A NOTE ON DYNAMIC STABILITY

By Kenneth J. Arrow and Maurice McManus[1]

1. INTRODUCTION

In a recent issue of *Econometrica*, A. C. Enthoven and K. J. Arrow [1] were interested in the following problem. Let $\emptyset(A)$ be the largest of the real parts of the characteristic roots of the real, square matrix A. Then A is said to be *stable* if and only if $\emptyset(A) < 0$. Now if A is stable, in what circumstances is DA stable, where D is diagonal? Their theorem is that if A has nonnegative off-diagonal elements, this being a generalized version of the type examined by L. A. Metzler [3], then DA is stable if and only if the diagonal elements of D are all positive. The purpose of this note is to examine the same problem for certain other classes of matrices.[2] The importance of the results for economic dynamics is discussed at the end of the paper.

2. THREE THEOREMS ON STABILITY

Definition 1: A real, square matrix M is called negative (resp., positive) quasi-definite if and only if $h'Mh$[3] is negative (resp., positive) for every real, non-null column vector h.[4]

Remarks: Definiteness is the special case of quasi-definiteness where M is symmetric; if M is negative (resp., positive) quasi-definite, it is nonsingular and its inverse is correspondingly quasi-definite, for

$$k'M^{-1}k = k'M^{-1\prime} M'M^{-1}k = h'M'h = h'Mh,$$

where $h = M^{-1}k$; the inverse of a nonsingular symmetric matrix S is also symmetric, since $S^{-1\prime} = S^{-1\prime} SS^{-1} = S'^{-1} S' S^{-1} = S^{-1}$.

Lemma 1: *Any real symmetric matrix S can be transformed by a real orthogonal matrix P into a diagonal matrix $D = P'SP$. The diagonal elements of D are the characteristic roots of S; they are real, and they are all positive (resp., negative) if and only if S is positive (resp., negative) definite.*

Lemma 2: *Negative quasi-definite matrices are stable.*

These lemmas are classical results. A proof of the second is given by Samuelson [7], p. 438. It also follows directly from the remarks and equation (2) below with $S = I$.

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[2] For a different approach to closely related questions see M. McManus [2].

[3] A prime after a vector or matrix denotes its transpose.

[4] See P. A. Samuelson [7, p. 140] for an equivalent definition.

DEFINITION 2: A matrix M is said to be S-stable if, for symmetric matrices S, SM is stable if and only if S is positive definite.

DEFINITION 3: A matrix M is said to be D-stable if, for diagonal matrices D, DM is stable if and only if $d_{ii} > 0$ for every i, where the d_{ii} denote the diagonal elements of D.

Remarks: S-stability implies D-stability, for diagonal matrices are symmetric and are positive definite if and only if all their diagonal elements are positive; S-stability or D-stability implies stability.

We are now in a position to state and prove

THEOREM 1: *Negative quasi-definite matrices are S-stable.*
The proof turns upon

LEMMA 3: *If A is negative quasi-definite and S is nonsingular and symmetric then no characteristic root of SA has a zero real part.*

PROOF OF LEMMA 3: If λ is any characteristic root of SA, there exists a non-null vector x such that $SAx = \lambda x$. Premultiply both sides by $\bar{x}'S^{-1}$, where $\bar{x}$ denotes the complex conjugate of x:

$$(1) \qquad \bar{x}' \, S^{-1} \, SAx = \bar{x}' \, Ax = \lambda \bar{x}' \, S^{-1} \, x.$$

Write $x = y + iz$, $\lambda = \alpha + i\beta$, where y, z, α and β are all real. Equating real parts of (1),

$$(2) \qquad y' \, Ay + z' \, Az = \alpha(y' \, S^{-1} \, y + z' \, S^{-1} \, z),$$

for, by the remarks, the real coefficient of β is zero. By hypothesis the left hand side of (2) is negative and so $\alpha \neq 0$.

PROOF OF THEOREM 1: Let A be negative quasi-definite. If S is singular it is not definite and SA is singular; hence det.$(SA - \lambda I) = 0$ for $\lambda = 0$. Therefore, the theorem is satisfied if det.$S = 0$. Now assume that S is not singular. If it is positive definite it follows from (2) that $\alpha < 0$. This proves the sufficiency part of the theorem, though both parts are simultaneously deduced in what follows. Define

$$A(t) = (1 - t)A - tI, \ 0 \leqq t \leqq 1.$$

$A(t)$ is negative quasi-definite for all t since $h'A(t)h = (1-t)h'Ah - th'h < 0$ because A is negative quasi-definite. Hence, by Lemma 3, $\emptyset[SA(t)] \neq 0$ for all t. By continuity, therefore, $\emptyset[SA(t)]$ is either positive for all t or else negative for all t. In particular,

$$\emptyset[SA(0)] < 0 \text{ if and only if } \emptyset[SA(1)] < 0.$$

But $A(0) = A$ and $A(1) = -I$. Hence the characteristic roots of $SA(1)$ are

those of $-S$. By Lemma 1 they are all real and are all negative if and only if S is positive definite. Thus $\emptyset(SA)$, i.e., $\emptyset[SA(0)]$, is negative if and only if S is positive definite.

The sufficiency part of Theorem 1 was stated by Samuelson [7], p. 275.[5] M. Sono [8] proved that if S is symmetric and if both A and SA are negative quasi-definite then S is positive definite. This is a special case of the necessity part of Theorem 1 above.[6]

Since S-stability implies D-stability, we immediately get the

COROLLARY: *Negative quasi-definite matrices are D-stable.*

This corollary is the counterpart of the Enthoven-Arrow Theorem, proved for the negative quasi-definite type of stable matrix instead of the "generalized Metzlerian" type, i.e., stable matrices with all off-diagonal elements nonnegative.[7]

A useful extension of the Enthoven-Arrow Theorem and the Corollary to Theorem 1 is provided by

THEOREM 2: *Let E be a nonsingular diagonal matrix. C is D-stable if and only if $A = ECE^{-1}$ is D-stable.*

The proof makes use of

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[5] Actually he claims that, ın terms of our notation, if A is negative quasi-definite then it is sufficient for the stability of, say, HA that H^{-1} (and so, by the remarks, H) be only positive *quasi*-definite. This, however, goes too far, as the following counter-example shows.

$$\text{Let } H = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}. \text{ Then } HA = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix}$$

is unstable, although H is positive quasi-definite and A negative quasi-definite.

[6] An example suffices to show that Sono's Theorem does not apply to as wide a class of matrices as does Theorem 1.

$$\text{Let } S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \text{ so that } SA = \begin{bmatrix} -2 & 0 \\ -3 & -1 \end{bmatrix}.$$

Now S is positive definite and A is negative quasi-definite, but SA, though stable, is not quasi-definite.

[7] The question may be raised as to whether or not stable generalized Metzlerian matrices are S-stable. The following two examples show that they are not.

$$(1) \; S_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \; M_1 = \begin{bmatrix} -1 & 3 \\ .2 & -1 \end{bmatrix};$$

S_1 is positive definite, M_1 is stable Metzlerian, but S_1M_1 is not stable.

$$(2) \; S_2 = \begin{bmatrix} -1 & -.5 \\ -.5 & -.27 \end{bmatrix}, \; M_2 = \begin{bmatrix} -1 & 4 \\ 1 & -5 \end{bmatrix};$$

S_2 is symmetric but not positive definite, M_2 is a stable Metzlerian matrix, yet S_2M_2 is stable.

LEMMA 4: *Any matrix B is stable if and only if GBG^{-1} is stable for non-singular G.*

PROOF: B and GBG^{-1} are *similar* matrices, having the same characteristic roots.

PROOF OF THEOREM 2: Since diagonal matrices are commutative in multiplication, $DA = DECE^{-1} = E(DC)E^{-1}$. By Lemma 4, therefore, DC is stable if and only if DA is. Hence if the positivity of the d_{ii} is decisive for the stability of DA it must be for the stability of DC too, and vice versa.

COROLLARY: *Let C be such that, for some diagonal matrix E, $A = ECE^{-1}$ is either negative (quasi-) definite or has all off-diagonal elements nonnegative and is stable. Then DC is stable if and only if every d_{ii} is positive.*

PROOF: By the corollary of Theorem 1 and the Enthoven-Arrow Theorem—as the case may be—the matrix A of the corollary has the property stipulated for A in Theorem 2.

Theorem 2 is important because if A is generalized Metzlerian, or symmetric, or quasi-definite, $E^{-1}AE = C$ need be none of these.[8]

The following theorem describes how certain product matrices using diagonal matrices as premultipliers are related to other matrix products using symmetric matrices as premultipliers.

THEOREM 3. *Given C and a certain symmetric matrix $\bar{S}$, let P be such that $P'\bar{S}P = \bar{D}$ where $\bar{D}$ is diagonal, and define $\bar{C} = P'CP$. Assume that $\bar{D}\bar{C}$ is stable if and only if all the diagonal elements of $\bar{D}$ are positive. For example, for some diagonal E, $A = EP'CPE^{-1}$ may be either negative quasi-definite or generalized Metzlerian and stable. Then $\bar{S}C$ is stable if and only if $\bar{S}$ is positive definite.*

PROOF: By Lemma 1, P is orthogonal and so $\bar{D}\bar{C} = P'\bar{S}PP'CP = P^{-1}(\bar{S}C)P$. The roots of $\bar{D}\bar{C}$ and $\bar{S}C$ are the same by dint of Lemma 4, and by hypothesis $\bar{D}\bar{C}$ is stable if and only if all the $\bar{d}_{ii}$ are positive. By Lemma 1, however, $\bar{D}$ is positive definite if and only if $\bar{S}$ is. Hence $\bar{S}C$ is stable if and only if $\bar{S}$ is positive definite.

In Theorem 3, $\bar{C}$ does not have to be D-stable, for the question of the sta-

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[8] We do not know how wide the class of D-stable matrices is. Certainly not all stable matrices are D-stable. For example,

$\begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$ is stable, yet $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$ is unstable and $\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$ is stable.

It may be that all D-stable matrices are of the form $E^{-1}AE$, where E is diagonal and A is either generalized Metzlerian and stable or else negative quasi-definite. If this is so, Theorem 2 and its corollary are equivalent.

bility of $D\bar{C}$ is confined to the case where D is the given matrix $\bar{D}$. In the same way C is not necessarily S-stable, whether or not $\bar{C}$ is D-stable. Theorem 3 adds something new to the results discussed previously, for they do not cover those cases where either $\bar{C}$ is not D-stable or where $\bar{S} \neq \bar{D}$ and either A is generalized Metzlerian and stable or, for some $E \neq I$, A is negative quasi-definite.[9]

3. ECONOMIC APPLICATION

The Enthoven-Arrow dynamic general equilibrium system can be written as

$$(3) \qquad \dot{p} = Kx; \quad x = Q(p - p^o) + B(p' - p^o); \quad p' = p + \eta\dot{p}.$$

Here p is the column vector of all the other prices in terms of the numéraire, the dot represents differentiation with respect to time, the superscript "o" denotes "equilibrium" levels and p' is the vector of expected future relative prices, while x is the vector of the corresponding excess-demands. K is the matrix of speeds of adjustment of prices to excess-demands, $Q = [dx/dp']^o$, $B = [dx/dp'']^o$ and η is the matrix of "extrapolative price-expectational coefficients." In the original model, K, B and η are all diagonal matrices. Eliminating x and p', (3) yields the set of linear differential equations,

$$(4) \qquad \dot{p} = (I - KB\eta)^{-1} K(Q + B) (p - p^o).$$

Enthoven and Arrow [1] have D and A refer to $(I - KB\eta)^{-1}$ and $K(Q + B)$, respectively. It will now be more convenient, however, to rewrite (4) as

$$(5) \qquad \dot{p} = (K^{-1} - B\eta)^{-1} (Q + B) (p - p^o),$$

and let D (or S) refer to $(K^{-1} - B\eta)^{-1}$ and A (or C) to $(Q + B)$. The Samuelsonian expectationless model is a special case with $\eta = B = 0$. Denote the elements of matrices by the corresponding small letters with appropriate row and column suffixes. Then the corollary to Theorem 2 tells us that if $E(Q + B)E^{-1}$ is either negative (quasi-) definite or generalized Metzlerian and stable for some E (including, of course, $E = I$), the expectationless system is stable if and only if all the k_{ii} are positive–the usual behavior. Moreover, for the same $(Q + B)$, the introduction of the expectational coefficients does not upset stability if and only if $1/k_{ii} > b_{ii} \eta_{ii}$ for all i.

Since the same conclusions do not hold for arbitrary stable $(Q + B)$,[10] it is fortunate that some of the types that are covered by the theorems are of particular interest to the economist. If income effects are either absent or else symmetrical, $Q + B$ is symmetric. Alternatively, if the system contains

[9] If $\bar{C}$ itself is negative quasi-definite then C is too (see Samuelson [7, pp. 140–1]), so Theorem 1 applies.

[10] See above, p. 451, fn. 8.

no gross complements[11] then $Q + B$ has nonnegative off-diagonal elements. In any actual economy, however, we must be prepared to find substantial, asymmetrical income effects and a goodly sprinkling of gross complementarity. It is desirable, therefore, to try to find other classes of matrices about which useful statements about stability can be made. An important stride in this direction was made by Samuelson [7] in his consideration of non-symmetric, negative quasi-definite matrices, and the E transformation provides a way of generating further types out of the two or three basic ones. If, for example, A is generalized Metzlerian and the elements of E are of mixed signs, then $E^{-1}AE$ is of the type which has been examined in detail by M. Morishima [4].

An additional interest in Theorems 1 and 3 lies in the fact that S need not be diagonal, only symmetric. This has an application in terms of the above model. The speed of price adjustment in any one market may depend upon the excess-demands in other markets as well as in its own, as has been argued cogently by D. Patinkin [6], p. 157; it is quite feasible that some excess demands partly depend upon expected prices in other markets; it may well be that expected prices are influenced by what is happening to several different current prices. In general, any combination of these makes $(I - KB\eta)^{-1}$ or $(K^{-1} - B\eta)^{-1}$ non-diagonal.

Samuelson [7, pp. 274–5] subjects the original variables p and x to a contragredient transformation $\bar{p} = c'p$; $x = c\bar{x}$.

The system (5) becomes

$$(6) \qquad \dot{p} = c'(K^{-1} - B\eta)^{-1} cc^{-1}(Q + B)c'^{-1} (\bar{p} - \bar{p}^o).$$

The transformation of Lemma 1 as used in Theorem 3 is a special case where c is the orthogonal P. Using this interpretation, SC and DC would refer to the *same* economic system, but one in which two different ways of describing and measuring the given number of commodities are considered.

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[11] The adjective "gross" indicates that income effects are being taken into account. The name is due to J. Mosak [5].

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