Continuous Mathematical Foundations: Introduction to Intergration

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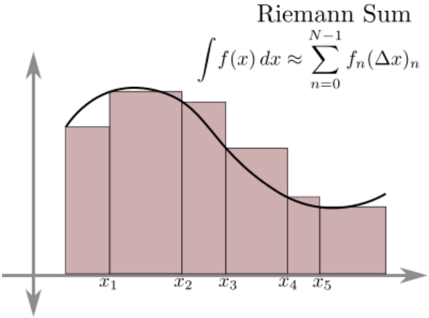
(Riemann) Integration

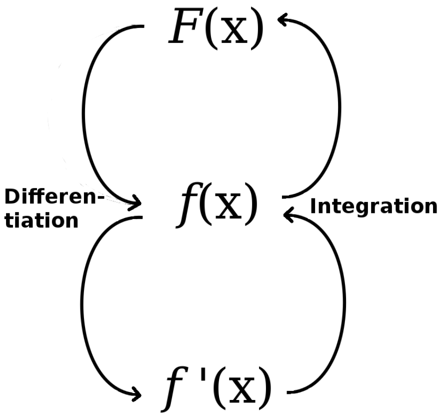
The integral for a continuous function $f : \mathbb{R} \to \mathbb{R}$ is defined as,

$$\int_X f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Integration in Geometry



Differentiation and Integration

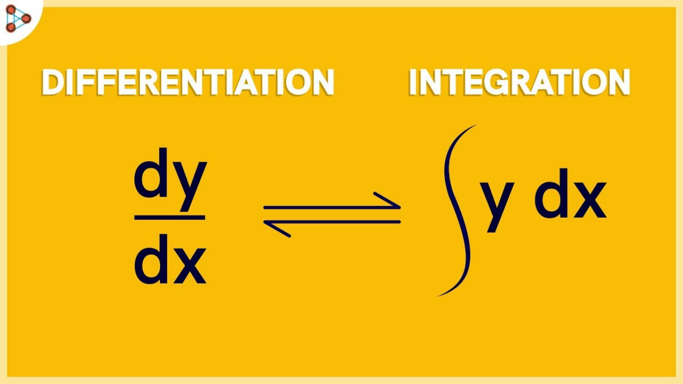


Differentiation and Integration in Physics

The rate (derivative) of movement is speed and the addition of speed with respect to time (integration) returns movement range.

Similarly, the rate of progress for a function is described by differentiation and the addition of the derivative with respect to the variable returns the function.

Differentiation and Integration, Leibniz notation



Indefinite Integrals

The notation for the indefinite integral is,

$$\int f(x)dx.$$

For a continuous function $f : \mathbb{R} \to \mathbb{R}$, with continuous derivative $f' : \mathbb{R} \to \mathbb{R}$,

$$\int f'(x)dx = f(x) + C.$$

Basic Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln(a)} + C$$

$$\int \sin(x) dx = -\cos(x) + C$$

$$\int \cos(x) dx = \sin(x) + C$$

Definite Integrals

The notation for the definite integral is,

$$\int_a^b f(x)dx.$$

For a continuous function $f : \mathbb{R} \to \mathbb{R}$, with continuous derivative $f' : \mathbb{R} \to \mathbb{R}$,

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Example 1

In the previous class we had mentioned that,

$$(x^n)' = nx^{n-1}.$$

As a result,

$$\int nx^{n-1}dx = \int (x^n)'dx = x^n + C,$$

and

$$\int_0^1 nx^{n-1}dx = \int_0^1 (x^n)'dx = 1^n - 0^n = 1.$$

Example 2

We calculate the following derivative,

$$(log(log(x))' = \frac{1}{xlog(x)}.$$

As a result,

$$\int \frac{1}{xlog(x)} dx = log(log(x)) + C.$$

For $f(x) = \int \frac{1}{xlog(x)} dx$ and $f(e) = 0$,

$$f(e) = log(log(e)) + C = log(1) + C = C,$$

as a result $C = 0$ and $f(x) = log(log(x)$.

Example 3

In the previous class we had mentioned that, $(sin(x))' = cos(x)$ and $(cosx(x))' = -sin(x)$. As a result,

$$\int \Big(\int -sin(x)dx \Big) dx = \int \Big(cos(x) \Big) dx = sin(x) + C.$$

Integration Properties

- $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$
- $\int kf(x)dx = k \int f(x)dx$
- $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

Addition Property Proof

$$\begin{aligned}
\int_X f(x) \pm g(x)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i) \pm g(x_i))\Delta x_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i)\Delta x_i \pm g(x_i)\Delta x_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i)\Delta x_i) \pm \sum_{i=1}^{n} (g(x_i)\Delta x_i) \\
&= \int_X f(x)dx \pm \int_X g(x)dx
\end{aligned}$$

Product Property Proof

$$
\begin{aligned}
\int_X kf(x)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} kf(x_i)\Delta x_i \\
&= k \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i \\
&= k \int_X f(x)dx
\end{aligned}
$$

Example 1

For $f(x) = x^{99} + x^{-99}$,

$$\begin{aligned}\int f(x)dx &= \int x^{99} + x^{-99} dx \\ &= \int x^{99} dx + \int x^{-99} dx \\ &= \frac{1}{100} x^{100} - \frac{1}{98} x^{-98} + C\end{aligned}$$

Example 2

For $f(x) = xe^x$,

$$\begin{aligned}
\int f(x)dx &= \int xe^x dx \\
&= \int x(e^x)' dx \\
&= xe^x - \int (x)' e^x dx \\
&= xe^x - \int e^x dx \\
&= xe^x - e^x + C
\end{aligned}$$

Example 3

For $f(x) = sin(x)e^x$,

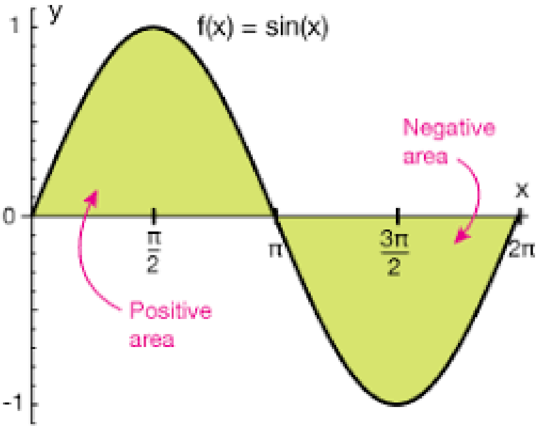
$$
\begin{aligned}
I := \int f(x)dx &= \int sin(x)e^x dx \\
&= \int sin(x)(e^x)' dx \\
&= sin(x)e^x - \int (sin(x))' e^x dx \\
&= sin(x)e^x - \int cos(x)e^x dx \\
&= sin(x)e^x - \int cos(x)(e^x)' dx \\
&= sin(x)e^x - cos(x)e^x + \int (cos(x))'(e^x) dx \\
&= sin(x)e^x - cos(x)e^x - \int sinx(e^x) dx \Rightarrow
\end{aligned}
$$

$$I = sin(x)e^x - cos(x)e^x - I \Rightarrow 2I = sin(x)e^x - cos(x)(e^x)$$

Area

- For a positive function $f(x) > 0$, the integral returns the area of the function in cartesian coordinates.
- For a negative function $f(x) < 0$, the integral returns the area of the function in cartesian coordinates in negative form.

Area of sine function



Volume

- For a positive function $f(x, y) > 0$, the double integral returns the volume of the function in cartesian coordinates.
- For a negative function $f(x, y) < 0$, the double integral returns the volume of the function in cartesian coordinates in negative form.

Continuous Distributions

For continuous distribution with probability density function $f(x)$,

- $P(a < X < b) = \int_a^b f(x)dx$,
- $\mu_X = \int_\Omega xf(x)dx$,
- $\sigma_X^2 = \int_\Omega (\mu_X - x)^2 f(x)dx$.

Goals of the Week

- Understand what integration is.
- Know how to calculate basic integrals.
- Spend time in your assignement.