CS 228 : Logic in Computer Science

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- To make sense out of a formula, we need structures

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- Structures also tell you the set of values your variables x_i can assume : these are the elements from the universe
- A structure in PL will just consist of the universe $\{0, 1\}$, since there is no signature. All variables assume values from this Boolean universe.

Satisfiability in PL and FO

- The satisfiability of a PL formula depends on the existence of an assignment satisfying it; likewise, the satisfiability of a FO formula φ over signature τ depends on the existence of a structure $\mathcal{A}$ of τ such that φ is true on $\mathcal{A}$.

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- $\varphi = P(x, y) \to \forall x((\forall y R(x, y)) \to Q(x, y))$
  - y is free in $Q(x, y)$ and bound in $R(x, y)$,
  - x is free in $P(x, y)$, and bound in $Q(x, y)$, $R(x, y)$

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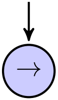
  - x is free in $P(x, y)$, and bound in $Q(x, y)$, $R(x, y)$
- Given φ, denote by $\varphi(x_1, \ldots, x_n)$, that $x_1, \ldots, x_n$ are the free variables of φ, also $free(\varphi)$
- A sentence is a formula φ none of whose variables are free

$P(x, y) \to \forall x((\forall y R(x, y)) \to Q(x, y))$



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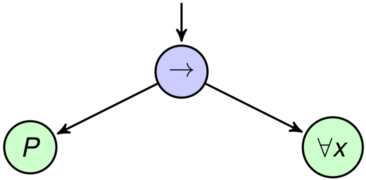

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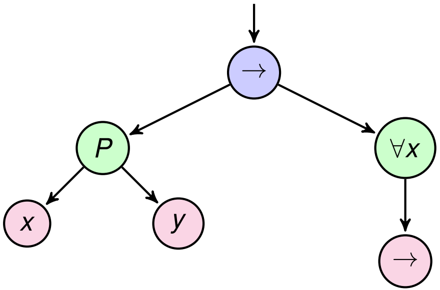

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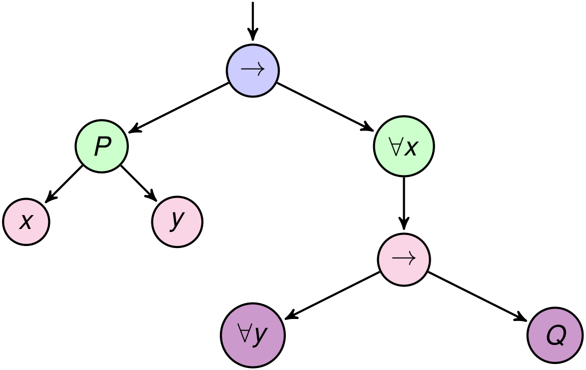

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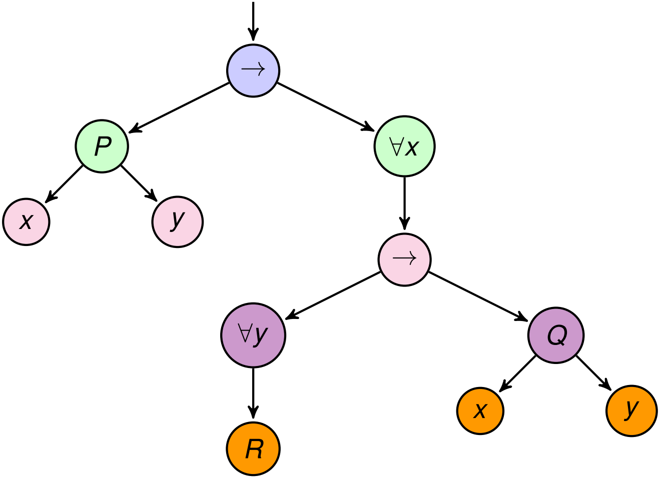

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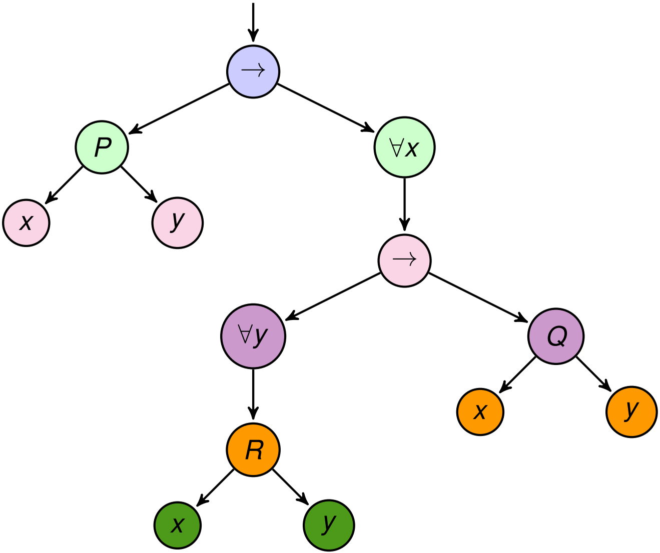
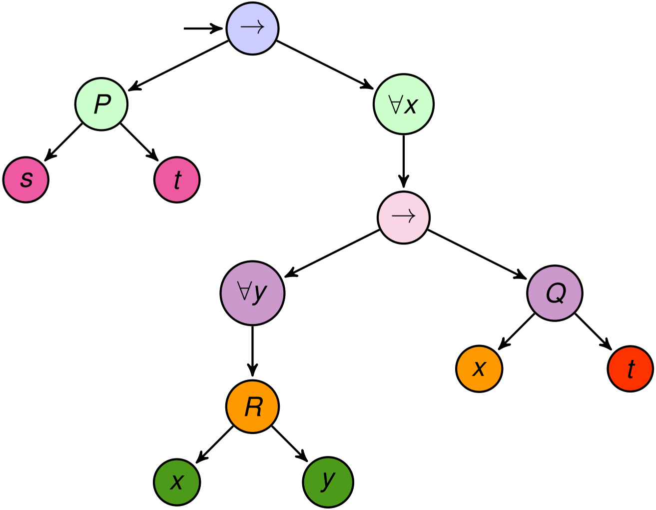


$P(x, y) \to \forall x((\forall y R(x, y)) \to Q(x, y))$

$\varphi(s, t) = P(s, t) \to \forall x((\forall y R(x, y)) \to Q(x, t))$



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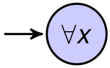

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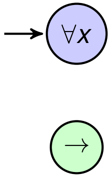

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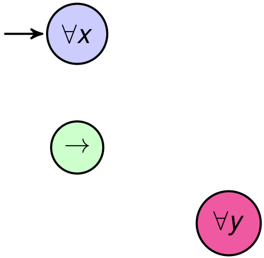

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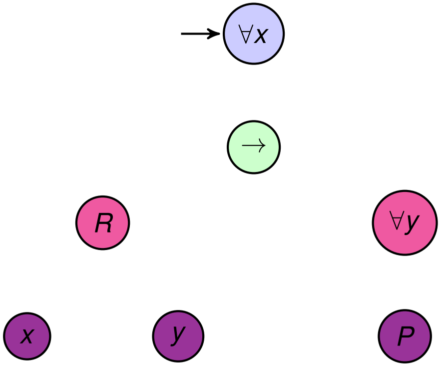

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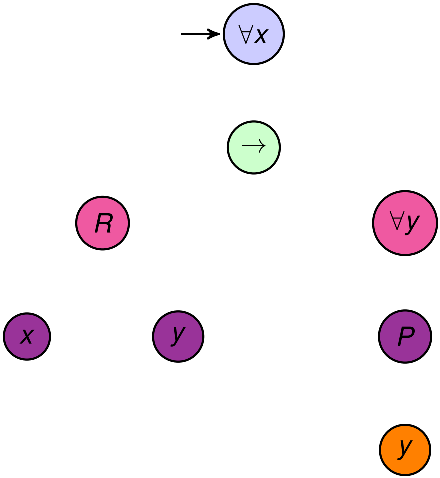

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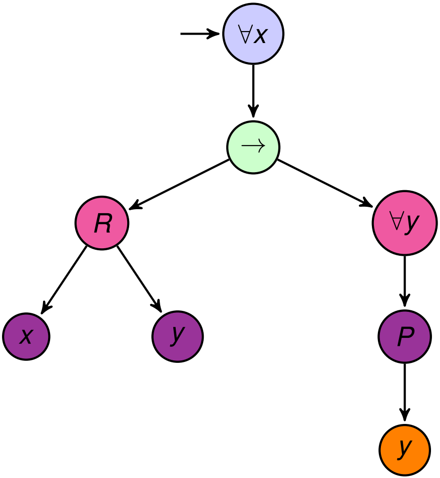

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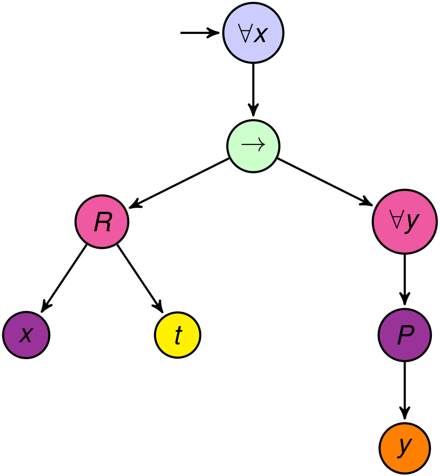

$\varphi(t) = \forall x(R(x,t) \to \forall y P(y))$

Assignments on τ-structures

Assignments

For a τ-structure $\mathcal{A}$, an assignment over $\mathcal{A}$ is a function $\alpha : \mathcal{V} \to u(\mathcal{A})$ that assigns every variable $x \in \mathcal{V}$ a value $\alpha(x) \in u(\mathcal{A})$. If t is a constant symbol c, then $\alpha(t)$ is $c^{\mathcal{A}}$

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Binding on a Variable

For an assignment α over $\mathcal{A}$, $\alpha[x \mapsto a]$ is the assignment

$$\alpha[x \mapsto a](y) = \begin{cases} \alpha(y), y \neq x, \\ a, y = x \end{cases}$$

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- $\mathcal{A} \models_\alpha (\exists x)\varphi$ iff there is some $a \in u(\mathcal{A})$, $\mathcal{A} \models_{\alpha[x \mapsto a]} \varphi$

Last two cases, α has no effect on the value of x. Thus, assignments matter only to free variables.

Example of Satisfaction

- $\mathcal{G} = (\{1, 2, 3\}, E^{\mathcal{G}} = \{(1, 2), (2, 1), (2, 3), (3, 2)\})$
  - For any assignment α, $\mathcal{G} \models_\alpha \forall x \forall y (E(x, y) \rightarrow E(y, x))$ iff for every $a, b \in \{1, 2, 3\}$, $\mathcal{G} \models_{\alpha[x \mapsto a, y \mapsto b]} (E(x, y) \rightarrow E(y, x))$

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  - Prove or disprove : $\mathcal{W} \models \exists x \forall y [Q_b(x) \wedge x < y \wedge Q_a(y)]$
  - Prove or disprove : $\mathcal{W} \models \exists x \forall y [Q_b(x) \wedge x < y \Rightarrow Q_a(y)]$

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- Formulae $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are equivalent denoted $\varphi \equiv \psi$ iff for every $\mathcal{A}$ and α, $\mathcal{A} \models_\alpha \varphi$ iff $\mathcal{A} \models_\alpha \psi$

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- Recap $\varphi_1(x) = \forall y R(x, y)$ and $\varphi_2 = \exists x \forall y R(x, y)$.
- It is clear that whenever φ_2 is satisfiable on $\mathcal{A}$, one can find an assignment α such that $\mathcal{A} \models_\alpha \varphi_1(x)$.
- Likewise, if $\mathcal{A} \models_\alpha \varphi_1(x)$, then $\mathcal{A} \models \varphi_2$.
- Thus, $\varphi_1(x), \varphi_2$ agree on satisfiability.

True or False?

For a formula φ and assignments α_1 and α_2 such that for every $x \in \mathit{free}(\varphi)$, $\alpha_1(x) = \alpha_2(x)$, $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

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- Evaluate for all $a, b \in u(\mathcal{A})$, $R(a, \alpha) \rightarrow P(b)$
- $\mathcal{A} \models_{\alpha_1} \varphi$ iff $\mathcal{A} \models_{\alpha_2} \varphi$

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No free variables!

Check SAT

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- $\varphi = \exists x[(\forall y E(x,y)) \wedge \forall z[(\forall y E(z,y)) \rightarrow z = x]]$. Does φ evaluate to true under some graph structure?
- $\psi = \exists x[Q_a(x) \wedge \forall y[(y < x \wedge Q_b(y)) \rightarrow (z < x \wedge y < z \wedge Q_a(z))]]$. Does ψ evaluate to true under some word structure?