Unit-4 Partial Differential Equation.

Textbook :- Advanced Engineering Mathematics

Topic 1 :- Formation of PDE :-

Ex:- 16.1 :-

ODE :- $F(x, y, c_1, c_2, ---, c_n) = 0$

$\Rightarrow F_1\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, --, \frac{d^ny}{dx^n}\right) = 0$

Formation of ODE :- The order of DE is equal to no. of arbitrary constant present in the expression.

Formation of PDE :- Case I :- If the expression will contain arbitrary constant :-

If the number of arbitrary constant & the no. of independent variables are same then order of partial diff. eq. should not exceed 1

$F(x, y, z, a, b) = 0$ $\qquad \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Case II :- If the expression containing arbitrary function

The order of PDE should not exceed the no. of arbitrary function.

Q: Eliminate the arbitrary constant to obtain PDE :- $Z = (x+c)(y+d)$ — (*)

Sol:-

$$\frac{\partial z}{\partial x} = (y+d)\frac{\partial}{\partial x}(x+c)$$

$$\boxed{\frac{\partial z}{\partial x} = (y+d)\,(1)} \quad \text{— (1)}$$

$$\boxed{\begin{aligned} \frac{dy}{dx} &= P \\ \frac{\partial z}{\partial x} &= p \\ \frac{\partial z}{\partial y} &= q \end{aligned}}$$

$$\frac{\partial z}{\partial y} = (x+c)\frac{\partial}{\partial y}(y+d)$$

$$\frac{\partial z}{\partial y} = (x+c)\,(1) \quad \text{— (2)}$$

(1) $\Rightarrow$ $p = y+d$,   (2) $\Rightarrow$ $q = x+c$.

$$\boxed{d = p-y} \qquad C = q-x$$

(*) $Z = (x+q-x)(y+p-y)$

$\boxed{Z = pq}$ which is required PDE.

Q: $Z = (x+ay)^2 + by$ — (1)

Sol: $\dfrac{\partial z}{\partial x} = 2(x+ay)\dfrac{\partial}{\partial x}(x+ay) + 0$

$\dfrac{\partial z}{\partial x} = 2(x+ay)(1) \Rightarrow \boxed{p = 2(x+ay)}$ —(2)

again $\dfrac{\partial z}{\partial y} = 2(x+ay)\cdot\dfrac{\partial}{\partial y}(x+ay) + b$

$= 2(x+ay)(0+a) + b$

$\Rightarrow \boxed{q = 2(x+ay)\cdot a + b}$ —(3)

eq (2) $p = 2(x+ay) \Rightarrow \dfrac{p}{2} = x+ay \Rightarrow \boxed{a = \dfrac{1}{y}\left(\dfrac{p}{2} - x\right)}$

eq (3) $q = 2(x+ay)\cdot a + b$

$\Rightarrow b = q - 2(x+ay)\cdot a$

$\Rightarrow \boxed{b = q - \dfrac{p}{y}\left(\dfrac{p}{2} - x\right)}$

(1) $\Rightarrow$ $z = \left(\dfrac{p}{2}\right)^2 + \left(q - \dfrac{p}{y}\left(\dfrac{p}{2} - x\right)\right)y$

$= \dfrac{p^2}{4} + qy - p\left(\dfrac{p}{2} - x\right)$

$z = \dfrac{p^2}{4} + qy - \dfrac{p^2}{2} + px$

$\Rightarrow \boxed{z = px + qy - \dfrac{p^2}{4}}$

Q1 $(x-a)^2 + y^2 + (z-b)^2 = 16.$ — (1)

(z as function of x & y)

$F(x,y,z) = c.$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Sol: Let $f(x,y,z) = (x-a)^2 + y^2 + (z-b)^2.$

$$\frac{\partial f}{\partial x} = 2(x-a), \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2(z-b)$$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = \frac{-\cancel{2}(x-a)}{\cancel{2}(z-b)} \Rightarrow \boxed{p = -\frac{(x-a)}{(z-b)}}$$

$$\frac{\partial z}{\partial y} = \frac{-f_y}{f_z} = -\frac{\cancel{2}y}{\cancel{2}(z-b)} \Rightarrow \boxed{q = -\frac{y}{z-b}}$$

$$\boxed{z-b = -\frac{y}{q}}$$

$$(x-a) = -p(z-b)$$

$$\Rightarrow x-a = -p\left(\frac{-y}{q}\right)$$

$$\Rightarrow \boxed{x-a = \frac{py}{q}}$$

(1) $\Rightarrow$ $$\left(\frac{py}{q}\right)^2 + y^2 + \left(\frac{-y}{q}\right)^2 = 16$$

$$\Rightarrow \frac{p^2y^2}{q^2} + y^2 + \frac{y^2}{q^2} = 16$$

$$\Rightarrow \frac{p^2 y^2}{q^2} + y^2 + \frac{y^2}{q^2} = 16$$

$$\Rightarrow \frac{p^2y^2 + q^2y^2 + y^2}{q^2} = 16$$

$$\Rightarrow \boxed{(p^2+q^2+1)\,y^2 = 16q^2}$$ which is required PDE.

Q) Eliminate the arbitrary function to obtain PDE. $x^2+y^2+z^2 = g(2x-y)$ — ① (z as function of x, y)

Sol. diff. of ① w.r.t x.

$$2x + 0 + 2z\frac{\partial z}{\partial x} = g'(2x-y)\cdot\frac{\partial}{\partial x}(2x-y)$$

$$\Rightarrow \cancel{2}(x+zp) = g'(2x-y)\cdot\cancel{2}.$$

$$\Rightarrow x+zp = g'(2x-y) \quad \text{—— ②}$$

diff of ① w.r.t y.

$$0 + 2y + 2z\frac{\partial z}{\partial y} = g'(2x-y)\cdot\frac{\partial}{\partial y}(2x-y)$$

$$\Rightarrow 2(y+zq) = g'(2x-y)(-1) \quad \text{—③}$$

from ② & ③

$$2(y+zq) = -(x+zp)$$

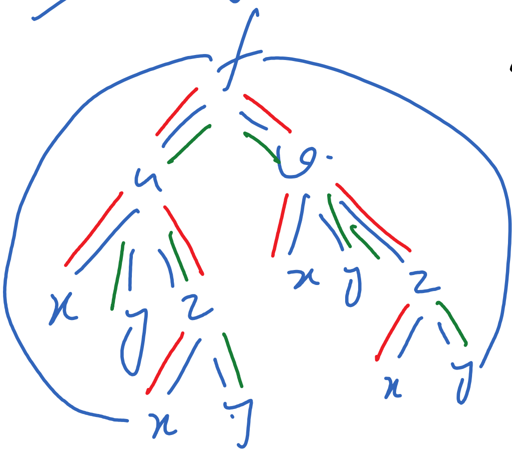
$$\Rightarrow 2y+2zq+x+zp = 0$$

$$\Rightarrow \quad 2y + 2zq + x + zp = 0$$

$$\Rightarrow \quad \boxed{x + 2y + z(p + 2q) = 0}$$

Q. $f(x+yz,\ x^2+y^2-z^2) = 0$

Sol. $f(u,v) = 0$, $u = x + yz$, $v = x^2+y^2-z^2$



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)$$

$$0 = \frac{\partial f}{\partial u}(1 + yp) + \frac{\partial f}{\partial v}(2x + (-2z)p)$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial u}(1+yp) + \frac{\partial f}{\partial v}(2x - 2zp) = 0} \quad -(1)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right)$$

$$0 = \frac{\partial f}{\partial u}(z + yq) + \frac{\partial f}{\partial v}(2y - 2zq)$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial u}(z+yq) + \frac{\partial f}{\partial v}(2y - 2zq) = 0} \quad -(2)$$

We know that a homogeneous system of equations have non-trivial sol. iff

$$\left| \begin{matrix} 1+yp & 2x-2zp \end{matrix} \right|$$

$$\begin{vmatrix} 1+yp & 2x-2zp \\ z+yq & 2y-2zq \end{vmatrix} = 0$$

$$\Rightarrow (1+yp)\,2(y-zq) - (z+yq)\,2(x-zp) = 0$$

$$\Rightarrow \boxed{(1+yp)(y-zq) - (z+yq)(x-zp) = 0}$$

$$\Rightarrow y - zq + y^2p - \cancel{yzpq} - zx + z^2p - xyq + \cancel{yzpq} = 0$$

$$\Rightarrow y - zq - zx + y^2p + z^2p - xyq = 0$$

$$\Rightarrow \boxed{y - z(x+q) + p(y^2+z^2) - xyq = 0}$$

Which is required PDE.