Method of separation of variables :-

given pde

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, --\right) = 0 \quad — (1)$$

Let the trial solution of eq (1) be.

$$\boxed{z = f(x)\, g(y)}$$

$$\frac{\partial z}{\partial x} = g(y) f'(x), \quad \frac{\partial z}{\partial y} = f(x) g'(y)$$

$$\frac{\partial^2 z}{\partial x^2} = g(y) f''(x), \quad \frac{\partial^2 z}{\partial y^2} = f(x) g''(y)$$

— — — -

eq (1) $\Rightarrow$ $F\left(x, y, f(x) g(y), f'(x) g(y), f(x) g'(y), -\right) = 0$

$\Rightarrow$ $F_1\left(x, f(x), f'(x), --\right) = F_2\left(y, g(y), g'(y) --\right)$

$= K$ (constant)

$\because$ x & y are independent.

$\Rightarrow$ $F_1\left(x, f(x), f'(x) --\right) = K$ — (2)

& $F_2\left(y, g(y), g'(y) --\right) = K$ — (3)

After solving diff. eq eq (2) & (3)

$f(x) =$ ______ , $g(y) =$ ______

sol. is $z = f(x)\, g(y)$.

Q1:- Solve the pde $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x,0) = 4e^{-x}$ using method of separation of variables.

Sol:- Clearly u is a function of x & y.
Let the final solution be $u(x,y) = f(x)\, g(y)$

$u = x^2 y^2$
$\frac{\partial u}{\partial x} = y^2 \frac{d}{dx} x^2$

$$\frac{\partial u}{\partial x} = f'(x)\, g(y), \quad \frac{\partial u}{\partial y} = f(x)\, g'(y)$$

$\frac{d}{dy}\sin y$
$\cos y$

The given diff. eq is $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ — ①

① $\Rightarrow$ $3f'(x)\, g(y) + 2f(x)\, g'(y) = 0$

$\Rightarrow$ $3f'(x)\, g(y) = -2f(x)\, g'(y)$

dividing both sides by $f(x)\, g(y)$

$$\frac{3f'(x)\,\cancel{g(y)}}{f(x)\,\cancel{g(y)}} = \frac{-2\,\cancel{f(x)}\, g'(y)}{\cancel{f(x)}\, g(y)}$$

or. $\frac{3f'(x)}{} = \frac{-2g'(y)}{g(y)}$ — ②

or: $$\frac{3f'(x)}{f(x)} = \frac{-2g'(y)}{g(y)} \quad \text{—} (*)$$

$\therefore$ LHS is a function of x alone & RHS is a function of y alone but x & y are independent. So they are equal only if both of them are equal to some constant 'K'.

$$\frac{3f'(x)}{f(x)} = -\frac{2g'(y)}{g(y)} = K \text{ (constant)}$$

$$\Rightarrow \frac{3f'(x)}{f(x)} = K, \quad \frac{-2g'(y)}{g(y)} = K.$$

On integrating both sides

$$3\int \frac{f'(x)}{f(x)}\,dx = K\int dx \quad \Bigg| \quad -2\int \frac{g'(y)}{g(y)}\,dy = K\int dy$$

$$\Rightarrow 3\ln|f(x)| = Kx + C, \quad -2\ln|g(y)| = Ky + C_1$$

$$\ln|f(x)| = \frac{Kx+C}{3}, \quad \ln|g(y)| = \frac{-Ky - C_1}{2}.$$

$$\text{or: } f(x) = e^{\frac{Kx+C}{3}}, \quad g(y) = e^{-\left(\frac{Ky+C_1}{2}\right)}$$

$$f(x) = e^{\frac{Kx}{3}}\, e^{C/3}, \quad g(y) = e^{-Ky/2}\cdot e^{-C_1/2}.$$

Let $A = e^{C/3}$ & $B = e^{-C_1/2}$.

$$\Rightarrow f(x) = e^{Kx/3}\cdot A, \quad g(y) = e^{-Ky/2}\, B$$

hence sol. is $u(x,y) = f(x).g(y) = Ae^{kx/3} \cdot Be^{\cdots}$

or $u(x,y) = AB\, e^{k\left(\frac{x}{3} - \frac{y}{2}\right)}$, let $AB = D$

sol is $\boxed{u(x,y) = De^{\frac{k}{6}(2x-3y)}}$ — ②

given $u(x,0) = 4e^{-x}$
i.e when $y = 0$, $u = 4e^{-x}$

② ⇒ $u(x,0) = De^{\frac{k}{\cancel{6}\,3}(\cancel{2}x)} = 4e^{-x}$.

$D = 4$, $\frac{k}{3} = -1 \Rightarrow k = -3$

Sol. is $u(x,y) = 4\, e^{-\frac{3}{6}(2x-3y)}$

$\boxed{u(x,y) = 4\,e^{-\frac{1}{2}(2x-3y)}}$

Or Solve the pde $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$ —① using method of separation of variables.

Sol: Let the trial solution be $u(x,y) = f(x)\, g(y)$.

$\frac{\partial u}{\partial x} = f'(x)\, g(y)$, $\frac{\partial u}{\partial y} = f(x)\, g'(y)$

① ⇒ $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$

① $\Rightarrow \quad y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0$

$$\Rightarrow \quad y \cdot f'(x) g(y) + x f(x) g'(y) = 0$$

$$\Rightarrow \quad y f'(x) g(y) = -x f(x) g'(y)$$

dividing both sides by $(x y f(x) g(y))$

$$\frac{\cancel{y} f'(x) \cancel{g(y)}}{x \cancel{y} f(x) \cancel{g(y)}} = -\frac{\cancel{x} \cancel{f(x)} g'(y)}{\cancel{x} y \cancel{f(x)} g(y)}$$

$$\Rightarrow \quad \frac{f'(x)}{x f(x)} = \frac{-g'(y)}{y g(y)} \quad \text{--- } (*)$$

LHS is a function of x alone & RHS is a function of y alone but x & y are independent.

$(*) \Rightarrow \quad \dfrac{f'(x)}{x f(x)} = \dfrac{-g'(y)}{y g(y)} = K$ (constant.)

$$\Rightarrow \quad \frac{f'(x)}{x f(x)} = K, \qquad \frac{-g'(y)}{y g(y)} = K.$$

$$\Rightarrow \int \frac{f'(x)}{f(x)} dx = \int K x \, dx, \qquad \int \frac{g'(y)}{g(y)} dy = -\int K y \, dy$$

$\int f(x) \qquad\qquad \int g \ldots$

$\Rightarrow \ln|f(x)| = K\frac{x^2}{2} + C$, $\ln|g(y)| = -K\frac{y^2}{2} + C_1$

$\Rightarrow f(x) = e^{\frac{Kx^2}{2} + C}$, $g(y) = e^{-\frac{Ky^2}{2} + C_1}$

or $f(x) = e^{\frac{Kx^2}{2}} e^{C}$, $g(y) = e^{-\frac{Ky^2}{2}} e^{C_1}$

Let $A = e^{C}$ & $B = e^{C_1}$

$f(x) = e^{Kx^2/2} \cdot A$, $g(y) = B e^{-Ky^2/2}$

$\therefore$ sol is $u(x,y) = f(x) \cdot g(y) = AB\, e^{\frac{Kx^2}{2}} \cdot e^{-Ky^2/2}$

or $u(x,y) = AB\, e^{\frac{K}{2}(x^2 - y^2)}$, let $AB = D$

Sol is $\boxed{u(x,y) = D e^{\frac{K}{2}(x^2 - y^2)}}$

Classification of second order homogeneous linear PDE :-

Consider the second order homogeneous linear PDE of the form

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2}} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

where A, B, C, D, E & F are function of x & y or real constant.

The PDE is said to be

① parabolic if $B^2-4AC=0$

② Hyperbolic if $B^2-4AC>0$

③ Elliptic if $B^2-4AC<0$.

Q1 Classify the following pde.

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x} = 0 \quad \text{— (1)}$$

Soln Compare (1) with $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$

$A=1, \quad B=0, \quad C=3$

Consider $B^2-4AC = (0)^2 - 4(1)(3)$
$= -12 < 0$

∴ The given eq is elliptic.

Q1 Classify the pde.

$$\frac{\partial^2 u}{\partial x \partial y} = 3\frac{\partial u}{\partial y}$$

(a) parabolic
(b) Elliptic
(c) Hyperbolic

$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$

Sol: $A = 0$, $B = 1$, $C = 0$

$$B^2 - 4AC = (1)^2 - 4(0)(0)$$
$$= 1 > 0$$

eq is Hyperbolic.

Q. Classify the pde. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x \partial y} = 0$ — (1)

(a) parabolic (b) elliptic (c) Hyperbolic

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

Sol: eq (1) is $A = 1$, $B = -1$, $C = 1$.

$$B^2 - 4AC = (-1)^2 - 4(1)(1)$$
$$= 1 - 4 = -3 < 0$$

$\Rightarrow$ eq is elliptic

Q. $(1-y)\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + (1+y)\frac{\partial^2 u}{\partial y^2} = 0$

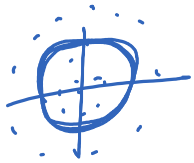
Sol: $A = 1-y$, $B = 2x$, $C = 1+y$.

$$B^2 - 4AC = (2x)^2 - 4(1-y)(1+y)$$
$$= 4x^2 - 4(1-y^2)$$

$= 4x^2 + 4y^2 - 4 = 4(x^2+y^2-1)$

(a) parabolic if $B^2 - 4AC = 0 \Rightarrow 4(x^2+y^2-1) = 0$

$\Rightarrow x^2+y^2 = 1.$



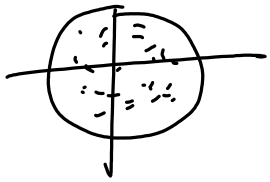
i.e eq is parabolic in the region

$R = \{(x,y);\ x^2+y^2=1\}$

(b) Elliptic if $B^2 - 4AC < 0$

$\Rightarrow 4(x^2+y^2-1) < 0$

$\Rightarrow x^2+y^2 < 1$

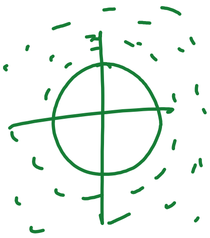


i.e eq. is elliptic in the region

$R_1 = \{(x,y);\ x^2+y^2 < 1\}$

(c) Hyperbolic if $B^2 - 4AC > 0$

$\Rightarrow 4(x^2+y^2-1) > 0 \Rightarrow x^2+y^2 > 1$



i.e eq. is hyperbolic in the region

$R_2 = \{(x,y);\ (x^2+y^2) > 1\}$