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Differential equation with variable coeff.

Euler-Cauchy Homogeneous LDE :-

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n y = r(x) \quad \text{—} (x)$$

Put: $\boxed{\ln x = z}$ $\Rightarrow \boxed{x = e^z}$ $\Rightarrow \frac{dz}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \Rightarrow x^1 \frac{dy}{dx} = \frac{dy}{dz} \quad \text{—} (1)$$

diff w.r.t x

$$\Rightarrow x \frac{d^2y}{dx^2} + \frac{dy}{dx}(1) = \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} = \frac{d^2y}{dz^2}\left(\frac{1}{x}\right)$$

$$\Rightarrow x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dz^2} \cdot \frac{1}{x}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dz^2}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \left(x\frac{dy}{dx}\right)$$

$\boxed{\frac{d}{dx} = D}$

$\frac{d}{dz} = \theta$

$$(1) \Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} = \theta y, \quad x^2\frac{d^2y}{dx^2} = \theta^2 y - \theta y = \theta(\theta-1)y$$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} = \theta y$, $x^2\frac{d^2y}{dx^2} = \theta(\theta-1)y$

$\Rightarrow x^3\frac{d^3y}{dx^3} = \theta(\theta-1)(\theta-2)y, \ldots$

$(*) \Rightarrow \{a_0(\theta(\theta-1)(\theta-2)\ldots(\theta-(n-1)) + a_1(\theta(\theta-1)\ldots(\theta-\overline{n-2}) + \ldots + a_n\}y = r(e^z)$

$$\Rightarrow \boxed{F(\theta)y = G(z)}, \quad \theta = \frac{d}{dz}$$

this is diff. eq. with constant coeff.

Q. Solve the diff. eq. $x^2y'' + 4xy' + 2y = 0$ — (1)

Sol: Given eq is Euler Cauchy homogeneous diff eq.

To reduce the given diff eq. into diff. eq. with constant coeff we make a substitution

$\ln x = z \Rightarrow x = e^z$.

$x\frac{dy}{dx} = \theta y$, $x^2\frac{d^2y}{dx^2} = \theta(\theta-1)y$, $\theta = \frac{d}{dz}$.

$(1) \Rightarrow (\theta(\theta-1) + 4\theta + 2)y = 0$, $\theta = \frac{d}{dz}$

$\Rightarrow (\theta^2 - \theta + 4\theta + 2)y = 0$

$\Rightarrow (\theta^2 + 3\theta + 2)y = 0.$

Let $y = e^{mz}$ be its sol.

A.E., $m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2$

sol., $y = c_1 e^{-z} + c_2 e^{-2z}$

$= c_1 (x)^{-1} + c_2 (x)^{-2}$

$$\boxed{y = \frac{c_1}{x} + \frac{c_2}{x^2}}$$

Q: find the general sol. of diff. eq.

$x^2 y'' - 3xy' + 3y = 2 + \ln x$ — (1)

Sol:- To reduce the given into diff. eq. with constant coeff put $\ln x = z$ or $x = e^z$.

$\Rightarrow x\frac{dy}{dx} = \theta y$, $x^2 \frac{d^2y}{dx^2} = \theta(\theta-1)y$, $\theta = \frac{d}{dz}$.

(1) $\Rightarrow \theta(\theta-1)y - 3\theta y + 3y = 2 + z$.

$\Rightarrow (\theta^2 - \theta - 3\theta + 3)y = 2 + z$, $\theta = \frac{d}{dz}$.

$\Rightarrow (\theta^2 - 4\theta + 3)y = 2 + z$.

For C.F., Consider the homogeneous diff. eq.

$(\theta^2 - 4\theta + 3)y = 0$.

Let $y = e^{mz}$ be the sol.

A.E: $m^2 - 4m + 3 = 0 \Rightarrow (m-3)(m-1) = 0$

$\Rightarrow m = 3, 1.$

$y_c = c_1 e^{3z} + c_2 e^{z}$. $\boxed{\because x = e^{z}}$

$= c_1 x^3 + c_2 x$

For P.I:

$$(\theta^2 - 4\theta + 3) y_p = 2 + z$$

$$y_p = \frac{1}{\theta^2 - 4\theta + 3} \cdot (2+z)$$

$$= \frac{1}{3\left(1 + \frac{\theta^2 - 4\theta}{3}\right)} (2+z)$$

$$= \frac{1}{3}\left(1 + \left(\frac{\theta^2 - 4\theta}{3}\right)\right)^{-1} (z+2)$$

$$= \frac{1}{3}\left(1 - \left(\frac{\theta^2 - 4\theta}{3}\right) + \cdots\right)(z+2)$$

$$= \frac{1}{3}\left((z+2) - \frac{1}{3}\left(0 - 4(1+0)\right) + 0\right)$$

$$y_p = \frac{1}{3}\left(z + 2 + \frac{4}{3}\right) = \frac{1}{3}\left(z + \frac{10}{3}\right)$$

$$\Rightarrow y_p = \frac{1}{3}\left(\ln x + \frac{10}{3}\right)$$

C.S: $\boxed{y = y_c + y_p = c_1 x^3 + c_2 x + \frac{\ln x}{3} + \frac{10}{9}}$

— ①

Q) Solve $4x^2y'' + 16xy' + 9y = 19\cos(\ln x) + 24x\ln x$.

Sol:- To reduce it into diff eq with constant coeff.

Put $\ln x = z \Rightarrow x = e^z$.

$xy' = \theta y$, $x^2y'' = \theta(\theta-1)y$, $\theta = \frac{d}{dz}$.

(1) $\Rightarrow$ $4(\theta(\theta-1)y) + 16(\theta y) + 9y = 19\cos z + 24e^z\cdot z$.

$\Rightarrow (4(\theta^2-\theta) + 16\theta + 9)y = 19\cos z + 24ze^z$.

$\Rightarrow (4\theta^2 + 12\theta + 9)y = 19\cos z + 24ze^z$

For CF: Consider the homogeneous diff eq.

$(4\theta^2 + 12\theta + 9)y = 0$, $\theta = \frac{d}{dz}$.

(a) $(c_1 + \cancel{c_2 x})x^{-3/2}$ (b) $(c_1 + \cancel{c_2 x})e^{-3/2x}$

(c) $(c_1 + c_2\ln x)x^{-3/2}$ (d) none of them.

Sol, Let $y = e^{mz}$ be the sol!

A.E:- $4m^2 + 12m + 9 = 0$

$4m^2 + 6m + 6m + 9 = 0$

$\Rightarrow 2m(2m+3) + 3(2m+3) = 0$

$\Rightarrow (2m+3)(2m+3) = 0 \Rightarrow m = -\frac{3}{2}, -\frac{3}{2}$

C.F, $y_c = (c_1 + c_2 z)e^{-3/2 z}$

$= (c_1 + c_2\ln x)x^{-3/2}$.

$= \left(C_1 + C_2 \ln x\right) x^{-3/2}.$

For P.I.:

$$y_p = \frac{1}{4\theta^2+12\theta+9}\left(19\cos z + 24 z e^z\right)$$

$$= 19 \cdot \frac{1}{4\theta^2+12\theta+9}\cos z + 24\,\frac{1}{4\theta^2+12\theta+9} z e^z$$

$$= 19\,\frac{1}{4(-1^2)+12\theta+9}\cos z + 24\,\frac{1}{(2\theta+3)^2} z e^z.$$

$$= 19\,\frac{1}{12\theta+5}\cos z + 24 e^z \frac{1}{(2(\theta+1)+3)^2} z.$$

$$= 19\,\frac{12\theta-5}{(12\theta+5)(12\theta-5)}\cos z + 24 e^z \frac{1}{(2\theta+5)^2} z$$

$$= 19\,\frac{(12\theta-5)}{144\theta^2-25}\cos z + \frac{24 e^z}{25\left(1+\frac{2\theta}{5}\right)^2} z.$$

$$= 19\,\frac{(12\theta-5)}{144(-1^2)-25}\cos z + \frac{24}{25} e^z \left(1+\frac{2\theta}{5}\right)^{-2} z.$$

$$= 19\,\frac{(12\theta-5)}{-169}\cos z + \frac{24}{25} e^z \left(1-2\left(\frac{2\theta}{5}\right)+\dots\right) z$$

$$= \frac{-19}{169}\left(-12\sin z - 5\cos z\right) + \frac{24}{25} e^z \left(z - \frac{4}{5}(1) + 0\right)$$

$$= \frac{19}{169}\left(12\sin z + 5\cos z\right) + \frac{24}{25} e^z \left(z - \frac{4}{5}\right)$$

$$\frac{1}{f(D)}\cos ax = \frac{1}{f(D)}_{D^2=-a^2}\cos ax.$$

$$\frac{1}{f(D)} e^{ax} \cdot V = e^{ax}\left(\frac{1}{f(D+a)} V\right)$$

$\theta = z\frac{d}{dz}$

$$\frac{19}{169}(12\sin z + 5\cos z) + \frac{24}{25} e^{z}\left(z - \frac{4}{5}\right)$$

$$y_p = \frac{19}{169}(12\sin(\ln x) + 5\cos(\ln x)) + \frac{24}{25} x\left(\ln x - \frac{4}{5}\right)$$

$$= \frac{19}{169}(12\sin(\ln x) \ldots$$

$\therefore$ $y = y_c + y_p$

$$= (c_1 + c_2 \ln x)x^{-3/2} + \frac{19}{169}(12\sin(\ln x) + 5\cos(\ln x)) + \frac{24}{25}x\left(\ln x - \frac{4}{5}\right).$$

Q) Solve the diff eq.

$$(3x+1)^2 y'' + (3x+1)y' + y = 6x \quad \text{--- (1)}$$

Sol:- $\ln(3x+1) = z \Rightarrow (3x+1) = e^z \Rightarrow x = \frac{e^z - 1}{3}$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{1}{(3x+1)}\cdot 3, \quad \theta = \frac{d}{dz}.$$

$$\Rightarrow (3x+1)\frac{dy}{dx} = 3\frac{dy}{dz} = 3\theta y$$

$$\Rightarrow (3x+1)^2\frac{d^2y}{dx^2} = 3^2\frac{d^2y}{dz^2} - 3^2\frac{dy}{dz}$$

$$= 3^2\,\theta(\theta-1)\,y$$

In general

If diff eq of the type –

$$A_0(ax+b)^n y^{(n)} + A_1(ax+b)^{n-1} y^{(n-1)} + \cdots + A_n y = R(x)$$

$A_0(ax+b)^n y^{(n)} + A_1(ax+b) y + \cdots + A_n y = X(x)$

Put $\ln(ax+b) = z \Rightarrow (ax+b) = e^z$.

$(ax+b)y' = a\theta y$, $(ax+b)^2 y'' = a^2\theta(\theta-1)y$

$(ax+b)^3 y''' = a^3\theta(\theta-1)(\theta-2)y - - - -$

$$3^2(\theta(\theta-1)y) + 3\theta y + y = 6\left(\frac{e^z-1}{3}\right)$$

$$\Rightarrow (9(\theta^2-\theta)+3\theta+1)y = 2e^z - 2$$

$$(9\theta^2-9\theta+3\theta+1)y = 2e^z-2$$

$$\Rightarrow (9\theta^2-6\theta+1)y = 2e^z-2, \quad \theta = \frac{d}{dz}$$

For C.F., Consider the homogeneous diff eq.

$$(9\theta^2-6\theta+1)y = 0 \Rightarrow$$

Let $y = e^{mz}$ be its sol.

A.E.: $9m^2-6m+1 = 0 \Rightarrow (3m-1)^2 = 0$

$\Rightarrow m = 1/3, 1/3$

$$y_c = (C_1+C_2 z)e^{z/3}$$

$$= (C_1 + C_2(\ln(3x+1)))(3x+1)^{1/3}$$

P.I.: $y_p = \dfrac{1}{(3\theta-1)^2} 2e^z - 2.$

$$= 2\frac{1}{(3\theta-1)^2}e^{z} - \frac{1}{(3\theta-1)^2}\cdot 2$$

$$= 2\frac{1}{(3(1)-1)^2}e^{z} - \frac{\cancel{2}\,1}{(3\theta-1)^2}\, e^{0z}$$

$$= \frac{2\;e^{z}}{4} - \frac{2}{(3(0)-1)^2}$$

$$= \frac{1}{2}e^{z} - 2. = \frac{1}{2}(3x+1) - 2.$$

$$= \frac{3x}{2} + \frac{1}{2} - 2 = \frac{3x}{2} - \frac{3}{2}.$$

$$= \frac{3}{2}(x-1)$$

$$y = y_c + y_p = \left(C_1 + C_2\ln(3x+1)\right)(3x+1)^{1/3} + \frac{3}{2}(x-1)$$