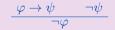
### **CS 228 : Logic in Computer Science**

Krishna, S

#### **Rules for Natural Deduction**

#### Another implies elimination rule or Modus Tollens MT



▶ Show that  $p \rightarrow \neg q, q \vdash \neg p$ 

1.  $p \rightarrow \neg q$  premise

2.

▶ Show that  $p \rightarrow \neg q, q \vdash \neg p$ 

- 1.  $p \rightarrow \neg q$  premise
- 2. q premise
- 3.

▶ Show that  $p \rightarrow \neg q, q \vdash \neg p$ 

1.	p  ightarrow  eg q	premise
2.	q	premise
3.	$\neg \neg q$	¬¬ <i>i</i> 2
4.		

▶ Show that  $p \rightarrow \neg q, q \vdash \neg p$ 

1.	$oldsymbol{ ho}  ightarrow  eg oldsymbol{q}$	premise
2.	q	premise
3.	$\neg \neg q$	¬¬ <i>i</i> 2
4.	$\neg p$	MT 1,3

▶ Thanks to MT, we have  $p \rightarrow q, \neg q \vdash \neg p$ .

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- ▶ Thanks to MT, we have  $p \rightarrow q, \neg q \vdash \neg p$ .
- ▶ Can we prove  $p \rightarrow q \vdash \neg q \rightarrow \neg p$ ?
- So far, no proof rule that can do this.
- ▶ Given  $p \rightarrow q$ , let us assume  $\neg q$ . Can we then prove  $\neg p$ ?
- ► Yes, using MT.

### The implies introduction rule $\rightarrow i$

1.	p  o q	premise
2.	$\neg q$	assumption

4. 
$$\neg q \rightarrow \neg p \rightarrow i \ 2-3$$

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

premise

- true
- 2.

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

- 1. true premise 2.  $q \rightarrow r$  assumption
- 3.

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	$q \rightarrow r$	assumption
3.	eg q  o  eg p	assumption
4.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q  o r	assumption
3.	eg q  ightarrow  eg p	assumption
4.	p	assumption
5.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	$q \rightarrow r$	assumption
3.	eg q  o  eg p	assumption
4.	p	assumption
5.		¬¬ <i>i</i> 4
6.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q  o r	assumption
3.	eg q  o  eg p	assumption
4.	p	assumption
5.	$  \   \   \ \neg \neg p$	¬¬ <i>i</i> 4
6.	$  \   \   \ \neg \neg q$	MT 3,5
7.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	$q \rightarrow r$	assumption
3.	eg q  ightarrow  eg p	assumption
4.	p	assumption
5.		¬¬ <i>i</i> 4
6.		MT 3,5
7.	q	¬¬ <i>e</i> 6
8.		

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q  o r	assumption
3.	eg q  ightarrow  eg p	assumption
4.	р	assumption
5.	$  \   \   \ \neg \neg \rho$	¬¬ <i>i</i> 4
6.	$  \   \   \ \neg \neg q$	MT 3,5
7.		¬¬ <i>e</i> 6
8.		MP 2,7

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q  o r	assumption
3.	eg q  ightarrow  eg p	assumption
4.	p	assumption
5.		¬¬ <i>i</i> 4
6.	$    \neg \neg q$	MT 3,5
7.	q	¬¬ <i>e</i> 6
8.	r	MP 2,7
9	$p \rightarrow r$	→ <i>i</i> 4-8

$$\blacktriangleright \vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$$

1.	true	premise
2.	q  o r	assumption
3.	eg q  o  eg p	assumption
4.	P	assumption
5.	$  \cdot   \cdot   \neg \neg p$	¬¬ <i>i</i> 4
6.	$  \cdot   \cdot   \neg \neg q$	MT 3,5
7.	q	¬¬ <i>e</i> 6
8.	r	MP 2,7
9.	ho  ightarrow r	→ <i>i</i> 4-8
10.	( eg q  ightarrow  eg p)  ightarrow (p  ightarrow r)	→ <i>i</i> 3-9

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11.

 $(q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)] \rightarrow i \text{ 2-10}$ 

 $\rightarrow$  *i* 4-8

 $\rightarrow$  *i* 3-9

6/24

9.

10.

11.

 $p \rightarrow r$ 

 $(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$ 

## **Transforming Proofs**

- $ightharpoonup (q 
  ightarrow r), (\neg q 
  ightarrow \neg p), p \vdash r$
- ► Transform any proof  $\varphi_1, \ldots, \varphi_n \vdash \psi$  to  $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow \ldots (\varphi_n \rightarrow \psi) \ldots))$  by adding n lines of the rule  $\rightarrow i$

▶ 
$$p \to (q \to r) \vdash (p \land q) \to r$$

1.  $p \to (q \to r)$  premise 2.

$$\begin{array}{c|cccc} \blacktriangleright & p \rightarrow (q \rightarrow r) \vdash (p \land q) \rightarrow r \\ & 1. & p \rightarrow (q \rightarrow r) & \text{premise} \\ & 2. & p \land q & \text{assumption} \\ & 3. & p & \land e_1 \ 2 \\ & 4. & q & \land e_2 \ 2 \\ & 5. & q \rightarrow r & \text{MP 1,3} \\ & 6. & \end{array}$$

▶ 
$$p \rightarrow (q \rightarrow r) \vdash (p \land q) \rightarrow r$$

1.  $p \rightarrow (q \rightarrow r)$  premise

2.  $p \land q$  assumption

3.  $p \land e_1 2$ 

4.  $q \land e_2 2$ 

5.  $q \rightarrow r \land P 1,3$ 

6.  $r \land P 4,5$ 

7.

$$\begin{array}{c|cccc} \blacktriangleright & p \rightarrow (q \rightarrow r) \vdash (p \land q) \rightarrow r \\ & 1. & p \rightarrow (q \rightarrow r) & \text{premise} \\ & 2. & p \land q & \text{assumption} \\ & 3. & p & \land e_1 \ 2 \\ & 4. & q & \land e_2 \ 2 \\ & 5. & q \rightarrow r & \text{MP 1,3} \\ & 6. & r & \text{MP 4,5} \\ & 7. & p \land q \rightarrow r & \rightarrow i \ 2\text{-}6 \end{array}$$

#### The or introduction rule $\vee i_1$

$$\frac{\varphi}{\varphi\vee\psi}$$

#### The or introduction rule $\vee i_2$

$$\frac{\psi}{\varphi \vee \psi}$$

#### The or elimination rule $\vee e$

$$\begin{array}{ccc} \varphi \lor \psi & \varphi \vdash \chi & \psi \vdash \chi \\ \hline \chi & \end{array}$$

# **Or Elimination Example**

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

- 1.  $q \rightarrow r$
- 2

premise

### Or Elimination Example

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumpt
3.		

### **Or Elimination Example**

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$oldsymbol{q}  ightarrow oldsymbol{r}$	premise
2.	$p \lor q$	assumption
3.	p	∨ <i>e</i> (1)
4.		

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	p	∨ <i>e</i> (1)
4.	$p \lor r$	√ <i>i</i> <sub>1</sub> 3
5.		

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

	q  o r	premise
	$p \lor q$	assumption
3.	p	∨ <i>e</i> (1)
	p∨r	∨ <i>i</i> <sub>1</sub> 3
<b>)</b> .	q	∨ e (2)
<b>.</b>		

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

1.	$oldsymbol{q}  ightarrow oldsymbol{r}$	premise
2.	$p \lor q$	assumption
3.	p	∨ <i>e</i> (1)
4.	$p \lor r$	∨ <i>i</i> ₁ 3
5.	q	∨ e (2)
6.	r	MP 1,5
7.		

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$oldsymbol{q} ightarrow oldsymbol{r}$	premise
2.	$p \lor q$	assumption
3.	p	∨ <i>e</i> (1)
4.	$p \lor r$	√ <i>i</i> <sub>1</sub> 3
5.	q	∨ e (2)
6.	r	MP 1,5
7.	p∨r	∨ <i>i</i> <sub>2</sub> 6

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	q  o r	premise
2.	$p \lor q$	assumption
3.	р	∨ <i>e</i> (1)
4.	p∨r	∨ <i>i</i> ₁ 3
5.	q	∨ <i>e</i> (2)
6.	r	MP 1,5
7.	p∨r	∨ <i>i</i> <sub>2</sub> 6
8.	p∨r	∨ <i>e</i> 2, 3-4, 5-7

$$P q \to r \vdash (p \lor q) \to (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	р	∨ <i>e</i> (1)
4.	<i>p</i> ∨ <i>r</i>	∨ <i>i</i> ₁ 3
5.	q	∨ e (2)
6.	r	MP 1,5
7.	p∨r	∨ <i>i</i> <sub>2</sub> 6
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9.		

$$P q \rightarrow r \vdash (p \lor q) \rightarrow (p \lor r)$$

1.	$q \rightarrow r$	premise
2.	$p \lor q$	assumption
3.	р	∨ <i>e</i> (1)
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6.	r	MP 1,5
7.	p∨r	∨ <i>i</i> <sub>2</sub> 6
8.	p∨r	∨ <i>e</i> 2, 3-4, 5-7
9.	$(p \lor q) \rightarrow (p \lor r)$	→ <i>i</i> 2-8

▶ 
$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

1.  $(p \lor q) \lor r$  premise

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$ 

```
1. (p \lor q) \lor r premise
2. p \lor q \lor e(1)
3.
```

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$ 

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	р	∨ <i>e</i> (1.1)

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ e (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5		

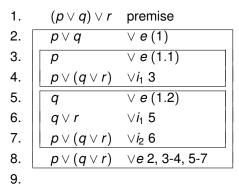
1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ <i>e</i> (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5.	q	∨ <i>e</i> (1.2)
6.		

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ <i>e</i> (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5.	q	∨ <i>e</i> (1.2)
6.	$q \vee r$	∨ <i>i</i> <sub>1</sub> 5
7.		

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$ 

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ <i>e</i> (1.1)
4.	$p \lor (q \lor r)$	√ <i>i</i> <sub>1</sub> 3
5.	q	∨ e (1.2)
6.	$  q \lor r$	∨ <i>i</i> <sub>1</sub> 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 6
8.		

 $(p \lor q) \lor r \vdash p \lor (q \lor r)$ 



1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ <i>e</i> (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5.	q	∨ e (1.2)
6.	$q \vee r$	∨ <i>i</i> <sub>1</sub> 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	∨ e (2)
0.		

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ e (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5.	q	∨ <i>e</i> (1.2)
6.	$  q \lor r$	∨ <i>i</i> <sub>1</sub> 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	∨ e (2)
0.	$q \lor r$	∨ <i>i</i> <sub>2</sub> 9
1.		

$$(p \lor q) \lor r \vdash p \lor (q \lor r)$$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ e (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5.	q	∨ <i>e</i> (1.2)
6.	$q \lor r$	∨ <i>i</i> <sub>1</sub> 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	∨ <i>e</i> (2)
10.	$q \vee r$	√ <i>i</i> <sub>2</sub> 9
11.	$p \lor (q \lor r)$	√ <i>i</i> <sub>2</sub> 10

$$\blacktriangleright (p \lor q) \lor r \vdash p \lor (q \lor r)$$

1.	$(p \lor q) \lor r$	premise
2.	$p \lor q$	∨ <i>e</i> (1)
3.	p	∨ <i>e</i> (1.1)
4.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>1</sub> 3
5.	q	∨ <i>e</i> (1.2)
6.	$q \lor r$	∨ <i>i</i> <sub>1</sub> 5
7.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 6
8.	$p \lor (q \lor r)$	∨ <i>e</i> 2, 3-4, 5-7
9.	r	∨ <i>e</i> (2)
10.	$q \vee r$	∨ <i>i</i> <sub>2</sub> 9
11.	$p \lor (q \lor r)$	∨ <i>i</i> <sub>2</sub> 10
12.	$p \lor (q \lor r)$	∨ <i>e</i> 1, 2-8, 9-11

#### **Basic Rules So Far**

- $ightharpoonup \land i, \land e_1, \land e_2$  (and introduction and elimination)
- $\rightarrow \neg \neg e, \neg \neg i$  (double negation elimination and introduction)
- ► MP (Modus Ponens)
- $ightharpoonup \rightarrow i$  (Implies Introduction : remember opening boxes)
- $\vee i_1, \forall i_2, \forall e$  (Or introduction and elimination)

▶ 
$$\vdash p \rightarrow (q \rightarrow p)$$

1. true

premise

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
2		

▶ 
$$\vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

	true	premise
2.	р	assumption
3.	q	assumption
ŀ.	р	copy 2

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	p	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	<i>→ i</i> 3-4

$$\blacktriangleright \vdash p \rightarrow (q \rightarrow p)$$

1.	true	premise
2.	р	assumption
3.	q	assumption
4.	р	copy 2
5.	$oldsymbol{q} ightarrow oldsymbol{p}$	→ <i>i</i> 3-4
6.	$p \rightarrow (q \rightarrow p)$	$\rightarrow$ i 2-5

▶ We have seen  $\neg \neg e$  and  $\neg \neg i$ , the elimination and introduction of double negation.

- ▶ We have seen  $\neg \neg e$  and  $\neg \neg i$ , the elimination and introduction of double negation.
- ▶ How about introducing and eliminating single negations?

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- How about introducing and eliminating single negations?
- ▶ We use the notion of contradictions, an expression of the form  $\varphi \land \neg \varphi$ , where  $\varphi$  is any propositional logic formula.

- We have seen ¬¬e and ¬¬i, the elimination and introduction of double negation.
- How about introducing and eliminating single negations?
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- ▶ Any two contradictions are equivalent :  $p \land \neg p$  is equivalent to  $\neg r \land r$ . Contradictions denoted by  $\bot$ .

- We have seen ¬¬e and ¬¬i, the elimination and introduction of double negation.
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- $ightharpoonup \perp \to \varphi$  for any formula  $\varphi$ .

### Rules with $\bot$

The  $\perp$  elimination rule  $\perp e$ 

$$\frac{\perp}{\psi}$$

The  $\perp$  introduction rule  $\perp i$ 

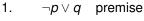
$$\frac{\varphi \qquad \neg \varphi}{\bot}$$

▶ 
$$\neg p \lor q \vdash p \rightarrow q$$

- 1.  $\neg p \lor q$  premise
- 2.

▶ 
$$\neg p \lor q \vdash p \rightarrow q$$

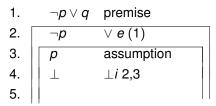
- 1.  $\neg p \lor q$  premise
- 2.  $\neg p \lor e(1)$
- 3.



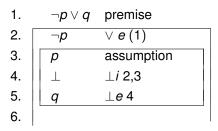
2. 
$$\neg p \lor e(1)$$

4.

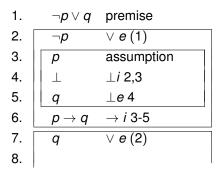
▶ 
$$\neg p \lor q \vdash p \rightarrow q$$



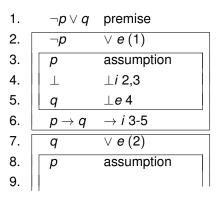
▶ 
$$\neg p \lor q \vdash p \rightarrow q$$



▶ 
$$\neg p \lor q \vdash p \rightarrow q$$



▶ 
$$\neg p \lor q \vdash p \rightarrow q$$



▶ 
$$\neg p \lor q \vdash p \rightarrow q$$

1.	$\neg p \lor q$	premise
2.	$\neg p$	∨ <i>e</i> (1)
3.	р	assumption
4.		<i>⊥i</i> 2,3
5.	q	⊥ <i>e</i> 4
6.	p  o q	→ <i>i</i> 3-5
7.	q	∨ e (2)
8.	р	assumption
9.	q	copy 7
0.	p  o q	→ <i>i</i> 8-9
1.	p  o q	∨ <i>e</i> 1, 2-6, 7-10

### **Introducing Negations (PBC)**

- In the course of a proof, if you assume  $\varphi$  (by opening a box) and obtain  $\bot$  in the box, then we conclude  $\neg \varphi$
- ▶ This rule is denoted  $\neg i$  and is read as  $\neg$  introduction.
- ► Also known as Proof By Contradiction

- 1.  $p \rightarrow \neg p$  premise
- 2.

۱.	p  ightarrow  eg p	premise

2. p assumption 3.

$$\blacktriangleright \ p \to \neg p \vdash \neg p$$

1.	p  ightarrow  eg p	premise
2.	р	assumption
3.	$\neg p$	MP 1,2
4.		

1.	$oldsymbol{ ho}  ightarrow  eg eta$	premise
2.	р	assumption
3.	$\neg p$	MP 1,2
4.		<i>⊥i</i> 2,3
5.	$\neg p$	<i>¬i</i> 2-4

### **The Last One**

Law of the Excluded Middle (LEM)



## **Summary of Basic Rules**

- $\rightarrow \land i, \land e_1, \land e_2,$
- ¬¬e
- ► MP
- $\rightarrow i$
- $\triangleright \forall i_1, \forall i_2, \forall e$
- ▶ Copy,  $\neg i$  or PBC
- **▶** ⊥*e*, ⊥*i*

### **Derived Rules**

- ▶ MT (derive using MP,  $\perp i$  and  $\neg i$ )
- $ightharpoonup \neg \neg i$  (derive using  $\bot i$  and  $\neg i$ )
- ▶ LEM (derive using  $\forall i_1, \bot i, \neg i, \forall i_2, \neg \neg e$ )

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- Formulae  $\varphi$  and  $\psi$  are semantically equivalent iff  $\varphi \models \psi$  and  $\psi \models \varphi$