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Equation reducible to exact form (Contd)

$2y\,dx - x\,dy$

Case V :- $f\,dx + g\,dy = 0$ is not exact i.e. $\frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}$

$$x^a y^b(my\,dx + nx\,dy) + x^{a'} y^{b'}(m'y\,dx + n'x\,dy) = 0 \quad —(1)$$

I.F. . $x^h y^k$

Multiply $x^h y^k$ to eq (1)

$$x^{a+h} y^{b+k}(my\,dx + nx\,dy) + x^{a'+h} y^{b'+k}(m'y\,dx + n'x\,dy) = 0 \quad —(2)$$

Now (2) is exact.

$$\Rightarrow \underbrace{\left(m x^{a+h} y^{b+k+1} + m' x^{a'+h} y^{b'+k+1}\right)}_{F} dx + \underbrace{\left(n x^{a+h+1} y^{b+k} + n' x^{a'+h+1} y^{b'+k}\right)}_{G} dy = 0$$

which is of form $F\,dx + G\,dy = 0$

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} \quad (\because \text{eq is exact})$$

$$\frac{\partial}{\partial y}\left(m x^{a+h} y^{b+k+1} + m' x^{a'+h} y^{b'+k+1}\right) = \frac{\partial}{\partial x}\left(n x^{a+h+1} y^{b+k} + n' x^{a'+h+1} y^{b'+k}\right)$$

$$\Rightarrow m x^{a+h}(b+k+1) y^{b+k} + m' x^{a'+h}(b'+k+1) y^{b'+k}$$
$$= n y^{b+k}(a+h+1) x^{a+h} + n' y^{b'+k}(a'+h+1) x^{a'+h}$$

$$\Rightarrow x^{a+h} y^{b+k}\left(m(b+k+1)\right) + x^{a'+h} y^{b'+k}\left(m'(b'+k+1)\right)$$

$\Rightarrow x^{a+h} y^{b+k}(m(b+k+1)) + x^{a'+h} y^{b'+k}(m'(b'+k+1))$

$= x^{a+h} y^{b+k}(n(a+h+1)) + x^{a'+h} y^{b'+k}(n'(a'+h+1))$

On comparing the coeff the like term on both sides

$m(b+k+1) = n(a+h+1)$ | $m'(b'+k+1) = n'(a'+h+1).$

or $\Rightarrow \dfrac{a+h+1}{m} = \dfrac{b+k+1}{n}.$ | or. $\dfrac{a'+h+1}{m'} = \dfrac{b'+k+1}{n'}.$

Q: Solve the differential eq. $(y^2+2x^2y)dx + (2x^3-xy)dy = 0.$ —(1)

Sol: Compare it with $f dx + g dy = 0$

$$f = y^2+2x^2y, \quad g = 2x^3-xy$$

$$\frac{\partial f}{\partial y} = 2y+2x^2, \quad \frac{\partial g}{\partial x} = 6x^2-y.$$

Clearly $\dfrac{\partial f}{\partial y} \neq \dfrac{\partial g}{\partial x} \Rightarrow$ eq is not exact.

$$\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = 3y-4x^2$$

Case I is not applicable
Case II is not applicable
Case III is not applic eq is not homogeneous
Case IV is not applicable

$(y^2dx - xy\,dy) + (2x^2y\,dx + 2x^3dy) = 0$

$x^0y^1(y\,dx - x\,dy) + x^2y^0(2y\,dx + 2x\,dy) = 0$

$$x^{0}y^{1}(y\,dx - x\,dy) + x^{2}y^{0}(2y\,dx + 2x\,dy) = 0$$

It is of the form.

$$x^{a}y^{b}(my\,dx + nx\,dy) + x^{a'}y^{b'}(m'y\,dx + n'x\,dy) = 0$$

$a = 0, b = 1, m = 1, n = -1, a' = 2, b' = 0, m' = 2, n' = 2.$

I.F.: $x^{h}y^{k}$, where

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

$$\Rightarrow \underbrace{\frac{0+h+1}{1} = \frac{1+k+1}{-1}} \quad \Big| \quad \frac{2+h+1}{\cancel{2}} = \frac{0+k+1}{\cancel{2}}.$$

(a) 2, 2 (b) −2, −2 (c). 3, 2 (d) $-\frac{3}{2}, -\frac{1}{2}$.

(e). None of these.

$$h+1 = -(k+2) \qquad h+3 = k+1$$

$$\Rightarrow \quad h+k+3 = 0. \qquad h-k+2 = 0$$

$$h-k+2 = 0$$

$$2h+5 = 0 \Rightarrow \boxed{h = -\frac{5}{2}}$$

$$h+k+3 = 0$$

$$k = -3-h$$

$$= -3+\frac{5}{2}$$

$$\boxed{k = -\frac{1}{2}}$$

I.F. $x^{h}y^{k} = x^{-5/2}y^{-1/2}.$

On Multiplying eq(1) with I.F. we get

$$x^{-5/2}y^{-1/2}(y^2+2x^2y)dx + x^{-5/2}y^{-1/2}(2x^3-xy)dy = 0$$

$$\Rightarrow \left(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2}\right)dx + \left(2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}\right)dy = 0$$

Which is of the form $F\,dx + G\,dy = 0$

$F = x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}$ | $G = 2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}$.

$$\frac{\partial F}{\partial y} = x^{-5/2}\,\frac{3}{2}y^{1/2} + 2x^{-1/2}\frac{1}{2}y^{-1/2}$$

$$= \frac{3}{2}x^{-5/2}y^{1/2} + x^{-1/2}y^{-1/2}$$

$$\frac{\partial G}{\partial x} = x^{-1/2}y^{-1/2} + \frac{3}{2}x^{-5/2}y^{1/2}.$$

Clearly $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} \Rightarrow$ eq. is exact.

Sol is. $\displaystyle\int_{y\text{ constant}} F\,dx + \int_{\substack{\text{Term of } G \\ \text{not containing } x}} G\,dy = C.$

$$-x^{-3/2}y^{3/2} + 2x^{1/2}y^{+1/2} = C.$$

$$\int_{y\text{ constant}} \left(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}\right) + \int 0\,dy = c.$$

$$\Rightarrow y^{3/2}\cdot\frac{x^{-3/2}}{-3/2} + \frac{2x^{1/2}}{1/2}y^{1/2} = C$$

$$\Rightarrow \boxed{-\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = c}$$

(i) $(x^3y^2 + x)dy + (x^2y^3 - y)dx = 0$ ~~Case III is not applied.~~

Sol: $f\,dx + g\,dy = 0$, $f = x^2y^3 - y$, $g = x^3y^2 + x$.

$$\frac{\partial f}{\partial y} = 3x^2y^2 - 1, \quad \frac{\partial g}{\partial x} = 3x^2y^2 + 1$$

$$\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = \cancel{3x^2y^2} - 1 - \cancel{3x^2y^2} - 1 = -2.$$

$$x\,dy - y\,dx + \underline{x^3y^2}\,dy + \underline{x^2y^3}\,dx = 0$$

$$\Rightarrow x\,dy - y\,dx + x^2y^2(x\,dy + y\,dx) = 0$$

$$\Rightarrow \underbrace{x\,dy - y\,dx} + \underbrace{x^2y^2\, d(xy)}_{(xy)^2 d(xy)} = 0$$

~~(a)~~ $\frac{1}{x^2}$ ~~(b)~~ $\frac{1}{y^2}$ ~~(c)~~ $\frac{1}{xy}$ (d) $\frac{1}{x^2+y^2}$ (e) $\frac{1}{x^2-y^2}$.

$$\frac{x\,dy - y\,dx}{xy} + \frac{(xy)^{\cancel{2}}}{\cancel{(xy)}}\,d(xy) = 0$$

$$\Rightarrow \frac{dy}{y} - \frac{dx}{x} + d(xy) = 0$$

Integrating

$y \qquad x$

Integrating

$$\int \frac{dy}{y} - \int \frac{dx}{x} + \int d(xy) = C.$$

$$\Rightarrow \ln|y| - \ln|x| + xy = C$$

$$\Rightarrow \ln\left|\frac{y}{x}\right| + xy = C.$$

$$x(x^2y^2+1)dy + y(x^2y^2-1)dx = 0$$

I.F. $\dfrac{1}{fx-gy} = \dfrac{1}{xy(\cancel{x^2y^2}+1) - xy(\cancel{x^2y^2}-1)}$

$$= \frac{1}{2xy}.$$

Q.: $(x^3y^2dy + x^2y^3dx) + (xdy - ydx) = 0.$

$x^2y^2\underline{(xdy+ydx)} + x^0y^0(xdy - ydx) = 0.$

$\cancel{(xy)^2(d(xy))}$

$x^ay^b(mydx + nxdy) + x^{a'}y^{b'}(m'ydx + n'xdy) = 0$

$a=2, b=2, m=1, n=1, \quad a'=0, b'=0, m'=-1, n'=1.$

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

$$\frac{\cancel{2}+h+\cancel{1}}{} = \frac{\cancel{2}+k+\cancel{1}}{}, \quad \frac{0+h+1}{} = \frac{0+k+1}{}$$

$$\frac{\cancel{2}+h+\cancel{1}}{\cancel{1}} = \frac{\cancel{2}+k+\cancel{1}}{\cancel{1}} \quad , \quad \frac{0+h+1}{-1} = \frac{0+k+1}{1}$$

$$\boxed{h=k}$$

$$h+1 = -(k+1)$$

$$h+k+2 = 0$$

$$2k+2 = 0$$

$$k = -1$$

$$h = -1$$

I.F $x^h y^k = x^{-1} y^{-1} = \frac{1}{xy}$.