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A new centrality ranking method for multilayer networks

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ABSTRACT

There are a good deal of centrality ranking evaluation methods for single-layer networks. However, the centrality evaluation method of a single-layer network cannot guarantee the accuracy of the measurement results on a multilayer network. Most of the existing methods for node centrality in multilayer networks ignore the effect of network interlayer relationships on the importance of nodes. Thereby, the problem of missing information between layers of multilayer networks occurs in the calculation process, resulting in inaccurate assessment results or differences between nodes that are too small to determine their significance. This paper proposes a new centrality measure for multilayer networks, which measures the importance of nodes by calculating the weighted local structure entropy of nodes. The proposed method not only considers the importance of connections within node layers, but also adds the influence of the number of network layers between node layers. Through the empirical analysis of real multilayer networks of different types and sizes, such as the Hubei transportation multilayer network, Lazega-Law-Firm multilayer network, CS-Aarhus multilayer network, CKM-Physicians-Innovation multilayer network, it is proved that the proposed method is optimal and general.

1. Introduction

With the richness and diversity of the types of information interactions in life, the research on network science is gradually developing from a single-layer network to a multilayer network [1–3]. The centrality of an independent single-layer network can no longer meet the needs of practical research. The evaluation of node importance in a multilayer network is also significantly more complex than a single-layer network [1]. For example, a person may have accounts registered on multiple different social platforms, but the level of activity on different social platforms is different (for example, active on WeChat and QQ, but relatively quiet on Weibo and Email), so it is too one-sided and inaccurate to measure the influence of this person from any single social platform. Therefore, the research on the centrality of multilayer networks is of great significance.

Effective centrality methods can help control public opinion transmission [4], optimize resource allocation [5], determine faster dissemination paths for information, epidemics, failures and congestion [6]. Some classical methods used to identify node centrality based on network topology include degree centrality [7], betweenness centrality [8], closeness centrality [9], eigenvector centrality [10], PageRank [11], K-shell [12], etc [13–15]. On this basis, there are many improved algorithms to evaluate the centrality of multilayer networks. The PageRank node centrality algorithm of a multilayer network depends not only on the quality and quantity of pointing to the node, but also on the relative importance of the layer where the node is located. The Refs. [16,17] defined the PageRank algorithm for multilayer networks based on the interactions between layers. In multiplex networks, the PageRank of multiplex networks was applied as a smooth distribution for calculation by defining some constraints in Ref. [18]. The Ref. [19] introduced the FMP (Functional Multiplex PageRank) method to obtain the importance of nodes by maximizing the centrality function. The Ref. [20] established a multilayer PageRank based on the biased random walk model, which directly considered the influence of the interaction between network layers on the node centrality. The tensor analysis method based on multilayer network is more used to study the centrality of layers and the centrality of nodes affected by the connection between layers. The Ref. [21] quantified the centrality of multilayer networks based on tensor decomposition methods. The Ref. [22] used the tensor framework to study the centrality of multilayer network eigenvectors, and considered the effect of intralayer couplings on the centrality of multilayer networks. The Ref. [23] introduced a new feature vector centrality-based measure based on the definition of tensor analysis methods. The Refs. [24,25] presented a measure of centrality for multiplex networks using an iterative algorithm of tensor equations. This iterative algorithm compressed and aggregated the multilayer network into a single-layer network to quantify the importance of edges, and then defined the centrality of a

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layer as the sum of the edge centrality of each layer in the multilayer network. Applying the iterative algorithm can not only calculate the node centrality but also the centrality of the layers in a multilayer network. In addition, there are several studies focusing on the topology information of connections within multiple network layers to quantify node centrality. The Ref. [26] combined the importance of layers and nodes to determine the layer centrality as well as the node centrality of a multilayer network. The basic idea of combining the importance of layers and nodes is that the more nodes with high node centrality in a network layer, the higher the centrality of the layer, correspondingly, the higher the centrality of the network layer, the higher the centrality of the nodes connected to that layer. However, these methods lack consideration of the mutual influence of connecting nodes in different multilayer networks, and ignore the influence of the number of layers connected by nodes on the centrality of nodes. It is easy to cause information loss and cannot completely retain the multilayer network structure information.

This paper proposes a new centrality measurement method for multilayer networks. Firstly, for the nodes connected between layers, assigning a centrality to the node using the betweenness centrality of the local single-layer network, and adding the number of interlayer connection edges of the multilayer network to the connection nodes from the overall structure. Secondly, the sum of the betweenness centrality of each layer of network nodes and the number of connecting edges between layers is defined as the weight of connecting nodes in each layer. Finally, the structure entropy is used to measure the node centrality after weighting. The core idea of this method is that the more layers a node is connected to and the larger the centrality of the connected nodes, the higher the centrality evaluation rank of the node in the overall multilayer network structure. Correspondingly, the centrality rank of a node in a layer of a multilayer network is high, but the centrality rank in the overall structure is not necessarily high. The new centrality measurement method in this paper not only combines the centrality of the target node and the node connection layer, but also considers the effect of the number of connection layers of the node in the multilayer network. The effectiveness and superiority of the method proposed in this paper are demonstrated by empirical analysis on four real multilayer networks of different types and scales.

The rest of the paper is organized as follows, some basic concepts, which including multilayer networks, several multilayer network centrality indicators and structure entropy, are introduced in Section 2. In Section 3, a new method to measure the node centrality of multilayer network is proposed. In Section 4, the effectiveness and generality of the method proposed in this paper are illustrated by comparing the measurement results of several multilayer network centrality measurement methods on real multilayer networks. Finally, Some conclusions are summarized in Section 5.

2. Preliminaries

In this section, some basic concepts, which including multilayer networks, several multilayer network centrality indicators and structure entropy, are introduced.

2.1. Multilayer networks

This section introduces the definition and types of multilayer networks.

2.1.1. Definition of multilayer network

A multilayer network can be represented by a binary group $M = (G, C)$ [1,27], where network $G = \{G^\alpha; \alpha \in \{1, 2, \ldots, L\}\}$ represents a set of (directed or undirected, weighted or unweighted) graphs $G^\alpha = (V^\alpha, E^\alpha)$. G^α is called layer α of the multilayer network M, $V^\alpha = \{v_1^\alpha, v_2^\alpha, \ldots, V_{N^\alpha}^\alpha\}$ represents the node set of layer G^α (N^α is the number of nodes in layer G^α), E^α represents the intralayer connections of layer G^α [28]. $C = \{E^{\alpha\beta} \in V^\alpha \times V^\beta; \alpha, \beta \in \{1, \ldots .L\}, \alpha \neq \beta\}$ is the set of interconnections between nodes of different layers G^α and G^β, while $E^{\alpha\beta}$ denotes a single interlayer link of layers G^α and G^β [1,29].

The intralayer adjacency matrix of layer G^α is expressed as $A^\alpha = (a_{ij}^\alpha) \in R^{N^\alpha \times N^\alpha}$, where,

$$a_{ij}^\alpha = \begin{cases} 1 & (v_i^\alpha, v_j^\alpha) \in E^\alpha \\ 0 & \text{others} \end{cases} \tag{1}$$

where $1 \leq i, j \leq N^\alpha$ and $1 \leq \alpha \leq L$. The interlayer adjacency matrix $C^{\alpha\beta} = (c_{ij}^{\alpha\beta}) \in R^{N^\alpha \times N^\beta}$ corresponding to $E^{\alpha\beta}$, in which,

$$c_{ij}^{\alpha\beta} = \begin{cases} 1 & (v_i^\alpha, v_j^\beta) \in E^{\alpha\beta} \\ 0 & \text{others} \end{cases} \tag{2}$$

Therefore, the multilayer network can be established as a mathematical model [1], which is expressed as $M = (V^M, E^M)$, in which,

$$V^M = \bigcup_{\alpha=1}^{L} V^\alpha \tag{3}$$

$$E^M = (\bigcup_{\alpha=1}^{L} E^\alpha) \cup (\bigcup_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{L} C^{\alpha\beta}) \tag{4}$$

2.1.2. Types of multilayer networks

From the structural point of view, we can simply classify multilayer networks into two types. The above mathematical model provides a theoretical research basis for studying these two types of multilayer networks in this paper.

• Multidimensional Multilayer Network [30]

Multidimensional multilayer network is a type of network in which the multilayer network M considers both G and C tuples, mathematically represented as $M = (G, C)$, satisfying $V^1 = \cdots = V^L = L$. Its intralayer couplings are denoted as $E^{\alpha\beta} = \{(v, v); v \in V\}(1 \leq \alpha \neq \beta \leq L)$.

This type of multilayer network is characterized by the existence of the same set of nodes in each layer but different edge attributes of nodes in each layer, and the existence of edges of the same nodes between layers. The network structure diagram of this type is shown in Fig. 1(a).

• Dependent Multilayer Network [31]

In a dependent multilayer network, non-corresponding nodes between different layers can also have edges, and nodes in different layers can have a one-to-many relationship. Dependent multilayer networks can be seen as an extension of multidimensional multilayer networks, which is also a special type of multilayer network $M = (G, C)$. The mathematical expression of the connection between the layers of the dependent multilayer network is $(\cup\{(v_i^\alpha, v_i^\beta); (v_i^\alpha, v_i^\beta) \in E^{\alpha\beta}\}) \cup (\cup\{(v_i^\alpha, v_j^\beta); (v_i^\alpha, v_j^\beta) \in E^{\alpha\beta}\})(1 \leq \alpha \neq \beta \leq \mathrm{L})$, the number of shared nodes in each layer satisfies $|\cap_\alpha X^\alpha| \leq |X^\alpha| (1 \leq \alpha \leq L)$. The network structure diagram of this type is shown in Fig. 1(b).

2.2. Several centrality indicators

This section describes several multilayer network centrality metrics that are compared in this paper.

2.2.1. Degree centrality [7]

The degree centrality of a node is a local concept, which measures the influence of local neighbor nodes on that node. The greater the degree of a node, the more important the node is in the network. The formula is defined as

$$DC(i) = \frac{\sum_j^N x_{ij}}{N - 1} \tag{5}$$

Given a multilayer network, the degree of node i is $k_i = \{k_i^\alpha; \alpha \in \{1, 2, \ldots, L\}\}$, where $k_i^\alpha = \sum_j^N x_{ij}^\alpha$ represents the degree value of node i in layer G^α. The overlapping degree of nodes in each layer is $o_i = \sum_{\alpha=1}^L k_i^\alpha$.

Among them, x_{ij} represents the connection relationship between node i and node j, and the value is 1 when the two nodes are connected, otherwise, it is 0. N denotes the total number of nodes in the network.

(a) Multidimensional Multilayer Network (b) Dependent Multilayer Network

Fig. 1. Two types of multilayer network structure diagrams.

(a) Multidimensional Multilayer Network (b) Dependent Multilayer Network

Fig. 2. Multilayer network node random walk scheme.

2.2.2. Betweenness centrality [8]

Unlike degree centrality, betweenness centrality is a global concept. It is used to measure the betweenness centrality of nodes in the whole network. The greater the number of shortest paths through a node in the network, the greater the influence of that node. Specifically defined as

$$BC(i) = \frac{2}{(N-1)(N-2)} \sum_{s \neq i \neq t} \frac{g_{st}(i)}{g_{st}} \tag{6}$$

where g_{st} is the number of all shortest paths from node s to node t, and $g_{st}(i)$ is the number of shortest paths through i among the shortest paths that node is the shortest path to node t. $\frac{2}{(N-1)(N-2)}$ is the formula used to normalize the betweenness, and N is the total number of network nodes.

2.2.3. Random walk betweenness centrality [13]

The number of shortest paths passing through a node in the process of random walk in the network can represent the importance of the node in the network. That is, the more times the shortest path is passed, the higher the weight is given to the node. In order to avoid the deviation caused by back and forth percolation in the network on the betweenness centrality of the same node, it is stipulated that the effects of percolation in opposite directions during one random tour on the betweenness centrality of that node cancel each other out. Therefore, the random walk betweenness centrality of node v_i can be expressed as

$$RWBC(i) = \frac{\sum_{s \neq t} I_{st}^{i}}{N(N-1)/2} \tag{7}$$

where I_{st}^{i} represents the number of times the node v_i is passed during the random walk from the source node v_s to the target node v_t.

The paths of random walks of nodes in a multilayer network have more choices than in a single-layer network [32]. In the multilayer network shown in Fig. 2, the random walk path of node v_i includes not only neighbor nodes within the same layer, but also homogeneous nodes on different layers.

2.2.4. Closeness centrality [9]

Closeness centrality is to use the average distance of nodes in the network to determine the importance of nodes. The smaller the average distance of a node to other nodes in the network, the greater the closeness centrality of the node, and the more important the node is in the network. The closeness centrality of node i in the network is defined as

$$CC(i) = \frac{1}{\sum_{i \neq j} d_{ij}} \tag{8}$$

among them, d_{ij} is the shortest distance from node i to other nodes.

Closeness centrality can also be used in multilayer networks to measure node importance. According to the multilayer network structure, accumulating the closeness centrality of homogeneous nodes in different layers. The calculation of the centrality of heterogeneous nodes in a multilayer network is similar to the random walk centrality, in which a corresponding number of adjacent nodes are added according to the location of the network layer.

2.2.5. A tensor-based framework for studying eigenvector multicentrality [22]

The Ref. [22] describes characterizing eigenvector multicentrality in general multilayer networks by developing a tensor-based framework, and demonstrated the uniqueness of the existence of eigenvector centrality in multilayer networks given the form of interlayer influence. This method considers the contribution of the interlayer interaction mode to the multilayer centrality in resource allocation, namely

$$H_{j\beta}^{i\alpha} \Phi_{i\alpha} = \lambda_{\beta} \Phi_{j\beta} \tag{9}$$

where λ_β is a coefficient related to layer β. The interaction of the layers is expressed as $H_{j\beta}^{i\alpha} = W_{\beta}^{\alpha} u_{j\beta}^{i\alpha}$, which is the product of the interaction $u_{j\beta}^{i\alpha}$ from the node i in layer α to the node j in layer β and the influence tensor W_{β}^{α}, where the influence tensor W_{β}^{α} is used to measure the interlayer influence from layer α to layer β. Intuitively, when the influence from layer α to layer β is greater than 1, the centrality scores propagating along the links from layer α to layer β will be magnified, and vice versa.

2.2.6. MRFNL centrality [25]

In Ref. [25], a tensor iteration equation was used to establish a measure of the centrality of multiplex networks, referred to as the $MRFNL$ centrality. The method obtains the centrality of nodes and layers by developing a new iterative algorithm to compute a set of tensor equations. Under some conditions, the existence and uniqueness of this centrality were proven by applying the Brouwer fixed point theorem. The tensor iteration equation is

$$\begin{cases} x_i = \sum_{\alpha=1}^{M}\sum_{j=1}^{N}\sum_{\beta=1}^{M} o_{i\alpha j\beta} x_j y_\alpha y_\beta \\ y_\alpha = \sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{\beta=1}^{M} o_{i\alpha j\beta} x_i x_j y_\beta \end{cases} \tag{10}$$

where $1 \le i, j \le N$; $1 \le \alpha, \beta \le M$; x_i and y_α distributions represent the node centrality and layer centrality of the multilayer network.

2.3. Structure entropy

Structure entropy is an important method in the study of node importance of complex networks [33,34]. Measuring the importance of nodes in a network is not only that the nodes themselves are important in a complex network, but also that the local network formed by nodes and neighboring nodes has a great influence in the whole complex network. Therefore, it is reasonable to use the local structure entropy of nodes to measure the importance of nodes. The definition of structure entropy is shown in the following Eq. (11).

$$E(i) = -\sum_{j=1}^{n} p_i(j) \log p_i(j), \quad p_i(j) = \frac{\deg ree(j)}{\sum_{j=1}^{n} \deg ree(j)} \tag{11}$$

Among them, p_i is the probability set of node i. n is the total number of local network nodes including node i.

3. A new method to measure the node centrality of multilayer network

3.1. The proposed method

In a multilayer network, when one or more social relations are from one domain to another and mixed with the original relations, the structural characteristics of the network are transformed. Aiming at the structural characteristics of multilayer network, this paper introduces the centrality of single-layer network into multilayer network, and proposes a new method to measure the centrality of multilayer network, which combining the number of network layers and the relationship between layers. It not only combines the global importance of each layer of network nodes, but also considers the influence between layers. The specific method steps are shown in Algorithm 1.

When the multilayer network has only two layers, the mathematical expression is as follows,

$$D^{\alpha}_{BC}(i) = \begin{cases} BC^{\alpha}(i) + BC^{\beta}(j) + C^{\alpha\beta}_{ij}, & C^{\alpha\beta}_{ij} = 1 \\ BC^{\alpha}(i), & C^{\alpha\beta}_{ij} = 0 \end{cases} \tag{12}$$

So on and so forth, when the same node in a multilayer network is connected to at most L layers, the mathematical expression is as follows,

$$D^{\alpha}_{BC}(i) = \begin{cases} BC^{\alpha}(i) + BC^{\beta}(j) + \cdots + BC^{L}(j) + L \times C^{\alpha\beta}_{ij}, & C^{\alpha\beta}_{ij} = 1 \\ BC^{\alpha}(i), & C^{\alpha\beta}_{ij} = 0 \end{cases} \tag{13}$$

where $1 \le i, j \le N^{\alpha}$, $1 \le \alpha \ne \beta \le \mathrm{L}$, $BC^{\alpha}(i) = \frac{2}{(N-1)(N-2)} \sum_{s \ne i \ne t} \frac{g^{\alpha}_{st}(i)}{g^{\alpha}_{st}}$.

After increasing the importance of "bridge" nodes in the network, the original multilayer network becomes a weighted multilayer network. In this paper, the weighted local structure entropy is used to measure the centrality of multilayer network nodes. The mathematical expression is as follows,

Input: Import the intralayer and interlayer adjacency matrix of the nodes of the multilayer network to be calculated
Output: $con(i), E^{\alpha}_{D}(i)$

$con(r_i) = \Phi$, $D^{\alpha}_{BC}(i) = \Phi$;
for $1 \le i, j \le N^{\alpha}$, $1 \le \alpha \ne \beta \le \mathrm{L}$ **do**
  $BC^{\alpha}(i) = \frac{2}{(N-1)(N-2)} \sum_{s \ne i \ne t} \frac{g^{\alpha}_{st}(i)}{g^{\alpha}_{st}}$;
  while $C^{\alpha\beta}_{ij} = 1$ **do**
    $D^{\alpha}_{BC}(i) = BC^{\alpha}(i) + BC^{\beta}(j) + C^{\alpha\beta}_{ij}$;
    if $C^{\alpha\beta}_{ij} = 0$ **then**
      $D^{\alpha}_{BC}(i) = BC^{\alpha}(i)$;
    end
  end
  $con(r_i) = con(r_i) \cup D^{\alpha}_{BC}(i)$;
end
return $con(r_i)$;
for $1 \le i, j \le N^{\alpha}$, $1 \le \alpha \le \mathrm{L}$ **do**
  if $C^{\alpha}_{ij} = 1$ **then**
    $degree(j) + 1$;
  end
end
$p^{\alpha}_{i}(j) = \frac{con(r_i) + \deg ree(j)}{\sum_{j=1}^{n^{\alpha}} (con(r_i) + \deg ree(j))}$;
$E^{\alpha}_{D}(i) = \sum_{j=1}^{n^{\alpha}} p^{\alpha}_{i}(j) \log p^{\alpha}_{i}(j)$;
return $E^{\alpha}_{D}(i)$;

Algorithm 1: Multilayer network centrality algorithm

$$E^{\alpha}_{D}(i) = \sum_{j=1}^{n^{\alpha}} p^{\alpha}_{i}(j) \log p^{\alpha}_{i}(j),$$

$$p^{\alpha}_{i}(j) = \frac{D^{\alpha}_{BC}(j) + \deg ree(j)}{\sum_{j=1}^{n^{\alpha}} (D^{\alpha}_{BC}(j) + \deg ree(j))} \tag{14}$$

$$E_D(i) = \sum_{\alpha=1}^{L} E^{\alpha}_{D}(i) \tag{15}$$

Among them, n^{α} is the total number of node i in layer G^{α} including node i connected to the neighbor nodes, $p^{\alpha}_{i}(j)$ is the probability set of the local network composed of node i as the center and the first-order neighbor nodes j in layer G^{α}. When the scale of the network is large enough, the complexity of the algorithm in this paper is $O(2N)$, where N is the total number of network nodes.

This method not only takes into account the global centrality inside each layer of the network in a multilayer network, but also solves the problem of how nodes in each layer affect the importance of nodes in other layers. As nodes gradually increase in the number of connected layers in a multilayer network, the importance of nodes is not only dependent on intralayer connections, but is more affected by nodes from other layers connected to and the number of connected layers. When the number of network layers is large, the method in this paper can reduce more information loss between layers and maintain the accuracy of node importance evaluation.

3.2. Numerical experiment

A multilayer network is formed by the interaction of multiple simple networks and nodes between layers. These nodes are equivalent to connecting to the propagation channel between different layers.

Table 1
Nodes centrality ranking.

Order	1	2	3	4	5	6	7	8
Node	$MEDICI$	$PERUZZI$	$CASTELLAN$	$BISCHERI$	$GUADAGNI$	$BARBADORI$	$LAMBERTES$	$TORNABUON$
$E_D(i)$	5.0069	4.3102	3.9884	3.9668	3.8245	3.7574	3.1677	2.6853
Order	9	10	11	12	13	14	15	16
Node	$GINORI$	$STROZZI$	$SALVIATI$	$RIDOLFI$	$ALBIZZI$	$PAZZI$	$ACCIAIUOL$	$PUCCI$
$E_D(i)$	2.4660	2.3127	2.1234	1.8986	1.8232	1.7258	0.5012	0.0000

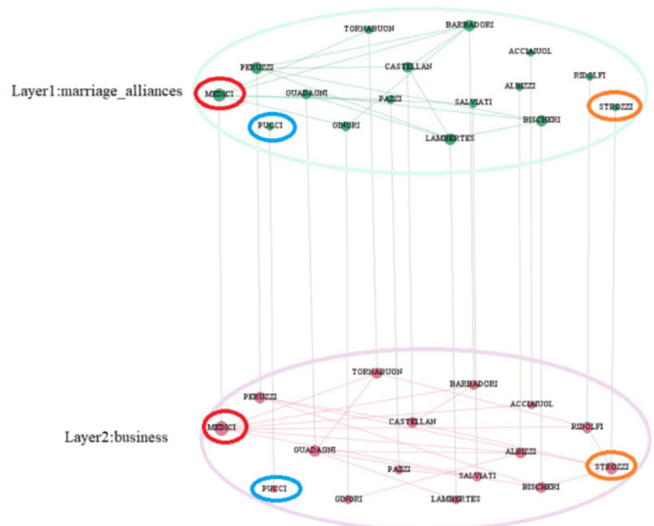

Fig. 3. Multilayer network diagram of different relationships between Renaissance Florentine families [35].

A multilayer network structure diagram is shown in Fig. 3, which illustrating different relationships between families in Renaissance Florence [35]. The multilayer network consists of 2 layers (marriage alliances and business relationships) describing florentine families in the Renaissance. The connection between the two layers is equivalent to a "bridge" connecting the two fields of marriage alliances and business relationships. The nodes connected between layers are multiple roles active in different fields.

Taking the multilayer network shown in Fig. 3 as an example [35], the multilayer network consists of 16 nodes and 35 edges. Algorithm 1 in this paper is used to calculate the centrality of each node, and the results are shown in Table 1.

From Table 1, the centrality ranking of the multilayer network is that the node $MEDICI$ rank the highest, and the node $PUCCI$ rank the lowest. From Fig. 3, the node $MEDICI$ has high and most close connections with other nodes in the two single-layer networks, so the centrality value of this node is the largest in the overall multilayer network. The node $PUCCI$ is an isolated node in the two single-layer networks, so the centrality value of the node in the multilayer network is 0. The node $STROZZI$ is an isolated node in layer 1, and the degree ranking of this node in layer 2 is second. However, in the overall multilayer network, the centrality value of this node measured by this method ranks tenth. These results indicate that the number of network layers connected by a node and the importance of the connected isomorphic nodes will affect the centrality ranking of nodes in multilayer networks.

By analyzing the node connection structure of the multilayer network in Fig. 3 and the calculation results in Table 1, it indicates that the method in this paper is reasonable and effective in measuring the node centrality of the multilayer network. To further demonstrate the effectiveness and generality of our method, in the next section, the proposed method will be applied to multilayer networks of different types and scales, and compared with the classical and recent multilayer network centrality methods.

Table 2
Four multilayer network node connection data.

Multilayer networks	Layers	Nodes	Edges
Hubei transportation network	3	17	214
Lazega-Law-Firm multilayer network [36]	3	71	2571
CS-Aarhus multilayer network [37]	5	61	620
CKM-Physicians-Innovation multilayer network [38]	3	246	1551

4. Applications

In this section, the method proposed in this paper is tested on an artificially constructed multilayer network and three real multilayer network datasets, and compared with other centrality measures results. The node connection data of these four multilayer networks are shown in Table 2.

4.1. Hubei transportation network

To verify the effectiveness of the method proposed in this paper, according to the real traffic data of Hubei Province of China, a multilayer transportation network in Hubei Province of China is constructed, which is composed of highway network, railway transportation network and aviation network. Each layer of the network contains 17 homogeneous nodes, representing 17 prefecture-level cities in Hubei Province. Whether the nodes in each layer of the network are connected depends on whether the two cities can reach each other in a certain mode of transportation. If there are two or more transportation options from the city to other cities in Hubei Province, the city will be connected to its own homogeneous node on the optional transportation network.

The schematic diagram of the transportation network structure in Hubei Province is shown in Fig. 4. The data of this transportation network comes from China Highway Network "https://www.chinahighway.com/", China Railway Network "http://www.gaotie.cn/" and China Civil Aviation Network "http://www.caacnews.com.cn/". Among them, the two cities in the highway transportation network can be connected by high-speed. The connection of the railway transportation network mainly considers whether there is a direct train between the two cities. The aviation network only considers flight information in Hubei Province. Only when there are direct non-stop flights between the two cities, the two cities are considered to be connected on the aviation network.

For the entire Hubei transportation network, the proposed method is tested with this multilayer network, and compared with the results of the three basic centrality measures mentioned above as well as two more recent methods of metrics. The measurement results of the proposed method and several other multilayer network centrality measurement methods on the transportation network in Hubei Province are shown in Table 3. The results in the table show that although the measurement results of several methods are slightly different, they all measure that $Wuhan$ has the highest centrality value, $Enshi$ is the second, and $Shiyan$, $Tianmen$ and $Yichang$ are all higher ranked nodes in terms of centrality. From the perspective of one mode of transportation in Hubei Province, $Wuhan$ is not the most developed highway transportation in the province. But combined with the interaction of the three-layer transportation network, $Wuhan$ ranks first. These results

Table 3
Centrality values measured by several methods.

Cities	$DC(i)$	Order	$RWBC(i)$	Order	$CC(i)$	Order	Ref. [22]	Order	Ref. [25]	Order	$E_D(i)$	Order
Wuhan	2.9375	1	0.6701	1	2.9412	1	2.9403	1	0.6243	1	9.5730	1
Xiaogan	1.7375	6	0.1689	7	1.7745	7	1.8150	6	0.1282	7	7.6221	6
Huanggang	1.4000	16	0.1524	14	1.6250	12	1.4586	16	0.1022	17	6.7609	16
Jingzhou	**1.6083**	**9**	0.1604	9	**1.6784**	**10**	1.6640	11	0.1205	10	7.3922	10
Ezhou	1.4792	12	**0.1563**	**12**	1.5921	16	1.5605	12	0.1115	13	7.1706	12
Suizhou	1.6042	11	0.1459	17	1.6912	9	1.6999	10	0.1189	12	7.3795	11
Xiangyang	1.6750	8	0.1577	10	1.7222	8	1.7485	8	0.1249	9	7.5149	8
Huangshi	**1.6083**	**9**	0.1576	11	**1.6784**	**10**	1.7001	9	0.1197	11	7.4170	9
Yichang	1.8750	3	0.2108	4	1.8889	4	1.8873	5	0.1408	5	7.7881	5
Jingmen	**1.4042**	**14**	0.1500	15	**1.5934**	**14**	1.5043	14	0.1031	16	6.9033	14
Xianning	1.4708	13	**0.1563**	**12**	1.6230	13	1.5324	13	0.1084	14	7.0801	13
Shiyan	**1.8708**	**4**	0.4084	3	2.2555	3	2.2039	3	0.2864	3	7.9902	3
Xiantao	**1.4042**	**14**	0.1480	16	**1.5934**	**14**	1.4719	15	0.1037	15	6.8753	15
Enshi	2.2666	2	0.4612	2	2.5375	2	2.5295	2	0.3159	2	8.8357	2
Qianjiang	1.7333	7	0.1680	8	1.7895	6	1.8097	7	0.1267	8	7.5883	7
Shennongjia	1.1458	17	0.1853	6	1.4421	17	1.4037	17	0.2263	4	4.4087	17
Tianmen	**1.8708**	**4**	0.2009	5	1.8787	5	1.8925	4	0.1391	6	7.8077	4

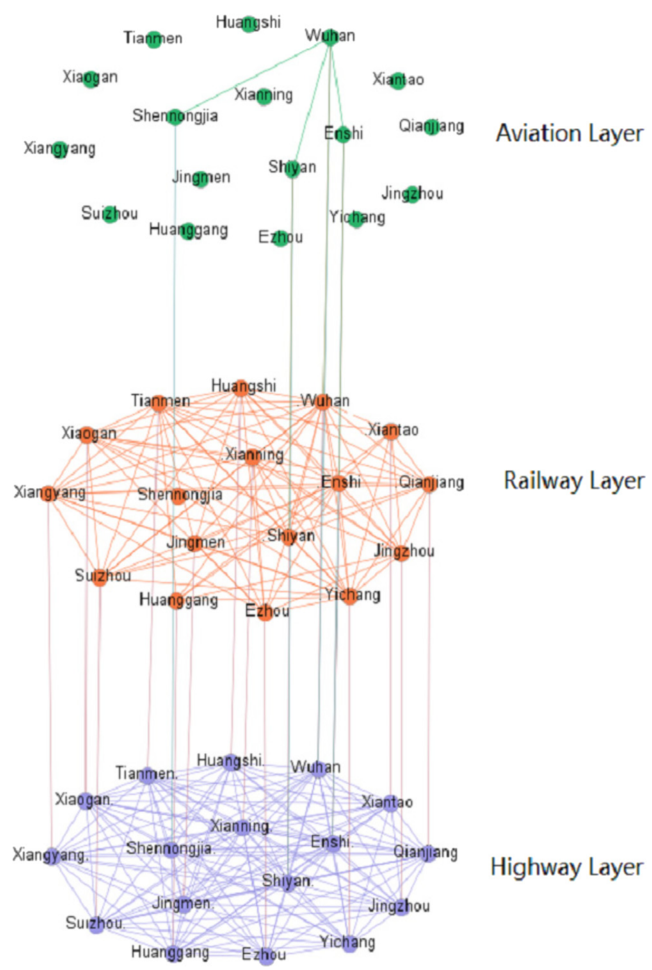

Fig. 4. Schematic diagram of transportation network structure in Hubei Province.

show that it is reasonable and feasible to consider the comprehensive effects of different network layers on nodes from the perspective of multilayer networks.

From the comparison of the results in the Table 3, the method proposed in this paper has a much larger order of magnitude of measurement results than the other five methods. The values of node centrality measured by the top five methods in Table 3 differ by an order of magnitude of 10^{-4}. There is even a situation where the centrality value of two nodes is the same, and the importance of the node in the network cannot be judged. However, the method in this paper can well measure the centrality value of each node of the multilayer network, and considers the number of layers and the influence of the node centrality between layers, which greatly reduces the information is lost. In order to prove the superiority of the method in this paper more clearly, this paper provides a comparison chart of the measurement results of different methods for the multilayer network of Hubei Province transportation network, as shown in Fig. 5. From the figure, the overall fluctuation trend of the results of several evaluation methods is consistent. However, the difference value of the node centrality obtained by the method in this paper is obviously larger than that of the other methods, and the importance of each node in the multilayer network can be clearly distinguished, which cannot be achieved by other centrality measurement methods.

4.2. Lazega-Law-Firm multilayer network

The Lazega-Law-Firm multilayer network consists of 3 kinds of (Co-work, Friendship and Advice) between partners and associates of a corporate law partnership [36]. There are 71 nodes in total, labeled with integer ID between 1 and 71, with 2571 connections. The centrality measurement results of the method proposed in this paper and several other multilayer network centrality measurement methods for the Lazega-Law-Firm multilayer network are shown in Fig. 6. Table 4 shows the top 20 measurement results.

From Fig. 6, there is a small difference between the results obtained by the method presented in Ref. [25] and the random walk intermediate centrality method, and the linear relationship is close to the same as shown in the figure. Degree centrality, closeness centrality and the method presented in Ref. [22] measure the centrality values between nodes all fluctuate, but they are all within a small range. These five methods are prone to the situation that the difference between the results of multiple nodes is extremely small or the same, and its importance cannot be judged. However, the importance value difference between nodes in the network measured by the method proposed in this paper is significantly greater than that of other methods, which can clearly identify the importance of each node. The results for the top 20 nodes of centrality values measured by these methods in the Lazega-Law-Firm multilayer network are shown in Table 4. In the table, the first nodes measured by several methods are nodes 26, 17, and 24, respectively, and these nodes are also ranked in the top position in other methods. Although the evaluation results of different methods differed slightly, the evaluation results of several methods in the multilayer network showed a similar trend from an overall perspective.

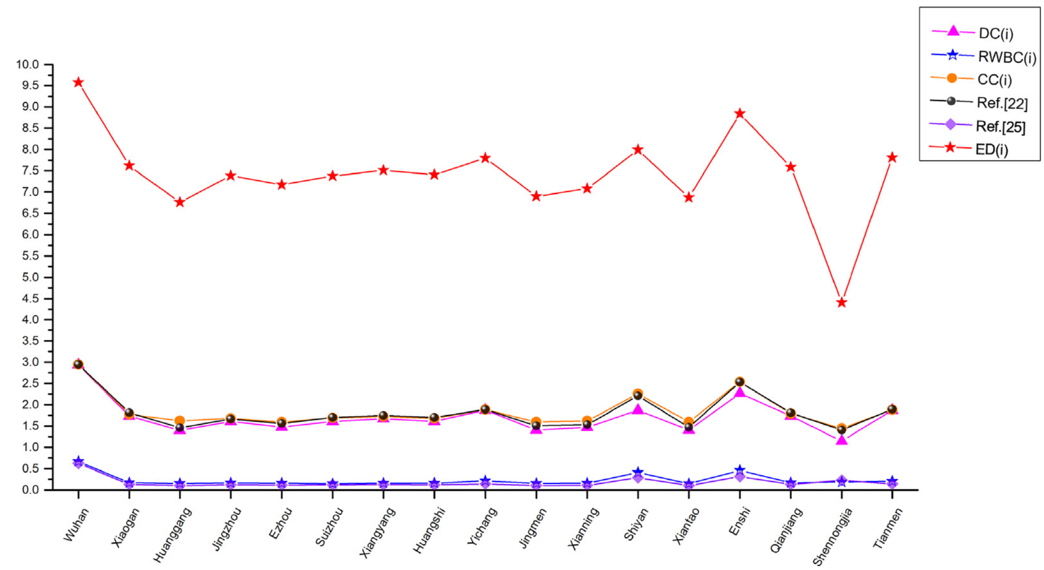

Fig. 5. Centrality measurement results of several methods in multilayer transportation networks.

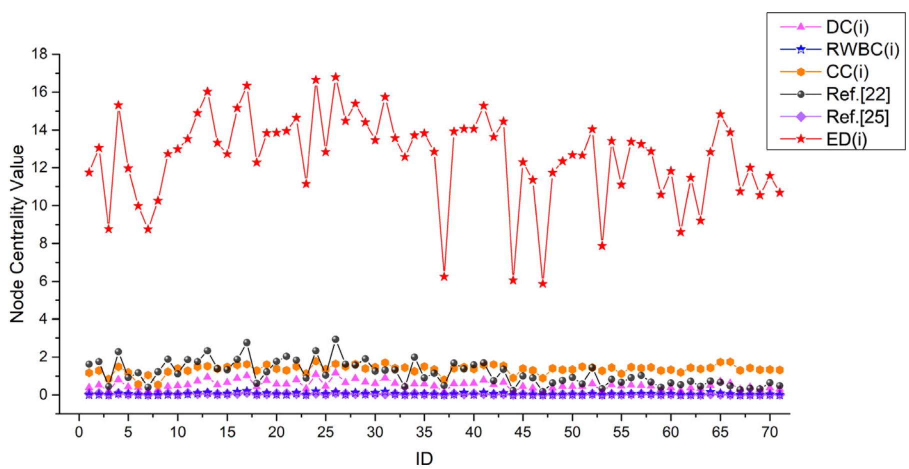

Fig. 6. Centrality measurement results of several methods in Lazega-Law-Firm multilayer network.

Table 4
The results of the top 20 nodes in the centrality value of the Lazega-Law-Firm multilayer network measured by several methods.

Order	$DC(i)$	ID	$RWBC(i)$	ID	$CC(i)$	ID	Ref. [22]	ID	Ref. [25]	ID	$E_D(i)$	ID
1	1.1571	26	0.1972	17	1.7707	24	2.9309	26	0.1206	17	16.7919	26
2	1.0786	24	0.1827	24	1.7381	66	2.7507	17	0.0975	26	16.6410	24
3	1.0071	17	0.1759	31	1.7335	65	2.3285	13	0.0862	11	16.3331	17
4	0.9357	13	0.1621	16	1.7028	31	2.3246	24	0.0795	24	16.0237	13
5	0.8857	31	0.1535	64	1.6397	26	2.2797	4	0.0778	13	15.7498	31
6	0.8786	16	0.1389	26	1.6226	28	2.0264	21	0.0725	9	15.3930	28
7	0.8714	28	0.1252	28	1.6137	17	1.9820	34	0.0722	2	15.3120	4
8	0.8071	4	0.1243	13	1.6058	42	1.8985	29	0.0722	4	15.2827	41
9	0.8071	22	0.1172	41	1.6035	19	1.8847	9	0.0720	21	15.1595	16
10	0.7857	41	0.1101	39	1.5699	16	1.8702	16	0.0714	16	14.8966	12
11	0.7571	19	0.1007	30	1.5695	41	1.8558	11	0.0673	20	14.8297	65
12	0.7286	12	0.0949	22	1.5378	43	1.8286	22	0.0671	1	14.6477	22
13	0.7071	65	0.0894	19	1.5159	13	1.7610	20	0.0657	34	14.4901	27
14	0.6714	29	0.0860	43	1.4979	22	1.7547	2	0.0616	29	14.4568	43
15	0.6643	43	0.0840	12	1.4970	27	1.7513	12	0.0614	12	14.4194	29
16	0.6571	15	0.0800	65	1.4917	35	1.6922	41	0.0606	22	14.0606	40
17	0.6429	27	0.0798	4	1.4863	4	1.6755	38	0.0600	41	14.0591	39
18	0.6143	32	0.0702	15	1.4843	12	1.6178	27	0.0585	27	14.0406	52
19	0.6071	40	0.0572	32	1.4814	51	1.6155	1	0.0541	28	13.9587	21
20	0.6071	66	0.0568	57	1.4769	15	1.5780	40	0.0525	40	13.9263	38

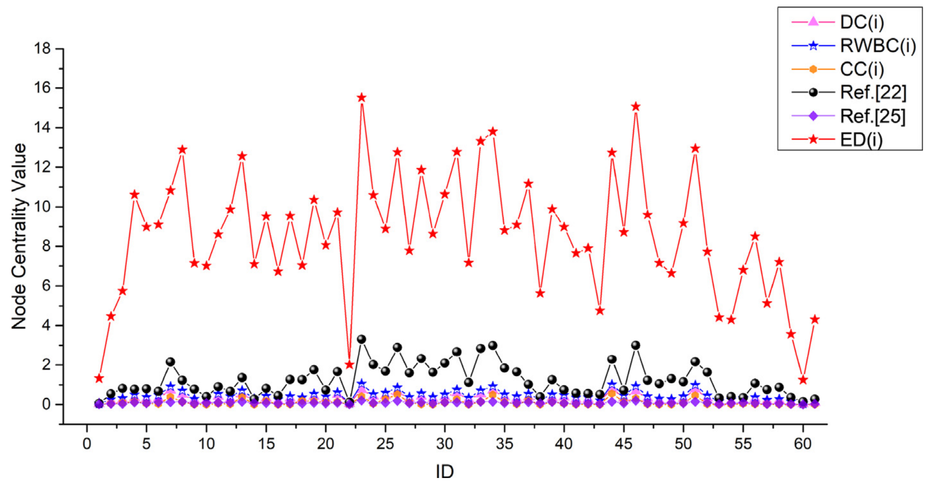

Fig. 7. Centrality measurement results of several methods in CS-Aarhus multilayer network.

4.3. CS-Aarhus multilayer network

The CS-Aarhus multilayer social network consists of five kinds of online and offline relationships (Facebook, Leisure, Work, Co-authorship, Lunch) between the employees of Computer Science department at Aarhus [37]. There are 61 nodes in total, labeled with integer ID between 1 and 61, with 620 connections. The results of the node centrality value measured by the method proposed in this paper and several other methods on the CS-Aarhus multilayer network are shown in Fig. 7.

From Fig. 7, the results obtained by the method presented in Ref. [25] have the least fluctuation in the figure. The results of degree centrality, random walk betweenness centrality, and closeness centrality all have small fluctuation, but the general trend is not obvious. The overall trend of the results obtained by the method presented in Ref. [22] and this paper is similar, but the results obtained by the method proposed in this paper are obviously orders of magnitude larger than the results of several other evaluation methods.

Table 5 gives the specific values of the top 20 nodes in the measurement results of these methods on the CS-Aarhus multilayer network. In Table 5, the hub nodes obtained by several methods are all relatively important nodes in the network. From the specific value of the results, the method in this paper can clearly distinguish the importance degree between nodes. The other methods more or less have too little difference between nodes or cannot distinguish the importance of nodes.

4.4. CKM-Physicians-Innovation multilayer network

The CKM-Physicians-Innovation multilayer network data collected by Coleman, Katz and Menzel on medical innovation, considering physicians in four towns in Illinois, Peoria, Bloomington, Quincy and Galesburg [38]. They were concerned with the impact of network ties on the physicians' adoption of a new drug, tetracycline. Three sociometric matrices (layers) were generated, based on the following questions:

1. When you need information or advice about questions of therapy where do you usually turn?
2. And who are the three or four physicians with whom you most often find yourself discussing cases or therapy in the course of an ordinary week-last week for instance?
3. Would you tell me the first names of your three friends whom you see most often socially?

The CKM-Physicians-Innovation multilayer network have 246 nodes in total, labeled with integer ID between 1 and 246, with 1551 connections. The comparison of node centrality measurement results obtained by several methods is shown in Fig. 8. Table 6 presents the centrality values of the top 20 nodes measured by several methods on the CKM-Physicians-Innovation multilayer network.

From Fig. 8, there is little difference between the node centrality measured by the degree centrality method, closeness centrality and the method presented in Ref. [25]. Both the random walk betweenness centrality and the method presented in Ref. [22] have small fluctuations. The method proposed in this paper measures the largest difference in importance between nodes, and the fluctuation range in the graph is the most significant. From Table 6, the results of degree centrality, Refs. [22,25] and the method proposed in this paper are relatively similar, and the ranking of nodes 122, 29, 36, and 141 is relatively high. Compared with other methods, the measurement results of random walk centrality and close centrality in the CKM-Physicians-Innovation multilayer network are slightly inaccurate.

From the experimental results of the above four multilayer networks, the important nodes obtained by the method proposed in this paper are largely consistent with the important nodes obtained by other methods, which shows that the method proposed in this paper is reasonable. From the comparison chart of the evaluation results of the four networks, the evaluation results obtained by the method proposed in this paper are consistent with the evaluation results obtained by several other methods, but the results obtained by the method proposed in this paper are orders of magnitude larger than those obtained by other methods. In the above results, the difference between nodes obtained by other methods may be less than 10^{-4}, then there will be a difference between nodes is small or the same, and its importance cannot be judged. In this case, the method in this paper is still applicable, and the method in this paper adds the number of connection layers of nodes, which can greatly reduce the loss of information. And with the increase of the number of network nodes and the connection scale, the evaluation results of the proposed method will be more accurate and reflect the differences between nodes.

5. Conclusion

From the existing research, the current research on the identification of influential nodes in complex networks mainly focuses on single-layer networks, while the research based on multilayer networks has just started. And the existing multilayer network node centrality evaluation methods ignore the existence of multiple network interactions in the actual complex system. In this paper, a new centrality measurement method for multilayer network is proposed, which can accurately and effectively calculate the centrality of multilayer network nodes. One of the main innovations of this method is to combine the structural characteristics of the multilayer network to increase

Table 5
The results of the top 20 nodes in the centrality value of the CS-Aarhus multilayer network measured by several methods.

Order	$DC(i)$	ID	$RW\,BC(i)$	ID	$CC(i)$	ID	Ref. [22]	ID	Ref. [25]	ID	$E_D(i)$	ID
1	0.8033	44	1.0476	23	0.5512	44	3.3117	23	0.2062	23	15.5157	23
2	0.7705	51	1.0126	44	0.5083	26	2.9956	46	0.1898	46	15.0566	46
3	0.7541	23	0.9838	51	0.4844	34	2.9882	34	0.1813	26	13.7997	34
4	0.7213	7	0.9311	46	0.4491	51	2.8933	26	0.1460	44	13.3179	33
5	0.7213	34	0.9172	34	0.4132	23	2.8437	33	0.1460	51	12.9518	51
6	0.6721	46	0.9050	7	0.3812	7	2.6714	31	0.1420	13	12.8918	8
7	0.6393	26	0.8579	26	0.3686	13	2.3255	28	0.1398	34	12.7757	31
8	0.5738	31	0.7470	31	0.3114	46	2.2813	44	0.1398	8	12.7603	26
9	0.5246	13	0.7114	33	0.2731	25	2.1584	51	0.1397	33	12.7418	44
10	0.5246	33	0.7057	13	0.2536	31	2.1498	7	0.1269	39	12.5585	13
11	0.5082	8	0.7015	8	0.1923	37	2.0956	30	0.1222	37	11.8617	28
12	0.4918	21	0.6119	21	0.1841	18	2.0203	24	0.1214	7	11.1741	37
13	0.4426	25	0.5854	25	0.1737	15	1.8392	35	0.1180	6	10.8352	7
14	0.4098	19	0.5533	28	0.1668	19	1.7544	19	0.1148	31	10.6352	30
15	0.4098	24	0.5386	11	0.1467	4	1.6781	25	0.1129	28	10.6190	4
16	0.4098	28	0.5340	37	0.1384	33	1.6562	21	0.1121	4	10.5959	24
17	0.3934	11	0.5324	19	0.1328	21	1.6495	36	0.1060	30	10.3628	19
18	0.3770	30	0.5171	24	0.1293	8	1.6259	52	0.1015	12	9.8790	39
19	0.3770	35	0.5092	39	0.1281	39	1.6193	29	0.1008	11	9.8753	12
20	0.3607	37	0.5045	35	0.1220	35	1.5954	27	0.0915	19	9.7196	21

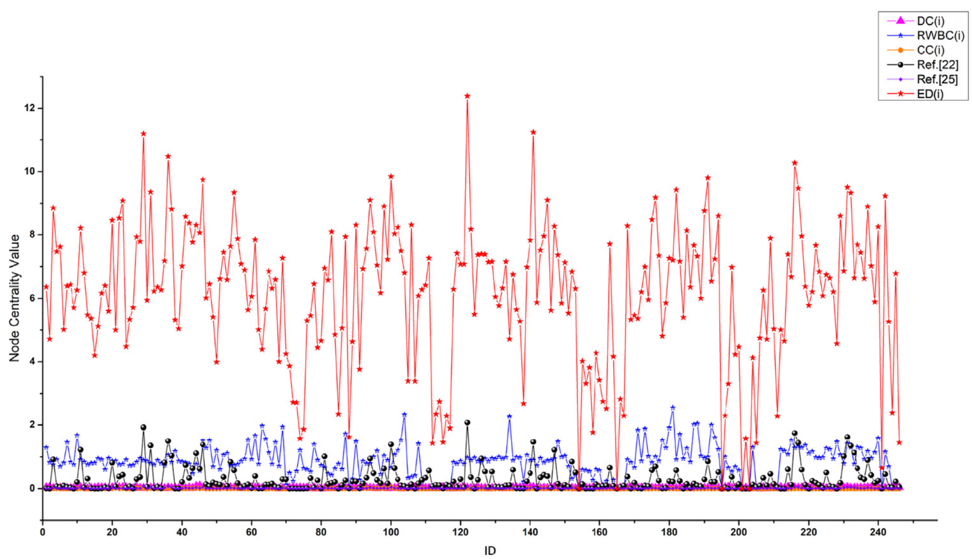

Fig. 8. Centrality measurement results of several methods in CKM-Physicians-Innovation multilayer network.

Table 6
The results of the top 20 nodes in the centrality value of the CKM-Physicians-Innovation multilayer network measured by several methods.

Order	$DC(i)$	ID	$RW\,BC(i)$	ID	$CC(i)$	ID	Ref. [22]	ID	Ref. [25]	ID	$E_D(i)$	ID
1	0.0224	122	2.5500	181	0.0449	100	2.0756	122	0.0728	29	12.3783	122
2	0.0143	29	2.3404	104	0.0251	52	1.9263	29	0.0697	36	11.2381	141
3	0.0469	141	2.2772	134	0.0242	29	1.7489	216	0.0564	122	11.1930	29
4	0.0306	36	2.0472	188	0.0241	44	1.6264	231	0.0503	216	10.4764	36
5	0.0306	216	2.0143	187	0.0240	36	1.4979	36	0.0466	100	10.2772	216
6	0.0184	100	2.0000	192	0.0223	21	1.4697	141	0.0455	46	9.8477	100
7	0.0224	46	1.9683	63	0.0200	46	1.4383	217	0.0451	231	9.8027	191
8	0.0245	191	1.9379	69	0.0190	27	1.3893	100	0.0443	31	9.7478	46
9	0.0184	55	1.9119	180	0.0180	81	1.3835	46	0.0436	141	9.5078	231
10	0.0224	217	1.8807	173	0.0177	93	1.3550	31	0.0434	217	9.4726	217
11	0.0469	231	1.8428	171	0.0173	98	1.3492	232	0.0415	232	9.4308	182
12	0.0245	31	1.7298	87	0.0164	122	1.2213	11	0.0383	94	9.3575	31
13	0.0184	232	1.7143	183	0.0161	55	1.2156	147	0.0381	11	9.3413	55
14	0.0184	94	1.6799	10	0.0149	94	1.1278	233	0.0372	44	9.3302	232
15	0.0143	145	1.6725	61	0.0142	31	1.1104	44	0.0358	37	9.2332	242
16	0.0163	182	1.6091	193	0.0141	101	1.0337	37	0.0335	20	9.1901	176
17	0.0224	242	1.5880	240	0.0136	11	1.0195	230	0.0335	233	9.1021	145
18	0.0224	98	1.5714	202	0.0136	77	1.0091	81	0.0325	230	9.1008	94
19	0.0184	176	1.5632	64	0.0132	92	0.9459	126	0.0324	3	9.0855	23
20	0.0408	237	1.5366	59	0.0128	23	0.9405	94	0.0316	147	8.9047	98

the importance of the interlayer connection nodes in the multilayer network. By calculating the betweenness centrality and the number of node connection layers of each layer of the network, the problem of the mutual influence of the centrality between the layers of the multilayer network is solved, and the problem of information loss in the multilayer network is greatly reduced. Different from the previous centrality measurement methods, this paper calculates the node centrality of the weighted network through the local structure entropy. This is another major innovation of this paper. Not only the node weighting degree is added from a local perspective, but also the influence of network node betweenness centrality is considered from a global perspective, and the original network structure information is maintained. The node centrality value measured by this method can accurately determine the importance of multilayer network nodes, and the value is within a suitable magnitude range, which is of great significance in the evaluation of multilayer network node centrality.

In addition, the method proposed in this paper is applied to real multilayer networks of different sizes in four different domains, and compared with several classical multilayer network centrality measures as well as the latest multilayer network centrality measures. The empirical results show that the method proposed in this paper is effective and general, and the computational complexity is low. When the network size is large enough, other methods may have small differences between nodes or inaccurate evaluation, but the proposed method can still maintain the accuracy of evaluation. Experimental results show that the proposed method not only has significant advantages in effectiveness and versatility, but also has lower computational complexity. Further, when the scale of the network is large enough, other methods may have small differences between nodes or inaccurate calculations, but the proposed method can still highlight the differences and the accuracy of the results.

CRediT authorship contribution statement

Dan Wang: Conceptualization, Methodology, Software, Data curation, Writing – original draft. **Feng Tian:** Visualization, Investigation. **Daijun Wei:** Reviewing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

I have shared the link to my data at the Attach File step

Padgett-Florence-Families_Multiplex_Social (Reference data) (Figshare)
CKM-Physicians-Innovation_Multiplex_Social (Reference data) (Figshare)
Lazega-Law-Firm_Multiplex_Social (Reference data) (Figshare)
CS-Aarhus_Multiplex_Social (Reference data) (Figshare)

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