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Diff. equation of first order & higher degree :-

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \quad -(1) \qquad , \boxed{p = \frac{dy}{dx}}$$

Case I :- eq (1) is in the form of

$$a_0 p^n + a_1 p^{n-1} + \cdots + a_n = 0, \quad (2)$$

here $a_0, a_1, \ldots$ are either constant or function of x & y.

$$(p - f_1(x,y))(p - f_2(x,y)) \cdots (p - f_n(x,y)) = 0$$

$$p - f_1(x,y) = 0, \quad p - f_2(x,y) = 0, \ldots, \quad p - f_n(x,y) = 0$$

here $\boxed{p = \frac{dy}{dx}}$

$\frac{dy}{dx} - f_1(x,y) = 0$	$\frac{dy}{dx} - f_2(x,y) = 0$	$\cdots$	$\frac{dy}{dx} - f_n(x,y) = 0$
Sol. $F_1(x, y, c) = 0$	$F_2(x, y, c) = 0$	$\cdots$	$F_n(x, y, c) = 0$

Complete sol. of eq (2)

$$F_1(x, y, c) \cdot F_2(x, y, c) \cdots F_n(x, y, c) = 0$$

Note :- general sol. should not contain more than one arbitrary const. $\because$ the given differential equation is of first order.

Q1. Solve the differential equation.

$$y\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}-x=0 \quad -(1)$$

Sol: Put $\frac{dy}{dx}=p$ in eq (1)

$$(1) \Rightarrow \quad yp^2+(x-y)p-x=0$$

$$\Rightarrow \underbrace{yp^2+xp}-\underbrace{yp-x}=0$$

$$\Rightarrow p[yp+x]-1[yp+x]=0$$

$$\Rightarrow [yp+x][p-1]=0$$

either $yp+x=0$ or $p-1=0$

$\Rightarrow p=\frac{-x}{y}$, $p=1$.

$\frac{dy}{dx}=\frac{-x}{y}$.	$\frac{dy}{dx}=1$.
$\Rightarrow y\,dy=-x\,dx$	$dy=dx$.

On integrating, we will get

$\int y\,dy=-\int x\,dx$	$\int dy=\int dx$.
$\frac{y^2}{2}=-\frac{x^2}{2}+c$	$y=x+c$.
$\Rightarrow y^2+x^2-2c=0$	$y-x-c=0$

$\Rightarrow\ y^2+x^2-2c=0 \quad\Big|\quad y-x-c=0$

General sol: $(y^2+x^2-2c)(y-x-c)=0$

Q. Solve the diff. eq.

$$p^2+py=x(x+y) \; ; \; p=\frac{dy}{dx}.$$

Sol. (a) $(p-x)(p-y)=0$ (b) $(p+x)(p-x+y)=0$

(c) $(p-x)(p+y)=0$ (d) $(p+x)(p+x+y)=0$

(e) none of these.

$\Rightarrow p^2+py=x^2+xy.$

$\Rightarrow \underbrace{p^2-x^2}+py-xy=0$

$\Rightarrow (p-x)(p+x)+y(p-x)=0$

$\Rightarrow (p-x)\left[p+x+y\right]=0$

$p-x=0$ or $p+x+y=0$

$\frac{dy}{dx}-x=0$ or $\frac{dy}{dx}+y+x=0$

$\Rightarrow \frac{dy}{dx}=x.$

$\Rightarrow \int dy=\int x\,dx.$

$\Rightarrow y=\frac{x^2}{2}+c.$

$(\Rightarrow)\ 2y-x^2-c=0$

$\Rightarrow \frac{dy}{dx}+y=-x.$

$\frac{dy}{dx}+Py=Q$, $P=1, Q=-x.$

$I.F.\ e^{\int P dx}=e^{\int 1 dx}=e^x.$

$e^x\frac{dy}{dx}+e^x y=-xe^x.$

$\Rightarrow \frac{d}{dx}(e^x y)=-x e^x.$

$h(x)=1$

$I.F.\ e^{\int h(x)dx}=e^x.$

$dy+(x+y)dx=0$

$f=x+y, g=1$

$\frac{\partial f}{\partial y}=1, \frac{\partial g}{\partial x}=0$

$\frac{\frac{\partial f}{\partial y}-\frac{\partial g}{\partial x}}{g}=\frac{1}{1}=h(x)$

$$\Rightarrow y - \frac{x^2}{2} - c = 0$$

$$y + (x-1) - C_1 e^{-x} = 0$$

complete sol. is ✗

$$\left(y - \frac{x^2}{2} - c\right)\left(y + (x-1) - C_1 e^{-x}\right) = 0$$

$$\Rightarrow \frac{d}{dx}(e^x y) = -x e^x$$

On integrating

$$\int \frac{d}{dx}(e^x y)\,dx = -\int x e^x\,dx$$

$$\Rightarrow e^x y = -(x e^x - (1)e^x) + C$$

$$\Rightarrow e^x y = -(x-1)e^x + C$$

$$\Rightarrow y = -(x-1) + C e^{-x}.$$

Constant should be same.

Complete sol. $\left(y - \frac{x^2}{2} - c\right)\left(y + (x-1) - c e^{-x}\right) = 0$

Case II: If $f(x, y, p) = 0$ can't be factorize as product of linear factor of p.

$$y = g(x, p)$$

$$p = \frac{dy}{dx} = G\left(x, p, \frac{dp}{dx}\right)$$

$$\Rightarrow G_1\left(x, p, \frac{dp}{dx}\right) = 0$$

$$\Rightarrow G_2(p, x, c) = 0$$

$$\Rightarrow p = G_3(x, c)$$

Sol. is $\boxed{y = g(x, G_3(x, c))}$

$$x = h(y, p)$$

$$\frac{1}{p} = \frac{dx}{dy} = H\left(y, p, \frac{dp}{dy}\right)$$

$$\Rightarrow H_1\left(y, p, \frac{dp}{dy}\right) = 0$$

$$\Rightarrow h_2(p, y, c) = 0$$

$$\Rightarrow p = H_3(y, c)$$

sol. is $\boxed{x = h(y, H_3(y, c))}$

if it is not possible to eliminate p

If it is not possible to eliminate p

$$\boxed{\begin{array}{l} x = K(p, c) \\ y = K_1(p, c) \end{array}}$$ is called parametric sol. of given diff-eq.

Q) Solve the differential equation.

$$\underbrace{y + px - x^4 p^2}_{} = 0 \qquad y = x^4 p^2 - px \quad \text{--- (1)}$$

Sol: (a) factorize ~~(b)~~ $y = f(x, p)$ (c) $x = g(y, p)$

$$y = \underbrace{x^4 p^2} - px.$$

diff. w.r.t x.

$$p = \frac{dy}{dx} = x^4\left(2p\frac{dp}{dx}\right) + p^2(4x^3) - \left(p(1) + x\frac{dp}{dx}\right)$$

$$\Rightarrow p = 2px^4\frac{dp}{dx} + 4p^2x^3 - p - x\frac{dp}{dx}.$$

$$\Rightarrow \frac{dp}{dx}x(2px^3 - 1) + 4p^2x^3 - 2p = 0$$

$$\Rightarrow (2px^3 - 1)x\frac{dp}{dx} + 2p(2px^3 - 1) = 0$$

$$\Rightarrow \underbrace{(2px^3 - 1)}_{\times}\boxed{\left(x\frac{dp}{dx} + 2p\right)} = 0$$

discarding the factor $(2px^3-1)$

$$x\frac{dp}{dx}+2p=0 \text{ or } x\frac{dp}{dx}=-2p$$

$$\Rightarrow \int\frac{dp}{p}=-2\int\frac{dx}{x}.$$

$$\Rightarrow \ln p=-2\ln x+c.$$

$$\Rightarrow \ln p=\ln x^{-2}+c$$

$$\Rightarrow p=e^{\ln x^{-2}+c}=x^{-2}.e^{c}$$

$$\Rightarrow \boxed{p=x^{-2}A}, \quad \boxed{A=e^{c}}$$

Sol: of eq (1) is given by:

$$y=p^2x^4-px=(x^{-2}A)^2.x^4-(x^{-2}A).x.$$

This is $\boxed{y=A^2-\frac{A}{x}}$, where A is an arbitrary constant.