CS 228 : Logic in Computer Science

Krishna. S

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- Each element in the group has a right inverse
- All words starting with the letter a, ending with the letter b, have even length

Signatures

- A vocabulary or signature τ is a set consisting of
  - constants $c_1, c_2, \ldots$
  - Relation symbols $R_1, R_2 \ldots$, each with some arity k, denoted R_i^k
- We look at finite signatures
- $\tau = (E^2, F^3)$ is a finite signature with two relations, E with arity 2 and F with arity 3

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- The symbols (and) called paranthesis

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- If φ is a wff and x is a variable, then $(\forall x)\varphi$ is a wff

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- Precedence of operators : $\neg > \wedge > \vee > \to > \forall$

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- $\forall x \forall y \forall z (R(x, y) \to (R(y, z) \to R(x, z)))$ Transitivity

First-Order Logic : Semantics

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  - The structure $\mathcal{A}$ is finite if A (or $u(\mathcal{A})$) is finite

Examples of Structures

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  - $\mathcal{G} = (V = \{1, 2, 3, 4\}, E^{\mathcal{G}} = \{(1, 2), (2, 3), (3, 4), (1, 1)\})$. We could just as well draw the graph for convenience.

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  - $\mathcal{O} = (O = \{1, 2, 4\}, <^{\mathcal{O}} = \{(1, 2), (1, 4), (2, 4)\}, S^{\mathcal{O}} = \{(1, 2)\})$
- Can you write a Partial Order as a structure, where the universe consists of all subsets of a given finite set?

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  - The structure with $u(\mathcal{W}) = \{0, 1, 2, \ldots, 8\}$, ${Q_a}^{\mathcal{W}} = \{0, 1, 4, 6, 8\}$, ${Q_b}^{\mathcal{W}} = \{2, 3, 5, 7\}$,
  - $<^{\mathcal{W}} = \{(0, 1), (0, 2), \ldots, (7, 8)\}$, $S^{\mathcal{W}} = \{(0, 1), (1, 2), \ldots, (7, 8)\}$ uniquely defines the word $W = aabbababa$.
  - For convenience, we can just write the word instead of the structure.