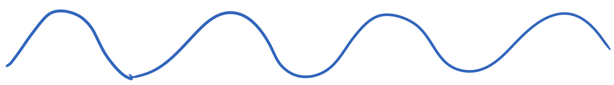
Solution of 1-D wave equation :-

1-D wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The solution of 1-D wave equation using the method of separation of variables are given by

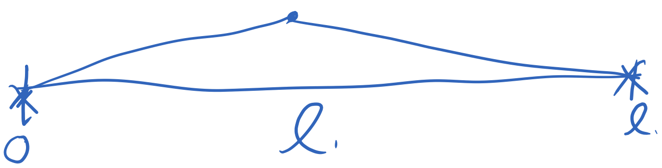
$$(*)\quad \begin{cases} ① \; u(x,t) = (C_1 + C_2 t)(C_3 + C_4 x) & K = 0 \\ ② \; u(x,t) = (C_5 e^{\lambda c t} + C_6 e^{-\lambda c t})(C_7 e^{\lambda x} + C_8 e^{-\lambda x}) & K = \lambda^2,\ K>0,\ \lambda \in \mathbb{R}-\{0\} \\ ③ \; u(x,t) = (C_9 \cos \lambda c t + C_{10} \sin \lambda c t)(C_{11} \cos \lambda x + C_{12} \sin \lambda x) & K = -\lambda^2,\ K<0,\ \lambda \in \mathbb{R}-\{0\} \end{cases}$$



Q :- Find the solution of wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0 \qquad ①$$

under the boundary conditions $u(0,t) = 0 = u(l,t)$ and the initial conditions $u(x,0) = f(x)$ & $u_t(x,0) = g(x)$.



Sol :- We know that the solution of eq ① is given by eq (*)

Consider $u(x,t) = (C_1 + C_2 t)(C_3 + C_4 x)$ be the solution of eq ① — ②

it is given $u(0,t) = 0$ i.e. when $x=0$, $u=0$

② ⇒ $u(0,t) = (C_1 + C_2 t)(C_3 + C_4(0))$

$\Rightarrow \quad 0 = \underbrace{(C_1 + C_2 t)}_{\times} \, C_3 \Rightarrow \boxed{C_3 = 0}$

eq ② $\Rightarrow \quad u(x,t) = (C_1 + C_2 t)\, C_4 x$ — ③

again $u(l,t) = 0 \Rightarrow$ when $x = l \Rightarrow u = 0$.

l = length of the string

eq ③ $\Rightarrow \quad u(l,t) = \underbrace{(C_1 + C_2 t)}_{\times} C_4 \underbrace{l}_{\times} = 0$

$\Rightarrow \boxed{C_4 = 0}$

eq ③ $\Rightarrow u(x,t) = 0$, which is not possible.
Hence we discard this solution.

Consider $u(x,t) = (C_5 e^{\lambda c t} + C_6 e^{-\lambda c t})(C_7 e^{\lambda x} + C_8 e^{-\lambda x})$ — ④
be the sol. of eq ①

$(u(0,t) = 0 = u(l,t))$

(a) $C_7 = 0, \; C_8 = -1$ (b) $C_7 = \frac{1}{e^{\lambda l} - e^{-\lambda l}}, \; C_8 = \frac{-1}{e^{\lambda l} + e^{-\lambda l}}$

(c) $C_7 = C_8 = \frac{1}{e^{\lambda l} - e^{-\lambda l}}$ ~~(d)~~ $C_7 = C_8 = 0$

when $x = 0, u = 0 \;\therefore u(0,t) = 0$

eq ④ $\quad u(0,t) = (C_5 e^{\lambda c t} + C_6 e^{-\lambda c t})(C_7(1) + C_8(1)) = 0$

$\Rightarrow \underbrace{(C_5 e^{\lambda c t} + C_6 e^{-\lambda c t})}_{\times} \underline{(C_7 + C_8)} = 0$

$\Rightarrow C_7 + C_8 = 0 \Rightarrow \boxed{C_8 = -C_7}$

eq ④ $\Rightarrow \quad u(x,t) = (C_5 e^{\lambda c t} + C_6 e^{-\lambda c t})(C_7 e^{\lambda x} - C_7 e^{-\lambda x})$

$\Rightarrow u(x,t) = (C_5 e^{\alpha c t} + C_6 e^{-\alpha c t}) C_7 (e^{\alpha x} - e^{-\alpha x})$ — (5)

$\because u(l,t) = 0 \Rightarrow$ when $x = l$, $u = 0$

$$u(l,t) = \underbrace{(C_5 e^{\alpha c t} + C_6 e^{-\alpha c t})}_{\times} C_7 \underbrace{(e^{\alpha l} - e^{-\alpha l})}_{\times} = 0$$

$\Rightarrow C_7 = 0$

eq. (5) $\Rightarrow$ $u(x,t) = 0$, which is not possible. hence we discard this sol.

— (6)

Consider $u(x,t) = (C_9 \cos\alpha c t + C_{10} \sin\alpha c t)(C_{11} \cos\alpha x + C_{12} \sin\alpha x)$

be the solution of eq (1).

again $u(0,t) = 0 \Rightarrow$ when $x = 0$, $u = 0$

eq (6) $\Rightarrow$ $u(0,t) = (C_9 \cos\alpha c t + C_{10} \sin\alpha c t)(C_{11} \cos 0 + C_{12} \sin 0)$

$$\Rightarrow 0 = \underbrace{(C_9 \cos\alpha c t + C_{10} \sin\alpha c t)}_{\times} C_{11}.$$

$\Rightarrow C_{11} = 0$

eq (6) $\Rightarrow$ $u(x,t) = (C_9 \cos\alpha c t + C_{10} \sin\alpha c t) C_{12} \sin\alpha x$ — (7)

again $u(l,t) = 0 \Rightarrow$ when $x = l$, $u = 0$

eq. (7) $\Rightarrow$ $u(l,t) = (C_9 \cos\alpha c t + C_{10} \sin\alpha c t) C_{12} \sin\alpha l.$

$$0 = \underbrace{(C_9 \cos\alpha c t + C_{10} \sin\alpha c t)}_{\times} \underbrace{C_{12}}_{\times} \sin\alpha l.$$

$\Rightarrow \sin\alpha l = 0 \Rightarrow \alpha l = n\pi, n \in \mathbb{Z}$ $\quad [\sin\theta = 0, \theta = n\pi.]$

$\Rightarrow \boxed{\alpha = \frac{n\pi}{l}}, n \in \mathbb{Z}.$

eq (7) $\Rightarrow$ $u(x,t) = \left(C_1 \cos\frac{n\pi ct}{l} + C_{10} \sin\frac{n\pi ct}{l}\right) C_{12} \sin\frac{n\pi x}{l}, n \in \mathbb{Z}$

By superposition principle,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \left(C_9 \cos\frac{n\pi ct}{l} + C_{10} \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l}}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l} \quad -(8)$$

$\boxed{\text{where } A_n = C_9 b_n,\ B_n = C_{10} b_n}$

It is given $u(x,0) = f(x)$, i.e. when $t=0$, $u=f(x)$

eq (8) $\Rightarrow$ $u(x,0) = \sum_{n=1}^{\infty} \left(A_n (1) + B_n (0)\right) \sin\frac{n\pi x}{l}$

$$\boxed{f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}}, \quad 0 < x < l.$$

which is half range fourier sine series

where $\boxed{A_n = \frac{2}{l} \int_0^l f(x) \sin\frac{n\pi x}{l}\, dx}$

Note. if $f(x) = 0 \Rightarrow A_n = 0$

Sol is given by $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$.

again $u_t(x,0) = g(x) \Rightarrow$ when $t=0$, $\frac{\partial u}{\partial t} = g(x)$;

Eq (8). diff w.r.t t, we get

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left(A_n\left(-\sin\frac{n\pi ct}{l}\left(\frac{n\pi c}{l}\right)\right) + B_n\left(\cos\frac{n\pi ct}{l}\left(\frac{n\pi c}{l}\right)\right)\sin\frac{n\pi x}{l}\right.$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} A_n(0) + B_n\left(1\cdot\frac{n\pi c}{l}\right)\cdot\sin\frac{n\pi x}{l}$$

$$\Rightarrow \boxed{g(x) = \sum_{n=1}^{\infty}\left(B_n\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l}}, \quad 0<x<l.$$

which is again half range fourier sine series

$$B_n\left(\frac{n\pi c}{\cancel{l}}\right) = \frac{2}{\cancel{l}}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

$$\Rightarrow \boxed{B_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx}$$

Remark: if $g(x) = 0 \Rightarrow B_n = 0$.

sol. is given by $\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}}$

Solution of 1-D wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$; $0<x<l$, $t>0$

under the boundary condition $u(0,t) = 0 = u(l,t)$ & initial condition $u(x,0) = f(x)$ & $u_t(x,0) = g(x)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}$$

where $A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx.$

$$B_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx.$$