Structured Network Games: Leveraging Relational Information in Equilibrium Analysis

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We study games with nonlinear best response functions played on a *structured* network, where a structure can naturally arise when agents form network communities or when agents' connections are multi-relational, i.e., their action space is multi-dimensional and each dimension may be governed by a different interaction graph. This significantly extends prior works on unstructured (mostly single-relational) network games without network structures, which have identified conditions to guarantee the uniqueness and stability of a pure strategy Nash equilibrium. We are interested in accomplishing the same for structured network games. It turns out that a parallel set of conditions exist and are stronger yet much easier to verify. Specifically, the network structures allow us to create partitions in an extended weighted adjacency matrix; this in turn allows us to establish conditions on the uniqueness and stability of Nash equilibria that can be verified using matrices potentially much lower in dimension, on the order of the number of partitions, compared to conventional analysis that uses matrices whose dimensions are given by the size of the network (i.e., number of agents) times the size of the action (i.e., number of action dimensions). We further introduce a new notion of degree centrality to measure the importance and influence of a partition in such a network, and show that this notion enables us to find new conditions for the uniqueness and stability of Nash equilibria. We compare our results with those of prior work on unstructured networks both analytically and through numerical simulations.

CCS Concepts: • **Theory of computation → Network games**; • **Networks → Network economics**; • **Applied computing** → *Economics*.

Additional Key Words and Phrases: equilibrium analysis, multi-relational networks, network structures

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1 INTRODUCTION

Network games are commonly used to model the strategic interactions among interdependent agents, where the interdependencies are typically described by a weighted and directed (interaction) network, which also maps into an adjacency matrix. The utility of an agent in these games depends on its own actions as well as the actions of other agents in its local neighborhood as defined by the interaction network. This framework can be used to capture different forms of interdependencies between agents' decisions, such as allowing the action of an agent to be a strategic substitute or complement to

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that of its neighbors. While these problems have been extensively studied in prior works, see e.g., [2, 3, 5, 7, 18, 21, 22, 24, 25, 30, 38], the focus has been largely on network games with a single, generic underlying network governing the relationship among agents. These will be referred to as *unstructured* network games in this paper; accordingly, their corresponding adjacency matrices will be referred to as *regular* adjacency matrices. A main focus of this line of work is in identifying sufficient conditions under which the Nash equilibrium (NE) of such a network game is unique and stable, see e.g., [24, 30].

In contrast, this paper focuses on network games with *structures* that arise naturally. Specifically, we consider two (non-mutually exclusive) families of structures in this study. (1) In the first, the underlying network enjoys certain special graphical properties, a prime example being agents forming *local communities*, thereby creating sub-graphs more strongly dependent/connected within themselves. (2) In the second case, agents enjoy *multi-relational* interactions, whereby they are connected over multiple (parallel) networks, each of which governs the dependency relationship of a different action dimension in a high-dimensional action space. Our goal is to understand how the existence of these types of structures affects the resulting Nash equilibria analysis, and how to exploit such structures, when they exist, to provide characterizations of conditions for existence, uniqueness, and stability of NEs in such network games. Similar to [2, 3, 24, 30, 38], we consider utility functions with nonlinear best-response functions and use the equivalent variational inequality (VI) representation to study the properties of the NE.

It turns out that such network structures can be used to our advantage in analyzing the NEs of these games, as well as in substantially reducing the computational requirement for equilibrium computation and for verifying the aforementioned conditions. This is done by adopting an *extended* or *generalized* form of adjacency matrix defined over the space of both agents and actions, and identifying *partitions* in this matrix as a result of the types of structures in the network and/or in the agents' dependent relationships described earlier.

Specifically, the computational effort in verifying the uniqueness and stability conditions derived using existing methods, also referred to as *unstructured conditions* in this paper, entails the study of the properties of a game Jacobian (this is the Jacobian matrix of the operator in the VI problem yielding the NE). This matrix is in general asymmetric, making the condition verification a co-NP-complete problem; this means the computational complexity grows faster than polynomial time in the size of the game (total number of agents for an unstructured network game, or total number of agent-action pairs for a structured network game). In the special case of a symmetric game Jacobian, the verification complexity of such conditions still grows in polynomial time in the size of the game.

In contrast, taking advantage of the partition structures in a structured network game, we derive a new set of conditions, also referred to as *structured conditions*, that depend only on the partitions in the game Jacobian and the size of the partitions. The verification complexity of these conditions, in both the asymmetric and symmetric cases, is reduced by at least one degree in the polynomial. Empirically, as we show in Section 7, the verification of the new condition takes only 2% of the CPU time needed to verify the unstructured conditions from existing literature and avoids memory overflow on large games.

Reducing the verification complexity is of great conceptual and practical interest: it allows us to obtain a high-level understanding of a game much faster in a more sustainable manner especially for large games, and enables early decisions. However, such computational efficiency gain does come at a cost, in that the structured conditions are stronger, i.e., they are sufficient conditions to the unstructured conditions. In other words, some games may satisfy the unstructured conditions (which guarantee the uniqueness and stability of the NE) but fail to satisfy the structured conditions. Our extensive numerical experiments show that this sufficiency gap is small in general, and the sample games that lead to such sufficiency gaps have concentrated features. We characterize such features and provide real-world interpretations.

In addition to the uniqueness and stability of the NE, we also introduce a new notion of centrality for structured network games, which can be viewed as a generalization of degree centrality. This new centrality measures the influence or importance of a partition in the network game. We show that this notion of centrality can help identify additional conditions for the uniqueness and stability of the NE.

Our main contributions are summarized as follows.

- We provide sufficient conditions for the existence, uniqueness, and stability of the Nash equilibrium in network games, by taking advantage of the partition structures which may arise due to, e.g., existence of communities or due to multi-relational interactions.
- We show that these conditions are sufficient conditions to those obtained in previous works, but that they are computationally much easier to verify than their counterparts obtained using conventional methods without utilizing the partition structures.
- The partition structure further allows us to define a new centrality measure, which can be used to verify the uniqueness and stability of the NE in these games.
- We conduct numerical experiments that compare our new conditions with conditions in previous works in terms of their verification complexity and strengths, which shed light on when to using these new conditions is advantageous.

The remainder of the paper is organized as follows. We provide motivating examples for the study of structured networks in Section 2. In Section 3, we introduce our model of structured network games. We present our results on the existence and uniqueness of Nash equilibria in Section 4, followed by results on the stability of Nash equilibria in Section 5. We propose a generalized notion of degree centrality for this class of games in Section 6. We present and discuss our numerical experiment results in Section 7, review related works in Section 8, and conclude in Section 9.

2 MOTIVATING EXAMPLES

We elaborate on the idea of structured networks through a running example consisting of a number of vendors/store owners (strategic agents) selling similar merchandize.

Communities due to stronger connectivity. Community formation stems from situations where the strengths of agents' strategic interactions are (statistically) different within certain groups compared to those between these groups. One type of community structure, commonly studied and discovered using spectral analysis [26, 27], is characterized by groups with much stronger connectivity (higher density of connections or existence of edges, as well as higher edge weights on those edges) within themselves, and much weaker connectivity (lower density of edges and smaller edge weights) between them.

Consider the case of three store owners, with agents a_1 and a_2 in close proximity of each other and a_3 located far away. Further, consider the store owners' single action of selecting business hours. Since a_1 and a_2 offer similar goods, their individual decisions on business hours (from complete overlap to mutually exclusive) will have direct consequence on the other's business volume and goods sold, resulting in a stronger dependence relationship between the two. Their dependence on (or influence of) a_3 may be far weaker. This is illustrated in the left of Figure 1, where the stronger relationship between a_1 and a_2 is indicated by a thicker edge; here a_1 and a_2 form a group or community.

The recognition of groups gives rise to a "block" view in the structured adjacency matrix shown in the right-hand side of Figure 1. Given there is only a single action, here every row and column is associated with an agent, resulting in a 3×3 interaction matrix. The colors of the diagonal entries match the identity of the agents depicted in the network as well as their indices. The diagonal

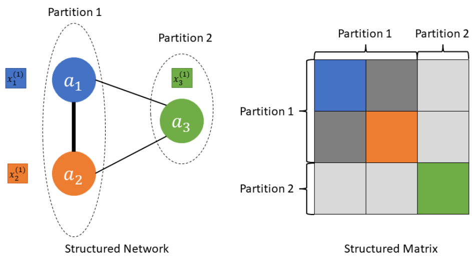

Fig. 1. Network with communities/groups: 3 agents and 1 action dimension; a_1 and a_2 form the first group, a_3 is a singleton group.

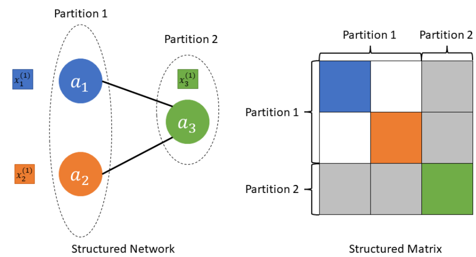

Fig. 2. Bipartite graph, where a_1 and a_2 are on one side, and exhibit the same type of dependence on a_3 on the other side.

elements in the regular interaction matrix represent an agent's self influence, whereas the off-diagonal elements represent the agents' mutual influence. Similarly, the diagonal blocks in the structured matrix represent a partition's self influence, whereas the off-diagonal blocks represent the partitions' mutual influence. These quantities will be precisely defined in the next section.

Communities due to similarity in function. Communities can also result from a logical relationship. As an example, consider again the three store owners, where a_1 and a_2 carry completely orthogonal merchandises (e.g., a bakery vs. a hardware store) but both rely on a_3 to provide store security and vehicle rental as needed, and consider the single action of staffing levels. In this case, the dependency only exists between a_1, a_3, and between a_2, a_3, but not between a_1, a_2, resulting in a bipartite network shown in Figure 2. Yet in this case it is still appropriate to view a_1 and a_2 as belonging to the same group, because they each exhibit very similar dependence on another group.

Partition on action dimensions. Our next example is more complex and introduces a high-dimensional action space. The general idea is that when actions are high-dimensional, the interaction/dependency relationships among agents can be different for different action dimensions, effectively resulting in multiple parallel networks superimposed on each other; this will also be referred to as a *multi-relational* network game.

Consider again the three store owners, each of which now has both physical and online sales. It is reasonable to expect that the decisions a seller makes about what goods to display in the window or offer for sampling may have more impact on other similar stores in its physical proximity (one set of agents), whereas its decisions on webpage design, layout, picture quality, and payment options may only impact its online competitors (another set of agents). In such multi-relational games, the action dimensions naturally create action-based structures on the network.

Suppose we capture the above with two action dimensions (in-store and online decisions), and further suppose in our case a_1 and a_2 are much more in direct competition in terms of physical-store sales due to their proximity, but that a_2 and a_3 are much more in direct competition in terms of their online sales due to high similarity in the goods they carry. This means that agents form different groups in different actions, as illustrated in Figure 3, where agents are represented by different colors and action dimensions are represented by different shades. In the first (solid color) action, a_1 and a_2 form a group whereas in the second (faded color) action, a_2 and a_3 form a group.

To capture the multiple action dimensions, we will define an *extended* adjacency/interaction matrix where each row and each column represent an agent-action dimension pair; the regular adjacency matrix is then a special case of this extended definition when the action dimension is 1. Under this definition, we have a 6×6 interaction matrix for this example. This is shown in Figure 4. Given such a matrix, partitions can emerge either by agents (similar sets of agents form the same group regardless of the action dimension) or by actions (interactions along different dimensions tend to be orthogonal), as is the case in this example and shown in Figure 4.

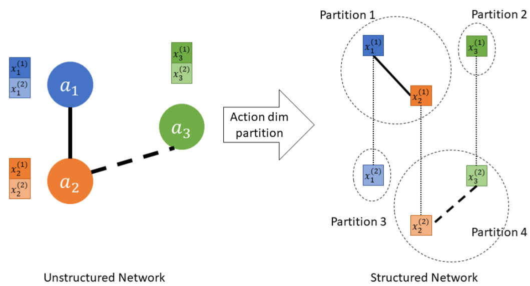

Fig. 3. Action dimension partitioned graph, with 3 agents and 2 actions. a_1 and a_2 are closely connected on action dimension 1, while a_2 and a_3 are closely connected on action dimension 2.

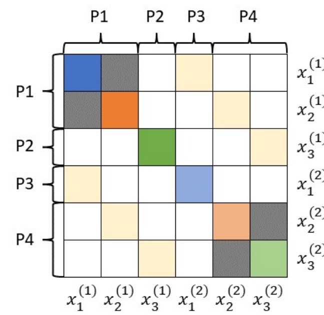

Fig. 4. Action dimension partitioned matrix for the network in Figure 3.

Arbitrary partitions in the extended adjacency matrix. It is important to note that the idea of a partitioned extended adjacency matrix is not restricted to the above example scenarios. In principle, this can be done across both agents and action dimensions in arbitrary ways.

It is equally important to note that in our analysis, we will be working with the Jacobian (∇F) of a best-response operator (F), defined precisely in subsequent sections, rather than the extended adjacency matrix (G) itself. This is because the former captures both first-order and second-order information about the utility functions, which is not captured in the latter. In particular, the extended adjacency matrix does not reveal cross-action dependencies induced by the utility functions. We will, however, also show that there is a direct correspondence between a partition on the extended adjacency matrix and that on the Jacobian. For this reason, the use of adjacency matrix in this section is for illustration purpose only, as it is much more intuitive and straightforward to visualize the described partition structures in an adjacency matrix.

3 MODEL AND PRELIMINARIES

3.1 The Structured Network Game Model

We consider a structured network game among N agents $\mathcal{N} = \{a_1, \ldots, a_N\}$, each with K action dimensions. The multi-relational network is represented by a multi-relational graph with extended adjacency matrix G. The adjacency matrix on the k-th action dimension is denoted as $G^{(k)}$, and is a submatrix of G. The edge weight $G_{ij}^{(k)} \in \mathbb{R}$ is a real number representing the strength of influence agent a_j has on agent a_i (or a_i's dependence on a_j) in the k-th action dimension.

We use $x_i^{(k)} \in \mathbb{R}$ to denote the k-th action of agent a_i, $\boldsymbol{x}_i = (x_i^{(k)})_{k=1}^K \in \mathbb{R}^K$ to denote the action vector of a_i, and $\boldsymbol{x}_i^{(-k)} = [x_i^{(1)}, \cdots, x_i^{(k-1)}, x_i^{(k+1)}, \cdots, x_i^{(K)}]^T$ to denote the action profile of a_i, excluding the k-th dimension. In addition, let $\boldsymbol{x}^{(k)} = (x_i^{(k)})_{i=1}^N \in \mathbb{R}^N$ be the action vector of all agents on the k-th action dimension, and $\boldsymbol{x}_{-i}$ denotes the action profile of all agents other than a_i.

Each agent a_i has an action constraint $\boldsymbol{x}_i \in Q_i = \prod_{k=1}^K Q_i^{(k)}$, where $Q_i^{(k)} := [0, B_i^{(k)}]$ such that $B_i^{(k)}$ captures the physical or financial constraints (budgets) on the k-th action dimension.

We consider games with utility functions consisting of an individual component and a network component:

$$u_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}, G) = d_i(\boldsymbol{x}_i) + f_i(\boldsymbol{x}_i, G^{(1)}\boldsymbol{x}^{(1)}, \dots, G^{(K)}\boldsymbol{x}^{(K)}). \tag{1}$$

Here, $d_i(\cdot)$ is the individual component, which only depends on a_i's own actions; conventionally, it contains a standalone benefit and cost of taking action $\boldsymbol{x}_i$. The network component $f_i(\cdot)$ depends on not only the agent's own but also others' actions. Here, the network influence on the k-th action dimension is captured by $G^{(k)}\boldsymbol{x}^{(k)}$. Throughout the paper, we make the following assumption on the game.

Assumption 1. *The utility functions $u_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}, G)$ are concave and twice continuously differentiable, for all i.*

An example of this type of utility function is given below.

Example 1. *The multi-relational extension of linear-quadratic utility functions, studied in e.g., [12], has the following form,*

$$u_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}, G) = \boldsymbol{x}_i^T \boldsymbol{b}_i + \sum_{k=1}^{K} \sum_{j \neq i} x_i^{(k)} g_{ij}^{(k)} x_j^{(k)} - \frac{1}{2}\boldsymbol{x}_i^T C_i \boldsymbol{x}_i,$$

where $\boldsymbol{x}_i^T \boldsymbol{b}_i - \frac{1}{2}\boldsymbol{x}_i^T C_i \boldsymbol{x}_i$ is the individual component containing the standalone benefit term and the cost term, and $\sum_{k=1}^{K} \sum_{j \neq i} x_i^{(k)} g_{ij}^{(k)} x_j^{(k)}$ is the (multi-relational) network influence component. (We can further expand this utility function with cross-relational influence terms $\sum_{k=1}^{K} \sum_{j \neq i} x_i^{(k)} g_{ij}^{(k,l)} x_j^{(l)}$, where $G^{(k,l)}$ is a submatrix in G modeling the network influence from the l-th action to the k-th action.)

A Nash equilibrium of the described structured network game can be found as a fixed point of the agents' *best response* mappings. The best response of an agent in the network game is defined as the action an agent takes to maximize its own utility, given other agents' actions and the network topology. For our model, we denote the best response of agent a_i as

$$BR_i(\boldsymbol{x}_{-i}, G) := \arg\max_{\boldsymbol{x}_i \in Q_i} \; u_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}, G). \tag{2}$$

We also define an operator F_i as follows,

$$F_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) = -\nabla_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}, G) \in \mathbb{R}^K. \tag{3}$$

Next, we introduce the Variational Inequality (VI) framework and its relation to Nash equilibria in network games.

3.2 The Variational Inequality (VI) Problem

Variational Inequalities (VIs) are a class of optimization problems with applications in game theory. In particular, the Nash equilibria of many games can be found as solutions to a corresponding VI problem [30, 33]. We state the VI problem formally below, using the set of notations introduced earlier so that the correspondence between the game model and the VI problem is clear.

Definition 1. *A variational inequality $VI(Q, F)$ consists of a set $Q \subseteq \mathbb{R}^N$ and a mapping $F : Q \to \mathbb{R}^N$, and is the problem of finding a vector $\boldsymbol{x}^* \in Q$ such that,*

$$(\boldsymbol{x} - \boldsymbol{x}^*)^T F(\boldsymbol{x}^*) \geq 0, \; \forall \boldsymbol{x} \in Q. \tag{4}$$

Finding the Nash equilibrium of a structured network game is equivalent to solving a variational inequality problem $VI(Q, F(\boldsymbol{x}))$ with the appropriate choice of Q and F. An example choice of Q and F is: $Q = Q_1 \times Q_2 \times \ldots \times Q_N \subseteq \mathbb{R}^{NK}$, with Q_i being the action space of a_i, and $F(\boldsymbol{x}) = (F_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}))_{i=1}^N \in \mathbb{R}^{NK}$, with $F_i(\cdot)$ given in (3). Then, since finding an NE is the problem of finding a fixed point of the best response mappings in (2), it is equivalent to solving the VI problem given in (4) with the choice of F and Q stated above (see, e.g., [11, Proposition 1.4.2]). Intuitively, if the equilibrium $\boldsymbol{x}_i^*$ is an interior point of Q_i, then $F_i(\boldsymbol{x}_i^*, \boldsymbol{x}_{-i}^*) = \boldsymbol{0}$ if and only if $\boldsymbol{x}_i^*$ is a best response to $\boldsymbol{x}_{-i}^*$; if $\boldsymbol{x}_i^*$ is on the boundary, we can still show $(\boldsymbol{x}_i - \boldsymbol{x}_i{}^*)^T F_i(\boldsymbol{x}^*) \geq 0$ holds. Note also that the above definition holds for Q and F that are permutations of the preceding example choice, as long as the permutations are consistent.

3.3 Partitions

We next formally define partitions in a structured network game. As shown in Section 2, a partition is essentially a set of indices, where each index corresponds to an agent-action pair (which also corresponds to a column or row in the extended adjacency/interaction matrix G). Figures 1, 2, and 4 show how, by grouping certain indices together (rearranging rows and columns), block structures may emerge in G.

It turns out a block structure in G translates to a similar block structure in the Jacobian of the operator F, when F is a suitable permutation of the operator $(F_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}))_{i=1}^N$ consistent with the partitions.

As defined in (3), we can think of operator F_i as the best response direction vector of a_i, where each element in F_i corresponds to an agent-action component. Therefore, operator $F(\boldsymbol{x})$ is the global best response direction vector containing all agent-action components; these components can be arranged in an arbitrary order, and $(F_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}))_{i=1}^N$ corresponds to a common order.

While F captures important first-order derivative information of the utility functions with respect to the agent-action components, the Jacobian of F, denoted as ∇F captures important second-order information of the utility functions and first-order information of the operator F. It is common, see e.g., [24, 30, 33], to use the properties of ∇F to derive unstructured conditions for equilibrium analysis.

When $K = 1$, $\nabla F \in \mathbf{R}^{N \times N}$, where the off-diagonal elements in ∇F measure how an agent's action influences another agent's best response. For $K > 1$, $\nabla F \in \mathbf{R}^{NK \times NK}$, and now the off-diagonal elements in ∇F measure how an agent-action component influences the best response direction of another agent-action component.

There is a one-to-one mapping between the dimensions in ∇F and G, since each dimension in ∇F and G corresponds to an agent-action component, and thus any partition on G can equally apply to ∇F, as partitions are nothing more than separating the agent-action components into disjoint sets. Using the same 3-agent, 2-action example in Figures 3 and 4, while $(F_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}))_{i=1}^N$ orders the agent-action components as $x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, x_3^{(1)}, x_3^{(2)}$, F orders them as $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_1^{(2)}, x_2^{(2)}, x_3^{(2)}$. Then, if we partition the network as shown in Figure 3, we can partition ∇F the same way, as illustrated in Figure 4.

We note that the partition based on (agent) group structure and the partition based on action dimensions are special cases. For the remainder of the paper, we will discuss network games with general, arbitrary partition structures. Specifically, for an arbitrary structure, we can partition all the $N \times K$ agent-action components (and their corresponding indices) into an arbitrary number (M) of disjoint sets, denoted by $\mathcal{P}_1, \ldots, \mathcal{P}_M$. Accordingly, we will also denote by $N_i = |\mathcal{P}_i|$ the size of the partitions, where $\sum_{i=1}^M N_i = NK$.

4 EXISTENCE AND UNIQUENESS

We now identify conditions for the existence and uniqueness of Nash equilibrium based on the VI formulation of the game.

4.1 Existence of NE

We first state the conditions under which a Nash equilibrium exists. From [33], we have the following theorem that guarantees the existence of NE:

THEOREM 4.1. *([33, Theorem 3]) If F is continuous on Q, and Q is nonempty, compact and convex, then $VI(Q, F)$ has a nonempty and compact solution set.*

This is because in our problem, $Q = Q_1 \times Q_2 \times \ldots \times Q_N$, with $Q_i = [0, B_i^{(1)}] \times \ldots \times [0, B_i^{(K)}]$, so that together with Assumption 1, the conditions of Theorem 4.1 are satisfied for the network game defined in Section 3.1.

4.2 Uniqueness of NE

We next introduce sufficient conditions under which the network game defined in Section 3.1 has a unique Nash equilibrium. We begin by introducing the following definitions.

DEFINITION 2. *P-Matrix: A matrix $A \in \mathbb{R}^{N \times N}$ is a P-matrix if every principal minor of A has a positive determinant.*

DEFINITION 3. *A mapping $F : Q \mapsto \mathbb{R}^N$, where $Q \subseteq \mathbb{R}^N$ is nonempty, compact and convex, and F is continuously differentiable on Q, is strongly monotone if there exists a constant $c > 0$ such that*

$$(\boldsymbol{x} - \boldsymbol{y})^T (F(\boldsymbol{x}) - F(\boldsymbol{y})) \geq c||\boldsymbol{x} - \boldsymbol{y}||_2^2, \ \forall \boldsymbol{x}, \boldsymbol{y} \in Q \ . \tag{5}$$

Further, $F = (F_1, F_2, \ldots, F_N)$ is a uniform block P-function w.r.t. the partition $Q = Q_1 \times Q_2 \times \cdots \times Q_N$ if there exists a constant $b > 0$ such that

$$\max_{i \in \mathbb{N}[1,M]} (\boldsymbol{x}_i - \boldsymbol{y}_i)^T [F_i(\boldsymbol{x}) - F_i(\boldsymbol{y})] \geq b||\boldsymbol{x} - \boldsymbol{y}||_2^2, \ \forall \boldsymbol{x}, \boldsymbol{y} \in Q \ . \tag{6}$$

By setting $b = c/N$, it is easy to see that strong monotonicity is a sufficient condition for the uniform block P-condition. Parise and Ozdaglar [30] show that if $F(\boldsymbol{x})$ is a uniform block P-function, then $VI(Q, F)$ has a unique solution, and the Nash equilibrium of the network game corresponding to $VI(Q, F)$ is unique.

Unfortunately, it is computationally costly to verify these conditions for the function $F(\boldsymbol{x})$, since checking whether a square matrix is a P-matrix is co-NP-complete [10], and typically, the complexity grows (faster than polynomial time) in the size of the matrix. Below, we show that it is possible to take advantage of the block structure when it is present, and identify conditions for the uniqueness of Nash equilibrium that are of lower computational complexity to verify.

To do so, we define a structured matrix Υ^S and its components, the *internal* and *external* impact levels of partitions, as follows:

DEFINITION 4. *We say a partition $\mathcal{P}_i$ receives internal impact, α_i^S, and external impact, β_{ij}^S, defined as follows:*

$$\begin{aligned} \alpha_i^S &= \inf_{\boldsymbol{x} \in Q} ||\nabla_i F_i(\boldsymbol{x})||_2, \ \forall i \in \mathbb{N}[1, M] \\ \beta_{ij}^S &= \sup_{\boldsymbol{x} \in Q} ||\nabla_j F_i(\boldsymbol{x})||_2, \ \forall i, j \in \mathbb{N}[1, M], \ i \neq j \ , \end{aligned} \tag{7}$$

where $\nabla_j F_i(\boldsymbol{x}) \in \mathbb{R}^{N_i \times N_j}$ is a matrix with k,l-th entry $\frac{\partial F_i^{(k)}(\boldsymbol{x})}{\partial x_j^{(l)}}$. The structured matrix Υ^S for the network game is defined accordingly as:

$$\Upsilon^S = \begin{bmatrix} \alpha_1^S & -\beta_{12}^S & \dots & -\beta_{1M}^S \\ -\beta_{21}^S & \alpha_2^S & \dots & -\beta_{2M}^S \\ \vdots & \vdots & \ddots & \vdots \\ -\beta_{M1}^S & -\beta_{M2}^S & \dots & \alpha_M^S \end{bmatrix}. \tag{8}$$

We note that these impact measures are only defined between partitions and not agents. Accordingly, in this definition the subscripts are indices of partitions instead of agents.

To motivate the above definition, it helps to understand the matrix Υ^S in the context of what is typically used in prior works for checking the uniqueness of the Nash equilibrium. If we ignore the structure in the network and simply view each agent-action pair as a singleton partition, then using existing methodology (such as in [24, 30, 33]) will give us the Jacobian Υ^U, which is a $NK \times NK$ matrix. Specifically, Υ^U contains the following elements:

$$\begin{aligned} \alpha_k^U &= \inf_{\boldsymbol{x} \in Q} |\nabla_k F_k(\boldsymbol{x})|, \ \forall k \in \mathbb{N}[1, NK] \\ \beta_{kl}^U &= \sup_{\boldsymbol{x} \in Q} |\nabla_l F_k(\boldsymbol{x})|, \ \forall k, l \in \mathbb{N}[1, NK], \ k \neq l. \end{aligned} \tag{9}$$

For clarify, these will be referred to as the *component-level* internal and external impact, respectively. Suppose we rearrange the rows and columns of Υ^U in the following way: group together rows whose action dimensions are in $\mathcal{P}_i$, $i = 1, \cdots, M$; group together columns whose agents are in $\mathcal{P}_j$, $j = 1, \cdots, M$.

We can now view this rearranged Υ^U in blocks/submatrices denoted by $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}$, and there are $M \times M$ blocks. The matrix Υ^S is essentially a condensed version of this rearranged matrix, summarizing or abstracting each block into a single quantity as defined in Definition 4: α_i^S for the diagonal block $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}$ and β_{ij}^S for the off-diagonal block $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}$.

This abstraction aims at capturing the dependence relationship between partitions rather than between individual agent-action pairs. In particular, β_{ij}^S represents the largest influence level of partition $\mathcal{P}_j$ on partition $\mathcal{P}_i$, and α_i^S represents the minimum influence level of $\mathcal{P}_i$ on itself. This is formally established in the following lemma.

LEMMA 4.2. *We have $||\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}||_2 \geq \beta_{ij}^S$ and $||\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}||_2 \leq \alpha_i^S$.*

For a detailed proof, see Appendix A, and please refer to Section 4.3 for additional interpretation.

In what follows, we show that the matrix Υ^S can be used to provide sufficient conditions on the uniqueness of NE in network games with partition structures. Such an abstraction takes advantage of the partition structure and reduces the dimension of the matrix used to check conditions for uniqueness from $NK \times NK$ to a single $M \times M$ matrix and M matrices of size $|\mathcal{P}_k| \times |\mathcal{P}_k|$, $k \in [1, M]$. This greatly reduces the complexity of condition verification, since as mentioned earlier, P-matrix verification is co-NP-complete [10], and the complexity typically grows (faster than polynomial) in the size of the matrix. For example, in a special case where the matrix is symmetric, P-matrix verification is equivalent to examining its positive definiteness and the complexity becomes $O(N^3)$ for an $N \times N$ matrix. If there are $N = 20$ agents on the network with $K = 5$ action dimensions, then the verification complexity of unstructured conditions is $O(10^6)$; if we partition the game by action dimensions, the complexity of the structured conditions is $5 \cdot 8 \cdot O(10^3) + O(5^3)$, much lower than $O(10^6)$. This complexity gap will only increase if the matrix is asymmetric.

However, we also note that the strengths of the conditions obtained from the structured network and unstructured network are not equivalent. Specifically, we will later show that the conditions obtained from the structured network are stronger (sufficient conditions to) their counterparts in the unstructured network. Numerical results presented in Section 7 also highlight the gap between these two sets of conditions.

The following result identifies a condition for the uniqueness of the Nash equilibrium in games with the structured network.

THEOREM 4.3. *If the following two conditions are satisfied,*

(1) Υ^S *is a P-matrix, and*
(2) $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}, \forall i$ *are P-matrices (the diagonal blocks of* Υ^U *are P-matrices),*

then the network game has a unique Nash equilibrium.

Moreover, when both conditions are satisfied, then Υ^U *is also a P-matrix; i.e., the uniqueness conditions (1 & 2) on the structured network are also a sufficient condition for uniqueness of the Nash equilibrium in the underlying unstructured network (i.e., one that ignores the partitioned structures).*

PROOF. (Sketch)

- First, we show that if Υ^S is a P-matrix, then $F(\boldsymbol{x})$ is a uniform block P-function with respect to the partitions $\mathcal{P}_1, \ldots, \mathcal{P}_M$.
- By [30], if $F(\boldsymbol{x})$ is a uniform block P-function with respect to the partitions, then $VI(Q, F)$ has a unique solution which implies that the network game has a unique equilibrium.
- Finally, using Lemma 4.2, we show that if $F(\boldsymbol{x})$ is a uniform block P-function with respect to the partitions and $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}, \forall i$ are P-matrices, then in the unstructured network, if we treat each agent-action component as a singleton partition, the counterpart of $F(\boldsymbol{x})$ is also a uniform block P-function with respect to the singleton partition.

For a detailed proof, see Appendix B. □

An interpretation of the above result is as follows. As noted earlier, β^S_{ij} represents the external impact of partition $\mathcal{P}_j$ on partition $\mathcal{P}_i$, while α^S_i is the internal impact of $\mathcal{P}_i$. Typically, when β^S_{ij} has a relatively small value compared to α^S_i, then Υ^S is a P-matrix. Moreover, when Υ^S is (row or column) diagonally dominant, then Υ^S is a P-matrix [36]. In these types of networks, partitions' action profiles have a bounded influence on each other. On the other hand, if at least one partition's action profile has an out-sized effect on other partitions, then its decision can shift the state of the network substantially and result in possibly multiple equilibria.

REMARK 1. *It is worth mentioning that the structured (resp. unstructured) network condition verification is of a much lower complexity if* Υ^S *(resp.* Υ^U*) is symmetric. By [23], a symmetric matrix is a P-matrix if and only if it is positive definite, which means that instead of checking the determinant for every principal minor, we only need to do an eigendecomposition. This reduces the complexity from solving a co-NP-complete problem down to polynomial time.*

We next present two corollaries of Theorem 4.3 which provide alternative ways for verifying the uniqueness of NE in structured network games. We define

$$\Gamma^S = \begin{bmatrix} 0 & -\beta^S_{12}/\alpha^S_1 & \ldots & -\beta^S_{1M}/\alpha^S_1 \\ -\beta^S_{21}/\alpha^S_2 & 0 & \ldots & -\beta^S_{2M}/\alpha^S_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\beta^S_{M1}/\alpha^S_M & -\beta^S_{M2}/\alpha^S_M & \ldots & 0 \end{bmatrix}, \tag{10}$$

where β_{ij}^S and α_i^S are as in Definition 4. Similar to before, by treating each agent as a singleton partition, we can obtain Γ^S's unstructured counterpart, $\Gamma^U \in \mathbb{R}^{NK\times NK}$. We also denote the spectral radius of Γ^S as $\rho(\Gamma^S)$. By [34], if $\rho(\Gamma^S) < 1$, then Υ^S is a P-matrix. Therefore, we have the following Corollary of Theorem 4.3.

COROLLARY 4.3.1. *Assume the following two conditions hold simultaneously*

(1) $\rho(\Gamma^S) < 1$,
(2) $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}, \forall i$ *are P-matrices.*

Then, the network game has a unique Nash equilibrium. Moreover, under these conditions Υ^U *is a P-matrix.*

Since Υ^S is a P-matrix under the conditions of Corollary 4.3.1, the sufficiency is a direct result of Theorem 4.3.

Our second corollary is on the special case where Γ^S is a symmetric matrix. As mentioned above, by [23], we know that a symmetric matrix is a P-matrix if and only if it is positive definite. Therefore, we have the following.

COROLLARY 4.3.2. *Assume* Γ^S *is a symmetric matrix. Denote the eigenvalues of* Γ^S *by* $\lambda_1(\Gamma^S) \leq \lambda_2(\Gamma^S) \leq \cdots \leq \lambda_M(\Gamma^S)$. *Then, the network game has a unique Nash equilibrium if the following two conditions hold simultaneously,*

(1) *Eigenvalues of* Γ^S *are larger than* -1, *i.e.,* $\lambda_1(\Gamma^S) > -1$,
(2) $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}, \forall i$ *are P-matrices.*

Moreover, under these conditions, Υ^U *is a P-matrix.*

The proof is given in Appendix C.

These corollaries provide two alternative ways to check for the uniqueness of the Nash equilibrium. In terms of complexity, both finding the spectral radius and the eigendecomposition are of complexity $O(N^3)$ for an $N\times N$ matrix; these corollaries' conditions are therefore computationally easier to verify than the co-NP-complete problem. However, the trade-off is that the conditions in Corollary 4.3.1 are stronger than those of Theorem 4.3, while Corollary 4.3.2 can only be used given a symmetric Γ^S (resp. Γ^U) on structured (resp. unstructured) networks. It is also worth mentioning that even when Υ^S (resp. Υ^U) are asymmetric, we can still have symmetric Γ^S (resp. Γ^U) matrices, and thus, Corollary 4.3.2 could still provide a computationally lighter alternative verification in these cases.

Lastly, we again note that $\Gamma^U \in \mathbb{R}^{NK\times NK}$ could be formed by treating each agent as a singleton partition. When we take this viewpoint, Corollary 4.3.2 reduces to Proposition 3 of [24]. On the other hand, by using the partition structure, Γ^S is an $M \times M$ matrix, and the conditions in Corollary 4.3.2 are computationally easier to verify as compared to those in Proposition 3 of [24]. Specifically, checking the eigenvalues of a matrix requires performing eigendecomposition over it. To elaborate on the comparison, suppose $N_i = \overline{N}$ for all $\mathcal{P}_i$ (all partitions have the same size). Then, using the unstructured network, the complexity of the eigendecomposition on Υ^U or Γ^U is $O(K^3N^3)$ while using the partition structure, the complexity on Υ^S or Γ^S is reduced to $O(M^3)$. Of course, using the partition structure, we have to compute the α_i^S and β_{ij}^S values as well, which is of complexity $O(M^2\overline{N}^3)$. Altogether, the complexity of checking whether Υ^S is a P-matrix under the conditions in the above corollary is $O(M^3 + M^2\overline{N}^3)$, which can be much lower than $O(K^3N^3)$ in the unstructured case.

4.3 Sufficiency Gaps on the Uniqueness Conditions

To close this section, we elaborate on the difference between using and not using information about partition structures in checking for uniqueness of Nash equilibria.

The first thing to note is that, as mentioned earlier, the set of sufficient conditions derived when accounting for partition structures are generally stronger than their counterparts derived without using information about the structure: as Theorem 4.3, Corollary 4.3.1, and Corollary 4.3.2 indicate, if a structured network satisfies these uniqueness conditions, then it also satisfies the corresponding unstructured uniqueness conditions, but the opposite is in general not true. This means that there is a sufficiency gap between the conditions obtained from the structured and unstructured networks. The most important reason behind this sufficiency gap has to do with the way partition structures are abstracted. There are two forms of abstractions made during the creation of the Υ^S matrix; both are for partitions and each summarizes the component (agent-action) level internal and external impact, respectively:

(1) The internal impact α_i^S of partition $\mathcal{P}_i$, is an abstraction of the component-level internal impact α_k^U for components (and corresponding indices) in $\mathcal{P}_i$, and component level external impact β_{kl}^U between different indices in $\mathcal{P}_i$;
(2) The external impact β_{ij}^S of $\mathcal{P}_j$ on $\mathcal{P}_i$, is an abstraction of component-level external impact of indices in $\mathcal{P}_j$ to indices in $\mathcal{P}_i$.

These types of abstractions inevitably introduce gaps in the sufficiency conditions. As mentioned in Lemma 4.2, the value of α_i^S is lower-bounded by $||\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}||$ and highly depends on the agents with component-level internal impact in $\mathcal{P}_i$. Meanwhile, the β_{ij}^S value is upper bounded by $||\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}||$, and highly depends on the strongest component-level external impact from one index in $\mathcal{P}_j$ to another in $\mathcal{P}_i$. The significance of α_i^S in verifying conditions for the uniqueness of the Nash equilibria is akin to the observation "a chain is as strong as its weakest link"; we refer to this as the "weakest link effect". Similarly, the significance of the β_{ij}^S values is referred to as the "strongest link effect".

Recall an earlier observation in Theorem 4.3 that when each agent has stronger component-level internal impact than component-level external impact, the NE is unique. Similarly, in terms of structure, when each partition has stronger internal impact than external impact, the NE is unique and the conditions in Theorem 4.3 and Corollaries 4.3.1, 4.3.2 are sufficient to guarantee such impact differentials.

In some games, the indices k with weak component-level internal impact α_k^U also have weak component-level external impact β_{kl}^U while the indices l with strong component-level internal impact α_l^U have similarly strong component-level external impact β_{lk}^U. While we may be able to guarantee the uniqueness of an NE using the unstructured network, we may not be able to do so using structures when a partition contains both types of (strong and weak) indices. This is because the abstraction becomes inaccurate as the partition's internal impact α_i^S is weak and the external impact β_{ij}^S strong; thus the conditions obtained from the structured network may fail to guarantee the uniqueness of NE. The following example highlights these observations.

EXAMPLE 2. *Consider a 4-agent, single-action dimension, 2-partition game where agents a_1, a_2 form $\mathcal{P}_1$ and agents a_3, a_4 form $\mathcal{P}_2$, with utility functions: $u_1(\boldsymbol{x}) = x_1(x_2 + 5x_3) - 5x_1^2, u_2(\boldsymbol{x}) = x_2(x_1 + x_4) - 2x_2^2, u_3(\boldsymbol{x}) = x_3(x_4 + 5x_1) - 5x_3^2, u_4(\boldsymbol{x}) = x_4(x_3 + x_2) - 2x_4^2$. Then, we have*

$$\Upsilon^U = \begin{bmatrix} 10 & -1 & -5 & 0 \\ -1 & 4 & 0 & -1 \\ -5 & 0 & 10 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \succ \mathbf{0}, \ \Upsilon^S = \begin{bmatrix} 3.838 & -5 \\ -5 & 3.838 \end{bmatrix} \prec \mathbf{0},$$

which means that the unstructured condition guarantees the uniqueness of the NE in this game, yet the game fails to satisfy the sufficient structured condition. This is an example when the internal impact of a partition is weak but the external impact between partitions is strong.

We end this section by summarizing the types of games and partition structures that result in small sufficiency gaps between the two sets of sufficient conditions. The sufficiency gap is small if partition members have similar component-level internal impact; or if members have similar connections to agents in other partitions and have similar component-level external impact level; or if the member with the weakest component-level internal impact also has the strongest component-level external impact. We present numerical experiments in Section 7 to further elaborate on these comparisons.

5 STABILITY

We next examine conditions for the stability of the Nash equilibrium in these games. When small changes occur to the underlying model parameters, a new Nash equilibrium may result. Intuitively, if the new Nash equilibrium is close enough to the original one, then we say the original Nash equilibrium is stable.

Formally, we generalize our utility functions in Eqn (1) to the family of parameterized functions $u_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}, \boldsymbol{p}_i)$, where $\boldsymbol{p}_i = [p_i^{(1)}, \cdots, p_i^{(K)}] \in \mathbb{R}^K$ is a vector valued *perturbation parameter* or *shock* on a_i, and $\boldsymbol{p} = [\boldsymbol{p}_1, \cdots \boldsymbol{p}_N] \in \mathbb{R}^{NK}$ denotes the vector of all perturbations/shocks. Moreover, let $\boldsymbol{x}^*(\boldsymbol{p})$ be the action profile at the Nash equilibrium of the game under perturbation vector $\boldsymbol{p}$ and $\boldsymbol{x}^*$ be the Nash equilibrium of the unperturbed game ($\boldsymbol{x}^* := \boldsymbol{x}^*(\boldsymbol{0})$).

We denote a ball of radius $r > 0$ centered at $\boldsymbol{x} \in \mathbb{R}^N$ by $B(\boldsymbol{x}, r) := \left\{\boldsymbol{y} \in \mathbb{R}^N : ||\boldsymbol{x} - \boldsymbol{y}||_2 < r\right\}$.

DEFINITION 5. *([19]) A Nash equilibrium $\boldsymbol{x}^*$ is stable if $\exists r > 0, d > 0$ such that $\forall \boldsymbol{p} \in B(\boldsymbol{0}, r)$, the Nash equilibrium $\boldsymbol{x}^*(\boldsymbol{p})$ exists and satisfies*

$$||\boldsymbol{x}^*(\boldsymbol{p}) - \boldsymbol{x}^*||_2 \le d||H(\boldsymbol{x}^*(\boldsymbol{p}), \boldsymbol{p}) - H(\boldsymbol{x}^*(\boldsymbol{p}), \boldsymbol{0})||_2 ,$$

where $H(\boldsymbol{x}, \boldsymbol{p}) = (H_i(\boldsymbol{x}, \boldsymbol{p}))_{i=1}^M$ with

$$H_i(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{x}_i - BR_i(\boldsymbol{x}_{-i}, G, \boldsymbol{p}) .$$

Definition 5 states that if an NE $\boldsymbol{x}^*$ is stable, the Nash equilibrium of the perturbed game ($\boldsymbol{x}^*(\boldsymbol{p})$) remains close to the Nash equilibrium of the unperturbed game ($\boldsymbol{x}^*(\boldsymbol{0})$).

5.1 Stability Condition Without Network Structure

In order to determine whether a Nash equilibrium $\boldsymbol{x}^*$ is stable, [24] proposed dividing the agents' action indices into three disjoint sets based on $\boldsymbol{x}^*$:

$$\begin{aligned} A(\boldsymbol{x}^*) :=& \{j = \psi(i,k) \mid x_i^{(k)*} > 0,\ x_i^{(k)*} = \tilde{BR}_i^{(k)}(\boldsymbol{x}_{-i}^*, u_i)\}, \\ I(\boldsymbol{x}^*) :=& \{j = \psi(i,k) \mid x_i^{(k)*} = 0,\ x_i^{(k)*} > \tilde{BR}_i^{(k)}(\boldsymbol{x}_{-i}^*, u_i)\}, \\ B(\boldsymbol{x}^*) :=& \{j = \psi(i,k) \mid x_i^{(k)*} = 0,\ x_i^{(k)*} = \tilde{BR}_i^{(k)}(\boldsymbol{x}_{-i}^*, u_i)\}, \end{aligned}$$

where $\tilde{BR}_i(\boldsymbol{x}_{-i}^*, u_i)$ is the *unbounded* best response and can take negative values, $\tilde{BR}_i^{(k)}$ denotes the k-th action dimension unbounded best response, and $\psi : \{1, \ldots, N\} \times \{1, \ldots, K\} \mapsto \{1, \ldots, KN\}$ maps the $N \times K$ agent-action indices pair to the KN indices in the unstructured operator F. $A(\boldsymbol{x}^*)$ is referred to as the set of active indices, $I(\boldsymbol{x}^*)$ the set of strictly inactive indices, and $B(\boldsymbol{x}^*)$ the set of borderline inactive indices. Intuitively, with a small parametric perturbation $\boldsymbol{p}$, agent action indices in $A(\boldsymbol{x}^*)$ remain active ($x_i^{(k)*}(\boldsymbol{p}) > 0$) and agent action indices in $I(\boldsymbol{x}^*)$ remain inactive ($x_i^{(k)*}(\boldsymbol{p}) = 0$), while agent action indices in $B(\boldsymbol{x}^*)$ can transform from inactive to active ($x_i^{(k)*}(\boldsymbol{p}) > x_i^{(k)*}(\boldsymbol{0}) = 0$).

Under these definitions, [24] established the following sufficient condition for the solution to $VI(Q, F)$ to be stable in the sense of Definition 5.

THEOREM 5.1. *([24]) Consider the matrix*

$$\nabla_{A,B}F_{A,B}(\boldsymbol{x}^*) = \begin{bmatrix} \nabla_A F_A(\boldsymbol{x}^*) & \nabla_B F_A(\boldsymbol{x}^*) \\ \nabla_A F_B(\boldsymbol{x}^*) & \nabla_B F_B(\boldsymbol{x}^*) \end{bmatrix} \tag{11}$$

where $\nabla_{S_1}F_{S_2}(\boldsymbol{x}^*)$ is a sub-matrix of $\nabla F(\boldsymbol{x}^*)$ with rows and columns corresponding to the agent action indices in sets S_1 and S_2 (not necessarily groups), respectively, and $\nabla_{A,B}F_{A,B}(\boldsymbol{x}^*)$ is generated by selecting rows and columns corresponding to $A \cup B$ from the game Jacobian $\nabla F(\boldsymbol{x}^*)$. If $\nabla_{A,B}F_{A,B}(\boldsymbol{x}^*)$ is positive definite on Q, then the solution $\boldsymbol{x}^*$ to $VI(Q, F)$ is stable.

Below we provide a condition for stability which is easier to verify as compared to that in Theorem 5.1 by taking the partition structure into account.

5.2 Stability Condition with Partition Structure

Similar to [24], we divide *partitions* into active, strictly inactive, and borderline inactive sets. Specifically: (1) a partition is active if at least one agent action index in that partition is active at NE $\boldsymbol{x}^*$; (2) if all agent action indices in a partition are strictly inactive, then the partition is strictly inactive; (3) if all agent action indices of a partition are inactive and at least one of them is borderline inactive, then the partition is considered as a borderline inactive partition. Formally, we have,

$$\begin{aligned} A_S(\boldsymbol{x}^*) &:= \{\mathcal{P}_i \mid \boldsymbol{x}^*_{\mathcal{P}_i} \neq \mathbf{0}\}, \\ I_S(\boldsymbol{x}^*) &:= \{\mathcal{P}_i \mid \boldsymbol{x}^*_{\mathcal{P}_i} = \mathbf{0}, \boldsymbol{x}^*_{\mathcal{P}_i} > \tilde{BR}_{\mathcal{P}_i}(\boldsymbol{x}^*)\}, \\ B_S(\boldsymbol{x}^*) &:= \{\mathcal{P}_i \mid \boldsymbol{x}^*_{\mathcal{P}_i} = \mathbf{0}\} - I_S(\boldsymbol{x}^*), \end{aligned} \tag{12}$$

where $A_S(\boldsymbol{x}^*)$, $I_S(\boldsymbol{x}^*)$, $B_S(\boldsymbol{x}^*)$ denote the set of active, strictly inactive and borderline inactive partitions, respectively. We use $\boldsymbol{x}^*_{\mathcal{P}_i}, \tilde{BR}_{\mathcal{P}_i}(\boldsymbol{x}^*)$ denote the vectors by choosing all indices in partition $\mathcal{P}_i$ from the NE $\boldsymbol{x}^*$, and the unbounded best response $\tilde{BR}(\boldsymbol{x}^*)$.

THEOREM 5.2. *Consider a Nash equilibrium $\boldsymbol{x}^*$ of the network game. Re-index all partitions in $A_S(\boldsymbol{x}^*) \cup B_S(\boldsymbol{x}^*)$ with indices $1, 2, \cdots, Z$, $Z = |A_S(\boldsymbol{x}^*)| + |B_S(\boldsymbol{x}^*)|$. Then, define*

$$G^S(\boldsymbol{x}^*) = \begin{bmatrix} \theta^S_1(\boldsymbol{x}^*) & -\delta^S_{12}(\boldsymbol{x}^*) & \dots & -\delta^S_{1Z}(\boldsymbol{x}^*) \\ -\delta^S_{21}(\boldsymbol{x}^*) & \theta^S_2(\boldsymbol{x}^*) & \dots & -\delta^S_{2Z}(\boldsymbol{x}^*) \\ \vdots & \vdots & \ddots & \vdots \\ -\delta^S_{Z1}(\boldsymbol{x}^*) & -\delta^S_{Z2}(\boldsymbol{x}^*) & \dots & \theta^S_Z(\boldsymbol{x}^*) \end{bmatrix}$$

where $\delta^S_{ij}(\boldsymbol{x}^*) = ||\nabla_j F_i(\boldsymbol{x}^*)||_2$, $\theta^S_i(\boldsymbol{x}^*) = ||\nabla_i F_i(\boldsymbol{x}^*)||_2$. If the following conditions hold simultaneously:

(1) $G^S(\boldsymbol{x}^*) \succ \mathbf{0}$,
(2) $\nabla_i F_i(\boldsymbol{x}^*) \succ \mathbf{0}, \forall \mathcal{P}_i \in A_S \bigcup B_S$,

then $\boldsymbol{x}^*$ is stable. Moreover, these conditions are sufficient for $\nabla_{A,B}F_{A,B}(\boldsymbol{x}^*) \succ \mathbf{0}$, i.e., the condition for stability on an unstructured network (Theorem 5.1) holds under these conditions.

For a detailed proof, see Appendix D.

Intuitively, the matrix $G^S(\boldsymbol{x}^*)$ captures the mutual influence between active and borderline inactive partitions at the current Nash equilibrium profile. The borderline inactive partitions can turn into active partitions under parametric perturbations. When such flips are significant, large fluctuations can appear in the network, which can be further amplified through rebounds and reflections. In this case, new equilibria may not exist, and even if they do, they may be far away from the original equilibrium. However, when $G^S(\boldsymbol{x}^*) \succ \mathbf{0}$ holds, the maximum impacts of flipping partitions from (borderline) inactive to active are bounded, and therefore the current Nash equilibrium remains stable.

In terms of complexity of verifying these conditions, note that G^S is a $Z \times Z$ matrix. Similar to the comparison shown in Section 4, if we denote $Y = |A(\boldsymbol{x}^*)| + |B(\boldsymbol{x}^*)|$, then the computational complexity of condition verification in Proposition 5.2 vs. Theorem 5.1 are $O(Z^3 + Z^2\overline{N}^3)$ vs. $O(K^3Y^3)$, where $\overline{N}$ is the average group size. Therefore, since $Z < KY$ ($Z\overline{N} \approx KY$), the computational complexity of condition verification in Proposition 5.2 is lower than that of Theorem 5.1.

We conclude this section with a condition on Υ^S leading to stable Nash equilibrium.

THEOREM 5.3. *Assume Υ^S is symmetric. Then if the following two conditions hold simultaneously:*

(1) $\Upsilon^S \succ \mathbf{0}$,

(2) $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}, \forall i$ *are P-matrices,*

the network game's Nash equilibrium is unique and stable.

For a detailed proof, see Appendix E.

6 CENTRALITY

In network games, notions of node centrality are used to measure the influence of individual nodes on network-level outcomes. Degree centrality is one of the centrality metrics which has gained attention in the literature [6, 28]. In a directed graph, two different measures of degree centrality are considered for each node: in-degree centrality, which is a count of edges directed to a given node, and out-degree centrality, which is the number of outward edges from the given node. In this section, we propose a generalization of the degree centrality measure for disjoint partitions.

Recall that we capture the influence of a partition using the Jacobian matrix $\nabla F(\boldsymbol{x})$. Matrix $\nabla_j F_i(\boldsymbol{x})$ measures the sensitivity of agents in partition i to the action profile of agents in partition j. Accordingly, we define our generalized centrality measure as follows.

DEFINITION 6. *Generalized Degree Centrality (GDC): Following Definition 4, denote $\beta^S_{ij} = \sup_{\boldsymbol{x}\in\mathcal{Q}} ||\nabla_j F_i(\boldsymbol{x})||_2$, and $\alpha^S_i = \inf_{\boldsymbol{x}\in\mathcal{Q}} ||\nabla_i F_i(\boldsymbol{x})||_2$. The generalized degree centralities for partition $\mathcal{P}_i$ are given by:*

$$D^{in}_i = \sum_{j:j\neq i} \frac{\beta^S_{ij}}{\alpha^S_i},\ D^{out}_i = \sum_{j:j\neq i} \frac{\beta^S_{ji}}{\alpha^S_j},\ \forall i,j \in \mathbb{N}[1,M]\ .$$

Moreover, the maximum GDCs are defined as follows:

$$D^{in}_{max} = \max_{i\in\mathbb{N}[1,M]} D^{in}_i,\ D^{out}_{max} = \max_{i\in\mathbb{N}[1,M]} D^{out}_i\ .$$

The above definition can be interpreted as follows: out-degree centrality measures the influence of a given partition $\mathcal{P}_i$ on the network based on three factors, (1) connectivity, or the number of links directed outward from $\mathcal{P}_i$, (2) the internal impact of the target partitions that receive impact from $\mathcal{P}_i$, and (3) the external impact $\mathcal{P}_i$ has for every target group. In Definition 6, D^{out}_i captures these factor through the summation of $\frac{\beta^S_{ji}}{\alpha^S_j}$. D^{in}_i can be interpreted similarly.

In addition to capturing importance due to their roles in the network, each partition can be endowed with certain *exogenous* (non-network related) importance (not to be confused with external impact from another partition). This will result in an extended centrality measure. The following is a generalization of the extended centrality measure defined in [31].

DEFINITION 7. *Generalized Extended Degree Centrality (GEDC): Let $\boldsymbol{e} \in \mathbb{R}^M_{>0}$ denote the vector of external importance, where $(\boldsymbol{e})_i = e_i > 0$ is $\mathcal{P}_i$'s external importance. The generalized extended* *degree centralities for $\mathcal{P}_i$ are given by*

$$D_i^{in}(\boldsymbol{e}) = \sum_{j:j\neq i} \frac{\beta_{ij}^S}{\alpha_i^S}\frac{e_j}{e_i},\ D_i^{out}(\boldsymbol{e}) = \sum_{j:j\neq i} \frac{\beta_{ji}^S}{\alpha_j^S}\frac{e_i}{e_j}, \forall i, j \in \mathbb{N}[1, M]$$

and the maximum GEDCs are defined as

$$D_{max}^{in}(\boldsymbol{e}) = \max_{i\in\mathbb{N}[1,M]} D_i^{in}(\boldsymbol{e}),\ D_{max}^{out}(\boldsymbol{e}) = \max_{i\in\mathbb{N}[1,M]} D_i^{out}(\boldsymbol{e})\ .$$

When $\boldsymbol{e} = \alpha\mathbf{1}, \alpha > 0$, Definitions 6 and 7 are equivalent. We now show the connection between our centrality measure and the uniqueness of the Nash equilibrium.

THEOREM 6.1. *If the following two conditions hold simultaneously:*

(1) $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}, \forall i$ *are P-matrices,*

(2) *there exists* $\boldsymbol{e} > \mathbf{0}$ *such that* $D_{max}^{in}(\boldsymbol{e}) < 1$*, or* $D_{max}^{in}(\boldsymbol{e}) < 1$*,*

then the Nash equilibrium is unique. If in addition Υ^S *is symmetric, the Nash equilibrium is unique and stable.*

For a detailed proof, see Appendix F.

Theorem 6.1 implies that if either the in-degree or out-degree GEDCs are bounded, then the Nash equilibrium is unique. On the other hand, if neither the indegree nor outdegree is bounded, then at least one partition has an outsized effect on the network. This partition's decision can change the state of the network significantly, resulting in possibly multiple equilibria.

Theorem 6.1 is similar to Proposition 7 in [33], but differs in the following aspect. In our work, the β_{ij}^S represent the influence of partitions on each other, while β_{ij} in [33] represents the component-level influence of agents on each other. Moreover, when both conditions in Theorem 6.1 hold, $\Upsilon^S > 0$; this then becomes a special case of the condition in Corollary 4.3.2 (where Υ^S is symmetric).

Moreover, Theorem 6.1 shows that if Υ^S is symmetric and the degree centrality of the partitions are bounded, then the unique Nash equilibrium is also stable. Intuitively, this is because if the degree centrality of a given partition is not bounded, the partition has a considerable impact on the network, and the Nash equilibrium may not be stable as a small perturbation affecting this partition can influence the network dramatically.

7 NUMERICAL RESULTS

In this section, we present numerical results that are closely related to the analytical results shown in previous sections. We first show the computational complexity and sufficiency gaps between the structured and unstructured conditions, using single-action dimension (single-relational) network games; the results are applicable to multi-relational network games as our analysis has shown. We then use a multi-relational network game example to demonstrate an interesting setting where some agents are important in one type of relationship but not in another. We will visualize this phenomenon with the corresponding centrality measures defined in Section 6.

7.1 Procedure for Game Instance Generation

Our results on computational complexity gaps and the sufficiency gaps are obtained from a large number of game instances randomly generated using the following procedure. A game is generated for a specified size (number of agents N, number of action dimensions $K = 1$, number of partitions M, and size of each partition N_i) and using utility functions $u_i(\boldsymbol{x}, G) = x_i(b_i + \sum_j g_{ij}x_j) - \frac{c_i}{2}x_i^2$. The rest of the game is given by the interaction matrix G, and the vectors $\mathbf{b} = (b_i)_{i=1}^N$ and $\mathbf{c} = (c_i)_{i=1}^N$. In generating a random G, the diagonal elements are set to 0 without loss of generality. Each partition is associated with consecutive agent indices and thus the diagonal blocks represent each partition and the off-diagonal blocks represent cross-partition interdependencies. The off-diagonal elements in a diagonal block are generated using a Bernoulli distribution with parameter P^{in}_{exist}, the probability for an connection (non-zero element) to exist between any pair of agent-action indices. If such a connection exists, its value (strength of the connection), g_{ij}, is drawn from a uniform distribution on the interval $[S^{in}_{low}, S^{in}_{high}]$. We generate g_{ij} and g_{ji} independently. These elements also determine the off-diagonal elements in the diagonal blocks of Υ^U and the diagonal elements α^S_i in the Υ^S matrix.

The off-diagonal blocks of G are generated using the same approach, with parameters P^{out}_{exist} and $[S^{out}_{low}, S^{out}_{high}]$. These determine the connection frequencies and strengths between groups. These elements also determine the off-diagonal blocks of the Υ^U matrix as well as the off-diagonal elements β^S_{ij} in the Υ^S matrix.

In subsequent numerical results, a *dense* matrix G refers to both $P^{in}_{exist} = 1$ and $P^{in}_{exist} = 1$, i.e., the entries in Υ^U and Υ^S are non-zero with probability 1.

The vector of individual cost $\boldsymbol{c}$ is generated by first choosing a fixed mean value $\bar{c} = \frac{1}{2}$ (so that $\mathbb{E}[\alpha^U_k] = 1$) for all the cost terms. We then generate a partition mean $\bar{c}_{\mathcal{P}_i}$ by sampling uniformly at random from an interval $[c^{low}, c^{high}]$, where $c^{low} + c^{high} = 1$. Next, within each group, we choose an interval $[c^{low}_{\mathcal{P}_i}, c^{high}_{\mathcal{P}_i}]$, where $(c^{low}_{\mathcal{P}_i} + c^{high}_{\mathcal{P}_i})/2 = \bar{c}_{\mathcal{P}_i}$ and then sample uniformly at random the individual cost terms from this. All individual cost terms are set to strictly positive values, otherwise neither structured nor unstructured conditions hold, making the game instances trivial. The individual benefit is set to $\boldsymbol{b} = \mathbf{1}$ as it does not affect either Υ^U or Υ^S. In generating these values we fix the global mean but change the variance.

Using game instances generated through this procedure, we will separately compare the verification complexity and sufficiency gaps on the two sets of structured and unstructured conditions. We first use games where both sets of conditions are satisfied to allow comparison of verification complexity. For the sufficiency gap comparison, we generate games using different parameter settings described above, and measure how frequently the structured conditions fail while the unstructured conditions are satisfied.

7.2 The Computational Complexity Gap

As discussed in Sections 4, the verification of uniqueness conditions on structured networks is of lower complexity. There are several factors that determine how big this complexity gap is, which we examine in this section.

The first factor is the size of the network (total number of agents). Specifically, the complexity gap increases with the number of agents, and is at least quadratic in the number of groups. Table 1 lists the verification complexity (in floating point operations, flops) of the conditions in Corollary 4.3.2 with a dense Γ^U matrix. We see that the verification complexity of structured conditions is orders of magnitude lower than that of the unstructured conditions, and the gap increases with the size of the network. Table 2 shows the difference in CPU times, where the results are averaged over 50 different instances. Moreover, the complexity reduction in verifying the conditions in Theorem 4.3 is much more significant than those in Corollary 4.3.2. For instance, the verification complexity of conditions in Theorem 4.3 on a game of size 10×10 (10 partitions of 10 agents each) on a dense Υ^U matrix is 1.08×10^{35} flops while that on the corresponding Υ^S matrix is 1.26×10^6 flops. We refer the interested reader to Appendix G for additional comparisons.

The second factor affecting the complexity gap is how (a given number of) agents are partitioned into groups. Figure 5 shows the complexity (in flops) of verifying the structured condition (of Υ^S being a P-matrix) in two games of size 50 and 100 agents, respectively, both with a dense Υ^U matrix,

Size ($N_i \times M$)	Unstructured	Structured
10×10	$6.67{\times}10^5$	$\mathbf{5.83{\times}10^4}$
20×20	$4.27{\times}10^7$	$\mathbf{1.99{\times}10^6}$
50×50	$1.04{\times}10^{10}$	$\mathbf{2.02{\times}10^8}$
100×100	$6.67{\times}10^{11}$	$\mathbf{6.57{\times}10^9}$
200×200	$4.27{\times}10^{13}$	$\mathbf{2.12{\times}10^{11}}$

Table 1. Verification complexity in number of FLOPs of conditions in Corollary 4.3.2 over number of agents.

Size ($N_i \times M$)	Unstructured (sec)	Structured (sec)
10×10	0.0031±0.0011	**0.003±0.0002**
20×20	0.0575±0.0074	**0.0213±0.0019**
50×50	5.851±0.197	**0.6768±0.0442**
100×100	332.8±1.6	**7.052±0.203**
200×200	Memory Overflow	**116.8±6.9**

Table 2. Verification complexity in CPU times of conditions in Corollary 4.3.2 over number of agents; all times are in seconds. All experiments were performed on a machine with A 6-core 2.60/4.50 GHz CPU with hyperthreaded cores, 12MB Cache, and 16GB RAM.

as we vary the number of groups. Here, we set to groups to be of equal size, rounding off to the nearest integer if needed; e.g., with 10 agents and 3 groups, the partition sizes are 3, 3, 4.[1]

We see that in each game the complexity has a V shape, reaching a minimum when $M \approx \sqrt{N}$. To explain this, note that we can approximate the complexity of LU decomposition by $\frac{2}{3}p^3$ for a matrix of size $p \times p$; for singular value decomposition the complexity can be approximated by $2pk^2 - \frac{2}{3}k^2$, $p > k$, ([13, p. 75], based on QR decomposition using Householder transformations). Then the complexity for checking the conditions in Theorem 4.3 is given by ($S = N/M$):

$$\binom{M}{2}\left(2S^3 - \frac{2}{3}S^2\right) + M\sum_{k=1}^{S}\binom{S}{k}\left(\frac{2}{3}k^3\right) + \sum_{k=1}^{M}\binom{M}{k}\left(\frac{2}{3}k^3\right).$$

As $\max\{M!, S!\}$ will be the dominant term in the (expanded) expressions, the minimum is achieved when $M = S = \sqrt{N}$.

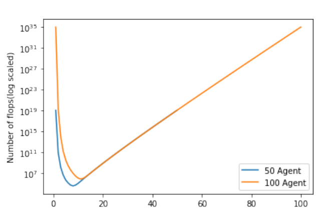

Fig. 5. Verification complexity (flops) of conditions in Theorem 4.3 over number of groups.

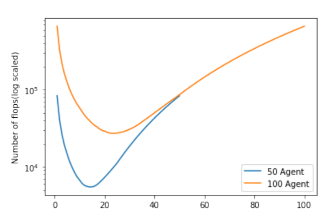

Fig. 6. Verification complexity (flops) of conditions in Corollary 4.3.2 over number of groups.

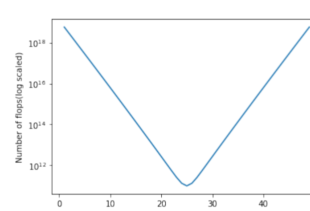

Fig. 7. Complexity (flops) over different partitions in games with 50 agents, 2 groups and a dense Υ^U matrix.

Note also that in Figure 5 the two curves overlap almost completely beyond the minimum. This is because verifying whether Υ^S is a P-matrix has two phases: generating the Υ^S matrix followed by an

[1]Here we are comparing the structured complexity between games of different sizes. For a given size N with M equal-sized partitions, the structured and unstructured conditions have the same complexity when $M = 1$ or $M = N$.

eigen-decomposition on it. When M is small the first phase is dominant, whereas when M is larger than $\sqrt{N}$ the second becomes dominant. Since Υ^S is of size $M \times M$, when M is large the complexity depends much more on the value M than on the network size N.

In another experiment, we verify the conditions in Corollary 4.3.2 on two games of sizes 50 and 100, respectively, both with a dense Υ^U matrix. The results are shown in Figure 6. This time the minimum occurs at some $M > \sqrt{N}$. This is because the approximated complexity is given by

$$\binom{M}{2}\left(2S^3 - \frac{2}{3}S^2\right) + \frac{2}{3}M^3 + \frac{2}{3}S^3,$$

which has a minimum at $M > \sqrt{N}$.

A third factor affecting the complexity gap is the size distribution of groups, given fixed N and M. Figure 7 shows the complexity of verifying whether Υ^S is a P-matrix in a 50-agent, 2-partition game with a dense Υ^U matrix, as we vary the size of the first group. We see that the complexity reaches its minimum when the two groups are equal sized.

More generally, when we have more action dimensions and create partitions based on them, we can expect that some of the off-diagonal elements in the structured matrix Υ^S will be computed with much lower complexity. To see why, consider two games, one with $N = 100$, $K = 1$ and 10 groups of size 10 each, the other with $N = 50$, $K = 2$ and 5 groups of size 10 each. If we create a partition for all agent-action components from the same group in the same action, we get 10 partitions for both games. While the first game is similar to the samples in this section, the second game has 20 off-diagonal elements in Υ^S computed from the coupled cost. Computing these 20 elements are easier than computing the 2-norm of a matrix since the corresponding block is a diagonal matrix. Moreover, if all agents have the same cost function, then these 20 elements are the same and based on one utility function, which makes the computation even easier.

7.3 The Sufficiency Gap

The structured conditions are stronger than their unstructured counterparts, thus may fail to discover the uniqueness and stability of an NE in a game that fails the former while satisfying the latter. We showed this using an example in Section 4. In what follows we will use numerical results to measure this sufficiency gap. We will focus on the uniqueness conditions, and note that comparison on stability conditions is very similar.

Each of the next set of figures is a heat map showing how often these two conditions in Corollary 4.3.2 (over Γ^S and Γ^U, respectively, with sample games that guarantee both Γ^S and Γ^U are symmetric) yield the same or different result. The former means either both are satisfied or not satisfied, while the latter necessarily means the structured condition fails and the unstructured condition holds. Each game is of size 400, 20 groups of 20 members each. For each cell in the heat map, 50 sample games are generated using a set of parameters corresponding to the cell indicated on the figure; the cell color indicates the fraction of these games that resulted in a difference (sufficiency gap), the higher the fraction, the darker the color. In each of the heat maps, we can see regions of darker cells (clearly) separating the map into two lighter regions. In general, the bottom left represents parameter settings where both structured and unstructured conditions are satisfied and the top right represents settings where both conditions do not hold.

Specifically, in Figures 8, 9, and 10 we hold component-level external impact fixed (at weak, medium, and strong levels, respectively, corresponding to $P^{in}_{exist}, P^{out}_{exist}$ at 0.2, 0.5, 0.8 respectively; $(S^{in}_{low} + S^{in}_{high})/2$ at $0.2/N_i, 0.5/N_i, 0.8/N_i$ respectively, for partition $\mathcal{P}_i$ (normalizing the strengths to make external and internal impact comparable); $(S^{out}_{low} + S^{out}_{high})/2$ fixed at $0.2/N, 0.5/N, 0.8/N$ (normalize the strengths, similar as above), while changing the variances of component-level internal impact (within-partition along the x-axis by changing the value of $c^{high}_{\mathcal{P}_i} - c^{low}_{\mathcal{P}_i}$, and between-partition along the y-axis by changing the value of $c^{high} - c^{low}$). We note that the normalized upper bound for the model parameters are chosen at 1 when we study the sufficiency gaps, because higher values cause both sets of conditions to fail thereby reducing the significance of such sample games. Please refer to Appendix H for more details.

Overall the sufficiency gap is quite low, i.e., the two types of conditions yield the same outcome in the vast majority of parameter settings (as evidenced by mostly 0 values/light-colored cells on these heat maps). The measured difference (obtained by adding the number of different results in every cell and dividing by the total number of sample games) in Figures 8, 9, and 10 are 0.26%, 5.58%, 0.05%, respectively.

Figures 8 and 9 show a similar pattern. In the top right corners where component-level internal impact has large variance both between and within groups, neither condition is true, while in the lower left corners both conditions are satisfied, giving rise to the broad agreement (light regions). In comparing the two, we see that the dark region expands and shifts leftward and downward in Figure 9, suggesting that the gap between the two conditions are bigger and triggered by lower variances in component-level internal impact when the component-level external impact increases from weak to medium. Furthermore, high within-partition variance combined with low between-partition variances in component-level internal impact results in the largest gap. The reason is the "weakest member effect" discussed in Section 4, where the α^S_i values depend highly on the minimum component-level internal impact α^U_k of members in $\mathcal{P}_i$, which makes the abstraction Υ^S inaccurate, causing a larger gap. Interestingly, an increase in between-partition variances in component-level internal impact can mitigate the above effect and reduce the gap. On the other hand, when the component-level external impact is sufficiently high, Figure 10 shows that the sufficiency gap all but disappears as now most game instances do not satisfy either condition.

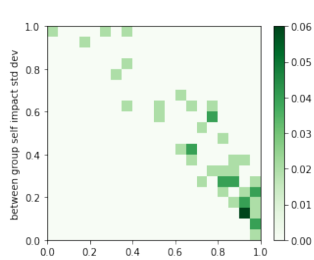

Fig. 8. Sufficiency gap frequency over the variance of internal impact, weak external impact.

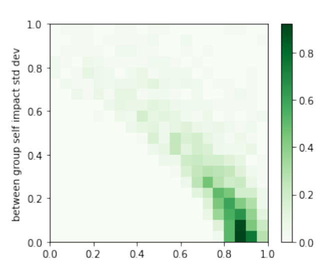

Fig. 9. Sufficiency gap frequency over the variance of internal impact, medium external impact.

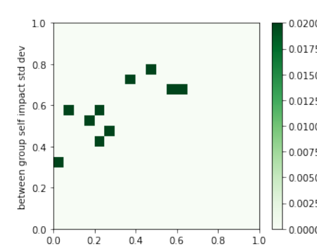

Fig. 10. Sufficiency gap frequency over the variance of internal impact, strong external impact.

We next fix the agent's component-level internal impact at 1, the between-partition (resp. within-group) component-level external impact at strong, and vary the within-partition (resp. between-group) component-level external impact strength (x-axis) and connection frequency (y-axis), shown in Figure 11 (resp. Figure 12). We see that the gap is generally small (2% gap for both). In both figures the disagreement is concentrated around a reciprocal curve, suggesting that when the product of the two parameters is around a critical level, the sufficiency gaps occur. They also suggest that the role of between-partition and within-partition external impact is very similar under these two sets of conditions. The dark regions in these two figures suggest that when individuals have homogeneous internal impact, games where the expected sum of component-level external impact is about 10%

higher than the expected sum of component-level internal impact are most likely to have sufficiency gaps. Please refer to Appendix H for detailed discussion on this phenomenon. Moreover, the dark cell curve in Figure 12 shows that when the between-partition connection frequency gets lower, the chance of having a sufficiency gap is higher. This is because a mismatch similar to Example 2 (an agent with weak component-level internal impact does not receive strong component-level external impact but a member in the same partition with strong component-level internal impact does) is more likely to happen when the between-partition connection frequency is lower.

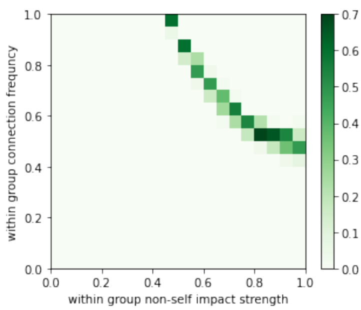

Fig. 11. Sufficiency comparison over the within partition connections, when every agent has the same internal impact and strong between partition influence.

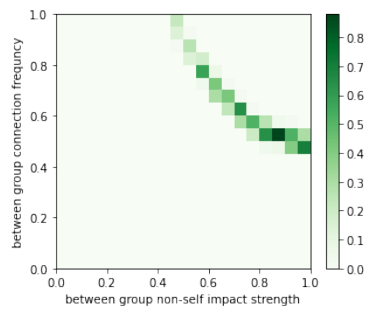

Fig. 12. Sufficiency comparison over the between partition connections when every agent has the same internal impact, medium within partition influence.

These numerical results suggest that in general, when network communities are formed around homogeneous agents, i.e., those with similar component-level internal impact, component-level external impact and connectivity, then the two types of conditions yield identical verification outcome.

More generally, when we have more action dimensions and create partitions based on them, we can expect some of the off-diagonal elements in the structured matrix Υ^S to be computed from the coupled cost function. Similar to Section 7.2, we compare the sufficiency gap in two games, one with $N = 100$, $K = 1$ and 10 groups of size 10 each, the other with $N = 50$, $K = 2$ and 5 groups of size 10 each, and create 10 partitions for both games. Again, the first game is similar to the samples in this section, while the second game has 20 off-diagonal elements in Υ^S computed from the coupled cost. When all agents have the same cost functions, these 20 elements do not suffer from the strongest link effect and thus further reduce the sufficiency gap.

7.4 Visualizing the Partition Centrality

We next create a relatively small multi-relational network game of $N = 30$, $K = 2$, and use this visualize the centrality measures defined in Section 6. We generate the game to have symmetric inter-dependencies so that $D^{in} = D^{out}$ and compute the centrality with exogenous importance set to $\boldsymbol{e} = \mathbf{1}$.

The agents form 3 (pre-defined and color-coded) groups, each with a size of 10: we generate interaction graphs where group 1 has high network influence levels on action dimension 1 but low influence on dimension 2; group 2 has low influence on action dimension 1 but high influence on dimension 2; group 3 has low influence levels on both action dimensions. Figures 13 and 14 depict the interaction graphs on each action dimension. As can be seen, the red group is highly connected to both the yellow and blue groups in action 1, while the blue group is highly connected to the red and yellow in action 2.

With 3 groups and 2 action dimensions, we create 6 partitions, each consisting of agent-action components of one group on one action dimension. A partition is illustrated as a colored circle in Figures 15 and 16: the color corresponds to the group identity of the agents in that partition, with a label ("1" or "2") indicating the action dimension. The size of the circle indicates the value of that partition's generalized degree centrality (GDC) given in Definition 6, the larger the circle the higher the centrality. The links (their existence and thickness) between two partitions mirror the off-diagonal entries in the Υ^S matrix.

Figures 15 and 16 each represents a different type of cost function, coupled and decoupled. In the coupled cost, the action dimensions influence each other through the cost function $c(x^{(1)}, x^{(2)}) = \frac{1}{2}(x^{(1)})^2 + \frac{1}{2}(x^{(2)})^2 + \frac{1}{10}x^{(1)}x^{(2)}$. In the decoupled cost, the cost function is $c(x^{(1)}, x^{(2)}) = \frac{1}{2}(x^{(1)})^2 + \frac{1}{2}(x^{(2)})^2$. We see that since the red group has frequent and strong connections in action 1, the GDC of the partition R1 is high, so is B2. When the cost is coupled, the partitions of the same group of agents on different action dimensions are connected; otherwise the structured graph consists of disconnected components and we can solve the subgames on each dimension independently.

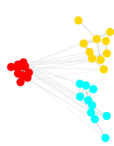

Fig. 13. Action 1 interaction graph for the experiments in Section 7.4. The red group has high influence on action dimension 1.

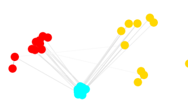

Fig. 14. Action 2 interaction graph for the experiments in Section 7.4. The blue group has high influence on action dimension 2.

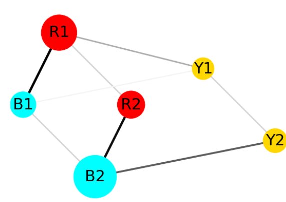

Fig. 15. Visualization of partitions' GDC with *coupled* cost. The largest nodes, "R1" and "B2", denoting the partition of the agent-action components from group red on action 1 and group blue on action 2, respectively, have the highest GDC.

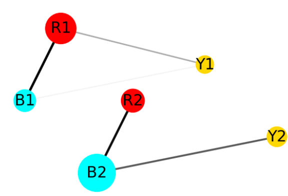

Fig. 16. Visualization of partitions' GDC with *decoupled* cost. The size of the nodes are again proportional to each partition's GDC. As the action dimension costs are decoupled, the graph consists of two subgraphs, one for each action dimension.

8 RELATED WORKS

Conventional (unstructured) network games and their equilibrium outcomes have been studied in a variety of application areas, including the private provision of public goods [3, 7, 18], security decision making in interconnected cyber-physical systems [15, 20], and shock propagation in financial markets [1]. A common line of research in this literature studied the effect of the network on the existence, uniqueness, and stability of the game equilibria (see [16] for a survey). In particular, unstructured network games with linear best-response functions have been studied in [5, 22, 25]. Bramoulle *et al.* [5] uncovered the importance of the lowest eigenvalue of the adjacency matrix of the network in determining the uniqueness and stability of the Nash equilibrium. Miura-Ko *et al.* [22] showed that if the adjacency matrix of the network is strictly diagonally dominant, then the Nash equilibrium is unique. Naghizadeh and Liu [25] identified necessary and sufficient conditions for the existence and uniqueness of Nash equilibria in network games with linear best-replies by establishing a connection to linear complementarity problems.

Unstructured network games with nonlinear best-response functions have been studied in [2, 3, 21, 24, 30, 38]. Allouch [3] introduces a sufficient condition for the uniqueness of Nash equilibrium called *network normality* which imposes lower and upper bounds on the derivative of Engel curves. Acemoglu *et al.* [2] consider a network game with idiosyncratic shocks and show that if the best response mapping is either a contraction with a Lipschitz constant smaller than one or a bounded non-expansive mapping, then the game has a unique Nash equilibrium. Zhou *et al.* [38] establish a connection between nonlinear complementary problems (NCP) and network games, and use existing results from the NCP literature to find sufficient conditions for the uniqueness of Nash equilibria. The works of [21, 24, 30] use the framework of variational inequalities to study network games with nonlinear best-replies. Naghizadeh and Liu [24] show that a sufficient condition for the uniqueness and stability of the Nash equilibrium can be determined by the lowest eigenvalue of matrices constructed based on the slope of the agents' interaction functions and the intensity of their interactions. Parise and Ozdaglar [30] identify an operator in the variational inequality problem which involves the derivative of the agent's cost with respect to its action, and show that various properties of this operator and its Jacobian will determine conditions for the existence, uniqueness, and stability of a Nash equilibrium.

An earlier version of this work appeared in [17]. Results of [17] have been generalized in the present paper in the following ways. First, [17] presented "structured conditions" on the uniqueness and stability of the Nash equilibrium and centrality measures for networks with a multipartite interaction graph only; the current paper generalizes the results to a much broader class of structured networks (which includes community-structured and multi-relational networks) with general interaction graphs, recovering the results of [17] as a special case. In addition, in the current paper we elaborate on the comparison between the structured conditions and the unstructured conditions on both the verification complexity and the condition strengths through both analytical results and numerical simulations. Finally, we outline the characteristics of network games with community structures for which the proposed structured conditions would be best suited.

The computational complexity of checking the uniqueness of Nash equilibria has been explored in a number of prior works; notably, [8, 14] show that verifying the uniqueness of Nash equilibria is in general an NP-hard problem. More recently, the complexity of checking the existence of a Nash equilibrium in (unstructured) games on networks has been explored in [29, 37]. Our work contributes to this literature by providing such results for structured network games. A computationally efficient method for verifying the existence and uniqueness of Nash equilibria can further serve as a precursor to algorithms for (efficient) computation of Nash equilibria, or before implementing iterative or distributed algorithms which require uniqueness of the Nash equilibrium for convergence to it (e.g., [32, 35]).

9 CONCLUSION

We introduced and studied a family of structured network games with non-linear best response functions. Prior works on network games have found sufficient conditions for uniqueness and stability of Nash equilibria which are mostly difficult to verify. In this work, we showed that the existence of structure in the network (e.g., in the form of communities, or when there are multi-relational dependencies between agents), helps us find alternatives for such conditions, which we refer to as "structured conditions" as opposed to the "unstructured conditions" in previous works. In particular, we show that the structured conditions for the uniqueness and stability of Nash equilibria are related to matrices which are possibly lower dimensional, with their dimensions depending on the number of partitions naturally arising in a network due to its structured nature. We also demonstrated both analytically and numerically that the structured conditions are sufficient conditions to the unstructured conditions, and that their verification is of much lower computational complexity. We used numerical experiment results to show that the sufficiency gap between the structured conditions and unstructured conditions is small in general and typically occurs in games with some specific characteristics. Moreover, we proposed a new notion of degree centrality to evaluate the influence of a partition in the network, and used it to identify additional conditions for uniqueness and stability.

ACKNOWLEDGMENTS

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A PROOF OF LEMMA 4.2

PROOF. We will prove that $||\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}||_2 \geq \beta^S_{ij}$. The proof establishing $||\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}||_2 \leq \alpha^S_i$ is similar.

From the definition of β^S_{ij} in Eqn (7) , we denote the action profile in Q that obtains the value of β^S_{ij} as $\hat{\boldsymbol{x}}(i,j)$, i.e.,

$$\beta^S_{ij} = \sup_{\boldsymbol{x}\in Q} ||\nabla_j F_i(\boldsymbol{x})||_2 = ||\nabla_j F_i(\hat{\boldsymbol{x}}(i,j))||_2,$$

where $\nabla_j F_i(\hat{\boldsymbol{x}}(i,j)) \in \mathbb{R}^{N_i\times N_j}$. We note here that since our action space Q is compact and ∇F is assumed to be continuous and differentiable on q, there exist maxima for $||\nabla_j F_i(\boldsymbol{x})||$ such that $\hat{\boldsymbol{x}} \in Q$. Therefore, the supremum equals the maximum and can be achieved. Since we will only focus on a specific pair of (i,j) values where $i \neq j$ in this part, we will simply use $\hat{\boldsymbol{x}}$ to denote this action profile for convenience.

Also, based on the definition of β^U_{kl}, we have

$$\beta^U_{kl} = \sup_{\boldsymbol{x}\in Q} |\nabla_l F_k(\boldsymbol{x})| \geq |\nabla_l F_k(\hat{\boldsymbol{x}})|,$$

where $\nabla_l F_k(\hat{\boldsymbol{x}}) \in \mathbb{R}$.

Next, we perform Singular Value Decomposition on $\nabla_j F_i(\hat{\boldsymbol{x}})$ such that $\nabla_j F_i(\hat{\boldsymbol{x}}) = USV^T$. From this, we can get the left singular vector $\boldsymbol{u} \in \mathbb{R}^{N_i}$ and the right singular vector $\boldsymbol{v} \in \mathbb{R}^{N_j}$ that correspond to the largest singular value β^S_{ij}, where $||\boldsymbol{u}|| = ||\boldsymbol{v}|| = 1$. Then

$$\boldsymbol{u}^T \nabla_j F_i(\hat{\boldsymbol{x}})\boldsymbol{v} = \beta^S_{ij} \cdot ||\boldsymbol{u}|| \cdot ||\boldsymbol{v}|| = \beta^S_{ij}.$$

To proceed with the proof, we introduce the following matrix A obtained from matrix $\nabla_j F_i(\hat{\boldsymbol{x}}(i,j))$ by replacing every element with its absolute value, formally,

$$A = (|\nabla_l F_k(\hat{\boldsymbol{x}})|)_{k,l:a_k\in\mathcal{P}_i,a_l\in\mathcal{P}_j}.$$

We also introduce vectors $\boldsymbol{u}^+$ and $\boldsymbol{v}^+$, where

$$\boldsymbol{u}^+ = (|u_k|)^{N_i}_{k=1},\ \boldsymbol{v}^+ = (|v_l|)^{N_j}_{l=1}.$$

Then, $||\boldsymbol{u}^+|| = ||\boldsymbol{v}^+|| = 1$, and it is easy to see that

$$(\boldsymbol{u}^+)^T A\boldsymbol{v}^+ \geq \boldsymbol{u}^T \nabla_j F_i(\hat{\boldsymbol{x}})\boldsymbol{v},$$

Next, consider the product $\boldsymbol{u}^T\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}\boldsymbol{v}$. We denote the index in $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}$ that corresponds to the element u_p and v_q as k_p and l_q respectively, so that,

$$\boldsymbol{u}^T\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}\boldsymbol{v} = \sum_{p=1}^{N_i}\sum_{q=1}^{N_j} \Upsilon^U_{k_p l_q} \cdot u_p \cdot v_q\ .$$

then we have

$$\begin{aligned}
||\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}|| &= ||-\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}|| \cdot ||\boldsymbol{u}^+|| \cdot ||\boldsymbol{v}^+|| \\
&\geq (\boldsymbol{u}^+)^T(-\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j})\boldsymbol{v}^+ \\
&= -\sum_{p=1}^{N_i}\sum_{q=1}^{N_j} \Upsilon^U_{k_p l_q} \cdot |u_p| \cdot |v_q| \\
&= \sum_{p=1}^{N_i}\sum_{q=1}^{N_j} \beta^U_{k_p l_q} \cdot |u_p| \cdot |v_q| \\
&\geq \sum_{p=1}^{N_i}\sum_{q=1}^{N_j} |\nabla_{l_q} F_{k_p}(\hat{\boldsymbol{x}})| \cdot |u_p| \cdot |v_q| \\
&= (\boldsymbol{u}^+)^T A \boldsymbol{v}^+ \\
&\geq \boldsymbol{u}^T \nabla_j F_i(\hat{\boldsymbol{x}})\boldsymbol{v} \\
&= \beta^S_{ij}.
\end{aligned}$$

This completes the proof. □

B PROOF OF THEOREM 4.3

PROOF. Given the community level partition $Q = \prod_{i=1}^{M} Q_i$, we denote $\nabla F_i(\boldsymbol{z}) = ((\nabla_i F_j(\boldsymbol{z}))_{j=1}^{M})^T \in \mathbb{R}^{N_i \times N}$.

We use the notation $L(\boldsymbol{x},\boldsymbol{y})$ to denote the line segment between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^N$. Formally,

$$L(\boldsymbol{x},\boldsymbol{y}) = \{\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} : 0 \leq \alpha \leq 1\}.$$

Under Assumption 1 in Section II, $F_i : Q \rightarrow \mathbb{R}^{N_i}, Q \subseteq \mathbb{R}^N$ is continuously differentiable on Q, and for $\forall \boldsymbol{x},\boldsymbol{y}$ in Q_i, $L(\boldsymbol{x},\boldsymbol{y}) \subseteq Q_i$. According to [4, Theorem 12.9] we know that for every vector $\boldsymbol{a}$ in $\mathbb{R}^{N_i}$, there is a point $\boldsymbol{z} \in L(\boldsymbol{x},\boldsymbol{y})$ such that:

$$\boldsymbol{a} \cdot (F_i(\boldsymbol{x}) - F_i(\boldsymbol{y})) = \boldsymbol{a} \cdot (\nabla F_i(\boldsymbol{z})(\boldsymbol{x}-\boldsymbol{y})). \tag{13}$$

Let $\boldsymbol{a}$ in equation (13) be $(\boldsymbol{x}_i - \boldsymbol{y}_i)^T$, and denote $\boldsymbol{l} = (l_j)_{j=1}^{M}$, where $l_j = ||\boldsymbol{x}_j - \boldsymbol{y}_j||_2, \forall j \in \mathbb{N}[1,M]$, then,

$$\begin{aligned}
& (\boldsymbol{x}_i - \boldsymbol{y}_i)^T(F_i(\boldsymbol{x}) - F_i(\boldsymbol{y})) \\
= \quad & (\boldsymbol{x}_i - \boldsymbol{y}_i)^T(\nabla F_i(\boldsymbol{z})(\boldsymbol{x}-\boldsymbol{y})) \\
= \quad & (\boldsymbol{x}_i - \boldsymbol{y}_i)^T[\sum_{j=1}^{M} \nabla_i F_j(\boldsymbol{z})(\boldsymbol{x}_j - \boldsymbol{y}_j)] \\
\geq \quad & (\boldsymbol{x}_i - \boldsymbol{y}_i)^T \nabla_i F_i(\boldsymbol{z})(\boldsymbol{x}_i - \boldsymbol{y}_i) \\
& -|\sum_{j\neq i}(\boldsymbol{x}_i - \boldsymbol{y}_i)^T \nabla_i F_j(\boldsymbol{z})(\boldsymbol{x}_j - \boldsymbol{y}_j)| \\
\geq \quad & \alpha_i^S (l_i)^2 - \sum_{j\neq i} \beta^S_{ij} \cdot l_i \cdot l_j \\
= \quad & l_i \cdot (\Upsilon^S \boldsymbol{l})_i
\end{aligned} \tag{14}$$

By [9, Theorem 3.3.4(b)], a real square matrix $M \in \mathbb{R}^{n\times n}$ is a P-matrix if it satisfies $\forall \boldsymbol{l} \in \mathbb{R}^n$

$$\max_{i\in\mathbb{N}[1,M]} l_i(M\boldsymbol{l})_i > 0$$

Denote $b = \max_{i\in\mathbb{N}[1,M]} \frac{l_i\cdot[\Upsilon^C \boldsymbol{l}]_i}{||\boldsymbol{l}||_2^2} > 0$. Then, we have

$$\max_{i\in\mathbb{N}[1,M]} (\boldsymbol{x}_i - \boldsymbol{y}_i)^T(F_i(\boldsymbol{x}) - F_i(\boldsymbol{y})) \geq \max_{i\in\mathbb{N}[1,M]} l_i \cdot [\Upsilon^C \boldsymbol{l}]_i \geq b \cdot ||\boldsymbol{x} - \boldsymbol{y}||_2^2$$

which, according to Definition 2.(b), shows that F satisfies uniform block P-condition. Therefore, by [30, Proposition 2 part (b)] and [11, Proposition 3.5.10 part (b)], the Nash equilibrium is unique.

Finally, we show that the two conditions

(1) Υ^S is a P-matrix,
(2) $\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}$ are P-matrices,

are sufficient for Υ^U to be a P-matrix. Using Lemma 4.2,

$$\begin{aligned}
& \Upsilon^S \in \mathbb{R}^{M\times M} \text{ is a P-matrix} \\
\Leftrightarrow\ & \max_j\ l_j(\Upsilon^S \boldsymbol{l})_j > 0, \forall \boldsymbol{l} \neq \boldsymbol{0}, \boldsymbol{l} \in \mathbb{R}^M, \text{let } i = \arg\max_j l_j(\Upsilon^C \boldsymbol{l})_j, \\
\Leftrightarrow\ & \boldsymbol{x}_i(\Upsilon^U_{\mathcal{P}_i,:} \cdot \boldsymbol{x})_{\mathcal{P}_i} \\
& \geq\ ||\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_i}|| \cdot ||\boldsymbol{x}_i||^2 - \sum_{j\neq i} ||\Upsilon^U_{\mathcal{P}_i,\mathcal{P}_j}|| \cdot ||\boldsymbol{x}_i|| \cdot ||\boldsymbol{x}_j|| \\
& \geq\ l_i \alpha^S_{ii} l_i - \sum_{j\neq i} l_i \beta^C_{ij} l_i \ =\ l_i(\Upsilon^C \boldsymbol{l})_i \ >\ 0, \\
\Rightarrow\ & \max_{k\in S_i}\ x_k(\Upsilon^U \boldsymbol{x})_k > 0, \\
\Rightarrow\ & \max_k\ x_k(\Upsilon^U \boldsymbol{x})_k > 0,\ \forall \boldsymbol{x} \neq \boldsymbol{0}, \boldsymbol{x} \in \mathbb{R}^N, \\
\Leftrightarrow\ & \Upsilon^U \in \mathbb{R}^{N\times N} \text{ is a P-matrix}
\end{aligned}$$

□

C PROOF OF COROLLARY 4.3.2

Proof. We can see that $I + \Gamma^S > \boldsymbol{0}$, iff $\lambda_1(\Gamma^S) > -1$, and from symmetry, $I + \Gamma^S$ is a P-matrix. The Υ^S matrix can be obtained by scaling the ith row of $I + \Gamma^S$ by α^S_i, which is a positive number. Therefore, the determinant of every principal minor of Υ^S has the same sign as the determinant of the corresponding principal minor of $I + \Gamma^S$, and thus Υ^S is also a P-matrix. By Theorem 4.3, the Nash equilibrium is unique and the sufficiency holds. □

D PROOF OF THEOREM 5.2

We prove for the $K = 1$ case and the proof generates to an arbitrary K.

Proof. When we consider borderline inactive and active groups, the set of all their members are a superset of the union of the borderline inactive and active set of agents.

We denote

$$Y = \{a_k | a_k \in \mathcal{P}_i, s.t. \mathcal{P}_i \in A_S(\boldsymbol{x}^*) \cup B_S(\boldsymbol{x}^*)\}$$

as the set of all agents that belong to communities in $A_S(\boldsymbol{x}^*) \cup B_S(\boldsymbol{x}^*)$, then we have $A(\boldsymbol{x}^*) \bigcup B(\boldsymbol{x}^*) \subseteq \mathcal{S}$. Similar to Theorem 5.1, we denote $\nabla_{\mathcal{S}} F_{\mathcal{S}}(\boldsymbol{x}^*)$ as a sub-matrix of $\nabla F(\boldsymbol{x}^*)$ whose columns and rows correspond to the agents in set $\mathcal{S}$.

Our proof proceeds by the following logic:

$$G^S(\boldsymbol{x}^*) \succ \mathbf{0} \Rightarrow \nabla_{\mathcal{S}} F_{\mathcal{S}}(\boldsymbol{x}^*) \succ \mathbf{0} \Rightarrow \nabla_{A,B} F_{A,B}(\boldsymbol{x}^*) \succ \mathbf{0}.$$

For the first part, we adopt techniques similar to those we used in Appendix B. For $\forall \boldsymbol{v} \in \mathbb{R}^{|S|}_{\geq 0}$, we denote $\boldsymbol{l} = (l_i)_{i=1}^{Z} \in \mathbb{R}^{Z}_{\geq 0}$, where $l_i = ||\boldsymbol{v}_i||_2, \forall i \in \mathbb{N}[1, Z]$. Then we have

$$\begin{aligned}
& \boldsymbol{v}^T \nabla_Y F_{\mathcal{S}}(\boldsymbol{x}^*) \boldsymbol{v} \\
= \quad & \sum_{i=1}^{Z} \sum_{j=1}^{Z} \boldsymbol{z}_j^T (\nabla_j F_i(\boldsymbol{x}^*)) \boldsymbol{v}_i \\
= \quad & \sum_{i=1}^{Z} \boldsymbol{v}_i^T (\nabla_i F_i(\boldsymbol{x}^*)) \boldsymbol{v}_i + \sum_{i=1}^{Z} \sum_{j=1, j\neq i}^{Z} \boldsymbol{v}_j^T (\nabla_j F_i(\boldsymbol{x}^*)) \boldsymbol{v}_i \\
\geq \quad & \sum_{i=1}^{Z} \theta_i^S \cdot ||\boldsymbol{v}_i||_2^2 - \sum_{i=1}^{Z} \sum_{j=1, j\neq i}^{Z} \delta_{ij}^S \cdot ||\boldsymbol{v}_i||_2 \cdot ||\boldsymbol{v}_j||_2 \\
= \quad & \sum_{i=1}^{Z} \theta_i^S \cdot {l_i}^2 - \sum_{i=1}^{Z} \sum_{j=1, j\neq i}^{Z} \delta_{ij}^S \cdot l_i \cdot l_j \\
= \quad & \boldsymbol{l}^T G^S(\boldsymbol{x}^*) \boldsymbol{l} \ > \ 0
\end{aligned}$$

which shows $\nabla_{\mathcal{S}} F_{\mathcal{S}}(\boldsymbol{x}^*) \succ \mathbf{0}$ and completes the proof of the first part.

For the second part, we know from the Sylvester's criterion that an $N \times N$ Hermitian matrix (for real valued matrices, it is symmetric) is positive definite if and only if every leading principal component of it (the top left $k \times k$ sub-matrices, for $k = 1, \dots, N$) has a positive determinant. Since $A(\boldsymbol{x}^*) \bigcup B(\boldsymbol{x}^*) \subseteq Y$, we know that $\nabla_{A,B} F_{A,B}(\boldsymbol{x}^*)$ is a leading principal minor of $\nabla_Y F_Y(\boldsymbol{x}^*)$ and thus $\nabla_{A,B} F_{A,B}(\boldsymbol{x}^*) \succ \mathbf{0}$, which completes the proof. □

E PROOF OF THEOREM 5.3

Proof. Since $\Upsilon^S \succ \mathbf{0}$, we know from Theorem 4.3 that the Nash equilibrium is unique. It remains to prove that this NE $\boldsymbol{x}^*$ is stable.

We denote $\Upsilon^S_{A,B} \in \mathbf{Z} \times \mathbf{Z}$ as the principal minor of Υ^S by picking out all the rows and columns corresponding to groups in $A_S(\boldsymbol{x}^*) \bigcup B_S(\boldsymbol{x}^*)$, and thus $\Upsilon^S_{A,B} \succ \mathbf{0}$.

Without loss of generality, suppose $\mathcal{P}_1, \mathcal{P}_2 \in A_S(\boldsymbol{x}^*) \bigcup B_S(\boldsymbol{x}^*)$, then we know their new indices in $G^S(\boldsymbol{x}^*)$ remain unchanged. We know from the definition that $\alpha_1^S \leq \theta_1^S$ and $\beta_{12}^S \geq \delta_{12}^S$ (similar comparison can generalize to all other elements), and thus for $\forall \boldsymbol{v} \in \mathbf{R}^Z$, we have

$$\begin{aligned}
& \boldsymbol{v}^T G^S(\boldsymbol{x}^*) \boldsymbol{v} \\
&= \sum_{i=1}^{Z} \sum_{j=1}^{Z} G_{ij}^S(\boldsymbol{x}^*) \cdot v_i \cdot v_j \\
&= \sum_{i=1}^{Z} \theta_i^S \cdot v_i^2 - \sum_{i=1}^{Z} \sum_{j=1, j\neq i}^{Z} \delta_{ij} \cdot v_i \cdot v_j \\
&\geq \sum_{i=1}^{Z} \alpha_i^S \cdot v_i^2 - \sum_{i=1}^{Z} \sum_{j=1, j\neq i}^{Z} \beta_{ij} \cdot v_i \cdot v_j \\
&= \boldsymbol{v}^T \Upsilon^S_{A,B} \boldsymbol{v} \ > \ 0,
\end{aligned}$$

which shows $G^S(\boldsymbol{x}^*) > \mathbf{0}$, and thus $\boldsymbol{x}^*$ is stable. □

F PROOF OF THEOREM 6.1

PROOF. First of all, when $\boldsymbol{e} = \mathbf{1}, \forall t > 0$, $D_{max}^{out}(\mathbf{1}) < 1$ implies

$$\alpha_i^S > \sum_{j \neq i} \beta_{ij}^S, \forall i = 1, \ldots, M,$$

and this shows that the Υ^S matrix is diagonally row dominant, and thus is a P-matrix [36] (Generating method 4.1, this matrix is a *positively stable* P-matrix).

For an arbitrary $\boldsymbol{e} > \mathbf{0}$, we define the following matrix E^S, where the ith column of E^S is equal to e_i times the ith column of Υ^S. Then since $e_i > 0$, every principal minor's determinant of E^S is equivalent to the corresponding principal minor's determinant of Υ^S and thus E^S is a P-matrix iff Υ^S is a P-matrix. Then, it's easy to see that $D_{max}^{out}(\boldsymbol{e}) < 1$ implies the diagonal row dominance of E^S, which is sufficient to show E^S is a P-matrix. This completes the proof of the out-degree part and the in degree part is similar. Moreover, the stability result follows from Theorem 5.3 and the in and out degree parts of this theorem.

□

G COMPLEXITY OF CONDITION VERIFICATION IN THEOREM 4.3

Size	Unstructured (sec)	Structured (sec)
10×10	1.08×10^{35}	$\mathbf{1.26} \times 10^{6}$
20×20	1.39×10^{127}	$\mathbf{1.69} \times 10^{10}$
50×50	4.90×10^{761}	$\mathbf{6.34} \times 10^{20}$

Table 3. Verification complexity in CPU times of conditions in Theorem 4.3 over number of agents.

H SUPPLEMENTARY REASONING IN SECTION 7

We begin with the claim that the upper bounds are large enough for the study of sufficiency gaps.

When the parameters that control the external impact are all chosen at their normalized upper bound, the expected sum of the off-diagonal elements will be twice as high as the mean value of the internal impact. For our choices of $S_{low}^{in}, S_{high}^{in}, S_{low}^{out}, S_{high}^{out}$, the Υ^U will not be a P-matrix with probability 1 and thus Υ^S will not be a P-matrix. For example, we assume that Υ^U is symmetric, and for a matrix with positive diagonal elements and non-positive off-diagonal elements, as long as the sum of the absolute values of the off-diagonal elements is greater than the sum of the diagonal elements (the sum of all its elements is negative), then we know $\mathbf{1}^T \Upsilon^U \mathbf{1} < 0$ and it is not positive definite and thus not a P-matrix. Increasing such upper bounds will only increase the upper right area above the curve formed by dark cells.

For the parameters that control the internal impact variances, the argument is similar, when the values of $c_{\mathcal{P}_i}^{high} - c_{\mathcal{P}_i}^{low}$ are $c^{high} - c^{low}$ close to 1, then the agent with the lowest internal impact will have an internal impact close to 0 and Υ^U will not be a P-matrix. Figures 19 and 20 shows the comparison after we double such normalized upper bounds, and support our argument above.

Next, we elaborate more on the "critical values" in Figures 11 and 12.

If agents have homogeneous internal impact, i.e., α_k^U are constant, then when $\mathbf{E}[\sum_{k\neq l}\beta_{kl}^U]$ is about 10% higher than $\sum_{k=1}^N \alpha_k^U$, the games are most likely to have sufficiency gaps. We discussed above that when Υ^U is symmetric, and $\sum_{k\neq l}\beta_{kl}^U > \sum_{k=1}^N \alpha_k^U$, then the unstructured condition is not satisfied, which is sufficient to conclude that the structured condition does not hold either. But when it's $\mathbf{E}[\sum_{k\neq l}\beta_{kl}^U] > \sum_{k=1}^N \alpha_k^U$, there are realizations (sample games under our choices of $S_{low}^{in}, S_{high}^{in}, S_{low}^{out}, S_{high}^{out}$) that have $\sum_{k\neq l}\beta_{kl}^U < \sum_{k=1}^N \alpha_k^U$ and thus are possible candidates for causing sufficiency gaps between the two sets of conditions. However, if $\mathbf{E}[\sum_{k\neq l}\beta_{kl}^U]$ is sufficiently larger (30% or more) $\sum_{k=1}^N \alpha_k^U$, all sample games will have neither set of conditions satisfied, which corresponds to the top right corner of the figures.

I SUFFICIENCY GAP HEATMAPS FOR MULTIPARTITE GRAPHS

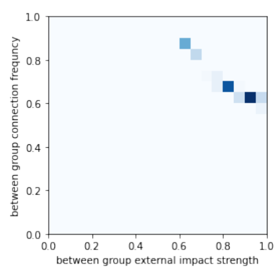

Fig. 17. Sufficiency gap frequency over the between group external impact, frequency 0.83%.

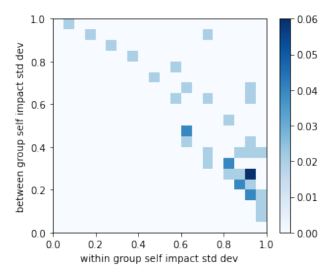

Fig. 18. Sufficiency gap frequency over the internal impact variances, with weak external impact, frequency 0.18%.

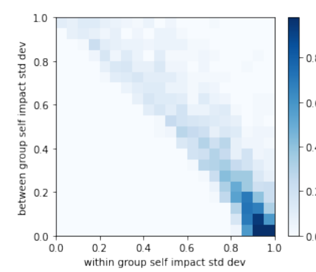

Fig. 19. Sufficiency gap frequency over the internal impact variances, with medium external impact, frequency 6.31%.

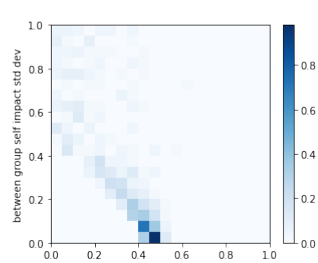

Fig. 20. Sufficiency gap frequency over the internal impact variances, with medium external impact, frequency 2.09%, doubled upper bounds.

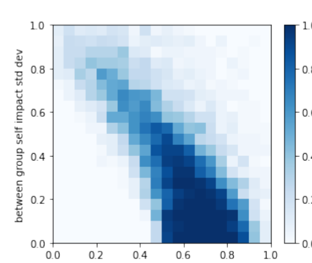

Fig. 21. Sufficiency gap frequency over the internal impact variances, with strong external impact, frequency 26.93%.