Continuous Mathematical Foundations: Introduction to Differentiation

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Differentiation

The derivative of a continuous function $f(x)$ in an x_0 point is calculated as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Differentiation in bivariate functions

The derivative of a continuous function $f(x, y)$ with respect to x, in an x_0 point is calculated as

$$\frac{df}{dx}(x_0, y) = \lim_{x \to x_0} \frac{f(x, y) - f(x_0, y)}{x - x_0}.$$

Differentiation of f(x)=c

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c - c}{x - x_0} = 0.$$

Differentiation of f(x)=x

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1.$$

Basic Derivatives (you need to know them)

$$(x^n)' = n(x^{n-1})$$
$$(a^x)' = a^x \, ln(a)$$
$$(e^x)' = e^x$$
$$(\ln x)' = \frac{1}{x}$$
$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$
$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$
$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Power Properties

- $x^k = xx \dots x \ (k \ times)$
- $x^{-k} = \frac{1}{x^k}$
- $x^{m-k} = \frac{x^m}{x^k}$
- $x^{1/2} = \sqrt{x}$

Example 1: Power function

- $(x^2)' = 2x$
- $(x^4)' = 4x^3$
- $(x^{1000})' = 1000x^{999}$
- $(x^{10^{10}})' = 10^{10}x^{(10^{10}-1)}$
- $(x^e)' = ex^{(e-1)}$
- $(x^{-5})' = -5x^{-6}$

Example 2: Double Differentiation

- $((x^2)')' = (2x)' = 2$
- $((x^{1000})')' = (1000x^{999})' = 1000 \times 999x^{998}$

Differentiation Properties

- $(f(x) + g(x))' = f'(x) + g'(x)$
- $(f(x) - g(x))' = f'(x) - g'(x)$
- $(kf(x))' = kf'(x)$
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- $f(g(x))' = f'(g(x))g'(x)$

Example 1: Polynomials

$$f'(x) = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)'$$
$$f'(x) = (a_n x^n)' + (a_{n-1}x^{n-1})' + \cdots + (a_1 x)' + (a_0)'$$
$$f'(x) = a_n (x^n)' + a_{n-1}(x^{n-1})' + \cdots + a_1 (x)' + (a_0)'$$
$$f'(x) = a_n n(x^{n-1}) + a_{n-1}(n-1)(x^{n-2}) + \cdots + a_1(1) + (0)$$
$$f'(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + \cdots + a_1$$

Example 2: Tripple Differentiation

$(((x^{1000})')')' = ((1000x^{999})')' = 1000((x^{999})')' = 1000(999x^{998})' =$
$1000 \times 999(x^{998})' = 1000 \times 999 \times 998x^{997}$

Example 3: Division

$$f'(x) = (\frac{g(x)}{h(x)})'$$

$$f'(x) = (g(x)\frac{1}{h(x)})'$$

$$f'(x) = g'(x)\frac{1}{h(x)} + g(x)(\frac{1}{h(x)})'$$

$$f'(x) = g'(x)\frac{1}{h(x)} - g(x)\frac{1}{h(x)^2}h'(x)$$

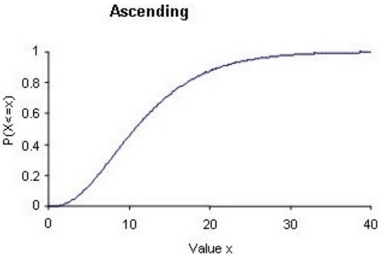
Why we need dervatives??

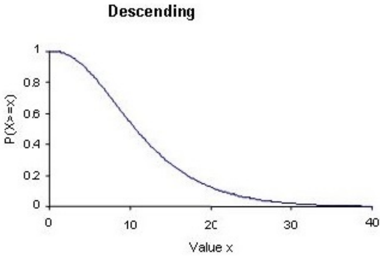
- Arrange functions' monotonicity.
- Find functions' optimal points.

Monotonicity

A function $f : \mathbb{R} \to \mathbb{R}$ is monotonic in an area $A \subset \mathbb{R}$ if in this area is only ascending or only descending.

Monotonicity





Derivatives in Monotonicity

- if $f'(x) < 0 \ \forall x \in A \Rightarrow f(x)$ descending in A
- if $f'(x) > 0 \ \forall x \in A \Rightarrow f(x)$ ascending in A

Optimal Point

An optimal point for a function $f : \mathbb{R} \to \mathbb{R}$ is a point $(x_0, f(x_0))$ in which the function has a maximum or a minimum.

Optimal Points

Derivatives in Optimal Points

The x_0 points where $f'(x_0) = 0$ are possible optimal points for $f(x)$.

If x_0 is optimal point $\Rightarrow f'(x_0) = 0$, but the vice versa case is not necessarily true.

Maximum or minimum?

You can check what happens to values close to $x_0 : f'(x_0) = 0$ in order to determing if it is maximum or minimum.

Example 1: Exponential

$$f(x) = e^x \Rightarrow f'(x) = e^x > 0$$

As a result $f(x)$ is ascending with no optimal points.

Example 2: Logarithm

$$f(x) = log(x),\ x > 0 \Rightarrow f'(x) = \frac{1}{x} > 0,\ as\ x > 0$$

As a result $f(x)$ is ascending with no optimal points.

Example 3: 3rd degree Polynomial

$$f(x) = x^3 - \frac{9}{2}x^2 + 6x - 1,\ x \in \mathbb{R} \Rightarrow f'(x) = 3x^2 - 9x + 6x$$

In order to determine $f'(x)$'s sign you have to use the discriminant (Δ).

Example 3: 3rd degree Polynomial

$$\underset{a}{3x^2} \underset{b}{- 9x} + \underset{\gamma}{6} \qquad \Delta = b^2 - 4a\gamma, \quad x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$\underline{\Delta} = (-9)^2 - 4 \cdot 36$$

$$= 81 - 72 = 9$$

$$\underline{x} = \frac{9 \pm \sqrt{9}}{2 \cdot 3} = \frac{9 \pm 3}{6} = \begin{cases} \frac{12}{6} = 2 \\ \frac{6}{6} = 1 \end{cases}$$

x	$-\infty$	1		2	$+\infty$
$f'(x)$	+	0	−	0	+
$f(x)$	↗		↘		↗

Optimal points: − 1 max

− 2 : min

L' Hopital Rule

If $f(x) \to \infty$ and $g(x) \to \infty$ for $x \to x_0$,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Example 1: Power

$$\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{(x)'}{(x^2+1)'} = \lim_{x\to\infty} \frac{1}{2x} = 0$$

Example 2: Exponential

$$\lim_{x\to\infty} \frac{e^x}{x+1} = \lim_{x\to\infty} \frac{(e^x)'}{(x+1)'} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

Optimization in Statistics

Mark and recapture is a common technique to study a population in Biostatistics. Biostatisticians mark a free faunal population with sensors and observe how they behave in short time intervals. For a lizard population observed one year after the marking, we noticed that $10/20$ survived and $10/20$ died. We will estimate the probability for a lizard to survive a year, based on the likelihood of the sample.

Solution

The likelihood of the sample is calculated as the probability (independent observations),

$$L(p) = P(L_1, L_2, \ldots, L_{20}) = P(L_1)P(L_2)\ldots P(L_{20}) = p^{10}(1-p)^{10}.$$

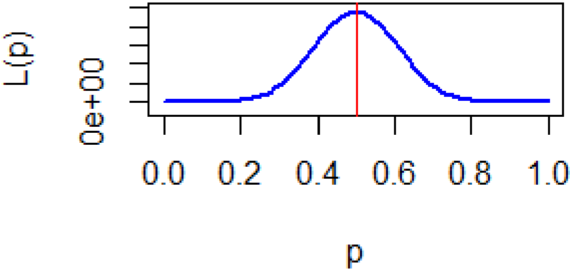
We will find the p that maximizes the likelihood $L(P)$ in order to make the estimation.

Differentiation of Likelihood

$$L'(p) = (p^{10}(1-p)^{10})' = 10p^9(1-p)^{10} - 10p^{10}(1-p)^9.$$

So the optimal value is for $p = \frac{1}{2}$ and it is maximum.

Likelihood Plot



(you can always use your computer to explore the function)

Goals of the Week

- Understand what differentiation is.
- Know how to calculate derivatives.
- Being able to apply L' Hopital Rule.
- Acknowledge the use of derivatives in Optimization.