CHAPTER

20 Projectiles

Contents

20.1. INTRODUCTION

In the previous chapters, we have been discussing the motion of bodies, either in horizontal or vertical directions. But we see that whenever a particle is projected upwards at a certain angle (but not vertical), we find that the particle traces some path in the air and falls on the ground at a point, other than the point of projection. If we study the motion of the particle, we find that the velocity, with which the particle was projected, has two components namely vertical and horizontal.

The function of the vertical component is to project the body vertically upwards, and that of the horizontal is to move the body horizontally in its direction. The combined effect of both the components is to move the particle along a parabolic path. A particle, moving under the combined effect of vertical and horizontal forces, is called a *projectile*. It may be noted that the vertical component of the motion is always subjected to gravitational acceleration, whereas the horizontal component remains constant.

20.2. IMPORTANT TERMS

The following terms, which will be frequently used in this chapter, should be clearly understood at this stage :

1. ***Trajectory***. The path, traced by a projectile in the space, is known as trajectory.
2. ***Velocity of projection***. The velocity, with which a projectile is projected, is known as the velocity of projection.
3. ***Angle of projection***. The angle, with the horizontal, at which a projectile is projected, is known as the angle of projection.
4. ***Time of flight***. The total time taken by a projectile, to reach maximum height and to return back to the ground, is known as the time of flight.
5. ***Range***. The distance, between the point of projection and the point where the projectile strikes the ground, is known as the *range*. It may be noted that the range of a projectile may be horizontal or inclined.

20.3. MOTION OF A BODY THROWN HORIZONTALLY INTO THE AIR

Consider a body at A thrown horizontally into the *air with a horizontal velocity (v) as shown in Fig. 20.1. A little consideration will show, that this body is subjected to the following two velocities :

1. Horizontal velocity (v), and

2. Vertical velocity due to gravitational acceleration.

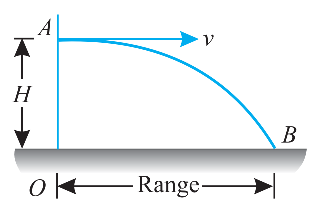

Fig. 20.1.

It is thus obvious, that the body will have some resultant velocity, with which it will travel into the air. We have already discussed in Art 20.1. that the vertical component of this velocity is always subjected to gravitational acceleration, whereas the horizontal component remains constant. Thus the time taken by the body to reach the ground, is calculated from the vertical component of the velocity, whereas the horizontal range is calculated from the horizontal component of the velocity. The velocity, with which the body strikes the ground at B, is the resultant of horizontal and vertical velocities.

Example 20.1. *An aircraft, moving horizontally at 108 km/hr at an altitude of 1000 m towards a target on the ground, releases a bomb which hits it. Estimate the horizontal distance of the aircraft from the target, when it released the bomb. Calculate also the direction and velocity with which the bomb hits the target. Neglect air friction.*

Solution. Given : Horizontal velocity of aircraft, (V) = 108 km/hr = 30 m/s

Horizontal distance of the aircraft from the target when it released the bomb

First of all, consider the vertical motion of the bomb due to gravitational acceleration only. In this case, initial velocity (u) = 0 and distance covered (s) = 1000 m.

Let t = Time required by bomb to reach the ground.

* For all types of calculation, the air resistance, until specified otherwise, is neglected.

We know that height of aircraft (s)

$$1000 = ut + \frac{1}{2}gt^2 = 0 + \frac{1}{2} \times 9.8t^2 = 4.9t^2$$

or $$t^2 = \frac{1000}{4.9} = 204.1 \quad \text{or} \quad t = 14.3 \text{ s}$$

∴ Horizontal distance of the aircraft from the target when it released the bomb,

$$H = V \times t = 30 \times 14.3 = 429 \text{ m} \quad \textbf{Ans.}$$

Direction and velocity with which the bomb hits the target

Let θ = Angle which the bomb makes with vertical when it hits the target.

We know that final velocity of the bomb in the vertical direction when it hits target (*i.e.* after 14.3 seconds),

$$v = u + gt = 0 + (9.8 \times 14.3) = 140.1 \text{ m/s}$$

∴ $$\tan \theta = \frac{30}{140.1} = 0.2141 \quad \text{or} \quad \theta = 12.1° \quad \textbf{Ans.}$$

and resultant velocity with which the bomb hits the target

$$= \sqrt{(140.1)^2 + (30)^2} = 143.3 \text{ m/s} \quad \textbf{Ans.}$$

Example 20.2. *A motor cyclist wants to jump over a ditch as shown in Fig. 20.2.*

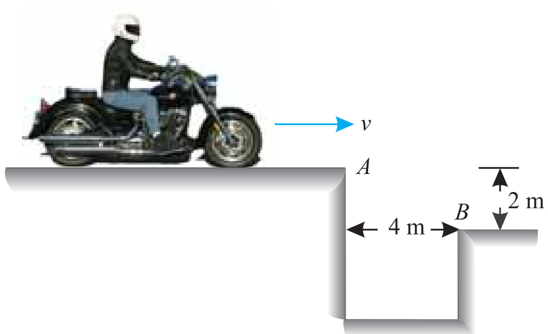

Fig. 20.2.

Find the necessary minimum velocity at A in km. p. hr. of the motor cycle. Also find the inclination and the magnitude of the velocity of the motor cycle just after clearing the ditch.

Solution. Given : Width of ditch (x) = 4 m and vertical distance between A and B (s) = 2 m.

Minimum velocity of motor cycle at A

Let u = Minimum velocity of motor cycle at A, and

t = Time taken by the motor cycle to clear the ditch.

First of all, let us consider the vertical motion of the motor cycle from A to B due to gravitational acceleration only. In this case, initial velocity of motor cycle (u) = 0.

We know that vertical distance between A and B (s),

$$2 = ut + \frac{1}{2}gt^2 = 0 + \frac{1}{2} \times 9.8\,t^2 = 4.9\,t^2$$

or $$t^2 = \frac{2}{4.9} = 0.41 \quad \text{or} \quad t = 0.64 \text{ s}$$

∴ Minimum velocity of the motor cycle at A

$$= \frac{4}{0.64} = 6.25 \text{ m/s} = 22.5 \text{ km.p.h.} \quad \textbf{Ans.}$$

Inclination and magnitude of the velocity of motor cycle just after clearing the ditch (*i.e. at B*)

Let θ = Inclination of the velocity with the vertical.

We know that final velocity of the motor cycle in the vertical direction at B (*i.e.*after 0.64 second)

$$v = u + gt = 0 + (9.8 \times 0.64) = 6.27 \text{ m/s}$$

$$\therefore \quad \tan\theta = \frac{6.25}{6.27} = 0.9968 \quad \text{or} \quad \theta = 44.9° \quad \textbf{Ans.}$$

and magnitude of the velocity of the motor cycle just after clearing the ditch

$$= \sqrt{(6.25)^2 + (6.27)^2} = 8.85 \text{ m/s} = 31.86 \text{ km.p.h.} \quad \textbf{Ans.}$$

Example 20.3. *An aeroplane is flying on a straight level course at 200 km per hour at a height of 1000 metres above the ground. An anti-aircraft gun located on the ground fires a shell with an initial velocity of 300 m/s, at the instant when the plane is vertically above it. At what inclination, to the horizontal, should the gun be fired to hit the plane ? What time after firing, the gun shell will hit the plane ? What will then be the horizontal distance of the plane from the gun ?*

Solution. Given : Aeroplane velocity = 200 km.p.h. = 55.56 m/s ; Height of plane = 1000 m and velocity of shell (u) = 300 m/s

Inclination of the gun

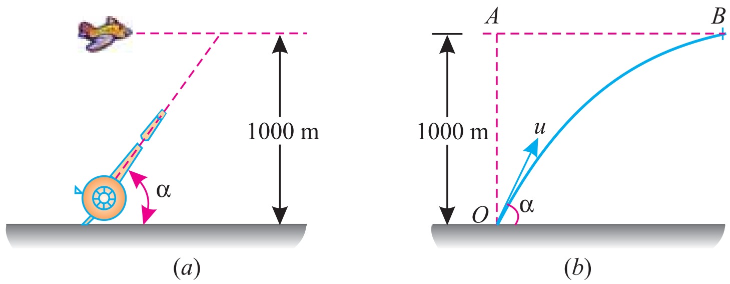

Fig. 20.3.

The actual position of the plane and anti-aircraft gun is shown in Fig. 20.3 (*a*). Let the anti-aircraft gun be located at O and the plane at the time of firing the shell be at A. Now after sometime, let the plane reach at B, when it is hit by the shell as shown in Fig. 20.3 (*b*).

Now let α = Inclination of gun with the horizontal, and

t = Time taken by the shell to hit the plane.

First of all, consider the motion of the plane. We know that the distance AB

$$= 55.56 \times t = 55.56\, t \text{ metres} \qquad ...(i)$$

Now consider motion of the plane. We know that horizontal component of the shell velocity

$$u_x = u \cos\alpha = 300 \cos\alpha$$

and distance $AB = 300 \cos\alpha.\, t$ ...(*ii*)

Equating equation (*i*) and (*ii*),

$$55.56\, t = 300 \cos\alpha.t$$

$$\cos\alpha = \frac{55.56}{300} = 0.1852 \quad \text{or} \quad \alpha = 79.3° \quad \textbf{Ans.}$$

Time after firing the shell will hit the plane

Now consider the vertical motion of the shell. We know that vertical component of the shell velocity,

$$u_y = 300 \sin 79.3° = 300 \times 0.9826 = 295 \text{ m/s}$$

We know that the vertical distance *OA* (*s*),

$$1000 = u_y\, t - \frac{1}{2}\, gt^2 = 295t - \frac{1}{2} \times 9.8\, t^2 = 295\, t - 4.9\, t^2$$

$4.9\, t^2 - 295\, t + 1000 = 0$

This is a quadratic equation in *t*.

$$\therefore \qquad t = \frac{+\,295 \pm \sqrt{(295)^2 - (4 \times 4.9 \times 1000)}}{2 \times 4.9} = 3.57 \text{ s} \qquad \textbf{Ans.}$$

Horizontal distance of the plane from the gun

We know that horizontal distance of the plane from the gun

$$AB = 55.56\, t = 55.56 \times 3.57 = 198.3 \text{ m} \qquad \textbf{Ans.}$$

EXERCISE 20.1

1. A bomber, flying horizontally at a height of 500 m with a velocity of 450 km.p.h., has aimed to hit a target. Find at what distance from the target, he should release the bomb in order to hit the target. (**Ans.** 1262.5 m)
2. A shot is fired horizontally from the top of a tower with a velocity of 100 m/s. If the shot hits the ground after 2 seconds, find the height of the tower and the distance from the foot of the tower, where the shot strikes the ground. (**Ans.** 19.2 m ; 200 m)
3. A helicopter is moving horizontally at 90 km.p.h. at a height of 200 m towards a target on the ground, which is intended to be shelled. Estimate the distance from the target, where the shell must be released in order to hit the target.

   Also find the velocity with which the shell hits the target and the direction of shell at the time of hitting the target. (**Ans.** 173.25 m ; 67.4 m/s ; 21° 46')

20.4. MOTION OF A PROJECTILE

Consider a particle projected upwards from a point *O* at an angle α, with the horizontal, with an initial velocity *u* m/sec as shown in Fig. 20.4.

Now resolving this velocity into its vertical and horizontal components,

$$V = u \sin \alpha \quad \text{and } H = u \cos \alpha$$

We know that the vertical component ($u \sin \alpha$) is subjected to retardation due to gravity. The particle will reach maximum height, when the vertical component becomes zero. After this the particle will come down, due to gravity, and this motion will be subjected to acceleration due to gravity.

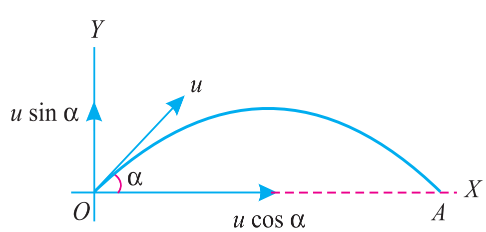

Fig. 20.4. Projectile on a horizontal plane.

The horizontal component ($u \cos \alpha$) will remain constant, since there is no acceleration or retardation (neglecting air resistance). The combined effect of the horizontal and the vertical components will be to move the particle, along some path in the air and then the particle falls on the ground at some point *A*, other than the point of projection *O* as shown in Fig. 20.4.

20.5. EQUATION OF THE PATH OF A PROJECTILE

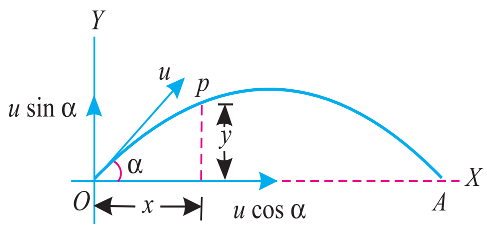

Fig. 20.5. Path of a projectile.

Consider a particle projected from a point O at a certain angle with the horizontal. As already discussed, the particle will move along certain path OPA, in the air, and will fall down at A as shown in Fig. 20.5.

Let u = Velocity of projection, and

α = Angle of projection with the horizontal.

Consider any point P as the position of particle, after t seconds with x and y as co-ordinates as shown in Fig. 20.5. We know that horizontal component of the velocity of projection.

$= u \cos \alpha$

and vertical component $= u \sin \alpha$

$$\therefore \quad y = u \sin \alpha t - \frac{1}{2} g t^2 \qquad \text{...}(i)$$

and $x = u \cos \alpha \, t$

or $$t = \frac{x}{u \cos \alpha}$$

Substituting the value of t in equation (i),

$$y = u \sin \alpha \left(\frac{x}{u \cos \alpha} \right) - \frac{1}{2} g \left(\frac{x}{u \cos \alpha} \right)^2$$

$$= x \tan \alpha - \frac{g x^2}{2 u^2 \cos^2 \alpha} \qquad \text{...}(ii)$$

Since this is the equation of a parabola, therefore path of a projectile (or the equation of trajectroy) is also a parabola.

Note. It is an important equation, which helps us in obtaining the following relations of a projectile :

1. Time of flight,
2. Horizontal range, and
3. Maximum height of a projectile.

20.6. TIME OF FLIGHT OF A PROJECTILE ON A HORIZONTAL PLANE

It is the time, for which the projectile has remained in the air. We have already discussed in Art. 20.5 that the co-ordinates of a projectile after time t.

$$y = u \sin \alpha t - \frac{1}{2} g t^2$$

We know that when the particle is at A, y is zero. Substituting this value of y in the above equation,

$$0 = u \sin \alpha t - \frac{1}{2} g t^2$$

or $$u \sin \alpha t = \frac{1}{2} g t^2$$

$$u \sin \alpha = \frac{1}{2} g t \quad \text{...(Dividing both sides by } t\text{)}$$

$\therefore$ $$t = \frac{2u \sin \alpha}{g}$$

Example 20.4. *A projectile is fired upwards at an angle of 30° with a velocity of 40 m/s. Calculate the time taken by the projectile to reach the ground, after the instant of firing.*

Solution. Given : Angle of projection with the horizontal (α) = 30° and velocity of projection (u) = 40 m/s.

We know that time taken by the projectile to reach the ground after the instant of firing,

$$t = \frac{2u \sin \alpha}{g} = \frac{2 \times 40 \sin 30°}{g} = \frac{80 \times 0.5}{9.8} = 4.08 \text{ s} \quad \textbf{Ans.}$$

20.7. HORIZONTAL RANGE OF A PROJECTILE

We have already discussed, that the horizontal distance between the point of projection and the point, where the projectile returns back to the earth is called horizontal range of a projectile. We have also discussed in Arts. 20.4 and 20.6 that the horizontal velocity of a projectile

$$= u \cos \alpha$$

and time of flight, $$t = \frac{2u \sin \alpha}{g}$$

$\therefore$ Horizontal range = Horizontal velocity × Time of flight

$$= u \cos \alpha \times \frac{2u \sin \alpha}{g} = \frac{2u^2 \sin \alpha \cos \alpha}{g}$$

$$R = \frac{u^2 \sin 2\alpha}{g} \quad \text{...}(\because \sin 2\alpha = 2 \sin \alpha \cos \alpha)$$

Note. For a given velocity of projectile, the range will be maximum when $\sin 2\alpha = 1$. Therefore

$$2\alpha = 90° \quad \text{or} \quad \alpha = 45°$$

or $$R_{max} = \frac{u^2 \sin 90°}{g} = \frac{u^2}{g} \quad \text{...}(\because \sin 90° = 1)$$

Example 20.5. *A ball is projected upwards with a velocity of 15 m/s at an angle of 25° with the horizontal. What is the horizontal range of the ball ?*

Solution. Given : Velocity of projection (u) = 15 m/s and angle of projection with the horizontal (α) = 25°

We know that horizontal range of the ball,

$$R = \frac{u^2 \sin 2\alpha}{g} = \frac{(15)^2 \times \sin (2 \times 25°)}{g} = \frac{225 \sin 50°}{g}$$

$$= \frac{225 \times 0.766}{9.8} = 17.6 \text{ m} \quad \textbf{Ans.}$$

20.8. MAXIMUM HEIGHT OF A PROJECTILE ON A HORIZONTAL PLANE

We have already discussed that the vertical component of the initial velocity of a projectile

$$= u \sin \alpha \qquad ...(i)$$

and vertical component of final velocity

$$= 0 \qquad ...(ii)$$

$\therefore$ Average velocity of (*i*) and (*ii*),

$$= \frac{u \sin \alpha + 0}{2} = \frac{u \sin \alpha}{2} \qquad ...(iii)$$

Let H be the maximum height reached by the particle and t be the time taken by the particle to reach maximum height *i.e.*,to attain zero velocity from ($u \sin \alpha$). We have also discussed that time taken by the projectile to reach the maximum height,

$$= \frac{u \sin \alpha}{g}$$

$\therefore$ Maximum height of the projectile,

$$H = \text{Average vertical velocity} \times \text{Time}$$

$$= \frac{u \sin \alpha}{2} \times \frac{u \sin \alpha}{g} = \frac{u^2 \sin^2 \alpha}{2g}$$

Example 20.6. *A bullet is fired with a velocity of 100 m/s at an angle of 45° with the horizontal. How high the bullet will rise ?*

Solution. Given : Velocity of projection (u) = 100 m/s and angle of projection with the horizontal (α) = 45°

We know that maximum height to which the bullet will rise,

$$H = \frac{u^2 \sin^2 \alpha}{2g} = \frac{(100)^2 \times \sin^2 45°}{2 \times 9.8} = \frac{10000 \times (0.707)^2}{19.6} \text{ m}$$

$$= 255.1 \text{ m} \quad \textbf{Ans.}$$

Example 20.7. *If a particle is projected inside a horizontal tunnel which is 5 metres high with a velocity of 60 m/s, find the angle of projection and the greatest possible range.*

Solution. Given : Height of the tunnel (H) = 5 m and velocity of projection (u) = 60 m/s.

Angle of projection

Let α = Angle of projection.

We know that height of tunnel (H)

$$5 = \frac{u^2 \sin^2 \alpha}{2g} = \frac{(60)^2 \sin^2 \alpha}{2 \times 9.8} = 183.7 \sin^2 \alpha$$

or

$$\sin^2 \alpha = \frac{5}{183.7} = 0.0272$$

$$\therefore \quad \sin \alpha = 0.1650 \quad \text{or} \quad \alpha = 9.5° \quad \textbf{Ans.}$$

Greatest possible range

We know that greatest possible range,

$$R = \frac{u^2 \sin 2\alpha}{g} = \frac{(60)^2 \sin (2 \times 9.5°)}{9.8} = \frac{(60)^2 \sin 19°}{9.8} \text{ m}$$

$$= \frac{3600 \times 0.3256}{9.8} = 119.6 \text{ m} \quad \textbf{Ans.}$$

Example 20.8. *A body is projected at such an angle that the horizontal range is three times the greatest height. Find the angle of projection.*

Solution. Given : Horizontal range $(R) = 3\,H$ (where H is the greatest height). ...(*i*)

Let α = Angle of projection.

We know that horizontal range,

$$R = \frac{u^2 \sin 2\alpha}{g}$$

and the greatest height

$$H = \frac{u^2 \sin^2 \alpha}{2g}$$

Substituting these values of R and H in the given equation (*i*),

$$\frac{u^2 \sin 2\alpha}{g} = 3 \times \frac{u^2 \sin^2 \alpha}{2g}$$

$$\frac{u^2 \times 2 \sin\alpha \cos\alpha}{g} = 3 \times \frac{u^2 \sin^2 \alpha}{2g} \qquad ...(\because\ 2\alpha = 2\sin\alpha\cos\alpha)$$

or

$$2\cos\alpha = 1.5 \sin\alpha$$

$\therefore$

$$\tan\alpha = \frac{2}{1.5} = 1.333 \quad \text{or} \quad \alpha = 53.1° \quad \textbf{Ans.}$$

Example 20.9. *A particle is thrown with a velocity of 5 m/s at an elevation of 60° to the horizontal. Find the velocity of another particle thrown at an elevation of 45° which will have* (*a*) *equal horizontal range*, (*b*) *equal maximum height, and* (*c*) *equal time of flight.*

Solution. Given : Velocity of projection of first particle $(u_1) = 5$ m/s ; Angle of projection of first particle with the horizontal $(\alpha_1) = 60°$ and angle of projection of second particle with the horizontal $(\alpha_2) = 45°$

Let u_2 = Velocity of projection of the second particle.

(*a*) *Velocity of the second particle for equal horizontal range*

We know that horizontal range of a projectile,

$$R = \frac{u^2 \sin 2\alpha}{g}$$

$\therefore$ For equal horizontal range

$$\frac{u_1^2 \sin 2\alpha_1}{g} = \frac{u_2^2 \sin 2\alpha_2}{g}$$

$$(5)^2 \sin(2 \times 60°) = u_2^2 \sin(2 \times 45°)$$

or

$$u_2^2 = 25 \times \frac{\sin 120°}{\sin 90°} = 25 \times \frac{0.866}{1.0} = 21.65$$

$\therefore$

$$u_2 = 4.65 \text{ m/s} \quad \textbf{Ans.}$$

(*b*) *Velocity of the second particle for equal maximum height*

We know that maximum height of a projectile,

$$H = \frac{u^2 \sin^2 \alpha}{2g}$$

$\therefore$ For equal maximum height

$$\frac{u_1^2 \sin^2 \alpha_1}{2g} = \frac{u_2^2 \sin^2 \alpha_2}{2g}$$

$$(5)^2 \sin^2 60° = u_2^2 \sin^2 45°$$

or

$$u_2^2 = 25 \times \frac{\sin^2 60°}{\sin^2 45°} = 25 \times \frac{(0.866)^2}{(0.707)^2} = 37.5$$

$\therefore$

$$u_2 = 6.12 \text{ m/s} \quad \textbf{Ans.}$$

(*c*) *Velocity of the second particle for equal time of flight*

We know that time of flight of a projectile

$$t = \frac{2u \sin \alpha}{g}$$

∴ For equal time of flight

$$\frac{2u_1 \sin \alpha_1}{g} = \frac{2u_2 \sin \alpha_2}{g}$$

$$2 \times 5 \sin 60° = 2u_2 \sin 45°$$

or $$u_2 = 5 \times \frac{\sin 60°}{\sin 45°} = 5 \times \frac{0.866}{0.707} = 6.12 \text{ m/s} \quad \textbf{Ans.}$$

Example 20.10. *A particle is projected from the base of a hill whose shape is that of a right circular cone with axis vertical. The projectile grazes the vertex and strikes the hill again at a point on the base. If θ be the semi-vertical angle of the cone, h its height, u the initial velocity of the projectile and α the angle of projection measured from the horizontal, show that :*

$$\tan \alpha = 2 \cot \theta \quad \text{and} \quad u^2 = gh\left(2 + \frac{1}{2}\tan^2 \theta\right).$$

where g is acceleration due to gravity.

Solution. Given : Semi-vertical angle = θ ; Initial velocity of the projectile = u and angle of projection with the horizontal = α

(*i*) We know that maximum height through which the particle will rise,

$$h = \frac{u^2 \sin^2 \alpha}{2g}$$

and horizontal range *OB* $$R = \frac{u^2 \sin 2\alpha}{g}$$

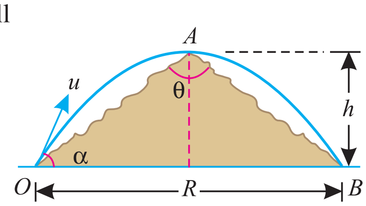

Fig. 20.6.

From the geometry of the figure we find that

$$\cot \theta = \frac{h}{\frac{R}{2}} = \frac{2h}{R} = \frac{2 \times \frac{u^2 \sin^2 \alpha}{2g}}{\frac{u^2 \sin 2\alpha}{g}} = \frac{\sin^2 \alpha}{\sin 2\alpha} = \frac{\sin^2 \alpha}{2 \sin \alpha \cos \alpha} = \frac{\tan \alpha}{2}$$

∴ $$\tan \alpha = 2 \cot \theta \quad \textbf{Ans.}$$

(*ii*) The above equation, may be written as

$$\frac{1}{\tan \alpha} = \frac{1}{2 \cot \theta} \quad \text{or} \quad \cot \alpha = \frac{1}{2} \tan \theta$$

∴ $$\cot^2 \alpha = \frac{1}{4} \tan^2 \theta$$ ...(Squaring both sides)

We know that maximum height through which the particle will rise,

$$h = \frac{u^2 \sin^2 \alpha}{2g}$$

∴ $$u^2 = \frac{2gh}{\sin^2 \alpha} = 2gh \operatorname{cosec}^2 \alpha$$

$$u^2 = 2gh(1 + \cot^2 \alpha) = 2gh\left(1 + \frac{1}{4}\tan^2 \theta\right)$$

$$\ldots\left(\text{Substituting } \cot^2\alpha = \frac{1}{4}\tan^2 \theta\right)$$

$$= gh\left(2 + \frac{1}{2}\tan^2 \theta\right) \quad \textbf{Ans.}$$

Example 20.11. *A projectile is aimed at a mark on the horizontal plane through the point of projection. It falls 12 metres short when the angle of projection is 15° ; while it overshoots the mark by 24 metres when the same angle is 45°. Find the angle of projection to hit the mark. Assume no air resistance.*

Solution. Given : When angle of projection with the horizontal $(\alpha_1) = 15°$, horizontal range $(R_1) = R - 12$ m and when angle of projection with the horizontal $(\alpha_2) = 45°$, horizontal range $(R_2) = R + 24$ m (where R is the horizontal range).

Let u = Velocity of projection, and

α = Angle of projection to hit the mark.

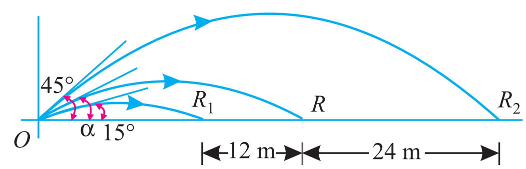

Fig. 20.7.

We know that horizontal range of the projectile when $\alpha = 15°$,

$$R_1 = \frac{u^2 \sin 2\alpha_1}{g} = \frac{u^2 \sin (2 \times 15°)}{g}$$

$$\therefore \quad (R - 12) = \frac{u^2 \sin 30°}{g} = \frac{u^2 \times 0.5}{g} \quad \ldots(i)$$

Similarly

$$R_2 = \frac{u^2 \sin 2\alpha_2}{g} = \frac{u^2 \sin (2 \times 45°)}{g}$$

$$(R + 24) = \frac{u^2 \sin 90°}{g} = \frac{u^2 \times 1}{g} \quad \ldots(ii)$$

Dividing equation (i) by (ii),

$$\frac{R - 12}{R + 24} = \frac{0.5}{1} \quad \text{or} \quad R - 12 = 0.5R + 12$$

$$\therefore \quad 0.5\,R = 12 + 12 = 24 \quad \text{or} \quad R = \frac{24}{0.5} = 48 \text{ m}$$

Substituting the value of R in equation (i),

$$48 - 12 = \frac{u^2 \times 0.5}{g} = \frac{u^2}{2g}$$

$$\therefore \quad u^2 = 36 \times 2g = 72g$$

We know that the horizontal distance between the point of projection and the mark (R),

$$48 = \frac{u^2 \sin 2\alpha}{g} = \frac{(72g)\sin 2\alpha}{g} = 72 \sin 2\alpha$$

or $\sin 2\alpha = \frac{48}{72} = 0.667$ or $2\alpha = 41.8°$

$\therefore$ $\alpha = 20.9°$ **Ans.**

Example 20.12. *A projectile fired from the edge of a 150 m high cliff with an initial velocity of 180 m/s at an angle of elevation of 30° with the horizontal. Neglecting air resistance find :*

1. *The greatest elevation above the ground reached by the projectile* ; *and*
2. *Horizontal distance from the gun to the point, where the projectile strikes the ground.*

Solution. Given : Height of cliff = 150 m ; Velocity of projection (u) = 180 m/s and angle of projection with the horizontal (α) = 30°.

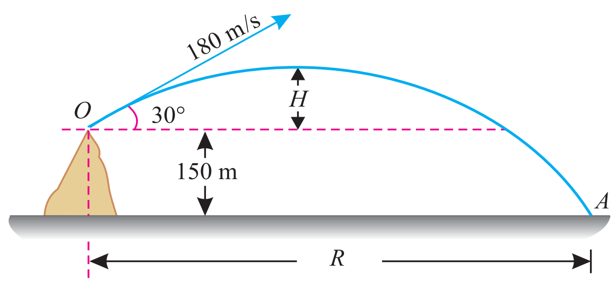

Fig. 20.8.

1. *The greatest elevation above the ground reached by the projectile*

We know that maximum height to which the projectile will rise above the edge O of the cliff,

$$H = \frac{u^2 \sin^2 \alpha}{2g} = \frac{(180)^2 \sin^2 30°}{2 \times 9.8} = \frac{(180)^2 \times (0.5)^2}{19.6} = 413.3 \text{ m}$$

$\therefore$ Greatest elevation above the ground reached by the projectile,

$s = 413.3 + 150 = 563.3$ m **Ans.**

2. *The horizontal distance from the gun to the point, where the projectile strikes the ground.*

First of all, consider motion of the projectile from the edge of the cliff to the maximum height. We know that the time taken by the projectile to reach maximum height from the edge of the cliff,

$$t_1 = \frac{u \sin \alpha}{g} = \frac{180 \sin 30°}{9.8} = \frac{180 \times 0.5}{9.8} = 9.2 \text{ s}$$

Now consider vertical motion of the projectile from the maximum height to the ground due to gravitational acceleration only. In this case, $u = 0$ and $s = 563.3$ m.

Let t_2 = Time taken by the projectile to reach the ground from the maximum height.

We know that the vertical distance (s),

$$563.3 = u t_2 + \frac{1}{2} g t_2^2 = 0 + \frac{1}{2} \times 9.8 t_2^2 = 4.9 t_2^2$$

or $t^2 = \frac{563.3}{4.9} = 115$ or $t_2 = 10.7$ s ...(*ii*)

$\therefore$ Total time taken by the projectile to reach the ground from the edge of the cliff

$$= t_1 + t_2 = 9.2 + 10.7 = 19.9 \text{ s}$$

and horizontal distance from the gun to the point, where the projectile strikes the ground,

$$R = \text{Horizontal components of velocity} \times \text{Time}$$

$$= 180 \cos 30° \times 19.9 = (180 \times 0.866) \times 19.9 \text{ m}$$

$$= 3102 \text{ m} = 3.102 \text{ km} \quad \textbf{Ans.}$$

Example 20.13. *A bullet is fired upwards at an angle of 30° to the horizontal from a point P on a hill, and it strikes a target which is 80 m lower than P. The initial velocity of bullet is 100 m/s. Calculate the actual velocity with which the bullet will strike the target.*

Solution. Given : Angle of projection with the horizontal (α) = 30° and initial velocity of projection (u) = 100 m/s

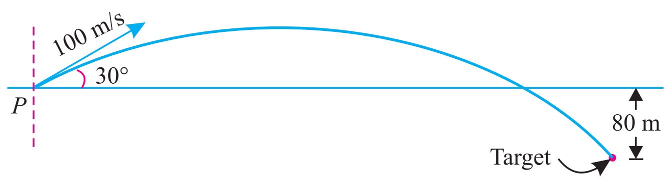

Fig. 20.9.

We know that the maximum height to which the bullet will rise above the horizontal,

$$H = \frac{u^2 \sin^2 \alpha}{2g} = \frac{(100)^2 \sin^2 30°}{2 \times 9.8} = \frac{(100)^2 \times (0.5)^2}{19.6} = 127.6 \text{ m}$$

First of all, consider motion of the bullet to reach maximum height. We know that time taken by the bullet to reach maximum height,

$$t_1 = \frac{u \sin \alpha}{g} = \frac{100 \sin 30°}{9.8} = \frac{100 \times 0.5}{9.8} = 5.1 \text{ s} \qquad ...(i)$$

Now consider vertical motion of the bullet from the maximum height to the target due to gravitational acceleration only.

In this case, initial velocity (u) = 0 and total distance (s) = 127.6 + 80 = 207.6 m.

Let t_2 = Time taken by the bullet to reach the target from the maximum height.

We know that the vertical distance (s),

$$207.6 = u\,t_2 + \frac{1}{2} g\,t_2^2 = (0 \times t_2) + \left(\frac{1}{2} \times 9.8 t_2^2\right) = 4.9 t_2^2$$

$$t_2^2 = \frac{207.6}{4.9} = 42.4 \quad \text{or} \quad t_2 = 6.5 \text{ s} \qquad ...(ii)$$

$\therefore$ Total time required for the flight of the bullet

$$= t_1 + t_2 = 5.1 + 6.5 = 11.6 \text{ s}$$

We know that final velocity of the bullet in the vertical direction, when it strikes the target (*i.e.* after 6.5 seconds),

$$v = u + gt = (0) + (9.8 \times 6.5) = 63.7 \text{ m/s}$$

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and horizontal component of the initial velocity of the bullet

$$= u \cos \alpha = 100 \cos 30° = 100 \times 0.866 = 86.6 \text{ m/s}$$

∴ Actual velocity with which the bullet will strike the target

$$= \sqrt{(63.7)^2 + (86.6)^2} = 107.5 \text{ m/s} \quad \textbf{Ans.}$$

Example 20.14. *A shot is fired with a velocity if 30 m/s from a point 15 metres in front of a vertical wall 6 metres high. Find the angle of projection, to the horizontal for the shot just to clear the top of the wall.*

Solution. Given : Initial velocity = 30 m/s; Distance of point of projection from wall (OB) = 15 m and height of the wall AB = 6 m.

Let α = Angle of projection.

∴ Vertical component of the velocity of the projection = $30 \cos \alpha$

First of all, consider vertical motion of the shot. Let the bullet take t seconds to cross the wall. In order to enable the shot just to clear the top of the wall, it must rise 6 metres high in t seconds. In this case, s = 6 m and $u = 30 \sin \alpha$

We know that vertical distance travelled by the shot (s),

$$6 = ut - \frac{1}{2} g t^2 = (30 \sin \alpha)t - \frac{1}{2} \times 9.8 t^2$$

$$= (30 \sin \alpha)\, t - 4.9\, t^2 \quad ...(i)$$

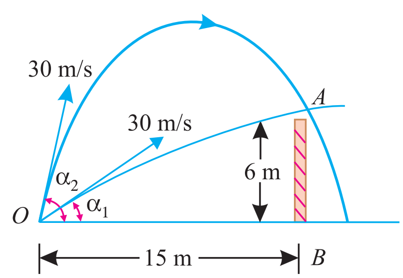

Fig. 20.10.

Now consider the horizontal motion of the shot. In order to enable the shot just to clear the top of the wall, it must traverse 15 m in t seconds.

∴ 15 = Horizontal velocity × Time = $(30 \cos \alpha)\, t$

or $$t = \frac{15}{30 \cos \alpha} = \frac{0.5}{\cos \alpha} \quad ...(ii)$$

Substituting the value of t in equation (i),

$$6 = 30 \sin \alpha \left(\frac{0.5}{\cos \alpha}\right) - 4.9 \left(\frac{0.5}{\cos \alpha}\right)^2$$

$$6 = 15 \tan \alpha - 1.225 \sec^2 \alpha \quad ...\left(\because \frac{1}{\cos^2 \alpha} = \sec^2 \alpha\right)$$

$$= 15 \tan \alpha - 1.225\,(1 + \tan^2 \alpha) \quad ...(\because \sec^2 \alpha = 1 + \tan^2 \alpha)$$

$$= 15 \tan \alpha - 1.225 - 1.225 \tan^2 \alpha$$

or $$1.225 \tan^2 \alpha - 15 \tan \alpha + 7.225 = 0$$

This is quadratic equation in $\tan \alpha$.

∴ $$\tan \alpha = \frac{+15 \pm \sqrt{15^2 - 4 \times 1.225 \times 7.225}}{2 \times 1.225} = 11.74 \quad \text{or} \quad 0.5$$

or $$\alpha = 85.1° \quad \text{or} \quad 26.6° \quad \textbf{Ans.}$$

Example 20.15. *Find the least initial velocity which a projectile may have, so that it may clear a wall 3.6 m high and 4.8 m distant (from the point of projection) and strike the horizontal plane through the foots of the wall at a distance 3.6 m beyond the wall. The point of projection is at the same level as the foot of the wall.*

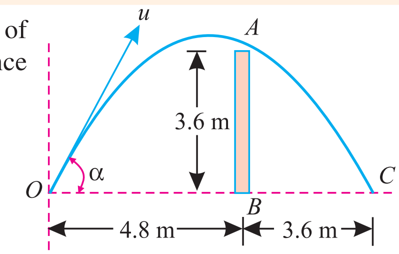

Fig. 20.11.

Solution. Given : Height of wall = 3.6 m ; Distance of the wall from the point of projection (OB) = 4.8 m and distance of strike point from the foot of the wall (BC) = 3.6 m.

Let u = Initial velocity of projection, and

α = Angle of projection.

We know that the range OC of the projectile (R),

$$4.8 + 3.6 = \frac{u^2 \sin 2\alpha}{g}$$

$$8.4 = \frac{u^2 \; 2 \sin \alpha \cos \alpha}{g} \qquad \text{...}(\because \sin 2\alpha = 2 \sin \alpha \cos \alpha)$$

$$\therefore \qquad u^2 = \frac{8.4g}{2 \sin \alpha \cos \alpha} = \frac{4.2g}{\sin \alpha \cos \alpha} \qquad \text{...}(i)$$

and equation of the path of trajectory,

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}$$

Substituting the value of x = 4.8 m and y = 3.6 m in the above equation,

$$3.6 = 4.8 \tan \alpha - \frac{g\,(4.8)^2}{2u^2 \cos^2 \alpha}$$

$$= 4.8 \tan \alpha - \frac{11.52\, g}{\cos^2 \alpha} \times \frac{1}{u^2}$$

Now substituting the value of u^2 from equation (i),

$$3.6 = 4.8 \tan \alpha - \frac{11.52\, g}{\cos^2 \alpha} \times \frac{1}{\dfrac{4.2g}{\sin \alpha \cos \alpha}}$$

$$= 4.8 \tan \alpha - 2.74 \tan \alpha = 2.06 \tan \alpha$$

$$\therefore \qquad \tan \alpha = \frac{3.6}{2.06} = 1.748 \qquad \text{or} \qquad \alpha = 60.2°$$

and now substituting the value of α in equation (i),

$$u^2 = \frac{4.2 \times 9.8}{\sin 60.2° \cos 60.2°} = \frac{41.16}{0.8681 \times 0.4965} = 95.5$$

or $u = 9.77$ m/s **Ans.**

Example 20.16. *Two guns are pointed at each other, one upward at an angle of 30°, and the other at the same angle of depreesion the muzzles being 30 m apart. If the guns are shot with velocities of 350 m/s upwards and 300 m/s downwards respectively, find when and where they will meet ?*

Solution. Given : Angle of projection of both the guns (α) = 30° ; Velocity of projection of first gun (v_A) = 350 m/s ; Velocity of projection of second gun (v_B) = 300 m/s and distance between the muzzles = 30 m

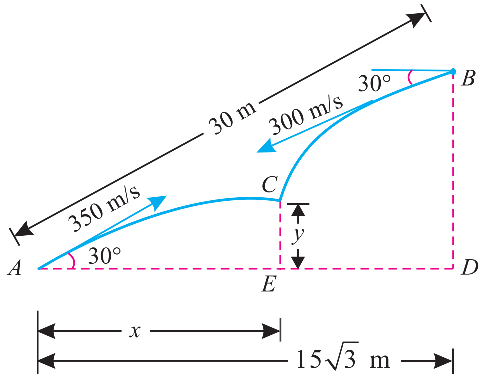

Fig. 20.12.

Time when the shots meet after they leave the guns

Let the two shots meet at C as shown in Fig. 20.12.

Now let t = Time in seconds, when the two shots meet after they leave the guns.

x = Horizontal distance between A and C

y = Vertical distance between A and C

$\therefore$ Horizontal distance between A and B (*i.e. AD*)

$$= 30 \cos 30° = \frac{30\sqrt{3}}{2} = 15\sqrt{3} \text{ m}$$

We know that distance covered by the shot A in t seconds,

$$x = \text{Horizontal component of } v_A \times t$$

$$= 350 \cos 30° \times t = 350 \frac{\sqrt{3}}{2} \times t = 175\sqrt{3}\, t \qquad ...(i)$$

Similarly, distance covered by the shot B in t seconds

$$(15\sqrt{3} - x) = 300 \cos 30° \times t = 300 \frac{\sqrt{3}}{2} \times t = 150\sqrt{3}\, t \qquad ...(ii)$$

Adding equation (*i*) and (*ii*),

$$15\sqrt{3} = 175\sqrt{3} \times t + 150\sqrt{3} \times t$$

or $$15 = 175\, t + 150\, t = 325\, t$$

$\therefore$ $$t = \frac{15}{325} = 0.046 \text{ s} \quad \textbf{Ans.}$$

Point where the two shots meet

Substituting this value of t in equation (*i*),

$$x = 175\sqrt{3} \times 0.046 = 13.94 \text{ m} \quad \textbf{Ans.}$$

We know that vertical component of v_A

$$= 350 \sin 30° = 300 \times 0.5 = 175 \text{ m/s}$$

$\therefore$ Vertical distance between A and C

$$y = ut - \frac{1}{2} gt^2 = (175 \times 0.046) - \left(\frac{1}{2} \times 9.8\,(0.046)^2\right) \text{m}$$

$$= 8.04 \text{ m} \quad \textbf{Ans.}$$

EXERCISE 20.2

1. A bullet is fired at an angle of 45° with the horizontal with a velocity of 275 m/s. How high the bullet will rise above the ground and what will be its horizontal range ? Take g = 9.8 m/s^2 (**Ans.** 1928.6 m ; 7716.8 m)
2. A bullet is fired at such an angle, over a horizontal plane, that its horizontal range is equal to its greatest height. Find the angle of projection. (**Ans.** 75° 58′)
3. Find the angle of projection which will give a horizontal range equal to 3/4 th of the maximum range for the same velocity of projection. (**Ans.** 24° 18′ ; 65° 42′)
4. A cricket ball, shot by a batsman from a height of 1.8 m at an angle of 30° with horizontal with a velocity of 18 m/s is caught by a fields man at a height of 0.6 m from the ground. How far apart were the two players ? (**Ans.** 30.56 m)
5. A jet of water, discharged from a nozzle, hits a screen 6 m away at a height of 4 m above the centre of a nozzle. When the screen is moved 4 m further away, the jet hits it again at the same point. Assuming the curve described by the jet to be parabolic, find the angle at which the jet is projected. (**Ans.** 46° 51′)
6. A bird is sitting on the top of a tree 10 m high. With what velocity should a person, standing at a distance of 25 m from the tree, throws a stone at an angle of 30° with the horizontal so as to hit the bird ? (**Ans.** 30.35 m/s)
7. A projectile is fired from a point at 125 m/s so as to strike a point at a horizontal distance of 1000 m and 200 m higher than the point of firing. Neglecting air resistance, calculate (*i*) the angle with the horizontal, at which the projectile should be fired in order to strike the point in minimum time, and (*ii*) time taken by the projectile to strike the point.
(**Ans.** 32° 46′ ; 9.5 s)

20.9. VELOCITY AND DIRECTION OF MOTION OF A PROJECTILE, AFTER A GIVEN INTERVAL OF TIME, FROM THE INSTANT OF PROJECTION

Consider a projectile projected from O as shown in Fig. 20.13.

Let u = Initial velocity of projection, and

α = Angle of projection with the horizontal.

After t seconds, let the projectile reach at any point P, as shown in Fig. 20.13.

Now let v = Velocity of the projectile at P, and

θ = Angle, which the projectile at P makes with the horizontal.

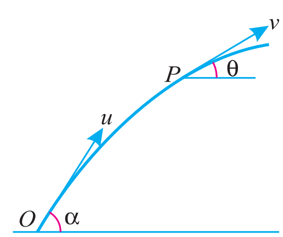

Fig. 20.13.

We know that the vertical component of the initial velocity

$$= u \sin \alpha$$

and vertical component of the final velocity after t seconds

$$= v \sin \theta$$

This change in velocity (*i.e.*, from $u \sin \alpha$ to $v \sin \theta$) is because of the retardation (g) due to gravity

$$\therefore \quad v \sin \theta = u \sin \alpha - gt \quad \text{...(i)}$$

...(Minus sign due to upward direction)

We also know that the horizontal component of these two velocities does not change. Therefore

$$v \cos \theta = u \cos \alpha \quad \text{...(ii)}$$

Squaring the equations (*i*) and (*ii*) and adding the same,

$$v^2 \sin^2\theta + v^2\cos^2\theta = u^2\sin^2\alpha + g^2t^2 - 2u(\sin\alpha)\,gt + u^2\cos^2\alpha$$

$$v^2(\sin^2\theta + \cos^2\theta) = u^2(\sin^2\alpha + \cos^2\alpha) + g^2t^2 - 2u(\sin\alpha)\,gt$$

or $$= u^2 + g^2t^2 - 2u(\sin\alpha)\,gt \qquad ...(\because \sin^2\alpha + \cos^2\alpha = 1)$$

$$\therefore \quad v = \sqrt{u^2 + g^2t^2 - 2u(\sin\alpha)\,gt}$$

The angle which the projectile makes with horizontal at *P* may be found out by dividing the equation (*i*) by (*ii*), *i.e.*

$$\frac{v\sin\theta}{v\cos\theta} = \frac{u\sin\alpha - gt}{u\cos\alpha}$$

$$\therefore \quad \tan\theta = \frac{u\sin\alpha - gt}{u\cos\alpha} = \frac{\text{Vertical velocity after } t \text{ seconds}}{\text{Horizontal component of initial velocity}}$$

Example 20.17. *A projectile is fired with a velocity of 80 m/s at an elevation of 65°. Find its velocity and direction after 5 seconds of firing.*

Solution. Given : Initial velocity of projection (u) = 80 m/s ; Angle of projection with the horizontal (α) = 65° and time (t) = 5 s.

Velocity of the projectile

We know that velocity of the projectile,

$$v = \sqrt{u^2 + g^2t^2 - 2u(\sin\alpha)\,gt.}$$

$$= \sqrt{80^2 + (9.8)^2 \times (5)^2 - 2\times 80 \times (\sin 65°)\times 9.8\times 5} \text{ m/s}$$

$$= \sqrt{6400 + 2401 - 160 \times 0.9063 \times 49} \text{ m/s}$$

$$= \sqrt{1696} = 41.2 \text{ m/s} \quad \textbf{Ans.}$$

Direction of the projectile

Let θ = Angle which the projectile makes with the horizontal.

We also know that $$\tan\theta = \frac{u\sin\alpha - gt}{u\cos\alpha} = \frac{80\sin 65° - 9.8\times 5}{80\cos 65°} = \frac{(80\times 0.9063) - 49}{80 \times 0.4226}$$

$$= 0.6952 \quad \text{or} \quad \theta = 34.8° \quad \textbf{Ans.}$$

Example 20.18. *A particle is projected upwards with a velocity of 100 m/s at an angle of 45° to the horizontal. When it reaches a certain point P, it is found to be moving at an angle of 30° to the horizontal. Find the time for the particle to reach the point P and distance OP.*

Solution. Given : Initial velocity of projectile (u) = 100 m/s ; Angle of projection (α) = 45° and angle of projection at point P (θ) = 30°

Let t = Time for the particle to reach the point P from O.

We know tha $$\tan\theta = \frac{u\sin\alpha - gt}{u\cos\alpha}$$

$$\tan 30° = \frac{100\sin 45° - 9.8t}{100\cos 45°}$$

$$0.5774 = \frac{(100\times 0.707) - 9.8t}{100 \times 0.707}$$

$$40.82 = 70.7 - 9.8\,t$$

$$\therefore \quad t = \frac{70.7 - 40.82}{9.8} = 3.05 \text{ s} \quad \textbf{Ans.}$$

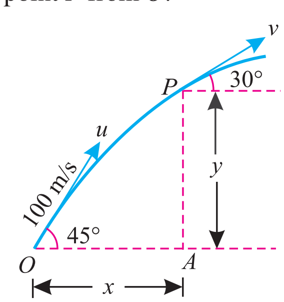

Fig. 20.14.

Distance OP

First of all, consider the horizontal motion of the particle. We know that the horizontal distance *OA*

$$x = \text{Horizontal component of velocity} \times \text{Time}$$
$$= 100 \cos 45° \times 3.05 = 100 \times 0.707 \times 3.05 = 215.6 \text{ m} \quad ...(i)$$

Now consider vertical motion of the particle. We know that vertical component of the velocity.

$$u_y = 100 \sin 45° = 100 \times 0.707 = 70.7 \text{ m}$$

and vertical distance *AP*, $y = u_y t - \frac{1}{2}.gt^2 = (70.7 \times 3.05) - \frac{1}{2} \times 9.8\,(3.05)^2$

$$= 215.6 - 45.6 = 170 \text{ m}$$

$\therefore$ Distance $OP = \sqrt{(215.6)^2 + (170)^2} = 274.6$ m **Ans.**

Example 20.19. *A projectile is fired with a velocity of 500 m/s at an elevation of 35°. Neglecting air friction, find the velocity and direction of the projectile moving after 29 seconds and 30 seconds of firing.*

Solution. Given : Velocity of projection (u) = 500 m/s and angle of projection (α) = 35°

Velocity of the projectile after 29 *and* 30 *seconds of firing*

We know that velocity of projectile after 29 seconds

$$= \sqrt{u^2 + g^2t^2 - 2u\,(\sin \alpha)\, gt}$$

$$v_{39} = \sqrt{\left[(500)^2 + (9.8)^2 \times (29)^2\right] - (2 \times 500 \times (\sin 35°) \times 9.8 \times 29)}$$

$$= \sqrt{(250\,000 + 80770) - (1000 \times 0.5736 \times 284.2)} \text{ m/s}$$

$= 409.58$ m/s **Ans.**

Similarly $$v_{30} = \sqrt{\left[(500)^2 + (9.8)^2 \times (30)^2\right] - (2 \times 500 \times (\sin 35°) \times 9.8 \times 30)}$$

$$= \sqrt{[250\,000 + 86440] - [1000 \times 0.5736 \times 294]} \text{ m/s}$$

$= 409.64$ m/s **Ans.**

Direction of the projectile after 29 *and* 30 *seconds*

We know that the angle, which the projectile makes with the horizontal after 29 seconds,

$$\tan \theta_{29} = \frac{u \sin \alpha - gt}{u \cos \alpha} = \frac{(500 \sin 35°) - (9.8 \times 29)}{500 \cos 35°}$$

$$= \frac{(500 \times 0.5736) - 284.2}{500 \times 0.8192} = 0.00635$$

or $\theta_{29} = 0.36°$ **Ans.**

Similarly $$\tan \theta_{30} = \frac{(500 \sin 35°) - (9.8 \times 30)}{500 \cos 35°} = \frac{(500 \times 0.5736) - 294.0}{500 \times 0.8192}$$

$= -0.0176$ or $\theta_{30} = -1°$

Note : Minus sign means that the projectile is moving downwards after reaching the highest point.

Example 20.20. *A particle is projected upwards with a velocity of 100 m/s at an angle of 30° with the horizontal.*

Find the time, when the particle will move perpendicular to its initial direction.

Solution. Given : Initial velocity of projection (u) = 100 m/s and angle of projection with the horizontal (α) = 30°

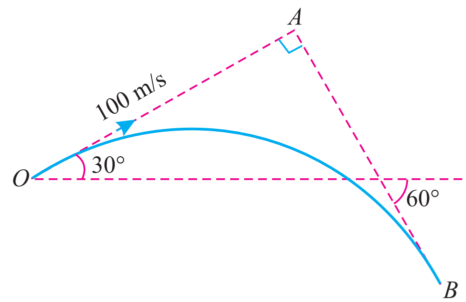

Fig. 20.15.

Let t = Time from the instant of projection, when the particle will move perpendicular to its initial direction.

We know that when the particle will move perpendicular to its initial direction, it will make an angle of 90° – 30° = 60° with the horizontal, but in the downward direction as shown in Fig. 20.14. Therefore actual angle,

$$\theta = (-60°) \qquad \text{...(Minus sign due to downward)}$$

We also know that the angle, which the particle makes with the horizontal after t seconds (θ),

$$\tan(-60°) = \frac{u \sin \alpha - gt}{u \cos \alpha} = \frac{100 \sin 30° - 9.8t}{100 \cos 30°}$$

$$-1.732 = \frac{(100 \times 0.5) - 9.8t}{100 \times 0.866} = \frac{50 - 9.8t}{86.6}$$

$$-150 = 50 - 9.8\,t$$

$$\therefore \qquad t = \frac{50 + 150}{9.8} = 20.4 \text{ s} \quad \textbf{Ans.}$$

20.10. VELOCITY AND DIRECTION OF MOTION OF A PROJECTILE, AT A GIVEN HEIGHT ABOVE THE POINT OF PROJECTION

Consider a projectile projected from O as shown in Fig. 20.16.

Let u = Initial velocity of projection, and

α = Angle of projection with the horizontal.

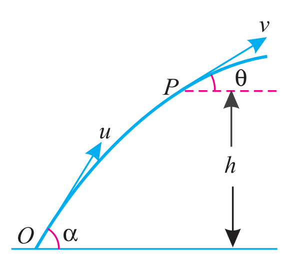

Fig. 20.16.

After reaching a height h, let the projectile reach at any point P as shown in Fig. 20.15.

Let v = Velocity of the projectile at P, and

θ = Angle, which the projectile at P makes with the horizontal.

We know that the vertical component of initial velocity

$$= u \sin \alpha$$

and vertical component of the final velocity after t seconds

$$= v \sin \theta$$

This change in velocity (*i.e.* from $u \sin \alpha$ to $v \sin \theta$) is becasue of the retardation (g) due to gravity.

$$\therefore \quad v^2 \sin^2 \theta = u^2 \sin^2 \alpha - 2gh \qquad \ldots(\because \; v^2 = u^2 - 2as) \ldots(i)$$

$$\text{or} \quad v \sin \theta = \sqrt{u^2 \sin^2 \alpha - 2gh} \qquad \ldots(ii)$$

We also know that the horizontal component of these two velocities does not change.

$$\therefore \quad v \cos \theta = u \cos \alpha \qquad \ldots(iii)$$

Squaring equation (*iii*) and adding to equation (*i*),

$$v^2 \sin^2 \theta + v^2 \cos^2 \theta = u^2 \sin^2 \alpha - 2gh + u^2 \cos^2 \alpha$$

$$v^2 (\sin^2 \theta + \cos \theta) = u^2 (\sin^2 \alpha + \cos^2 \alpha) - 2gh$$

$$\text{or} \quad v^2 = u^2 - 2gh \qquad \ldots(\because \; \sin^2 \theta + \cos^2 \theta = 1)$$

$$\therefore \quad v = \sqrt{u^2 - 2gh}$$

This angle which the particle makes with the horizontal at P may be found out by dividing the equation (*ii*) by (*iii*) *i.e.*,

$$\frac{v \sin \theta}{v \cos \theta} = \frac{\sqrt{u^2 \sin^2 \alpha - 2gh}}{u \cos \alpha}$$

$$\therefore \quad \tan \theta = \frac{\text{Vertical velocity at a height } h}{\text{Horizontal component of initial velocity}}$$

Example 20.21. *A body is projected upwards with a velocity of 50 m/s at angle of 50° with the horizontal. What will be its (i) velocity and (ii) direction at a height of 30 m from the point of projection.*

Solution. Given : Initial velocity of projection (u) = 50 m/s ; Angle of projection (α) = 50° and height (h) = 30 m.

(*i*) *Velocity of the projectile*

We know that velocity of the projectile,

$$v = \sqrt{u^2 - 2gh} = \sqrt{(50)^2 - (2 \times 9.8 \times 30)} = 43.7 \text{ m/s} \quad \textbf{Ans.}$$

(*ii*) *Direction of the projectile*

Let θ = Angle which the projectile makes with the horizontal.

We also know that

$$\tan \theta = \frac{\sqrt{u^2 \sin^2 \alpha - 2gh}}{u \cos \alpha} = \frac{\sqrt{(50)^2 \sin^2 50° - 2 \times 9.8 \times 30}}{50 \cos 50°}$$

$$= \frac{\sqrt{2500 \times (0.766)^2 - 588}}{50 \times 0.6428} = 0.9224$$

$$\text{or} \quad \theta = 42.7° \quad \textbf{Ans.}$$

Example 20.22. *The velocity of a particle, at its greatest height is $\sqrt{2/5}$ times of its velocity at half of its greatest height. Show that the angle of projection is 60°.*

Solution. Given : Velocity of the particle at its greatest height $= \sqrt{2/5} \times$ Velocity at half of its greatest height.

Let u = Initial velocity of projection, and

α = Angle of projection with the horizontal.

We know that velocity of a projectile at its greatest height

= Horizontal component of velocity of projection

$= u \cos \alpha$ ...(*i*)

We also know that the maximum height of projection.

$$= \frac{u^2 \sin^2 \alpha}{2g}$$

and half of the greatest height $\quad h = \dfrac{1}{2} \times \dfrac{u^2 \sin^2 \alpha}{2g} = \dfrac{u^2 \sin^2 \alpha}{4g}$

$\therefore$ Velocity of projectile at half of the greatest height,

$$v = \sqrt{u^2 - 2gh} = \sqrt{u^2 - 2g \times \frac{u^2 \sin^2 \alpha}{4g}}$$

$$= \sqrt{u^2 - \frac{u^2 \sin^2 \alpha}{2}} \qquad \text{...(ii)}$$

Now substituting the values from equations (*i*) and (*ii*) in the given equation,

$$u \cos \alpha = \sqrt{\frac{2}{5}} \times \sqrt{u^2 - \frac{u^2 \sin^2 \alpha}{2}} = \sqrt{\frac{2}{5}} \times \sqrt{\frac{2u^2 - u^2 \sin^2 \alpha}{2}}$$

Squaring both sides,

$$u^2 \cos^2 \alpha = \frac{2}{5} \times \frac{2u^2 - u^2 \sin^2 \alpha}{2} = \frac{u^2 (2 - \sin^2 \alpha)}{5}$$

$$5 \cos^2 \alpha = 2 - \sin^2 \alpha$$

or $\quad 5 (1 - \sin^2 \alpha) = 2 - \sin^2 \alpha \qquad \text{...}(\sin^2 \alpha + \cos^2 \alpha = 1)$

$$4 \sin^2 \alpha = 3 \quad \text{or} \quad \sin^2 \alpha = \frac{3}{4} = 0.75$$

$\therefore \quad \sin \alpha = 0.866 \quad$ or $\quad \alpha = 60°$ **Ans.**

20.11. TIME OF FLIGHT OF A PROJECTILE ON AN INCLINED PLANE

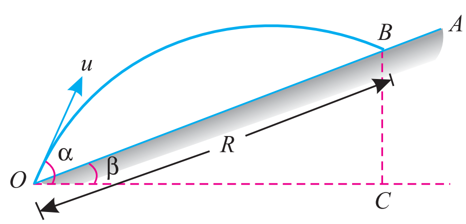

Fig. 20.17. Projectile on an inclined plane.

Consider a projectile projected from O on an upward inclined plane OA. Let the projectile strike B as shown in Fig. 20.17.

Let u = Initial velocity of projection,

α = Angle of projection with the horizontal.

β = Inclination of the plane OA with the horizontal,

R = Range of flight from O to B, and

t = Time of flight from O to B.

$\therefore$ Component of initial velocity, normal to the plane OA

$$= u \sin (\alpha - \beta) \quad ...(i)$$

We know that acceleration due to gravity normal to the plane OA

$$= g \cos \beta \quad ...(ii)$$

and acceleration due to gravity along the plane OA

$$= g \sin \beta \quad ...(iii)$$

Now consider the motion of the projectile normal to the plane. We know that distance covered by the projectile normal to the plane OA is zero. Therefore substituting these values in the general equation of motion, *i.e.*

$$s = ut - \frac{1}{2} g t^2$$

$$0 = u \sin (\alpha - \beta)\, t - \frac{1}{2} (g \cos \beta) t^2$$

or
$$0 = u \sin (\alpha - \beta) - \frac{1}{2} g (\cos \beta) t \quad \text{... (Dividing both sides by } t)$$

$\therefore$
$$t = \frac{2u \sin (\alpha - \beta)}{g \cos \beta}$$

Note : When the projectile is projected on a downward inclined plane, the time of flight may be found out by substituting $-\beta$ instead of $+\beta$ in the above equation. Therefore time of flight in this case,

$$t = \frac{2u \sin (\alpha + \beta)}{g \cos \beta} \quad \text{... } [\because \cos (-\beta) = \cos \beta]$$

Example 20.23. *A ball is projected from a point with a velocity of 10 m/s on an inclined plane. the angle of projection and inclination of the plane are 35° and 15° respectively with the horizontal. Find the time of flight of the ball, when it is projected upwards and downwards the plane.*

Solution. Given : Velocity of projection (u) = 10 m/s ; Angle of projection with the horizontal (α) = 35° and inclination of the plane (β) = 15°

Time of flight when the ball is projected upwards

We know that time of flight when the ball is projected upwards,

$$t_1 = \frac{2u \sin (\alpha - \beta)}{g \cos \beta} = \frac{2 \times 10 \sin (35° - 15°)}{9.8 \cos 15°} = \frac{20 \sin 20°}{9.8 \cos 15°} \text{ s}$$

$$= \frac{20 \times 0.342}{9.8 \times 0.9659} = 0.72 \text{ s} \quad \textbf{Ans.}$$

Contents

Time of flight when the ball is projected downwards.

We also know that time of flight when the ball is projected downwards,

$$t_2 = \frac{2u \sin(\alpha + \beta)}{g \cos \beta} = \frac{2 \times 10 \sin(35° + 15°)}{9.8 \cos 15°} = \frac{20 \sin 50°}{9.8 \cos 15°}$$

$$= \frac{20 \times 0.766}{9.8 \times 0.9659} = 1.62 \text{ s} \quad \textbf{Ans.}$$

EXERCISE 20.3

1. A shot is fired with a velocity of 420 m/s at an elevation of 32°. Find the velocity and direction of the shot after 20 seconds of its firing. (**Ans.** 4° 16')
2. A stone is projected with a velocity 21 m/s at an angle of 30° with the horizontal. Find its velocity at a height of 5 m from the point of projection. Also find the interval of time between two points at which the stone has the same velocity of 20 m/s.
(**Ans.** 18.52 m/s ; 1.69 s)

20.12. RANGE OF PROJECTILE ON AN INCLINED PLANE

We have already discussed in Art. 20.11. that the time of flight,

$$t = \frac{2u \sin(\alpha - \beta)}{g \cos \beta}$$

and horizontal components of the range,

$$OC = \text{Horizontal component of velocity} \times \text{Time}$$

$$= u \cos \alpha \times \frac{2u \sin(\alpha - \beta)}{g \cos \beta} = \frac{2u^2 \sin(\alpha - \beta) \cos \alpha}{g \cos \beta}$$

∴ Actual range on the inclined plane,

$$R = \frac{OC}{\cos \beta} = \frac{2u^2 \sin(\alpha - \beta) \cos \alpha}{g \cos \beta \times \cos \beta} = \frac{2u^2 \sin(\alpha - \beta) \cos \alpha}{g \cos^2 \beta}$$

$$= \frac{u^2}{g \cos^2 \beta} [2 \cos \alpha \sin(\alpha - \beta)]$$

$$= \frac{u^2}{g \cos^2 B} [\sin(2\alpha - \beta) - \sin \beta] \quad ...(i)$$

...[∵ $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$]

From the above equation, we find that for the given values of u and β, the range will be maximum, when $\sin(2\alpha - \beta)$ is maximum (as the values of u, g and β are constant). We know that for maximum value of sine of any angle, the angle must be equal to 90° or π/2.

$$\therefore \quad (2\alpha - \beta) = \frac{\pi}{2} \quad \text{or} \quad \alpha = \left(\frac{\pi}{4} + \frac{\beta}{2}\right)$$

Or in other words, the range on the given plane is maximum, when the direction of projection bisects the angle between the vertical and inclined plane.

Now for maximum range, substituting the value of α in equation (i),

$$R_{max} = \frac{u^2}{g\cos^2\beta}\left\{\sin\left[2\left(\frac{\pi}{4}+\frac{\beta}{2}\right)-\beta\right]-\sin\beta\right\}$$

$$= \frac{u^2}{g\cos^2\beta}\left[\sin\left(\frac{\pi}{2}+\beta-\beta\right)-\sin\beta\right]$$

$$= \frac{u^2}{g\cos^2\beta}\left[\sin\left(\frac{\pi}{2}\right)-\sin\beta\right]$$

$$= \frac{u^2}{g\cos^2\beta}[1-\sin\beta] \qquad \ldots\left[\because \sin\left(\frac{\pi}{2}\right)=1\right]$$

$$= \frac{u^2(1-\sin\beta)}{g(1-\sin^2\beta)} \qquad \ldots(\because \sin^2\beta+\cos^2\beta=1)$$

$$= \frac{u^2}{g(1+\sin\beta)}$$

Notes : 1. When the projectile is projected on a downward inclined plane, then the range of flight will be given by substituting $-\beta$ instead of $+\beta$ in the above equation. Therefore range of flight in this case,

$$R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha+\beta)+\sin\beta]$$

2. When the projectile is projected on a downward inclined plane, the range will be maximum, when

$$\alpha = \left(\frac{\pi}{2}-\frac{\beta}{2}\right)$$

3. When the projectile is projected on a downward inclined plane,the value of maximum range will be

$$R_{max} = \frac{u^2}{g(1-\sin\beta)}$$

Example 20.24. *A particle is projected from a point, on an inclined plane,with a velocity of 30 m/s. The angle of projection and the angle of plane are 55° and 20° to the horizontal respectively. Show that the range up the plane is maximum one for the given plane. Find the range and the time of flight of the particle.*

Solution. Given : Velocity of projection (u) = 30 m/s ; Angle of projection with the horizontal (α) = 55° and angle of plane (β) = 20°

Maximum Range

We know that for maximum range, the angle of projection,

$$\alpha = \frac{\pi}{4}+\frac{\beta}{2} = \frac{180°}{4}+\frac{20°}{2} = 55°$$

Since the given angle of projection is 55°, therefore range up the plane is maximum one for the given plane. **Ans.**

Range of the projectile

We know that range of the projectile,

$$R = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta] \text{ m}$$

$$= \frac{(30)^2}{9.8\cos^2 20°}[\sin(2\times 55° - 20°) - \sin 20°] \text{ m}$$

$$= \frac{900}{9.8\,(0.9397)^2}[\sin 90° - \sin 20°] \text{ m}$$

$$= 104.0\,(1 - 0.3420) = 68.43 \text{ m} \quad \textbf{Ans.}$$

Time of flight

We also know that the time of flight,

$$t = \frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{2\times 30\sin(55° - 20°)}{9.8\cos 20°} = \frac{60\times\sin 35°}{9.8\cos 20°} \text{ s}$$

$$= \frac{60\times 0.5736}{9.8\times 0.9397} = 3.74 \text{ s} \quad \textbf{Ans.}$$

Note : Since the angle of projection is for the maximum range, therefore the range may also be found out from the relation :

$$R = \frac{u^2}{g\,(1+\sin\beta)} = \frac{(30)^2}{9.8\,(1+\sin 20°)} = \frac{900}{9.8\,(1+0.3420)} \text{ m}$$

$$= 68.43 \text{ m} \quad \textbf{Ans.}$$

Example 20.25. *A plane has a rise of 5 in 12. A shot is projected with a velocity of 200 m/s at an elevation of 30°. Find the range of the plane, if (a) the shot is fired up the plane, (b) the shot is fired down the plane.*

Solution. Given : $\tan\beta = 5/12 = 0.4167$ or $\beta = 22.6°$; Velocity of projection with the horizontal (u) = 200 m/s and angle of projection (α) = 30°.

(*a*) *Range of the plane, when the shot is fired up the plane*

We know that range of the plane, when the shot is fired up the plane,

$$R_1 = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta) - \sin\beta]$$

$$= \frac{(200)^2}{9.8\cos^2 22.6°}[\sin(2\times 30° - 22.6°) - \sin 22.6°]$$

$$= \frac{40\,000}{9.8\,(0.9231)^2}[\sin 37.4° - \sin 22.6°]$$

$$= 4790\,(0.6072 - 0.3846) = 1066 \text{ m} \quad \textbf{Ans.}$$

(*b*) *Range of the plane, when the shot is fired down the plane*

We know that range of the plane, when the shot is fired down the plane,

$$R_2 = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha+\beta) + \sin\beta]$$

$$R_2 = \frac{(200)^2}{9.8 \cos^2 22.6°} \left[\sin (2 \times 30° + 22.6°) + \sin 22.6°\right]$$

$$= \frac{40\ 000}{9.8\ (0.9231)^2} \left[\sin 82.6° + \sin 22.6°\right] \text{ m}$$

$$= 4790\ (0.9917 + 0.3846) = 6592 \text{ m} \qquad \textbf{Ans.}$$

EXERCISE 20.4

1. A player can throw a cricket ball 100 m on a level ground. Find the distance through which he can throw the same ball from the top of hill at angle of 52° 30', if slope of the hill is 15°. (**Ans.** 120.5 m)
2. A shot is fired with a velocity of 100 m/s at an angle of 45° with the horizontal on a plane inclined at an angle of 30° with the horizontal. Find the maximum range of the shot. (**Ans.** 680.3 m)
3. A projectile is projected up a plane of inclination (β) with an initial velocity of (u) at an angle (α) to the horizontal. Show that condition for the projectile to strike the inclined plane at right angles is

$$\cot \beta = 2 \tan (\alpha - \beta).$$

QUESTIONS

1. What is a projectile ? Give an example of a projectile.
2. Define the terms : velocity of projection and angle of projection.
3. Obtain an equation for the trajectory of a projectile, and show that it is a parabola.
4. Derive an expression for the maximum height and range of a projectile traversed by a stone, thrown with an initial velocity of u and an inclination of α.
5. At what angle,the projectile should be projected in order to have maximum range ? Justify your answer by calculations.
6. Derive a relation for the velocity and direction of motion of a projectile :
   (*a*) after a given interval of time t from the instant of projection.
   (*b*) at a given height h above the point of projection.
7. How would you find out (*a*) time of flight (*b*) range of a projectile, when projected upwards on an inclined plane ?

   What happens to the above equations, when the same projectile is projected on the same plane, but in a downward direction ?

OBJECTIVE TYPE QUESTIONS

1. The path of a projectile is not a parabola.
   (*a*) True (*b*) False
2. The time of flight of a projectile on a horizontal plane is
   (*a*) $\frac{2u \sin \alpha}{g}$ (*b*) $\frac{2u \cos \alpha}{g}$ (*c*) $\frac{2u \sin \alpha}{2g}$ (*d*) $\frac{u \cos 2\alpha}{2g}$

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3. The horizontal range of a projectile is

(a) $\dfrac{u \sin 2\alpha}{g}$ (b) $\dfrac{u^2 \sin 2\alpha}{g}$ (c) $\dfrac{u \sin 2\alpha}{2g}$ (d) $\dfrac{u^2 \sin 2\alpha}{2g}$

4. The horizontal range of a projectile is maximum when the angle of projectile is

(a) 30° (b) 45° (c) 60° (d) 75°

5. The maximum height of a projectile on a horizontal range is

(a) $\dfrac{u^2 \sin 2\alpha}{2g}$ (b) $\dfrac{u^2 \sin \alpha}{2g}$ (c) $\dfrac{u^2 \sin^2 2\alpha}{2g}$ (d) $\dfrac{u^2 \sin^2 \alpha}{2g}$

6. The time of flight of a projectile on un upward inclined plane depends upon

(a) angle of projection (b) angle of inclination of the plane

(c) both 'a' and 'b' (d) none of the above

7. The range of projectile on a downward inclined plane is ....the range on upward inclined plane for the same velocity of projection and angle of projection.

(a) less than (b) equal to (c) more than

ANSWERS

1. (b)	2. (a)	3. (b)	4. (b)	5. (d)	6. (c)	7. (c)

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