Problem Set 1

1. There are three suspects for a murder: A, B, and C.

   - A says "I did not do it. The victim was an old acquaintance of B's. But C hated him."
   - B states "I did not do it. I did not even know the guy. Besides I was out of town all that week."
   - C says "I did not do it. I saw both A and B downtown with the victim that day; one of them must have done it."

   Assume that the two innocent men are telling the truth, but that the guilty man might not be. Who did it? Deduce the answer by encoding in propositional logic and finding a solution.

2. In an island, there are three tribes : the Knights, Knaves and Normals. The Knights always speak truth, the Knaves always lie, while the Normals lie sometimes and speak truth sometimes. On a visit to this island, I met two inhabitants A and B. A told me that B is a knight and B told me that A is a knave. Prove, using natural deduction, that one of them told the truth but is not a knight, or that one of them told a lie but is not a knave.

3. Let F, G and H be formulas and let $\mathbf{S}$ be a set of formulas. Which of the following statements are true? Justify your answer.

   (a) If F is unsatisfiable, then $\neg F$ is valid.

   (b) If $F \to G$ is satisfiable and F is satisfiable, then G is satisfiable.

   (c) $P_1 \to (P_2 \to (P_3 \to \ldots (P_n \to P_1)\ldots))$ is valid.

   (d) $S \models F$ and $S \models \neg F$ cannot both hold.

   (e) If $\mathbf{S} \models F \vee G$, $\mathbf{S} \cup \{F\} \models H$ and $\mathbf{S} \cup \{G\} \models H$, then $\mathbf{S} \models H$.

4. The Pigeon Hole Principle states that if there are $n+1$ pigeons sitting amongst n holes then there is atleast one hole with more than one pigeon sitting in it. For $i \in \{1, 2, .., n+1\}$ and $j \in \{1, 2, ..., n\}$, let the atomic proposition $P(i, j)$ indicate that the i-th pigeon is sitting in the j-th hole.

   Write out a propositional logic formula that states the Pigeon Hole Principle.

5. Prove formally $\vdash [(p \to q) \to q] \to [(q \to p) \to p]$

6. Let $\mathcal{H}$ be a given set of premises. If $\mathcal{H} \vdash (A \to B)$ and $\mathcal{H} \vdash (C \vee A)$, then show that $\mathcal{H} \vdash (B \vee C)$ where A, B, C are wffs.

7. Let $\mathcal{H}$ be a given set of premises. If $\mathcal{H} \vdash (A \to C)$ and $\mathcal{H} \vdash (B \to C)$, then show that $\mathcal{H} \vdash ((A \vee B) \to C)$. Here, A, B and C are wffs.

8. Show that a truth assignment α satisfies the wff

$$(\ldots(x_1 \leftrightarrow x_2) \leftrightarrow \ldots \leftrightarrow x_n)$$

iff $\alpha(x_i) = false$ for an even number of i's, $1 \leq i \leq n$.

9. Of the following three formulae, which tautologically imply which?

   (a) $x \leftrightarrow y$
   (b) $(\neg((x \leftrightarrow y) \rightarrow (\neg y \rightarrow x))))$
   (c) $((\neg x \vee y) \wedge (x \vee \neg y))$

10. Let $\mathcal{L}$ be a formulation of propositional logic in which the sole connectives are negation and disjunction. The rules of natural deduction corresponding to disjunction and negation (also includes double negation) are available. For any wffs A, B and C, let $\neg(A \vee B) \vee (B \vee C)$ be an axiom of $\mathcal{L}$. Show that any wff of $\mathcal{L}$ is a theorem of $\mathcal{L}$.

11. Let $\mathcal{P}$ denote propositional logic. Suppose we add to $\mathcal{P}$ the axiom schema $(A \rightarrow B)$ for wffs A, B of $\mathcal{P}$. Comment on the consistency of the resulting logical system obtained. A logic system $\mathcal{P}$ is inconsistent if it is capable of producing $\perp$ using the rules of natural deduction.