Subject: Calculus (03019101BS01)
Semester: 1st Sem, B. Tech
Lecture Notes: Unit-4, Vector Calculus

Module-4

Vector-valued function, velocity and acceleration, the gradient of a scalar function, directional derivatives, divergence and curl of a vector-valued function. Parameterization of curves and surfaces, vector fields, line integrals, Green's theorem, surface integrals, Gauss divergence theorem and Stokes' theorems with applications

Vector-Valued Function

A vector-valued function assigns a vector to each value of the parameter t. In 3D space, it is usually written as:

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

where $(x(t), y(t), z(t))$ are scalar functions of t.

Geometrically, $\vec{r}(t)$ represents the position vector of a moving particle at time t.

The curve traced by $\vec{r}(t)$ as t varies is called a space curve.

Note: The vector-valued function $\vec{r}(t)$ is said to be differentiable at t, if all functions $x(t),\ y(t),$ $and\ z(t)$ are differentiable at t.

Examples:

1. The velocity of a moving fluid at any instant.
2. The gravitational force.
3. The electric and magnetic field intensity.

Scalar Point Function

If corresponding to each point $\mathrm{P}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ of a region R in space, there corresponds a unique scalar function $\phi = \phi(\mathrm{x}, \mathrm{y}, \mathrm{z})$ then ϕ is called a scalar point function and R It is called a scalar field.

Examples:

1. The temperature field in a body.
2. The pressure field of the air in the Earth's atmosphere.
3. The density of a body.

Note: A scalar field that is independent of time is called a stationary or steady-state scalar field.

Velocity

The velocity vector is the derivative of position with respect to time

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}.$$

The direction of velocity is tangent to the path of the particle.

Speed

Speed is the magnitude of the velocity. It is a scalar quantity, defined by

$$|\vec{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

Acceleration

The acceleration vector is the derivative of velocity (or the second derivative of position):

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2}.$$

In components:

$$\vec{a}(t) = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} + \frac{d^2z}{dt^2}\hat{k}.$$

Acceleration describes how velocity (magnitude and/or direction) changes with time.

Example:

Let $\vec{r}(t) = t^2\hat{\imath} + t^3\hat{\jmath} + \sin t\ \hat{k}$. Find the velocity, speed and acceleration at time $t = 0$.

Solution:

Given the position vector is

$$\vec{r}(t) = t^2\hat{\imath} + t^3\hat{\jmath} + \sin t\ \hat{k}.$$

Velocity:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = 2t\hat{\imath} + 3t^2\hat{\jmath} + \cos t\ \hat{k}.$$

Speed :

$$|\vec{v}(t)| = 2t\hat{\imath} + 3t^2\hat{\jmath} + \cos t\ \hat{k}|$$

$$= \sqrt{(2t)^2 + (3t^2)^2 + (cos\, t)^2}$$

$$= \sqrt{4t^2 + 9t^4 + \cos^2 t}$$

Acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = 2\hat{\imath} + 6t\hat{\jmath} - \sin t\ \hat{k}.$$

At $t = 0$:

$$\vec{v}(0) = \hat{k},$$

$$speed = 1,$$

$$\vec{a}(0) = 2\hat{\imath}.$$

Exercise:

1. If $\vec{r}(t) = (t^2 + 1)\hat{\imath} + (2t - 3)\hat{\jmath} + (t^3)\hat{k},$ find the velocity and acceleration at $t = 2$.

   Answer: $\vec{v}(2) = 4\hat{\imath} + 2\hat{\jmath} + 12\hat{k},\ \vec{a}(2) = 2\hat{\imath} + 0\hat{\jmath} + 12\hat{k}.$

2. For $\vec{r}(t) = \cos t\ \hat{\imath} + \sin t\ \hat{\jmath} + t\,\hat{k}$, find velocity, speed, and acceleration.

   Answer: $\vec{v}(t) = -\sin t\ \hat{\imath} + \cos t\ \hat{\jmath} + \hat{k}, \quad |\vec{v}(t)| = \sqrt{2}, \quad \vec{a}(t) = -\cos t\ \hat{\imath} - \sin t\ \hat{\jmath}.$

3. A particle moves with a position vector $\vec{r}(t) = e^t\hat{\imath} + e^{-t}\hat{\jmath} + t^2\hat{k}.$ Find velocity and acceleration at t=0.

   Answer: $\vec{v}(0) = \hat{\imath} - \hat{\jmath} + 0\hat{k},\ \vec{a}(0) = \hat{\imath} + \hat{\jmath} + 2\hat{k}.$

Vector Differential Operator

The vector differential operator is denoted by $\vec{\nabla}$ (del or nabla) and is defined as

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}.$$

Gradient of a Scalar Field

For a given scalar function $\phi(x, y, z)$ the gradient of ϕ is denoted by $grad\ \phi$ or $\vec{\nabla}\phi$. It is defined by

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k}.$$

Remark: The direction of the gradient of the scalar field $\phi(x, y, z)$ is the normal to the surface $\phi(x, y, z) = constant$ at (x, y, z) .

Example 1:

Find the gradient of $\phi = 3x^2y - y^3z^2$ at the point $(1, -2, 1)$.

Solution:

Given:

$$\phi = 3x^2y - y^3z^2$$

We are required to find the gradient of ϕ at the point (1, -2, 1).

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k}.$$

Partial derivatives

$$\frac{\partial \phi}{\partial x} = 6xy,$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - 3y^2z^2,$$

$$\frac{\partial \phi}{\partial z} = -2y^3z.$$

Therefore,

$$\vec{\nabla}\phi = 6xy\ \hat{\imath} + (3x^2 - 3y^2z^2)\ \hat{\jmath} + (-2y^3z)\ \hat{k}.$$

Substituting (x, y, z) = (1, -2, 1), we have,

$$\vec{\nabla}\phi(1, -2, 1) = -12\ \hat{\imath}\ - 9\ \hat{\jmath} +\ 16\ \hat{k} = (-12,\ -9,\ 16).$$

Example 2:

Evaluate $\vec{\nabla}e^{r^2}$ where $r^2 = x^2 + y^2 + z^2$.

Solution:

We know that

$$\vec{\nabla}e^{r^2} = \frac{\partial(e^{r^2})}{\partial x}\hat{\imath} + \frac{\partial(e^{r^2})}{\partial y}\hat{\jmath} + \frac{\partial(e^{r^2})}{\partial z}\hat{k}.$$

Partial differentiations are,

$$\frac{\partial e^{r^2}}{\partial x} = \frac{\partial e^{(x^2+y^2+z^2)}}{\partial x} = 2xe^{(x^2+y^2+z^2)} = 2xe^{r^2},$$

$$\frac{\partial e^{r^2}}{\partial y} = 2ye^{r^2},$$

$$\frac{\partial e^{r^2}}{\partial z} = 2ze^{r^2}.$$

Thus,

$$\vec{\nabla}e^{r^2} = \frac{\partial(e^{r^2})}{\partial x}\hat{\imath} + \frac{\partial(e^{r^2})}{\partial y}\hat{\jmath} + \frac{\partial(e^{r^2})}{\partial z}\hat{k}.$$

Therefore,

$$\vec{\nabla}e^{r^2} = 2e^{r^2}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right).$$

Example 3:

Find a unit normal vector to the surface $\boldsymbol{x^3} + \boldsymbol{y^3} + \boldsymbol{3xyz} = \boldsymbol{3}$ at the point $(\boldsymbol{1}, \boldsymbol{2}, -\boldsymbol{1})$

Solution:

Let $\phi(x, y, z) = x^3 + y^3 + 3xyz - 3$

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \hat{\imath}(3x^2 + 3yz) + \hat{\jmath}(3y^2 + 3xz) + \hat{k}(3xy).$$

At the point $(1, 2, -1)$,

$$\vec{\nabla}\phi = -3\hat{\imath} + 9\hat{\jmath} + 6\hat{k}.$$

Therefore the unit vector is

$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{-3\hat{\imath} + 9\hat{\jmath} + 6\hat{k}}{\sqrt{126}}.$$

Exercise:

1. Find a unit normal vector to the surface $x^2y + 3xz^2 = 8$ at $(1,0,2)$.

   Ans: $\hat{n} = \frac{1}{17}\left(12\,\hat{\imath} + \hat{\jmath} + 12\,\hat{k}\right)$.

2. Find the unit normal to the surface $x^2 + xy + y^2 + xyz$ at $(1, -2, 1)$.

   Ans: $\hat{n} = -\frac{1}{\sqrt{3}}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)$.

Directional Derivative

The directional derivative of a scalar point function $\phi(x, y, z)$ in the direction of the vector $\vec{a}$ ($\hat{a}$ is the unit vector along) is

$$D_{\vec{a}}\phi = \vec{\nabla}\phi \cdot \hat{a}.$$

Now by the definition of dot product, we have

$$|D_{\vec{a}}\phi| = |\vec{\nabla}\phi|\,|\hat{a}|\cos\theta = |\vec{\nabla}\phi|\cos\theta,$$

where θ is the angle between the gradient vector $\vec{\nabla}\phi$ and the direction vector $\hat{a}$.

Since $\cos\theta$ has its maximum value 1 when $\theta = 0$, the maximum value of the directional derivative is

$$(D_{\vec{a}}\phi)_{max} = |\vec{\nabla}\phi|.$$

Hence, the directional derivative of ϕ is maximum in the direction of the gradient vector $\vec{\nabla}\phi$, and its maximum value equals the magnitude of the gradient.

Example 1:

Find the directional derivative of $\phi = x^2y + yz^3$ at the point $(1, -2, 1)$ in the direction of the vector $\vec{a} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$.

Solution:

The gradient of ϕ is,

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= 2xy\hat{\imath} + (x^2 + z^3)\hat{\jmath} + 3yz^2\hat{k}.$$

At the point (1,-2,1),

$$\vec{\nabla}\phi = (-4,\ 2,\ -6).$$

The unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{\boldsymbol{a}}|} = \frac{2\hat{\imath} - 1\hat{\jmath} + 2\hat{k}}{\sqrt{(2)^2 + (-1)^2 + (2)^2}}.$$

Therefore,

$$\hat{\boldsymbol{a}} = \frac{1}{3}(2,\ -1,\ 2).$$

Hence, the directional derivative of ϕ in the direction of $\vec{\boldsymbol{a}}$ is

$$D_{\vec{\boldsymbol{a}}}\phi = \vec{\nabla}\phi \cdot \hat{a} = (-4,\ 2,\ -6) \cdot \frac{1}{3}(2,\ -1,\ 2)$$

$$= \frac{1}{3}[-8-2-12] = -\frac{22}{3}.$$

Therefore,

$$\boxed{D_{\vec{a}}\phi(1,-2,1) = -\frac{22}{3}.}$$

Example 2:

Find the directional derivative of $\phi = 6x^2y + 24y^2z - 8z^2x$ at $(1,1,1)$ in the direction of $\vec{v} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}$. Hence, find the maximum value.

Solution:

$$\vec{\nabla}\phi = (12xy - 8z^2)\hat{\imath} + (6x^2 + 48yz)\hat{\jmath} + (24y^2 - 16zx)\hat{k}.$$

At $(1,1,1)$,

$$\vec{\nabla}\phi = 4\hat{\imath} + 54\hat{\jmath} + 8\hat{k}.$$

Maximum value of the directional derivative $= ||\ \vec{\nabla}\phi\ ||$.

$$||\vec{\nabla}\phi|| = \sqrt{4^2 + 54^2 + 8^2} = \sqrt{16 + 2916 + 64} = \sqrt{2996}.$$

Therefore

$$\max D\,\phi = \sqrt{2996} \approx 54.74.$$

Exercise:

1. Find the directional derivative of $\phi(x,y,z) = xy^2 + yz^3$ at $P(2,-1,1)$ in the direction of PQ, where $Q = (3,1,3)$. Ans: $D_{\overrightarrow{PQ}}\phi(2,-1,1) = -\frac{11}{3}$.

2. Find the directional derivative of $\phi = xyz$ in the direction of the outer upward normal to the surface $z = xy$ at $(3,1,3)$. Ans: $D_{\hat{n}}\phi(3,1,3) = -\frac{27}{\sqrt{11}}$.

3. Find the directional derivative of $\phi = xy + yz + zx$ at $(1,2,0)$ in the direction of $\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$. Ans: $D_{\hat{a}}\phi(1,2,0) = \frac{10}{3}$.

Vector Field

A *vector-valued function* is a function whose output is a vector. It may depend on one or more scalar variables. A vector field is a vector-valued function of position, i.e., it assigns a vector to every point in space.

If each point (x, y, z) has a vector

$$\vec{F}(x,y,z) = F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{k}$$

then $\vec{F}$ is called a vector field.

Remark: Every vector field is a type of vector-valued function, but not every vector-valued function is a vector field. Only those defined over spatial coordinates (x, y, z) represent vector fields.

Differentiable vector field

$\vec{F}: D \to R^3$ is called differentiable if each of its component functions is differentiable.

That is, if

$$\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\imath} + F_2(x,y,z)\,\hat{\jmath} + F_3(x,y,z)\,\hat{k},$$

then $\vec{F}$ is differentiable if $F_1,\ F_2,\ \text{and}, F_1$ are differentiable functions on D.

Divergence of a Vector Field

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$. Then

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Note: If $\vec{\nabla}\cdot\vec{F} = 0$, then $\vec{F}$ is called $solenoidal$ or incompressible.

Example 1:

If $\vec{F} = x^2z\hat{\imath} - 2y^3z^3\hat{\jmath} + xy^2z\hat{k}$, find the divergence at $(1,-1,1)$.

Solution:

We have

$$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$$

with

$$F_1 = x^2z, \quad F_2 = -2y^3z^3, \quad F_3 = xy^2z.$$

The divergence is

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Now, $\frac{\partial F_1}{\partial x} = \frac{\partial}{\partial x}(x^2z) = 2xz, \quad \frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}(-2y^3z^3) = -6y^2z^3, \quad \frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}(xy^2z) = xy^2.$

Thus,

$$\vec{\nabla}\cdot\vec{F} = 2xz - 6y^2z^3 + xy^2.$$

At the point (1,-1,1): $\vec{\nabla}\cdot\vec{F} = 2(1)(1) - 6(1)(1) + 1(1) = 2 - 6 + 1 = -3.$

Hence

$$\vec{\nabla} \cdot \vec{F} = -3.$$

Example 2:

Show that $\vec{A} = 3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} - 3x^2y^2\hat{k}$ is solenoidal.

Solution:

We have

$$A_1 = 3y^4z^2, \quad A_2 = 4x^3z^2, \quad A_3 = -3x^2y^2.$$

Now,

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}.$$

$$\frac{\partial A_1}{\partial x} = 0, \quad \frac{\partial A_2}{\partial y} = 0, \quad \frac{\partial A_3}{\partial z} = 0.$$

Hence,

$$\vec{\nabla} \cdot \vec{A} = 0.$$

$$\boxed{\vec{A} \text{ is solenoidal.}}$$

Examples for Practice:

1. Determine the constant a such that $\vec{A} = (ax^2y + yz)\hat{\imath} + (xy^2 + xz^2)\hat{\jmath} + (2xyz - 2x^2y^2)\hat{k}$ is solenoidal. Ans: $a = -2$.
2. Find div $\vec{F}$, where $\vec{F} = \vec{\nabla}(x^3 + y^3 + z^3 - 3xyz)$. Ans: div $\vec{F} = 6(x + y + z)$.
3. If $\vec{F} = xy^2\hat{\imath} + 2x^2yz\hat{\jmath} - 3yz^2\hat{k}$, find $\vec{\nabla} \cdot F$ at $(1, -1, 1)$. Ans: $\left(\vec{\nabla} \cdot \vec{F}\right)_{(1,-1,1)} = 9$.
4. If $\vec{F} = (x^2 - y^2 + 2xz)\hat{\imath} + (xz - xy + yz)\hat{\jmath} - (z^2 + x^2)\hat{k}$, then find $\vec{\nabla} \cdot \vec{F}$. Ans: $\vec{\nabla} \cdot \vec{F} = x + z$.
5. If $\vec{F} = \sin x\ \hat{\imath} + \cos y\ \hat{\jmath} + z^2\hat{k}$, compute $\vec{\nabla} \cdot F$ at $(\pi/2, 0, 1)$. Ans: $\left(\vec{\nabla} \cdot \vec{F}\right)_{\left(\frac{\pi}{2}, 0, 1\right)} = 2$.

Curl

Let $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$. Then

$$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F} = det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}.$$

Note: If $\vec{\nabla} \times \vec{F} = \vec{0}$, then $\vec{F}$ is call $\boldsymbol{irrotational}$ or conservative.

Example 1:

If $\vec{F} = xz^3\hat{\imath} - 2x^2yz\hat{\jmath} + 2yz^4\hat{k}$, then find curl at $(1,-1,1)$.

Solution:

We have

$$F_1 = xz^3, \quad F_2 = -2x^2yz, \quad F_3 = 2yz^4.$$

The curl of the function $\vec{F}$ is defined by

$$Curl\ \vec{F} = det\begin{pmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \dfrac{\partial}{\partial\, x} & \dfrac{\partial}{\partial\, y} & \dfrac{\partial}{\partial\, z} \\ \mathrm{F}_1 & \mathrm{F}_2 & \mathrm{F}_3 \end{pmatrix} = det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial\, x} & \dfrac{\partial}{\partial\, y} & \dfrac{\partial}{\partial\, z} \\ xz^3 & -2x^2yz & 2yz^4 \end{pmatrix}$$

Expanding,

$$\vec{\nabla} \times \vec{F} = \hat{\imath}\left(\frac{\partial(2yz^4)}{\partial y} - \frac{\partial(-2x^2yz)}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial(2yz^4)}{\partial x} - \frac{\partial(xz^3)}{\partial z}\right) + \hat{k}\left(\frac{\partial(-2x^2yz)}{\partial x} - \frac{\partial(xz^3)}{\partial y}\right)$$

Now compute each term

$$\frac{\partial}{\partial y}(2yz^4) = 2z^4, \quad \frac{\partial}{\partial z}(-2x^2yz) = -2x^2y, \quad \frac{\partial}{\partial x}(2yz^4) = 0,$$

$$\frac{\partial}{\partial z}(xz^3) = 3xz^2, \quad \frac{\partial}{\partial x}(-2x^2yz) = -4xyz, \quad \frac{\partial}{\partial y}(xz^3) = 0.$$

Thus,

$$\text{Curl}\ \vec{F} = \vec{\nabla} \times \vec{F} = \hat{\imath}(2z^4 + 2x^2y) + \hat{\jmath}(3xz^2) + \hat{k}(-4xyz).$$

Hence, Curl $\vec{F}\ (1,-1,1) = 3\hat{\jmath} + 4\hat{k}$.

Example 2:

Show that $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is irrotational.

Solution:

We have

$$F_1 = x, \quad F_2 = y, \quad F_3 = z.$$

Now,

$$\vec{\nabla} \times \vec{r} = det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial\, x} & \dfrac{\partial}{\partial\, y} & \dfrac{\partial}{\partial\, z} \\ x & y & z \end{pmatrix}$$

$$= \hat{\imath}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = \vec{0}$$

Hence,

$$\vec{\nabla} \times \vec{r} = \vec{0}.$$

Therefore, $\vec{r}$ is irrotational.

Example 3:

Find the curl of $\vec{A} = e^{xyz}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)$ at (1,2,3).

Solution:

$$\text{curl of } \vec{A} = \vec{\nabla} \times \vec{A} = det\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{pmatrix}$$

$$= e^{xyz}\left[x(z-y)\hat{\imath} - y(z-x)\hat{\jmath} + z(y-x)\hat{k}\right].$$

At (1,2,3),

$$\vec{\nabla} \times \vec{A} = e^{6}\left(\hat{\imath} - 4\hat{\jmath} + 3\hat{k}\right).$$

Examples for Practice:

1. If $\vec{F} = xy^2\hat{\imath} + 2x^2yz\hat{\jmath} - 3yz^2\hat{k}$, find $\vec{\nabla} \times \vec{F}$ at $(1,-1,1)$. Ans: $-\hat{\imath} - 2\hat{k}$.
2. If $\vec{F} = (x^2 - y^2 + 2xz)\hat{\imath} + (xz - xy + yz)\hat{\jmath} - (z^2 + x^2)\hat{k}$, then find $\vec{\nabla} \times \vec{F}$.

   Ans: $(-x - y)\,\hat{\imath} + 4x\,\hat{\jmath} + (y + z)\,\hat{k}$.
3. Find div(grad ϕ) and curl(grad ϕ) at (1,1,1) for $\phi = x^2y^3z^4$.

   Ans: $\text{div}(\text{grad}\,\phi)_{(1,1,1)} = 20,\ \ \text{curl}(\text{grad}\,\phi)_{(1,1,1)} = \vec{0}$.

Parametrisation of Curves and Surfaces:

A parametrisation expresses a curve or a surface in terms of one or more parameters. A vector-valued function of a single parameter t can represent a curve in space:

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}, \quad t \in I,$$

where ($x(t), y(t), z(t)$) are continuous functions defining the coordinates of the curve.

A surface in space can be represented by a vector-valued function of two parameters u and v:

$$\vec{r}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}, \quad (u,v) \in D,$$

where ($x(u,v),\ y(u,v),\ z(u,v)$) define the coordinates of each point on the surface.

Example 1:

Find the parametrisation of the circle $x^2 + y^2 = 9$ in the xy-plane.

Solution:

Let the parameter be t (angle in radians).

$$x = 3\cos t, \qquad y = 3\sin t, \qquad z = 0$$

Hence, the vector form of the curve is

$$\vec{r}(t) = 3\cos t\ \hat{\imath} + 3\sin t\ \hat{\jmath} + 0\,\hat{k}, \qquad 0 \le t \le 2\pi.$$

Example 2:

Find the parametrisation of the line joining the points A(1,2,3) and B(4,5,6).

Solution:

Let t vary from 0 to 1. The vector equation of a line joining two points is

$$\vec{r}(t) = \overrightarrow{r_A} + t(\overrightarrow{r_B} - \overrightarrow{r_A})$$

Substituting $\overrightarrow{r_A} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\overrightarrow{r_B} = 4\hat{\imath} + 5\hat{\jmath} + 6\hat{k}$, we get

$$\vec{r}(t) = (1 + 3t)\hat{\imath} + (2 + 3t)\hat{\jmath} + (3 + 3t)\hat{k}, \qquad 0 \le t \le 1.$$

Example 3:

Find a parametrisation of the cylinder $x^2 + y^2 = 9$.

Solution:

Let the height along the z-axis be the parameter v, and let the circular cross-section be described by the angle u,

$x = 3\cos u, \qquad y = 3\sin u, \qquad z = v.$

Hence, the vector form of the surface is

$$\vec{r}(u, v) = 3\cos u\ \hat{\imath} + 3\sin u\ \hat{\jmath} + v\,\hat{k},$$

Where, $0 \le u < 2\pi$ and $v \in R$.

Exercise:

1. Find a parametrisation of the sphere $x^2 + y^2 + z^2 = 9$.
   Ans: $\vec{r}(\theta, \phi) = 3\sin\theta\cos\phi\ \hat{\imath} + 3\sin\theta\sin\phi\ \hat{\jmath} + 3\cos\theta\ \hat{k}$

2. Find the parametrisation of the plane $x + 2y + 3z = 6$.
   Ans:: $\overrightarrow{r(s,t)} = (6 - 2s - 3t)\hat{\imath} + s\hat{\jmath} + t\hat{k}$

Line Integral

Definition:

Let $\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\imath} + F_2(x,y,z)\,\hat{\jmath} + F_3(x,y,z)\,\hat{k}$ be a vector field defined on a smooth curve C. Suppose the curve C is parametrised by

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}, \quad a \le t \le b.$$

Then the line integral of $\vec{F}$ along C is defined as

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{dr}{dt}\,dt = \int_a^b \left(F_1(x,y,z)\,\frac{dx}{dt} + F_2(x,y,z)\,\frac{dy}{dt} + F_3(x,y,z)\,\frac{dz}{dt}\right)dt.$$

A line integral is a generalisation of a definite integral along a curve C in space.

If C is a closed curve, the symbol is replaced by $\oint_C \vec{F}\cdot d\vec{r}$.

Example 1:

If $\vec{\mathrm{F}} = 3xy\hat{\imath} - y^2\hat{\jmath}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $C\!: y = 2x^2$ from (0,0) to (1,2).

Solution:

$$\vec{\mathrm{F}} = 3\mathrm{xy}\hat{\mathrm{\imath}} - \mathrm{y}^2\hat{\mathrm{\jmath}}, \quad \mathrm{C}:\ \mathrm{y} = 2\mathrm{x}^2,\ (0,0) \text{ to } (1,2).$$

Parametrise the curve by $x = t,\ y = 2t^2,\ 0 \le t \le 1,$ then

$$d\vec{r} = (\hat{\imath} + 4t\hat{\jmath})dt,$$

Along the curve,

$$\vec{F}(t) = 6t^3\hat{\imath} - 4t^4\hat{\jmath}.$$

Therefore

$$\vec{F}\cdot d\vec{r} = (6t^3 - 16t^5)\,dt.$$

Hence

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (6t^3 - 16t^5)\,dt = \left[\frac{3}{2}t^4 - \frac{8}{3}t^6\right]_0^1 = -\frac{7}{6}.$$

Example 2:

Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}$, moves a particle from (0,0) to (1,1) along $y^2 = x$.

Solution:

We have $\vec{F} = (x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}$, $C: y^2 = x$ from $(0,0)$ to $(1,1)$.

Let $y = t$, $x = t^2$, $0 \le t \le 1$. Then $\vec{r}(t) = t^2\hat{\imath} + t\hat{\jmath}$.

Therefore,

$$d\vec{r} = (2t\hat{\imath} + \hat{\jmath})\, dt.$$

Along the the given curve

$$\vec{F}(t) = (t^4)\hat{\imath} - (2t^3 + t)\hat{\jmath}.$$

Therefore

$$\vec{F} \cdot d\vec{r} = \big((t^4)(2t)\big) + \big((-2t^3 - t)(1)\big)$$
$$= 2t^5 - 2t^3 - t.$$

The line integral is,

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 2t^5 - 2t^3 - t\ dt$$
$$= \left[\frac{t^6}{3} - \frac{t^4}{2} - \frac{t^2}{2}\right]_0^1$$
$$= -\frac{2}{3}.$$

1. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (x + y)\hat{\imath} + (x - y)\hat{\jmath}$, and C is the curve $y = x^2$ from $(0,0)$ to $(1,1)$. Ans: 1
2. Find the work done by the force $\vec{F} = (x^2 + y)\hat{\imath} + (y^2 + x)\hat{\jmath}$, in moving a particle along the straight line joining (0,0) to (1,2). Ans: 5
3. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = y\hat{\imath} + x\hat{\jmath}$, and C is the upper half of the circle $x^2 + y^2 = 1$ from (-1,0) to (1,0). Ans: 0
4. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = (x^2 - y)\hat{\imath} + (y^2 + x)\hat{\jmath}$, and C is the parabola $y = 2x^2$ from (0,0) to (1,2). Ans: $\frac{11}{3}$.

Surface Integral

Definition:

An integral evaluated over a surface S is called a *surface integral*. It represents the flux of a vector field $\vec{F}$ through the surface S. If $\hat{n}$ is the unit outward normal to the surface at a point $P \in S$, then the normal component of $\vec{F}$ at P is $\vec{F} \cdot \hat{n}$. The surface integral of $\vec{F}$ over S is defined $\iint_S \vec{F} \cdot \hat{n}\, ds$, where ds denotes an infinitesimal area element of the surface.

EVALUATION OF SURFACE INTEGRAL

If R_1 be the projection of S on the xy-plane, $\hat{k}$ is the unit vector normal the xy-plane then $ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_1} \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\,\cdot\,\hat{k}|}.$$

If R_2 be the projection of S on the yz-plane, $\hat{\imath}$ is the unit vector normal the yz-plane then $ds = \frac{dz\,dy}{|\hat{n}\cdot\hat{\imath}|}$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_1} \vec{F}\cdot\hat{n}\,\frac{dz\,dy}{|\hat{n}\,\cdot\,\hat{\imath}|}$$

If R_3 be the projection of S on the xz-plane, $\hat{\jmath}$ is the unit vector normal the xz-plane then $ds = \frac{dx\,dz}{|\hat{n}\cdot\hat{\jmath}|}$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_1} \vec{F}\cdot\hat{n}\,\frac{dx\,dz}{|\hat{n}\,\cdot\,\hat{\jmath}|}$$

Example 1:

Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ if $\vec{F} = (x+y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

Solution:

Given $\vec{F} = (x+y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}$

Let $\varphi = 2x + y + 2z - 6$

$$\vec{\nabla}\varphi = \hat{\imath}\frac{\partial\varphi}{\partial x} + \hat{\jmath}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z} = 2\hat{\imath} + 1\hat{\jmath} + 2\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|} = \frac{2\hat{\imath} + 1\hat{\jmath} + 2\hat{k}}{3}$$

$$\vec{F}\cdot\hat{n} = \left[(x+y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}\right]\cdot\frac{2\hat{\imath} + 1\hat{\jmath} + 2\hat{k}}{3}$$

$$= \frac{1}{3}[2(x+y^2) - 2x + 4yz]$$

$$= \frac{2}{3}[y^2 + 2yz]$$

$$= \frac{2}{3}y[y + 2z]$$

$$= \frac{2}{3}y[y + 6 - 2x - y][\because 2z = 6 - 2x - y]$$

$$= \frac{2}{3}y[6 - 2x]$$

$$= \frac{4}{3}y[3 - x]$$

Let R be the projection of S on the xy–plane

$$\therefore ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\hat{n}\cdot\hat{k} = \left(\frac{2\hat{\imath}+1\hat{\jmath}+2\hat{k}}{3}\right)\cdot\hat{k} = \frac{2}{3}$$

Now we have

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \vec{F}\cdot\hat{n}\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_R \frac{4}{3}y[3-x]\cdot\frac{dx\,dy}{\left(\frac{2}{3}\right)}$$

$$= 2\iint_R y[3-x]\ dx\,dy$$

In $R_1(2x+y=6), x$ varies from 0 to $\frac{6-y}{2}$ and y varies from 0 to 6

$$= 2\int_0^6\int_0^{\frac{6-y}{2}} y(3-x)\,dx\,dy$$

$$= 2\int_0^6\left[3x-\frac{x^2}{2}\right]_0^{\frac{6-y}{2}} y\,dy$$

$$= 2\int_0^6 [\frac{1}{2}(18y-3y^2)-\frac{1}{8}(6-y)^2y]dy$$

$$= \frac{2}{2}\left[\frac{18y^2}{2}-\frac{3y^3}{3}-\frac{(6-y)^3}{8(3)(-1)}\right]$$

$$= \left[9(6)^2-(6)^3+\frac{1}{12}(0)\right]-\left[0-0+\frac{1}{12}.6^2\right]$$

$$= 81\ units.$$

Example 2:

Evaluate $\iint_S \boldsymbol{6xy\,ds}$ where S is the portion of the plane $\boldsymbol{x+y+z=1}$ that lies in front of yz plane.

Solution:

We are looking for a portion of the plane ABC that lies in front of the yz – plane, therefore, we write equation of the surface in the form $\boldsymbol{x=f(y,z)}$.

For the points on the surface, we have $\boldsymbol{x=1-y-z}$

$$\iint_S 6xy\ ds = \iint_S 6(1-y-z)y\sqrt{3}$$

$$= 6\sqrt{3}\int_0^1\int_0^{1-y}(1-y-z)y\,dz\,dy$$

$$= 6\sqrt{3}\int_0^1 \left[yz - y^2z - \frac{1}{2}yz^2\right]_0^{1-y} dy$$

$$= 6\sqrt{3}\left[\tfrac{1}{4}y^2 - \tfrac{1}{3}y^3 + \tfrac{1}{8}y^4\right]_0^1 = \tfrac{\sqrt{3}}{4}.$$

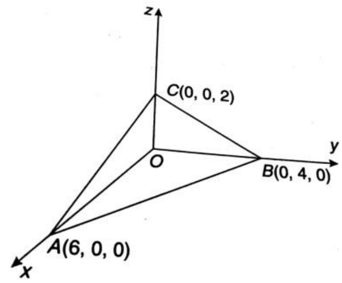
Example 3:

Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = \mathbf{18z\hat{i} - 12\hat{j} + 3y\hat{k}}$ and S is the part of the plane $\mathbf{2x + 3y + 6z = 12}$ in the first octant.

Solution:

The given surface is the plane $2x + 3y + 6z = 12$ in the first octant.

Let $\phi = 2x + 3y + 6z - 12$



$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4 + 9 + 36}}$$

$$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}$$

Let R be the projection of the plane $2x + 3y + 6z = 12$ on the xy-plane, which is a triangle OAB bounded by the lines $y = 0, x = 0$ and $2x + 3y = 12$

$$dS = \frac{dxdy}{|\hat{n}.\hat{k}|} = \frac{7}{6}dxdy$$

Along the vertical strip PQ, y varies from 0 to $\frac{12-2x}{3}$ and in the region R, x varies from 0 to 6

$$\iint_S \vec{F}\cdot\hat{n}\,ds, = \iint_R (18z\hat{i} - 12\hat{j} + 3y\hat{k}).\left(\frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}\right)\frac{7}{6}\mathrm{dxdy}$$

$$= \tfrac{1}{6}\iint_R (36z - 36 + 18y)\mathrm{dxdy}$$

$$= 3\iint_R \left[2\left(\tfrac{12-2x-3y}{6}\right) - 2 + y\right]\mathrm{dxdy}$$

$$= \int_0^6\int_0^{\frac{12-2x}{3}} (6 - 2x)\mathrm{dydx}$$

$$= 2\int_0^6 (3 - x)[y]_0^{\frac{12-2x}{3}}$$

$$= 2\int_0^6 (3 - x)\left(\tfrac{12-2x}{3}\right)dx$$

$$= \tfrac{4}{3}\int_0^6 (x^2 - 9x + 18)\mathrm{dx} = 24.$$

Exercise:

1. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$, if $\vec{F} = (x + y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}$ and S is the surface $2x + y + 2z = 6$ in the first octant. Ans: 81 units.
2. $\iint_S \vec{F} \cdot \hat{n}\ dS$, where $\vec{F} = (2x + 3y)\,\hat{\imath} - y\,\hat{\jmath} + 4z\,\hat{k}$ and S is the portion of the plan $x + 2y + 3z = 6$ in the first octant. Ans: 39 units.

VOLUME INTEGRAL:

A volume integral of a scalar or vector function over a three-dimensional region V is defined as

$$\iiint_V \varphi\ dv = \iiint_V \varphi(x, y, z)\ dx\, dy\, dz,$$

where V denotes the volume in 3-dimensional space, $\varphi(x, y, z)$ is a scalar (or vector) function defined on V, and $dv\ =\ dx\, dy\, dz$ is the elementary volume element.

For a scalar field F, the volume integral represents the total quantity of F within V: $\iiint_V F\ dv$.

Example 1:

If $\varphi = 45x^2y$ then evaluate $\iiint_v \varphi dv$ where v denote the closed region bounded by the planes $4x + 2y + z = 8, x = 0, y = 0, z = 0$.

Solution:

$$\iiint_{\mathrm{v}} \varphi dv = \int_0^2 \int_0^{4-2\mathrm{x}} \int_0^{8-4\mathrm{x}-2\mathrm{y}} 45\ \mathrm{x}^2\mathrm{y}\ \mathrm{dx\ dy\ dz}$$

$$= 45 \int_0^2 \int_0^{4-2\mathrm{x}} \mathrm{x}^2\mathrm{y}\ (8 - 4\mathrm{x} - 2\mathrm{y})\mathrm{dy\ dx}$$

$$= 45\ \int_0^2 \frac{1}{3}\mathrm{x}^2(4 - 2\mathrm{x})^3\ \mathrm{dx}\ \ = 128.$$

Example 2:

Evaluate the volume integral

$$I = \iiint_V (x^2 + y^2 + z^2)\ dV,$$

where V is the solid ball bounded by the sphere $x^2 + y^2 + z^2 = a^2$.

Solution:

Use spherical coordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

with $0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi < 2\pi$. The Jacobian is $dV = r^2 \sin\theta\ dr\, d\theta\, d$.

Since $x^2 + y^2 + z^2 = r^2$, the integral becomes

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2\ (r^2\sin\theta)\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\ d\theta \int_0^{a} r^4\ dr$$

$$= (2\pi)\left(\int_0^{\pi}\sin\theta\ d\theta\right)\left(\frac{a^5}{5}\right)$$

$$= 2\pi\cdot 2\cdot\frac{a^5}{5}$$

$$= \frac{4\pi a^5}{5}.$$

Exercise:

1. Evaluate $\int_0^1\int_0^x\int_0^{x+y} e^{2x+2y+2z}\ dz\, dy\, dx.$ Ans: $\frac{1}{64}e^8 - \frac{3}{32}e^4 + \frac{1}{8}e^2 - \frac{3}{64}.$
2. Evaluate $I = \iiint_V (x+y+z)\ dV$, where V does the cube bound the region $0\ \le x, y, z\ \le 1$.

   Ans: $\frac{3}{2}$.

GREEN'S THEOREM

Statement: If $M(x,y),\ N(x,y),\ \frac{\partial M}{\partial y},\ \frac{\partial N}{\partial x}$ be continuous everywhere in a region R of xy plane bounded by a closed curve c, then

$$\oint_c (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

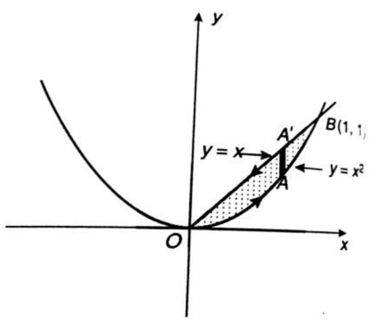
Example 1:

Verify Green's Theorem for $\oint_c Mdx + Ndy = \iint_R [(\mathrm{x}^2 - 2\mathrm{xy})\mathrm{dx} + (\mathrm{x}^2\mathrm{y} + 3)\mathrm{dy}]$ where C is the boundary of the region bounded by the parabola $\mathrm{y} = \mathrm{x}^2$ and the line $\mathrm{y} = \mathrm{x}$.

Solution:

The points of intersection of the parabola $\mathrm{y} = \mathrm{x}^2$ and the line $\mathrm{y} = \mathrm{x}$ are obtained as $\mathrm{x} = \mathrm{x}^2,\ \mathrm{x} = 0{,}1$ and $\mathrm{y} = 0{,}1$

Hence, $\mathrm{O}(0{,}0)$ and $\mathrm{B}(1{,}1)$ are the points of intersection.



$$\mathrm{M} = \mathrm{x}^2 - 2\mathrm{xy}, \mathrm{N} = \mathrm{x}^2\mathrm{y} + 3$$

$$\frac{\partial \mathrm{M}}{\partial \mathrm{y}} = -2\mathrm{x}, \qquad \frac{\partial \mathrm{N}}{\partial \mathrm{x}} = 2\mathrm{xy}$$

$\oint_C(Mdx + Ndy) = \oint_{OAB}(Mdx + Ndy) + \oint_{BO}(Mdx + Ndy)$

Along OAB: $\mathrm{y = x^2, dy = 2xdx}$

x varies from 0 to 1

$$\oint_{OAB}(Mdx + Ndy) = \oint_{OAB}[(\mathrm{x^2 - 2xy})\mathrm{dx} + (\mathrm{x^2y + 3})\mathrm{dy}]$$

$$= \int_0^1[(\mathrm{x^2 - 2x.x^2})\mathrm{dx} + (\mathrm{x^2.x^2 + 3}).\,\mathrm{2x\,dx}]$$

$$= \int_0^1(\mathrm{x^2 - 2x^3 + 2x^5 + 6x})\mathrm{dx} = \frac{19}{6}$$

Along BO: $\mathrm{y = x, dy = dx}$

x varies from $\mathrm{x = 1}$ to 0

$$\oint_{BO}(Mdx + Ndy) = \oint_{BO}[(\mathrm{x^2 - 2xy})\mathrm{dx} + (\mathrm{x^2y + 3})\mathrm{dy}]$$

$$= \int_1^0[(\mathrm{x^2 - 2x^2})\mathrm{dx} + (\mathrm{x^3 + 3})\mathrm{dx}] = -\frac{35}{12}$$

Substituting in (1), $\oint_C(Mdx + Ndy) = \frac{19}{6} - \frac{35}{12} = \frac{1}{4}$

Let R be the region bounded by the line $\mathrm{y = x}$ and the parabola $\mathrm{y = x^2}$

Along the vertical strip AA', y varies from $\mathrm{x^2}$ to x and in the region R, x varies from 0 to 1

$$\iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \int_0^1\int_{\mathrm{x^2}}^{\mathrm{x}}(\mathrm{2xy + 2x})\mathrm{dy\,dx}$$

$$= \int_0^1 \left|\mathrm{xy^2 + 2xy}\right|_{\mathrm{x^2}}^{\mathrm{x}}$$

$$= \int_0^1(\mathrm{x^3 + 2x^2 - x^5 - 2x^3})\mathrm{dx} = \frac{1}{4}$$

We have,

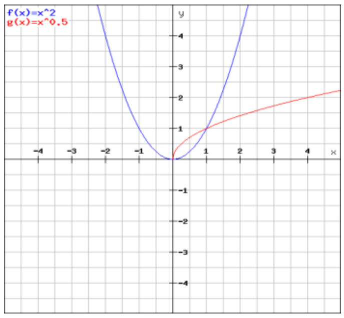
$$\oint_C(Mdx + Ndy) = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \frac{1}{4}$$

Hence, Green's theorem is verified.

Example 2:

Using Green's Theorem, evaluate $\oint_C(\mathrm{3x^2 - 8y^2})\mathrm{dx} + (\mathrm{4y - 6xy})\mathrm{dy}$ where C is the boundary of the region bounded by $\mathrm{y^2 = x}$ and $\mathrm{y = x^2}$.

Solution:



$y^2 = x$ and $y = x^2$ are two parabolas intersecting at (0,0) and (1,1)

Here, $M = 3x^2 - 8y^2, N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y, \qquad \frac{\partial N}{\partial x} = -6y$$

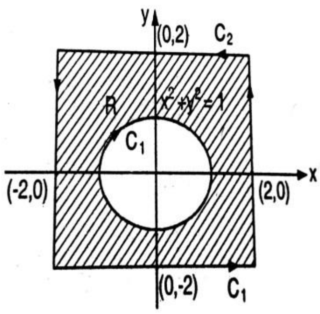
$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y \, dy \, dx$$

$$= \int_0^1 5(y^2)_{x^2}^{\sqrt{x}} \, dx$$

$$= 5\int_0^1 (x - x^4) dx = \frac{3}{2}$$

Example 3:



Evaluate $\oint_C \left[-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right]$, where $C = C_1 \cup C_2$, with $C_1: x^2 + y^2 = 1$ and $C_2: x = \pm 2, y = \pm 2$.

Solution:

Here, $M = -\frac{y}{x^2+y^2}, N = \frac{x}{x^2+y^2}$

$$\frac{\partial M}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad \frac{\partial N}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Which are continuous on the region R bounded by C.

$$\oint_c (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$\oint_C \left[-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right] = \iint_R \left(\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right) dxdy = 0$$

Example 4:

State Green's Theorem and use it to find the work done by $\vec{F} = (4x - 2y)\vec{\imath} + (2x - 4y)\vec{\jmath}$ in moving a particle once clockwise around the circle $(x-2)^2 + (y-2)^2 = 4$

Solution:

$$\vec{F}.d\vec{r} = (4x - 2y)dx + (2x - 4y)dy$$

$$= Mdx + Ndy$$

$$M = 4x - 2y, \qquad N = 2x - 4y$$

$$\frac{\partial M}{\partial y} = -2, \quad \frac{\partial N}{\partial x} = 2$$

By Green's Theorem, $\oint_C \vec{\mathrm{F}}.d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$= \iint_R (2+2)dxdy = 4\iint_R dxdy = 4(4\pi)$$

$$= 16\pi\ (anti\ clockwise)$$

But the motion is **clockwise**, i.e., the negative orientation.

Therefore,

$$\oint_C \vec{\mathrm{F}}.d\vec{r} = -16\pi.$$

Exercise:

1. Evaluate using Green's Theorem:

$$\oint_C (y^2 + 3x)\ dx + (2xy + 4y)\, dy,$$

   where C is the boundary of the region enclosed by $y = x^2$ and $y = 4$.

   Ans: 0

2. Use Green's Theorem to find the area enclosed by the curve

$$x = a\cos^3 t, \qquad y = a\sin^3 t, \qquad 0 \le t \le 2\pi.$$

   Ans: $\frac{3\pi a^2}{8}$

3. Verify Green's Theorem for

$$F = (x^2 - y^2)\hat{\imath} + 2xy\,\hat{\jmath}$$

   around the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$. Ans: $\frac{2}{3}$

4. Apply Green's Theorem to evaluate

$$\oint_C (x^2 - y^2)\ dx + 2xy\, dy,$$

   where C is the circle $x^2 + y^2 = 4$ oriented counterclockwise. Ans: 0

5. Use Green's Theorem to find the work done by the force field

$$F = (2x - y)i + (x + 3y)j$$

   in moving a particle once around the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ counterclockwise. Ans: 12π

DIVERGENCE THEOREM: (Convert surface integral to volume integral)

Statement: If $\bar{F}$ be a vector point function having continuous partial derivatives in the region bounded by a closed surface S, then

$$\iint_s \vec{F}.\hat{n}\, ds = \iiint_v div\, \vec{F}\ \ dv.$$

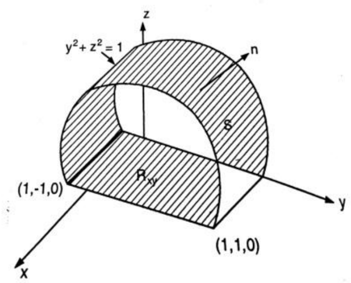
Example 1:

Find the flux of $\vec{F} = yz\hat{\jmath} + z^2\hat{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1, z \geq 0$ by the plane $x = 0$ & $x = 1$.

Solution:

Let $g(x, y, z) = y^2 + z^2$. The gradient of the fuction is

$$\vec{\nabla}g = 2y\hat{\jmath} + 2z\hat{k}.$$



The outward normal field on S is calculated from the gradient of $g(x, y, z) = y^2 + z^2$ to be

$$\hat{n} = \frac{\vec{\nabla}g}{|\vec{\nabla}g|} = \frac{2y\hat{\jmath} + 2z\hat{k}}{\sqrt{4y^2 + 4z^2}}$$

$$= \frac{2y\hat{\jmath} + 2z\hat{k}}{2\sqrt{1}} = y\hat{\jmath} + z\hat{k}.$$

So projecting the surface onto the xy-**plane** gives

$$dS = \frac{\vec{\nabla}g}{|\vec{\nabla}g.\hat{k}|}\ dx\, dy$$

$$= \frac{2}{|2z|} dx\, dy = \frac{1}{z}\ dx\, dy.$$

Since $z \geq 0$ on S

$$\vec{F}.\hat{n} = \left(yz\hat{\jmath} + z^2\hat{k}\right).\left(y\hat{\jmath} + z\hat{k}\right)$$

$$= y^2z + z^3$$

$$= z(y^2 + z^2) = z$$

Therefore, the flux F outward through S is

$$\iint_s \vec{F}.\hat{n}\,ds == \int_{x=0}^{1}\int_{y=-1}^{1} 1\,dy\,dx$$

$$= 2.$$

Example 2:

Find the flux of $\vec{F} = 4xz\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1, y = 1, z = 1$

Solution:

Here $\vec{F} = 4xz\hat{\imath} - y^2\hat{\jmath} + yz\hat{k}$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y$$

$$\therefore \vec{\nabla}\cdot\vec{F} = 4z - y$$

Over the interior of the cube:

$$\text{Flux} = \iint \vec{F}.\hat{n}\,\text{ds}$$

$$= \iint \vec{\nabla}\cdot\vec{F}\,\text{dV}$$

$$= \int_0^1\int_0^1\int_0^1 (4z - y)\,\text{dxdydz}$$

$$= \int_0^1\int_0^1 (4z - y)\,dy\,dz$$

$$= \int_0^1 \left[4zy - \frac{1}{2}y^2\right]_{y=0}^{1} dz$$

$$= \int_0^1 \left(4z - \frac{1}{2}\right) dz$$

$$= \left[2z^2 - \frac{1}{2}z\right]_0^1$$

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2}.$$

Exercise:

1. Evaluate $\iiint_v div\,\vec{F}\ dv$, where $\vec{F} = (x + y)\hat{\imath} + yz\hat{\jmath} + z\hat{k}$, through the surface of the cube $0 \le x, y, z \le 1.$ Ans: $\frac{5}{2}$

2. Find the flux of $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ outward through the sphere $x^2 + y^2 + z^2 = a^2$.

   Ans: $4\pi a^3$.

3. Evaluate $\iint \vec{\nabla} \cdot \vec{F}\ \mathrm{dV}$ for $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$ through the unit cube $0 \le x, y, z \le 1$.

   Ans: 0.

4. Evaluate $\iint \vec{F}.\hat{n}$ ds, where $\vec{F} = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$, through the sphere $x^2 + y^2 + z^2 = R^2$.

   Ans: 0.

STOKE'S THEOREM:

If s is an open two-sided surface bounded by a closed non-intersecting curve, and if a vector function $F(x, y, z)$ has continuous first partial derivatives in a domain in a space containing s. Then

$$\oint_C \vec{F}.d\vec{r} = \iint_S (curl\ \vec{F}).\hat{n}\ ds = \iint_S (\vec{\nabla} \times \vec{F})\ ds$$

Where c is described in a positive (anti-clockwise) direction and $\hat{n}$ is a unit positive (outward drawn) normal to s.

Example 1:

Verify Stokes' theorem for $\vec{\mathrm{A}} = (2\mathrm{x} - \mathrm{y})\hat{\imath} - \mathrm{yz}^2\hat{\jmath} - \mathrm{y}^2\mathrm{z}\hat{k}$, where S is the upper half surface of the sphere $\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2 = 1$ and C is its boundary.

Solution:

The boundary C of S is a circle in the xy-plane of radius unity and centre at the origin. Let $x = \cos t, y = \sin t,\ z = 0, 0 \le t \le 2\pi$ be the parametric equations of C

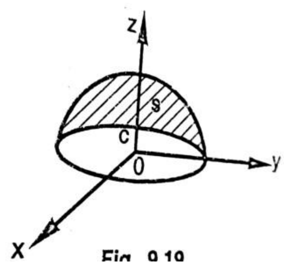


Fig 9.10

Then, $\oint_C \vec{\mathrm{A}}.\mathrm{d}\vec{\mathrm{r}} = \oint_C [(2\mathrm{x} - \mathrm{y})\mathrm{dx} - \mathrm{yz}^2\mathrm{dy} - \mathrm{y}^2\mathrm{zdz}]$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt$$

$$= \int_0^{2\pi} (-2\sin t \cos t + \sin^2 t)dt \quad = \pi$$

Also, $\vec{\nabla} \times \vec{\mathrm{A}} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}} \\ 2\mathrm{x} - \mathrm{y} & -\mathrm{yz}^2 & -\mathrm{y}^2\mathrm{z} \end{vmatrix} = \hat{k}$

$$\mathrm{curl}\,\vec{\mathrm{A}} \cdot \hat{n} = \hat{k} \cdot \hat{k} = 1$$

$\iint_S (curl\ \vec{A}).\hat{n}\ dS = \iint_R dx\ dy$ where R is the projection of S on the xy-plane

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx$$

$$= \int_{-1}^{1} 2\sqrt{1-x^2}\, dx$$

$$= 4 \int_0^1 \sqrt{1-x^2} dx = \pi$$

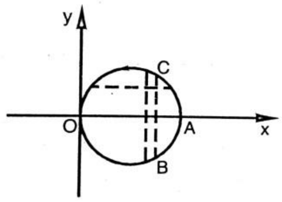
Hence, Stoke's Theorem is verified.

Example 2:

Evaluate $\iint_R (\vec{\nabla} \times \vec{F}).dS$ taken over the portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the bounding curve in the plane z=0 and $\vec{F} = (y^2 + z^2 - x^2)\hat{\imath} + (z^2 + x^2 - y^2)\hat{\jmath} + (x^2 + y^2 - z^2)\hat{k}$

Solution:

The given surface meets the plane $z = 0$ in the circle $x^2 + y^2 - 2ax = 0, z = 0$



$$\vec{F} = (y^2 + z^2 - x^2)\hat{\imath} + (z^2 + x^2 - y^2)\hat{\jmath} + (x^2 + y^2 - z^2)\hat{k}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 - x^2 & z^2 + x^2 - y^2 & x^2 + y^2 - z^2 \end{vmatrix}$$

$$= (2y - 2z)\hat{\imath} + (2z - 2x)\hat{\jmath} + (2x - 2y)\hat{k}$$

The surface integral of $\vec{\nabla} \times \vec{F}$ over the given surface is the same as the surface integral of $\vec{\nabla} \times \vec{F}$over the area of the circle $x^2 + y^2 - 2ax = 0, z = 0$

$$d\vec{S} = \hat{n}dS = \hat{k}dxdy$$

$$\iint_R (\vec{\nabla} \times \vec{F}).dS = \int_0^{2a} \int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} (2x - 2y) dy\, dx$$

$$= \int_0^{2a} \int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} 2x dy\, dx - \int_0^{2a} \int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} 2y\, dy\, dx$$

$$= 2 \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} 2x\, dy\, dx$$

$$= 4 \int_0^{2a} [y]_0^{\sqrt{2ax-x^2}} x dx = 4 \int_0^{2a} \sqrt{2ax - x^2}\, x dx$$

$$= 4\int_0^{2a} x\sqrt{a^2-(x-a)^2}\ dx.$$

Let

$$t = \frac{x-a}{a}, \qquad x = a(1+t), \qquad dx = a\,dt.$$

Whern$(x\colon 0 \to 2a), we\ get\ (t\colon -1 \to 1)$.

Also

$$\sqrt{a^2-(x-a)^2} = a\sqrt{1-t^2}.$$

Thus,

$$I = 4\int_{-1}^{1} a(1+t)\cdot a\sqrt{1-t^2}\cdot a\,dt$$

$$= 4a^3\int_{-1}^{1} (1+t)\sqrt{1-t^2}\ dt.$$

$$= 4a^3\int_{-1}^{1} \sqrt{1-t^2}\ dt + 4a^3\int_{-1}^{1} t\sqrt{1-t^2}\ dt.$$

$$= 4a^3\int_{-1}^{1} \sqrt{1-t^2}\ dt = 2a^3\pi.$$

Exercise:

1. Verify the Stokes' theorem for $\vec{F} = (x^2+y^2)\hat{\imath} + 2xy\hat{\jmath}$ in the rectangular region in the xy-plane given by $(0,0)$, $(a,0)$, $(0,b)$, (a,b). Ans: 0.
2. Verify the Stokes' theorem for $\vec{F} = (2x-y)\hat{\imath} + yz^2\hat{\jmath} - y^2z\hat{k}$ over the upper half of the surface of $x^2+y^2+z^2 = 1$ bounded by its projection on the xy −plane.
   Ans: π.
3. Verify Stokes' theorem for the vector field $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$ where S is the surface of the triangle in the plane $x+y+z = 1$ bounded by the coordinate axes.
   Ans:$-\frac{3}{2}$