④ (a) for $E(X)$ is finite & $E(X^2)$ is not finited

$E(X)$ ~~$P(X=n) \cdot n$~~

we need, to find $P(X=x)$ such that

$$E(X) = \sum_{\forall x} P(X=x) \cdot x < \infty \ \&$$

$$E(X^2) = \sum_{\forall x} P(X=x) \cdot x^2 \text{ is not finite}$$

atleast, $\Rightarrow P(X=x)$ is the form of $\frac{1}{x^3}$

$\Rightarrow P(X=x)$ can be $\frac{1}{z} \cdot \frac{1}{n^{1+\epsilon}}$ $\quad \epsilon \in (1,2)$

here z is req const, such that $\frac{1}{z} \sum_{\forall x} P(X=x) = 1$

$\boxed{\text{aproved}}$

(b) $E(x)$ is finite & $E(x^2)$ is not finite

we need to find $f(x)$ such that

$$E(x) = \int_{\forall x} f(x) \cdot x \, dx < \infty \ \&$$

$$E(x^2) = \int_{\forall x} f(x) \cdot x^2 \, dx \text{ is not finite}$$

alleast ~~$P(X=$~~ $f(x)) \simeq \frac{1}{x^3}$

$\Rightarrow f(x)$ can be $\frac{1}{z} \frac{1}{x^{1+\epsilon}}$ $\quad \epsilon \in (1,2)$

where z is req. const, such that

$$\frac{1}{z} \int_{\forall x} f(x) \, dx = 1$$

$\boxed{\text{proved}}$

(c) e^{-x} is convex,

Jensen inequality says

$$E(e^{-x}] \geq e^{-E(x)} = e^{-1} = \frac{1}{e}$$

$$E(e^{-x}) \geq \frac{1}{e} > \frac{1}{3}$$

so we cannot have $E(e^{-x}) < \frac{1}{3}$

if $E(X) = 1$

disproved

⑥ ~~Let X,Y be a uniformly~~

Let X, Y be random variables which are independent & uniformly distributed over $[0, d]$

$\Rightarrow$ $f(x) = \begin{cases} \frac{1}{d}, & 0 \le x \le d \\ 0, & \text{o.w} \end{cases}$ lly $f(y) = \begin{cases} \frac{1}{d}, & 0 \le y \le d \\ 0, & \text{o.w} \end{cases}$

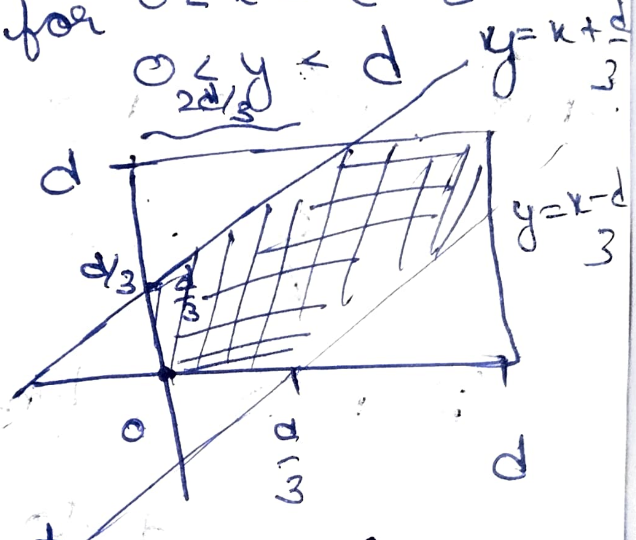
we want $P\left(|X-Y| < \frac{d}{3}\right)$

$|X-Y|$ give $\frac{d}{3}$ line segment

$f(x,y) = \frac{1}{d^2}$ for $0 < x < d$ & $0 \le y < d$

$|X-Y| < \frac{d}{3}$

$y = x + \frac{d}{3}$ $\qquad$ $y = x - \frac{d}{3}$



area of shaded part

$$= d^2 - \frac{1}{2}\left(\frac{2d}{3}\right)\left(\frac{2d}{3}\right) - \frac{1}{2}\left(\frac{2d}{3}\right)\left(\frac{2d}{3}\right)$$

$$= \cancel{\frac{7d^2}{9}} \; \frac{5d^2}{9}$$

probability = volume ~~below~~ of shaded part below $f(x,y)$

$$= \frac{1}{d^2} \cdot \frac{5d^2}{9} = \frac{5}{9}$$

$\Rightarrow$ req prob $= \frac{5}{9}$

(4) by def

$$H(x) = \int_{-\infty}^{\infty} F(x-y)\, G(y)\, dy$$

$H(x)$ should be → (1) non decreasing

(2) right continuous

(3) $\lim_{x\to-\infty} H(x) = 0$ & $\lim_{x\to\infty} H(x) = 1$

(1) for $x_1 < x_2$

F is also non decreasing ⇒ $F(x_1 - y) < F(x_2 - y)$

$$\Rightarrow H(x_1) < H(x_2)$$

(2) $\lim_{x\to x_0^+} H(x) = \int_{-\infty}^{\infty} \lim_{x\to x_0^+} F(x-y)\, G(y)\, dy$

lt can be taken inside $\int$ & F is right continuous

$$= H(x_0)$$

(3) $\lim_{x\to-\infty} H(x) = \int_{-\infty}^{\infty} \lim_{x\to-\infty} F(x-y)\, G(y)\, dy = \int_{-\infty}^{\infty} 0\, G(y)\, dy$

$$= 0$$

$$\Rightarrow \lim_{x\to-\infty} H(x) = 0.$$

$\lim_{x\to+\infty} H(x) = \int_{-\infty}^{\infty} \lim_{x\to\infty} F(x-y)\, G(y)\, dy = \int_{-\infty}^{\infty} 1\cdot G(y)\, dy$

$$= 1$$

$$\Rightarrow \lim_{x\to\infty} H(x) = 1$$

⇒ $H(x)$ is also a distributive function

Similarly can be prove for discrete case too

(11) (i) $E(e^{ux}) = e^{u\mu + \frac{1}{2}u^2\sigma^2}$

$E(e^{ux})$ is moment generating function

$\Rightarrow E(e^{ux}) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{ux} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$\frac{x-\mu}{\sigma} = z$

$\Rightarrow \frac{dx}{\sigma} = dz$

$\Rightarrow E(e^{ux}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\mu+\sigma z)u} e^{-z^2/2} dz$

$E(e^{ux}) = \frac{e^{\mu u}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2\sigma z u + (\sigma u)^2 - (\sigma u)^2)} dz$

$E(e^{ux}) = \frac{e^{\mu u + \sigma^2 u^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma u)^2} dz$

$E(e^{ux}) = e^{\mu u + \frac{\sigma^2 u^2}{2}}$ (proved

no need to prove it is just MGF of $\underline{\underline{N(\mu,\sigma)}}$

(ii) $E[\phi(x)] \geq \phi[E(x)]$ Jensen's inequality

$E[e^{ux}] \geq e^{u[E[x]]} \geq e^{u\mu}$

$\Rightarrow e^{u\mu + \frac{1}{2}u^2\sigma^2} > e^{u\mu}$

$\because \sigma > 0$ we have

$E[e^{ux}] \geq e^{u[E(x)]}$ (inequality holds)