Continuous Mathematical Foundations: Week 3 - Introduction to Probability Theory

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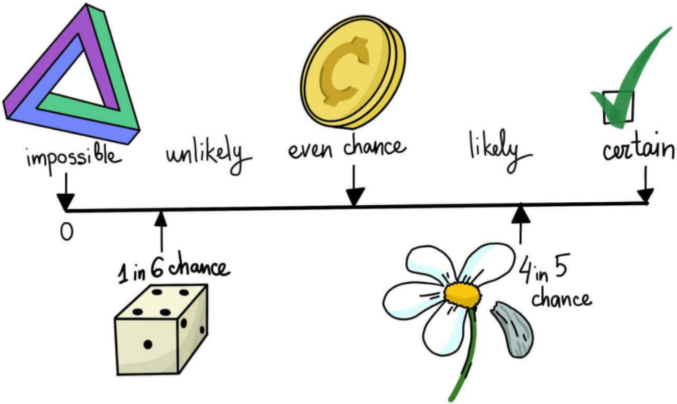
What is Probability?



What is Probability?

"I wish we hadn't learned probability 'cause I don't think our odds are good."

Probability Definition

The word probability derives from the Latin probabilitas, which can also mean "probity", a measure of the authority of a witness in a legal case in Europe, and often correlated with the witness's nobility.

In a sense, this differs much from the modern meaning of probability, which in contrast is a measure of the weight of empirical evidence, and is arrived at from inductive reasoning and statistical inference.

Hacking, I. (2006)

Probability Space

In probability theory, a probability space or a probability triple $(\Omega, \mathcal{F}, P)$ is a mathematical construct that provides a formal model of a random process or "experiment".

Sample Space Ω

Sample Space Ω is the set of all possible outcomes.

Event Space $\mathcal{F}$

Event space is the set of events $\mathcal{F}$, an event being a set of outcomes in the sample space.

In event space, the empty set and the sample space are always included, or else $\emptyset, \Omega \in \mathcal{F}$.

Probability Function P

A probability (set) function assigns each event in the event space a probability, which is a number between 0 and 1.

Probability Definition

The probability of a set A in comparison with the all the possible outcomes Ω, is conservatively defined as

$$P(A) = \frac{|A|}{|\Omega|},$$

where $|\cdot|$ returns the number of elements in a set.

Probability in Trivial Sets

- $P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$
- $P(\emptyset) = \frac{|\emptyset|}{|\Omega|} = \frac{0}{|\Omega|} = 0$
- $P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{|\Omega|-|A|}{|\Omega|} = \frac{|\Omega|}{|\Omega|} - \frac{|A|}{|\Omega|} = 1 - P(A)$
- $P(\Omega_{1/2}) = \frac{|\Omega_{1/2}|}{|\Omega|} = \frac{\frac{|\Omega|}{2}}{|\Omega|} = \frac{1}{2}$

($\Omega_{1/2}$ is a subset of Ω with half of Ω elements)

Probability Space Example 1

In a fair coin toss,

- $\Omega = \{\text{“}heads\text{”}, \text{“}tails\text{”}\}$,
- $\mathcal{F} = \{\emptyset, \text{“}heads\text{”}, \text{“}tails\text{”}, \Omega\}$,
- $P(\text{“}heads\text{”}) = P(\text{“}tails\text{”}) = \frac{1}{2}$.

Probability Space Example 2

In an unfair coin toss,

- $\Omega = \{\text{“}heads\text{”}, \text{“}tails\text{”}\}$,
- $\mathcal{F} = \{\emptyset, \text{“}heads\text{”}, \text{“}tails\text{”}, \Omega\}$,
- $P(\text{“}heads\text{”}) = \frac{99}{100}$, $P(\text{“}tails\text{”}) = \frac{1}{100}$.

Probability function

P is a set function, transforming sets to probability. If sets are replaced by numbers, we get p, a normal real number function,

$$P : \mathcal{F} \to [0, 1],$$

$$p : \mathbb{R} \to [0, 1].$$

Probability function Examples

In a fair coin toss, we can set $\{$"*heads*"$\} := 1$, $\{$"*tails*"$\} := 0$. As a result,

- $\Omega = \{1, 0\}$,
- $\mathcal{F} = \{\emptyset, 1, 0, \Omega\}$,
- $p(1) = p(0) = \frac{1}{2}$.

Probability Notation

In the interior of the parenthesis of $p(\cdot)$ we only write single numbers.

In the interior of the parenthesis of $P(\cdot)$ we write sets with set notation $(A, B, \Omega, \dots)$ or set description (“$Coin = heads$”, “$Dice = 2''$, …).

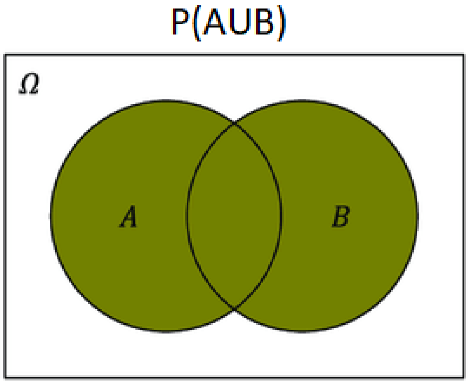
We separate sequential set assumptions with commas, i.e. $P(Coin = \text{“}heads\text{”}, Dice = 2, Card = \text{“}ace\text{”}, Roulette = 34)$.

In general in math, set functions are more robust to describe real world phenomena, yet number functions are easier to work with and bear more properties as we will see in next weeks.

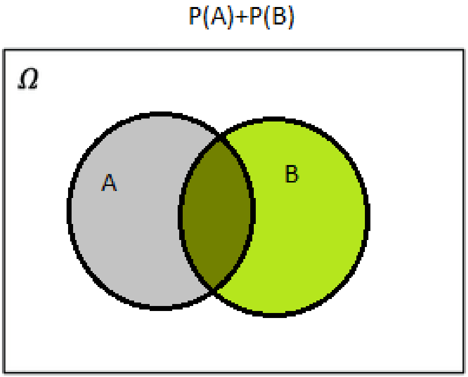
Addition Rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Addition Rule in Venn Diagram



Sum in Venn Diagram



Independency

If two sets A, B are independent,

$$P(A \cap B) = P(A) \times P(B).$$

Two sets being independent means that the result of any them doesn't affect the result of the other.

Independent Phenomena examples

- Independent Phenomena: The result of a dice toss and of a coin toss (there is no mechanism that connects the two happenings).
- Not Independent Phenomena: Two cards from a card deck (The first draw affects the probability of the second draw).

Empty Set Independency

Empty set $\emptyset$ is independent with anything,

$$P(\emptyset \cap A) = P(\emptyset) = 0, \ (1)$$

$$P(\emptyset) \times P(A) = 0 \times P(A) = 0, \ (2)$$

$$(1), (2) \Rightarrow P(\emptyset \cap A) = P(\emptyset) \times P(A) = 0.$$

Applied Independency

We usually use independency for calculations. If two events are independent the joined probability is equal to the multiplication of the corresponding probabilities (multiplication rule).

Independency Example

The probability for two fair coin tosses (obviously independent) to be heads is equal to,

$$\begin{aligned} P(Coin_1 = \text{``}heads\text{''}, Coin_2 = \text{``}heads\text{''}) = \\ &= P(Coin_1 = \text{``}heads\text{''}) \times P(Coin_2 = \text{``}heads\text{''}) \\ &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}. \end{aligned}$$

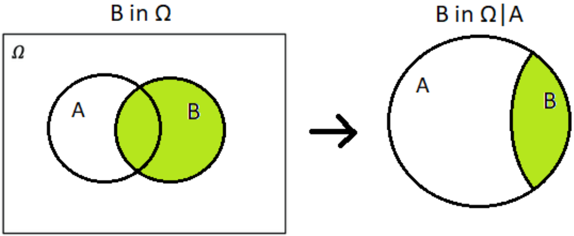
Conditional Probability

Conditional Probability is a measure of the probability of an event occurring (A), given that another event (B) (by assumption, presumption, assertion or evidence) has already occurred,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Conditionality in Venn Diagram

We use Conditional Probability to restrict measure spaces in smaller sets we want emphasize more.



Example

- In two coin tosses we check the probability of both coins.
- In two coin tosses if we know the first, then we only look for the probability of the second.

Same Example in Algebra

- In two (independent) coin tosses,

$$P(Coin_1, Coin_2) = P(Coin_1) \times P(Coin_2) = \frac{1}{4}.$$

- In two coin tosses if we know the first,

$$P(Coin_2|Coin_1) = \frac{P(Coin_1, Coin_2)}{P(Coin_1)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$

Goals of the Week

- Understand what probability is.
- Being able to model probabilistic phenomena such as a coin toss, a dice roll or card games.
- Being able to work in problems with probability properties such as independency and conditionality.