Theorem :- If the coeff. $a_0(x), a_1(x), ---, a_n(x)$ are continuous on interval I & $a_0 \neq 0$. then the linear differential equation

$$a_0 y^n + a_1 y^{n-1} + \cdots + a_n y = 0$$

has n linearly independent solution. i.e. if $y_1(x), y_2(x), --, y_n(x)$ are 'n' linearly independent solution then their linear combination i.e $c_1y_1 + c_2 y_2 + --- + c_n y_n$ is also the solution of the differential eq. (general sol.)

The 'n' linearly independent solutions ie $\{y_1, y_2, --, y_n\}$ are called fundamental solution of diff. eq. on I. This set of fundamental sol. form basis of the linear diff. eq.

$\{y_1, y_2, --, y_n\}$ is basis to diff. eq..

Q: Show that $\{x^{1/4}, x^{5/4}\}$ form a set of fundamental solution (basis) to the differential eq.

$$16x^2y'' - 8xy' + 5y = 0, \quad x > 0$$

Sol:- Let $y_1 = x^{1/4}$, $y_2 = x^{5/4}$.

$y_1' = \frac{1}{4}x^{-3/4}$, $y_2' = \frac{5}{4}x^{1/4}$

$y_1'' = \frac{-3}{16}x^{-7/4}$, $y_2'' = \frac{5}{16}x^{-3/4}$.

y_1 & y_2 are L.I. or not

y_1 & y_2 are L.I. or not

$\Leftarrow W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{1/4} & x^{5/4} \\ \frac{1}{4}x^{-3/4} & \frac{5}{4}x^{1/4} \end{vmatrix}$

(Wronskian)

$= \frac{5}{4}x^{1/2} - \frac{1}{4}x^{1/2} = x^{1/2} > 0$

$\neq 0$

$\therefore W(y_1, y_2) \neq 0 \Rightarrow y_1$ & y_2 are L.I.

To prove: y_1 & y_2 are solution of diff. eq.

$$16x^2y'' - 8xy' + 5y = 0$$

Consider $16x^2y_1'' - 8xy_1' + 5y_1$

$$= \cancel{16}x^2\left(\frac{-3}{\cancel{16}}x^{-7/4}\right) - \overset{2}{\cancel{8}}x\left(\frac{1}{\cancel{4}}x^{-3/4}\right) + 5\left(x^{1/4}\right)$$

$$= -3x^{2-7/4} - 2x^{1-3/4} + 5x^{1/4}$$

$$= -3x^{1/4} - 2x^{1/4} + 5x^{1/4} = 0$$

$\Rightarrow y_1$ satisfies the given diff. eq.

Hence y_1 is solution of given diff.

Consider $16x^2y_2'' - 8xy_2' + 5y_2$

$$= \cancel{16}x^2\left(\frac{5}{\cancel{16}}x^{-3/4}\right) - \overset{2}{\cancel{8}}x\left(\frac{5}{\cancel{4}}x^{1/4}\right) + 5x^{5/4}$$

$$= 5x^{2-3/4} - 10x^{1+1/4} + 5x^{5/4}$$

$$= 5x^{5/4} - 10x^{5/4} + 5x^{5/4} = 0$$

$\Rightarrow$ y_2 is also solution of diff. eq.

$\Rightarrow$ y_1 & y_2 are L.I. solution of diff. eq.

Let $y_3 = c_1 y_1 + c_2 y_2$ (To prove y_3 is also one sol.)

$y_3' = c_1 y_1' + c_2 y_2'$

$y_3'' = c_1 y_1'' + c_2 y_2''$

Consider. $16x^2 y_3'' - 8x y_3' + 5y_3$

$= 16x^2 (c_1 y_1'' + c_2 y_2'') - 8x (c_1 y_1' + c_2 y_2') + 5(c_1 y_1 + c_2 y_2)$

$= c_1 [16x^2 y_1'' - 8x y_1' + 5y_1) + c_2 [16x^2 y_2'' - 8x y_2' + 5y_2)$

$= c_1 (0) + c_2 (0)$ [$\because$ y_1 & y_2 are solution of given diff eq (proved)]

$= 0$

$\Rightarrow$ y_3 is also the solution of the given diff. eq.

Hence $y_3 = c_1 x^{1/4} + c_2 x^{5/4}$ is general sol. of diff eq.

Q:- Let $a_0(x) y'' + a_1(x) y' + a_2(x) y = 0$ be diff. eq. normal in I. & y_1 & y_2 be linearly independent solution of the diff. eq. Show that wronskian of y_1 & y_2 satisfies the diff. eq. $\boxed{a_0(x) W'(x) + a_1(x) W(x) = 0}$

Also show that $\boxed{W(x) = C e^{-\int \frac{a_1(x)}{a_0(x)} dx}}$ (Abel's formula)

Sol:- $\because$ y_1 & y_2 are L.I. solution of diff. eq.

$a_0 y'' + a_1 y' + a_2 y = 0$

$\Rightarrow$ $a_0 y_1'' + a_1 y_1' + \cancel{a_2 y_1} = 0 \quad \big] \times y_2,$

& $-\, a_0 y_2'' - a_1 y_2' - \cancel{a_2 y_2} = 0. \quad \big] \times y_1$

$a_0(y_1'' y_2 - y_2'' y_1) + a_1(y_1' y_2 - y_2' y_1) = 0$ — (*)

$\because$ y_1 & y_2 are L.I. $\Rightarrow$ $W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= y_1 y_2' - y_2 y_1' \neq 0$

$W(x) = y_1 y_2' - y_2 y_1' = -(y_1' y_2 - y_2' y_1)$

$W'(x) = y_1 y_2'' + \cancel{y_2' y_1'} - (y_2 y_1'' + \cancel{y_1' y_2'})$

$= y_1 y_2'' - y_2 y_1'' = -(y_1'' y_2 - y_2'' y_1)$

Eq (*) $\Rightarrow$ $a_0(x)(-W'(x)) + a_1(x)(-W(x)) = 0$

$\Rightarrow -[a_0(x) W'(x) + a_1(x) W(x)) = 0$

$\Rightarrow a_0(x) W'(x) + a_1(x) W(x) = 0$

hence. proved the result (1)

$a_0(x) W'(x) = -a_1(x) W(x)$

$\Rightarrow \int \frac{W'(x)}{W(x)} dx = -\int \frac{a_1(x)}{a_0(x)} dx.$

$\Rightarrow \ln|W(x)| = -\int \frac{a_1(x)}{a_0(x)} dx + C.$

$-\int \frac{a_1(x)}{a_0(x)} dx + C.$

$\int \frac{W'(x)}{W(x)}$

$\ln|W(x)| = -\int \frac{a_1(x)}{a_0(x)}dx + C.$

$\Rightarrow W(x) = e^{-\int \frac{a_1(x)}{a_0(x)}dx + C.}$

$W(x) = e^{-\int \frac{a_1(x)}{a_0(x)}dx} \cdot e^{C}$, $\boxed{A = e^{C}}$

$$\boxed{W(x) = A e^{-\int \frac{a_1(x)}{a_0(x)}dx}}$$

Q1:- $16x^2y'' - 8xy' + 5y = 0$

$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$

$a_0(x) = 16x^2$, $a_1(x) = -8x.$

$y_1 = x^{1/4}$

$y_2 = x^{5/4}$

$\boxed{W(y_1, y_2) = \sqrt{x}}$

$W(x) = C e^{+\int \frac{+\cancel{8x}}{\cancel{16}_2 x^{\cancel{2}}}dx} = C e^{\frac{1}{2}\int \frac{1}{x}dx}$

$= C e^{\frac{1}{2}\ln x} = C e^{\ln\sqrt{x}}.$

$$\boxed{W(x) = C\sqrt{x}.}$$

(a) x^2 (b) ✓ $\sqrt{x}$ (c) $x^{3/2}$ (d) $x^{-1/2}$.

Theorem:- If $a_0(x), a_1(x), \ldots, a_n(x)$ are continuous over interval I & $a_0(x) \neq 0$ on I. Let $y_1, y_2, \ldots, y_n$ be the n solution of diff eq. $a_0(x)y^n + a_1 y^{n-1} + \ldots + a_n y = 0$

then

(1). $W(x) = W(y_1, y_2, \ldots, y_n) \neq 0$ $\forall x \in I$ $\Leftrightarrow$ $y_1, y_2, \ldots, y_n$ are linear independent solution on I.

--- y_n are linear independent solutions.

② $W(x) = W(y_1, y_2, --, y_n) = 0$ for any $x_0 \in I$.

i.e $\boxed{W(x_0) = 0}$ where x_0 is any fixed point.

then $\Rightarrow$ $W(x) = 0$ $\forall x \in I$ & functions are linearly dependent. i.e $y_1, y_2 --, y_n$ are linearly dependent.