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Sufficient Conditions for D-Stability

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Sufficient conditions for an n by n matrix to be D-stable are surveyed. Use is made of some transformations under which the D-stables are invariant and relations among the conditions are given. The verifiability of the thirteen conditions cited is also discussed. The lack of an effective characterization of D-stability motivates the discussion.

I. INTRODUCTION

The notion of D-stability was introduced in the late 1950's by Arrow and McManus [2] and Enthoven and Arrow [10] in their study of dynamic equilibria. They defined a real n by n matrix A to be D-stable if, for $D = \text{diag}\{d_1, \ldots, d_n\}$, DA is stable if and only if $d_i > 0$, $i = 1, \ldots, n$. In recent work, the "only if" portion of the definition has been omitted (yielding a larger set to be studied), but the D-stable matrices have not been adequately characterized using either definition. However, since the original work, a variety of sufficient conditions involving a number of diverse matrix theoretic concepts have been developed. In this survey note we assemble and relate these conditions—including several which have not been explicitly delineated in the literature. In the process several important classes of matrices which historically have arisen in varied contexts are linked by having D-stability in common. Practical interest in these sufficient conditions stems from the lack of an effective method for verifying the definition of D-stability. The verifiability of the diverse conditions given will be discussed.

Arrow and McManus [2] also employed the notion of S-stability (A is S-stable if for symmetric matrices S, SA is stable if and only if S is positive definite). S-stability involves more mathematical structure than D-stability, and, in particular, the two alternate S-stability definitions yield the same class of matrices. The S-stability problem has been solved by characterizations due to Carlson and Schneider [7, 9].

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II. Definitions and Preliminaries

We shall denote the real and complex n by n matrices and the n by n diagonal matrices with positive diagonal entries by $M_n(R)$, $M_n(C)$, and D_n respectively. For the sake of simplicity of exposition we define stability in the positive sense. A matrix $A \in M_n(C)$ is *stable* if all its eigenvalues have positive real parts.

Definition. $A \in M_n(C)$ is *D-stable* if DA is stable for all $D \in D_n$. Since for $D \in D_n$, DA and AD are similar ($DA = D(AD)D^{-1}$), it is irrelevant in defining D-stability whether multiplication from D_n is performed on the right or left. Thus the D-stables are just those matrices which remain stable under any relative reweighting of the rows or columns. Some simple observations are of use.

Observation (i). *A condition on matrices which implies stability and which is preserved under positive diagonal multiplication is sufficient for D-stability.*

Observation (ii). *If $A \in M_n(C)$ is D-stable, then A is nonsingular and each of the following is D-stable:*

(a) A^{-1};

(b) P^TAP *where P is any permutation matrix*;

(c) DAE; $D, E \in D_n$;

(d) A^*.

Observation (iii) [19]. *If $A \in M_n(C)$ is D-stable, then any k by k principal submatrix of A is in the Euclidean closure of the k by k D-stable matrices, $k = 1,..., n$.*

Remark. A well known [20, 25] necessary condition for $A \in M_n(R)$ to be D-stable is that all principal minors of A be nonnegative and that for each order $k = 1,..., n$, at least one k by k principal minor be positive. Let P_0^+ denote the set of all such $A \in M_n(R)$ [14, 21]. The question then arises that supposing $A \in M_n(R) \cap P_0^+$ what additional conditions on A imply D-stability. Some answers are given in III. Alternatively, assuming that $A \in M_n(R)$ and that for some permutation matrix P, each of the leading principal minors of P^TAP is positive, it has been shown [3] that there always exists some $D \in D_n$ such that DA is stable.

An alternate view of D-stability may be given in the following way.

OBSERVATION (iv). *Suppose that $A \in M_n(C)$ and that there is an $F \in D_n$ such that FA is stable. Then A is D-stable if and only if $A \pm iD$ is nonsingular for all $D \in D_n$.*

Proof. Suppose A is D-stable so that EA is stable for all $E \in D_n$. Then $\pm i$ is an eigenvalue of no matrix EA, $E \in D_n$ and thus $EA \pm iI$ is nonsingular for all $E \in D_n$. Equivalently $A \pm iD\ (= D(D^{-1}A \pm iI))$ is nonsingular for all $E^{-1} = D \in D_n$. Suppose conversely that A is not D-stable. Then FA is stable while EFA is not stable for some $E \in D_n$. It follows by continuity that for some $0 < t \leqslant 1$ and some $\alpha > 0$, either value, $\pm i$, is an eigenvalue of

$$\frac{1}{\alpha}(tE + (1 - t)\,I)\,FA.$$

Thus $A \pm iD$ is singular where $D = \alpha(tE + (1 - t)\,I)^{-1} \in D_n$ which completes the proof.

In the case of $A \in M_n(R)$, the "$\pm$" symbols in the above may be shortened to "$+$" since, for a real matrix, any complex eigenvalues occur in conjugate pairs.

A similar approach to D-stability is pursued in greater detail in [1]. Unfortunately the nonsingularity of all $A + iD$ is not an effective criterion and is no more verifiable than the definition itself. However, theoretical use of this approach has been made in [8] and [1] and there may be potential for further use. Progress would be made if conditions for the simultaneous solvability in D of

$$\mathrm{Re}(\det(A + iD)) = 0$$

and

$$\mathrm{Im}(\det(A + iD)) = 0$$

could be found. Both are algebraic expressions in the principal minors of A and the diagonal entries of D.

III. SUFFICIENT CONDITIONS FOR D-STABILITY

In discussing these conditions we shall omit many proofs which occur in the literature. Each condition is sufficient on the n by n matrix A to imply D-stability. Unfortunately none of these conditions is, in general, a necessary condition for D-stability. We mention some of the better known conditions first and indicate which are restricted to $M_n(R)$.

(1) There is a $D \in D_n$ such that $DA + A^*D$ is positive definite [2, 17, 25]. This is essentially the condition that Arrow and McManus

originally gave [2]. That it is equivalent to the general complex analogue of their condition is noted in [17] where a development of this condition is also given. One of several ways of verifying that such an A is D-stable is via Lyapunov's theorem and observation (i). Lyapunov's theorem [5] characterizes the stable matrices in the following way. $A \in M_n(C)$ is stable if and only if $GA + A^*G$ is positive definite for some positive definite Hermitian $G \in M_n(C)$. In our case D plays the role of G and since condition (1) is evidently invariant under multiplication from D_n we have by observation (i) that (1) is sufficient for D-stability. Unfortunately, condition (1) may not necessarily be verified for a given A in a finite number of steps. However, several of the conditions which follow turn out to be special cases of (1) and are finitely verifiable. Another way of stating (1) is that some positive diagonal multiple of A has a positive definite real part. (The real part of $B \in M_n(C)$ is $\frac{1}{2}(B + B^*)$.) The matrices with positive definite real part are, of course, stable and also play an extremely important role in stability theory. The more restrictive condition that A itself have positive definite real part is finitely verifiable; for instance, form $A + A^*$ and check its leading principal minors for positivity. That (1) is not necessary for D-stability is shown by an example given in [19].

(2) *$A \in M_n(R)$ is an M-matrix.* The notion of an M-matrix was advanced by Ostrowski [24], but we use the slightly stronger definition given by Fiedler and Ptak [13] which is in common current usage. A matrix $A \in M_n(R)$ is of class M if and only if its off-diagonal entries are nonpositive and all of its principal minors are positive. It is straightforward that this condition is invariant under multiplication from D_n and since M-matrices are well-known to be stable [13], it follows that they are D-stable.

(3) *There is a $D \in D_n$ such that $AD = B = (b_{ij})$ satisfies*

$$\operatorname{Re}(b_{ii}) > \sum_{\substack{j=1 \\ j \neq i}}^{n} |\, b_{ij} \,|, \qquad i = 1,\ldots, n.$$

The matrices A satisfying this condition were called *quasidominant diagonal* in [26], although only the real case was considered. The definition we have given with respect to right diagonal multiplication and rows could equivalently be given in terms of left diagonal multiplication and columns. The same class of matrices results [17, 18]. That the matrix B above is stable follows quite easily from Gersgorin's theorem [17, 18, 27]. Since the condition on B is invariant under left multiplication from D_n, it follows that B (and thus $A = BD^{-1}$) is D-stable. Condition (3) is finitely verifiable using its special case, condition (2), [13, 17]. This observation employs one of the several useful conditions equivalent to (2) given in [13]. The matrix A satisfies (3) if and only if the matrix $C = (c_{ij})$ defined by

$$c_{ij} = -\mid a_{ij} \mid, \qquad i \neq j$$

and

$$c_{ii} = \operatorname{Re}(a_{ii}), \qquad i = 1,..., n,$$

is an M-matrix. Whether or not C is an M-matrix is, of course, determined by checking its principal minors for positivity (in ascending order).

(4) $A = (a_{ij})$ *is triangular and* $\operatorname{Re}(a_{ii}) > 0$, $i = 1,..., n$. This is the most straightforward of the conditions sufficient for D-stability. Condition (4) is clearly invariant under multiplication from D_n and since the eigenvalues of a triangular matrix occur on the diagonal, the given condition is sufficient for stability and thus for D-stability.

(5) $A \in M_n(R)$ *is sign-stable.* The sign-stable matrices, as introduced by Quirk and Ruppert [25], are those real matrices which are stable by virtue of the sign pattern, $+$, 0, $-$, of their entries alone; i.e. A is sign stable if and only if any matrix B for which the sign of b_{ij} is the same as the sign of a_{ij}, $i, j = 1,..., n$, is stable. Since the sign pattern of a matrix is unchanged under multiplication from D_n, the sign-stable matrices are D-stable. Though the sign stables are a fairly narrow class, they contain some important types of matrices; for instance, $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right]$ is sign stable. The sign stable matrices have been characterized [23, 25] in such a way that a given matrix may be easily checked. It is assumed without loss of generality that $A = (a_{ij}) \in M_n(R)$ is indecomposable (for, if not, we decompose A into its indecomposable parts each of which must satisfy the following conditions). Then A is sign stable if and only if

(a) $a_{ii} \geqslant 0$, $i = 1,..., n$, and at least one $a_{ii} > 0$;

(b) $a_{ij}a_{ji} \leqslant 0$ for all $i \neq j$; and

(c) every cycle [23] in A of length greater than two is 0.

Sign stability is an essentially combinatorial quality and the most obvious examples of sign stable matrices are the real matrices satisfying condition (4) above.

(6) *Each principal minor of $A \in M_n(R)$ is positive and A is tridiagonal.* $A = (a_{ij})$ is tridiagonal (the term Jacobi matrix is also used) if $a_{ij} = 0$ whenever $\mid i - j \mid > 1$. The tridiagonal matrices have frequently been discussed in the literature in relation to stability [4, 16, 27], but condition (6) does not seem to have been pinpointed as sufficient for stability. That (6) is sufficient for stability (and thus D-stability by (i)) may be

proved in the following manner. We may assume without loss of generality that $a_{ij} \neq 0$ for $|i-j| = 1$. We may then choose $D \in D_n$ so that $DA = B = (b_{ij})$ satisfies $|b_{ij}| = |b_{ji}|$ if $|i-j| = 1$. Then B and also $B + B^T$ satisfy (6). But this means $B + B^T$ is positive definite which means A satisfies condition (1) and is D-stable. Structurally the tridiagonal matrices closely generalize the 2 by 2 matrices (for which D-stability is easily characterized, see (11) below). Condition (6) may be generalized somewhat in two ways. First, the assumption that all principal minors of A are positive may be relaxed to the assumption that each indecomposable component of A is in P_0^+. This is then necessary and sufficient for a tridiagonal matrix to be D-stable. Secondly, the assumption that A is tridiagonal may be replaced by the assumption that A is quasi-Jacobi [4]. Under either (or both) relaxations A must still be D-stable.

(7) *$A \in M_n(R)$ is oscillatory.* A matrix A is called oscillatory [15] if A is totally nonnegative (i.e., *each* minor is nonnegative) and some power of A is totally positive (i.e., each minor is positive). The oscillatory matrices arise classically in mechanics [15] and have the following extraordinary properties. Their eigenvalues are real (arising from the total nonnegativity), positive and distinct (because of the total positivity of some power). Since the oscillatory property is preserved under multiplication from D_n, it follows that the oscillatories are D-stable.

We next turn to some conditions which have been developed more recently.

(8) *For each $0 \neq x \in C^n$ there is a $D \in D_n$ such that* $\mathrm{Re}(x^*DAx) > 0$. This condition was recently developed as a sufficient condition for D-stability in [20]. Since the D depends on x, (8) is, in a sense, a local analog of (1). Note that if A and B both satisfy (8) and if for each $0 \neq x \in C^n$,

$$\{D \in D_n: \mathrm{Re}(x^*DAx) > 0\} \cap \{D \in D_n: \mathrm{Re}(x^*DBx) > 0\} \neq \varnothing,$$

then $A + B$ is D-stable. That (8) is not necessary for D-stability has been shown in [20] and unfortunately condition (8) is not in general finitely verifiable.

(9) *The Hadamard product of P and A is stable for each positive definite symmetric matrix P.* The Hadamard (entry-wise) product of two matrices $X = (x_{ij})$ and $Y = (y_{ij}) \in M_n(C)$ is defined to be $X \circ Y = (x_{ij}y_{ij})$ [19]. The matrices satisfying (9) were called Schur stable and proved to be included in the Euclidean closure of the D-stables. Practical interest in the Hadamard product arises in differential equations and statistics and

theoretical interest stems in part from Schur's old result that the positive definite matrices are Schur stable. Though (9) is proven one of the most general sufficient conditions for D-stability, it is neither necessary [19] nor finitely verifiable.

(10) *Each principal minor of A is positive and A is (strongly) sign-symmetric.* This is a new sufficient condition for stability provided by Carlson [8]. He terms a matrix A (strongly) sign symmetric if (a) the principal minors of A are positive and (b) each pair of symmetrically placed minors has nonnegative product. This condition is clearly preserved under multiplication from D_n so that (10) is sufficient for D-stability also. The condition has the advantage of being finitely verifiable, and it also relates some well known stability classes [11]. However, it is conjectured (see (13) below) that (10) may be significantly generalized.

We finally turn to some very specific conditions which have been developed for certain particular size matrices.

(11) $A \in M_2(R) \cap P_0^+$. It is easily verified [21] that (11) characterizes the 2 by 2 real D-stable matrices. Unfortunately, no such simple characterization is known for higher dimensions.

$$(12) \quad A \in M_3(R) \cap P_0^+ \textit{ and } A = \begin{bmatrix} x & a & b \\ \alpha & y & c \\ \beta & \gamma & z \end{bmatrix} \textit{ satisfies } xyz > \frac{ac\beta + \alpha\gamma b}{2}.$$

This condition which is developed and shown not to be necessary in [21] seems to be the best general 3 by 3 condition derivable from the Routh-Hurwitz conditions for stability [15, 21]. It is of interest since it is easily checked and is the first condition for D-stability known not to be included in (1).

(13) *$A \in M_n(R) \cap P_0^+$ satisfies the GKK condition, $n \leqslant 4$.* If two index sets I_1 and $I_2 \subseteq \{1, 2,..., n\}$ have equal cardinality and have intersection of cardinality at most one less, then the minor of A which they determine is called almost principal. If A has positive principal minors and each pair of symmetrically placed almost principal minors has nonnegative product, then A is said to satisfy the *GKK* condition [11, 12]. The *GKK* condition is a weaker version of the sign symmetry used in (10) and arises in a number of classical matrix inequalities. In [6, 8] the *GKK* matrices were related to stability including the computational verification that (13) is sufficient for stability. Since it is preserved under multiplication from D_n, (13) is also sufficient for D-stability. It is conjectured [8] that (13) may be generalized as a sufficient condition for D-stability by allowing

n to be arbitrary. Both theoretically and as far as the ease of verification, this would be a substantial improvement over (10).

Because of observation (ii) some of the conditions we have mentioned actually imply that additional matrices are D-stable. For instance conditions (4) and (6) are not preserved under permutational equivalence while each of the other conditions is. Thus by (iib) any matrix permutationally equivalent to one satisfying (4) or (6) is D-stable. Conditions (1), (4), (5), (6), (8) and (11) are preserved under inversion while (2), (3), (7) and (12) are known not to be. Thus matrices whose inverses satisfy these latter conditions are D-stable by (iia). The inverse invariance of conditions (9), (10) and (13) is apparently not yet known. Each of the mentioned conditions is preserved under Hermitian transposition and multiplication from D_n ; and except for (5) each is preserved under the extraction of principal submatrices.

IV. Relations Among the Sufficient Conditions

A number of more or less subtle interrelations among the conditions (1)–(13) are known. We enumerate here as many of these relations as possible.

(a) *The class of matrices defined by each of the conditions* (2), (3), (4), (5 *when each* $a_{ii} > 0$), (6) *and* (11) *is contained in that defined by* (1). For (2) and (3) this is shown in [17, 18] and for (4) in [17]. For (5) the containment is given in [25] and for (6) we have given the verification above. In the case of (11), $n = 2$, it is straightforward to verify the containment. Though (1) is not an easy condition to verify, these containments show its theoretical import. It is apparently not known if the classes given by (7), (8), (9), (10) and (13) are contained in that given by (1), and this suggests the importance of (12) which is the only one known not to be a special case of (1).

(b) *The class defined by* (1) *is, however, contained in those given by* (8) *and* (9). This has been verified in [20] and [19] respectively.

Each of the following classwise containments is either straightforward or has been mentioned earlier.

(c) (2) $\subseteq$ (3);

(d) (4) $\subseteq$ (3) [17];

(e) (4) $\subseteq$ (5) *for real matrices*;

(f) (7) $\subseteq$ (10);

(g) *for* $n \leqslant 4$, (10) $\subseteq$ (13).

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