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Clairaut's Equation :-

It is a differential equation of the form

$$y = px + f(p).$$

Solution of differential equation is obtained by replacing $\boxed{p = c}$ where c is an arbitrary constant

Sol. is $\boxed{y = cx + f(c)}$

Proof:- $\because$ differential eq. is of form.

$$y = px + f(p) = f(p, x)$$

diff. w.r.t x.

$$p = \frac{dy}{dx} = p(1) + x\frac{dp}{dx} + f'(p).\frac{dp}{dx}$$

$$\Rightarrow \cancel{p} = \cancel{p} + \frac{dp}{dx}(x + f'(p))$$

$$\Rightarrow \frac{dp}{dx}(x + f'(p)) = 0$$

discarding the factor $(x + f'(p))$

$$\Rightarrow \frac{dp}{dx} = 0 \Rightarrow p = c \quad \text{(constant)}$$

Sol is $\boxed{y = cx + f(c)}$

Q. Solve the differential eq

$$xp^2 - yp + a = 0.$$

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Sol: $yp = xp^2 + a.$

$$\Rightarrow y = \frac{xp^2 + a}{p} = px + \frac{a}{p}$$

which is of the form $y = px + f(p)$

i.e. it is Clairaut's equation.

Hence the solution is obtained by replacing $p = c$

i.e sol. is $\boxed{y = cx + \frac{a}{c}}$

Q: Solve the diff. eq. $y + 2\left(\frac{dy}{dx}\right)^2 = (x+1)\frac{dy}{dx}$. — (1)

Sol:- Here put $\boxed{\frac{dy}{dx} = p}$

$$(1) \Rightarrow \quad y + 2p^2 = (x+1)p$$

$$\Rightarrow 2p^2 - (x+1)p + y = 0$$

$$\text{or} \quad \boxed{2p^2 - px - p} + y = 0$$

$$\Rightarrow y = px + p - 2p^2.$$

which is of form $y = px + f(p)$ [Clairaut's eq.]

solution is obtained by replacing $p = c$.

Hence sol. is $\boxed{y = cx + c - 2c^2}$

Hence sol. is $\boxed{y = Cx + C - 2C^2}$

Q: Solve the differential equation.
$p^3 - 4xyp + 8y^2 = 0.$

Sol:-

$$8y^2 - 4xyp = -p^3$$

$$\Rightarrow 8y\left[y - \frac{px}{2}\right] = -p^3$$

$$\Rightarrow y - \frac{px}{2} = -\frac{p^3}{8y}$$

$$\Rightarrow y = \frac{px}{2} - \frac{p^3}{8y} \quad \text{— (1)} \qquad p = \frac{dy}{dx}$$

Put $y^{1/2} = Y \Rightarrow \boxed{y = Y^2}$

$$p = \boxed{\frac{dy}{dx} = \frac{2Y\,dY}{dx}}$$

$$Y^2 = \frac{\cancel{2}Y\,dY}{\cancel{2}\,dx}x - \frac{1}{8Y^2}\left(\frac{2Y\,dY}{dx}\right)^3$$

$$Y^2 = Yx\frac{dY}{dx} - \frac{1}{\cancel{8Y^2}} \times \cancel{8Y^3}\left(\frac{dY}{dx}\right)^3$$

$$\Rightarrow Y^{\cancel{2}} = \cancel{Y}\left(x\frac{dY}{dx} - \left(\frac{dY}{dx}\right)^3\right)$$

dx   dx

$$\Rightarrow \boxed{Y = \frac{dY}{dx}\cdot x - \left(\frac{dY}{dx}\right)^3} \quad -(*)$$

It is of the form $\boxed{Y = px + f(p)}$ (Clairaut's form)

Sol^n of (*) is given by, $\boxed{Y = cx - c^3}$

$$\Rightarrow \boxed{\sqrt{y} = cx - c^3}$$

Or Solve the diff. eq. $x^2\left(\frac{dy}{dx}\right)^4 + 2x\left(\frac{dy}{dx}\right) - y = 0$. $-(1)$

Sol^n: $\frac{dy}{dx} = p$

$$\Rightarrow x^2p^4 + 2xp - y = 0.$$

$$\Rightarrow y = 2px + x^2p^4$$

$p = \frac{dy}{dx}$

~~(a)~~ $x^{1/2}$ ~~(b)~~ x^2 ~~(c)~~ $y^{1/2}$ ~~(d)~~ y^2 (e) none of these.

Put $x^{1/2} = X \Rightarrow x = X^2$.

$dx = 2X\,dX$

$$y = 2\frac{dy}{dx}x + x^2\left(\frac{dy}{dx}\right)^4$$

$$y = \cancel{2}\frac{dy}{\cancel{2X}\,dX}X^2 + \cancel{(X^2)^2}\left(\frac{dy}{2\cancel{X}\,dX}\right)^4$$

$$\Rightarrow y = \frac{dy}{dX}X + \frac{1}{16}\left(\frac{dy}{dX}\right)^4 \quad -(2)$$

It is of the form $\boxed{y = px + f(p)}$

Solution of eq ② is given by

$$y = cX + \frac{1}{16}c^4.$$

$\therefore \boxed{x^{1/2} = X}$

$$\boxed{y = c\sqrt{x} + \frac{1}{16}c^4}$$

Q Solve the differential eq.

$$(px - y)(x + py) = 2p.$$

Sol:- (a) x (b) y (c) x & y.

$$f(y - px) = -\frac{2p}{x + py}$$

$$\Rightarrow \quad y = px - \frac{2p}{x + py} \quad \text{—} \circledast$$

$$\frac{x + \frac{dy}{dx}y}{x\,dx + y\,dy}$$

$$x^2 + y^2$$

put $x^2 = X, \; y^2 = Y$

$$\Rightarrow \quad x = \sqrt{X}, \; y = \sqrt{Y}$$

$$dx = \frac{dX}{2\sqrt{X}}, \quad dy = \frac{dY}{2\sqrt{Y}}$$

$$\boxed{p = \frac{dy}{dx} = \frac{2\sqrt{X}}{2\sqrt{Y}}\frac{dY}{dX} = \sqrt{\frac{X}{Y}}\frac{dY}{dX}}$$

eq (*)

$$\sqrt{Y} = \sqrt{\frac{X}{Y}}\left(\frac{dY}{dX}\right)\sqrt{X} - \frac{2\sqrt{\frac{X}{Y}}\frac{dY}{dX}}{\sqrt{X}+\sqrt{\frac{X}{\cancel{Y}}}\frac{dY}{dX}\cancel{\sqrt{Y}}}$$

$$\Rightarrow \sqrt{Y} = \frac{X}{\sqrt{Y}}\left(\frac{dY}{dX}\right) - 2\frac{\sqrt{\frac{\cancel{X}}{Y}}\frac{dY}{dX}}{\sqrt{\cancel{X}}\left(1+\frac{dY}{dX}\right)}$$

$$\Rightarrow \sqrt{Y}\sqrt{Y} = X\frac{dY}{dX} - 2\frac{\frac{dY}{dX}}{1+\frac{dY}{dX}}$$

Let $\boxed{\frac{dY}{dX} = P}$

$$\Rightarrow Y = XP - \frac{2P}{1+P} \quad \text{--- (1)}$$

Clearly it is of the form $\boxed{y = px + f(p)}$

Sol. of eq (1) is $Y = cX - \frac{2c}{1+c}$

$$\Rightarrow \boxed{y^2 = cx^2 - \frac{2c}{1+c}}$$

Sol. of given diff. eq.