22 January 2021 08:38

Unit-I :- Ordinary differential equation of I order.

Exact Differential equation :-

A differential equation $f\,dx + g\,dy = 0$ is said to be an exact differential equation, if it can be expressed as $du = 0$

i.e $f\,dx + g\,dy = du$, where u is a differentiable function & having continuous partial derivatives

Sol: $\boxed{u(x, y) = c}$ c is constant of integration.

Necessary & Sufficient condition for exactness of differential equation :- $f\,dx + g\,dy = 0$

Let $f\,dx + g\,dy = 0$ is exact

$\Rightarrow f\,dx + g\,dy = du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$

$\Rightarrow f = \frac{\partial u}{\partial x}, \quad g = \frac{\partial u}{\partial y}.$

$\frac{\partial f}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial g}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$

$\because$ partial derivatives are continuous

$\boxed{\frac{\partial^2 u}{\quad} = \frac{\partial^2 u}{\quad}} \Rightarrow \boxed{\frac{\partial f}{\quad} = \frac{\partial g}{\quad}}$

$$\boxed{\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}} \Rightarrow \boxed{\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}}$$

If $\boxed{\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}} \Rightarrow$ $f\,dx + g\,dy = 0$ is an exact diff. eq.

Proof: Let $\boxed{u = \int_{y\text{ const}} f\,dx}$ $\qquad u = \int f\,dx.$ $\quad \frac{du}{dx} = f.$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = f}$$

$$\Rightarrow \frac{\partial f}{\ } \quad \frac{\ }{\partial y \partial x} = \frac{\partial g}{\partial x}.$$

$$\frac{\partial g}{\partial x} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \quad (\because \text{partial derivatives are continuous})$$

$$\Rightarrow \frac{\partial g}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

$$\Rightarrow g = \int_{y\text{ const}} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) dx + K(y)$$

$$\Rightarrow \boxed{g = \frac{\partial u}{\partial y} + K(y).}$$

Given diff eq is $f\,dx + g\,dy = 0.$

$$\Rightarrow \frac{\partial u}{\partial x} dx + \left(\frac{\partial u}{\partial y} + K(y)\right) dy = 0$$

$$\frac{\partial x}{\partial x}dx + \left(\frac{\partial}{\partial y} \ldots\right)$$

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy} \Rightarrow \underbrace{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy}_{} + K(y)dy = 0$$

$$\Rightarrow du + K(y)dy = 0$$

$$du + d\left(\int K(y)dy\right) = 0, \quad d\left(\underline{u + \int K(y)dy}\right) = 0$$

$$\Rightarrow f\,dx + g\,dy = 0 \text{ is an exact diff. eq.}$$

$$\sin y\,dy = -d(\cos y) = d\left(\int \sin y\,dy\right)$$

$f\,dx + g\,dy = 0$ is an exact diff. eq.

$$d\left(u + \int K(y)dy\right) = 0$$

on integrating, we get

$$u + \int K(y)dy = C.$$

sol. is

$$\boxed{\int_{y\ \text{const}} f\,dx + \int_{\substack{\text{Term of } g \\ \text{not containing } x}} g\,dy = C}$$

Q: Check whether the eq. is exact if so find the general sol.

$$\underline{(3x^2y + y/x)}\,dx + (\underbrace{x^3 + \ln x}_{g})\,dy = 0 \quad \text{— (i)}$$

$$(3x^2y + y/x)\,dx + (x^3 + \ln x)\,dy = 0 \quad \text{—(1)}$$

Sol; Compare it with $f\,dx + g\,dy = 0$

$$f = 3x^2y + y/x, \quad g = x^3 + \ln x.$$

$$\frac{\partial f}{\partial y} = 3x^2(1) + \frac{1}{x}(1) = 3x^2 + 1/x. \quad \Bigg| \quad \frac{\partial g}{\partial x} = 3x^2 + \frac{1}{x}$$

clearly $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Rightarrow$ eq(1) is exact.

sol. is

$$\underset{y\ \text{constant}}{\int} f\,dx + \underset{\substack{\text{Term of } g \\ \text{not containing } x}}{\int} g\,dy = C.$$

$$\Rightarrow \underset{y\ \text{constant}}{\int} \left(3x^2y + y/x\right)dx + \int 0\,dy = C.$$

$$\Rightarrow y \underset{y\ \text{constant}}{\int} \left(3x^2 + \frac{1}{x}\right)dx = C.$$

$$\Rightarrow \boxed{y\left(x^3 + \ln x\right) = C}$$

Q. $(x e^{xy} + 2y)\,dy + (y e^{xy})\,dx = 0$ —(1) is exact ??

Sol, Compare (1) with $f\,dx + g\,dy = 0$

$$f = y e^{xy}, \quad g = x e^{xy} + 2y$$

$f = y e^{xy}$, $g = x e^{xy} + 2y$

$$\frac{\partial f}{\partial y} = y \, e^{xy} \frac{\partial}{\partial y}(xy) + e^{xy}(1)$$

$$= xy \, e^{xy}(1) + e^{xy} = e^{xy}(xy+1)$$

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(x e^{xy} + 2y\right) = x \, e^{xy} \frac{\partial}{\partial x}(xy) + e^{xy}(1) + 0$$

$$= xy \, e^{xy}(1) + e^{xy} = e^{xy}(xy+1)$$

$\because \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Rightarrow$ eq (1) is exact

hence sol. is given by

$$\underset{y \text{ constant}}{\int} f\,dx + \int \underset{\text{Term of } g \text{ not containing } x}{g \, dy} = c.$$

$$\underset{y \text{ constant}}{\int} y e^{xy} \, dx + \int 2y \, dy = c$$

$$\Rightarrow y \left[\frac{e^{xy}}{\cancel{y}} \right] + \cancel{2} \frac{y^2}{\cancel{2}} = c$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + c$$

$$\Rightarrow \boxed{e^{xy} + y^2 = c}$$

$$(x e^{xy} + 2y)dy + y e^{xy} dx = 0$$

$$\Rightarrow \underline{y e^{xy} dx + x e^{xy} dy} + 2y \, dy = 0$$

$$= e^{xy}(y\,dx + x.dy) + 2y\,dy = 0$$

$$\Rightarrow e^{xy}\,(d(xy)) + 2y\,dy = 0.$$

$$\Rightarrow d\left(\int e^{xy}\,d(xy) + \int 2y\,dy\right) = 0$$

$$\Rightarrow \boxed{e^{xy} + y^2 = C}$$

Q) $\sinh x \cos y\, dx - \cosh x \sin y\, dy = 0$ — ①

is exact ??

$\frac{d}{dx}(\sinh x) = \frac{e^x + e^{-x}}{2} = \cosh x$, $\frac{d}{dx}(\cosh x) = \frac{e^x - e^{-x}}{2} = \sinh x.$

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}.$

Sol, Compare ① with $f\,dx + g\,dy = 0$

$f = \sinh x \cos y$, $g = -\cosh x \sin y$

$\frac{\partial f}{\partial y} = \sinh x\,(-\sin y)$, $\frac{\partial g}{\partial x} = -\sin y\,(\sinh x)$

Clearly $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, hence eq ① is exact

sol. is $\int_{y\ \text{Constant}} f\,dx + \int \underset{\substack{\text{Term of } g \\ \text{not containing } x}}{g}\,dy = C.$

$$\Rightarrow \int_{y\ \text{Constant}} \sinh x \cos y\,dx + \int 0\,dy = C.$$

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$$\Rightarrow \boxed{\cos y \;\cosh x = c}$$

$$\Rightarrow \sinh x \cos y \, dx - \cosh x \sin y \, dy = 0$$

$$\cos y \, \sinh x \, dx - \cosh x \, \sin y \, dy = 0$$

$$\cos y \, d(\cosh x) + \cosh x \, d(\cos y) = 0$$

$$\Rightarrow d(\cosh x \cos y) = 0$$ on integrating,

$$\Rightarrow \boxed{\cosh x \cos y = c}$$