CS 228 : Logic in Computer Science

Krishna. S

Recap

- Syntax of Propositional Logic
- Encoding puzzles as formulae
- Natural deduction

A Proof Engine for Natural Deduction

- If it rains, Tia is outside and does not have any raingear with her, she will get wet. $\varphi = (\mathit{R} \wedge \mathit{TiaOut} \wedge \neg \mathit{RG}) \rightarrow \mathit{TiaWet}$
- It is raining, and Tia is outside, and is not wet. $\psi = (\mathit{R} \wedge \mathit{TiaOut} \wedge \neg \mathit{TiaWet})$
- So, Tia has her rain gear with her. *RG*
- Thus, $\chi = \varphi \wedge \psi \rightarrow \mathit{RG}$.
- Given φ, ψ, can we “prove” *RG*?

A Proof Engine

- Given a formula φ in propositional logic, how to "prove" φ if φ is valid?
- What is a proof engine?
- Show that this proof engine is sound and complete
  - Completeness: Any fact that can be captured using propositional logic can be proved by the proof engine
  - Soundness: Any formula that is proved to be valid by the proof engine is indeed valid

Natural Deduction

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- $\varphi_1, \ldots, \varphi_n \vdash \psi$: This is called a sequent. $\varphi_1, \ldots, \varphi_n$ are premises, and ψ, the conclusion.
- Given $\varphi_1, \ldots, \varphi_n$, we can deduce or prove ψ. What was the sequent in Tia's case?

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- For example, $\neg p \to q, q \to r, \neg r \vdash p$ is a sequent. How do you prove this?

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- In natural deduction, we have a collection of proof rules
- These proof rules allow us to infer formulae from some given formulae
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- Given $\varphi_1, \ldots, \varphi_n$, we can deduce or prove ψ. What was the sequent in Tia's case?
- For example, $\neg p \rightarrow q, q \rightarrow r, \neg r \vdash p$ is a sequent. How do you prove this?
- Proof rules to be carefully chosen, for instance you should not end up proving something like $p \wedge q \vdash \neg q$

The Rules of the Proof Engine

Rules for Natural Deduction

The and introduction rule denoted $\wedge i$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}$$

Rules for Natural Deduction

The and elimination rule denoted $\wedge e_1$

$$\frac{\varphi \wedge \psi}{\varphi}$$

The and elimination rule denoted $\wedge e_2$

$$\frac{\varphi \wedge \psi}{\psi}$$

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.		

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.	r	premise
3.		

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- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$ 1
4.		

A first proof using $\wedge i$, $\wedge e_1$, $\wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$ 1
4.	$q \wedge r$	$\wedge i$ 3,2

Rules for Natural Deduction

The rule of double negation elimination $\neg\neg e$

$$\frac{\neg\neg\varphi}{\varphi}$$

The rule of double negation introduction $\neg\neg i$

$$\frac{\varphi}{\neg\neg\varphi}$$

Rules for Natural Deduction

The implies elimination rule or Modus Ponens MP

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.		

Another Proof

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Another Proof

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1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
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Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.		

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.		

Another Proof

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1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.		

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.	r	$\neg\neg e$ 6

Rules for Natural Deduction

Another implies elimination rule or Modus Tollens MT

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}$$

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.		

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.		

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2
4.		

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2
4.	$\neg p$	MT 1,3

More Rules

- Thanks to MT, we have $p \to q, \neg q \vdash \neg p$.

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- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?

More Rules

- Thanks to MT, we have $p \to q, \neg q \vdash \neg p$.
- Can we prove $p \to q \vdash \neg q \to \neg p$?
- So far, no proof rule that can do this.
- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?
- Yes, using MT.

The implies introduction rule $\rightarrow i$

- $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1.	$p \rightarrow q$	premise
2.	$\neg q$	assumption
3.	$\neg p$	MT 1,2
4.	$\neg q \rightarrow \neg p$	$\rightarrow i$ 2-3

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	*true*	premise
2.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	*true*	premise
2.	$q \rightarrow r$	assumption
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More on $\rightarrow i$

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1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
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1.	$true$	premise
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3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.		

More on $\rightarrow i$

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1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	*true*	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.		

More on $\rightarrow i$

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1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8

More on $\rightarrow i$

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1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8
10.	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	$\rightarrow i$ 3-9
11.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8
10.	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	$\rightarrow i$ 3-9
11.	$(q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$	$\rightarrow i$ 2-10

Transforming Proofs

- $(q \rightarrow r), (\neg q \rightarrow \neg p), p \vdash r$
- Transform any proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ to $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi) \ldots))$ by adding n lines of the rule $\rightarrow i$

More Examples

- $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r)$	premise
2.		

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More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.		

More Examples

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1.	$p \to (q \to r)$	premise
2.	$p \land q$	assumption
3.	p	$\land e_1$ 2
4.	q	$\land e_2$ 2
5.		

More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.	q	$\wedge e_2$ 2
5.	$q \to r$	MP 1,3
6.		

More Examples

- $p \to (q \to r) \vdash (p \land q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \land q$	assumption
3.	p	$\land e_1$ 2
4.	q	$\land e_2$ 2
5.	$q \to r$	MP 1,3
6.	r	MP 4,5
7.		

More Examples

- $p \to (q \to r) \vdash (p \wedge q) \to r$

1.	$p \to (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge e_1$ 2
4.	q	$\wedge e_2$ 2
5.	$q \to r$	MP 1,3
6.	r	MP 4,5
7.	$p \wedge q \to r$	$\to i$ 2-6

More Rules

The or introduction rule $\vee i_1$

$$\frac{\varphi}{\varphi \vee \psi}$$

The or introduction rule $\vee i_2$

$$\frac{\psi}{\varphi \vee \psi}$$

More Rules

The or elimination rule $\vee e$

$$\frac{\varphi \vee \psi \qquad \varphi \vdash \chi \qquad \psi \vdash \chi}{\chi}$$

Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

1.	$q \to r$	premise
2.		

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1.	$q \to r$	premise
2.	$p \lor q$	assumption
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Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

1.	$q \to r$	premise
2.	$p \lor q$	assumption
3.	p	$\lor e$ (1)
4.		

Or Elimination Example

- $q \to r \vdash (p \lor q) \to (p \lor r)$

1.	$q \to r$	premise
2.	$p \lor q$	assumption
3.	p	$\lor e$ (1)
4.	$p \lor r$	$\lor i_1$ 3
5.		

Or Elimination Example

- $q \rightarrow r \vdash (p \vee q) \rightarrow (p \vee r)$

1.	$q \rightarrow r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee\ e\ (1)$
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee\ e\ (2)$
6.		

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee e$ (1)
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee e$ (2)
6.	r	MP 1,5
7.		

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee\ e\ (1)$
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee\ e\ (2)$
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee\ e\ (1)$
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee\ e\ (2)$
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7

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1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee\ e$ (1)
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee\ e$ (2)
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7
9.		

Or Elimination Example

- $q \to r \vdash (p \vee q) \to (p \vee r)$

1.	$q \to r$	premise
2.	$p \vee q$	assumption
3.	p	$\vee e$ (1)
4.	$p \vee r$	$\vee i_1$ 3
5.	q	$\vee e$ (2)
6.	r	MP 1,5
7.	$p \vee r$	$\vee i_2$ 6
8.	$p \vee r$	$\vee e$ 2, 3-4, 5-7
9.	$(p \vee q) \to (p \vee r)$	$\to i$ 2-8

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
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Associativity Using Or Elimination

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1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.		

Associativity Using Or Elimination

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1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

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2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee e$ (1.2)
6.		

Associativity Using Or Elimination

- $(p \lor q) \lor r \vdash p \lor (q \lor r)$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	$\lor e$ (1)
3.	p	$\lor e$ (1.1)
4.	$p \lor (q \lor r)$	$\lor i_1$ 3
5.	q	$\lor e$ (1.2)
6.	$q \lor r$	$\lor i_1$ 5
7.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee e$ (1.2)
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee\, e\,(1)$
3.	p	$\vee\, e\,(1.1)$
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee\, e\,(1.2)$
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.		

Associativity Using Or Elimination

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2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee e$ (1.2)
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	$\vee e$ (2)
10.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

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2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee e$ (1.2)
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	$\vee e$ (2)
10.	$q \vee r$	$\vee i_2$ 9
11.		

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee e$ (1.2)
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	$\vee e$ (2)
10.	$q \vee r$	$\vee i_2$ 9
11.	$p \vee (q \vee r)$	$\vee i_2$ 10

Associativity Using Or Elimination

- $(p \vee q) \vee r \vdash p \vee (q \vee r)$

1.	$(p \vee q) \vee r$	premise
2.	$p \vee q$	$\vee e$ (1)
3.	p	$\vee e$ (1.1)
4.	$p \vee (q \vee r)$	$\vee i_1$ 3
5.	q	$\vee e$ (1.2)
6.	$q \vee r$	$\vee i_1$ 5
7.	$p \vee (q \vee r)$	$\vee i_2$ 6
8.	$p \vee (q \vee r)$	$\vee e$ 2, 3-4, 5-7
9.	r	$\vee e$ (2)
10.	$q \vee r$	$\vee i_2$ 9
11.	$p \vee (q \vee r)$	$\vee i_2$ 10
12.	$p \vee (q \vee r)$	$\vee e$ 1, 2-8, 9-11