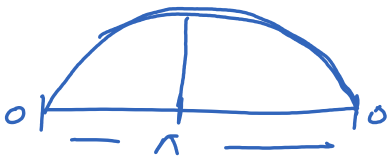

Q: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $0 < x < \pi$, $t > 0$

$\boxed{u(0,t) = 0 = u(\pi,t)}$, & $u(x,0) = \sin x$, $0 < x < \pi$.

Sol: We know that

$u(x,t) = A + Bx$. — (1)

(a) $A = 0, B = 1$, (b) $A = 0, B = \pi$

(c) $A = \pi, B = 0$, (d) $A = 0, B = 0$

when $x = 0$, $u = 0$, $(\because u(0,t) = 0)$

(1) $\Rightarrow$ $u(0,t) = A + B(0)$

$\boxed{0 = A}$

(1) $\Rightarrow$ $u(x,t) = Bx$ — (2)

$u(\pi,t) = 0 \Rightarrow$ when $x = \pi$, $u = 0$

(2) $u(\pi,t) = B\pi = 0$

$\Rightarrow B\pi = 0 \Rightarrow$ $B = 0$

$\Rightarrow$ (2), $u = 0$ which is not possible

Hence we discard this sol.

Consider $u(x,t) = e^{\lambda^2 c^2 t}(A_1 e^{\lambda x} + B_1 e^{-\lambda x})$ — (3)

$\boxed{u(0,t) = 0 \ \& \ u(\pi,t) = 0}$

$\boxed{u(0,t)=0 \ \& \ u(\pi,t)=0}$

(a) $A_1 = e^{\lambda\pi}$, $B_1 = e^{-\lambda\pi}$. (b) $A_1 = \dfrac{1}{e^{\lambda\pi}-e^{-\lambda\pi}}$

$B_1 = \dfrac{-1}{e^{\lambda\pi}-e^{-\lambda\pi}}$

~~(c)~~ $A_1 = 0,\ B_1 = 0$ (d) none of these.

$(K>0,\ K=\lambda^2,\ \lambda\in\mathbb{R}-\{0\})$

eq (2) $u(x,t) = e^{\lambda^2c^2t}\left(A_1e^{\lambda x}+B_1e^{-\lambda x}\right)$ — (3)

when $x=0, u=0$ ie $u(0,t)=0$

$u(0,t)=0 = e^{\lambda^2c^2t}\left(A_1e^{0}+B_1e^{0}\right)$

$\boxed{0 = e^{\lambda^2c^2t}\left(A_1+B_1\right)}$

$\Rightarrow A_1+B_1=0 \Rightarrow B_1=-A_1$

(3) $\Rightarrow u(x,t) = e^{\lambda^2c^2t}\left(A_1e^{\lambda x}-A_1e^{-\lambda x}\right)$

$u(x,t) = e^{\lambda^2c^2t}A_1\left(e^{\lambda x}-e^{-\lambda x}\right)$ — (4)

Now $u(\pi,t)=0 \Rightarrow$ when $x=\pi \Rightarrow u=0$

(4) $\Rightarrow u(\pi,t)=0 = \boxed{e^{\lambda^2c^2t}A_1\left(e^{\lambda\pi}-e^{-\lambda\pi}\right)}$ $(e^{\lambda\pi}-e^{-\lambda\pi}\neq 0$

$\because e^{\lambda\pi}-e^{-\lambda\pi}=0$

$\lambda\pi=-\lambda\pi$

$\lambda=0$

$\Rightarrow A_1 = 0$

(4) $\Rightarrow u(x,t)=0$ which is not possible.

Consider $u(x,t) = e^{-\lambda^2c^2t}\left(A_2\cos\lambda x+B_2\sin\lambda x\right)$ — (5)

$u(\pi,t)=0$, $u(x,0) = \sin x$, $0<x<\pi$

Consider $u(x,t) = e^{\ldots}$ ...

$u(0,t)=0$, $u(\pi,t)=0$, $u(x,0)=\sin x$, $0<x<\pi$

$\lambda = ??$

(a) $\lambda = n^2\pi^2$ (b) $\lambda = \frac{1}{n^2}$.

~~(c)~~ $\lambda = n$. (d) none of these.

When $x=0$, $u=0$ ($\because u(0,t)=0$)

(5) $\Rightarrow$ $u(0,t) = 0 = e^{-\lambda^2c^2t}(A_2\cos 0 + B_2 \sin 0)$

$0 = e^{-\lambda^2c^2t}(A_2(1) + B_2(0))$

$\therefore$ $0 = e^{-\lambda^2c^2t} \cdot A_2$. $\Rightarrow A_2 = 0$

(5) $\Rightarrow$ $u(x,t) = e^{-\lambda^2c^2t} B_2 \sin\lambda x$. — (6)

When $x=\pi$, $u=0$, ($\because u(\pi,t)=0$)

(5) $\Rightarrow$ $u(\pi,t) = e^{-\lambda^2c^2t} B_2 \sin\lambda\pi = 0$

$\Rightarrow e^{-\lambda^2c^2t} B_2 \sin\lambda\pi = 0$, $B_2 \neq 0$

$\sin\theta = 0$
$\theta = n\pi$

$\Rightarrow \sin\lambda\pi = 0 \Rightarrow$

$\Rightarrow \lambda\cancel{\pi} = n\cancel{\pi} \rightarrow \lambda = n$, $n \in \mathbb{Z}$

(6) $\Rightarrow$ $u(x,t) = e^{-n^2c^2t} B_2 \sin nx$, $n \in \mathbb{Z}$

By superposition principle.

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2c^2t} b_n \sin nx. \quad \text{— (7)}$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2c^2t}\, b_n \sin nx . \quad (7)$$

$u(x,0) = \sin x, \quad 0 < x < \pi$

$b_n = ??$

(a) $b_n = \frac{(-1)^n}{n^2\pi^2}$ (b) $b_n = \frac{1}{n}$

$b_n = 0, n \neq 1, b_1 = 1$

(c) $\frac{1-(-1)^n}{n\pi}$ (d) none of here.

Q (P)

$$u(x,t) = \sum_{n=1}^{\infty} e^{-n^2c^2t}\, b_n \sin nx .$$

when $t = 0$, $u = \sin x$ ($\because u(x,0) = \sin x$)

$$u(x,0) = \sin x = \sum_{n=1}^{\infty} e^{0}\, b_n \sin nx .$$

$$\Rightarrow \sin x = \sum_{n=1}^{\infty} b_n \sin nx .$$

$\Rightarrow \sin x = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$

$b_1 = 1$, $b_2 = 0, b_3 = 0, b_4 = 0, \cdots$

$b_n = 0, n \neq 1$

$$\int_c^{c+2l} \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\, dx = \begin{cases} l, & n = m \\ 0, & n \neq m \end{cases}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx \, dx .$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx \, dx.$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin x \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\cos(n-1)x - \cos(n+1)x\right)dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}, \; n \neq 1$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\pi}{n+1}\right]$$

$$= 0.$$

$$b_n = 0 \quad , n \neq 1$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \sin x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin^2 x \, dx = \frac{\cancel{2}}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{\cancel{2}}\right)dx$$

$$= \frac{1}{\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi - \left(\frac{\sin 2\pi}{2} - 0\right)\right]$$

$$\boxed{b_1 = \frac{1}{\pi} \times \pi = 1.}$$

$$u(x,t) = e^{-c^2 t} b_1 \sin x + \sum_{n=2}^{\infty} e^{-n^2 c^2 t} \, b_n \sin nx$$

$$= e^{-c^2 t} \sin x + 0$$

$$u(x,t) = e^{-c^2 t} \sin x$$