BUILDING A SIMULATOR

① FORWARD DYNAMICS : given τ find $\ddot{q}$

$$
\begin{array}{l} (1) \\ (2) \end{array}
\begin{bmatrix} M & -J_c^T \\ J_c & 0 \end{bmatrix}
\begin{bmatrix} \ddot{q} \\ f \end{bmatrix}
=
\begin{bmatrix} S^T\tau - h \\ -\dot{J}\dot{q} \end{bmatrix}
\rightarrow \text{RIGID CONTACT ASSUMPTION}
$$

we have also contact forces f as unknown

let's find a way to eliminate them ...

② solve (1) for $\ddot{q}$ and insert it into (2)

$$M\ddot{q} + h = S^T\tau + J_c^T f$$

$$\ddot{q} = M^{-1}(-h + S^T\tau + J_c^T f)$$

$$J_c M^{-1}(-h + S^T\tau + J_c^T f) + \dot{J}\dot{q} = 0$$

③ solve for contact forces f

$$J_c M^{-1} J_c^T f + J_c M^{-1}(-h + S^T \tau) + \dot{J}\dot{q} = 0$$

$$f = (J_c M^{-1} J_c^T)^{-1} \left[-\dot{J}_c \dot{q} + J_c M^{-1} (h - S^T \tau) \right]$$

Λ_c : joint inertia reflected at the contact

④ plug f back into dynamics (1)

$$M\ddot{q} + h = S^T \tau + J^T \Lambda \left[-\dot{J}\dot{q} + J M^{-1} (h - S^T \tau) \right]$$

$$(3) \quad M\ddot{q} + h = S^T \tau + J^T (J M^{-1} J^T)^{-1} \dot{J}\dot{q} + J^T (J M^{-1} J^T)^{-1} J M^{-1} h$$

$$- \underbrace{J^T (J M^{-1} J^T)^{-1} J M^{-1}}_{J^{T\#} \text{ weighted with } M^{-1}} S^T \tau$$

$J^{T\#}$ weighted with M^{-1}

(DYNAMICALLY CONSISTENT PSEUDOINVERSE) $J_{M}^{T\#}$

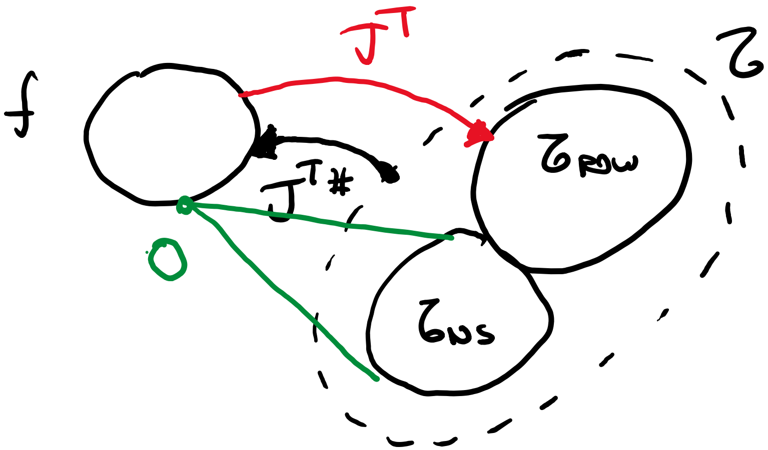
$$J^T = A = [\quad] \qquad A^\# = [\quad]$$

$$A^\# = (A^T A)^{-1} A^T$$

$$A_W^\# = (A^T W A)^{-1} A^T W \quad \rightarrow \quad J_W^{T\#} = (J W J^T)^{-1} J W$$

cfg. $A A_W^\# = I_m$ ok

For forces?



$$\tau_{Row} = J^T J^{T\#} \tau$$

$$\tau_{NULL} = (I - J^T J^{T\#}) \tau$$

$$\boxed{N = I - J^T J^{T\#}}$$

NULL SPACE PROJECTOR

Torques that do not produce any contact force but only joint accelerations

Note N requires J to be full rank

The weighed version N_W is not symmetric ($N_W \neq N_W^T$)

(5) From (3) collect $N_{\bar{M}}$

$$M\ddot{q} + R\underbrace{\left[I - J^T J_{\bar{M}}^{T\#}\right]}_{N_{\bar{M}}} + J^T \Lambda (\dot{J}\dot{q}) = \underbrace{\left[I - J^T J_{\bar{M}}^{T\#}\right]}_{N_{\bar{M}}} S^T \tau$$

$$\boxed{M\ddot{q} + N_{\bar{M}} R + J^T \Lambda (\dot{J}\dot{q}) = N_{\bar{M}} S^T \tau} \quad (1)$$

CONTACT CONSISTENT JOINT SPACE DYNAMICS

6) Replacing $-\dot{J}\dot{q} = J\ddot{q}$ [HUTTER PHD THESIS 2.3]

$$M\ddot{q} + N_{\bar{M}} h + J^T (J M^{-1} J^T)^{-1} J \ddot{q} = N_{\bar{M}} S^T \tau$$

$$\Big[\underbrace{I - J^T (J M^{-1} J^T)^{-1} J M^{-1}}_{N_{\bar{M}}}\Big] M \ddot{q} + N_{\bar{M}} h = N_{\bar{M}} S^T \tau$$

$$\boxed{N_{\bar{M}} (M\ddot{q} + h) = N_{\bar{M}} S^T \tau} \quad (2)$$

projection of dynamics through N

The contact force disappeared from equation

$$\underbrace{N_{\bar{M}} J^T}_{\text{null space}} f = 0$$

Note: $N_{\bar{\mu}}$ is a linear operator that maps the dynamics onto a constrained manifold such that EOM is independent of contact forces

$\Rightarrow$ any operator s.t. $P J^T = 0$ would work

7) Integrate

$$\ddot{q} = (N \; M)^{-1} \; N^T (S^T \tau - h)$$

we cannot do that!

$\hookrightarrow$ is not invertible, most of times is singular

- f disappears but we also lose informations on accelerations that we cannot recover because we project them onto a lower dimensional space with (2), and the result won't be correct.

for now we use:

$$\boxed{\ddot{q} = M^{-1}\left[N\,(S^T\tau - h) - J^T \Lambda \dot{J} \dot{q}\right]}$$

- with this we do not lose informations because we multiply $\ddot{q}$ for M that is positive definite

WITH forward Euler integration:

$$\dot{q}_{k+1} = \dot{q}_k + dT\,\ddot{q}_k$$

$$q_{k+1} = q_k + dT\,\dot{q}_{k+1} + \frac{dT^2}{2}\,\ddot{q}_{k+1}$$

NOTE: $\dot{J}\dot{q} \simeq 0$ for non redundant manipulators because they are completely constrained

NOTE: $\dot{q}^+$? constraint consistent joint velocity?

⊖ ... we need to consider impact dynamics

LIFT OFF: $F \downarrow 0$, $\ddot{x}_c > 0$

TOUCH DOWN: $x_c = \text{cost}$, $\dot{x}_c, \ddot{x}_c = 0$, $F \uparrow 0$

↳ collision at landing:

- foot point is immediately brought to rest
- instantaneous change in velocity to avoid penetration
- to compute it we need to integrate the dynamics of collision (inelastic impact)

CONTACT DYNAMICS

① Integration of Lagrangian dynamics on singletons

$$\int_{t^-}^{t^+} [M\ddot{q} + h - J^T f]\, dt = M(\dot{q}^+ - \dot{q}^-) - J^T F_c = 0 \qquad (1)$$

impulse

- all non impulsive forces drop out
- after collision the foot is at rest

$$\dot{x}_f^+ = \emptyset = J\dot{q}^+ \qquad \text{(inelastic impact)} \qquad (2)$$

② solving for $\dot{q}^+$ and inserting into (2)

$$\dot{q}^+ = \dot{q}^- + M^{-1} J^T F_c$$

$$\dot{x}_f^+ = J(M^{-1} J^T F_c + \dot{q}^-) = 0$$

③ we can compute the impulse:

$$F_c = -(J M^{-1} J^T)^{-1} J \dot{q}^- \qquad F_c^z > 0$$

④ we compute post-impact joint velocity

$$\dot{q}^+ = M^{-1} J^T \left[-(J M^{-1} J^T)^{-1} J \dot{q}^- \right] + \dot{q}^-$$

$$= \underbrace{\left[I - M^{-1} J^T (J M^{-1} J^T)^{-1} J \right]}_{I - J^{\#}_{M} J \, = \, N_{vel}} \dot{q}^-$$

$$\boxed{\dot{q}_c^+ = N_{vel} \dot{q}^-}$$

ONLY AT IMPACT!

Note:

This Null space projector is acting at the velocity level and is different from the previous one that acts at Torque level

ENERGY LOST IN COLLISIONS

0

$$E_{LOSS} = E_{KIN} = \frac{1}{2} \Delta \dot{q}^T M \Delta q$$

$\Downarrow$ Task space

$$\Delta \dot{x}_c = 0 - \dot{x}_c = J \Delta \dot{q}$$

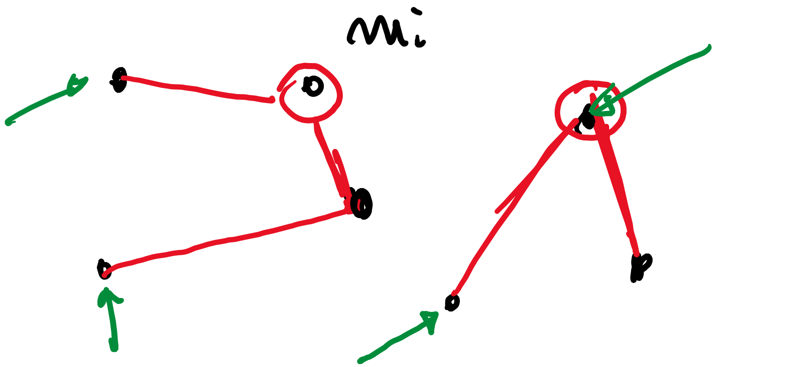
$$\Delta \dot{q} = J^{-1} \Delta \dot{x}_c$$

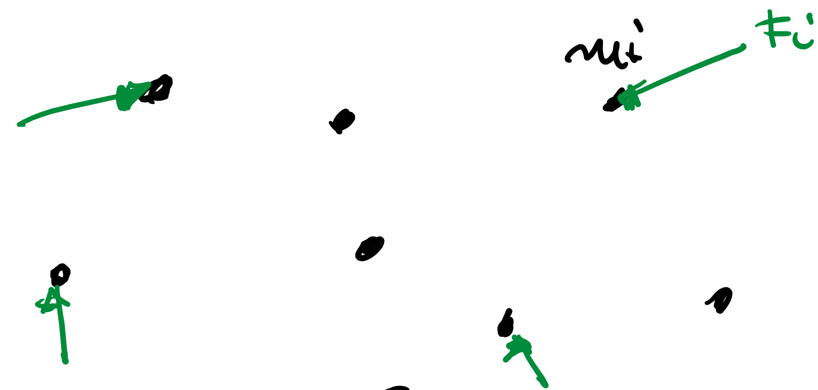
$$E_{LOSS} = E_{KIN} = \frac{1}{2} \Delta \dot{x}_c^T J^{-T} M J^{-1} \Delta \dot{x}_c$$

$$= \frac{1}{2} \Delta \dot{x}_c^T \Lambda_c \Delta \dot{x}_c$$

$\uparrow$ end-effector inertia

GAUSS PRINCIPLE OF LEAST CONSTRAINT





collection of particles with some constraints

unconstrained equivalent

$\Rightarrow$ The accelerations deviate from the unconstrained ones in a LS sense:

$$\ddot{q}_c = \min_{\ddot{q}} \frac{1}{2}\left[\ddot{q} - \ddot{q}_{uc}\right]^T M(q) \left[\ddot{q} - \ddot{q}_{uc}\right]$$

s.t. $J\ddot{q} + \dot{J}\dot{q} = 0$ with $\ddot{q}_{uc} = M^{-1}(S^T\tau - h)$

$$\rightarrow \frac{1}{2}\ddot{q}^T M \ddot{q} \underbrace{- \frac{1}{2}\ddot{q}^T M \ddot{q}_{uc} - \frac{1}{2}\ddot{q}_{uc}^T M \ddot{q}}_{-\ddot{q}_{uc}^T M \ddot{q}} + \cancel{\ddot{q}_{uc}^T \ddot{q}_{uc}}$$

$$\frac{1}{2}\ddot{q}^T M \ddot{q}$$

SOLVE THE QP

$$\ddot{q}_c = \min_{x} \quad \frac{1}{2} x^T G x \quad + \quad g x$$

$$\text{s.t.} \qquad A x = b$$

WITH :

$x = \ddot{q}$

$G = M$

$g = - \ddot{q}_{uc}^T M$

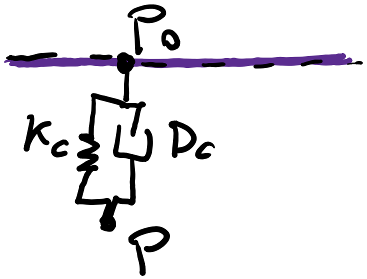
$A = J$

$b = - \dot{J} \dot{q}$

SOFT CONTACT MODEL

Many simulation environments use them

The contact force is given by:



$$f = K_c(P_0 - P) - D_c\dot{P}$$

$$M\ddot{q} + h = S^T\tau + J^T f$$

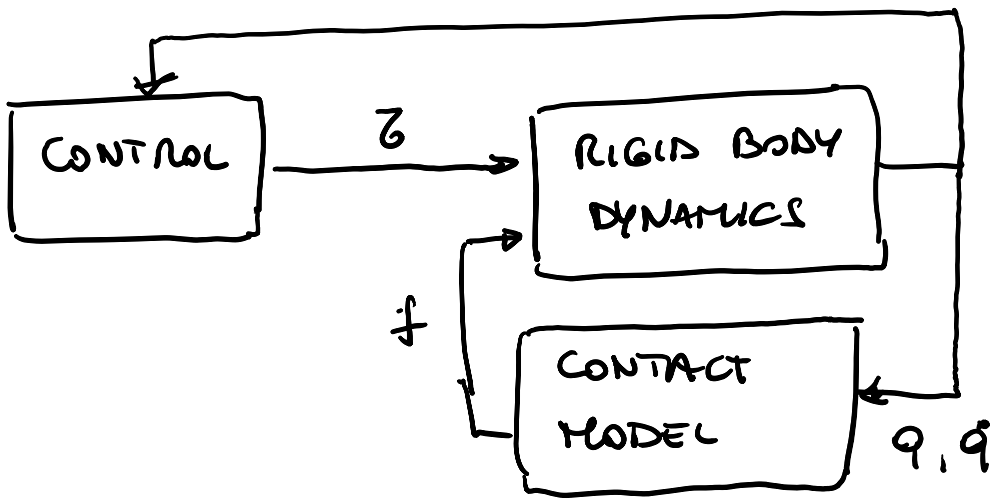
- different directions:

$$f_z = K\,(P_{0z} - P_z) - D\dot{P}_z$$

$$f_x = K\,(P_{0x} - P_x) - D\dot{P}_x$$

- generic law extension (e.g. non linear spring)

$f = \text{law}(p, \dot{p})$



COMPLIANT VS RIGID MODELS

COMPLIANT

⊕ easy to compute

⊖ stiff springs ⇒ ODE numerically stiff ⇒ small integration steps

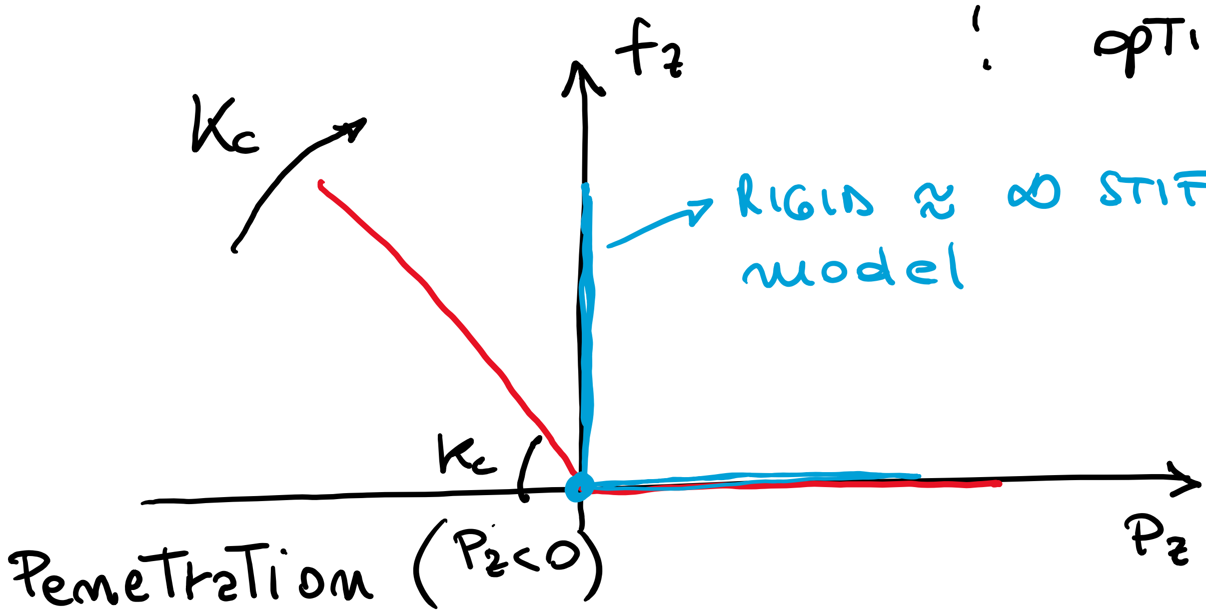
SLOW SIMULATIONS

RIGID

⊕ straightforward to integrate analize

⊕ Preferred in stability analysis

⊖ Pathological to differentiate (e.g. in optimization)



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