Continuous Mathematical Foundations: Introduction to Limits

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Function Definition

A function f from a set X to a set Y is an assignment of an element of Y to each element of X. The set X is called the domain of the function and the set Y is called the codomain of the function.

Function Notation

A function $f : X \to Y$ assigns elements from X to Y, or else
$f : x \in X \to f(x) \in Y$.

Function use

In order to observe the function-assignment mechanism, we use values on x and see what happens to $f(x)$.

Example 1

If $f(x) = x + 1, \; x \in \mathbb{R}$, then,

$$\text{for } x = 0, \; f(0) = 0 + 1 = 1,$$

$$\text{for } x = 1, \; f(1) = 1 + 1 = 2,$$

$$\text{for } x = 2, \; f(2) = 1 + 2 = 3.$$

Example 2

If $f(x) = x^2, \ x \in \mathbb{R}$, then,

$$\textit{for } x = 0, \ f(0) = 0^2 = 0,$$

$$\textit{for } x = 1, \ f(1) = 1^2 = 1,$$

$$\textit{for } x = 2, \ f(2) = 2^2 = 4.$$

Example 3

If $f(x) = x^2 + 1, \; x \in \mathbb{R}$, then,

$$\begin{aligned} &\textit{for } x = 0, \; f(0) = 0^2 + 1 = 1, \\ &\textit{for } x = 1, \; f(1) = 1^2 + 1 = 2, \\ &\textit{for } x = 2, \; f(2) = 2^2 + 1 = 5. \end{aligned}$$

Limit Definition

In mathematics, a limit is the value that a function f approaches as the x-variable approaches some value (x_0).

Limit Notation

The notation for the limit of f in x_0 is

$$\lim_{x \to x_0} f(x).$$

Limit Calculation

In order to calculate the limits, we replace x values with x_0.

Limit Example 1

If $f(x) = x + 1,\ x \in \mathbb{R}$, then,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x + 1) = 0 + 1 = 1,$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x + 1) = 1 + 1 = 2,$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x + 1) = 2 + 1 = 3.$$

Limit Example 2

If $f(x) = x^2,\ x \in \mathbb{R}$, then,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0^2 = 0,$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} x^2 = 1^2 = 1,$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 2^2 = 4.$$

Limit Example 3

If $f(x) = x^2 + 1,\ x \in \mathbb{R}$, then,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 + 1) = 0^2 + 1 = 1,$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 1) = 1^2 + 1 = 2,$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2 + 1) = 2^2 + 1 = 5.$$

Limit and function Difference

Functions are about reshaping sets,
Limits show functions' behavior in certain areas (mobility).

In a nutshell

Limits can take values equal to infinity or be equal to infinity.
Functions cannot.

Note

When the limit doesn't contain infinities or division with 0, more often than not it does not differ with how we calculate functions.

What is Infinity??

The unlimited extent of a variable.

Infinity ∞

https://www.youtube.com/watch?v=TKPBvUnOgyk

Infinity Operations

- $\infty + c = \infty$
- $-\infty + c = -\infty$
- $\infty + \infty = \infty$
- $-\infty - \infty = -\infty$
- $c \times \infty = sign(c) \times \infty$
- $\frac{1}{\infty} = 0$
- $\frac{1}{0} = \pm\infty$

Infinity NOT Operations

- $\infty - \infty$
- $0 \times \infty$
- $\frac{\infty}{\infty}$
- ∞^0
- 0^∞
- 0^0
- 1^∞

Indeterminate Forms

The operations in the previous slide are called Indeterminate and they don't provide any results. They actually represent every possible number.

https://www.youtube.com/watch?v=oc0M1o8tuPo

Undefined Limits

Undefined Limits are the limits of functions that in infinity repeat patterns instead of converging to values, i.e.

$$\lim_{x \to \infty} sin(x), \ \lim_{x \to \infty} cos(x).$$

Identity function

$$f(x) = x, \; x \in \mathbb{R},$$

$$\lim_{x \to \infty} x = \infty,$$

$$\lim_{x \to -\infty} x = -\infty.$$

Square function

$$f(x) = x^2, \; x \in \mathbb{R},$$

$$\lim_{x \to \infty} x^2 = \infty,$$

$$\lim_{x \to -\infty} x^2 = \infty.$$

Power function

$$f(x) = x^k, \; x \in \mathbb{R},$$

$$\lim_{x \to \infty} x^k = \infty,$$

$$\lim_{x \to -\infty} x^k = (-1)^k \infty.$$

Exponential function

$$f(x) = e^x, \ x \in \mathbb{R},$$

$$\lim_{x \to \infty} e^x = \infty,$$

$$\lim_{x \to -\infty} e^x = 0.$$

Logarithmic function

$$f(x) = log(x), \; x \in (0, \infty),$$

$$\lim_{x \to \infty} log(x) = \infty,$$

$$\lim_{x \to 0} log(x) = -\infty.$$

Polynomials

$$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0} = \lim_{x\to\pm\infty} \frac{a_n x^n}{a_m x^m}.$$

Exponential-Polynomial fraction

$$\lim_{x \to \infty} \frac{e^x}{a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0} = \infty.$$

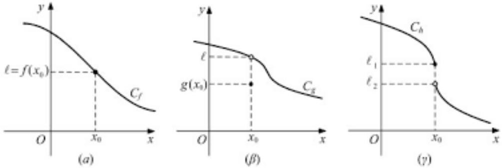
Continuity

A function is called continuous when

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Practically, for a function to be continuous means that its representation in $x - y$ axes is not separated.

Examples



Non-continuity Examples

- $f(x) = \frac{1}{x}$ $(x = 0)$.
- functions with different form among shapes.

Goals of the Week

- Understand the application of function and limits.
- Know how to do calculations with infinity and when something is indeterminate.
- Know what does it mean for a function to be continuous.