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Equation of first order & higher degree

Q. Solve the differential eq.

$$y = x\left[p+\sqrt{1+p^2}\right] = f(p, x)$$

(a) equation solvable for p (b) solvable for y, $y = f(x,p)$ (c) solvable for x, $\boxed{x = f(y,p)}$

$$x = \frac{y}{p+\sqrt{1+p^2}} = g(y,p)$$

Sol 1. $y = x\left(p+\sqrt{1+p^2}\right)$

diff w.r.t x.

$$f\left(p, x, \frac{dp}{dx}\right) = 0$$

$$p = \frac{dy}{dx} = x\left(\frac{dp}{dx} + \frac{1}{2\sqrt{1+p^2}}\, 2p\frac{dp}{dx}\right) + \left(p+\sqrt{1+p^2}\right) \quad (1)$$

$$\Rightarrow \cancel{p} = x\left(1+\frac{p}{\sqrt{1+p^2}}\right)\frac{dp}{dx} + \cancel{p} + \sqrt{1+p^2}.$$

$$\Rightarrow 0 = x\left(1+\frac{p}{\sqrt{1+p^2}}\right)\frac{dp}{dx} + \sqrt{1+p^2}.$$

$$\Rightarrow 0 = x\left(1+\frac{p}{\sqrt{1+p^2}}\right)dp + \sqrt{1+p^2}\, dx \quad (1)$$

eq. is of the form $\boxed{f\,dx + g\,dp = 0}$

$$f = \sqrt{1+p^2}, \quad g = x\left(1+\frac{p}{\sqrt{1+p^2}}\right)$$

$$\frac{\partial f}{\partial p} = \frac{1}{\cancel{2}\sqrt{1+p^2}}\cancel{2}p = \frac{p}{\sqrt{1+p^2}}, \quad \frac{\partial g}{\partial x} = 1 + \frac{p}{\sqrt{1+p^2}}.$$

Clearly $\frac{\partial f}{\partial p} \neq \frac{\partial g}{\partial x}$ eq is not exact.

$$\frac{\frac{\partial f}{\partial p} - \frac{\partial g}{\partial x}}{f} = \frac{-1}{\sqrt{1+p^2}} = K(p)$$

$$I.F. \quad e^{-\int K(p)\,dp} = e^{\int \frac{1}{\sqrt{1+p^2}}dp} = e^{\ln|p+\sqrt{1+p^2}|}$$

$$= \left(p + \sqrt{1+p^2}\right)$$

Multiply I.f to eq (1)

$$\left(p+\sqrt{1+p^2}\right)\left(\sqrt{1+p^2}\,dx + x\left(1 + \frac{p}{\sqrt{1+p^2}}\right)dp\right) = 0$$

$$\Rightarrow \left(p+\sqrt{1+p^2}\right)\sqrt{1+p^2}\,dx + x\,\frac{\left(p+\sqrt{1+p^2}\right)^2}{\sqrt{1+p^2}}\,dp = 0$$

which is of the form $F\,dx + G\,dp = 0$

$$F = p\sqrt{1+p^2} + 1 + p^2, \quad G = x\left(\frac{p^2 + 1 + p^2 + 2p\sqrt{1+p^2}}{\sqrt{1+p^2}}\right)$$

$$= x\left(\frac{p^2}{\sqrt{1+p^2}} + \sqrt{1+p^2} + 2p\right)$$

$= x\left(\frac{1}{\sqrt{1+p^2}} + \ldots\right)$

$$\frac{\partial F}{\partial p} = p \cdot \frac{1}{\cancel{2}\sqrt{1+p^2}} \cancel{2}p + \sqrt{1+p^2}\,(1) + 0 + 2p$$

$$= \frac{p^2}{\sqrt{1+p^2}} + \sqrt{1+p^2} + 2p \; , \quad \frac{\partial G}{\partial x} = \frac{p^2}{\sqrt{1+p^2}} + \sqrt{1+p^2} + 2p$$

$\therefore \quad \frac{\partial F}{\partial p} = \frac{\partial G}{\partial x}$.

Sol. $\displaystyle\int_{y \text{ constant}} F\,dx + \int_{\text{Term of } G \text{ not containing } x} G\,dp = C.$

$\Rightarrow \displaystyle\int_{p\text{ constant}} \left(p\sqrt{1+p^2} + p^2 + 1\right) dx + \int 0\,dp = C$

$\Rightarrow \left(p\sqrt{1+p^2} + 1 + p^2\right) x = C.$

$$\Rightarrow \boxed{x = \frac{C}{\sqrt{1+p^2}\,\left(p + \sqrt{1+p^2}\right)}} \quad \text{— (2)}$$

Substituting the value of x in given eq., we have.

$$\boxed{y = \frac{C\,\left(\cancel{\sqrt{1+p^2} + p}\right)}{\sqrt{1+p^2}\,\cancel{\left(p + \sqrt{1+p^2}\right)}} = \frac{C}{\sqrt{1+p^2}}} \quad \text{— (3)}$$

eq (2) & (3) constitute together to give solution of given diff eq.

eq (2) & (3) constitute together to give the [illegible]

Q. $y = x\left[p + \sqrt{1+p^2}\right]$

$\Rightarrow \dfrac{y}{x} = p + \sqrt{1+p^2} \Rightarrow \dfrac{y}{x} - p = \sqrt{1+p^2}.$

Squaring both sides $\left(\dfrac{y}{x} - p\right)^2 = 1 + p^2.$

$\Rightarrow (y - px)^2 = x^2(1+p^2)$

$\Rightarrow y^2 + \cancel{p^2x^2} - 2pyx = x^2 + \cancel{p^2x^2}$

$\Rightarrow y^2 - x^2 = 2pyx$

$\Rightarrow p = \dfrac{y^2 - x^2}{2yx}.$

Put $p = \dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}.$

$\Rightarrow 2xy\,dy = (y^2 - x^2)\,dx.$

$\Rightarrow (x^2 - y^2)\,dx + 2xy\,dy = 0$

Which is of the form $f\,dx + g\,dy = 0$

$f = x^2 - y^2, \quad g = 2xy.$

$\dfrac{\partial f}{\partial y} = -2y, \quad \dfrac{\partial g}{\partial x} = 2y.$

Clearly $\dfrac{\partial f}{\partial y} \neq \dfrac{\partial g}{\partial x}$ eq is not exact.

$$\frac{\frac{\partial f}{\partial y}-\frac{\partial g}{\partial x}}{g} = \frac{-4y}{2xy} = -\frac{2}{x} = h(x)$$

$$\text{I.F.} = e^{\int h(x)\,dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}}.$$

$$= x^{-2} = \frac{1}{x^2}.$$

$$\left(\frac{x^2-y^2}{x^2}\right)dx + \frac{2xy}{x^2}\,dy = 0$$

$$\Rightarrow \left(1-\frac{y^2}{x^2}\right)dx + \frac{2y}{x}\,dy = 0$$

$\Rightarrow$ Compare it with $F\,dx + G\,dy = 0$

$$F = 1-\frac{y^2}{x^2}, \quad G = \frac{2y}{x}.$$

$$\frac{\partial F}{\partial y} = 0-\frac{1}{x^2}(2y), \quad \frac{\partial G}{\partial x} = 2y\left(-\frac{1}{x^2}\right)$$

$$= \frac{-2y}{x^2}, \qquad = -\frac{2y}{x^2}.$$

$\Rightarrow \because \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} \Rightarrow$ eq. is exact.

sol. $\underset{y\text{ constant}}{\int} F\,dx + \int \underset{\substack{\text{Term of } G \\ \text{not containing } x}}{G\,dy} = c$

$$\Rightarrow \underset{y\text{ constant}}{\int}\left(1-\frac{y^2}{x^2}\right)dx + \int 0\,dy = c$$

y constant

$\Rightarrow x - y^2\left(\frac{-1}{x}\right) = c.$

$\Rightarrow \boxed{x + \frac{y^2}{x} = c}$ ✓

$\int \sin^2 x \, dx$

$\frac{1}{2}\int 2\sin x \cos x \, dx.$

$= \frac{\sin^2 x}{2} + C$

$= \frac{1}{2}\int \sin 2x \, dx = -\frac{\cos 2x}{4} + C.$

$x^2 + y^2 = a^2$, $x = a\cos t$
$y = a\sin t.$

Q) Solve the diff. eq. $x - yp = ap^2.$

(a) factorization ~~(b)~~ $x = f(y, p)$ ~~(c)~~ $y = g(x, p)$

Sol:

$$\underbrace{x = yp + ap^2}_{\boxed{x = g(y,p)}} \quad (1)$$

$x = yp + ap^2$
$x - ap^2 = yp$

diff (1) w.r.t y.

$$\frac{dx}{dy} = y\frac{dp}{dy} + p(1) + a\,2p\frac{dp}{dy}.$$

$$\Rightarrow \frac{1}{p} = p + (y + 2ap)\frac{dp}{dy}.$$

$\left(p - \frac{1}{?}\right) = 0$

or.. $(y+2ap)\frac{dp}{dy} + \left(p-\frac{1}{p}\right) = 0$

or. $(y+2ap)\,dp + \left(p-\frac{1}{p}\right)dy = 0.$

It is of the form $f\,dp + g\,dy = 0$

$$f = y+2ap, \quad g = p-\frac{1}{p}$$

$$\frac{\partial f}{\partial y} = 1, \quad \frac{\partial g}{\partial p} = 1+\frac{1}{p^2}.$$

Clearly $\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial p}$, $\dfrac{\frac{\partial f}{\partial y} - \frac{\partial g}{\partial p}}{g} = \dfrac{-\frac{1}{p^2}}{p-\frac{1}{p}} = K(p)$

$$K(p) = \frac{-1}{p^{2}\left(\frac{p^2-1}{p}\right)} = \frac{1}{p(1-p^2)}$$

I.F. $e^{\int K(p)dp}$.

(a) $\frac{1}{\sqrt{1-p^2}}$ ~~(b)~~ $\frac{p}{\sqrt{1-p^2}}$ (c) $\frac{p}{1-p^2}$ (d) none of these.

$$\int K(p)\,dp = \int \frac{dp}{p(1-p^2)} = \frac{-1}{2}\int \frac{-2\,dp}{p^3(p^{-2}-1)}$$

$$= \frac{-1}{2}\int \frac{dt}{t} \qquad \text{Put } p^{-2}-1 = t, \quad -2p^{-3}dp = dt.$$

$$= \frac{-1}{2}\ln|t| = \ln(t)^{-1/2}.$$

$= \frac{1}{2}\ln(\ldots) \ldots \ln(c)$

$$= \ln \frac{1}{\sqrt{p^{-2}-1}} = \ln \frac{1}{\sqrt{\frac{1}{p^2}-1}}$$

$$= \ln\left(\frac{p}{\sqrt{1-p^2}}\right)$$

$$I.F = e^{\int K(p)\,dp} = e^{\ln \frac{p}{\sqrt{1-p^2}}} = \frac{p}{\sqrt{1-p^2}}$$

$$(y+2ap)\frac{p}{\sqrt{1-p^2}}\,dx + \left(\frac{p^2-1}{\cancel{p}}\right)\frac{\cancel{p}}{\sqrt{1-p^2}}\,dp = 0$$

$$\Rightarrow (y+2ap)\frac{p}{\sqrt{1-p^2}}\,dx - \sqrt{1-p^2}\,dp = 0$$

Compare it with $F\,dx + G\,dp = 0$

$$F = (y+2ap)\frac{p}{\sqrt{1-p^2}}, \quad G = -\sqrt{1-p^2}$$

$$\frac{\partial F}{\partial y} = \frac{p}{\sqrt{1-p^2}}(1+0), \quad \frac{\partial G}{\partial p} = +\frac{1 \times \cancel{2}p}{\cancel{2}\sqrt{1-p^2}}$$

$\Rightarrow \frac{\partial F}{\partial y} = \frac{\partial G}{\partial p} \Rightarrow$ eq is exact.

Sol. is

$$\underbrace{\int F\,dp}_{y \text{ constant}} + \underbrace{\int G\,dy}_{\substack{\text{Term of } G \\ \text{not containing } p}} = C.$$

$$\Rightarrow \int (y+2ap)\frac{p}{\sqrt{1-p^2}}\,dp + \int 0\,dy = C$$

$$\Rightarrow \int (y+2ap)\frac{1}{\sqrt{1-p^2}} \ldots$$

y constant.

$$\Rightarrow \frac{y}{-2}\underset{y\ \text{const}}{\int}\frac{-2p}{\sqrt{1-p^2}}\,dp + \frac{2a}{-1}\int\frac{p^2}{\sqrt{1-p^2}}\,dp = C.$$

$$\Rightarrow -\frac{y}{2}\int -2p\,(1-p^2)^{-1/2}\,dp - 2a\int\frac{1-p^2-1}{\sqrt{1-p^2}}\,dp = C.$$

$$\Rightarrow -\frac{y}{\cancel{2}}\cdot\frac{(1-p^2)^{1/2}}{\cancel{1/2}} - 2a\int\left(\sqrt{1-p^2} - \frac{1}{\sqrt{1-p^2}}\right)dp = C$$

$$= -y\sqrt{1-p^2} - 2a\left(\frac{p}{2}\sqrt{1-p^2} - \frac{1}{2}\ln\left|p+\sqrt{1-p^2}\right|\right) + 2a\ln\left|p+\sqrt{1-p^2}\right| = C.$$

$$\Rightarrow -\cancel{2}a\,\frac{p}{\cancel{2}}\sqrt{1-p^2} + \frac{\cancel{2}a}{\cancel{2}}\ln\left|p+\sqrt{1-p^2}\right| + 2a\ln\left|p+\sqrt{1-p^2}\right| = C + y\sqrt{1-p^2}$$

$$\Rightarrow y = -\frac{a\cancel{\sqrt{1-p^2}}\,p}{\cancel{\sqrt{1-p^2}}} + 3a\,\frac{\ln\left|p+\sqrt{1-p^2}\right|}{\sqrt{1-p^2}} - C.$$

$$\boxed{y = -ap + 3a\,\frac{\ln\left|p+\sqrt{1-p^2}\right|}{\sqrt{1-p^2}} - C} \quad (2)$$

$$x = yp + ap^2.$$

$$x = -\cancel{ap^2} + 3ap\,\frac{\ln\left|p+\sqrt{1-p^2}\right|}{\sqrt{1-p^2}} - C + \cancel{ap^2}$$

$\sqrt{1-p^2}$

$$\boxed{x = 3ap \ln \frac{\left| p + \sqrt{1-p^2} \right|}{\sqrt{1-p^2}}} \quad -(3)$$

eq (2) & (3) constitute together to give sol. of diff. eq.