# Continuous Mathematical Foundations: Introduction to Linear Algebra

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# Why linear?

Because we will treat linear objects.

# Why algebra?

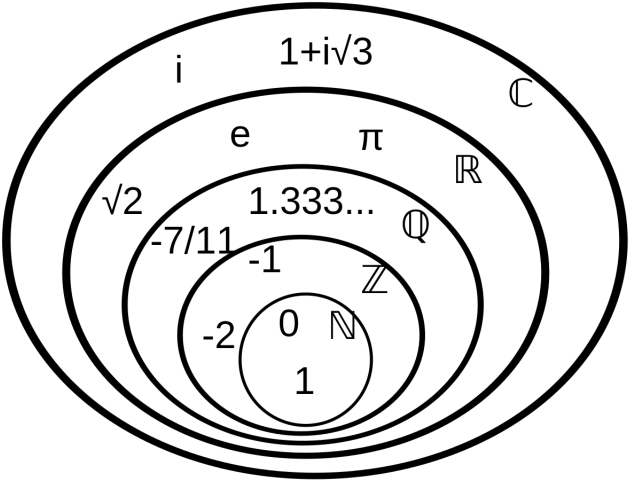
Because we discuss about operations in the objects.

# Vectors

An $\mathbf{x}$ $n$-dimensional vector is written as,

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n).$$

# Number Sets



# Vectors as elements of a Set

An $\mathbf{x}$ $n$-dimensional vector of real number elements is written as,

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n).$$

Where $x_1, x_2, x_3, \ldots, x_n \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

# Vectors as elements of a Set

Obviously, the elements of the vector and subsequently the whole vector can be from any Set of choice.

Yet, in this class we care only about vectors in $\mathbb{R}^n$.

# Sum of vectors

If $\mathbf{x} \in \mathbb{R}^n$ is

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$$

and $\mathbf{y} \in \mathbb{R}^n$ is

$$\mathbf{y} = (y_1, y_2, y_3, \ldots, y_n),$$

then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots, x_n + y_n).$$

# Multiplication on vectors

If $\mathbf{x} \in \mathbb{R}^n$ is

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$$

and $\lambda \in \mathbb{R}$, then

$$\lambda\mathbf{x} = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots, \lambda x_n).$$

# Inner product

If $\mathbf{x} \in \mathbb{R}^n$ is

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$$

and $\mathbf{y} \in \mathbb{R}^n$ is

$$\mathbf{y} = (y_1, y_2, y_3, \ldots, y_n),$$

then

$$\mathbf{xy} = x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

# Note

You can only sum and calculate inner products for vectors of equal dimension.

# Matrices

An $\mathbf{X}$ $m, n$-dimensional matrix is written as,

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \dots \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}.$$

The number of rows of the matrix is equal to $m$ and the number of columns to $n$.

# Matrices as elements of a Set

An $\mathbf{X}$ $m, n$-dimensional matrix of real number elements has elements $x_{11}, x_{12}, \ldots, x_{mn} \in \mathbb{R}$ and $\mathbf{X} \in \mathbb{R}^{m \times n}$.

# Sum of Matrices

If

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \dots \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}$$

and

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} & y_{13} & \dots & y_{1n} \\ y_{21} & y_{22} & y_{23} & \dots & y_{2n} \\ \dots \\ y_{m1} & y_{m2} & y_{m3} & \dots & y_{mn} \end{bmatrix},$$

then

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} x_{11} + y_{11} & x_{12} + y_{12} & x_{13} + y_{13} & \dots & x_{1n} + y_{1n} \\ x_{21} + y_{21} & x_{22} + y_{22} & x_{23} + y_{23} & \dots & x_{2n} + y_{2n} \\ \dots \\ x_{m1} + y_{m1} & x_{m2} + y_{m2} & x_{m3} + y_{m3} & \dots & x_{mn} + y_{mn} \end{bmatrix}.$$

# Multiplication on Matrices

If

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \dots \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}$$

and $\lambda \in \mathbb{R}$ then

$$\lambda\mathbf{X} = \begin{bmatrix} \lambda x_{11} & \lambda x_{12} & \lambda x_{13} & \dots & \lambda x_{1n} \\ \lambda x_{21} & \lambda x_{22} & \lambda x_{23} & \dots & \lambda x_{2n} \\ \dots \\ \lambda x_{m1} & \lambda x_{m2} & \lambda x_{m3} & \dots & \lambda x_{mn} \end{bmatrix}$$

# Matrix times Matrix

If

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \dots & & & & \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}$$

and

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} & y_{13} & \dots & y_{1m} \\ y_{21} & y_{22} & y_{23} & \dots & y_{2m} \\ \dots & & & & \\ y_{n1} & y_{n2} & y_{n3} & \dots & y_{nm} \end{bmatrix},$$

then

$$\mathbf{XY} = \left[ \sum_{k=1}^{m} x_{ik} y_{kj} \right]_{\{i,j=1,2,\dots,n\}}.$$

# Note

You can only sum matrices of equal dimension and multiply matrices with equality between rows and columns.

# Vector times Matrix

Think of vectors as a matrix with dimension of rows or columns equal to 1, depending the case.

# Identity Matrix of dimension $n$

$$I_{nn} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

# Identity Matrix

It is called Identity Matrix as for any matrix $\mathbf{X}$ of respective dimensions,

$$\mathbf{X}I = I\mathbf{X} = \mathbf{X}.$$

# Diagonal Matrix of dimension $n$

$$\mathbf{X}_{nn} = \begin{bmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & x_n \end{bmatrix}$$

# Transpose Matrix

A Transpose Matrix $\mathbf{X}^T$ is defined by an initial $\mathbf{X}$ matrix, by changing places between rows and columns, i.e.

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \dots & & & & \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}$$

and

$$\mathbf{X}^T = \begin{bmatrix} x_{11} & x_{21} & x_{31} & \dots & x_{m1} \\ x_{12} & x_{22} & x_{32} & \dots & x_{m2} \\ \dots & & & & \\ x_{1n} & x_{2n} & x_{3n} & \dots & x_{mn} \end{bmatrix}$$

# Inverse Matrix

An inverse matrix $\mathbf{X}^{-1}$ is is defined by an initial $\mathbf{X}$ matrix when

$$\mathbf{X}\mathbf{X}^{-1} = \mathbf{X}^{-1}\mathbf{X} = I.$$

# Note

You can always calculate the transpose matrix, by an inverse matrix does not always exists.

# Linear Equation System

A linear equation system is of the form,

$$
\begin{aligned}
a_{11}x_1 + a_{11}x_2 + \ldots + a_{1n}x_n &= y_1 \\
a_{21}x_1 + a_{21}x_2 + \ldots + a_{2n}x_n &= y_2 \\
&\ldots \\
a_{n1}x_1 + a_{n1}x_2 + \ldots + a_{nn}x_n &= y_n
\end{aligned}
$$

# Linear System in Matrix form

Please note that you can write the system as

$$\mathbf{A}\mathbf{x} = \mathbf{y}.$$

# Linear System Solution

Please note that the previous relation is equivalent to

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}.$$

# Determinant for $2, 2$-matrix

$$det(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}) = a_{11}a_{22} - a_{12}a_{21}.$$

# Eigenvalues

Eigenvalues of an $\mathbf{X}$ matrix are the $\lambda$ values that solve the equation

$$det(\mathbf{X} - \lambda I) = 0.$$

# Eigenvectors

Eigenvectors of an $\mathbf{X}$ matrix are the $u$ vectors that solve the equation

$$\lambda \mathbf{X} = \lambda u,$$

where $\lambda$ are the eigenvalues.

# Properties of Linear Equation Systems

- A system can have a unique solution, infinite solutions or no solution at all.
- If the $\mathbf{A}$'s rows are less than its columns then the system can't have unique solution.
- Determinants are only positive or 0.
- If $\mathbf{A}$ has eigenvalue equal to 0, $\mathbf{A}$ cannot be inverted.
- If the number of unique, positive eigenvalues of $\mathbf{A}$ is the same with the $\mathbf{x}$'s dimensions, then the system has unique solution.

# Goals of the Week

- Understand vectors and how to do operations with them.
- Understand matrices and how to do operations with them.
- Learn to solve linear systems.