Ninth Edition

CHAPTER

17

VECTOR MECHANICS FOR ENGINEERS:
DYNAMICS

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Lecture Notes:

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Plane Motion of Rigid Bodies: Energy and Momentum Methods

Introduction

- Method of work and energy and the method of impulse and momentum will be used to analyze the plane motion of rigid bodies and systems of rigid bodies.

- Principle of work and energy is well suited to the solution of problems involving displacements and velocities.

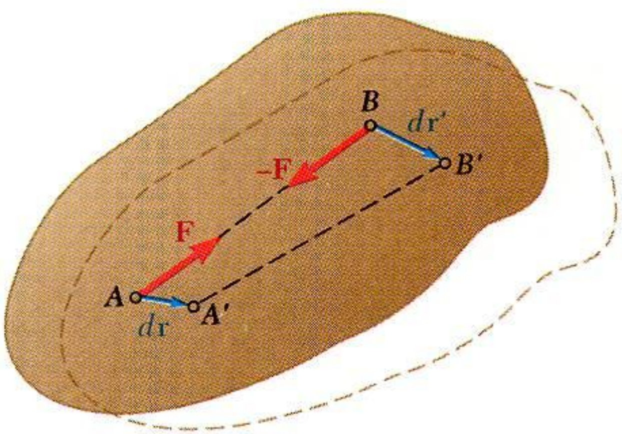
$$T_1 + U_{1\to 2} = T_2$$

- Principle of impulse and momentum is appropriate for problems involving velocities and time.

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F}\,dt = \vec{L}_2 \qquad \left(\vec{H}_O\right)_1 + \sum \int_{t_1}^{t_2} \vec{M}_O\,dt = \left(\vec{H}_O\right)_2$$

- Problems involving eccentric impact are solved by supplementing the principle of impulse and momentum with the application of the coefficient of restitution.

Principle of Work and Energy for a Rigid Body



- Method of work and energy is well adapted to problems involving velocities and displacements. Main advantage is that the work and kinetic energy are scalar quantities.
- Assume that the rigid body is made of a large number of particles.

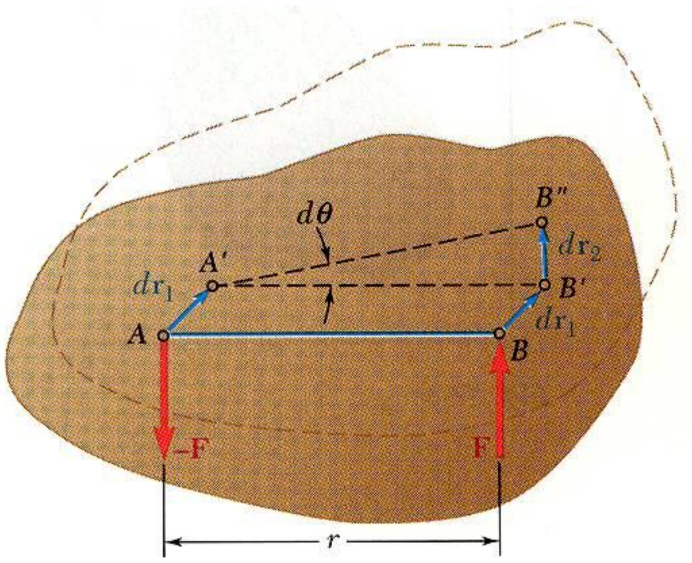
$$T_1 + U_{1\to 2} = T_2$$

$T_1, T_2 =$ initial and final total kinetic energy of particles forming body

$U_{1\to 2} =$ total work of internal and external forces acting on particles of body.

- Internal forces between particles *A* and *B* are equal and opposite.
- In general, small displacements of the particles *A* and *B* are not equal but the components of the displacements along *AB* are equal.
- Therefore, *the net work of internal forces is zero.*

Work of Forces Acting on a Rigid Body



- Work of a force during a displacement of its point of application,

$$U_{1\to 2} = \int_{A_1}^{A_2} \vec{F} \cdot d\vec{r} = \int_{s_1}^{s_2} (F \cos\alpha)\, ds$$

- Consider the net work of two forces $\vec{F}$ and $-\vec{F}$ forming a couple of moment $\vec{M}$ during a displacement of their points of application.

$$\begin{aligned} dU &= \vec{F} \cdot d\vec{r}_1 - \vec{F} \cdot d\vec{r}_1 + \vec{F} \cdot d\vec{r}_2 \\ &= F\, ds_2 = Fr\, d\theta \\ &= M\, d\theta \end{aligned}$$

$$\begin{aligned} U_{1\to 2} &= \int_{\theta_1}^{\theta_2} M\, d\theta \\ &= M(\theta_2 - \theta_1) \quad \text{if } M \text{ is constant.} \end{aligned}$$

Work of Forces Acting on a Rigid Body

Forces acting on rigid bodies which do no work:

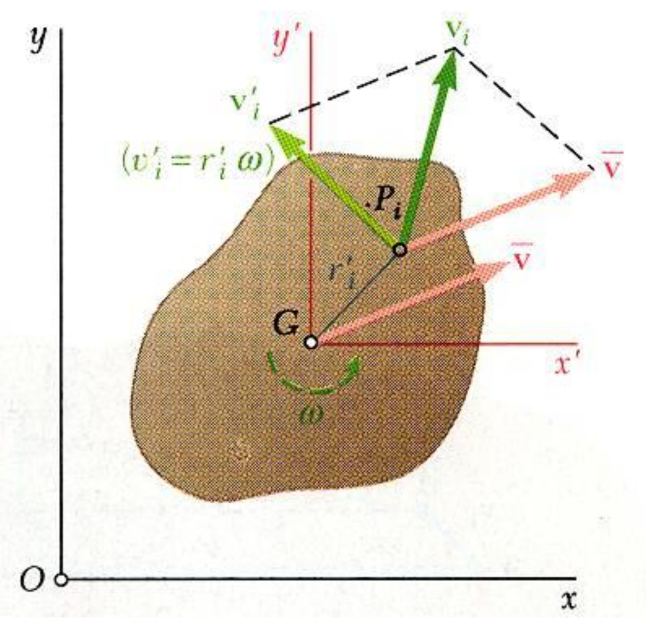
- Forces applied to fixed points:
  - reactions at a frictionless pin when the supported body rotates about the pin.

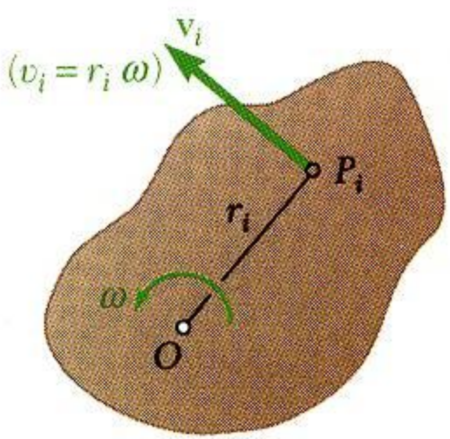
- Forces acting in a direction perpendicular to the displacement of their point of application:
  - reaction at a frictionless surface to a body moving along the surface
  - weight of a body when its center of gravity moves horizontally

- Friction force at the point of contact of a body rolling without sliding on a fixed surface.

$$dU = F\,ds_C = F\left(v_c dt\right) = 0$$

Kinetic Energy of a Rigid Body in Plane Motion





- Consider a rigid body of mass *m* in plane motion.

$$\begin{aligned} T &= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\sum \Delta m_i v_i'^2 \\ &= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\left(\sum r_i'^2 \Delta m_i\right)\omega^2 \\ &= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2 \end{aligned}$$

- Kinetic energy of a rigid body can be separated into:
  - the kinetic energy associated with the motion of the mass center *G* and
  - the kinetic energy associated with the rotation of the body about *G*.

- Consider a rigid body rotating about a fixed axis through *O*.

$$\begin{aligned} T &= \tfrac{1}{2}\sum \Delta m_i v_i^2 + \tfrac{1}{2}\sum \Delta m_i (r_i\omega)^2 + \tfrac{1}{2}\left(\sum r_i^2 \Delta m_i\right)\omega^2 \\ &= \tfrac{1}{2} I_O \omega^2 \end{aligned}$$

Systems of Rigid Bodies

- For problems involving systems consisting of several rigid bodies, the principle of work and energy can be applied to each body.

- We may also apply the principle of work and energy to the entire system,

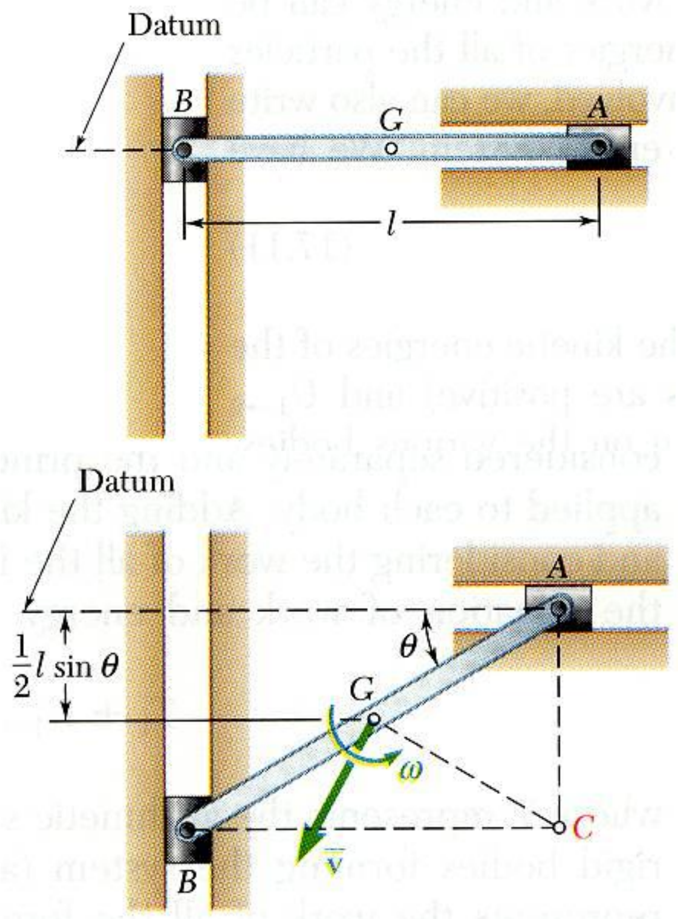
$$T_1 + U_{1\rightarrow 2} = T_2$$

T_1, T_2 = arithmetic sum of the kinetic energies of all bodies forming the system

$U_{1\rightarrow 2}$ = work of all forces acting on the various bodies, whether these forces are internal or external to the system as a whole.

- For problems involving pin connected members, blocks and pulleys connected by inextensible cords, and meshed gears,
  - internal forces occur in pairs of equal and opposite forces
  - points of application of each pair move through equal distances
  - net work of the internal forces is zero
  - work on the system reduces to the work of the external forces

Conservation of Energy



- mass m
- released with zero velocity
- determine ω at θ

- Expressing the work of conservative forces as a change in potential energy, the principle of work and energy becomes

$$T_1 + V_1 = T_2 + V_2$$

- Consider the slender rod of mass *m*.

$$T_1 = 0, \quad V_1 = 0$$

$$T_2 = \tfrac{1}{2} m \bar{v}_2^2 + \tfrac{1}{2} \bar{I} \omega_2^2$$

$$= \tfrac{1}{2} m \left(\tfrac{1}{2} l \omega\right)^2 + \tfrac{1}{2}\left(\tfrac{1}{12} m l^2\right)\omega^2 = \frac{1}{2}\frac{ml^2}{3}\omega^2$$

$$V_2 = -\tfrac{1}{2} W l \sin\theta = -\tfrac{1}{2} mgl \sin\theta$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 = \frac{1}{2}\frac{ml^2}{3}\omega^2 - \frac{1}{2} mgl \sin\theta$$

$$\omega = \left(\frac{3g}{l}\sin\theta\right)$$

Power

- Power = rate at which work is done

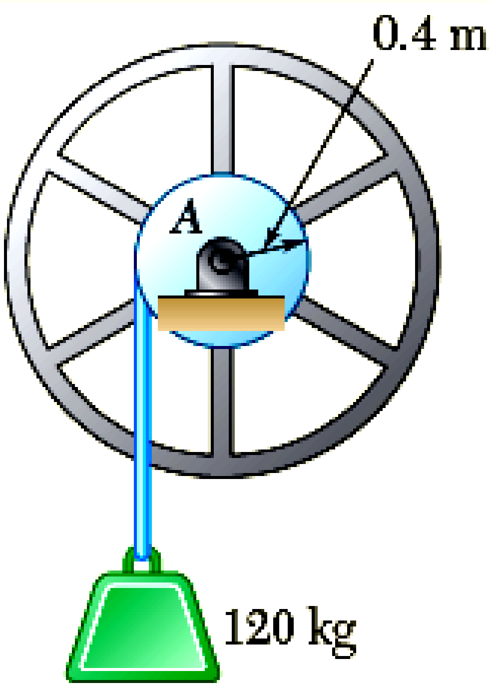
- For a body acted upon by force $\vec{F}$ and moving with velocity $\vec{v}$,

$$\text{Power} = \frac{dU}{dt} = \vec{F} \cdot \vec{v}$$

- For a rigid body rotating with an angular velocity $\vec{\omega}$ and acted upon by a couple of moment $\vec{M}$ parallel to the axis of rotation,

$$\text{Power} = \frac{dU}{dt} = \frac{M\,d\theta}{dt} = M\omega$$

Sample Problem 17.1



For the drum and flywheel, $\bar{I} = 16\,\mathrm{kg \cdot m^2}$. The bearing friction is equivalent to a couple of $90\,\mathrm{N \cdot m}$. At the instant shown, the block is moving downward at 2 m/s.

Determine the velocity of the block after it has moved 1.25 m downward.

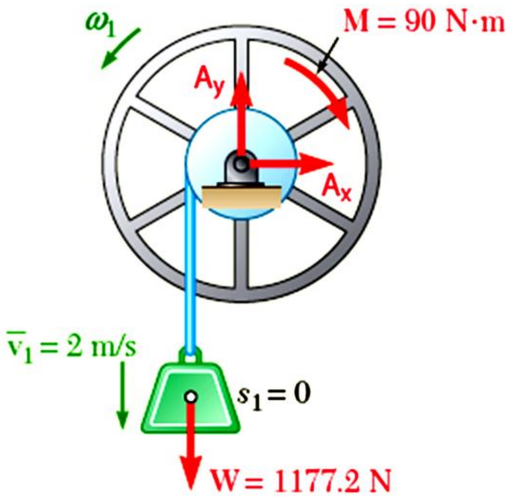
SOLUTION:

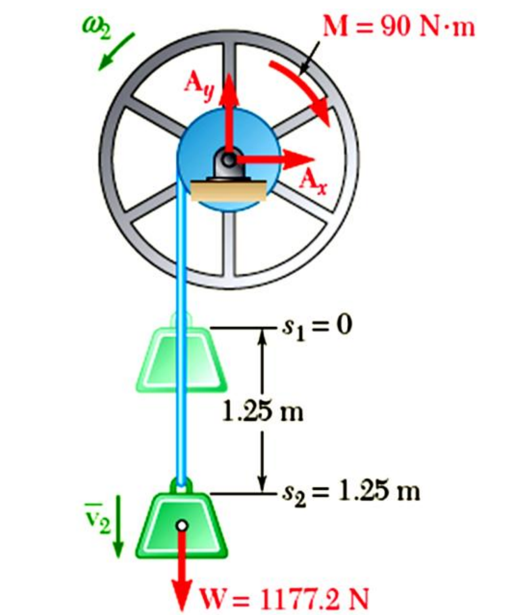
- Consider the system of the flywheel and block. The work done by the internal forces exerted by the cable cancels.
- Note that the velocity of the block and the angular velocity of the drum and flywheel are related by

$$\bar{v} = r\omega$$

- Apply the principle of work and kinetic energy to develop an expression for the final velocity.

Sample Problem 17.1





<u>SOLUTION</u>:

- Consider the system of the flywheel and block. The work done by the internal forces exerted by the cable cancels.
- Note that the velocity of the block and the angular velocity of the drum and flywheel are related by

$$\bar{v} = r\omega \qquad \omega_1 = \frac{\bar{v}_1}{r} = \frac{2\,\mathrm{m/s}}{0.4\,\mathrm{m}} = 5\,\mathrm{rad/s} \qquad \omega_2 = \frac{\bar{v}_2}{r} = \frac{\bar{v}_2}{0.4}$$

- Apply the principle of work and kinetic energy to develop an expression for the final velocity.

$$T_1 = \tfrac{1}{2} m v_1^2 + \tfrac{1}{2} \bar{I}\omega_1^2$$

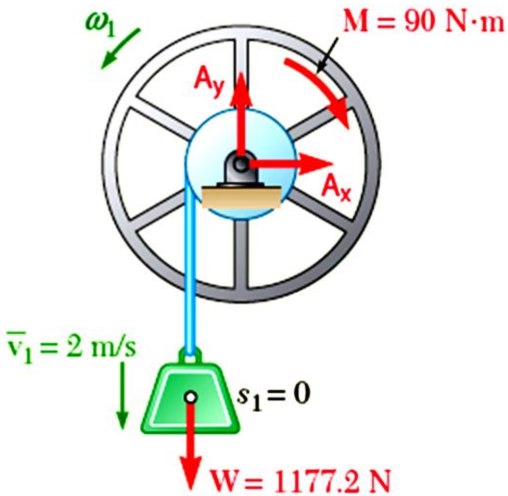
$$= \frac{1}{2}(120\,\mathrm{kg})(2\,\mathrm{m/s})^2 + \frac{1}{2}(16\,\mathrm{kg\cdot m})(5\,\mathrm{rad/s})^2$$

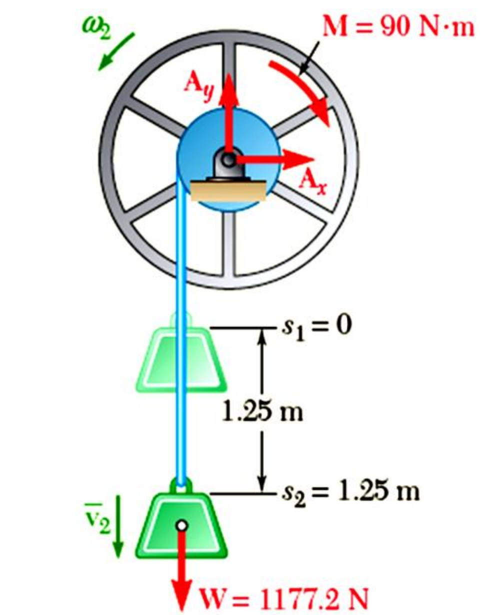
$$= 440\mathrm{J}$$

$$T_2 = \tfrac{1}{2} m \bar{v}_2^2 + \tfrac{1}{2} \bar{I}\omega_2^2$$

$$= \frac{1}{2}(120)\bar{v}_2^2 + \frac{1}{2}(16)\left(\frac{\bar{v}_2}{0.4}\right)^2 = 110\bar{v}_2^2$$

Sample Problem 17.1





$$T_1 = \tfrac{1}{2} m v_1^2 + \tfrac{1}{2} \bar{I} \omega_1^2 = 440\text{J}$$

$$T_2 = \tfrac{1}{2} m \bar{v}_2^2 + \tfrac{1}{2} \bar{I} \omega_2^2 = 110 v_2^2$$

- Note that the block displacement and pulley rotation are related by

$$\theta_2 = \frac{s_2}{r} = \frac{1.25\,\text{m}}{0.4\,\text{m}} = 3.125\text{rad}$$

Then,

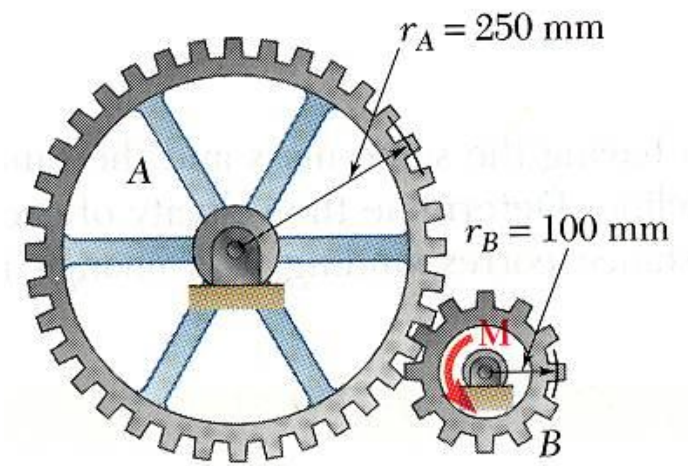
$$U_{1\to 2} = W(s_2 - s_1) - M(\theta_2 - \theta_1)$$
$$= (120\text{kg})(9.81\text{m/s}^2)(1.25\,\text{m}) - (90\,\text{N}\cdot\text{m})(3.125\text{rad})$$
$$= 1190\text{J}$$

- Principle of work and energy:

$$T_1 + U_{1\to 2} = T_2$$
$$(440\text{J}) + (1190\text{J}) = 110\bar{v}_2^2$$
$$\bar{v}_2 = 3.85\text{m/s}$$

$$\boxed{\bar{v}_2 = 3.85\text{m/s} \downarrow}$$

Sample Problem 17.2



$$m_A = 10\,\text{kg} \qquad \bar{k}_A = 200\,\text{mm}$$
$$m_B = 3\,\text{kg} \qquad \bar{k}_B = 80\,\text{mm}$$

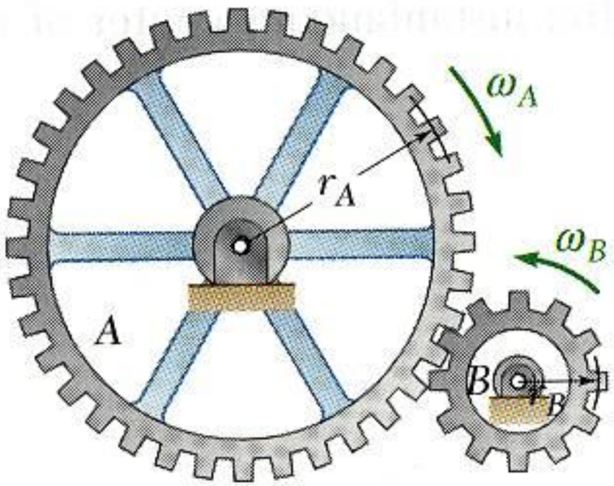
The system is at rest when a moment of $M = 6\,\text{N}\cdot\text{m}$ is applied to gear *B*.

Neglecting friction, *a*) determine the number of revolutions of gear *B* before its angular velocity reaches 600 rpm, and *b*) tangential force exerted by gear *B* on gear *A*.

SOLUTION:

- Consider a system consisting of the two gears. Noting that the gear rotational speeds are related, evaluate the final kinetic energy of the system.
- Apply the principle of work and energy. Calculate the number of revolutions required for the work of the applied moment to equal the final kinetic energy of the system.
- Apply the principle of work and energy to a system consisting of gear *A*. With the final kinetic energy and number of revolutions known, calculate the moment and tangential force required for the indicated work.

Sample Problem 17.2



SOLUTION:

- Consider a system consisting of the two gears. Noting that the gear rotational speeds are related, evaluate the final kinetic energy of the system.

$$\omega_B = \frac{(600\,\text{rpm})(2\pi\,\text{rad/rev})}{60\,\text{s/min}} = 62.8\,\text{rad/s}$$

$$\omega_A = \omega_B \frac{r_B}{r_A} = 62.8\frac{0.100}{0.250} = 25.1\,\text{rad/s}$$

$$\bar{I}_A = m_A \bar{k}_A^2 = (10\,\text{kg})(0.200\,\text{m})^2 = 0.400\,\text{kg}\cdot\text{m}^2$$

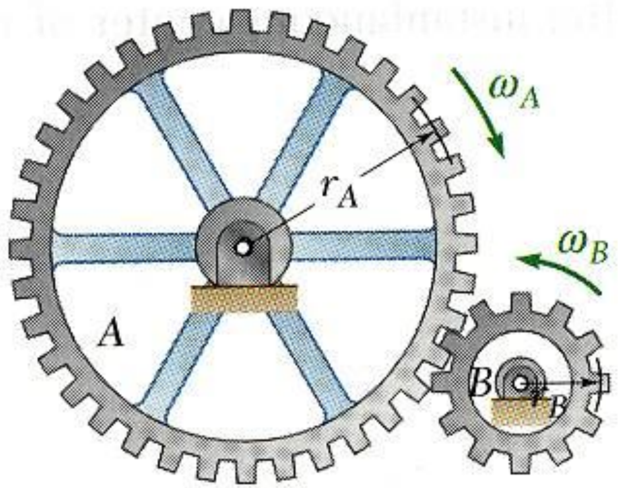
$$\bar{I}_B = m_B \bar{k}_B^2 = (3\,\text{kg})(0.080\,\text{m})^2 = 0.0192\,\text{kg}\cdot\text{m}^2$$

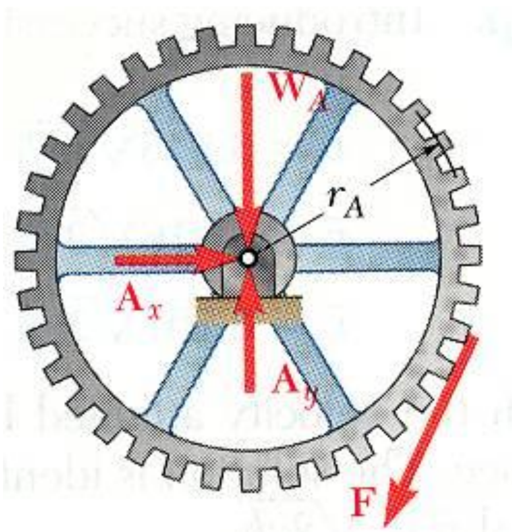
$$T_2 = \tfrac{1}{2}\bar{I}_A\omega_A^2 + \tfrac{1}{2}\bar{I}_B\omega_B^2$$

$$= \tfrac{1}{2}(0.400)(25.1)^2 + \tfrac{1}{2}(0.0192)(62.8)^2$$

$$= 163.9\,\text{J}$$

Sample Problem 17.2





- Apply the principle of work and energy. Calculate the number of revolutions required for the work.

$$T_1 + U_{1\to 2} = T_2$$
$$0 + (6\theta_B)\mathrm{J} = 163.9\mathrm{J}$$
$$\theta_B = 27.32\,\mathrm{rad}$$

$$\boxed{\theta_B = \frac{27.32}{2\pi} = 4.35\,\mathrm{rev}}$$

- Apply the principle of work and energy to a system consisting of gear *A*. Calculate the moment and tangential force required for the indicated work.

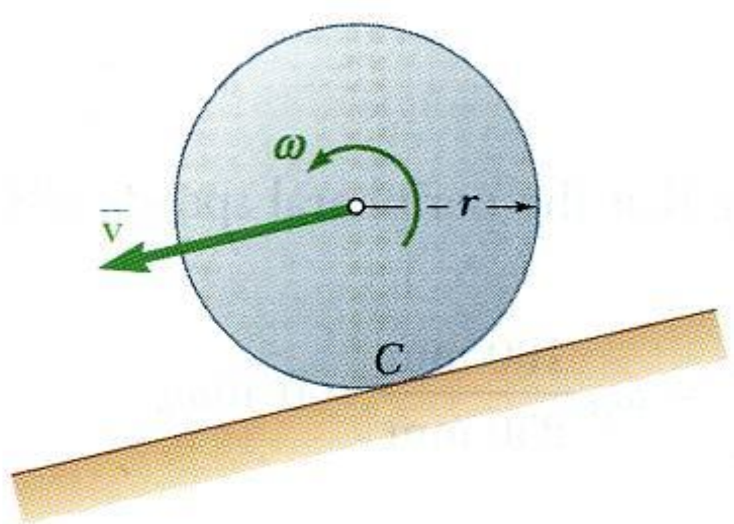
$$\theta_A = \theta_B \frac{r_B}{r_A} = 27.32\,\frac{0.100}{0.250} = 10.93\,\mathrm{rad}$$

$$T_2 = \tfrac{1}{2}\bar{I}_A\omega_A^2 = \tfrac{1}{2}(0.400)(25.1)^2 = 126.0\,\mathrm{J}$$

$$T_1 + U_{1\to 2} = T_2$$
$$0 + M_A(10.93\,\mathrm{rad}) = 126.0\mathrm{J}$$
$$M_A = r_A F = 11.52\,\mathrm{N\cdot m}$$

$$\boxed{F = \frac{11.52}{0.250} = 46.2\,\mathrm{N}}$$

Sample Problem 17.3

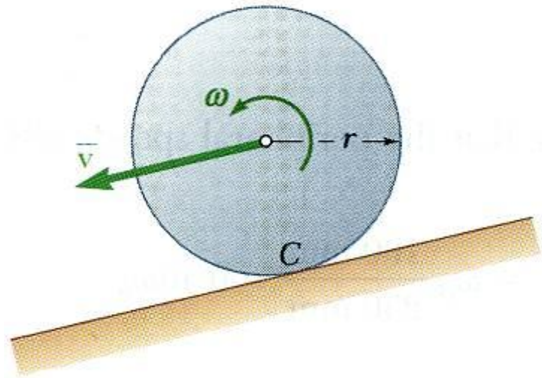


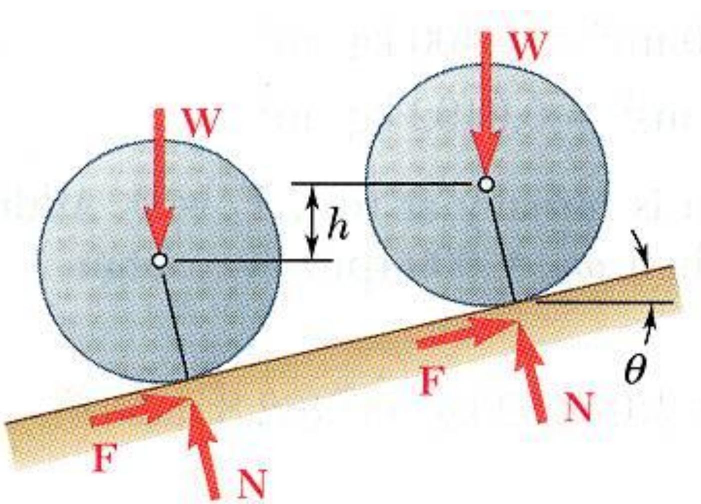
A sphere, cylinder, and hoop, each having the same mass and radius, are released from rest on an incline. Determine the velocity of each body after it has rolled through a distance corresponding to a change of elevation *h*.

<u>SOLUTION</u>:

- The work done by the weight of the bodies is the same. From the principle of work and energy, it follows that each body will have the same kinetic energy after the change of elevation.
- Because each of the bodies has a different centroidal moment of inertia, the distribution of the total kinetic energy between the linear and rotational components will be different as well.

Sample Problem 17.3





<u>SOLUTION</u>:

- The work done by the weight of the bodies is the same. From the principle of work and energy, it follows that each body will have the same kinetic energy after the change of elevation.

With $\omega = \dfrac{\bar{v}}{r}$

$$T_2 = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2 = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\left(\frac{\bar{v}}{r}\right)^2$$

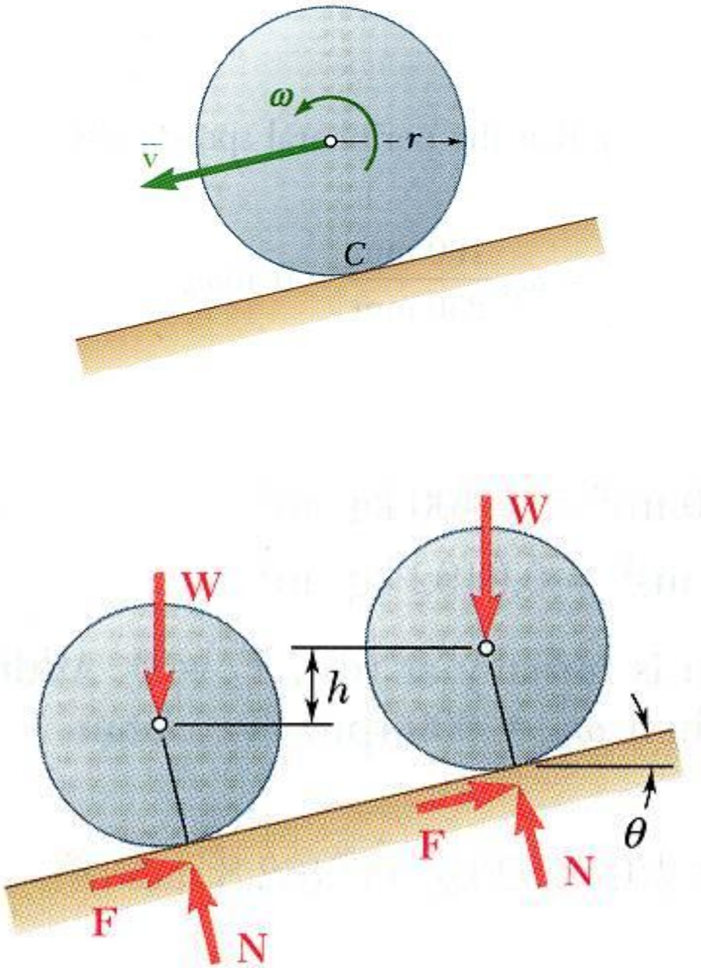
$$= \tfrac{1}{2}\left(m + \frac{\bar{I}}{r^2}\right)\bar{v}^2$$

$$T_1 + U_{1\to 2} = T_2$$

$$0 + Wh = \tfrac{1}{2}\left(m + \frac{\bar{I}}{r^2}\right)\bar{v}^2$$

$$\bar{v}^2 = \frac{2Wh}{m + \bar{I}/r^2} = \frac{2gh}{1 + \bar{I}/mr^2}$$

Sample Problem 17.3



- Because each of the bodies has a different centroidal moment of inertia, the distribution of the total kinetic energy between the linear and rotational components will be different as well.

$$\bar{v}^2 = \frac{2gh}{1 + \bar{I}/mr^2}$$

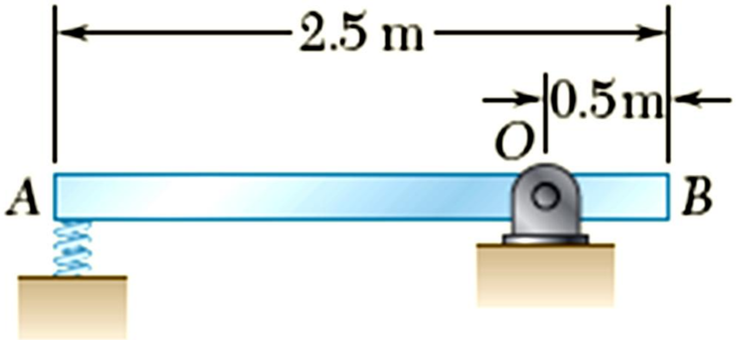
$$\begin{array}{lll} Sphere: & \bar{I} = \frac{2}{5}mr^2 & \bar{v} = 0.845\sqrt{2gh} \\ Cylinder: & \bar{I} = \frac{1}{2}mr^2 & \bar{v} = 0.816\sqrt{2gh} \\ Hoop: & \bar{I} = mr^2 & \bar{v} = 0.707\sqrt{2gh} \end{array}$$

NOTE:

- For a frictionless block sliding through the same distance, $\omega = 0, \quad \bar{v} = \sqrt{2gh}$
- The velocity of the body is independent of its mass and radius.
- The velocity of the body does depend on

$$\bar{I}/mr^2 = \bar{k}^2/r^2$$

Sample Problem 17.4



A 15-kg slender rod pivots about the point *O*. The other end is pressed against a spring (*k* = 300 kN/m) until the spring is compressed one inch and the rod is in a horizontal position.

If the rod is released from this position, determine its angular velocity and the reaction at the pivot as the rod passes through a vertical position.

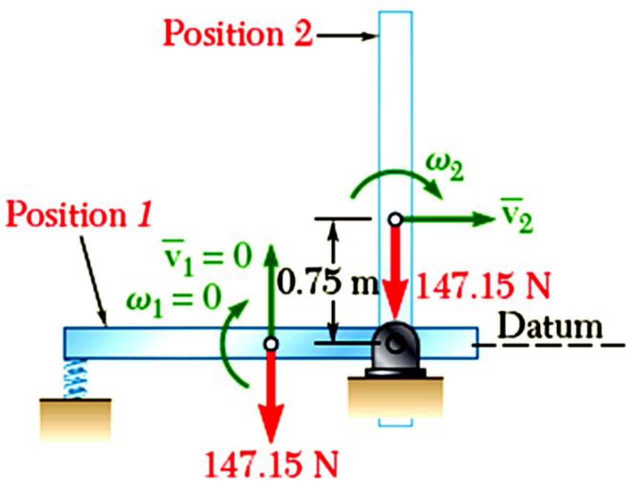
SOLUTION:

- The weight and spring forces are conservative. The principle of work and energy can be expressed as

$$T_1 + V_1 = T_2 + V_2$$

- Evaluate the initial and final potential energy.
- Express the final kinetic energy in terms of the final angular velocity of the rod.
- Based on the free-body-diagram equation, solve for the reactions at the pivot.

Sample Problem 17.4



$$\bar{I} = \tfrac{1}{12} m l^2$$
$$= \frac{1}{12}(15\,\text{kg})(2.5\,\text{m})^2$$
$$= 7.81\,\text{kg-m}^2$$

SOLUTION:

- The weight and spring forces are conservative. The principle of work and energy can be expressed as

$$T_1 + V_1 = T_2 + V_2$$

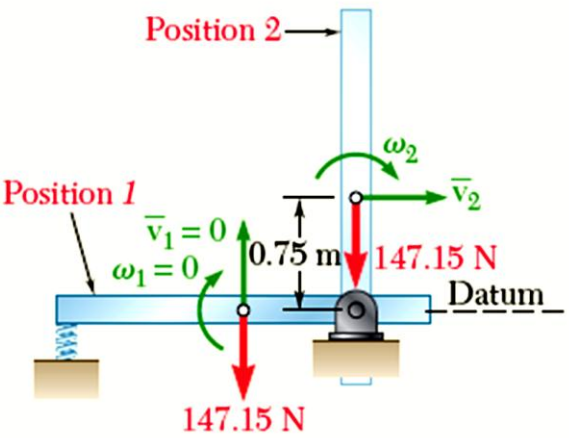
- Evaluate the initial and final potential energy.

$$V_1 = V_g + V_e = 0 + \tfrac{1}{2} k x_1^2 = \tfrac{1}{2}(300000\,\text{N/m})(0.040\,\text{m})^2$$
$$= 240\,\text{J}$$
$$V_2 = V_g + V_e = Wh + 0 = (147.15\,\text{N})(0.75\,\text{m})$$
$$= 110.4\,\text{J}$$

- Express the final kinetic energy in terms of the angular velocity of the rod.

$$T_2 = \tfrac{1}{2} m \bar{v}_2^2 + \tfrac{1}{2} \bar{I} \omega_2^2 = \tfrac{1}{2} m (r\omega_2)^2 + \tfrac{1}{2} \bar{I} \omega_2^2$$
$$= \frac{1}{2} \times (15)(0.75\omega_2)^2 + \tfrac{1}{2}(7.81)\omega_2^2 = 8.12\omega_2^2$$

Sample Problem 17.4



From the principle of work and energy,

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 240\,\mathrm{J} = 8.12\omega_2^2 + 110.4\,\mathrm{J}$$

$$\boxed{\omega_2 = 3.995\,\mathrm{rad/s} \circlearrowright}$$

- Based on the free-body-diagram equation, solve for the reactions at the pivot.

$$\bar{a}_n = \bar{r}\omega_2^2 = (0.75\,\mathrm{m})(3.995\,\mathrm{rad/s})^2 = 11.97\,\mathrm{m/s^2} \qquad \vec{\bar{a}}_n = 11.97\,\mathrm{m/s^2} \downarrow$$

$$\bar{a}_t = r\alpha \qquad \vec{\bar{a}}_t = r\alpha \rightarrow$$

$$+\circlearrowright \sum M_O = \sum (M_O)_{eff} \qquad 0 = \bar{I}\alpha + m(\bar{r}\alpha)\bar{r} \qquad \alpha = 0$$

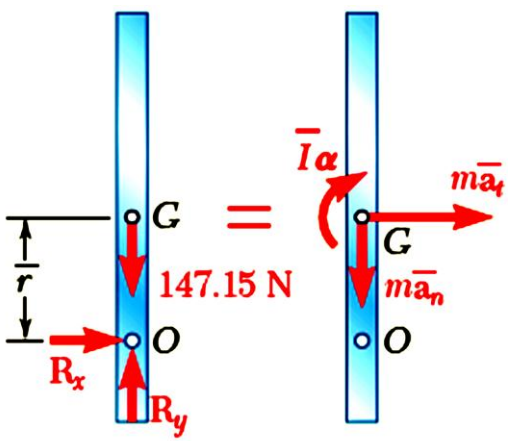
$$\overset{+}{\rightarrow} \sum F_x = \sum (F_x)_{eff} \qquad R_x = m(\bar{r}\alpha) \qquad R_x = 0$$

$$+\uparrow \sum F_y = \sum (F_y)_{eff} \qquad R_y - 147.15\,\mathrm{N} = -m a_n$$

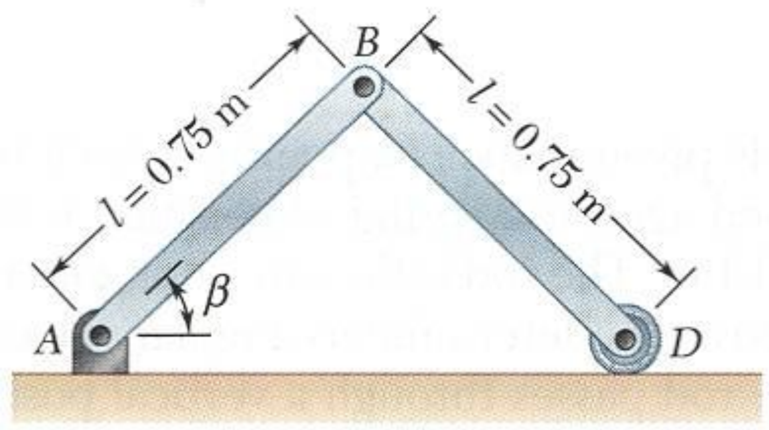
$$= -(15\,\mathrm{kg})(11.97\,\mathrm{m/s^2})$$

$$R_y = -32.4\,\mathrm{N}$$

$$\boxed{\vec{R} = 32.4 \uparrow}$$



Sample Problem 17.5



Each of the two slender rods has a mass of 6 kg. The system is released from rest with $\beta = 60°$.

Determine *a*) the angular velocity of rod *AB* when $\beta = 20°$, and *b*) the velocity of the point *D* at the same instant.

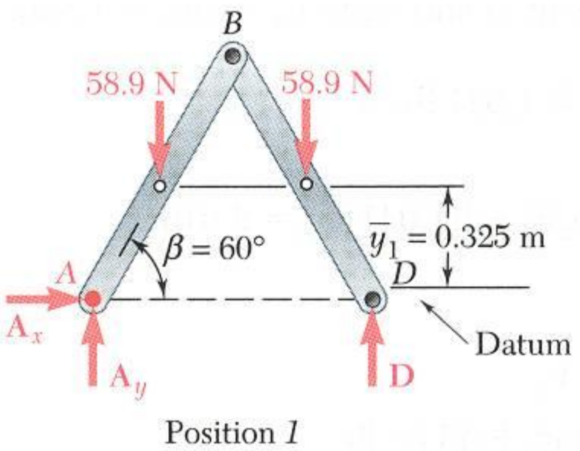
SOLUTION:

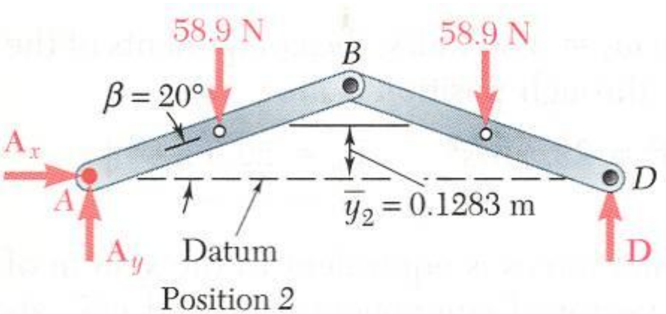
- Consider a system consisting of the two rods. With the conservative weight force,

$$T_1 + V_1 = T_2 + V_2$$

- Evaluate the initial and final potential energy.
- Express the final kinetic energy of the system in terms of the angular velocities of the rods.
- Solve the energy equation for the angular velocity, then evaluate the velocity of the point *D*.

Sample Problem 17.5





$$W = mg = (6\,\text{kg})(9.81\,\text{m/s}^2) = 58.86\,\text{N}$$

SOLUTION:

- Consider a system consisting of the two rods. With the conservative weight force,

$$T_1 + V_1 = T_2 + V_2$$

- Evaluate the initial and final potential energy.

$$V_1 = 2W\bar{y}_1 = 2(58.86\,\text{N})(0.325\,\text{m}) = 38.26\,\text{J}$$

$$V_2 = 2W\bar{y}_2 = 2(58.86\,\text{N})(0.1283\,\text{m}) = 15.10\,\text{J}$$

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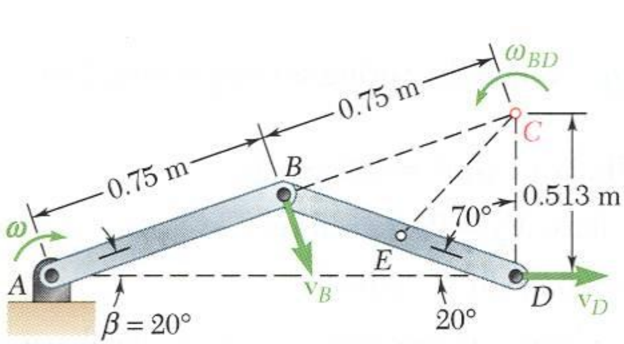

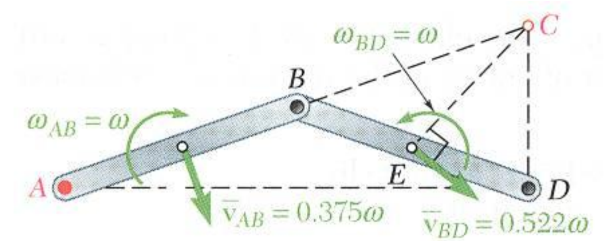

- Express the final kinetic energy of the system in terms of the angular velocities of the rods.

$$\vec{v}_{AB} = (0.375\,\mathrm{m})\omega \searrow$$

Since $\vec{v}_B$ is perpendicular to AB and $\vec{v}_D$ is horizontal, the instantaneous center of rotation for rod BD is C.

$$BC = 0.75\,\mathrm{m} \qquad CD = 2(0.75\,\mathrm{m})\sin 20° = 0.513\,\mathrm{m}$$

and applying the law of cosines to CDE, EC = 0.522 m

Consider the velocity of point B

$$v_B = (AB)\omega = (BC)\omega_{AB} \qquad \vec{\omega}_{BD} = \omega \circlearrowright$$

$$\vec{v}_{BD} = (0.522\,\mathrm{m})\omega \searrow$$

For the final kinetic energy,

$$\bar{I}_{AB} = \bar{I}_{BD} = \tfrac{1}{12}ml^2 = \tfrac{1}{12}(6\,\mathrm{kg})(0.75\,\mathrm{m})^2 = 0.281\,\mathrm{kg}\cdot\mathrm{m}^2$$

$$T_2 = \tfrac{1}{12}m\bar{v}_{AB}^2 + \tfrac{1}{2}\bar{I}_{AB}\omega_{AB}^2 + \tfrac{1}{12}m\bar{v}_{BD}^2 + \tfrac{1}{2}\bar{I}_{BD}\omega_{BD}^2$$

$$= \tfrac{1}{12}(6)(0.375\omega)^2 + \tfrac{1}{2}(0.281)\omega^2 + \tfrac{1}{12}(6)(0.522\omega)^2 + \tfrac{1}{2}(0.281)\omega^2$$

$$= 1.520\omega^2$$

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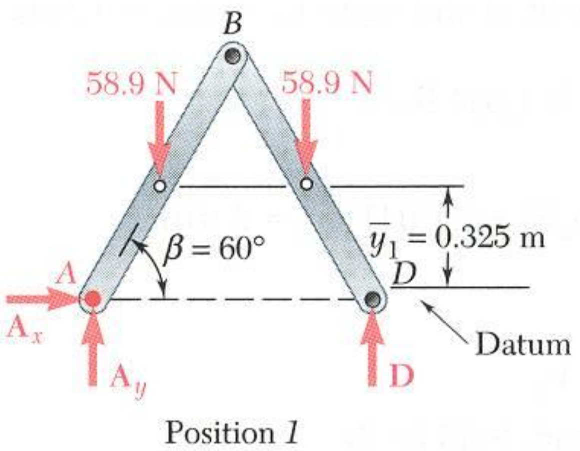

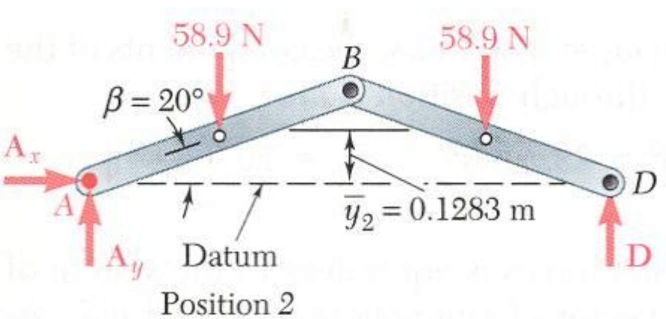

- Solve the energy equation for the angular velocity, then evaluate the velocity of the point *D*.

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 38.26\,\mathrm{J} = 1.520\,\omega^2 + 15.10\,\mathrm{J}$$

$$\omega = 3.90\,\mathrm{rad/s}$$

$$\boxed{\vec{\omega}_{AB} = 3.90\,\mathrm{rad/s}}$$

$$v_D = (CD)\omega$$

$$= (0.513\,\mathrm{m})(3.90\,\mathrm{rad/s})$$

$$= 2.00\,\mathrm{m/s}$$

$$\boxed{\vec{v}_D = 2.00\,\mathrm{m/s} \rightarrow}$$