Method of undetermined Coeff :

Consider the n^{th} order non-homogeneous LDE with constant coeff.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = r(x)$$

Let $y_1, y_2, \ldots, y_n$ be L.I solution of homogeneous part of d.e.

i.e $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$.

For y_p,

Case I, if $r(x) = e^{ax}$.

Let the trial sol. be $y_p = A_1 e^{ax} x^k$ k = multiplicity of root ($m = a$) in A.E.

eg: $y'' - 3y' + 2y = e^{3x}$.

Operator form is $(D^2 - 3D + 2)y = e^{3x}$, $D = \frac{d}{dx}$

Consider the homogeneous part of diff eq.

$$(D^2 - 3D + 2)y = 0,$$

Let $y = e^{mx}$ be its sol.

A.E : $m^2 - 3m + 2 = 0$

$\Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2$

$$y_c = c_1 e^{x} + c_2 e^{2x}.$$

For y_p: Let $y_p = Ae^{3x}$ | $y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x}$.

For y_1; Let $y_p = Ae^{3x}$ | $y_p'' = 9Ae^{3x}$

$\therefore$ y_p is P.I.

$\Rightarrow$ $y_p'' - 3y_p' + 2y_p = e^{3x}$.

$\Rightarrow$ $9Ae^{3x} - 3(3Ae^{3x}) + 2Ae^{3x} = e^{3x}$.

$\Rightarrow$ $\cancel{9Ae^{3x}} - \cancel{9Ae^{3x}} + 2Ae^{3x} = e^{3x}$

$\Rightarrow$ $2Ae^{3x} = e^{3x} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

$\Rightarrow$ $\boxed{y_p = \frac{1}{2}e^{3x}}$

General sol. is $\boxed{y = y_c + y_p = c_1e^{x} + c_2e^{2x} + \frac{1}{2}e^{3x}}$

Consider $\boxed{y'' - 3y' + 2y} = e^{2x}$.

$y_c = c_1e^{x} + c_2e^{2x}$.

Let $y_p = Ae^{2x} \cdot x^{k}$

k = multiplicity of solution in complimentary function.

here $k = 1$

$\Rightarrow$ $y_p = Ae^{2x}x$.

$y_p' = A(2xe^{2x} + e^{2x}(1))$

$= A(2x+1)e^{2x}$

$y_p'' = A[2(2x+1)e^{2x} + e^{2x} \cdot 2]$

$\boxed{\begin{array}{l} y'' - 4y' + 4y = 0 \\ \text{roots of A.E.} = 2, 2. \\ e^{mx}. \\ y = (c_1 + c_2x)e^{2x}. \end{array}}$

$= A(4x+4)e^{2x}$

$\therefore y_p$ is P.I

$y_p'' - 3y_p' + 2y_p = e^{2x}$.

$\Rightarrow A(4x+4)e^{2x} - 3A(2x+1)e^{2x} + 2Axe^{2x} = e^{2x}$.

$\Rightarrow A\{\cancel{4x}+4 - \cancel{6x} - 3 + \cancel{2x}\}e^{2x} = e^{2x}$.

$\Rightarrow Ae^{2x} = e^{2x} \Rightarrow A = 1$

Hence $y_p = xe^{2x}$.

Complete sol. is $y = y_c + y_p$

$= C_1e^{x} + C_2e^{2x} + xe^{2x}$.

Case II: If $r(x) = \sin ax$ or $\cos ax$.

$y_p = (A\cos ax + B\sin ax)x^{k}$.

k = multiplicity of root $m = \pm ia$ in Auxiliary equation

Or Solve the diff. eq. $y'' + 3y' + 2y = \cos x + \sin x$.

Sol: Consider the homogeneous diff. eq.

$y'' + 3y' + 2y = 0$.

Operator form is $(D^2 + 3D + 2)y = 0$, $D \equiv \frac{d}{dx}$.

Let $y = e^{mx}$ be its sol.

A.E $\quad m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0 \Rightarrow m = -1, -2.$

C.F. $\quad y_c = c_1 e^{-x} + c_2 e^{-2x}.$

P.I. Let the trial solution be

$y_p = (A\cos x + B\sin x)\, x^0$ [as there is no such function present in C.F]

$= A\cos x + B\sin x$

$\because y_p$ is P.I. $\therefore$ $y_p'' + 3y_p' + 2y_p = \cos x + \sin x.$ ①

$y_p' = -A\sin x + B\cos x \;;\; y_p'' = -A\cos x - B\sin x.$

① $\Rightarrow$ $(-A\cos x - B\sin x) + 3(-A\sin x + B\cos x)$
$+ 2(A\cos x + B\sin x) = \cos x + \sin x$

$\Rightarrow \cos x(-A + 3B + 2A) + \sin x(-B - 3A + 2B) = \cos x + \sin x$

$\Rightarrow \cos x\,(A + 3B) + \sin x\,(3A + B) = \cos x + \sin x$

$A + 3B = 1$
$3A + B = 1$ [On comparing the like term on both sides]

$\dfrac{A}{3-1} = \dfrac{B}{3-1} = \dfrac{-1}{1-9}.$

$\Rightarrow \dfrac{A}{2} = \dfrac{B}{2} = \dfrac{1}{8} \Rightarrow A = \dfrac{1}{4},\ B = \dfrac{1}{4}.$

$y_p = \frac{1}{4}(\cos x + \sin x)$

Hence complete solution is

$$\boxed{y = y_c + y_p = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{4}(\cos x + \sin x)}$$

Q. $y'' + 25y = 50\cos 5x$.

Sol: Consider the homogeneous diff eq

$y'' + 25y = 0$

C.F: (a) $C_1 e^{5x} + C_2 e^{-5x}$ (b) $(C_1 + C_2 x)e^{-5x}$

(c) $C_1\cos 5x + C_2 \sin 5x$ (d) none of them.

operator form. $(D^2 + 25)y = 0$, $D = \frac{d}{dx}$

Let $y = e^{mx}$ be its sol.

A.E: $m^2 + 25 = 0 \Rightarrow m^2 = -25$

$m = 0 \pm i5$

C.F: $y_c = C_1\cos 5x + C_2 \sin 5x$

P.I: (a) $y_p = A\cos 5x + B\sin 5x$ (b) $y_p = A\cos 5x$

(c) $y_p = A\sin 5x$ (d) $(A\cos 5x + B\sin 5x)x$.

(e) $(A\cos 5x + B\sin 5x)x^2$.

The trial sol. $y_p = (A\cos 5x + B\sin 5x)x^1$

$(A\cos 5x + B\sin 5x)$

$y_p' = x(-5A\sin 5x + 5B\cos 5x) + (A\cos 5x + B\sin 5x)$

$y_p'' = x(-25A\cos 5x - 25B\sin 5x) + (-5A\sin 5x + 5B\cos 5x)$

$+ (-5A\sin 5x + 5B\cos 5x)$

$y_p'' = -25x(A\cos 5x + B\sin 5x) + (-10A\sin 5x + 10B\cos 5x)$

$\because y_p$ is P.I. $\therefore$ $y_p'' + 25y_p = 50\cos 5x$

$\Rightarrow -25x\cancel{(A\cos 5x + B\sin 5x)} + (-10A\sin 5x + 10B\cos 5x)$
$+ 25\cancel{(A\cos 5x + B\sin 5x)}x = 50\cos 5x$

$\Rightarrow -10A\sin 5x + 10B\cos 5x = \boxed{50\cos 5x}$.

on compare L.h.s & R.h.s

$-10A = 0$, $\quad 10B = 50$

$A = 0$ $\quad B = 5$

$\therefore y_p = (5\sin 5x)x = 5x\sin 5x$.

hence complete sol. is

$\boxed{y = y_c + y_p = C_1\cos 5x + C_2\sin 5x + 5x\sin 5x}$

$r(x) = x^2 + 1$

$y_p = A_0x^2 + A_1x + A_2$.

Case III: If $r(x) = x^m$.

Let the trial solution be -

$y_p = (A_0x^m + A_1x^{m-1} + \cdots + A_m)x^k$.

Let the ...

$$y_p = (A_0 x^m + A_1 x^{m-1} + \dots + A_m) x^k.$$

where k represent the multiplicity of root ($\alpha = 0$) in the A.E.

If $m = 0, 0$
$y = (C_1 + C_2 x) e^{0x}$
$m = 0, 0, 0$
$y = (C_1 + C_2 x + C_3 x^2) e^{0x}.$

Example Solve the diff. eq. $3y'' + 2y' - y = e^{-2x} + x.$

Soln. Consider the homogeneous diff eq.

$$3y'' + 2y' - y = 0$$

operator form. $(3D^2 + 2D - 1)y = 0, \quad D \equiv \frac{d}{dx}$

Let $y = e^{mx}$ be its sol.

A.E, $\quad 3m^2 + 2m - 1 = 0$

$\Rightarrow 3m^2 + 3m - m - 1 = 0$

$\Rightarrow 3m(m+1) - (m+1) = 0$

$\Rightarrow (3m-1)(m+1) = 0$

$\Rightarrow m = 1/3, -1.$

C.F., $\quad y_c = C_1 e^{x/3} + C_2 e^{-x}$

P.I., The trial sol. for P.I will be.

$$y_p = (A_1 e^{-2x}) + (A_2 x + A_3)$$

$$y_p' = -2A_1 e^{-2x} + A_2.$$

$$y_p'' = 4A_1 e^{-2x}.$$

$\because y_p$ is P.I. $\therefore 3y_p'' + 2y_p' - y_p = e^{-2x} + x.$

$\Rightarrow 3\left(4A_1e^{-2x}\right) + 2\left(-2A_1e^{-2x} + A_2\right) - \left(A_1e^{-2x} + A_2x + A_3\right) = e^{-2x} + x.$

$\Rightarrow 12A_1e^{-2x} - 4A_1e^{-2x} + 2A_2 - A_1e^{-2x} - A_2x - A_3 = e^{-2x} + x$

$\Rightarrow e^{-2x}\left(7A_1\right) + \left(2A_2 - A_3\right) - A_2x = e^{-2x} + x$

On comparing the like term.

$7A_1 = 1$, $-A_2 = 1 \Rightarrow A_2 = -1$.

$2A_2 - A_3 = 0 \Rightarrow A_3 = 2A_2 = -2.$

$\therefore A_1 = 1/7,\ A_2 = -1,\ A_3 = -2.$

$y_p = \frac{1}{7}e^{-2x} - x - 2.$

Complete sol. is $\boxed{y = y_c + y_p = c_1e^{\frac{1}{3}x} + c_2e^{-x} + \frac{1}{7}e^{-2x} - x - 2.}$

Q7: $y^{iv} - 16y'' = 8x + 16.$

P.I.: (a) $Ax + B$ (b) $(Ax+B)x$

(c) $(Ax+B)x^2$ (d) $(Ax^2 + Bx + C)$

(e) none of here.

Consider the homogeneous diff. eq. $y^{iv} - 16y'' = 0$, operator form is

$y^{iv} - 16y'' = 0$, operator form is

$(D^4 - 16D^2)y = 0$, $D \equiv \frac{d}{dx}$

Let $y = e^{mx}$ be its sol.

A.E: $m^4 - 16m^2 = 0 \Rightarrow m^2(m^2-16) = 0$

$\Rightarrow m = 0, 0, \pm 4.$

C.F: $y_c = (C_1 + C_2 x)e^{0x} + C_3 e^{4x} + C_4 e^{-4x}.$

P.I:- Let the trial sol be

$y_p = (Ax+B)x^2 = Ax^3 + Bx^2$

$y_p' = 3Ax^2 + 2Bx.$

$y_p'' = 6Ax + 2B$, $y_p''' = 6A$, $y_p^{iv} = 0$

$\because y_p$ is P.I $\Rightarrow$ $y_p^{iv} - 16y_p'' = 8x + 16$

$\Rightarrow 0 - 16(6Ax + 2B) = 8x + 16$

$\Rightarrow -96Ax - 32B = 8x + 16$

$\Rightarrow -96A = 8$, $-32B = 16$

$A = -\frac{1}{12}$, $B = -\frac{1}{2}.$

$y_p = \left(-\frac{1}{12}x - \frac{1}{2}\right)x^2 = -\frac{x^3}{12} - \frac{1}{2}x^2.$

Complete sol, $y = y_c + y_p$

$= (C_1 + C_2 x) + C_3 e^{4x} + C_4 e^{-4x} - \frac{x^3}{12} - \frac{x^2}{2}.$

If $f(x) = e^{ax}\cos bx$ or $e^{ax}\sin bx$

Case IV: If $r(x) = e^{ax}\cos bx$ or $e^{ax}\sin bx$.

Let the trial sol. be.

$$y_p = e^{ax}(A_1\cos bx + B_1\sin bx)\cdot x^K.$$

* K represents multiplicity of root $m = a \pm ib$ in the A.E.

Q1: $y'' - 6y' + 13y = 6e^{3x}\sin x\cos x.$

$$= 3e^{3x}(2\sin x\cos x)$$

$$= 3e^{3x}\sin 2x$$

C.F.: Consider the homogeneous diff eq,

$$y'' - 6y' + 13y = 0.$$

operator form as $(D^2 - 6D + 13)y = 0$, $D = \frac{d}{dx}$.

Let $y = e^{mx}$ be the sol.

A.E.: $m^2 - 6m + 13 = 0$

$$m = \frac{6 \pm \sqrt{36-52}}{2(1)} = \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm 4i}{2} = 3 \pm 2i.$$

$$y_c = e^{3x}(C_1\cos 2x + C_2\sin 2x)$$

P.I.: The trial solution.

$$y_p = e^{3x}(A_1\cos 2x + A_2\sin 2x)\,x.$$