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Equation reducible to Exact form. (Contd):-

Case I:- $f\,dx + g\,dy = 0$ is not exact

i.e $\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}$,

① $\dfrac{\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}}{f} = K(y)$

$$I.F. \quad e^{-\int K(y)\,dy}$$

or. $\dfrac{\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}}{f} = K_1(y)$, IF $e^{\int K_1(y)\,dy}$

② $\dfrac{\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}}{g} = h(x)$, IF. $e^{\int h(x)\,dx}$.

Case II. We have a term $y\,dx - x\,dy$ in diff. eq.

IF (a) $\frac{1}{x^2}$ (b) $\frac{1}{y^2}$ (c) $\frac{1}{xy}$ (d) $\frac{1}{x^2+y^2}$ (e) $\frac{1}{x^2-y^2}$

Case III. $f\,dx + g\,dy = 0$, $\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}$.

(eq is not exact)

but eq. is homogeneous in x & y.

y f & g are homogeneous function of x & y & are of same degree. 'n'.

i.e $\boxed{f = x^n g\left(\frac{y}{x}\right) \quad \& \quad g = x^n h\left(\frac{y}{x}\right)}$

I.F. $= \boxed{\dfrac{1}{fx+gy}}$.

on multiplying I.F. to given eq we get

$$\frac{f\,dx}{fx+gy} + \frac{g\,dy}{fx+gy} = 0. \quad (\text{eq. should be exact})$$

$\dfrac{\partial}{\partial x}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{f}{fx+gy}\right) = \frac{\partial}{\partial x}\left(\frac{g}{fx+gy}\right)$$

$$\Rightarrow \frac{(fx+gy)\frac{\partial f}{\partial y} - f\left(x\frac{\partial f}{\partial y} + g(1) + y\frac{\partial g}{\partial y}\right)}{\cancel{(fx+gy)^2}} = \frac{(fx+gy)\frac{\partial g}{\partial x} - g\left(x\frac{\partial f}{\partial x} + f(1) + y\frac{\partial g}{\partial x}\right)}{\cancel{(fx+gy)^2}}$$

$$\Rightarrow \cancel{xf\frac{\partial f}{\partial y}} + gy\frac{\partial f}{\partial y} - \cancel{fx\frac{\partial f}{\partial y}} - \cancel{fg} - fy\frac{\partial g}{\partial y}$$

$$= fx\frac{\partial g}{\partial x} + \cancel{gy\frac{\partial g}{\partial x}} - gx\frac{\partial f}{\partial x} - \cancel{fg} - \cancel{gy\frac{\partial g}{\partial x}}$$

(1) & (2) (3) & (4)

$$g\left(y\frac{\partial f}{\partial y} + x\frac{\partial f}{\partial x}\right) = f\left(x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y}\right) \quad — (*)$$

$\because$ f & g are homogeneous functions of x & y & are of degree 'n'.

By Euler's theorem.

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf}, \quad \boxed{x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = ng}$$

$$\boxed{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf}\ ,\quad \boxed{x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}=ng}$$

from eq (B)

$$\boxed{g(nf) = f(ng)} \text{ — verified}$$

Q1. Solve the diff. eq. $(x^2+y^2)\,dx - 2xy\,dy = 0$ — (1)

Sol. Compare it with $f\,dx + g\,dy = 0$

$$f = \underline{x^2+y^2}\ ,\quad g = \underline{-2xy}\ .$$

$$\frac{\partial f}{\partial y} = 2y\ ,\quad \frac{\partial g}{\partial x} = -2y\ .\qquad \frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}.$$

eq. is not exact.

Clearly f & g are homogeneous function of x & y & are of same degree.

I.F.:
$$\frac{1}{fx+gy} = \frac{1}{(x^2+y^2)x-(2xy)y}$$

$$= \frac{1}{x^3+xy^2-2xy^2} = \frac{1}{x^3-xy^2}.$$

$$= \frac{1}{x(x^2-y^2)}$$

On multiplying eq (1) with I.F, we will get.

$$\frac{(x^2+y^2)}{x(x^2-y^2)}\,dx - \frac{2\cancel{x}y}{\cancel{x}(x^2-y^2)}\,dy = 0$$

Compare it with $F\,dx + G\,dy = 0$

Compare it with $F\,dx + G\,dy = 0$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{x(x^2-y^2)}\right) = \frac{1}{x}\left[\frac{(x^2-y^2)(2y)+(x^2+y^2)(2y)}{(x^2-y^2)^2}\right]$$

$$= \frac{1}{x}\left[\frac{2x^2y - \cancel{2y^3} + 2x^2y + \cancel{2y^3}}{(x^2-y^2)^2}\right]$$

$$= \frac{1}{x} \times \frac{4x^2y}{(x^2-y^2)^2} = \frac{4xy}{(x^2-y^2)^2}.$$

$$\frac{\partial G}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{2xy}{\cancel{x}(x^2-y^2)}\right) = +2y\cdot\left(+\frac{1}{(x^2-y^2)^2}\,2x\right)$$

$$= \frac{4xy}{(x^2-y^2)^2}.$$

Clearly $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$ $\Rightarrow$ eq is exact.

Soln $\displaystyle\int_{y\text{ constant}} F\,dx + \int_{\substack{\text{Term of } G \\ \text{not containing } x}} G\,dy = C.$

$$\Rightarrow \int_{y\text{ constant}} \frac{x^2+y^2}{x(x^2-y^2)}\,dx + \int 0\,dy = C$$

$$\Rightarrow \frac{1}{2}\int_{y\text{ const}} \frac{2x^2\,dx}{x(x^2-y^2)} + y^2\int \frac{dx}{x(x^2-y^2)} = C$$

$$\Rightarrow \frac{1}{2}\ln(x^2-y^2) + y^2\left(\int \frac{1}{\quad} + \frac{1}{\quad} + \frac{1}{\quad}\right)dx.$$

$$\Rightarrow \frac{1}{2}\ln|x^2-y^2| + y^2 \int_{y\,\text{const}} \left(\frac{1}{x\cdot(-y^2)} + \frac{1}{(x-y)2y^2} + \frac{1}{(x+y)2y^2}\right)dx = C.$$

$$\Rightarrow \frac{1}{2}\ln|x^2-y^2| + \frac{y^2}{y^2}\int_{y\,\text{const}} \left(-\frac{1}{x} + \frac{1}{2(x-y)} + \frac{1}{2(x+y)}\right)dx = C$$

$$\Rightarrow \frac{1}{2}\ln|x^2-y^2| - \ln|x| + \frac{1}{2}\ln|x-y| + \frac{1}{2}\ln|x+y| = C.$$

$$= \frac{1}{2}\ln|x^2-y^2| - \ln|x| + \frac{1}{2}\ln|(x-y)(x+y)| = C.$$

$$\Rightarrow \ln|x^2-y^2| - \ln|x| = C.$$

$$\Rightarrow \ln\left|\frac{x^2-y^2}{x}\right| = C.$$

Case IV :- $f\,dx + g\,dy = 0$, which is not exact

i.e. $\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}$.

but it is of the form.

$$y f_1(xy)\,dx + x f_2(xy)\,dy = 0$$

eg: $f(x) = x^2 + x^3 + 1$

$f(xy) = x^2y^2 + x^3y^3 + 1$

IF: $\frac{1}{fx - gy}$.

Q) Solve the diff. eq.

$$(x^2y^3+xy^2+y)dx+(x^3y^2-x^2y+x)dy=0$$

Sol, Compare it with $f\,dx+g\,dy=0$

$f=x^2y^3+xy^2+y$, $g=x^3y^2-x^2y+x$.

$\frac{\partial f}{\partial y}=3x^2y^2+2xy+1$, $\frac{\partial g}{\partial x}=3x^2y^2-2xy+1$

Clearly $\frac{\partial f}{\partial y}\neq\frac{\partial g}{\partial x}$. eq is not exact.

$$\frac{\partial f}{\partial y}-\frac{\partial g}{\partial x}=\cancel{3x^2y^2}+2xy+\cancel{1}-\cancel{3x^2y^2}+2xy-\cancel{1}$$
$$=4xy$$

$\Rightarrow$ $\underline{y(x^2y^2+xy+1)}dx+\underline{x(x^2y^2-xy+1)}dy=0$ — (1)

which is of the form $y f_1(xy)dx+x f_2(xy)dy=0$

I.F; $\frac{1}{fx-gy}=\frac{1}{xy(\cancel{x^2y^2}+xy+\cancel{1})-xy(\cancel{x^2y^2}-xy+\cancel{1})}$

$$=\frac{1}{2x^2y^2}$$

$$\frac{\cancel{y}(x^2y^2+xy+1)\,dx}{\cancel{2}x^2y^{\cancel{2}}}+\frac{\cancel{x}}{\cancel{2}x^{\cancel{2}}y^2}(x^2y^2-xy+1)dy=0$$

$\Rightarrow$ $\left(y+\frac{1}{\ }+\frac{1}{\ }\right)dx+\left(x-\frac{1}{\ }+\frac{1}{\ }\right)dy=0$

$\Rightarrow \left(y + \frac{1}{x} + \frac{1}{x^2 y}\right)dx + \left(x - \frac{1}{y} + \frac{1}{x y^2}\right)dy = 0$

which is of the form $F\,dx + G\,dy = 0$

$F = y + \frac{1}{x} + \frac{1}{x^2 y}$, $G = x - \frac{1}{y} + \frac{1}{x y^2}$.

$\frac{\partial F}{\partial y} = 1 + 0 + \frac{1}{x^2}\left(-\frac{1}{y^2}\right)$, $\frac{\partial G}{\partial x} = 1 + 0 + \frac{1}{y^2}\left(-\frac{1}{x^2}\right)$

$\frac{\partial F}{\partial y} = 1 - \frac{1}{x^2 y^2}$, $\frac{\partial G}{\partial x} = 1 - \frac{1}{x^2 y^2}$

$\therefore \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} \Rightarrow$ eq is exact

Sol: $\underset{y\ \text{constant}}{\int F\,dx} + \int \underset{\substack{\text{Term of } G \\ \text{not containing } x}}{G\,dy} = C.$

$\Rightarrow \int_{y\ \text{cnt}} \left(y + \frac{1}{x} + \frac{1}{x^2 y}\right) dx + \int \left(-\frac{1}{y}\right) dy = C$

$\Rightarrow xy + \ln x + \frac{1}{y}\left(-\frac{1}{x}\right) - \ln|y| = C$

$$\Rightarrow \boxed{xy + \ln\frac{x}{y} - \frac{1}{xy} = C}$$