# Recitation 1 (1/20)

- linear systems
- derivatives
- Taylor series

## Dynamic Systems:

$$\dot{x} = f(\underset{\text{state}}{x}, \underset{\text{control}}{u})$$ continuous time, ODE

$$x_{k+1} = f(x_k, u_k)$$ discrete time, difference eq

## 1st order ODEs

$$\ddot{x} = Fx \Rightarrow \begin{matrix} \dot{x} = \dot{x} \\ \ddot{x} = Fx \end{matrix}$$

we can convert any order ODE to a 1st order ODE

$$\underbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}_{\dot{z}} = \begin{bmatrix} 0 & I \\ F & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}_{z}$$

## Is it Linear? In x, u

$$\dot{x} = A(t)x + B(t)u$$ if it fits in this form, it is linear

$$\dot{x} = \begin{bmatrix} \cos(t) & t^2 \\ 1 & e^t \end{bmatrix} x + Bu$$

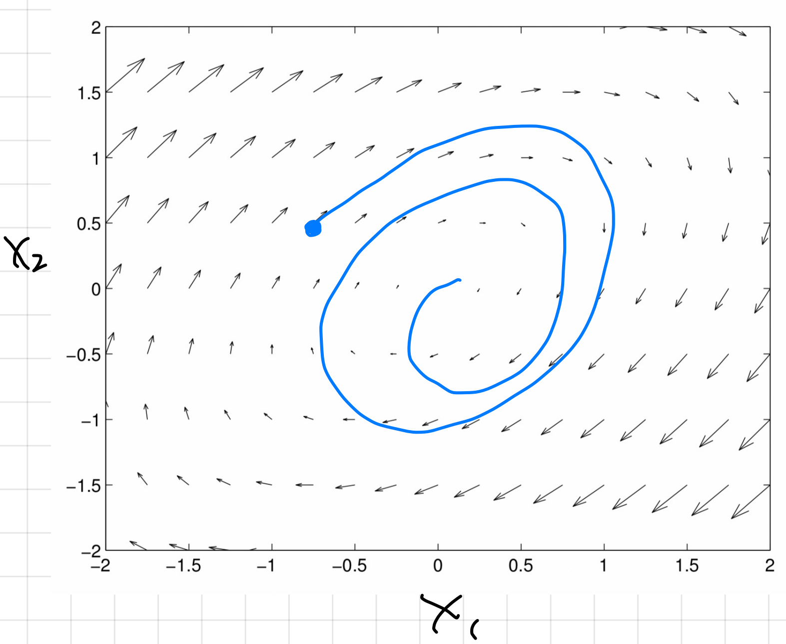
example of a system that is linear in x and u (but not t)

# LDS (Linear dynamical System)

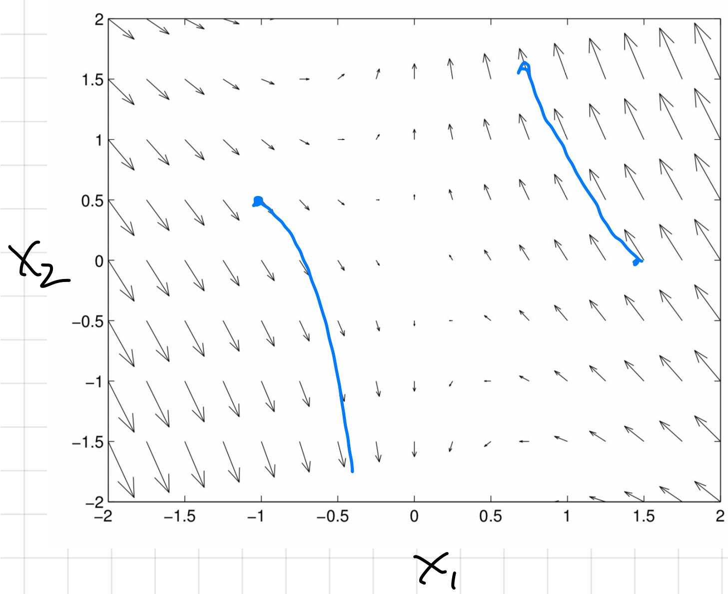
$$\dot{x} = Ax \quad , x \in \mathbb{R}^N$$

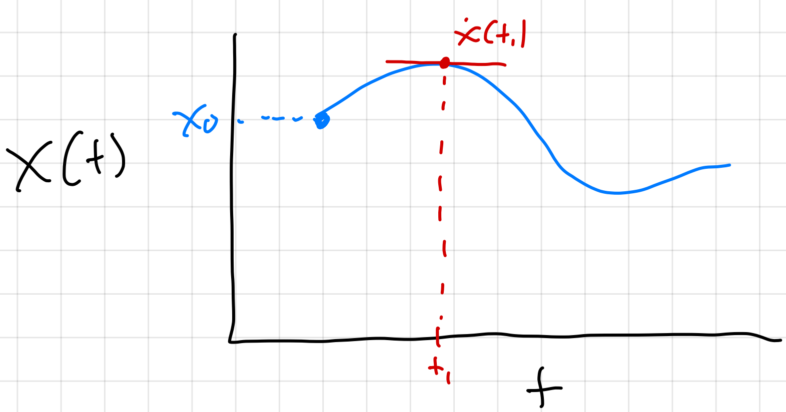
ODE's are vector fields

$$\dot{x} = \begin{bmatrix} -0.5 & 1 \\ -1 & 0.5 \end{bmatrix} x$$



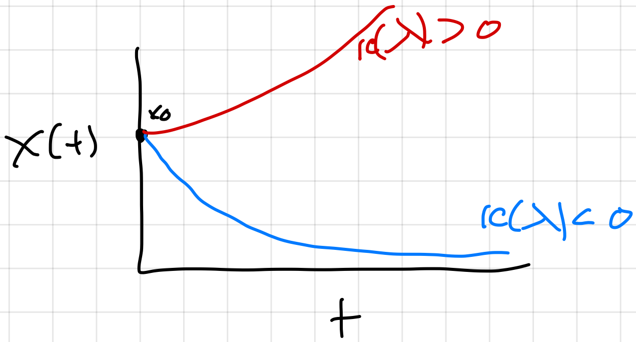

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} x$$



# Stability of $\dot{x} = Ax$

$\dot{x} = Ax$ is stable if $re(\text{eigvals}(A)) < 0$




## Solution to Linear ODE

Solve a scalar linear ODE

$$\dot{x} = ax$$

$$\frac{dx}{dt} = ax$$

$$\int \frac{1}{x}\, dx = \int a\, dt$$

$$\ln x = at + c$$

$$x = e^{at+c}$$

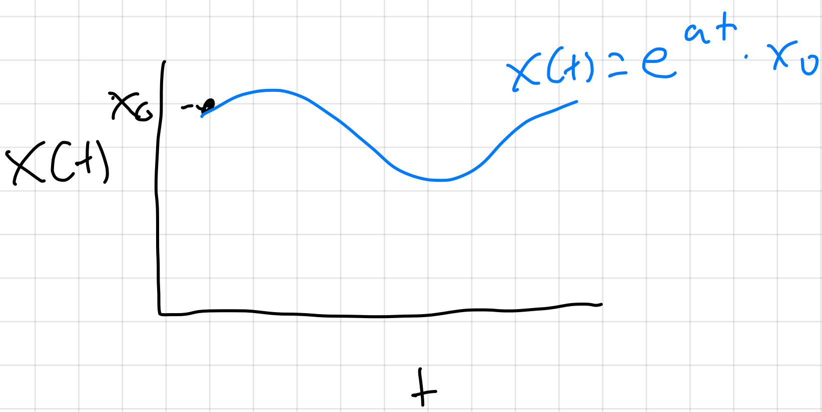
$$= e^{at} e^{c}$$

$$x = e^{at} c$$

$$x_0 = e^{0} c$$

$$x = e^{at} \cdot x_0$$

$$\dot{x} = ax$$



## extension to Linear systems in multiple variables

$$\dot{x} = Ax$$

$$x(t) = e^{At} x_0$$

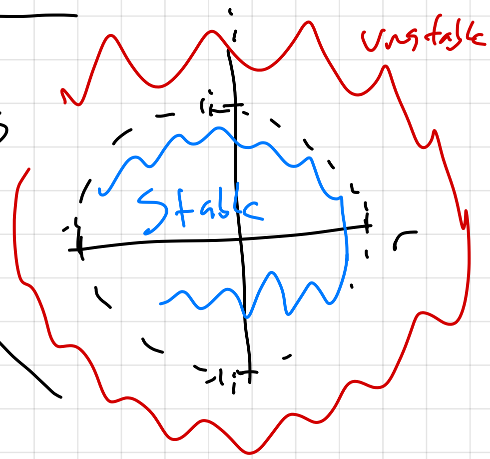
Matrix exponential

$$x_{t+1} = \underbrace{\left(e^{A\Delta t}\right)}_{A_d} x_t$$

### Is it stable?

$$|\text{eigvals}(A_d)| < 1$$

$\lambda s$

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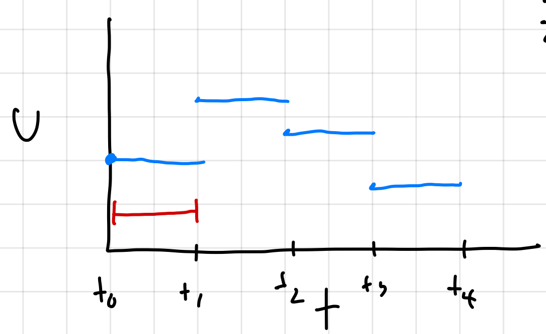
$$\dot{x} = Ax + Bu$$

ZOH for control




$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{u} &= 0 \end{aligned} \quad \overset{\text{equiv.}}{\Longrightarrow} \quad \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

0's to make square

← this is another Linear system!

$$\begin{bmatrix} x_{t+1} \\ u_{t+1} \end{bmatrix} = \left( e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \Delta t} \right) \begin{bmatrix} x_t \\ u_t \end{bmatrix}$$

← we can solve this in the same way as any other $\dot{x} = Ax$

$$\begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} = e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \Delta t}$$

$$x_{t+1} = A_d x_t + B_d u_t$$

Linear, discrete time dynamics

extension to $\dot{x} = Ax + Bu + d$

**Discretization of Linearized Dynamics**

In order to discretize the continuous system (14), the matrix exponential will be used. For a generic homogeneous linear ODE of the form $\dot{x} = Ax$, the solution for $x$ after a time $\delta t$, can be expressed using the matrix exponential and the initial condition:[13,14]

$$\dot{x} = Ax, \tag{15}$$
$$x(t_0 + \delta t) = \exp(A \cdot \delta t)x(t_0). \tag{16}$$

For a forced affine ODE, where the control input and affine forcing term are assumed constant over a time step, the state can simply be augmented with these terms,

$$\begin{bmatrix} \dot{x} \\ \dot{u} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} Ax + Bu + d \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A & B & I_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \\ d \end{bmatrix}, \tag{17}$$

and this system can be discretized with a sample time of $\delta t$ in the same way as (16)

$$\begin{bmatrix} x_{t+1} \\ u_{t+1} \\ d_{t+1} \end{bmatrix} = \exp\left( \begin{bmatrix} A & B & I_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \delta t \right) \begin{bmatrix} x_t \\ u_t \\ d_t \end{bmatrix}. \tag{18}$$

Finally, we obtain in the following difference equation,

$$x_{t+1} = A_d x_t + B_d u_t + D_d d_t, \tag{19}$$

where the transition matrices come from the matrix exponential,

$$\begin{bmatrix} A_d & B_d & D_d \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \exp\left( \begin{bmatrix} A & B & I_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \delta t \right). \tag{20}$$

## Derivative Formalism:

$$f: \mathbb{R}^N \to \mathbb{R}^M, \quad y = f(x)$$

$y \in \mathbb{R}^M$

$x \in \mathbb{R}^N$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}\right)$$

### Jacobian

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_N} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y_M}{\partial x_1} & \frac{\partial y_M}{\partial x_2} & \cdots & \frac{\partial y_M}{\partial x_N} \end{bmatrix}$$

# rows = # outputs

# cols = # inputs

## Gradient

$$y = f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}\right)$$

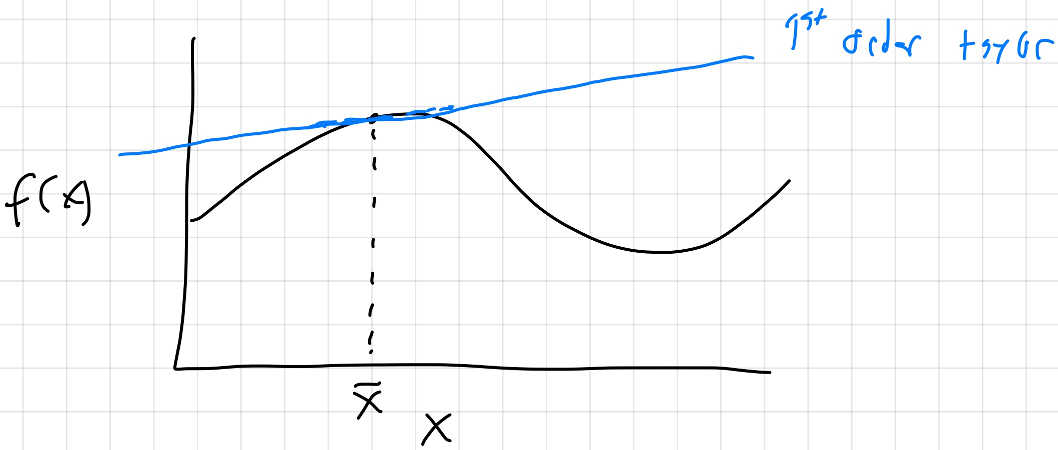
$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \cdots & \frac{\partial y}{\partial x_N} \end{bmatrix}$$ Row vector

$$\nabla_x f = \left(\frac{\partial f}{\partial x}\right)^T$$

## Hessian

$$\nabla_x^2 f = \overbrace{\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}}^{\text{Symmetric}},$$

# 1st order Taylor Series




$$f(x) \approx f(\bar{x}) + \left(\frac{\partial f}{\partial x}\bigg|_{\bar{x}}\right)(x-\bar{x})$$

$$x = \bar{x} + \Delta x$$

$$f(\bar{x}+\Delta x) \approx f(\bar{x}) + \left(\frac{\partial f}{\partial x}\bigg|_{\bar{x}}\right)\Delta x$$

## Taylor for dynamics (2 inputs)

$\dot{x} = f(x, u)$ Linearize about $\bar{x}, \bar{u}$

$$A = \frac{\partial f}{\partial x}\bigg|_{\bar{x},\bar{u}}, \quad B = \frac{\partial f}{\partial u}\bigg|_{\bar{x},\bar{u}}$$

1 $f(x,u) \approx f(\bar{x},\bar{u}) + A(x-\bar{x}) + B(u-\bar{u})$

$x = \bar{x} + \Delta x, \quad u = \bar{u} + \Delta u$

2 $f(x,u) \approx f(\bar{x},\bar{u}) + A\Delta x + B\Delta u$