① let A_i be the set of permutations where letter i is in the correct envelope.

then our required probability

$A = \dfrac{|A_1 \cup A_2 \cup \cdots \cup A_n|}{N!}$

By using Inclusion - Exclusion principle,

$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k|$
$\cdots + (-1)^{r+1} \sum |A_{i_1} \cap \cdots \cap A_{i_r}| + \cdots$

$|A_i| = (N-1)!$; $|A_i \cap A_j| = (N-2)!$ - - -

$|A_1 \cup A_2 \cup \cdots \cup A_n| = \binom{N}{1}(N-1)! - \binom{N}{2}(N-2)! + \cdots + (-1)^{N}\binom{N}{N}(N-N)!$

$= (N)! \left[1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{N+1} \cdot \frac{1}{N!} \right]$

probability $= 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{N+1} \cdot \frac{1}{N!}$

For $n = 50$.

For large n we can approximate above expression as $1/e$.

$\rightarrow$ probability $= 1/e$.

⑤. let X be the maximum of n Random variables $X_1, X_2, \ldots, X_n$ where each X_i is uniformly distributed over $\{1, 2, \ldots, N\}$.

$\Rightarrow$ CDF of X

$(F_X(k)) = P(X \leq k) = P(\forall X_i \leq k)$

$P(X_i \leq k) = \frac{k}{N}$.

$P(\forall X_i \leq k) = \left(\frac{k}{N}\right)^n$

$P(X = k) = P(X \leq k) - P(X \leq k-1) = \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n$

Expected value $E(X) = \sum_{k=1}^{N} k \cdot P(M=k)$

$$= \sum_{k=1}^{N} k\left[\left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n\right]$$

$$E(X) = \left(\frac{1}{N}\right)^n \cdot \sum_{k=1}^{N} k\left(k^n - (k-1)^n\right)$$

(7). a) No return to originator (1 person per step):

⇒ There are $(n-1)$ people other than the current teller and originator for every step.

probability of not choosing originator at one step $= \frac{n-1}{n}$.

probability for r-steps $= \left(\frac{n-1}{n}\right)^r$.

b) No repetition to any person (1 person per step):

At step-1 → there will be n-chances

step-2 → $n-1$ - chances.

⋮

step-r → $n-r+1$ chances.

$$\text{probability} = \left(\frac{n}{n}\right)\cdot\left(\frac{n-1}{n}\right)\cdot\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-r+1}{n}\right)$$

$$= \frac{n!}{(n-r)!\,n^r}.$$

c) No originator in group (N people per step)

⇒ There are $n-1$ people other than the current teller and originator for every step.

⇒ We have to select group of N from this $n-1$ people.

⇒ probability that originator not in group at each step $= \frac{\binom{n-1}{N}}{\binom{n}{N}} = \frac{n-N}{n}$

For r-steps probability $= \left(\frac{n-N}{n}\right)^r$.

9) No repeats to any person in total (N people per step).

At step-1 → we have to select N people from n people

step-2 → select ~~n~~ N people from n-N people

⋮

step-r → select N ~~-(r-1)N~~ people from N-(r-1)N people 4

$$\text{probability} = \frac{\binom{n}{N}}{\binom{n}{N}}\cdot\frac{\binom{n-N}{N}}{\binom{n}{N}} \cdots \frac{\binom{n-(r-1)N}{N}}{\binom{n}{N}}$$

$$= \frac{\binom{n}{N}\cdot\binom{n-N}{n}\cdots\binom{n-(r-1)N}{N}}{\binom{n}{N}^r} \qquad rN \le n.$$

$$0 \qquad \text{otherwise}$$

8) To prove :

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) \le e^{-P(A_1)-P(A_2)-\cdots-P(A_n)}$$

$$\Rightarrow P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} P(A_i^c). = \prod_{i=1}^{n}(1-P(A_i)).$$

We know that $1-x \le e^{-x}$ $\forall x \in [0,1]$.

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^{-x} - (1-x) = \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\Rightarrow \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \ge 0 \quad \forall \quad x \ge 0.$$

$$\Rightarrow e^{-x} \ge (1-x).$$

then $e^{-P(A_i)} \ge 1 - P(A_i).$

$$\Rightarrow \prod_{i=1}^{n}(1-P(A_i)) \le \prod_{i=1}^{n} e^{-P(A_i)}$$

$$\le e^{-P(A_1)-P(A_2)-\cdots-P(A_n)}$$

(10) prove,

$$E[X] = \int_0^\infty (1 - F(x))\, dx.$$

$$\Rightarrow \quad X(\omega) = \int_0^\infty 1_{[0, X(\omega)]}(x) \cdot dx.$$

This is simply the area under the indicator function from $x = 0$ to $x = X(\omega)$.

$$E[X] = E\left[\int_0^\infty 1_{[0, X(\omega))}(x) \cdot dx\right]$$

$$= \int_\Omega \int_0^\infty 1_{[0, X(\omega))}(x) \cdot dx \cdot dP(\omega)$$

$$= \int_0^\infty \left(\int_\Omega 1_{[0, X(\omega))}(x) \cdot dP(\omega)\right) \cdot dx.$$

$$\Rightarrow \int_\Omega 1_{[0, X(\omega))}(x) \cdot dP(\omega) = P(X > x) = 1 - F(x).$$

$$E[X] = \int_0^\infty \left(\int_\Omega 1_{[0, X(\omega))}(x) \cdot dP(\omega)\right) \cdot dx$$

$$\boxed{E[X] = \int_0^\infty (1 - F(x)) \cdot dx.}$$

We have shown both.

$$E[X] = \int_\Omega \int_0^\infty I_{[0, X(\omega))}(x) \cdot dx \; dP(\omega) = \int_0^\infty (1 - F(x)) \cdot dx.$$