① Vector differentiation :

Q. If $\vec{u}(t) = \sin(2t)\,\hat{i} - \cos(2t)\,\hat{j} + t\,\hat{k}$

& $\vec{v}(t) = \cos(2t)\,\hat{i} - \sin(2t)\,\hat{j} + t^2\,\hat{k}$

find $[\vec{u}\cdot\vec{v}]'$

Sol:
$$\frac{d}{dt}[\vec{u}\cdot\vec{v}] = \vec{u}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\cdot\vec{v}$$

$$\frac{d\vec{u}}{dt} = 2\cos(2t)\,\hat{i} + 2\sin(2t)\,\hat{j} + \hat{k}$$

$$\frac{d\vec{v}}{dt} = -2\sin(2t)\,\hat{i} - 2\cos(2t)\,\hat{j} + 2t\,\hat{k}$$

$$\frac{d}{dt}[\vec{u}\cdot\vec{v}] = (\sin(2t)\,\hat{i} - \cos(2t)\,\hat{j} + t\,\hat{k})\cdot[-2\sin(2t)\,\hat{i} - 2\cos(2t)\,\hat{j} + 2t\,\hat{k}]$$

$$+ [2\cos(2t)\,\hat{i} + 2\sin(2t)\,\hat{j} + \hat{k}]\cdot[\cos(2t)\,\hat{i} - \sin(2t)\,\hat{j} + t^2\,\hat{k}]$$

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1, \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

$$\frac{d}{dt}[\vec{u}\cdot\vec{v}] = -2\sin^2(2t) + 2\cos^2(2t) + 2t^2$$
$$+ 2\cos^2(2t) - 2\sin^2(2t) + t^2$$

$$\frac{d}{dt}[\vec{u}\cdot\vec{v}] = 4[\cos^2(2t) - \sin^2(2t)] + 3t^2$$
$$= 4\cos(4t) + 3t^2$$

Q) Evaluate $[t^2 \bar{u}(t^2)]'$

Sol:- $\frac{d}{dt}\left(t^2 \bar{u}(t^2)\right) = t^2 \frac{d}{dt}\bar{u}(t^2) + \bar{u}(t^2)\frac{d}{dt}t^2$

$= t^2\left[\bar{u}'(t^2)\,2t\right] + \bar{u}(t^2)\,(2t)$

$= 2t\left[\bar{u}(t^2) + \bar{u}'(t^2).t^2\right]$

Q) $\frac{d}{dt}\left[u(t).\left(u'(t) \times u''(t)\right)\right]$

$= u(t).\frac{d}{dt}\left(u'(t) \times u''(t)\right) + \frac{d\bar{u}}{dt}.\left(u'(t) \times u''(t)\right)$

$= u(t).\left(u'(t) \times \frac{d}{dt}u''(t) + \frac{d}{dt}u'(t) \times u''(t)\right]$

$+ u'(t).\left(u'(t) \times u''(t)\right]$ $\overset{0}{=}$

$= \left[u(t).\left(u'(t) \times u'''(t)\right)\right] + u(t).\left[u''(t) \times u''(t)\right]$

$+ \left[\underline{u'(t).\left(u'(t) \times u''(t)\right)}\right]$

$\overset{0}{=}$

$\bar{a} \times \hat{a} = 0$

$[\bar{a}\ \bar{b}\ \hat{c}]$

$[\hat{a}\ \hat{a}\ \hat{c}] = 0$

$= [\hat{a}$

$= [u(t)\ \ u'(t)\ \ u'''(t)]$

Q. If a particle moves with a constant speed c, then show that its acceleration vector is perpendicular to the velocity vector.

Sol: speed = constant = 0

$$(\vec{a} \cdot \hat{a} = |\vec{a}|^2$$

To prove: $\vec{v}(t) \perp \vec{a}(t)$

$\vec{v} \cdot \vec{a} = 0$

$$\left(\vec{a} = \frac{d\vec{v}}{dt}\right)$$

We know that $|\vec{v}| = \text{speed} = c.$

$$|\vec{v}|^2 = c^2$$

$$\vec{v} \cdot \vec{v} = c^2$$

$$\Rightarrow \frac{d}{dt}(\vec{v} \cdot \vec{v}) = 0$$

$$\left(\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}\right)$$

$$\Rightarrow \vec{v} \cdot \frac{d\vec{v}}{dt} + \frac{d\vec{v}}{dt} \cdot \vec{v} = 0$$

$$\Rightarrow 2\vec{v} \cdot \frac{d\vec{v}}{dt} = 0 \text{ or } \vec{v} \cdot \frac{d\vec{v}}{dt} = 0$$

$$\Rightarrow \vec{v} \cdot \vec{a} = 0$$

Hence velocity vector is $\perp$ar to acceleration vector.

Gradient of Scalar field (del operator):-

Let $f(x,y,z)$ be a real valued function defining a scalar field. To define the gradient of scalar field, we will introduce del operator. (∇)

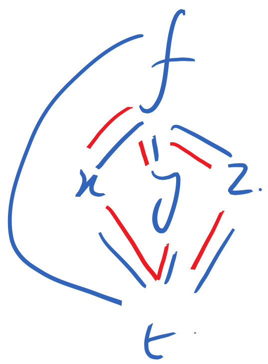
$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\text{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Geometrical representation of gradient:-

Let $f(x,y,z) = c$ be a level surface & its parametric representation is given by

$$f(x(t), y(t), z(t)) = c.$$



$$\frac{d}{dt} f(x(t), y(t), z(t)) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) = 0$$

$$\Rightarrow \nabla f \cdot \frac{d\vec{r}}{dt} = 0$$

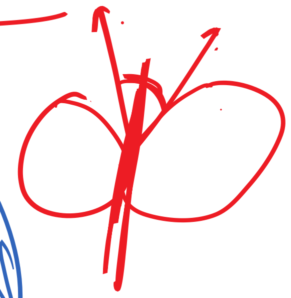
2) ∇f is orthogonal to $\frac{d\vec{r}}{dt}$

3) ∇f is vector normal to the surface f
i.e $f(x,y,z) = c$.

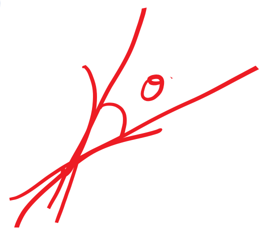
① grad f = normal vector
i.e normal to the surface f.

② normal unit vector $\hat{n} = \frac{\vec{\nabla f}}{|\vec{n}|} = \frac{\nabla f}{|\nabla f|}$.

③ <u>Angle between two surfaces</u>



(angle between two curves is equal to angle between their tangent at the pt. of intersection)



Angle between the surfaces is equal to the angle between their normal at the pt. of intersection. if f & g be two surfaces

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f|\,|\nabla g|}$$

(angle between two $\vec{a}$ & $\vec{b}$

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$)

Q: Compute the gradient of scalar function grad f = ln $(x^2+y^2+z^2)$ at $(3,-4,5)$

Sol: $\text{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$= \frac{2x}{x^2+y^2+z^2}\hat{i} + \frac{2y}{x^2+y^2+z^2}\hat{j} + \frac{2z}{x^2+y^2+z^2}\hat{k}$$

$$\text{grad} f = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{x^2+y^2+z^2}$$

$$(\text{grad} f)_{(3,-4,5)} = \frac{2(3\hat{i}-4\hat{j}+5\hat{k})}{(3)^2+(-4)^2+(5)^2}$$

$$= \frac{\cancel{2}(3\hat{i}-4\hat{j}+5\hat{k})}{\cancel{50}\ 25}$$

$$(\nabla f)_{(3,-4,5)} = \frac{3\hat{i}-4\hat{j}+5\hat{k}}{25}$$

Q: Find the normal vector & unit normal vector to the surface $f = x^2+2y^2+z^2 = 4$ at $(1,1,1)$.

Sol: normal vector $= \nabla f = \text{grad} f$

$$\text{grad} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 2x\hat{i} + 4y\hat{j} + 2z\hat{k}$$

normal vector $\vec{n} = (\text{grad} f)_{(1,1,1)} = 2\hat{i} + 4\hat{j} + 2\hat{k}$

unit normal vector $\Rightarrow$ $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{(2)^2 + (4)^2 + (2)^2}}$

$$= \frac{2(\hat{i} + 2\hat{j} + \hat{k})}{\sqrt{24}} = \frac{\cancel{2}(\hat{i} + 2\hat{j} + \hat{k})}{\cancel{2}\sqrt{6}}$$

$$\hat{n} = \frac{\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{6}}$$

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