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Lecture Notes on

# Algorithmic Algebra

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Draft—August 15, 2015 and forever

# Todo list

1: JS says: reviewed this lecture's notes until here . . . . .	12
2: JS says: This is a sample comment . . . . .	21

# List of Scribes

Lecture 1	<i>K Dinesh</i>	1
Lecture 2	<i>K Dinesh</i>	4
Lecture 3	<i>K Dinesh</i>	7
Lecture 4	<i>Ramya C</i>	10
Lecture 5	<i>Ameya Panse</i>	15
Lecture 6	<i>Ameya Panse</i>	18
Lecture 42	<i>*Student name*</i>	21
Lecture 43	<i>*Student name*</i>	22

# Table of Contents

<b>Lecture 1 Introduction, Motivation and the Language</b>	<b>1</b>
1.1 Overview of the course. Administrative, Academic policies . . . . .	1
1.2 Introduction and Motivation . . . . .	2
1.3 Overview of the course . . . . .	3
<b>Lecture 2 Algebraic Approach to Primality Testing</b>	<b>4</b>
2.1 Application to Number Theory . . . . .	4
<b>Lecture 3 Algebraic Approach to finding Perfect Matchings in Graphs</b>	<b>7</b>
3.1 Application to Graph algorithms . . . . .	7
<b>Lecture 4 Graphs, Groups and Generators</b>	<b>10</b>
4.1 Graph Isomorphism, Automorphism and Rigidity . . . . .	10
4.2 Graph Isomorphism, Automorphism and Rigidity . . . . .	10
4.3 Groups and Generators . . . . .	11
4.3.1 Lagrange's theorem . . . . .	13
<b>Lecture 05 Orbit Stabilizer Lemma</b>	<b>15</b>
5.1 Group Action and Orbits . . . . .	15
5.1.1 An Equivalence Relation . . . . .	16
5.1.2 Orbit Stabilizer Lemma . . . . .	16
5.2 Graph Automorphism and Graph Isomorphism . . . . .	17
5.3 Informal Reduction . . . . .	17
<b>Lecture 06 A Closer Look at Graph Isomorphism and Automorphism</b>	<b>18</b>
6.1 Another related Problem . . . . .	18
6.2 Relations Among the Problems . . . . .	18
6.2.1 $GI \leq CGI$ . . . . .	18
6.2.2 $CGI \leq GI$ . . . . .	18
6.2.3 Computing Isomorphism $\leq GI$ . . . . .	19
6.2.4 $GI \leq GA$ . . . . .	19
<b>Lecture 42 *Title of Lecture*</b>	<b>21</b>
<b>Lecture 43 *Title of Lecture*</b>	<b>22</b>
43.1 Add details of next lecture . . . . .	22

# Preface

This lecture notes are produced as a part of the course *CS6842: Algorithmic Algebra* which was a course offered during August to November semester in 2013 and 2015 at the CSE Department of IIT Madras.

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CS6842 – Algorithmic Algebra

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*Date:* Aug 3, 2015

LECTURE

**1**

## Introduction, Motivation and the Language

### 1.1 Overview of the course. Administrative, Academic policies

## 1.2 Introduction and Motivation

Main theme of this course is to use algebra to solve computational problems. Let us consider the following two problems :

**Plagiarism check** Given two  $C$  programs  $P_1$  and  $P_2$ , check if they are the same under renaming of variables.

**Molecule detection** Given two chemical molecules check if they have the same structure.

Consider the following simpler variant of plagiarism checking where the program submitted has just its variables renamed and all other portions of the program are the same. In this case, given the two versions of the program, we need to check if there is a way to rename the variables of one program to get the other one.

In the second case one could view the given molecules as graphs. The problem is again similar problem, where we want to see if there is way to rename the vertex labels of the graph so that under the relabelling both the graphs are the same.

Our aim in both cases is to check whether the two structures are same under renaming. We are interested in solving this problem on graphs.

**Definition 1.1** (Graph Isomorphism). *Two graphs  $X_1(V_1, E_1)$ ,  $X_2(V_2, E_2)$  are said to be isomorphic if there is a bijective map  $\sigma : V_1 \rightarrow V_2$  such that  $\forall (u, v) \in V_1 \times V_1$ ,*

$$(u, v) \in E_1 \iff (\sigma(u), \sigma(v)) \in E_2$$

**Problem 1.2.** *The graph isomorphism problem is the decision problem of checking if given two graphs  $X_1, X_2$  are isomorphic.*

We are also interested in the following special case of the above problem called graph automorphism problem.

**Definition 1.3** (Graph Automorphism). *For a graph  $X(V, E)$ , an automorphism of  $X$  is a renaming of the vertices of  $X$  given by a bijective map  $\sigma : V \rightarrow V$  such that  $\forall (u, v) \in V \times V$ ,*

$$(u, v) \in E \iff (\sigma(u), \sigma(v)) \in E$$

We are interested in the set of all bijections such that they are automorphisms of  $X$ . We denote this by  $Aut(X)$ .

**Definition 1.4.** *For a graph  $X$  on  $n$  vertices,  $Aut(X) = \{\sigma \mid \sigma : [n] \rightarrow [n], \sigma \text{ is an automorphism of } X\}$*

Note that an identity map which takes a vertex to itself always belongs to  $Aut(X)$  for all graphs  $X$ . Hence the question is : are there any bijections other than the identity map as automorphism of  $X$ .

**Problem 1.5** (Graph Automorphism Problem). *Given a graph  $X$  does  $Aut(X)$  has any element other than the identity element.*

One way to see bijections is via permutations. This is because, every bijection define a permutation and vice versa.

Let  $X$  be an  $n$  vertex graph. Denote  $S_n$  to be the set of all permutations on  $n$  elements. Hence  $Aut(X)$  can be defined as  $\{\sigma \mid \sigma \in S_n \text{ and } \sigma \text{ is an automorphism of } X\}$ .

We now show that the set  $Aut(X)$  has some nice properties. Given  $\sigma_1, \sigma_2 \in Aut(X)$ , we can compose two permutations as  $\sigma_1 \circ \sigma_2 = (\sigma_1(\sigma_2(1)), \sigma_1(\sigma_2(2)), \dots, \sigma_1(\sigma_2(n)))$ . This is same as applying  $\sigma_2$  on identity permutation and then applying  $\sigma_1$  to the result. We show that  $Aut(X)$  along with the composition operation  $\circ$  gives us many nice properties.

- If  $\sigma_1, \sigma_2 \in Aut(X)$ , then  $\sigma_1 \circ \sigma_2$  is also an automorphism of  $X$ . The reason is that for any  $(u, v) \in X \times X$ ,  $(u, v) \in E \iff (\sigma_2(u), \sigma_2(v)) \in E$ . Now applying  $\sigma_1$  on the previous tuple, we get that  $(\sigma_2(u), \sigma_2(v)) \iff (\sigma_1 \circ \sigma_2(u), \sigma_1 \circ \sigma_2(v)) \in E$ . Hence  $(u, v) \in E \iff (\sigma_1 \circ \sigma_2(u), \sigma_1 \circ \sigma_2(v)) \in E$ . This tells that  $Aut(X)$  is *closed* under  $\circ$ .
- The composition operation is also *associative*.
- *Identity* permutation belongs to  $Aut(X)$  as observed before.
- Since we are considering bijections, it is natural to consider *inverse* for a permutation  $\sigma$  denoted  $\sigma^{-1}$  with the property that  $\sigma \circ \sigma^{-1}$  is identity permutation.

We gave a definition of inverse for an arbitrary permutation. But then a natural question is : given  $\sigma \in Aut(X)$ , is it true that  $\sigma^{-1}$  also belongs to  $Aut(X)$ . It turns out that it is true.

**Claim 1.6.** For the graph  $X(V, E)$   $\sigma \in Aut(X) \iff \sigma^{-1} \in Aut(X)$

*Proof.* Recall that  $\sigma \in Aut(G)$  iff  $\forall (u, v) \in V \times V$ ,  $(u, v) \in E \iff (\sigma(u), \sigma(v)) \in E$ . In particular, this must be true for  $(\sigma^{-1}(u), \sigma^{-1}(v))$  also. This means,

$$(\sigma^{-1}(u), \sigma^{-1}(v)) \in E \iff (\sigma(\sigma^{-1}(u)), \sigma(\sigma^{-1}(v))) \in E$$

By definition of  $\sigma^{-1}$  we get that  $(\sigma^{-1}(u), \sigma^{-1}(v)) \in E \iff (u, v) \in E$ . This shows that  $\sigma^{-1} \in Aut(X)$   $\square$

Objects which satisfy these kind of properties are called groups.

### 1.3 Overview of the course

There are two major themes.

- Algorithms for permutation groups.
- Algorithms for polynomials.



## Algebraic Approach to Primality Testing

In this lecture, we will see an algebraic approach to solving a fundamental problem in Number Theory.

### 2.1 Application to Number Theory

Following is an algorithmic question that we are interested.

**Problem 2.1.** *Given a number  $n$  in its binary representation, check if it is a prime or not in time  $O(\text{poly}(\log N))$ .*

**Note 2.2.** *The trivial algorithms that we can think of will depend on  $n$  and hence takes time exponential in its input representation.*

Consider the following property about prime number proved by Fermat which is of interest in this context.

**Theorem 2.3** (Fermat's Little Theorem). *If  $N$  is a prime, then  $\forall a, 1 \leq a \leq N - 1$ ,*

$$a^N = a \pmod{N}$$

*Proof.* Fix an  $a \in \{1, 2, \dots, N - 1\}$ . Now consider the sequence  $a, 2a, \dots, (N - 1)a$ . The question we ask is : can any two of the numbers in this sequence be the same modulo  $N$ . We claim that this cannot happen. We give a proof by contradiction : suppose that there are two distinct  $r, s$  with  $1 \leq r < s \leq N - 1$  and  $sa = ra \pmod{N}$ . Then clearly  $N \mid (s - r)a$  which means  $N \mid a$  or  $N \mid (s - r)$ . But both cannot happen as  $a, s - r$  are strictly smaller than  $N$ .

This gives that all the  $N$  numbers in our list modulo  $N$  are distinct. Hence all numbers from  $1, 2, \dots, N - 1$  appear in the list when we go modulo  $N$ . Taking product of the list and the list modulo  $N$ , we get

$$(N - 1)!a^{N-1} = (N - 1)! \pmod{N}$$

By cancelling  $(N - 1)!$ , we get that  $a^{N-1} = 1 \pmod{N}$ . □

This tells that the above condition is necessary for a number to be prime. If this test is also sufficient (i.e, if the converse of the above theorem is true), we have a test for checking primality of a number. But it turns out that this is not true due to the existence of Carmichael numbers which are not prime numbers but satisfy the above test.

So one want a necessary and sufficient condition which can be used for primality testing. For this, we need the notion of polynomials.

**Definition 2.4.** A polynomial  $p(x) = \sum_{i=0}^d a_i x^i$  with  $a_d \neq 0$  denotes a polynomial in one variable  $x$  of degree  $d$ . Here  $a_i$ s are called coefficients and each term excluding the coefficient is called a monomial. A polynomial is said to be identically zero, if all the coefficients are zero.

A polynomial time algorithm for this problem has been found in 2002 by Manindra Agarwal, Neeraj Kayal and Nitin Saxena (called as the AKS algorithm). In their result, they used the following polynomial characterisation for a prime number.

**Theorem 2.5** (Polynomial formulation (Agarwal-Biswas 1999)). Let  $N \geq 1$  be an integer. Define a polynomial  $p_N(z) = (1+z)^N - 1 - z^N$ . Then

$$p_N(z) \equiv 0 \pmod{N} \iff N \text{ is prime}$$

Hence checking if  $N$  is prime or not boils down to checking if  $p_N(z)$  is identically 0 or not except for the fact that the underlying operations are done modulo  $N$ .

Proof of the theorem is as follows.

*Proof.* Note that  $p_N(z) = \sum_{i=1}^{N-1} \binom{N}{i} z^i$  and  $\binom{N}{i} = \frac{N(N-1)\dots(N-i+1)}{1 \cdot 2 \dots i}$  with  $1 \leq i \leq N-1$ .

If  $N$  is prime, then  $\binom{N}{i} = N \times k_i$  for some integer  $k_i$  as none of  $1, 2, \dots, i$  divides  $N$ . Hence  $\binom{N}{i} \pmod{N} = 0$  for every  $i$  and  $p_N(z) \equiv 0 \pmod{N}$  since the polynomial has all coefficients as zero.

If  $N$  is composite, we need to show that  $p_N(z) \not\equiv 0 \pmod{N}$  which means that there is at least one non-zero coefficient  $p_N(z) \pmod{N^1}$ . Since  $N$  is composite, there exists a prime  $p$  such that  $p \mid N$ . Let  $p^k \mid N$  where  $k \geq 1$  is the largest exponent of  $p$  in the prime factorisation of  $n$ . Hence  $p^{k+1} \nmid N$ .

We first show that  $p_N(z)$  is a non zero polynomial by showing that the coefficient of  $z^i$  for  $i = p$ , which is  $\binom{N}{p}$ , is non zero modulo  $p^k$  showing<sup>2</sup>  $p_N(z)$  is not a zero polynomial modulo  $N$ . Let  $N = g \times p^k$ . In  $\binom{N}{p}$ ,  $p$  of the denominator divides  $N$ . Note that  $p$  cannot divide  $g$ , for if it does, then  $p^{k+1} \mid N$  which is not possible by choice of  $p$  and  $k$ .

$$\binom{N}{p} = \frac{g \times p^k (N-1) \dots (N-p+1)}{1 \cdot 2 \dots p} = \frac{g \times p^{k-1} (N-1) \dots (N-p+1)}{1 \cdot 2 \dots p-1} \quad (2.1)$$

Hence it must be that  $p^{k-1}$  divides  $\binom{N}{p}$ . This is because, there is no  $p$  left now in the denominator to divide  $N$ . Now  $p^k \nmid \binom{N}{p}$  because, none of the terms  $(N-1), \dots, (N-p+1)$  can have  $p$  as a factor since all these terms are obtained by subtracting at most  $p-1$  times from  $N$ . Hence  $\binom{N}{p} \not\equiv 0 \pmod{p^k}$ . This completes the proof.  $\square$

Note that Fermat's Little Theorem is a special case of Agarwal-Biswas formulation of primality testing.

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<sup>1</sup>In class we asked the following question of finding an  $a \in \{1, 2, \dots, N-1\}$  such that  $p_N(a) \not\equiv 0 \pmod{N}$ . This has a counter example. Consider the case where  $N$  is a Carmichael number. By definition, Carmichael numbers are composite numbers that satisfy Fermat's Little theorem. Hence if  $N$  is Carmichael,  $\forall a \in \{1, 2, \dots, N-1\}$ , we get  $a^N = a \pmod{N}$ . This gives that  $\forall a \in \{1, 2, \dots, N-1\}$

$$p_N(a) = ((a+1)^N - a^N - 1) \pmod{N} = ((a+1) - a - 1) \pmod{N}$$

which is zero modulo  $N$ .

<sup>2</sup>Note that if  $N \mid \binom{N}{i}$  then  $p^k \mid \binom{N}{i}$ . Taking contrapositive, we get that  $\binom{N}{i} \not\equiv 0 \pmod{p^k}$  implies  $\binom{N}{i} \not\equiv 0 \pmod{N}$

**Claim 2.6.** *Fermat's Little Theorem is a special case of Agarwal-Biswas theorem.*

*Proof.* To prove Fermat's Little theorem, we need one direction of implication of Agarwal-Biswas theorem :

$$\text{"If } N \text{ is prime, then } (1+z)^N = (1+z^N) \pmod{N} \text{"}$$

Note that Fermat's Little theorem asks about  $a^N \pmod{N}$  for  $a \in \{1, 2, \dots, N-1\}$ . Since the implication talks about  $z^N \pmod{N}$  and  $(1+z)^N \pmod{N}$ , this suggests an induction strategy on  $a$ .

Let  $N$  be prime. We check the base case : for  $a = 1$ , the  $1^N = 1 \pmod{N}$ . Hence the base case is true. By induction, suppose that  $a^N = a \pmod{N}$  for  $a \in \{1, 2, \dots, N-1\}$ . Now,

$$\begin{aligned} (a+1)^N &= (a^N + 1) \pmod{N} && \text{[By Agarwal-Biswas as } N \text{ is prime]} \\ &= (a+1) \pmod{N} && \text{[By inductive hypothesis]} \end{aligned}$$

This completes the inductive case. □

This shows that checking if the polynomial  $p_N(z)$  is a zero polynomial is an if and only if check for primality of  $N$ . There are two naive ways to do it. One is to evaluate it at all the numbers from 1 to  $N$  or to expand out the polynomial and see if it is identically zero. But both operations are costly.

So checking primality of a number now boils down to checking if a polynomial is identically zero or not. This is a fundamental problem of polynomial identity testing. We will be discussing about polynomial identity testing and AKS algorithm in our next theme.

## Algebraic Approach to finding Perfect Matchings in Graphs

### 3.1 Application to Graph algorithms

Consider the following problem of finding perfect matching.

**Definition 3.1** (Finding Perfect Matching). *Given a bipartite graph  $G(V_1, V_2, E)$ , we need to come up with an  $E' \subseteq E$  such that  $\forall u \in V_1 \cup V_2$ , there is exactly one edge incident to it in  $E'$ .*

We shall give a polynomial formulation for the problem. Given  $G(V_1, V_2, E)$  with vertex sets  $|V_1| = |V_2| = n$ . Define an  $n \times n$  matrix  $A$  where  $A(i, j) = 1$  if  $(i, j) \in E$  and is 0 otherwise for all  $(i, j) \in V_1 \times V_2$ . Recall the determinant of  $A$  given by

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n A_{i, \sigma(i)}$$

where

$$\text{sign}(\sigma) = \begin{cases} -1 & \text{if } \text{inv}(\sigma) \text{ is even} \\ 1 & \text{if } \text{inv}(\sigma) \text{ is odd} \end{cases}$$

and  $\text{inv}(\sigma)$  is defined as  $|\{(i, j) \mid i < j \text{ and } \sigma(i) > \sigma(j), 1 \leq i < j \leq n\}|$ . We denote  $f(x) \equiv 0$  to denote that polynomial  $f(x)$  is the zero polynomial.

**Lemma 3.2.** *For the matrix  $A$  as defined as before,  $\det(A) \neq 0 \Rightarrow G$  has a perfect matching*

*Proof.* Let  $\det(A) \neq 0$ . Hence there exists a  $\sigma \in S_n$  such that  $\prod_{i=1}^n A_{i, \sigma(i)} \neq 0$ . Hence the edge set  $E' = \{(i, \sigma(i)) \mid 1 \leq i \leq n\}$  exists in  $G$  and since  $\sigma(i) = \sigma(j)$  iff  $i = j$  for every  $i, j$ ,  $E'$  form a perfect matching.

□

Note that converse of this statement is not true. For example, consider the bipartite graph whose  $A$  matrix is  $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . It can be verified that  $\det(A) = 0$  but the bipartite graph associated has a perfect matching.

Hence the next natural question, similar to our primality testing problem, is to ask for some kind of modification so that converse of previous lemma (lemma 3.2) is true. In the example considered, there were two perfect matchings in  $G$  having opposite sign due to which the determinant became 0. So the modification should ensure that perfect matchings of opposite signs does not cancel off in the determinant.

One way to achieve this is as follows. Define a matrix  $T$  as

$$T(i, j) = \begin{cases} x_{ij} & (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

where  $(i, j) \in V_1 \times V_2$ . This matrix is called as the Tutte matrix. Now, if we consider determinant of this matrix, we can see that the monomials corresponding a  $\sigma \in S_n$  can be a product of at most  $n$  variables. Hence  $\det(T)$  is a polynomial in  $n^2$  variables with degree at most  $n$ .

Also in the expansion of determinant, we can observe that each term picks exactly one entry from every row and column meaning each of the entries picked is from a distinct row and column. Hence each of the them is a bijection and hence is a permutation on  $n$  elements. Also corresponding to any permutation on  $n$  elements, we can get a term. This gives us the following observation.

**Observation 3.3.** *Set of monomials in  $\det(T)$  is in one-one correspondence with set of all permutations on  $n$ .*

We now give polynomial formulation for the problem of checking perfect matching in a bipartite graph.

**Claim 3.4.** *For the matrix  $T$  as defined before,  $\det(T) \neq 0 \iff G$  has a perfect matching*

*Proof.* By  $\det(T) \equiv 0$ , we mean that the polynomial  $\det(T)$  has all coefficients as zero. ( $\Rightarrow$ ) Since  $\det(T) \neq 0$ , there exists a  $\sigma \in S_n$  such that the monomial corresponding is non-zero and by definition of determinant it must be expressed as product of some  $n$  set of variables  $\prod_i T(i, \sigma(i))$ . The variable indices gives a matching and since there are  $n$  variables this is a perfect matching.

( $\Leftarrow$ ) Suppose  $G$  has a perfect matching given by a  $M$ . Let  $\tau$  denote the permutation corresponding to  $M$ . We now need to show that  $\det(T) \neq 0$ . To prove this, it suffices to show that there is a substitution to  $\det(T)$  which evaluates to a non-zero value.

Consider the following assignment,  $\forall i, j$

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in M \\ 0 & \text{otherwise} \end{cases}$$

From the formula of determinant,

$$\begin{aligned} \det(T) &= \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n T_{i, \sigma(i)} \\ &= \text{sign}(\tau) \prod_{i=1}^n A_{i, \tau(i)} + \sum_{\sigma \in S_n \setminus \{\tau\}} \text{sign}(\sigma) \prod_{i=1}^n T_{i, \sigma(i)} \end{aligned}$$

Now substituting  $x_{ij} = a_{ij}$ , we get that the first term evaluates to  $\text{sign}(\tau)$  since all the entries are 1. The second term evaluates to 0 since for all  $\sigma \neq \tau$ , there must be a  $j$  such that  $\sigma(j) \neq \tau(j)$ . Hence it must be that  $a_{j, \sigma(j)} = 0$  and hence the product corresponding to  $\sigma$  goes to 0.  $\square$

Hence to check if the bipartite graph  $G$  has a perfect matching or not, it suffices to check if the polynomial  $\det(T)$  is identically zero or not. Checking if a polynomial is identically zero or not is one of the fundamental question in this area.

Note that this problem becomes easy if the polynomial is given as sum of monomial form. In most of the cases, the polynomial will not be given this way. For example if we consider our problem, we are just given the  $T$  matrix and  $\det(T)$  is the required polynomial. Trying to expand  $\det(T)$  and simplifying will involve dealing with  $n!$  monomials which is not feasible.

Hence the computational question again boils down to checking if a polynomial is identically zero or not.

## Graphs, Groups and Generators

In this lecture we will pose three graph theoretic questions and find answers using approaches in algebra. `iiiiiii HEAD`

### 4.1 Graph Isomorphism, Automorphism and Rigidity

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### 4.2 Graph Isomorphism, Automorphism and Rigidity

**Definition 4.1.** (*Graph Isomorphism.*) Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be graphs. We say  $G_1 \cong G_2$  (read as  $G_1$  is isomorphic to  $G_2$ ) if there exists a bijection  $\sigma : V_1 \rightarrow V_2$  such that  $\forall (u, v) \in V_1 \times V_2$  we have `iiiiiii HEAD =====`

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$$(u, v) \in E_1 \iff (\sigma(u), \sigma(v)) \in E_2$$

In other words, we say a graph  $G_1$  is isomorphic to  $G_2$  if there exists a relabeling of the vertices in  $G_1$  such that the the adjacency and non-adjacency relationships in  $G_2$  is preserved.

**Observation 4.2.** If  $|V_1| \neq |V_2|$ , then  $G_1$  is not isomorphic to  $G_2$ . `iiiiiii HEAD =====`  
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The graph isomorphism problem is stated as follows.

**Problem 4.3 (Graph Isomorphism Problem).** Given two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , test if  $G_1 \cong G_2$  or not.

A natural question to ask in this setting is that if there is an isomorphism from a graph  $G$  to itself.

Let  $[n] = \{1, 2, \dots, n\}$ . Let  $S_n$  denote the set of all permutations from the set  $[n]$  to  $[n]$ . Let  $G = (V, E)$  be a graph. An automorphism of  $G$  is a bijection  $\sigma : V \rightarrow V$  such that  $\sigma(G) = G$ . Let

$$\text{Aut}(G) = \{\sigma \mid \sigma \in S_n \text{ and } \sigma(G) = G\}$$

be the set of all automorphisms of  $G$ . Ideally, we would like to compute the set of automorphism of a graph to itself.

**Problem 4.4 (Graph Automorphism Problem).** *Given a graph  $G$ , list the elements of  $\text{Aut}(G)$ .*

The above problem can be expected to be solved in polynomial time only if the output expected is polynomial in length. This brings up the size of the  $\text{Aut}(G)$  into question. Unfortunately, if  $G$  is the complete graph on  $n$  vertices. Then  $|\text{Aut}(G)| = n!$ . Hence the above question is not well-formulated.

Since identity permutation is trivially an automorphism for any graph, the  $\text{Aut}(G)$  is always a non-empty subset of  $S_n$  where  $n$  is the number of vertices. Hence, one can ask a natural decision variant of the above problem, namely the graph rigidity problem.

Formally, the graph rigidity problem is stated as follows.

**Problem 4.5 (Graph Rigidity Problem).** *Given a graph  $G$ , test if  $\text{Aut}(G)$  is trivial. That is, whether  $\text{Aut}(G)$  contains only the identity permutation or not.*

More than the size, in the first lecture of this course, we have seen that  $\text{Aut}(G)$  forms a *subgroup* of  $S_n$ . To utilize this structure, we first refresh the definition of an abstract group. From now on, we will use the letter  $X$  to denote a graph and  $G$  to denote a group.

### 4.3 Groups and Generators

**Definition 4.6.** (Groups.) *A set  $G$  together with a binary operation  $*$  is said to be a group if the following four conditions are met*

- **Closure** :  $a, b \in G$ , the element  $a * b \in G$ .
- **Associative** : For any  $a, b, c \in G$ , we have  $(a * b) * c = a * (b * c)$ .
- **Existence of Identity** : For any  $a \in G$  there exists a unique element  $e \in G$  such that  $a * e = e * a = a$ .
- **Existence of Inverse** : For any  $a \in G$  there exists a unique element  $b \in G$  (denoted by  $a^{-1}$ ) such that  $a * b = b * a = e$ .

**Example 4.7.** •  $S_n$  forms a group under composition.

- $(\mathbb{Z}_5, +)$  is a group.

iiiiii HEAD From now on, we will use the letter  $X$  to denote a graph and  $G$  to denote a group.

**Remark 4.8.** Let  $(G, *)$  be a group. Let  $H \subseteq G$  such that  $(H, *)$  also forms a group. We say  $H$  is a subgroup of  $G$  and denote by  $H \leq G$ .

**Exercise 4.9.** For any graph  $X$ , the set  $\text{Aut}(X)$  forms a group under the composition operation. That is,  $\text{Aut}(G) \leq M$  where  $M = (S_n, \circ)$  is the permutation group.

===== Let  $(G, *)$  be a group. Let  $H \subseteq G$  such that  $(H, *)$  also forms a group. We say  $H$  is a *subgroup* of  $G$  and denote by  $H \leq G$ . We showed in the first lecture of the course that, for any graph  $X$ , the set  $\text{Aut}(X)$  forms a group under the composition operation. That is,  $\text{Aut}(G) \leq M$  where  $M = (S_n, \circ)$  is the symmetric group. 43d9d1f90ad33333710d87eda7fb17a437fb405e



Let  $(G, +)$  be a finite group and  $g \in G$  be an element. Let  $g^2 = g * g, g^3 = g * g * g$ . Similarly  $g^k = \underbrace{g * g * \dots * g}_{k \text{ times}}$ . Now consider the set  $H = \{g, g^2, g^3, \dots\}$ . Since  $(G, *)$  is a finite group there must exist a  $k$  such that  $g^k = g$  in  $H$ .

**Lemma 4.10.** *Let  $(G, +)$  be a finite group and  $g \in G$  be an element. Let  $H = \{g, g^2, g^3, \dots\}$  be a set of elements. The unique identity  $e$  of  $G$  is in  $H$ .*

*Proof.* Since  $(G, *)$  is a finite group there must exist a  $k$  such that  $g^k = g$  in  $H$ . By definition,  $g^k = g^{k-1} * g = g$ . As  $(G, *)$  is a group,  $g^{-1}$  exists in  $G$ . Therefore,

$$g^{k-1} * g * g^{-1} = g * g^{-1} = e$$

□

iiiiiii HEAD

**Definition 4.11 (Generator).** *Let  $(G, +)$  be a finite group and  $g \in G$  be an element. We say an element  $g \in G$  is a generator of the set  $H$  if for every element  $h \in H$  there exists a  $m$  such that  $h = g^m$ . (denoted by  $H = \langle g \rangle$ ).*

**Observation 4.12.**  $H = \langle g \rangle$  is a subgroup of  $G$ . That is,  $H \leq G$ . =====

**Definition 4.13 (Generator).** *Let  $(G, +)$  be a finite group and  $g \in G$  be an element. We say an element  $g \in G$  is a generator of the set  $H$  if for every element  $h \in H$  there exists a  $m$  such that  $h = g^m$ . (denoted by  $H = \langle g \rangle$ ).*

**Observation 4.14.**  $H = \langle g \rangle$  is a subgroup of  $G$ . That is,  $H \leq G$ . ðððððððððð 43d9d1f90ad33333710d87eda7fb17a437f

A quick example is that 1 is a generator for  $(\mathbb{Z}_5, +)$

iiiiiii HEAD

**Definition 4.15.** *An group  $(G, *)$  that can be generated by a single element is called a cyclic group. For instance,  $(\mathbb{Z}_5, +)$ . =====*

**Definition 4.16.** *An group  $(G, *)$  that can be generated by a single element is called a cyclic group. For instance,  $(\mathbb{Z}_5, +)$ . ðððððððððð 43d9d1f90ad33333710d87eda7fb17a437fb405e*

Not every group is cyclic. For instance  $S_3$  is not cyclic.

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**Definition 4.17. (Generating set.)** *Let  $S \subseteq G$  be the set  $\{u_1, \dots, u_k\}$ .  $S$  is said to be generating if  $\langle S \rangle = G$ .*

Having observed that  $\text{Aut}(X)$  could have potentially be of exponenetial size, it is natural to look for generating sets of small size.

Given a graph  $X = (V, E)$  where  $|V| = n$ , does  $\text{Aut}(X)$  have a generating set of size  $\text{poly}(n)$ ?

### 4.3.1 Lagrange's theorem

Let  $(G, *)$  be a group. Let  $H \leq G$ . For any  $g \in G$ , define the right coset of  $H$  in  $G$  to be

$$Hg = \{hg \mid h \in H\}$$

Let  $g_1, g_2 \in G$  and  $Hg_1, Hg_2$  be the corresponding right cosets. Are there elements that belong to more than one coset of  $H$  in  $G$ ? Is  $Hg_1 \cap Hg_2 \neq \phi$ ? If yes, then the set of right cosets of  $H$  in  $G$  form a partition of the ground set of  $G$ . In that case, how many such cosets are required to cover the entire set  $G$ ? Let us answer these two questions.

**Lemma 4.18.** Let  $g_1, g_2 \in G$  and  $Hg_1 = \{hg_1 \mid h \in H\}, Hg_2 = \{hg_2 \mid h \in H\}$ . Then

$$Hg_1 = Hg_2 \text{ or } Hg_1 \cap Hg_2 = \phi.$$

*Proof.* If  $g_1 = g_2$ , then by definition  $Hg_1 = Hg_2$ . Therefore let  $g_1 \neq g_2$ . We will prove : If  $Hg_1 \cap Hg_2 \neq \phi$  then  $Hg_1 = Hg_2$ . Let  $Hg_1 \cap Hg_2 \neq \phi, g \in Hg_1 \cap Hg_2$  we will show

(i)  $Hg_1 \subseteq Hg_2$  ; and

(ii)  $Hg_2 \subseteq Hg_1$ .

Since  $g \in Hg_1$  we know that there exists a  $h_1 \in H$  such that  $g = h_1g_1$ . Similarly  $g \in Hg_2$  suggests that there exists a  $h_2 \in H$  such that  $g = h_2g_2$ .

$$h_1g_1 = h_2g_2 = g$$

As  $(H, *)$  is a group by itself,  $h_1^{-1}$  and  $h_2^{-1}$  exists.

$$g_1 = h_1^{-1}h_2g_2 \tag{4.2}$$

$$g_2 = h_2^{-1}h_1g_1 \tag{4.3}$$

(i)  $Hg_1 \subseteq Hg_2$

Let  $g' \in Hg_1$ . This implies there exists a  $h' \in H$  such that  $g' = h'g_1$ . Therefore,

$$\begin{aligned} g' &= h'g_1 \\ g' &= h'(h_1^{-1}h_2g_2) \quad [\text{By equation (4.2)}] \end{aligned}$$

By closure property in  $(H, *)$ , we have  $h'' = h'h_1^{-1}h_2 \in H$ . Therefore  $g' = h''g_2, g' \in Hg_2$ .

(ii)  $Hg_2 \subseteq Hg_1$

Let  $g' \in Hg_2$ . This implies there exists a  $h' \in H$  such that  $g' = h'g_2$ . Therefore,

$$\begin{aligned} g' &= h'g_2 \\ g' &= h'(h_2^{-1}h_1g_1) \quad [\text{By equation (4.3)}] \end{aligned}$$

By closure property in  $(H, *)$ , we have  $h'' = h'h_2^{-1}h_1 \in H$ . Therefore  $g' = h''g_1, g' \in Hg_1$ .

□

**Lemma 4.19.** *For every  $g \in G$ ,  $|Hg| = |H|$ .*

*Proof.* By construction, for every element in  $H$  there exists an element in  $Hg$ . So  $|Hg| \leq |H|$ .

Let us show :  $|H| \leq |Hg|$ . Suppose not,  $|Hg| < |H|$ . Then there exists  $h_1, h_2 \in H, h_1 \neq h_2$  such that  $h_1g = h_2g$ . Since  $(G, *)$  is a group,  $g^{-1}$  exists. We have  $h_1gg^{-1} = h_2gg^{-1}$  which implies  $h_1 = h_2$ , a contradiction.  $\square$

**Theorem 4.20.** *(Lagrange's theorem.) Let  $(G, *)$  be a group and  $H \leq G$ . Then  $|H|$  divides  $|G|$ .*

*Proof.* Direct consequence of Lemmas 4.18 and 4.19  $\square$

**Observation 4.21.** *Let  $H = \langle g \rangle$  and  $H' = \langle H, g' \rangle$  where  $g' \in G \setminus H$ . Then  $H \leq H' \leq G$ . We have  $g \in H' \setminus H$ , therefore  $|H'| > |H|$ .  $H' \leq H$ . By Theorem 4.20,  $|H'| \geq 2|H|$ . This shows that every group has a generating set of size  $\log |G|$ .*

**Remark 4.22.** *We know that  $\text{Aut}(X) \leq S_n$  for any graph  $X = (V, E)$ . By Observation 4.21  $\text{Aut}(X)$  has a generating set of size  $\log |S_n| = \log(n!) \in \mathcal{O}(n \log n)$  by Stirling's approximation.*  
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## Orbit Stabilizer Lemma

In this lecture we will understand and prove the Orbit Stabilizer Lemma, while defining the various terms associated with it.

**Definition 5.1.** (Order of a Group) Order of a group  $G$  is number of elements in the group, that is  $\|G\|$

**Definition 5.2.** (Right Co-set) Let  $H \leq G$ ,  $g \in G$  Right co-set of  $H$  in  $G$  is defined as

$$Hg = \{hg | h \in H\}$$

**Definition 5.3.** (Left Co-set) Let  $H \leq G$ ,  $g \in G$  Left co-set of  $H$  in  $G$  is defined as

$$gH = \{gh | h \in H\}$$

**Note 5.4.** In general, it is not necessary that the left and right co-sets are the same.

**Definition 5.5.** (Normal SubGroup) Let  $H \leq G$ , if we have

$$\forall g \in G, Hg = gH$$

then  $H$  is called a Normal SubGroup

Let  $H$  be a Normal SubGroup of  $G$ . Divide  $G$  into co-sets of  $H$  and take one element from each of these as a representative of the set.

**Claim 5.6.** These elements have a group structure among them.

This will be proved in later classes.

### 5.1 Group Action and Orbits

Let  $G$  be a SubGroup of  $S_n$ . Let  $\alpha \in [n]$  and  $g \in G$ .

**Note 5.7.**  $\alpha^g$  is the image of  $\alpha$  in the permutation  $g$ .

**Definition 5.8.** (Orbit)

$$\alpha^G = \{\alpha^g | g \in G\}$$

### 5.1.1 An Equivalence Relation

Consider the following relation,

$$\alpha \sim \beta \leftrightarrow \exists g \in G, \alpha^g = \beta$$

**Claim 5.9.** *The above relation is an Equivalence Relation.*

**Reflexive:**  $e \in G$ , where  $e$  is the identity element. Hence,  $\alpha \sim \alpha$

**Symmetric:** Let  $\alpha \sim \beta$ . Thus  $\exists g \in G, \alpha^g = \beta$ . Hence,  $\alpha = \beta^{g^{-1}}$ . Thus,  $\beta \sim \alpha$

**Transitive:** Let  $\alpha \sim \beta, \beta \sim \gamma$ . Thus  $\exists g_1, g_2 \alpha^{g_1} = \beta, \beta^{g_2} = \gamma$ . By composition of permutations,  $(\alpha^{g_1})^{g_2} = \gamma$ . Hence,  $\alpha \sim \gamma$

**Definition 5.10.** (Stabilizer of  $\alpha$ )

$$G_\alpha = \{g \mid \alpha^g = \alpha\}$$

**Observation 5.11.**  $G_\alpha$  is a SubGroup of  $G$ .

### 5.1.2 Orbit Stabilizer Lemma

$$\forall \alpha \in [n] \parallel \alpha^G \parallel * \parallel G_\alpha \parallel = \parallel G \parallel$$

**Claim 5.12.** *There exists a bijection from  $\alpha^G$  to Co-sets of  $G_\alpha$  in  $G$ .*

**Corollary 5.13.** *Number of Co-sets of  $G_\alpha = \parallel \alpha^G \parallel$ .*

And, since  $G_\alpha$  forms a SubGroup, applying Lagrange's Theorem, we get the Orbit Stabilizer Lemma.

**Proof of Claim:** The idea is that the set of permutations that send  $\alpha$  to  $\beta$ , form a Co-set of permutations that send  $\alpha$  to  $\alpha$ .

Let  $\beta \in \alpha^G$  and  $h \in G, \alpha^h = \beta$ .

Consider  $\{g \in G \mid \beta = \alpha^g\}$  We have to show that this is a Co-set

$$= \{g \in G \mid \alpha^h = \alpha^g\}$$

$$= \{g \in G \mid \alpha^{gh^{-1}} = \alpha\}$$

$$\text{Thus } gh^{-1} \in G_\alpha$$

$$= \{g \in G \mid gh^{-1} \in G_\alpha\}$$

$$= \{g \in G \mid g \in G_\alpha h\}$$

**Exercise 5.14.** Complete the above proof by showing a bijection between  $\beta$ 's and the Co-sets of  $G_\alpha$ .

## 5.2 Graph Automorphism and Graph Isomorphism

We will now define and analyse various problems related to GI.

**Graph Isomorphism [GI]:** Given graphs  $G_1, G_2$ ,

Output 1

if  $\exists \sigma : V_1 \rightarrow V_2, \forall (u, v) \in E_1, (\sigma(u), \sigma(v)) \in E_2$

else Output 0.

**Definition 5.15** (Automorphism). A Graph  $G$  is said to be automorphic if it is non-trivially isomorphic to itself.

**Definition 5.16** ( $\text{Aut}(G)$ ).

$$\text{Aut}(G) = \{\sigma \in S_n | G \cong \sigma(G)\}$$

**Graph Automorphism [GA] :** Given  $G$ , output a Generating Set for  $\text{Aut}(G)$ .

**Graph Rigidity [GR] :** Given  $G$ , test if  $\text{Aut}(G)$  is non-trivial.

**Number of Isomorphisms [#GI] :** Given  $G_1, G_2$ , output the number of Isomorphisms from  $G_1$  to  $G_2$ .

**Number of Automorphisms [#GA] :** Given  $G$ , output the size of  $\text{Aut}(G)$ .

**Computing Isomorphism** Given  $G_1, G_2$ , output the permutation that morphs  $G_1$  to  $G_2$ .

**Computing Automorphism** Given  $G$ , output a non-trivial element in  $\text{Aut}(G)$ .

## 5.3 Informal Reduction

Given two problems  $A, B$ , we say that  $A \leq B$  ( $A$  reduces to  $B$ ), if given a polytime algorithm for  $B$ , we can give out a polytime algorithm for  $A$ .

In the next lecture we will talk about the relation among the above defined problems.

## A Closer Look at Graph Isomorphism and Automorphism

### 6.1 Another related Problem

Here the vertex set is divided into  $c$  color classes by the function

$$\Psi : V(X) \rightarrow [c],$$

where  $i \in [c]$  denotes a color and  
the  $i^{\text{th}}$  color class is  $\Psi^{-1}(i)$ .

**Colored Graph Isomorphism [CGI]:** Given two  $C$ -colored Graphs  $(X_1, \Psi_1)$  and  $(X_2, \Psi_2)$

Output 1

if  $\exists \sigma : V(X_1) \rightarrow V(X_2)$  such that  $\forall (u, v) \in V(X_1)(X_2), (u, v) \in E(X_1)$  if and only if  $(\sigma(u), \sigma(v)) \in E(X_2)$

and  $\forall u \in V(X_1), \Psi_1(u) = \Psi_2(\sigma(u))$  .

### 6.2 Relations Among the Problems

#### 6.2.1 $GI \leq CGI$

Set  $c = 1$ , and color all the vertices with the same color.

#### 6.2.2 $CGI \leq GI$

Given  $(X_1, \Psi_1)$  and  $(X_2, \Psi_2)$ .

[Gadget] :  $\forall u \in V(X_1)$  such that  $u \in \Psi_1^{-1}(i)$ :

1. Add  $n_i$  extra vertices to  $X_1$ .
2. Add edges from each of the extra vertices to  $u$  to get the graph  $X'_1$ .

Do the same for  $(X_2, \Psi_2)$  to get  $X'_2$ .

Now run  $GI$  on  $X'_1, X'_2$ .

**Correctness :**

Let  $(X_1, \Psi_1)$  and  $(X_2, \Psi_2) \in CGI$ . Hence  $\exists \sigma : V(X_1) \rightarrow V(X_2)$  such that  $\forall (u, v) \in V(X_1) \times V(X_2), (u, v) \in E(X_1)$  if and only if  $(\sigma(u), \sigma(v)) \in E(X_2)$  and  $\forall u \in V(X_1), \Psi_1(u) = \Psi_2(\sigma(u))$ . Additionally map the extra vertices added correspondingly. Thus  $X'_1 \cong X'_2$ .

Let  $X'_1 \cong X'_2$ , hence  $\exists \sigma : V(X'_1) \rightarrow V(X'_2), \forall (u, v) \in E(X'_1), (\sigma(u), \sigma(v)) \in E(X'_2)$ . If possible let there exist  $u \in v(X_1), u \in \Psi^{-1}(i)$  such that  $\sigma(u) \notin V(X_2)$ . That is,  $u$  is mapped to one of the extra vertices. But  $u \in X'_1$  and  $\deg(u) \geq ni$ , whereas degree of any extra vertex is 1. Hence, we have a contradiction.

Now, if possible let  $u \notin \Psi_2^{-1}(i)$ . Hence  $\sigma(u) \in \Psi_2^{-1}(j)$  and  $j \neq i$ . Note that  $ni + n > \deg(u) \geq ni \Rightarrow n(i+1) > \deg(u) \geq ni$ . And,  $\sigma(u) \in \Psi_2^{-1}(j) \Rightarrow n(j+1) > \deg(u) \geq nj$ . Since both of these can not be true simultaneously, we have a contradiction.

Hence,  $\sigma(u) \in V(X_2), \sigma(u) \in \Psi_2^{-1}(i)$ . Thus,  $(X_1, \Psi_1) \cong (X_2, \Psi_2)$ .

**Time Complexity :** We have made only one query to GI. Hence, the reduction is polytime.

**Hence Proved.**

### 6.2.3 Computing Isomorphism $\leq GI$

Given  $X_1, X_2$  Output a permutation that morphs  $X_1$  to  $x_2$ . The Reduction is as follows:

1. Check if  $X_1 \cong X_2$ . If NO, end.
2. For each vertex  $i \in V_1$  Color  $i$  with color  $C_i$ .
  - Color  $j \in V_2 - [\text{Already Colored Vertices}]$  with  $C_i$ , temporarily.
  - Query to CGI.
  - Repeat on  $j$  untill you get a yes answer. Fix color of  $j$  as  $C_i$ .

Output the permutation.

Here each vertex is colored with a different color, and hence we have a permutation.

Also, if the graphs are isomorphic, then there will exists an  $j \in V_2$ , such that we get a yes answer.

**Time Complexity:** We make atmost  $O(n^2)$  queries to CGI which in turn makes a single query to GI.

Thus the reduction is polytime.

### 6.2.4 $GI \leq GA$

Take  $X = X_1 \cup X_2$ .

Let  $S$  be the generating set of  $\text{Aut}(G)$ ..



**Claim 6.1.**  $X_1 \cong X_2$  if and only if  $\exists \sigma \in S$  such that  $\sigma$  maps atleast one vertex in  $X_1$  to a vertex in  $X_2$ .

For the time being assume that the two graphs are connected graphs.

**Forward Direction** Assume that  $X_1 \cong X_2$ .

$\exists \tau$  which is an isomorphism between  $X_1, X_2$ .  $\tau \in \text{Aut}(X)$ . Hence, there is a  $\sigma$  that maps a vertex in  $X_1$  to a vertex in  $X_2$ .

**Backward Direction :**  $\exists \sigma$  that maps  $u \in X_1$  to  $\sigma(u) \in X_2$ . Let  $v \in X_1$  be such that  $\sigma(v) \in X_1$ . Since  $X_1$  is connected  $u, v$  are connected. But  $\sigma(u) \in X_2, \sigma(v) \in X_1$  are not connected. Hence, we have a contradiction. Thus,  $\sigma$  maps all vertices in  $X_1$  to  $X_2$ . Thus  $X_1 \cong X_2$ .

In case the two are not connected, add an extra vertex to both the graphs that is adjacent to all the vertices in the corresponding graphs. Since, the new vertices have degree  $n$ , they can be mapped only to each other.

**Hence Proved.**

*CS6842 – Algorithmic Algebra*

*Instructor: Jayalal Sarma*

*Scribe: \*Student name\**

*Date: \*Month, XX 2015\**

# LECTURE 42

**\*Title of Lecture\***

*Regarding environments and general latex usage, check `sample-scribe.tex`.*

**Note 42.1.** *Enter date, lecture number, title and your name. Also change the lecture number from 42 to appropriate number.*

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*CS6842 – Algorithmic Algebra*

*Instructor: Jayalal Sarma*

*Scribe: \*Student name\**

*Date: \*Month, XX 2015\**

# LECTURE 43

**\*Title of Lecture\***

## **43.1 Add details of next lecture**