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Linear combination of the functions :-

Let $f_1, f_2, f_3, ----, f_n$ be 'n' functions, then the linear combination of functions is given by

$$c_1f_1 + c_2f_2 + --- + c_nf_n$$, where $c_1, c_2, --, c_n$ are arbitrary constants

eg:- x^2, $x-1$, x^3

Linear combination is $c_1x^2 + c_2(x-1) + c_3x^3$

Linearly dependent & independent functions :-

Let $f_1, f_2, f_3, -- , f_n$ be n functions. Then these functions are said to be linearly independent over interval I if their linear combination equal to zero i.e.

$$c_1f_1 + c_2f_2 + ---. + c_nf_n = 0$$

implies $c_1 = c_2 = -- = c_n = 0$

& if $\exists$ some constants (not all zero) such that $\boxed{c_1f_1} + c_2f_2 + -- + c_nf_n = 0$, then $f_1, f_2, ---, f_n$ are said to linearly dependent

Or:- If any one or more functions can be expressed as linear combination of others

i.e. Let $c_1 \neq 0$

$$c_1f_1 = -(c_2f_2 + c_3f_3 + -- + c_nf_n)$$

$c_1 f_1 = -(c_2 f_2 + c_3 f_3 + \dots + c_n f_n)$

$$\boxed{f_1 = -\frac{1}{c_1}(c_2 f_2 + c_3 f_3 + \dots + c_n f_n)}$$

Eg: $3x+2, \quad 5x+3, \quad 7x+8$

Consider. $f_1 = 3x+2, \; f_2 = 5x+3, \; f_3 = 7x+8$

Let $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$ — (*)

$\Rightarrow c_1(3x+2) + c_2(5x+3) + c_3(7x+8) = 0$

$\Rightarrow x(3c_1 + 5c_2 + 7c_3) + (2c_1 + 3c_2 + 8c_3) = 0$

$\Rightarrow 3c_1 + 5c_2 + 7c_3 = 0$, Let c_3 be parameter.

$2c_1 + 3c_2 + 8c_3 = 0 \Rightarrow$

$2\times \quad 3c_1 + 5c_2 = -7c_3$ — (1)

$3\times \quad 2c_1 + 3c_2 = -8c_3$ — (2)

(1) $\Rightarrow \quad \cancel{6c_1} + 10c_2 = -14c_3$

(2) $\Rightarrow \quad \underline{-\cancel{6c_1} \mp 9c_2 = \mp 24c_3}$

$$\boxed{c_2 = 10c_3}$$

(2) $\Rightarrow \quad 2c_1 + 3(10c_3) = -8c_3$

$2c_1 = -8c_3 - 30c_3 = -38c_3$

$$\boxed{c_1 = -19c_3}$$

(*) $\Rightarrow \quad c_1(3x+2) + c_2(5x+3) + c_3(7x+8) = 0$

$\Rightarrow -19c_3(3x+2) + 10c_3(5x+3) + c_3(7x+8) = 0$

$\Rightarrow \cancel{c_3}\left(-19(3x+2)+10(5x+3)+1(7x+8)\right)=0$

$\Rightarrow -19,\ 10\ \&\ 1$

$\cancel{-57x}\ \cancel{-38}+\cancel{50x}+\cancel{30}+\cancel{7x}+\cancel{8}$

$\Rightarrow$ $(3x+2)$, $(5x+3)$ & $(7x+8)$ are linearly dependent.

Consider $c_1f_1+c_2f_2+\cdots+c_nf_n=0$

here $c_1, c_2, \ldots, c_n$ are unknown constant.

$\because$ Here $f_1, f_2, \ldots, f_n$ are n functions of x.

& if they are continuously differentiable.

$$c_1f_1+c_2f_2+\cdots+c_nf_n=0 \quad \text{--- } \circledast$$

diff eq $\circledast$ w.r.t x.

$$c_1f_1'+c_2f_2'+\cdots+c_nf_n'=0$$

$$c_1f_1''+c_2f_2''+\cdots+c_nf_n''=0$$

we have to differentiate $(n-1)$ times

$$\text{— — — — — — —}$$

$$c_1f_1^{n-1}+c_2f_2^{n-1}+\cdots+c_nf_n^{n-1}=0$$

$$\Rightarrow \quad Ax=0$$

$$A=\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ - & - & - & - \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1}\end{bmatrix},\quad x=\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n\end{bmatrix}$$

A homogeneous system of eq. ie $Ax=0$

A homogeneous system of eq. i.e $AX=0$ has trivial sol iff $|A| \neq 0$.

$\because |A| \neq 0 \Rightarrow A^{-1}$ exist

$AX=0$

$\boxed{X = A^{-1}0 = 0}$

for infinitely many sol. $|A| = 0$

If $W(f_1, f_2, \dots, f_n) = W(x) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ f_1^{n-1} & f_2^{n-1} & \dots & f_n^{n-1} \end{vmatrix}$

is called Wronskian of $f_1, f_2, \dots, f_n$

(a) if $W(f_1, f_2, \dots, f_n) = 0 \Rightarrow f_1, f_2, \dots, f_n$ are linearly dependent.

(b) if $W(f_1, f_2, \dots, f_n) \neq 0 \Rightarrow f_1, f_2, \dots, f_n$ are linearly independent.

Q. Let $f_1 = 3x+2$, $f_2 = 5x+3$, $f_3 = 7x+8$

Check linearly dependent or independent??

Sol. $W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$

Yes — L.D.
No — L.I.

$$= \begin{vmatrix} 3x+2 & 5x+3 & 7x+8 \\ 3 & 5 & 7 \\ 0 & 0 & 0 \end{vmatrix}$$

$= 0$

$\Rightarrow W(f_1, f_2, f_3) = 0 \Rightarrow f_1, f_2 \,\&\, f_3$ are linearly dependent

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Q: $f_1 = x^2 - x$, $f_2 = 3x^2 - x + 2$, $f_3 = 9x^2 + x + 8$

L.D or L.I.

Sol: $W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$

$$= \begin{vmatrix} x^2 - x & 3x^2 - x + 2 & 9x^2 + x + 8 \\ 2x - 1 & 6x - 1 & 18x + 1 \\ 2 & 6 & 18 \end{vmatrix}$$

$C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 9C_1$

$$= \begin{vmatrix} x^2 - x & 2x + 2 & 10x + 8 \\ 2x - 1 & 2 & 10 \\ 2 & 0 & 0 \end{vmatrix}$$

Expand through R_3

$= 2\left((2x+2)10 - 2(10x+8)\right)$

$= 2\left(2\cancel{0}x + 20 - \cancel{20}x - 16\right)$

$= 2(2\cancel{0}x+20 - \cancel{20}x-16)$

$= 8 \neq 0$

$\therefore W(f_1,f_2,f_3) \neq 0 \Rightarrow f_1, f_2 \& f_3$ are linearly independent.

Q: $f_1 = \ln x$, $f_2 = \ln x^2$, $f_3 = \ln x^3$

(a) L.D (b) L.I.

Sol. $f_1 = \ln x$, $f_2 = 2\ln x$, $f_3 = 3\ln x$

$3\ln x = 2\ln x + \ln x$.

$\ln x^3 = \ln x^2 + \ln x$.

$\boxed{f_3 = f_2 + f_1}$

$\Rightarrow f_1, f_2 \& f_3$ are linearly dependent

$$W(f_1,f_2,f_3) = \begin{vmatrix} \ln x & 2\ln x & 3\ln x \\ \frac{1}{x} & \frac{2}{x} & \frac{3}{x} \\ -\frac{1}{x^2} & -\frac{2}{x^2} & -\frac{3}{x^2} \end{vmatrix}$$

$$= 2\times 3 \begin{vmatrix} \ln x & \ln x & \ln x \\ 1/x & 1/x & 1/x \\ -1/x^2 & -1/x^2 & -1/x^2 \end{vmatrix}$$

$C_1 \& C_2 \& C_3$ are identical

~0.