23 January 2021 16:20

Equation reducible to exact form :-

$$f\,dx + g\,dy = 0$$

$\Rightarrow \frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x} \Rightarrow$ eq is not exact.

Case I :-> $\dfrac{\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}}{f} = K(y)$, I.F. $e^{-\int K(y)\,dy}$.

$\Rightarrow \dfrac{\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}}{f} = K_1(y)$, I.F. $e^{\int K_1(y)\,dy}$

$\boxed{\dfrac{\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}}{g} = h(x) \text{ , I.F. } e^{\int h(x)\,dx}}$.

Q) $y(1+xy^2)\,dx + 2(x^2y^2+x+y^4)\,dy = 0$ — (1)

Compare it with $f\,dx + g\,dy = 0$

$f = y(1+xy^2) = y + xy^3$

$g = 2(x^2y^2+x+y^4)$.

$\frac{\partial f}{\partial y} = 1+3xy^2$, $\frac{\partial g}{\partial x} = 2(2xy^2+1)$

$\therefore \frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x} \Rightarrow$ eq is not exact.

$$\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = (1+3xy^2) - (2xy^2+2)$$

$$= -1 - xy^2 = -(1+xy^2)$$

$$\Rightarrow \frac{\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}}{f} = \frac{(1+\cancel{xy^2})}{y(1+\cancel{xy^2})} = \frac{1}{y} = K(y)$$

$$I.F. \quad e^{\int K(y)dy} = e^{\int \frac{1}{y}dy} = e^{\ln y}$$

$$\boxed{I.F. = y}$$

Multiply I.F. to (1)

$$y^2(1+xy^2)dx + 2y(x^2y^2+x+y^4)dy = 0$$

$$\Rightarrow (y^2+xy^4)dx + (2x^2y^3+2xy+2y^5)dy = 0 \quad \text{--- (2)}$$

Compare it with $Fdx + Gdy = 0$

$$F = y^2 + xy^4, \quad G = 2x^2y^3 + 2xy + 2y^5$$

$$\frac{\partial F}{\partial y} = 2y + 4xy^3, \quad \frac{\partial G}{\partial x} = 4xy^3 + 2y + 0$$

Clearly $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$, eq (2) is exact.

Sol is $\underset{y \text{ constant}}{\int F\,dx} + \underset{\substack{\text{Term of } G \\ \text{not containing } x}}{\int G\,dy} = C.$

$$\Rightarrow \underset{y \text{ constant}}{\int (y^2+xy^4)\,dx} + \int 2y^5\,dy = C$$

y constant

$\Rightarrow \quad xy^2 + \frac{x^2}{2}y^4 + \frac{\cancel{2}y^6}{\cancel{6}\,3} = c$

$\Rightarrow \quad \boxed{xy^2 + \frac{1}{2}x^2y^4 + \frac{y^6}{3} = c}$

Case II: Inspection methods

$\boxed{y\,dx + x\,dy = d(xy)}$

$y\,dx - x\,dy.$

(1) $\frac{1}{x^2}$ (2) $\frac{1}{y^2}$ (3) $\frac{1}{xy}$ (4) $\frac{1}{x^2 \pm y^2}$

(1) $\frac{y\,dx - x\,dy}{x^2} = \frac{y}{x^2}dx - \frac{1}{x}dy$

$= -\left(\frac{1}{x}dy + y\left(\frac{-1}{x^2}\right)dx\right)$

$\boxed{\frac{y\,dx - x\,dy}{x^2} = -d\left(\frac{y}{x}\right)}$

(2) $\frac{y\,dx - x\,dy}{y^2} = \frac{1}{y}dx - \frac{x}{y^2}dy = d\left(\frac{x}{y}\right)$

(3) $\frac{y\,dx - x\,dy}{xy} = d\left(\frac{dx}{x}\right) - d\left(\frac{dy}{y}\right) = d(\ln x) - d(\ln y)$

$= d\left(\ln \frac{x}{y}\right)$

(4) $\frac{y\,dx - x\,dy}{x^2 \pm y^2} = \frac{\frac{1}{x^2}(y\,dx - x\,dy)}{\left(1 \pm \left(\frac{y}{x}\right)^2\right)} = \frac{-d\left(\frac{y}{x}\right)}{1 \pm \left(\frac{y}{x}\right)^2}$

$$= -d\left(\int \frac{d\left(\frac{y}{x}\right)}{1 \pm \left(\frac{y}{x}\right)^2}\right)$$

$$= \frac{1}{y^2}\left(\frac{y\,dx - x\,dy}{\left(\frac{x}{y}\right)^2 \pm 1}\right) \Rightarrow \frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y}\right)^2 \pm 1}.$$

$$= d\left(\int \frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y}\right)^2 \pm 1}\right)$$

Q) $x\,dy - y\,dx + y^2\,dx = 0$

$$\Rightarrow x\,dy + (y^2 - y)\,dx = 0$$

$\frac{1}{x^2}, \frac{1}{y^2}$

$$f\,dx + g\,dy = 0, \quad f = y^2 - y, \quad g = x.$$

$\frac{1}{xy}, \frac{1}{x^2 \pm y^2}$

$$\boxed{\frac{\partial f}{\partial y} = 2y - 1, \quad \frac{\partial g}{\partial x} = 1}$$

T.F. $\frac{1}{y^2}$.

$$\frac{x\,dy - y\,dx}{y^2} + dx = 0$$

$$\frac{x}{y^2}\,dy - \frac{1}{y}\,dx + dx = 0$$

$$\Rightarrow -d\left(\frac{x}{y}\right) + dx = 0$$

$$\Rightarrow d\left(x - \frac{x}{y}\right) = 0 \Rightarrow \text{eq is exact.}$$

$$\Rightarrow \boxed{x - \frac{x}{y} = c}$$