Continuous Mathematical Foundations: Week 3 - Discrete Probability Distributions

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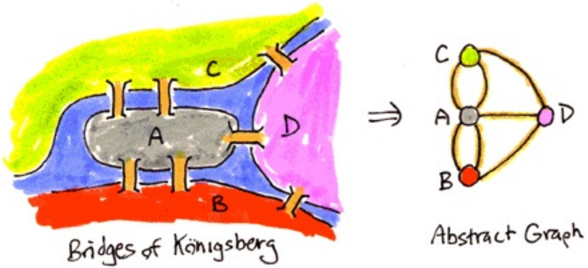
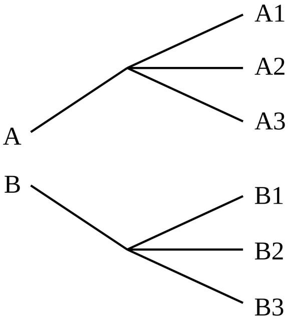
Combinatorics

Combinatorics is an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures.



Rule of product

If there are a ways of doing something and b ways of doing another thing, then there are $a \cdot b$ ways of performing both actions.



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What happens when there are more than 2 ways and cross cases are eliminated?? For example how would you calculate all possible password fora service, if you know that it is only of letters and can not be repeated??

Factorial

$$1! = 1 = 1$$

$$2! = 2 \times 1 = 2$$

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

Factorial Properties

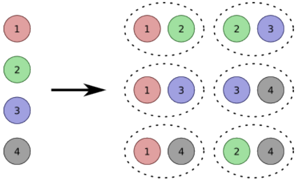
- $k! = 1 \times 2 \times 3 \times 4 \ldots \times k$
- $0! = 1$
- $\frac{n!}{(n-1)!} = n$

Combination

How many combinations do we have when we want to separate 4 students in teams of 2?

Combination

In order to calculate combinations of n elements in groups of r elements $C(n, r) = \frac{n!}{r!(n-r)!}$.



Combination Properties

- $C(n, 0) = 1$
- $C(n, 1) = n$
- $C(n, n) = 1$

Examples

- You want to count all the possible plates for vehicles if you know it's 3 character and 3 numbers.
- You want to count all the possible drawings from a classic card deck, for the 5 first cards.
- You want to count how many choices you have to park 3 cars in six parking places.

Distribution in Probability

A probability distribution is the set of all the probabilities of occurrence of all possible outcomes for an experiment.

Distribution in Coin Toss

$$D_{coin} = \left\{ P(Coin = HEADS) = \frac{1}{2},\ P(Coin = TAILS) = \frac{1}{2} \right\}$$

Distribution in Dice

$$D_{dice} = \left\{ P(Dice = 1) = \frac{1}{6},\ P(Dice = 2) = \frac{1}{6}, \dots, P(Dice = 6) = \frac{1}{6} \right\}$$

Discrete Probability Distribution

Let $f(x)$ be a function of x that satisfies

$$\sum_{\Omega} f(x) = 1 \qquad \text{and} \qquad f(x) > 0,\ x \in \mathcal{A}$$

For some set $A \subset \mathbf{\Omega}$ whenever $P(A)$ can be expressed

$$P(A) = P(X \in A) = \sum_{A} f(x)$$

then X is a discrete random variable and $f(x)$ is called a *probability mass function*. Other common terms for $f(x)$:

- probability density function
- probability function

Discrete Distribution Example

Example 1. The probability mass function for X, the number of heads that appear in two tosses of a fair coin, is

$$f(x) = \begin{cases} 1/4 & x = 0 \\ 1/2 & x = 1 \\ 1/4 & x = 2 \end{cases}$$

Measures of Discrete Distributions

- mean (E(X)): $\mu = \sum_{x \in \Omega} \ x \ f(x)$
- variance (V(X)): $\sigma^2 = \sum_{x \in \Omega} \ (x - \mu)^2 f(x)$

Discrete Distributions

X	X Counts	$p(x)$	Values of X	$E(x)$	$V(x)$
Discrete uniform	Outcomes that are equally likely (finite)	$\frac{1}{b-a+1}$	$a \leq x \leq b$	$\frac{b+a}{2}$	$\frac{(b-a+2)(b-a)}{12}$
Binomial	Number of sucesses in n fixed trials	$\binom{n}{x} p^x (1-p)^{n-x}$	$x = 0,1,\ldots,n$	np	$np(1-p)$
Poisson	Number of arrivals in a fixed time period	$\frac{e^{-\lambda}\lambda^x}{x!}$	$x = 0,1,2,\ldots$	λ	λ
Geometric	Number of trials up through 1st success	$(1-p)^{x-1}p$	$x = 1,2,3,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative Binomial	Number of trials up through kth success	$\binom{x-1}{k-1}(1-p)^{x-k}p^k$	$x = k, k+1,\ldots$	$\frac{k}{p}$	$\frac{k(1-p)}{p^2}$
Hyper - geometric	Number of marked individuals in sample taken without replacement	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$	$\max(0, M+n-N) \leq x \leq \min(M,n)$	$n * \frac{M}{N}$	$\frac{nM(N-M)(N-n)}{N^2(N-1)}$

Comments

- Discrete uniform distribution gives equal probability to all outcomes (coin, dice, etc).
- Hypergeometric distribution provides probability for an experiment that is repeated but each outcome is not repeatable (sequential card drawing from a card deck).
- Binomial distribution provides probability for an experiment that is repeated under the same condition (sequential coin tosses or dice rolls).
- Negative Binomial distribution provides probability for an experiment that is repeated until something happens (sequential coin tosses until 3 HEADs).

Goals of the Week

- Know the properties of discrete distributions.
- Know how to calculate measures of discrete distributions.
- Know the use of Binomial, Hypergeometric and Geometric distributions.