Recent Directions in Matrix Stability

Daniel Hershkowitz
Mathematics Department
Technion— Israel Institute of Technology
Haifa 32000, Israel

Submitted by Hans Schneider

ABSTRACT

Matrix stability has been intensively investigated in the past two centuries. We review work that has been done in this topic, focusing on the great progress that has been achieved in the last decade or two. We start with classical stability criteria of Lyapunov, Routh and Hurwitz, and Liénard and Chipart. We then study recently proven sufficient conditions for stability, with particular emphasis on P-matrices. We investigate conditions for the existence of a stable scaling for a given matrix. We review results on other types of matrix stability, such as D-stability, additive D-stability, and Lyapunov diagonal stability. We discuss the weak principal submatrix rank property, shared by Lyapunov diagonally semistable matrices. We also discuss the uniqueness of Lyapunov scaling factors, maximal Lyapunov scaling factors, cones of real positive semidefinite matrices and their applications to matrix stability, and inertia preserving matrices.

1. INTRODUCTION

A complex square matrix A is said to be stable if the spectrum of A lies in the open left or right half plane. This, and other related types of matrix stability, play an important role in various applications. For this reason, matrix stability has been intensively investigated in the past two centuries. In this paper we review work that has been done on this topic, focusing on the great progress that has been achieved in the last decade or two.

We now give a brief summary of the subjects that are covered in the paper.

Lyapunov studied the asymptotic stability of solutions of differential systems. In 1892 he proved a theorem that was restated (first, apparently, by

655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$5.00

Gantmacher in 1953) as a necessary and sufficient condition for stability of a matrix. In 1875, Routh introduced an algorithm that provides a criterion for stability. An independent solution was given by Hurwitz. This solution is known nowadays as the Routh-Hurwitz criterion for stability. Another criterion for stability, which has a computational advantage over the Routh-Hurwitz criterion, was proved in 1914 by Liénard and Chipart. These classical criteria for stability are discussed in Section 2.

In Section 3 we study recent stability results. The abovementioned studies have motivated an intensive search for conditions for matrix stability. A plausible way for finding necessary and/or sufficient conditions for matrix stability is to examine classes of matrices that are known to be stable, and to identify common properties of these classes. It would then be of interest to check whether such properties are necessary or sufficient conditions for stability. Indeed, three well-known classes of stable matrices share three properties: They are P-matrices, they are weakly sign symmetric, and they have eigenvalue monotonicity. None of these three properties is necessary for stability; none is sufficient for stability. It is just plausible to ask whether a combination of any two properties implies stability. The combination weak sign symmetry + eigenvalue monotonicity does not imply stability, while the combinations P-matrix + weak sign symmetry and P-matrix + eigenvalue monotonicity are conjectured to imply stability. If we replace the weak sign symmetry by the stronger sign symmetry property, then it was asserted by Carlson that P-matrix + sign symmetry implies stability. In Section 3 we provide a corrected proof of this result.

These conjectures and results raise the natural question: How far is a P-matrix A from being stable? This question may be answered in terms of the width of a wedge around the negative x-axis which is free from eigenvalues of A, or in terms of the number of eigenvalues of A in the open right half plane, or a combination of both. Indeed, the spectrum of a P-matrix cannot lie in a certain wedge around the negative x-axis. If we take the extreme case of a P-matrix A that has exactly one eigenvalue in the open right half plane, then the "forbidden wedge" in the left half plane is wider.

An interesting question, related to stability, is the following one: *Given a square matrix A. Can we find a diagonal matrix D such that the matrix DA is stable*? This question can be asked in full generality, as suggested above, or with some restrictions on the matrix D, such as positivity of the diagonal elements. Several theorems assert that, under certain conditions, there exists a (positive) diagonal matrix D such that DA is stable. Such questions on stable scalings are discussed in Section 4.

A problem related to finding conditions for the existence of a stable scaling is characterizing matrices A such that for *every* positive diagonal matrix D the matrix DA is stable. Such matrices are called D-stable matrices. This type of matrix stability, as well as two other related types, namely additive D-stability and Lyapunov diagonal (semi)stability, have important applications in many disciplines. Thus they are very important to characterize. These types of matrix stability are studied in Section 5. While regular stability is a spectral property, so that it is always possible to check whether a given matrix is stable or not by evaluating its eigenvalues, none of the other three types of matrix stability can be characterized by the spectrum of the matrix. This problem has been solved for certain classes of matrices. For example, for Z-matrices all the stability types are equivalent. Other cases in which these characterization problems have been solved are the cases of acyclic matrices and of H-matrices.

In the study of H-matrices, an interesting property shared by Lyapunov diagonally semistable matrices was discovered, namely, the weak principal submatrix rank property. In the H-matrix case, as well as in the acyclic case, the weak principal submatrix rank property is also sufficient for Lyapunov diagonal semistability. Section 6 discusses the weak principal submatrix rank property.

Section 7 reviews Lyapunov scaling factors. The search for characterizations of Lyapunov diagonally semistable matrices resulted in interesting conditions on Lyapunov scaling factors. The uniqueness of Lyapunov scaling factors appear to be of importance, and has been discussed in several papers. The study of maximal Lyapunov scaling factors lead to a result that reduces the problem of characterizing block triangular Lyapunov diagonally semistable matrices with p diagonal blocks to the problem of characterizing block triangular Lyapunov diagonally semistable matrices with just two diagonal blocks.

Important tools in studying the stability of a matrix are three cones of real positive semidefinite matrices determined by the matrix. A matrix is known to be Lyapunov diagonally stable if and only if two of the cones consist of the zero matrix only. A similar characterization holds for completely reducible Lyapunov diagonally semistable matrices. These cones and their applications to matrix stability are discussed in Section 8.

Finally, we devote Section 9 to a recently introduced subject, which is related to D-stability and Lyapunov diagonal stability: the subject of inertia preserving matrices. Inertia preserving matrices are D-stable, and Lyapunov diagonally stable matrices are strongly inertia preserving. An acyclic matrix is strongly inertia preserving if and only if it is Lyapunov diagonally stable, and an irreducible acyclic matrix is D-stable if and only if it is inertia preserving.

This is mainly a survey paper; however, we have taken the opportunity to include a couple of original results on stable scalings of complex matrices in Section 4 (see Theorems 4.4 and 4.6) and to provide positive answers to two open questions in Section 9 in the acyclic case (see Theorems 9.7 and 9.10).

Most of this article is based on an invited talk given in the Second SIAM Conference on Linear Algebra in Signals, Systems and Control, San Francisco (1990). Other parts were presented in an invited minisymposium talk in the SIAM Conference on Linear Algebra in Signals, Systems and Control, Boston (1986), in an invited special session talk in the International Conference on Linear Algebra and Applications, Valencia (1987), and in an invited session talk in the International Symposium on the Mathematical Theory of Networks and Systems (MTNS-89), Amsterdam (1989).

2. CLASSICAL CRITERIA FOR STABILITY

DEFINITION 2.1. A complex square matrix A is said to be *negative stable* [*positive stable*] *if the spectrum of* A *lies in the open left* [*right*] *half plane*.

CONVENTION 2.2. We shall use the term "stable matrix" for "positive stable matrix."

NOTATION 2.3. We denote by $\langle n\rangle$ the set $\{1,\dots,n\}$. We denote by $|\alpha|$ the cardinality of a set α.

Lyapunov, called by Gantmacher "the founder of the modern theory of stability," studied the asymptotic stability of solutions of differential systems. In 1892 he proved in his paper [40] a theorem which yields the following necessary and sufficient condition for stability of a real matrix. The matrix formulation of Lyapunov's theorem is apparently due to Gantmacher [20].

THEOREM 2.4. *A complex square matrix* A *is stable if and only if there exists a positive definite Hermitian matrix* H *such that the matrix* $AH + HA^*$ *is positive definite*.

REMARK 2.5. Theorem 2.4 was proved in [20] for a real matrix A; however, as also remarked in [20], the generalization to the complex case is immediate.

In 1875, Routh [43], using Sturm's theorem and the theory of Cauchy indices, introduced an algorithm to determine the number of roots of a real polynomial in the right half plane. In particular, this algorithm provides a criterion for stability. An independent solution, based on Hermite's paper [25], was given by Hurwitz [34]. This solution is known nowadays as the Routh-Hurwitz theorem, (e.g. [20, Vol. II, p. 194]).

THEOREM 2.6 (Routh-Hurwitz). *Let A be an $n \times n$ complex matrix, and let E_k be the sum of all principal minors of A of order k, $k \in \langle n \rangle$. Let $\Omega(A)$ be the $n \times n$ matrix*

$$\begin{pmatrix} E_1 & E_3 & E_5 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 1 & E_2 & E_4 & & & & & \cdot & \cdot \\ 0 & E_1 & E_3 & \cdot & & & & \cdot & \cdot \\ 0 & 1 & E_2 & \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & E_1 & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & & & \cdot & \cdot & E_n & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & E_{n-1} & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & E_{n-2} & E_n \end{pmatrix},$$

and assume that $\Omega(A)$ is real. Then A is stable if and only if all leading principal minors of $\Omega(A)$ are positive.

Another criterion for stability was proved in 1914 by Liénard and Chipart [39]. Their criterion has an advantage over the Routh-Hurwitz criterion, since the number of determinantal inequalities in the Liénard-Chipart criterion is roughly half of that in the Routh-Hurwitz criterion.

DEFINITION 2.7. Let $a(x)$ and $b(x)$ be two polynomials with real coefficients of degree n and m respectively, $n \geqslant m$. The *Bezoutiant* defined by $a(x)$ and $b(x)$ is the bilinear form

$$\frac{a(x)b(y)-a(y)b(x)}{x-y} = \sum_{i,k=0}^{n-1} b_{ik}x^i y^k.$$

The symmetric matrix $(b_{ik})_0^{n-1}$ associated with this bilinear form is called the *Bezout matrix*, and is denoted by $B_{a,b}$.

THEOREM 2.8 (Liénard-Chipart). *Let $f(x) = x^n - a_n x^{n-1} - \cdots - a_1$ be a polynomial with real coefficients, and let $a_{n+1} = -1$. Define the polynomials*

$$h(u) = -a_1 - a_3 u - \cdots,$$

$$g(u) = -a_2 - a_4 u - \cdots.$$

The polynomial $f(x)$ is negative stable if and only if the Bezout matrix $B_{h,g}$ is positive definite and $a_i < 0$ for all $i \in \langle n \rangle$.

The equivalence of the Routh-Hurwitz and Liénard-Chipart criteria was observed by Fujiwara [19].

3. RECENT STABILITY RESULTS

A plausible way for finding necessary and/or sufficient conditions for matrix stability is to examine classes of matrices that are known to be stable and to identify common properties of these classes. It would then be of interest to check whether such properties are necessary or sufficient conditions for stability. Three such classes are the class of positive definite Hermitian matrices; the class of totally positive matrices, that is, matrices all of whose minors are positive; and the class of nonsingular M-matrices, that is, matrices of the form $\alpha I - A$, where A is nonnegative entrywise and α is greater than the spectral radius of A. We shall denote the union of these three classes by $\mathscr{S}$. Matrices in the first two classes are known to have positive eigenvalues, while the stability of M-matrices follows from the Perron-Frobenius spectral theory for nonnegative matrices. Indeed, in 1958 Taussky [47] posed the question what are the common properties of matrices in $\mathscr{S}$. We now review three such properties.

Recall that matrix is said to be a P-matrix if it has positive principal minors. The validity of the following proposition for positive definite Hermitian matrices and totally positive matrices follows from their definition. As to nonsingular M-matrices, it follows from the Perron-Frobenius spectral theory for nonnegative matrices.

PROPOSITION 3.1. *The matrices in $\mathscr{S}$ are P-matrices.*

NOTATION 3.2. Let A be an $n \times n$ matrix, and let α and β be nonempty subsets of $\langle n \rangle$. We denote by $A[\alpha|\beta]$ the submatrix of A whose rows are indexed by α and whose columns are indexed by β in their natural order.

DEFINITION 3.3. An $n \times n$ matrix A is said to be *weakly sign symmetric* if it satisfies

$$\det(A[\alpha|\beta])\det(A[\beta|\alpha]) \geqslant 0 \qquad \forall \alpha, \beta \subseteq \langle n \rangle,\ |\alpha| = |\beta| = |\alpha \cap \beta| + 1.$$

The following proposition also holds for positive definite Hermitian matrices as well as for totally positive matrices by their definition. It is proved for nonsingular M-matrices in [11].

PROPOSITION 3.4. *The matrices in $\mathscr{S}$ are weakly sign symmetric.*

Finally, motivated by Taussky's question, Engel and Schneider [16] discussed eigenvalue monotonicity.

NOTATION 3.5. Let A be a square matrix. We denote by $l(A)$ the minimal real eigenvalue of A. If A has no real eigenvalues, then we agree that $l(A)=\infty$.

DEFINITION 3.6. An $n\times n$ matrix A is said to have *eigenvalue monotonicity* if the diagonal elements of A are real and

$$\alpha\subseteq\beta \;\Rightarrow\; l(A[\beta|\beta])\leqslant l(A[\alpha|\alpha]) \qquad \forall\alpha,\beta\subseteq\langle n\rangle.$$

Such a matrix is called an *ω-matrix*.

We remark that ω-matrices and τ-matrices [that is, ω-matrices A such that $l(A)\geqslant 0$] were named by Engel and Schneider after Olga Taussky, whose research problem motivated their study.

The following proposition follows for Hermitian matrices from Cauchy [14] and for M-matrices from the Perron-Frobenius theory. It was proved for totally positive matrices by Friedland [18].

PROPOSITION 3.7. *The matrices in $\mathscr{S}$ have eigenvalue monotonicity.*

We remark that none of these three properties is necessary for stability, as demonstrated by the following example.

EXAMPLE 3.8. Let A be the matrix

$$\begin{pmatrix} -1 & 1\\ -5 & 3\end{pmatrix}.$$

The eigenvalues of A are $1\pm i$, and so A is stable. Nevertheless, A is neither a P-matrix, nor weakly sign symmetric, nor an ω-matrix.

It is also easy to verify that none of these three properties is sufficient for stability, as demonstrated by the following two matrices.

EXAMPLE 3.9. The matrix

$$\begin{pmatrix} 1 & 0 & 3 \\ 3 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

is a P-matrix but not stable. The matrix $-I$ is both weakly sign symmetric and an ω-matrix but not stable.

It is natural to ask whether a combination of any two properties implies stability. The combination weak sign symmetry + eigenvalue monotonicity is ruled out in Example 3.9. The combination P-matrix + weak sign symmetry is conjectured by Carlson [12] to imply stability.

CONJECTURE 3.10 (Carlson). Weakly sign symmetric P-matrices are stable.

Conjecture 3.10 is known to be true for $n \leqslant 4$. Its proof for $n \leqslant 3$ is immediate. Its proof for $n = 4$ uses the Fischer-Hadamard determinantal inequalities. The conjecture is open for $n \geqslant 5$.

Engel and Schneider [16] ask whether the combination P-matrix + eigenvalue monotonicity implies stability. It is mentioned in [16] that the answer to this question is positive whenever $n \leqslant 3$. Varga [48] suggests that P-matrix + eigenvalue monotonicity implies even more than stability.

CONJECTURE 3.11 (Varga). Let A be an $n \times n$ ω-matrix. Then every eigenvalue λ of A, $\lambda \neq l(A)$, satisfies

$$|\text{argument}(\lambda - l(A))| \leqslant \frac{\pi}{2} - \frac{\pi}{n}.$$

We remark that, as observed in [16], an ω-matrix A is a P-matrix if and only if $l(A) > 0$. Therefore, Conjecture 3.11, if true, yields that P-matrix + eigenvalue monotonicity implies stability.

Conjecture 3.11 is known to be true for $n \leqslant 3$. Its proof for $n \leqslant 2$ is immediate. The case $n = 3$ was proved independently in Hershkowitz and Berman [29] and in Mehrmann [41]. The conjecture is open for $n \geqslant 4$.

In view of our discussion, it is natural to ask whether all three properties together imply stability. For the sake of completeness, we pose the following conjecture, which is clearly weaker than Conjectures 3.10 and 3.11.

CONJECTURE 3.12. Weakly sign symmetric P-matrices that have the eigenvalue monotonicity property are stable.

The positive definite Hermitian matrices and the totally positive matrices share a property which is stronger than weak sign symmetry, and is defined as follows.

DEFINITION 3.13. An $n \times n$ matrix A is said to be *sign symmetric* if it satisfies

$$\det(A[\alpha|\beta])\det(A[\beta|\alpha]) \geqslant 0 \qquad \forall \alpha, \beta \subseteq \langle n \rangle, |\alpha| = |\beta| > 0.$$

The following theorem is proved by Carlson in [12]. Its proof is simple, and it is given here, since it is related to several results mentioned in the sequel.

THEOREM 3.14. *Sign symmetric P-matrices are stable.*

Proof. Let A be a sign symmetric $n \times n$ P-matrix. By the Binet-Cauchy formula (e.g. [20, Vol. I, p. 9]), for every nonempty subset α of $\langle n \rangle$ we have

$$\det(A^2[\alpha|\alpha]) = \sum_{\substack{\beta \subseteq \langle n \rangle \\ |\beta| = |\alpha|}} \det(A[\alpha|\beta])\det(A[\beta|\alpha]). \tag{3.15}$$

Since A is a sign symmetric P-matrix, it now follows from (3.15) that A^2 is a P-matrix. By Theorem 3.16 below, A^2 does not have a nonpositive real eigenvalue. Therefore, A does not have an eigenvalue on the imaginary axis. Since A is a (complex) P-matrix, it follows by Theorem 4.4 or Theorem 4.6 below that there exists a positive diagonal matrix D such that the matrix DA is stable. For every real t, $0 \leqslant t \leqslant 1$, we define the positive diagonal matrix $D_t = (1-t)I_n + tD$, where I_n is the identity matrix of order n. Observe that D_tA is a sign symmetric P-matrix, and as such, as proved above, it does not have an eigenvalue on the imaginary axis. Since $D_0A = A$ and $D_1A = DA$, and since DA is stable, it follows by continuity arguments that A is a stable matrix. ■

Conjectures 3.10, 3.11, and 3.12 and Theorem 3.14 raise the natural question of how far a P-matrix A is from being stable. This question may be answered in terms of the width of the wedge around the negative x-axis which is free from eigenvalues of A, or in terms of the number of eigenvalues of A in the open right half plane, or a combination of both. Indeed, the following theorem, due to Kellogg [38], asserts that the spectrum of a P-matrix cannot lie in a certain wedge around the negative x-axis.

THEOREM 3.16. *Let A be a P-matrix, and let λ be an eigenvalue of A. Then*

$$|\text{argument}(\lambda)| < \pi - \frac{\pi}{n}.$$

It is easy to verify that a P-matrix must have at least one eigenvalue in the open right half plane. If we take the extreme case of a P-matrix A that has exactly one eigenvalue in the open right half plane, then it is interesting to check whether the "forbidden wedge" in the left half plane is bigger. In their paper [28], Hershkowitz and Berman proved

THEOREM 3.17. *Let A be a P-matrix, and let λ be an eigenvalue of A. If A has just one eigenvalue in the open right half plane then*

$$|\text{argument}(\lambda)| < \frac{2\pi}{3}.$$

It is also conjectured in [28] that if A has just two eigenvalues in the open right half plane, then eigenvalues λ of A satisfy

$$|\text{argument}(\lambda)| < \frac{5\pi}{6}.$$

We are now informed that this conjecture has been proven by Shmidel (private communication) to be false.

4. STABLE SCALINGS

An interesting question, related to stability, is the following one: *Given a square matrix A. Can we find a diagonal matrix D such that the matrix DA is stable?* This question can be asked in full generality, as suggested above, or with some restrictions on the matrix D, such as positivity of the diagonal elements. This question is not independent of the discussion in the previous section. In proving Theorem 3.14, Carlson uses the following two theorems.

THEOREM 4.1 (Fisher-Fuller [17]). *Let A be a real square matrix with positive leading principal minors. Then there exists a positive diagonal matrix D such that DA has simple positive eigenvalues.*

THEOREM 4.2 (Ballantine [2]). *Let A be a complex square matrix with positive leading principal minors. Then there exists a complex diagonal matrix D such that DA has simple positive eigenvalues.*

The result used in [12] is that for a complex square matrix A with positive leading principal minors, there exists a positive diagonal matrix D such that DA is stable. Such a result does not follow from Theorems 4.1 and 4.2, as Theorem 4.1 applies only to real matrices A and Theorem 4.2 asserts the existence of a complex diagonal matrix D, and so, technically, the proof in [12] is incorrect. Theorem 4.2 cannot be strengthened by replacing "complex diagonal matrix D" by "positive diagonal matrix D," as demonstrated by the following example.

EXAMPLE 4.3. Let A be the complex matrix

$$\begin{pmatrix} 1 & i-1 \\ 1 & i \end{pmatrix}.$$

Since for a diagonal matrix $D = \mathrm{diag}(d_1, d_2)$ the trace of DA is $d_1 + id_2$, it follows that, although the matrix A has positive leading principal minors, there exists no positive diagonal matrix D such that the eignvalues of DA are positive.

For the sake of completeness, we now prove two new theorems that provide the result needed in [12].

THEOREM 4.4. *Let A be a complex square matrix with positive leading principal minors, and let ϵ be any positive number. Then there exists a positive diagonal matrix D such that the eigenvalues of DA are simple, and the argument of every such eigenvalue is less in absolute value than ϵ.*

Proof. We prove our assertion by induction on the order n of the matrix A. For $n = 1$ there is nothing to prove. Assume the claim holds for $n < m$, where $m > 1$, and let $n = m$. We partition the matrix A as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is an $(n-1)\times(n-1)$ block. By the inductive assumption there exists a positive diagonal matrix $D_1 = \mathrm{diag}(d_1, \dots, d_{n-1})$ such that the eigenvalues $\alpha_1, \dots, \alpha_{n-1}$ of $D_1 A_{11}$ are simple and have arguments in absolute value less that $\delta = \epsilon/(n-1)$. Now, let $D(d)$ be the diagonal matrix

$\operatorname{diag}(d_1,\dots,d_{n-1},d)$ where d is a parameter. Since the eigenvalues of $D(0)A$ are $\alpha_1,\dots,\alpha_{n-1},0$, it follows by continuity arguments that for $d>0$ sufficiently small, the eigenvalues $\lambda_1,\dots,\lambda_n$ of $D(d)A$ are simple and satisfy

$$|\text{argument}(\lambda_i)| < \delta, \qquad i \in \langle n-1 \rangle. \tag{4.5}$$

Let $\lambda_k = r_k e^{i\theta_k}$, $k \in \langle n \rangle$, where $r_k = |\lambda_k|$ and where $|\theta_k| \leqslant \pi$. Since $\det A > 0$ and since $D(d)$ is a positive diagonal matrix, it follows that

$$\det[D(d)A] = \prod_{k=1}^{n} \lambda_k = \left(\prod_{k=1}^{n} r_k\right) e^{i(\theta_1 + \cdots + \theta_n)} > 0.$$

Therefore, we have $\theta_1 + \cdots + \theta_n = 0$, or, equivalently, $\theta_n = -(\theta_1 + \cdots + \theta_{n-1})$. By (4.5) we now have

$$|\text{argument}(\lambda_n)| = |\theta_n| \leqslant \sum_{k=1}^{n-1} |\theta_k| < (n-1)\delta = \epsilon.$$

Thus, together with (4.5) we have $|\text{argument}(\lambda_k)| \leqslant \epsilon$, $k \in \langle n \rangle$. ■

THEOREM 4.6. *Let A be a complex square matrix with real principal minors and positive leading principal minors. Then there exists a positive diagonal matrix D such that DA has simple positive eigenvalues.*

Proof. We outline the proof of this theorem, which is very similar to the proof of Theorem 4.4, and omit the details. In the inductive assumption we assert that DA has simple positive eigenvalues. The positive parameter d is chosen to be sufficiently small that the real parts $\lambda_1,\dots,\lambda_n$ are distinct. Since $D(d)A$ has real principal minors, it has a real characteristic polynomial, and hence its eigenvalues appear in conjugate pairs. Since they have distinct real parts, it follows that they are all positive. ■

5. OTHER TYPES OF MATRIX STABILITY

In the previous section we discussed matrices A for which there exists a positive diagonal matrix D such that DA is stable. A related case is the one of matrices A such that for *every* positive diagonal matrix D the matrix DA is stable.

Definition 5.1. A real square matrix A is said to be *D-stable* if DA is stable for every positive diagonal matrix D.

The D-stable matrices appear in various applications such as chemical networks and economics (e.g. [42]).

Similarly to multiplicative perturbations as in Definition 5.1, we can discuss additive perturbations.

Definition 5.2. A real square matrix A is said to be *additive D-stable* if $A + D$ is stable for every nonnegative diagonal matrix D.

Applications of additive D-stability may be found in linearized biological systems (e.g. [23]).

The intersection of the classes of D-stable matrices and of additive D-stable matrices contains the following matrices (see [15]). Their definition takes us back to the Lyapunov stability criterion.

Definition 5.3. A real square matrix A is said to be *Lyapunov diagonally stable* if there exists a positive diagonal matrix D such that $AD + DA^T$ is positive definite.

Lyapunov diagonally stable matrices play an important role in many disciplines, such as predator-prey systems in ecology (e.g. [22]), economics (e.g. [36]), and dynamic systems (e.g. [1]).

The implication relations between these types of stability is demonstrated in the following diagram (e.g. [15] and [9]):

$$
\begin{tikzcd}
 & A \text{ is Lyapunov diagonally stable} \arrow[dl] \arrow[d] \arrow[dr] & \\
A \text{ is a } P\text{-matrix} & A \text{ is } D\text{-stable} \arrow[d] \arrow[dr] & A \text{ is additive } D\text{-stable} \arrow[dl] \arrow[d] \\
 & A \text{ is stable} & A \text{ is a } P_0^+\text{-matrix} ,
\end{tikzcd}
$$

where a P_0^+-matrix is a matrix with nonnegative principal minors, and where at least one principal minor of each order is positive.

The problem of characterizing the various types of matrix stability is, in general, a hard open problem, and it has been solved only for matrices of order less than or equal to 4; see Cain [10], Cross [15], Goh [21], and Johnson [35]. While regular stability is a spectral property, and so it is always possible to check whether a given matrix is stable or not by evaluating its eigenvalues, neither of the other three types of matrix stability can be characterized by the spectrum of the matrix, as demonstrated by the following easy example.

Example 5.4. The matrices

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix}-1 & 2\\ -2 & 3\end{pmatrix}$$

have the same spectrum. Nevertheless, while A has all the abovementioned types of matrix stability, B has none, since it is not a P_0^+-matrix.

This problem has been solved for certain classes of matrices. For example, in the important class of Z-matrices (that is, matrices with nonpositive off-diagonal elements), all the stability types are equivalent, as demonstrated in the following diagram (e.g. [8]):

$$\begin{tikzcd}
 & A \text{ is Lyapunov diagonally stable} \arrow[dl, Leftrightarrow] \arrow[d, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\
A \text{ is a } P\text{-matrix} \arrow[d, Leftrightarrow] & A \text{ is } D\text{-stable} \arrow[d, Leftrightarrow] & A \text{ is additive } D\text{-stable} \arrow[dl, Leftrightarrow] \\
A \text{ is a nonsingular } M\text{-matrix} & A \text{ is stable} &
\end{tikzcd}$$

A related case is the case of H-matrices.

DEFINITION 5.5. An $n \times n$ matrix A is said to be a *H-matrix* if the matrix $M(A)$ defined by

$$[M(A)]_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is an M-matrix.

We have the following theorem, proved in Hershkowitz and Schneider [31].

THEOREM 5.6. *Let A be a H-matrix. Then A is Lyapunov diagonally stable if and only if A is nonsingular and the diagonal elements of A are nonnegative.*

Another important case where these characterization problems have been completely solved is the case of acyclic matrices, that is, matrices whose undirected graph contains no cycle other than loops. Presentation of these apparently complicated results would require more graph theoretic development than appropriate to this survey. We therefore avoid it here, and only give the relevant references. The D-stability in the special case of tridiagonal matrices was treated in [13], while the complete solution is given in the sequence of papers [4], [5], [26], and [27]; see also the survey in [6]. Acyclic-3 matrices, also called quasi-Jacobi matrices, are discussed in [37].

6. THE WEAK PRINCIPAL SUBMATRIX RANK PROPERTY

In the study of H-matrices, an interesting property shared by Lyapunov diagonally stable matrices was discovered. We first define another stability type, closely related to Lyapunov diagonal stability.

DEFINITION 6.1. A real square matrix A is said to be *Lyapunov diagonally semistable* if there exists a positive diagonal matrix D, called a *Lyapunov scaling factor of* A, such that $AD + DA^T$ is positive semidefinite.

We now define the rank properties that appear to play an important role in studying stability.

DEFINITION 6.2.

(i) An $n \times n$ matrix A is said to have the *principal submatrix rank property* if for all $\varnothing \neq \alpha \subseteq \langle n \rangle$ we have

$$\operatorname{rank}(A[\alpha|\langle n \rangle]) = \operatorname{rank}(A[\langle n \rangle|\alpha]) = \operatorname{rank}(A[\alpha|\alpha]).$$

(ii) An $n \times n$ matrix A is said to have the *weak principal submatrix rank property* if for all $\varnothing \neq \alpha \subseteq \langle n \rangle$ we have

$$A[j|\alpha] = 0 \quad \Rightarrow \quad A[\alpha|j] \in \operatorname{range}(A[\alpha|\alpha]),$$

$$A[\alpha|j] = 0 \quad \Rightarrow \quad A[j|\alpha]^T \in \operatorname{range}\left(A[\alpha|\alpha]^T\right).$$

We remark that positive semidefinite Hermitian matrices are known to have the principal submatrix rank property.

In [31], the following necessary condition for Lyapunov diagonal semistability was proved.

THEOREM 6.3. *Let A be a Lyapunov diagonally semistable matrix. Then A has the weak principal submatrix rank property.*

This condition is not sufficient. Clearly, every P-matrix has even the stronger principal submatrix rank property; however, not every P-matrix is Lyapunov diagonally semistable, as demonstrated by the following example.

EXAMPLE 6.4. Let A be the P-matrix

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that for a Lyapunov diagonally semistable B, the matrix $B + \epsilon I$ is Lyapunov diagonally stable for every $\epsilon > 0$. Since the matrix A has the eigenvalues $\{4, -0.5+2.5981i, -0.5-2.5981i\}$, it follows that for $0 < \epsilon < 0.5$ the matrix $A + \epsilon I$ is not stable and hence not Lyapunov diagonally stable. Therefore, A is not Lyapunov diagonally semistable.

In the H-matrix case, however, the weak principal submatrix rank property is also sufficient for Lyapunov diagonal semistability, as proved in [31].

THEOREM 6.5. *Let A be a H-matrix with nonnegative diagonal elements. Then the following are equivalent:*

(i) *A is Lyapunov diagonally semistable.*
(ii) *A has the principal submatrix rank property.*
(iii) *A has the weak principal submatrix rank property.*

Another case in which the weak principal submatrix rank property appears to be a necessary and sufficient condition for Lyapunov diagonal semistability is the following one, proved in [27].

THEOREM 6.6. *Let A be an acyclic matrix with nonnegative principal minors. Then A is Lyapunov diagonally semistable if and only if A has the weak principal submatrix rank property.*

7. LYAPUNOV SCALING FACTORS

The search for characterizations of Lyapunov diagonally semistable matrices resulted in interesting conditions on Lyapunov scaling factors. The following theorem is proved in [31].

THEOREM 7.1. *An irreducible H-matrix A with nonnegative diagonal elements is Lyapunov diagonally semistable. Furthermore, the following are equivalent:*

(i) *A is not Lyapunov diagonally stable.*
(ii) *A is singular.*
(iii) *A has a unique Lyapunov scaling factor (up to a scalar multiplication).*
(iv) *There exists a unique (up to a scalar multiplication) (positive) diagonal matrix D such that* $\mathrm{kernel}(AD) = \mathrm{kernel}(A^T)$.

The uniqueness of Lyapunov scaling factors was first discussed in [30], where the following observation is made.

LEMMA 7.2. *Let D be a Lyapunov scaling factor of A. Then* $\mathrm{kernel}(AD) = \mathrm{kernel}(A^T) \subseteq \mathrm{kernel}(AD + DA^T)$.

Consequently, the study of scalings that equalize the kernels of A and A^T leads to a sufficient condition and to a similar necessary condition for the uniqueness of the Lyapunov scaling factor. The conditions are formulated in terms of connectedness of a certain graph associated with the echelon form of a basis for kernel(A). The gap between the sufficient condition and the necessary condition was bridged in [44] and [45].

Another interesting study that has led to some important results is the study of maximal Lyapunov scaling factors.

NOTATION 7.3. Let D be a Lyapunov scaling factor of a Lyapunov diagonally semistable matrix A. The vector space Range$(AD + DA^T)$ is denoted by $V(D, A)$.

The following theorem is proved in [46].

THEOREM 7.4. *Let A be a Lyapunov diagonally semistable matrix, and let D_0 be a Lyapunov scaling factor of A such that $V(D_0, A)$ is of maximal dimension. Then for every Lyapunov scaling factor D of A we have $V(D, A) \subseteq V(D_0, A)$.*

Motivated by Theorem 7.4, we define

DEFINITION 7.5. A Lyapunov scaling factor D_0 of A is said to be a *maximal Lyapunov scaling factor of* A if for every Lyapunov scaling factor D of A we have $V(D, A) \subseteq V(D_0, A)$. $V(D_0, A)$ is called the *Lyapunov range of* A, and is denoted by V_A.

EXAMPLE 7.6. Let A be the matrix

$$\begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

The diagonal matrix $D_0 = \mathrm{diag}(2, 2, 1)$ is a maximal Lyapunov scaling factor of A, and the Lyapunov range is $V_A = V(D_0, A) = \{(x, x, y)^T : x, y \in \mathbf{R}\}$. The identity matrix I is also a Lyapunov scaling factor of A, and $V(I, A) = \{(x, x, x)^T : x \in \mathbf{R}\}$, which is contained in V_A.

Among the interesting properties of maximal Lyapunov scaling factors proved in [46] are the following two theorems.

THEOREM 7.7. *Let D_0 be a maximal Lyapunov scaling factor of A, and let D be a Lyapunov scaling factor of A. Then $D_0 + D$ is a maximal Lyapunov scaling factor of A. Also, D_0^{-1} is a maximal Lyapunov scaling factor of A^T.*

THEOREM 7.8. *Let A be a Lyapunov diagonally semistable matrix. We have* $V_A \subseteq \mathrm{Range}(A)$.

Let A be a Lyapunov diagonally semistable matrix in a $p \times p$ block triangular form with square diagonal blocks. Clearly, it follows that the diagonal blocks too are Lyapunov diagonally semistable matrices. The above-mentioned theorems imply results on the relations between maximal Lyapunov scaling factors of A and maximal Lyapunov scaling factors of the diagonal blocks A_{ii}, $i \in \langle p \rangle$. In particular it yields the following well-known theorem, e.g. [4].

THEOREM 7.9. *Let A be a matrix in a* $p \times p$ *block triangular form with square diagonal blocks* $A_{11}, \dots, A_{pp}$. *Then A is Lyapunov diagonally stable if and only if* $A_{11}, \dots, A_{pp}$ *are Lyapunov diagonally stable.*

Theorem 7.9 does not hold when "Lyapunov diagonally stable" is replaced by "Lyapunov diagonally semistable," as demonstrated by the following example.

EXAMPLE 7.10. Let A be the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Consider A as a 2×2 block triangular matrix, where all the blocks are 1×1. It is easy to verify that although the diagonal blocks of A are Lyapunov diagonally semistable, the matrix A is not Lyapunov diagonally semistable.

Nevertheless, the investigation of maximal Lyapunov scaling factors results in the following important theorem, proved in [46].

THEOREM 7.11. *Let A be a matrix in a* $p \times p$ *(upper) block triangular form with square diagonal blocks* $A_{11}, \dots, A_{pp}$. *Then A is Lyapunov diagonally semistable if and only if for every* $i, j \in \langle p \rangle$, $i < j$, *the matrix*

$$\begin{pmatrix} A_{ii} & A_{ij} \\ 0 & A_{jj} \end{pmatrix}$$

is Lyapunov diagonally semistable.

It follows from Theorem 7.11 that the problem of characterizing block triangular Lyapunov diagonally semistable matrices with p diagonal blocks can be reduced to the problem of characterizing block triangular Lyapunov diagonally semistable matrices with just two diagonal blocks. This observation might be very useful in the investigation of reducible Lyapunov diagonally semistable matrices.

We conclude this section by referring the reader to the intensive study [32] of the structure of stability factors of stable matrices A, that is, positive definite Hermitian matrices H such that the matrix $AH + HA^*$ is positive definite.

8. CONES OF REAL POSITIVE SEMIDEFINITE MATRICES ASSOCIATED WITH MATRIX STABILITY

The use of cones of real positive semidefinite matrices in characterizing the various types of matrix stability originated in a paper by Barker, Berman, and Plemmons [3]. Since then, this tool has been developed and used to produce many results.

NOTATION 8.1. Let A be a square real matrix.

(i) We denote by $B_{-0}(A)$ the cone consisting of all real symmetric positive semidefinite matrices B such that the diagonal elements of BA are nonpositive.

(ii) We denote by $B_{-}(A)$ the set consisting of all real symmetric positive semidefinite matrices B such that the diagonal elements of BA are negative. Observe that $B_{-}(A) \cup \{0\}$ is a cone.

(iii) We denote by $B_0(A)$ the cone consisting of all real symmetric positive semidefinite matrices B such that the diagonal elements of BA are zero.

The following is a restatement of a theorem proved in [3].

THEOREM 8.2. *A real square matrix A is Lyapunov diagonally stable if and only if* $B_0(A) = B_{-0}(A) = \{0\}$.

In view of Theorem 7.9, in the Lyapunov diagonal stability case it is enough to handle the irreducible matrices. The following theorem, proved in [33], improves Theorem 8.2.

THEOREM 8.3. *Let A be an irreducible real matrix with nonnegative diagonal elements. Then the following are equivalent*:

(i) *A is Lyapunov diagonally stable.*
(ii) $B_0(A) = B_{-0}(A) = \{0\}$.
(iii) $B_0(A) = \{0\}$.

Theorems 8.2 and 8.3 raise the natural question concerning the existence of a similar characterization for Lyapunov diagonally semistable matrices. Indeed, in one direction we have the following theorem, proved in [33].

THEOREM 8.4. *If A is a Lyapunov diagonally semistable matrix, then $B_0(A) = B_{-0}(A)$ (and consequently $B_-(A)$ is empty).*

The converse of Theorem 8.4 is not true, as demonstrated by the following example.

EXAMPLE 8.5. The matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

satisfies $B_0(A) = B_{-0}(A)$. However, A does not have the weak principal submatrix rank property, since $A[\{3\}|\{1\}] = 0$ while $A[\{1\}|\{3\}]^T \notin \text{range}(A[\{3\}|\{3\}]^T)$, and hence, by Theorem 6.3, A is not Lyapunov diagonally semistable.

We now define a weaker type of matrix stability.

DEFINITION 8.6. A real square matrix A is said to be *Lyapunov diagonally weakly semistable* if there exists a nonzero nonnegative diagonal matrix D such that $AD + DA^T$ is positive semidefinite.

Here we have the following characterization, proved in [9].

THEOREM 8.7. *A real square matrix A is weakly diagonally semistable if and only if $B_-(A)$ is empty.*

This theorem helps also in characterizing Lyapunov diagonal semistability, since we have the following relation, proved in [33].

THEOREM 8.8. *Let A be an irreducible real square matrix. Then A is Lyapunov diagonally semistable if and only if A is weakly diagonally semistable.*

From here one can obtain the following characterization of completely reducible (that is, a direct sum of irreducible) Lyapunov diagonally semistable matrices; see [33].

THEOREM 8.9. *Let A be a completely reducible real matrix. Then the following are equivalent:*

(i) *A is Lyapunov diagonally semistable.*
(ii) $B_{-0}(A) = B_0(A)$.
(iii) *For every component A_{ii} of A, the cone $B_-(A_{ii})$ is empty.*

9. INERTIA PRESERVING MATRICES

We conclude this article by reviewing a recently introduced subject, which is related to D-stability and Lyapunov diagonal stability.

DEFINITION 9.1. The *inertia* of a square matrix A is defined as the triple $(i_+(A), i_0(A), i_-(A))$, where

$i_+(A)$ = the number of eigenvalues of A with positive real part.
$i_0(A)$ = the number of eigenvalues of A on the imaginary axis.
$i_-(A)$ = the number of eigenvalues of A with negative real part.

DEFINITION 9.2.

(i) A real matrix A is said to be *inertia preserving* if inertia (AD) = inertia(D) for every nonsingular real diagonal matrix D.

(ii) A real matrix A is said to be *strongly inertia preserving* if inertia(AD) = inertia(D) for every (not necessarily nonsingular) real diagonal matrix D.

The following statement is immediate.

THEOREM 9.3. *Inertia preserving matrices are D-stable.*

It is shown in [7] that the converse of Theorem 9.3 is not true, using the following example from [24].

EXAMPLE 9.4. The matrix

$$\begin{pmatrix} 1 & 0 & -50 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

is D-stable. However, for $D = \operatorname{diag}(-1, 3, -1)$ the matrix AD is stable, and hence A is not inertia preserving.

In fact, it is shown in [7] that even P-matrices that are both D-stable and Lyapunov diagonally semistable are not necessarily inertia preserving. However, the following theorem is proved in [7].

THEOREM 9.5. *Lyapunov diagonally stable matrices are strongly inertia preserving.*

It is asked in [7] whether the converse of Theorem 9.5 is true.

QUESTION 9.6. Is every strongly inertia preserving matrix Lyapunov diagonally stable?

We are now informed that Question 9.6 has been answered negatively by Berman and Shasha (private communication). Nevertheless, we are able to answer this question positively in the acyclic case.

THEOREM 9.7. *Let A be a strongly inertia preserving acyclic matrix. Then A is Lyapunov diagonally stable.*

Proof. Clearly, a strongly inertia preserving matrix is a P-matrix. By Theorem 2 in [4] it follows that A is Lyapunov diagonally stable. ■

A similar question posed in [7] is

QUESTION 9.8. Is every irreducible inertia preserving matrix Lyapunov diagonally semistable?

The concluding result in [7], which is proved by using the cones discussed in the previous section, is

THEOREM 9.9. *An irreducible acyclic matrix is D-stable if and only if it is inertia preserving.*

Theorem 9.9 enables us to provide a positive answer to Question 9.8 too in the acyclic case.

THEOREM 9.10. *Let A be an irreducible inertia preserving acyclic matrix. Then A is Lyapunov diagonally semistable.*

Proof. Let A be an irreducible inertia preserving acyclic matrix. By Theorem (9.9) A is D-stable, and by Theorem 3 in [26] it follows that A is Lyapunov diagonally semistable. ■

The author is grateful to Professor David Carlson, to Professor Hans Schneider, and to the referee for their comments.

REFERENCES

1 M. Araki, Applications of M-matrices to the stability problems of composite dynamical systems, *J. Math. Anal. Appl.* 52:309–321 (1975).
2 C. S. Ballantine, Stabilization by a diagonal matrix, *Proc. Amer. Math. Soc.* 25:728–734 (1970).
3 G. P. Barker, A. Berman, and R. J. Plemmons, Positive diagonal solutions to the Lyapunov equations, *Linear and Multilinear Algebra* 5:249–256 (1978).
4 A. Berman and D. Hershkowitz, Matrix diagonal stability and its implications, *SIAM J. Algebraic Discrete Methods* 4:377–382 (1983).
5 A. Berman and D. Hershkowitz, Characterization of acyclic D-stable matrices, *Linear Algebra Appl.* 58:17–31 (1984).
6 A. Berman and D. Hershkowitz, Graph theoretical methods in studying stability, *Contemp. Math.* 47:1–6 (1985).
7 A. Berman and D. Shasha, Inertia preserving matrices, *SIAM J. Matrix Anal. Appl.*, to appear.
8 A. Berman, R. S. Varga, and R. C. Ward, *ALPS*: Matrices with nonpositive off-diagonal entries, *Linear Algebra Appl.* 21:233–244 (1978).
9 A. Berman and R. C. Ward, *ALPS*: Classes of stable and semipositive matrices, *Linear Algebra Appl.* 21:163–174 (1978).
10 B. E. Cain, Real 3×3 D-stable matrices, *J. Res. Nat. Bur. Standards Sect. B* 80:75–77 (1976).
11 D. Carlson, Weakly sign-symmetric matrices and some determinantal inequalities, *Colloq. Math.* 17:123–129 (1967).
12 D. Carlson, A class of positive stable matrices, *J. Res. Nat. Bur. Standards Sect. B* 78:1–2 (1974).
13 D. Carlson, B. N. Datta, and C. R. Johnson, A semidefinite Lyapunov theorem and the characterization of tridiagonal D-stable matrices, *SIAM J. Algebraic Discrete Methods* 3:293–304 (1982).
14 A. Cauchy, Sur l'équation à l'aide de laquelle on détermine les inégalités seculaires de mouvement des planètes, *Oeuv. Comp.* 9(2):174–195 (1829).
15 G. W. Cross, Three types of matrix stability, *Linear Algebra Appl.* 20:253–263 (1978).

16 G. M. Engel and H. Schneider, The Hadamard-Fischer inequality for a class of matrices defined by eigenvalue monotonicity, *Linear and Multilinear Algebra* 4:155–176 (1976).
17 M. E. Fisher and A. T. Fuller, On the stabilization of matrices and the convergence of linear iterative processes, *Proc. Cambridge Philos. Soc.* 54:417–425 (1958).
18 S. Friedland, Weak interlacing property of totally positive matrices, *Linear Algebra Appl.* 71:95–100 (1985).
19 M. Fujiwara, On algebraic equations whose roots lie in a circle or in a half plane, *Math. Z.* 24:161–169 (1926).
20 F. R. Gantmacher, *The Theory of Matrices*, Chelsea, New York, 1960.
21 B. S. Goh, Global stability in two species interactions, *J. Math. Biol.* 3:313–318 (1976).
22 B. S. Goh, Global stability in many species systems, *Amer. Nat.* 111:135–143 (1977).
23 K. P. Hadeler, Nonlinear diffusion equations in biology, in *Proceedings of the Conference on Differential Equations*, Dundee, 1976, Springer Lecture Notes.
24 D. J. Hartfiel, Concerning the interior of the D-stable matrices, *Linear Algebra Appl.* 30:201–207 (1980).
25 C. Hermite, Sur le nombre des racines d'une équation algébrique comprise entre des limites données, *J. Reine Angew. Math.* 52:39–51 (1856).
26 D. Hershkowitz, Stability of acyclic matrices, *Linear Algebra Appl.* 73:157–169 (1986).
27 D. Hershkowitz, Lyapunov diagonal semistability of acyclic matrices, *Linear and Multilinear Algebra* 22:267–283 (1988).
28 D. Hershkowitz and A. Berman, Localization of the spectra of P- and P_0-matrices, *Linear Algebra Appl.* 52/53:383–397 (1983).
29 D. Hershkowitz and A. Berman, Notes on ω- and τ-matrices, *Linear Algebra Appl.* 58:169–183 (1984).
30 D. Hershkowitz and H. Schneider, Scalings of vector spaces and the uniqueness of Lyapunov scaling factors, *Linear and Multilinear Algebra* 17:203–226 (1985).
31 D. Hershkowitz and H. Schneider, Lyapunov diagonal semistability of real H-matrices, *Linear Algebra Appl.* 71:119–149 (1985).
32 D. Hershkowitz and H. Schneider, Semistability factors and semifactors, *Contemp. Math.* 47:203–216 (1985).
33 D. Hershkowitz and D. Shasha, Cones of real positive semidefinite matrices associated with matrix stability, *Linear and Multilinear Algebra* 23:165–181 (1988).
34 A. Hurwitz, Über die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt, *Math. Ann.* 46:273–284 (1895).
35 C. R. Johnson, Second, third and fourth order D-stability, *J. Res. Nat. Bur. Standards Sect. B* 78:11–13 (1974).
36 C. R. Johnson, Sufficient conditions for D-stability, *J. Econom. Theory* 9:53–62 (1974).

37 E. Kaszkurewicz and L. Hsu, On two classes of matrices with positive diagonal solutions to the Lyapunov equation, *Linear Algebra Appl.* 59:19–27 (1984).
38 R. B. Kellogg, On complex eigenvalues of M and P matrices, *Numer. Math.* 19:170–175 (1972).
39 Liénard and Chipart, Sur la signe de la partie réelle des racines d'une équation algébrique, *J. Math. Pures Appl.* (6) 10:291–346 (1914).
40 A. M. Lyapunov, *Le Problème Général de la Stabilité du Mouvement*, Ann. Math. Stud. 17, Princeton U.P., 1949.
41 V. Mehrmann, On Classes of Matrices Containing M-Matrices, Totally Nonnegative and Hermitian Positive Semidefinite Matrices, Ph.D. Dissertation, Bielefeld, 1982.
42 L. Metzler, Stability of multiple markets: The Hick conditions, *Econometrica* 13:277–292 (1945).
43 E. J. Routh, *A Treatise on the Stability of a Given State of Motion*, Macmillan, London, 1877.
44 D. Shasha and A. Berman, On the uniqueness of the Lyapunov scaling factors, *Linear Algebra Appl.* 91:53–63 (1987).
45 D. Shasha and A. Berman, More on the uniqueness of the Lyapunov scaling factors, *Linear Algebra Appl.* 107:253–273 (1988).
46 D. Shasha and D. Hershkowitz, Maximal Lyapunov scaling factors and their applications in the study of Lyapunov diagonal semistability of block triangular matrices, *Linear Algebra Appl.* 103:21–39 (1988).
47 O. Taussky, Research problem, *Bull. Amer. Math. Soc.* 64:124 (1958).
48 R. S. Varga, Recent results in linear algebra and its applications (in Russian), in *Numerical Methods in Linear Algebra* (Proceedings of the Third Seminar on Methods of Numerical Applied Mathematics, Novosibirsk, 1978), Akad. Nauk SSSR Sibirsk. Otdel. Vychisl. Tsentr, Novosibirsk, 1978, pp. 5–15.

Received 7 January 1991; final manuscript accepted 30 May 1991