The first new vector is the same as the first old vector: $v_1 = u_1$.

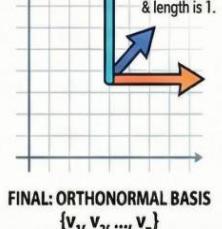
HOW THE PROCESS WORKS IN 3 STEPS



Starting with a basis $\{u_1, u_2, \dots, u_n\}$, we create a new orthonormal basis $\{v_1, v_2, \dots, v_n\}$.

STEP 2: SUBTRACT THE PROJECTIONS
For each next vector, subtract its projection onto the previously created vectors.

STEP 3: NORMALIZE TO UNIT LENGTH
Divide each new orthogonal vector by its own length to make its final length equal to 1.



4.0 Inner Product Space :

Inner Product:

Let u , v , and w be vectors in a vector space V , and let c be any scalar. An inner product on V is a function that associates a real number $\langle u, v \rangle$ with each pair of vectors u and v and satisfies the following axioms.

1. $\langle u, v \rangle = \langle v, u \rangle$
2. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
3. $c\langle u, v \rangle = \langle cv, u \rangle$
4. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$.

A vector space V with an inner product is called an **inner product space**.

Note: If $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$ are vectors in R^n then Euclidean Inner Product (dot product) $u \cdot v = u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n$ Satisfies all the four axioms of inner product

Inner Product Space

Inner Product Definition

Types of Inner Products

Norm and Distance

Orthogonality

Gram-Schmidt Process

space .Hence any vector space with respect to Euclidean inner product is an inner product space.

Example:1 If $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be vectors in \mathbb{R}^2 .Verify that the inner product

$$\langle u, v \rangle = u_1 v_1 + 2u_2 v_2 \text{ satisfies the four inner product axioms.}$$

Solution: (1) $\langle u, v \rangle = u_1 v_1 + 2u_2 v_2 = v_1 u_1 + 2v_2 u_2 = \langle v, u \rangle$

(2) If $w = (w_1, w_2)$ then

$$\begin{aligned}\langle u + v, w \rangle &= (u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\ &= (u_1 w_1 + v_1 w_1) + 2(u_2 w_2 + v_2 w_2) \\ &= (u_1 w_1 + 2u_2 w_2) + (v_1 w_1 + 2v_2 w_2) \\ &= \langle u, w \rangle + \langle v, w \rangle\end{aligned}$$

$$(3) c\langle u, v \rangle = c(u_1 v_1 + 2u_2 v_2)$$

$$= (cu_1)v_1 + 2(cu_2)v_2$$

$$= \langle cu, v \rangle$$

$$(4) \langle v, v \rangle = v_1^2 + v_2^2 \geq 0$$

$$\langle v, v \rangle = 0 \Rightarrow v_1^2 + v_2^2 = 0$$

$$\Rightarrow v_1 = v_2 = 0$$

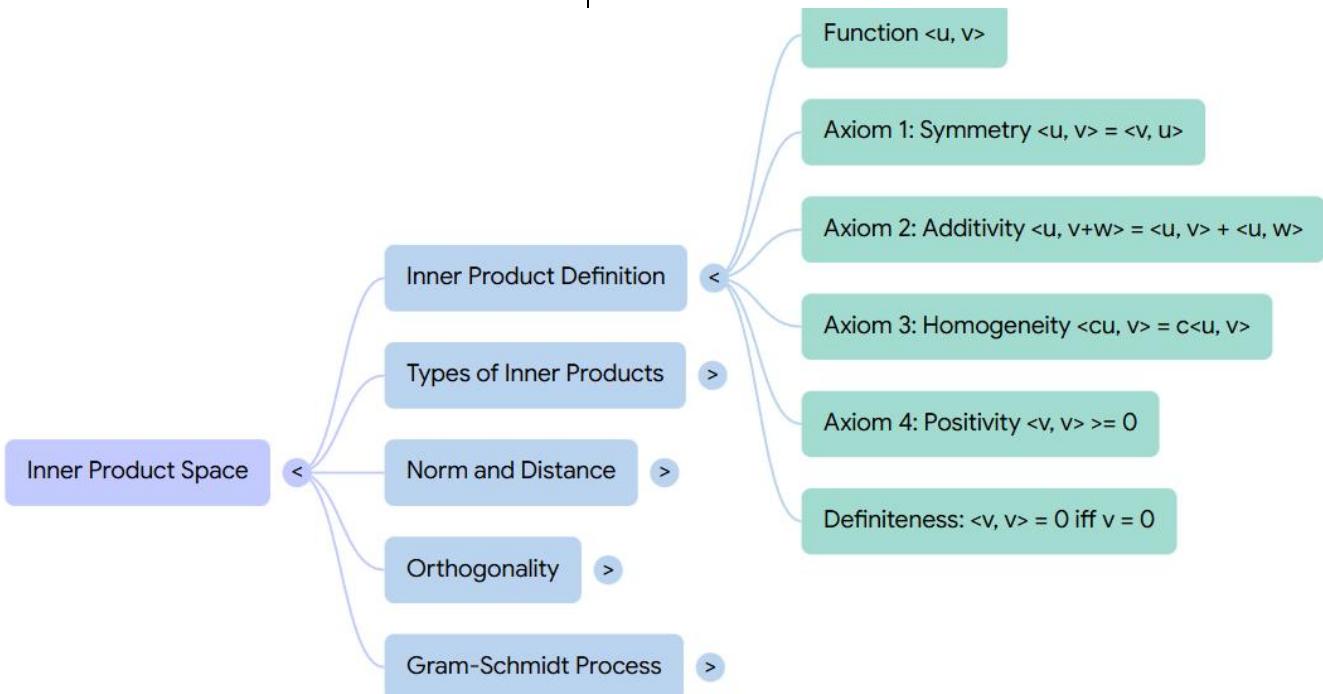
Example:2 Show that the following function is not an inner product space in \mathbb{R}^3 .

$$\langle u, v \rangle = u_1 v_1 - 2u_2 v_2 + u_3 v_3$$

Solution : Let $v = (1, 2, 1)$ then $\langle v, v \rangle = -6 < 0$

Therefore Axiom 4 is not satisfied.

Thus this function is not an inner product on \mathbb{R}^3 .



4.1 Properties of Inner Products :

Let u , v and w be vectors in an inner product space V , and let c be any real number.

$$1. \langle 0, v \rangle = \langle v, 0 \rangle = 0$$

$$2. \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$$

$$3. \langle u, cv \rangle = c\langle u, v \rangle$$

4.2 Inner product generated by Matrices:

Let u and v be vectors in \mathbb{R}^n expressed as $n \times 1$ matrices and A be an $n \times n$ invertible matrix. If $u \cdot v$ is the Euclidean inner product on \mathbb{R}^n then $\langle u, v \rangle = Au \cdot Av \dots (1)$ represent the inner product on \mathbb{R}^n

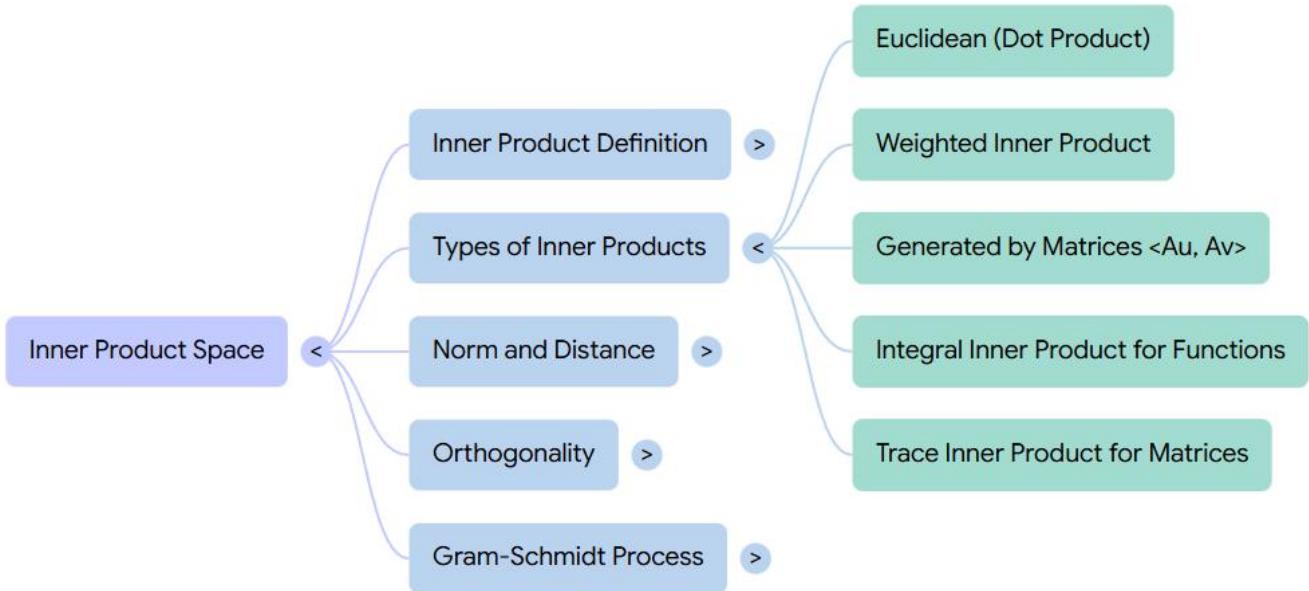
generated by matrix A . If u and v are in matrix form , $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$ then

$$u \cdot v = u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n$$

$$= [v_1 \quad v_2 \quad v_3 \quad \dots \quad v_n] \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = v^T u$$

Applying this formula in equation (1)

$$\langle u, v \rangle = Au \cdot Av = (Av)^T Av = v^T A^T Au = v$$



Example:2 Show that $\langle u, v \rangle = 9u_1v_1 + 4u_2v_2$ is the inner product on \mathbb{R}^2 generated by the matrix $A =$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Solution: Inner product generated by A is $\langle u, v \rangle = Au \cdot Av$ where $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\begin{aligned} \langle u, v \rangle &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 3u_1 \\ 2u_2 \end{bmatrix} \cdot \begin{bmatrix} 3v_1 \\ 2v_2 \end{bmatrix} = 3u_13v_1 + 2u_22v_2 = 9u_1v_1 + 4u_2v_2 \end{aligned}$$

Example:3 $\langle u, v \rangle = 9u_1v_1 + 4u_2v_2$ is the inner product on \mathbb{R}^2 generated by the matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

Solution: Inner product generated by A is $\langle u, v \rangle = Au \cdot Av$ where $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\begin{aligned} \langle u, v \rangle &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} 3u_1 \\ 2u_2 \end{bmatrix} \cdot \begin{bmatrix} 3v_1 \\ 2v_2 \end{bmatrix} = 3u_13v_1 + 2u_22v_2 = 9u_1v_1 + 4u_2v_2 \end{aligned}$$

4.3 Norm (length) in Inner product spaces :

$$\|u\| = \sqrt{\langle u, u \rangle} \text{ and } \|u\|^2 = \langle u, u \rangle$$

4.3.1 Unit vector: Let u be a vector in an inner product space V . If $\|u\| = 1$ then u is called a unit vector in V .

4.3.2 Properties of Length :

1. $\|u\| \geq 0$
2. $\|u\| = 0$ iff $u = 0$
3. $\|cu\| = |c|\|u\|$
4. $\|u + v\| \leq \|u\| + \|v\|$ (Triangular inequality)

4.3.3 Distance between u and v :

$$d(u, v) = \|u - v\| = \sqrt{\langle u - v, u - v \rangle}$$

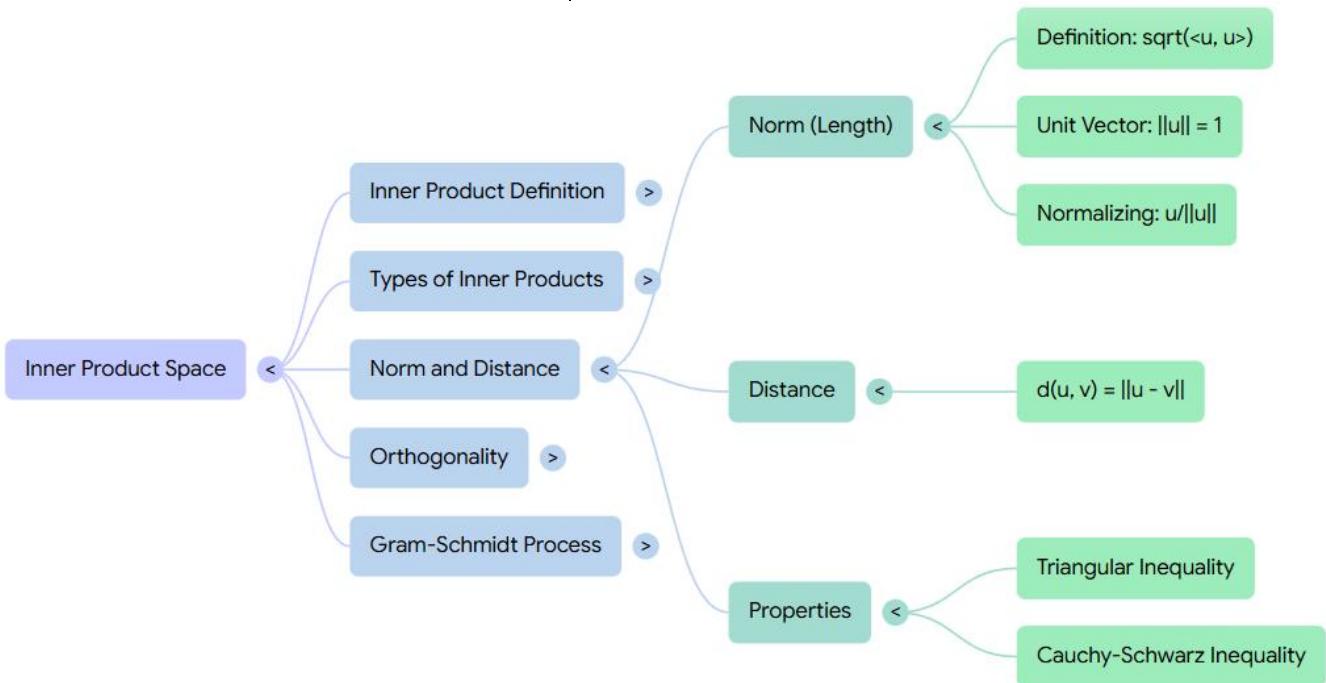
4.3.3.1 Properties of distance:

1. $d(u, v) \geq 0$
2. $d(u, v) = 0$ iff $u = v$
3. $d(u, v) = d(v, u)$
4. $d(u, v) \leq d(u, w) + d(w, v)$ (Triangular inequality)

4.3.3.2 Angle between two non zero vectors u and v :

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\|\|v\|}, 0 \leq \theta \leq \pi$$

4.3.4 Orthogonal vectors: Two vectors u and v are **orthogonal** if $\langle u, v \rangle = 0$. It is denoted by $u \perp v$.



Examples-1: Find the $\|u\|$ and $d(u, v)$ where $u = (-1, 2)$ and $v = (2, 5)$ using the Euclidean inner product $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$ where $u = (u_1, u_2)$ and $v = (v_1, v_2)$

The inner product generated by the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

Solution: $u - v = (-1, 2) - (2, 5) = (-3, -3)$

$$(i) \|u\| = \langle u, u \rangle^{1/2} = (u_1^2 + u_2^2)^{1/2} = (1 + 4)^{1/2} = \sqrt{5}$$

$$d(u, v) = \|u - v\| = \|(-3, -3)\| = [(-3, -3)(-3, -3)]^{1/2} = \sqrt{18} = 3\sqrt{2}$$

$$(ii) \|u\| = \langle u, u \rangle^{1/2} = (3u_1v_1 + 2u_2v_2)^{1/2} = (3(-1)^2 + 2(2)^2)^{1/2} = \sqrt{11}$$

$$d(u, v) = \|u - v\| = \|(-3, -3)\| = \langle(-3, -3)(-3, -3) \rangle^{1/2} = \sqrt{45} = 3\sqrt{5}$$

(iii) Inner product generated by the matrix A is

$$\begin{aligned} \langle u, v \rangle &= Au \cdot Av = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1 + 2u_2 \\ -u_1 + 3u_2 \end{bmatrix} \begin{bmatrix} v_1 + 2v_2 \\ -v_1 + 3v_2 \end{bmatrix} \\ &= (u_1 + 2u_2)(v_1 + 2v_2)(-u_1 + 3u_2)(-v_1 + 3v_2) \\ &= u_1v_1 + 2u_2v_1 + 2u_1v_2 + 4u_2v_2 + u_1v_1 - 3u_1v_2 - 3u_2v_1 + 9u_2v_2 \\ &= 2u_1v_1 - u_2v_1 - u_1v_2 + 13u_2v_2 \end{aligned}$$

$$\|u\| = \langle u, u \rangle^{1/2} = (2u_1v_1 - u_2v_1 - u_1v_2 + 13u_2v_2)^{1/2}$$

$$= (2(-1)^2 - (-1)(-2) - (2)(-1) + 13(2)^2)^{1/2} = \sqrt{58}$$

$$\begin{aligned} d(u, v) &= \|u - v\| = \|(-3, -3)\| = \langle(-3, -3)(-3, -3)\rangle^{1/2} \\ &= (2(-3)^2 - (-3)(-3) - (-3)(-3) + 13(-3)^2)^{1/2} \\ &= \sqrt{117} = 3\sqrt{13}. \end{aligned}$$

(2) Find the $\|P_1\|$ and $d(P_1, P_2)$ if $P_1 = 3 - x + x^2$, $P_2 = 2 + 5x^2$ and weighted inner product

$$\langle P_1, P_2 \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 \text{ where } P_1 = a_0 + a_1 x + a_2 x^2, P_2 = b_0 + b_1 x + b_2 x^2$$

Solution: $\|P_1\| = \langle P_1, P_1 \rangle^{1/2} = [3^2 + (-1)^2 + (1)^2]^{1/2} = \sqrt{11}$

$$P_1 - P_2 = (3 - x + x^2) - (2 + 5x^2) = 1 - x + 4x^2$$

$$d(P_1, P_2) = \|P_1 - P_2\| = [1^2 + (-1)^2 + (-4)^2]^{1/2} = \sqrt{18} = 3\sqrt{2}.$$

(3) Find $\langle f, g \rangle$ if $f = f(x) = 1 - x + x^2 + 5x^3$ and $g = g(x) = x - 3x^2$ and the inner product.

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx.$$

Angle between two non zero vectors u and v:

$$\cos\theta = \frac{\langle u, v \rangle}{\|u\|\|v\|}, 0 \leq \theta \leq \pi$$

Orthogonal vectors: Two vectors u and v are **orthogonal** if $\langle u, v \rangle = 0$. It is denoted by $u \perp v$.

4.3.5 Inequalities:

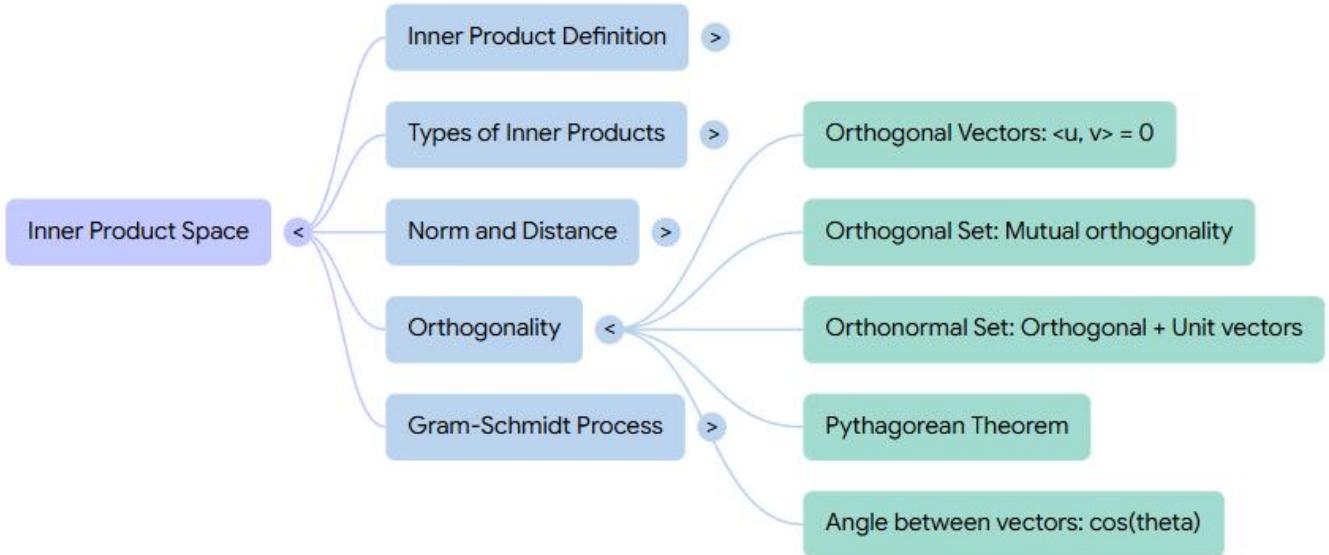
4.3.5.1 Pythagorean Theorem: If u and v are orthogonal vectors in an inner product space V then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

4.3.5.2 Cauchy-Schwarz inequality: If u and v are vectors in an inner product space V then

$$|\langle u, v \rangle| \leq \|u\|\|v\|. .$$

4.3.5.3 Triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$



Examples:

(4) Determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

- (i) $u = (-1, 3, 2), v = (4, 2, -1)$
- (ii) $u = (-4, 6, -10, 1), v = (2, 1, -2, 9)$

Solution:

$$(i) \langle u, v \rangle = u \cdot v = (-1, 3, 2)(4, 2, -1) = -4 + 3 \cdot 2 + 2 \cdot -1 = 0$$

Hence, u and v are orthogonal.

$$(ii) \langle u, v \rangle = u \cdot v = (-4, 6, -10, 1)(2, 1, -2, 9) = -8 + 6 + 12 + 9 = 27 \neq 0$$

Hence, u and v are not orthogonal.

(5) Show that the matrices $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ are orthogonal with respect to the inner product $\langle A, B \rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$ where, $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

$$\text{Solution : } \langle A, B \rangle = 2 \cdot 1 + 1 \cdot 1 + (-1) \cdot 0 + 3 \cdot (-1) = 0$$

Hence, A and B are orthogonal.

(6) Verify that the Cauchy-Schwarz inequality holds for the following vectors:

$$(i) u = (-2, 1) \text{ and } v = (1, 0) \text{ where, } \langle u, v \rangle = 3u_1v_1 + 2u_2v_2$$

$$(ii) A = \begin{bmatrix} -1 & 2 \\ 6 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 3 & 3 \end{bmatrix} \text{ using the inner product } \langle A, B \rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$$

where, $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

(i) $\langle u, v \rangle = \langle (-2, 1), (1, 0) \rangle = -6$

$$|\langle u, v \rangle| = |-6| = 6$$

$$\|u\| = \langle u, u \rangle^{1/2} = (3u_1^2 + 2u_2^2)^{1/2} = (3(-2)^2 + 2(1)^2)^{1/2} = \sqrt{14}.$$

$$\|v\| = \langle v, v \rangle^{1/2} = (3v_1^2 + 2v_2^2)^{1/2} = (3(1)^2 + 2(0)^2)^{1/2} = \sqrt{3}$$

$$\therefore \|u\| \cdot \|v\| = \sqrt{42} = 6.48$$

Since, $|\langle u, v \rangle| < \|u\| \cdot \|v\|$

\therefore Cauchy –Schwartz's inequality is verified.

(ii) $\langle A, B \rangle = -1 \cdot 1 + 2 \cdot 0 + 3 \cdot 6 + 1 \cdot 3 = 20$

$$\|A\| = \langle A, A \rangle^{1/2} = \sqrt{(-1)^2 + 2^2 + 6^2 + 1} = \sqrt{42}$$

$$\|B\| = \langle B, B \rangle^{1/2} = \sqrt{(-1)^2 + 2^2 + 6^2 + 1} = \sqrt{19}$$

$$\therefore \|A\| \cdot \|B\| = \sqrt{798} = 28.25$$

Since, $|\langle A, B \rangle| < \|A\| \cdot \|B\|$

\therefore Cauchy –Schwartz's inequality is verified.

(iii) $\langle P_1, P_1 \rangle = -1 \cdot 2 + 2 \cdot 0 + 1 \cdot (-4) = -6$

$$|\langle P_1, P_1 \rangle| = |-6| = 6$$

$$\|P_1\| = \langle P_1, P_1 \rangle^{1/2} = \sqrt{(-1)^2 + 2^2 + 1^2} = \sqrt{6}$$

$$\|P_2\| = \langle P_2, P_2 \rangle^{1/2} = \sqrt{(2)^2 + 0^2 + (-4)^2} = \sqrt{20}$$

$$\|P_1\| \|P_2\| = \sqrt{120} = 10.95$$

Since, $|\langle P_1, P_2 \rangle| < \|P_1\| \cdot \|P_2\|$

\therefore Cauchy –Schwartz's inequality is verified.

4.3.6 Orthogonal and Orthonormal set:

A set $s = \{u_1, u_2, \dots, u_p\}$ of vectors in an inner product space V is called an orthogonal set if each pair of distinct vectors in S are orthogonal. i.e $\langle u_i, u_j \rangle = 0$ for $i \neq j$.

An orthonormal set of unit vectors (norm is 1) is called orthonormal.

If $\langle u_i, u_j \rangle = 0$ for $i \neq j$ and $\langle u_i, u_j \rangle = 1$ for $i = 1, 2, \dots, p$

The Process of dividing a non zero vector u by its norm is called normalizing u i.e $\frac{u}{\|u\|}$.

Note: If S is a basis ,then it is called an orthogonal basis or an Orthonormal basis.

Example:1 Show that the following set S is an orthonormal basis.

$$S = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right), \left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3} \right), \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right) \right\}.$$

Solution: first we show the three vectors are mutually orthogonal.

$$\nu_1 \cdot \nu_2 = \frac{-1}{6} + \frac{1}{6} + 0 = 0 ; \nu_1 \cdot \nu_3 = \frac{-2}{3\sqrt{2}} + \frac{2}{3\sqrt{2}} + 0 = 0 ;$$

$$\nu_2 \cdot \nu_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

Secondly Show that each vector is of length 1.

$$\|\nu_1\| = \sqrt{\nu_1, \nu_1} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1 ;$$

$$\|\nu_2\| = \sqrt{\nu_2, \nu_2} = \sqrt{\frac{2}{36} + \frac{2}{36} + \frac{8}{9}} = 1 ;$$

$$\|\nu_3\| = \sqrt{\nu_3, \nu_3} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1 .$$

Thus S is an orthonormal set.

Example:7 In $P_3(x)$ with the inner product $\langle v_i, v_j \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$. The standard basis $B = \{1, x, x^2\}$ is orthonormal.

Solution : We have $\nu_1 = 1 + 0x + 0x^2$, $\nu_2 = 0 + x + 0x^2$, $\nu_3 = 0 + 0x + x^2$

$$\text{Then } \langle \nu_1, \nu_2 \rangle = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0,$$

$$\langle \nu_1, \nu_3 \rangle = 1 \cdot 0 + 0 \cdot 0 + 0 \cdot 1 = 0,$$

$$\langle \nu_2, \nu_3 \rangle = 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0.$$

$$\text{And } \|\nu_1\| = 1; \|\nu_2\| = 1; \|\nu_3\| = 1$$

Hence, B is orthonormal.

Example:8 Show that the vectors $u_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right)$, $u_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right)$, $u_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right)$ are orthonormal with respect to the Euclidean inner product on \mathbb{R}^3 .

4.3.7 Gram-Schmidt Orthonormalization process:

The **Gram-Schmidt Orthonormalization process** is a method for converting a set of linearly independent vectors into an orthonormal set with respect to an inner product space. This process is particularly useful in constructing an orthonormal basis for a subspace.

Gram-Schmidt Process: Let V be any non zero n-dimensional inner product space and $s_1 = \{u_1, u_2, \dots, u_n\}$ in an arbitrary basis for V . The process of constructing an orthogonal basis

$S_2 = \{v_1, v_2, v_3, \dots, v_n\}$ from S_1 is as follows:

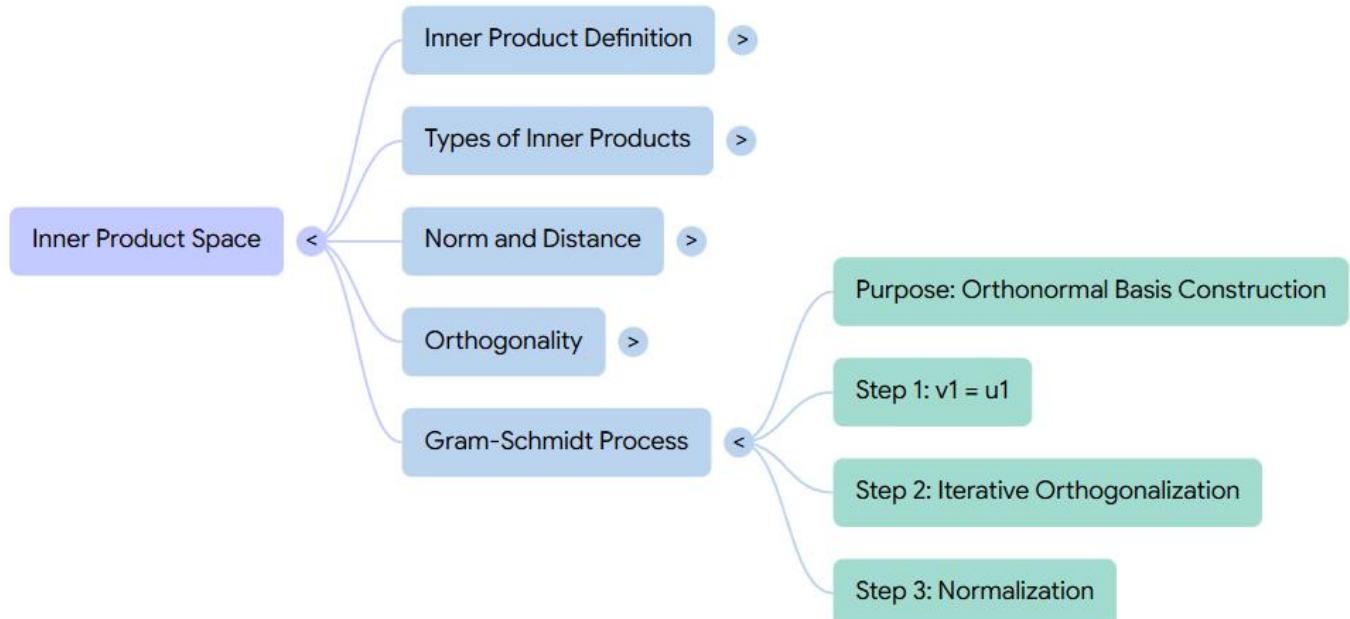
Step-1: Let $v_1 = u_1$

Step-2: Find the vector v_2, v_3, \dots, v_n successively using the formula

$$v_i = u_i - \frac{\langle u_i, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_i, v_2 \rangle}{\|v_2\|^2} v_2 - \dots - \frac{\langle u_i, v_{i-1} \rangle}{\|v_{i-1}\|^2} v_{i-1}$$

The set S_2 of vectors $v_1, v_2, v_3, \dots, v_n$ is an orthogonal set. Since every orthogonal set is linearly independent. S_2 is linearly independent and also has n vectors ($\dim V$) thus the set S_2 is an orthogonal basis for V .

Note: The orthogonal basis S_2 can be transformed to orthonormal basis by normalizing all the vectors of S_2 .



Example:9 Find an orthonormal basis for R^3 containing the vectors $v_1 = (3, 5, 1)$, $v_2 = (2, -2, 4)$ Using Euclidean inner product.

Solution : $\langle v_1, v_2 \rangle = v_1 \cdot v_2$

$$= (3, 5, 1) \cdot (2, -2, 4) = 0$$

Thus, v_1 and v_2 are orthogonal.

Basis for R^3 will have 3 non zero vectors.

Let $v_3 = (b_1, b_2, b_3)$ be the third vector of the basis such that

$$\langle v_1, v_3 \rangle = 0 \text{ & } \langle v_2, v_3 \rangle = 0$$

$$(3, 5, 1) \cdot (b_1, b_2, b_3) = 0$$

$$3b_1 + 5b_2 + b_3 = 0 \dots\dots(1)$$

$$\text{And } (2, -2, 4) \cdot (b_1, b_2, b_3) = 0$$

$$2b_1 - 2b_2 + 4b_3 = 0 \dots\dots(2)$$

Using the augmented matrix of the system of equations From (1) and (2) is

$$\begin{bmatrix} 3 & 5 & 1 & 0 \\ 2 & -2 & 4 & 0 \end{bmatrix}$$

Reducing the augmented matrix to row echelon form

$$R_1 \rightarrow R_1 - R_2$$

$$\begin{bmatrix} 1 & 7 & -3 & 0 \\ 2 & -2 & 4 & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 7 & -3 & 0 \\ 0 & -16 & 10 & 0 \end{bmatrix}$$

$$\Rightarrow b_1 + 7b_2 - 3b_3 = 0 ; -16b_2 + 10b_3 = 0$$

$$\Rightarrow b_2 = \frac{10}{16}b_3$$

$$\text{Let } b_3 = 8 \Rightarrow b_2 = 5$$

$$\therefore b_1 = -11$$

$$\Rightarrow v_3 = (b_1, b_2, b_3) = (-11, 5, 8)$$

The vectors v_1, v_2, v_3 from an orthogonal basis for \mathbb{R}^3 .

Normalizing the vectors v_1, v_2, v_3

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{(3, 5, 1)}{\sqrt{9+25+1}} = \left(\frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{1}{\sqrt{35}} \right)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{(2, -2, 4)}{\sqrt{4+4+16}} = \left(\frac{2}{\sqrt{24}}, \frac{-2}{\sqrt{24}}, \frac{4}{\sqrt{24}} \right)$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{(-11, 5, 8)}{\sqrt{121+25+64}} = \left(\frac{\sqrt{11}}{\sqrt{210}}, \frac{5}{\sqrt{210}}, \frac{8}{\sqrt{210}} \right)$$

Therefore vectors w_1, w_2, w_3 form an orthonormal basis for \mathbb{R}^3 .

Example:10 Let \mathbb{R}^3 have the Euclidean inner product .Use the Gram-Schmidt process to transform the basis vectors $u_1 = (1, 0, 0)$, $u_2 = (3, 7, -2)$, $u_3 = (0, 4, 1)$ into an orthonormal basis.

Solution: Step-1: Let $v_1 = u_1 = (1, 0, 0)$

$$\text{Step-2: } v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= (3, 7, -2) - \frac{\langle (3, 7, -2), (1, 0, 0) \rangle}{1} (1, 0, 0)$$

$$= (3, 7, -2) - 3(1, 0, 0)$$

$$= (0, 7, -2)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$v_3 = (0, 4, 1) - \frac{\langle (0, 4, 1), (1, 0, 0) \rangle}{1} (1, 0, 0) - \frac{\langle (0, 4, 1), (0, 7, -2) \rangle}{53} (0, 7, -2)$$

$$= (0, 4, 1) - 0 - \frac{26}{53} (0, 7, -2)$$

$$= (0, 4, 1) - \frac{15}{53} (0, 7, -2)$$

$$= \frac{15}{53} (0, 2, 7)$$

The vectors v_1, v_2, v_3 from an orthogonal basis for \mathbb{R}^3 .

Normalizing the vectors v_1, v_2, v_3

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{(1, 0, 0)}{\sqrt{1+0+0}} = (1, 0, 0)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{(0, 7, -2)}{\sqrt{49+4}} = \left(0, \frac{7}{\sqrt{53}}, \frac{-2}{\sqrt{53}}\right)$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{\frac{15}{53}(0, 2, 7)}{\frac{15}{53}\sqrt{49+4}} = \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)$$

Therefore vectors w_1, w_2, w_3 form an orthonormal basis for \mathbb{R}^3 .

Example:11 Let \mathbb{R}^3 have the inner product

$\langle (x_1, x_2, x_3)(y_1, y_2, y_3) \rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3$. Use the Gram-Schmidt process to transform the basis vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 1, 0)$, $u_3 = (1, 0, 0)$ into an orthonormal basis.

Example:12 Use the Gram-Schmidt method to transform the basis $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of M_{22} into an orthogonal basis if $\langle A, B \rangle = \text{tr}(AB^T)$.

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