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Strategic Interaction and Networks†

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Geography and social links shape economic interactions. In industries, schools, and markets, the entire network determines outcomes. This paper analyzes a large class of games and obtains a striking result. Equilibria depend on a single network measure: the lowest eigenvalue. This paper is the first to uncover the importance of the lowest eigenvalue to economic and social outcomes. It captures how much the network amplifies agents' actions. The paper combines new tools—potential games, optimization, and spectral graph theory—to solve for all Nash and stable equilibria and applies the results to R&D, crime, and the econometrics of peer effects. (*JEL* C72, D83, D85, H41, K42, O33, Z13)

In many economic settings, who interacts with whom matters. When deciding whether to adopt a new crop, farmers rely on information from neighbors and friends. Adolescents' consumption of tobacco and alcohol is affected by their friends' consumption. Firms' investments depend on the actions of other firms producing substitute and complementary goods. All these interactions can be represented formally through a network, interaction matrix, or graph. Because linked agents interact with other linked agents, the outcomes ultimately depend on the entire network structure. The major interest, and challenge, is uncovering how this network structure shapes outcomes. Networks are complex objects and answering this question is generally difficult even in otherwise simple settings. We study the large set of games where agents have linear best replies.[1] We bring to bear a new combination of tools

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[1] This class includes investment, crime, belief formation, public good provision, social interaction, and oligopoly. See, for example, Angeletos and Pavan (2007); Ballester, Calvó-Armengol, and Zenou (2006); Bergemann and Morris (2009); Calvó-Armengol, Patacchini, and Zenou (2009); Bénabou (2009); Bramoullé and Kranton (2007); İlkiliç (2008); Glaeser and Scheinkman (2003); Goyal and Moraga-González (2001); and Vives (1999).

and characterize the Nash and stable equilibria for any graph structure and for any impact of players' actions on others' payoffs.

We have a striking result. Equilibrium outcomes depend on a single number: the lowest eigenvalue of the network matrix. The lowest eigenvalue, we find, captures the cumulative effects of agents' actions on others. The highest eigenvalue, which is positive, is important in games of pure complements; the lowest eigenvalue is negative, and we show it is key whenever substitutes are present. For example, when one person provides more public goods, their neighbors free-ride and provide less, which forces their neighbors to provide more, and so on. There is a direct effect of one agent on another, and there is the cumulative effect of all the ups and downs along network paths. When the lowest eigenvalue is large in magnitude, these ups and downs can lead in several directions, and there can be multiple equilibria. When it is small in magnitude, the network dampens the ups and downs in a single direction, and there is a unique equilibrium.

This paper is the first to show the importance of the lowest eigenvalue to social and economic outcomes,[2] and we relate it to patterns of network links. The lowest eigenvalue depends on the two-sidedness of the network. A network most amplifies substitutabilities when agents are divided into two distinct sets, and they have links to agents in the other set, but not their own. Actions then rebound from one side to another.

We study all possible strategic interactions, with a focus on games of strategic substitutes. This focus helps us see the force of network interactions. With strategic substitutes, the indirect effects which work their way through a network involve complexities not present with strategic complements. Agents' actions tend to go in opposite directions along network paths. Multiple equilibria cannot generally be neatly ordered to conduct comparative statics. We tackle these problems by using a new combination of tools—optimization, potential games, and spectral graph theory—to study Nash and stable equilibria and how they relate to network structure. The analysis directly applies to a variety of strategic substitutes settings and more generally applies to any mixture of substitutes and complements.

We consider three specific applications in the paper. First, we study experimentation and R&D as a local public good, as occurs in many industries. When firms learn the results of others' research, there is the potential for free-riding. In the oil and natural resource industry, for example, Hendricks and Porter (1996) find that information spillovers reduce firms' exploratory drilling. In agriculture, Foster and Rosenzweig (1995) find evidence of free-riding when new seeds were introduced during the Green Revolution in India. The information spillovers depend on proximity and social and communication links. The question is then: how does the overall network structure determine the pattern and level of experimentation? We study two scenarios, reflecting the empirical literature on R&D spillovers. In one situation, agents have many links to firms in their local region. In another situation, agents have both local links and links to distant firms. According to our results, the network with more global links amplifies strategic substitutabilities (i.e., the lowest eigenvalue is large in magnitude) and all stable equilibria involve small numbers of

[2] For a compendium of network measures, see Wasserman and Faust (1994) and Scott (2004).

experimenters. At the other extreme, "industrial clusters," where all firms engage in some innovation, are outcomes when links are largely local.

Second, we study crime patterns. In a stylized model, individual costs of engaging in crime are lower when friends also engage in crime. However, higher crime levels overall can crowd out the gain to any individual. We find that an exogenous increase in the cost of crime (e.g., a harsher punishment) has two countervailing effects. It has the classic effect of reducing the baseline level of crime. But it enhances the value of social networks. When the network has agents sufficiently connected to each other but isolated from other agents, gangs can emerge. Overall crime level drops, but gang members have higher levels of criminal activity. The key is again how the relevant network amplifies substitutabilities.

Third, we indicate how our model can advance the econometrics of peer and spatial effects. The underlying mathematical structure of our games matches the linear regressions that measure, for example, how students' academic achievement relate to their friends or how jurisdictions set tax rates. We show that the commonly used empirical techniques can misestimate these dependencies, as multiple equilibria are not considered. Our results show how to construct more robust empirical models and connect the estimation to the econometrics of games with multiple equilibria.

This paper makes at least four general contributions.

First, this paper advances the analysis of n-person simultaneous-move games with continuous action spaces. Any n-player game can be written as a network game. But traditionally researchers restrict attention to (i) the case where all agents interact equally with all other agents, (ii) symmetric equilibria, and (iii) interior equilibria. This paper gives the tools to study any possible interaction structure, the entire action space, and the full set of equilibrium outcomes. New conclusions are possible, and we develop general intuitions about strategic play. In particular, we show that the usual focus on symmetric equilibria is with significant loss of generality. Even when agents have the same payoffs and symmetric network positions, the symmetric equilibrium may be unstable and coexist with stable asymmetric equilibria.

Second, our analysis alters our understanding of games played on networks. Previous analyses have taken two directions. One tack has been to reduce analytical complexity by presuming incomplete information about network links (Galeotti et al. 2010). The present paper takes a more traditional approach and solves for Nash equilibria given a network structure and parameters of the game. Previous studies in this vein have focused on the extremes of the direct-effects parameter. Ballester, Calvó-Armengol, and Zenou (2006) study small direct effects: there is unique, interior equilibrium and agents' actions are proportional to their Bonacich centralities in the original network. Bramoullé and Kranton (2007) study large direct effects: there are generally multiple equilibria, and stable equilibria involve all-or-nothing play corresponding to maximal independent sets of the graph. Outcomes in these two cases are radically different, and the literature did not provide a general, unifying analysis.

This paper builds a common framework and provides this unifying analysis. By solving for equilibria for any level of direct effects, we can determine the precise applicability of previous results. They turn out to be rather limited. Interiority and uniqueness are special properties and depend on small direct effects. In general, equilibrium actions are not proportional to Bonacich centralities in the original network and the results of Ballester, Calvó-Armengol, and Zenou (2006) do not apply. Similarly, Bramoullé and

Kranton's (2007) independent set result is a special property of large direct effects. All-or-nothing equilibria often do not exist in the general case. Nonetheless, we can rescue some insights from previous studies. Bonacich centrality still plays a role, but on subgraphs of the network. Stable outcomes always involve some agents playing nothing when the network sufficiently amplifies strategic play.

Third, the paper engages the new intersection of economics and computer science. We characterize equilibrium outcomes as solutions to well-defined quadratic programming problems. We exploit Monderer and Shapley's (1996) theory of potential games that computer scientists have used to study congestion and other issues (e.g., Roughgarden and Tardos 2002; Chien and Sinclair 2011). In economics, the theory of potential games has been applied to finite games, including a few network settings.[3] We use potential functions and exploit these crossovers to illuminate strategic interaction in networks.[4]

Finally, beyond the applications summarized above, we provide a new set of results that can be implemented in empirical and experimental work. Researchers conducting laboratory experiments (as in, e.g., Kearns, Suri, and Montfort 2006) can manipulate network structures to vary the lowest eigenvalue and study how outcomes relate to the structure. Given data on information networks (as in Conley and Udry 2010), the results indicate specific, testable predictions about innovation patterns. The lowest eigenvalue should be measured and studied alongside classic network statistics such as diameter and clustering.

The paper proceeds as follows. The next section presents the basic model and characterizes Nash equilibria. Section II studies unique and stable equilibria and finds the lowest eigenvalue is the key measure of the network. Section III relates the lowest eigenvalue to network link patterns. We conduct comparative statics in Section IV. Section V applies our results to R&D spillovers, crime, and empirical estimation of spatial and peer effects. Section VI shows how the results extend to a general model. Section VII concludes.

I. Games on a Network

A. *Basic Model*

There are n agents. Each agent i simultaneously chooses an action $x_i \geq 0$. Let $\mathbf{x} = (x_1, \dots, x_n)$ denote the actions for all agents. Let $\mathbf{x}_{-i}$ denote the actions of all agents other than i.

Agent i's action determines his payoffs and the payoffs of agents to whom he is linked. For any two agents i and j, $g_{ij} = g_{ji} = 1$ indicates that i and j are linked; otherwise $g_{ij} = g_{ji} = 0$. An agent j who is linked to i is called i's *neighbor*. A payoff parameter $\delta \geq 0$ measures how much i and j affect each others' payoffs, given they are linked. The collection of links form a *graph*, *network*, or *interaction matrix*,

[3] Blume (1993) studies finite games when players are situated on a lattice. Neyman (1997) studies potential games with continuous actions and finds conditions for a unique correlated equilibrium. Young (1998) considers coordination games with two actions and general networks. Bramoullé (2007) studies anti-coordination games with two actions and general networks.

[4] We also construct an (necessarily exponential time) algorithm to identify all equilibria and show that solving for certain focal equilibria is directly related to a well-known NP hard problem.

represented by an $n \times n$ matrix $\mathbf{G}$. Agent i's payoffs depend on his own action x_i, the actions of others $\mathbf{x}_{-i}$, the graph $\mathbf{G}$, and the payoff parameter δ. We denote agent i's payoffs as $U_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G})$.

Each agent chooses his action to maximize his individual payoffs. Given others' actions, let $x_i = f_i(\mathbf{x}_{-i}; \delta, \mathbf{G})$ denote agent i's *best reply*, the action that maximizes U_i given $\mathbf{x}_{-i}$, δ, and $\mathbf{G}$.

Our objective is to solve for and characterize outcomes as they depend on the payoff parameter δ and on the network $\mathbf{G}$. For a given $\mathbf{G}$, we will say a property holds for *almost any* δ if it holds for every δ except possibly a finite number of values. We consider Nash equilibrium outcomes $\mathbf{x}^*$, which are the profiles that satisfy the best replies of all agents; that is, $x_i^* = f_i(\mathbf{x}_{-i}^*; \delta, \mathbf{G})$ for all agents $i = 1, \ldots, n$.[5] We further consider the subset of Nash equilibria that are *asymptotically stable*; that is, starting with an equilibrium vector $\mathbf{x}^*$ and changing play by a little bit, we determine whether the system of best replies ultimately leads back to $\mathbf{x}^*$. (We define formally *asymptotically stable* and the best reply dynamics below.)

B. *Games with Linear Best Replies*

We study canonical economics settings, where the payoff functions yield best reply functions $f_i(\mathbf{x}_{-i}; \delta, \mathbf{G})$ that are linear in $\mathbf{x}_{-i}$. Many games in economics and the network literature fall in this class. For example, Bramoullé and Kranton (2007) study the private provision of public goods, as in Bergstrom, Blume, and Varian (1986). Here, the public goods are local; individual i benefits from the goods provided by his neighbors, scaled by δ which gives the substitutability between i's and his neighbors' goods. Agent i's payoff is

$$\hat{U}_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = b_i(x_i + \delta \sum_j g_{ij} x_j) - \kappa_i x_i,$$

where for all i $b_i(\cdot)$ is differentiable, strictly increasing, and concave in x_i, $\kappa_i > 0$ is i's marginal cost, and $b_i'(0) > \kappa_i > b_i'(+\infty)$ for all i. Maximizing $\hat{U}_i$ with respect to x_i gives the best reply:

$$f_i(\mathbf{x}_{-i}, \delta, \mathbf{G}) = \overline{x}_i - \delta \sum_j g_{ij} x_j \quad \text{if} \quad \delta \sum_j g_{ij} x_j < \overline{x}_i \tag{1}$$

and

$$f_i = 0 \quad \text{if} \quad \delta \sum_j g_{ij} x_j \geq \overline{x}_i,$$

where $\overline{x}_i$ equates i's marginal benefits and marginal costs; $b_i'(\overline{x}_i) \equiv \kappa_i$. Agent i aims to reach at least $\overline{x}_i$, through the combination of his own and his neighbors' actions. If the weighted sum of neighbors' actions is less than $\overline{x}_i$, agent i makes up the difference. If the weighted sum is higher than $\overline{x}_i$, agent i makes no contribution.

[5] Throughout the paper we study pure-strategy equilibria. For many games in our class, $U_i(x_i, \mathbf{x}_{-i}; \boldsymbol{\delta}, \mathbf{G})$ is strictly concave in x_i, and hence there are no mixed strategy Nash equilibria.

A second game represents negative externalities—an agent is hurt by his neighbors' actions (Ballester, Calvó-Armengol, and Zenou 2006). Individual i's payoff is quadratic:

$$\tilde{U}_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = \overline{x}_i x_i - \frac{1}{2}x_i^2 - \delta \sum_j g_{ij} x_i x_j .$$

Maximizing with respect to x_i, this game also yields best reply (1).

A third game is Cournot competition, where $g_{ij} = g_{ji} = 1$ indicates firm i's and firm j's products are substitutes and δ is the degree of substitutability. Let firm i earn price $p_i = a - s \cdot (x_i + 2\delta \sum_j g_{ij} x_j)$, where $a > 0$ and $s > 0$.[6] This demand, along with constant marginal cost d, yields the following payoffs for firm i :

$$\Pi_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = x_i\left(a - s \cdot \left(x_i + 2\delta \sum_j g_{ij} x_j\right)\right) - dx_i.$$

These profits are quadratic; maximizing Π_i with respect to x_i yields the linear best reply.

We consider all games with best reply (1).[7] If $\overline{x}_i = \overline{x}$ for all players, we can normalize $\overline{x} \equiv 1$ and write this best reply as $f_i(\mathbf{x}) \in [0, 1]$:

$$f_i(\mathbf{x}, \delta, \mathbf{G}) = \max\left(0, 1 - \delta \sum_{j=1}^{n} g_{ij} x_j\right). \tag{2}$$

Note that $\mathbf{f} = (f_1, \ldots, f_n)$ is continuous from $[0, 1]^n$ to itself and hence existence of a Nash equilibrium is guaranteed by Brouwer's fixed point theorem.

This best reply is the base case we study, with $g_{ij} \in \{0, 1\}$ and $\delta \in [0, 1]$. All players' actions are strategic substitutes. Section VI shows how results extend to heterogeneous $\overline{x}_i$, heterogeneous payoff impacts δ_i, weighted graphs $g_{ij} = g_{ji} \in \mathbb{R}$ and $\delta \in \mathbb{R}$ which capture any mix of substitutes and complements, and a class of directed graphs.

C. *Basics of Nash Equilibria*

Our first proposition gives the full set of Nash equilibria for any δ and $\mathbf{G}$. We distinguish between those agents who are *active*—agents whose actions are strictly positive—and agents who are inactive. Let $\mathbf{x}_A$ denote the vector of actions of active agents; let $\mathbf{G}_A$ be the links connecting active agents, and let $\mathbf{G}_{N-A,A}$ be the links connecting inactive agents to active agents. Let $\mathbf{I}$ denote the identity matrix. Parsing the agents in this way and using matrix notation to express the best-replies in (1) leads directly to the following result.

[6] Linear inverse demand functions derive from consumers with strictly concave quadratic utility functions over substitute products (Vives 1999).

[7] Any game with payoff function $U_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = V_i(x_i - \overline{x}_i + \delta \sum_j g_{ij} x_j) + W_i(\mathbf{x}_{-i})$ where V_i is increasing on $]-\infty, 0]$ and decreasing on $[0, +\infty[$ has best reply (1).

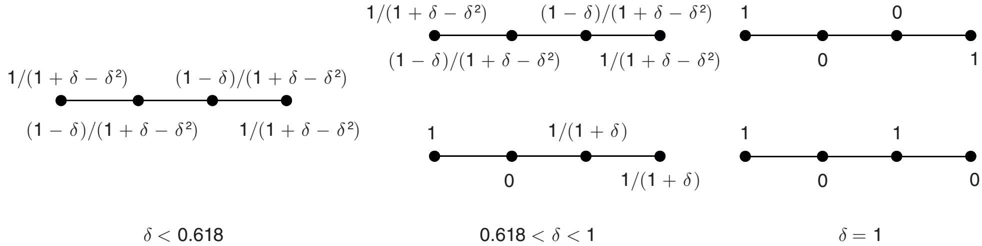

Figure 1. Equilibria in a Line Network

PROPOSITION 1: *Actions* $\mathbf{x}$ *with active agents A is a Nash equilibrium if and only if*

$$(i) \qquad (\mathbf{I} + \delta \mathbf{G}_A)\mathbf{x}_A = \mathbf{1}$$

$$(ii) \qquad \delta \mathbf{G}_{N-A,A}\mathbf{x}_A \geq \mathbf{1}.$$

PROOF:

The Appendix provides full proofs of all results.

With Proposition 1, we can find all the equilibria for any graph $\mathbf{G}$ and any δ.[8] Thus, this analysis supersedes previous studies of particular games in our class, which analyzed either low or high values of δ. The following examples illustrate and show the equilibrium set for two canonical network structures: lines and stars.

Example 1 (Equilibria for a Line): Consider a line with four agents. Figure 1 shows the unique equilibrium for $\delta < (\sqrt{5} - 1)/2 \approx 0.618$: all agents are active, the middle agents play $(1 - \delta)/(1 + \delta - \delta^2)$ and the agents at the end play $1/(1 + \delta - \delta^2)$. For $(\sqrt{5} - 1)/2 < \delta < 1$, this interior equilibrium coexists with two equilibria in which one end agent plays 1, his neighbor is inactive and the two remaining agents play $1/(1 + \delta)$. For $\delta = 1$, all-or-nothing equilibria arise, involving maximal independent sets of agents (active agents are not linked to any other active agents). This example shows the limits of Bramoullé and Kranton's (2007) result for $\delta = 1$. All-or-nothing play is not pertinent for $\delta < 1$.

Example 2 (Equilibria for a Star): Consider a star with n players, illustrated in Figure 2 for $n = 4$. For $\delta < 1/(n - 1)$ there is a unique equilibrium where all players are active: the center plays $[1 - (n - 1)\delta]/[1 - (n - 1)\delta^2]$ and each peripheral agent plays $(1 - \delta)/[1 - (n - 1)\delta^2]$. For $1/(n - 1) \leq \delta < 1$, there is a unique equilibrium, and it involves both active and inactive agents. The center plays 0 and peripheral agents play 1. For $\delta = 1$, there are two equilibria: (i) the center plays 0 and peripheral agents play 1, and (ii) the center plays 1 and the peripheral agents play 0. Stars show the limits of Ballester, Calvó-Armengol, and Zenou's (2006) δ-small results. They find there is a unique equilibrium and all agents are active for

[8] In the Appendix, we show that the number of equilibria is finite for any $\mathbf{G}$ and almost any δ, and we propose an algorithm to compute all equilibria. Describing all equilibria is computationally demanding, since the number of equilibria can increase exponentially with n (Bramoullé and Kranton 2007).

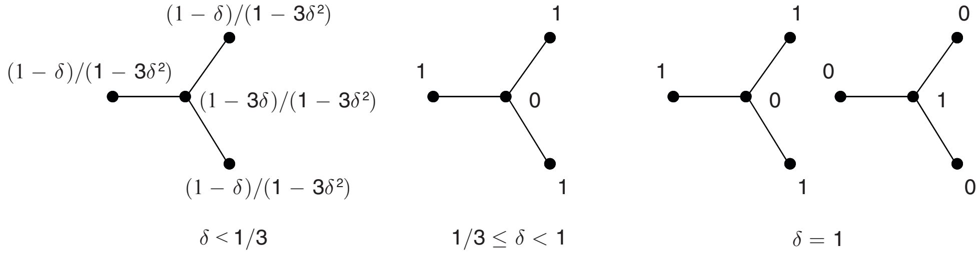

FIGURE 2. EQUILIBRIA IN A STAR NETWORK

$\delta < 1/(1 + \lambda_{\max}(\mathbf{C} - \mathbf{G}))$, where $\mathbf{C}$ is the complete graph (all agents are linked to all other agents) and $\lambda_{\max}$ denotes the highest eigenvalue; agents' actions are proportional to their Bonacich centralities in $\mathbf{C} - \mathbf{G}$.[9] For the star, their condition is $\delta < 1/(n-1)$. Our results demonstrate a star has a unique equilibrium for all $\delta < 1$. For $1/(n-1) \leq \delta < 1$, some agents are inactive, and actions are not proportional to Bonacich centrality in $\mathbf{C} - \mathbf{G}$.

II. The Shape of Nash Equilibria and Stable Equilibria

We now address the major challenge: how networks generally shape equilibrium outcomes.

We find the equilibrium set depends on the lowest eigenvalue of network graphs. To develop an intuition, note that for any $\mathbf{G}$, $\lambda_{\min}(\mathbf{G}) < 0$. The larger it is in magnitude (the more negative it is), the more agents' actions rebound in the network. The parameter δ tell us how much one agent's actions substitute for his neighbors' actions. The lowest eigenvalue tell us how much the network amplifies these substitutabilities. Thus, as we will show, stable equilibria occur when the lowest eigenvalue is sufficiently small in magnitude relative to δ, so that the network mutes the overall ups and downs of agents' actions.

Our main results are: (i) If $|\lambda_{\min}(\mathbf{G})| < 1/\delta$, there is a unique Nash equilibrium. (ii) An equilibrium with active agents A (and other agents are strictly inactive) is stable if and only if $|\lambda_{\min}(\mathbf{G}_A)| < 1/\delta$. (iii) If $|\lambda_{\min}(\mathbf{G})| > 1/\delta$, there are possibly multiple equilibria and all stable equilibria involve inactive agents.[10]

We obtain these results by constructing a proxy maximization problem whose conditions yield all the Nash equilibria. We use the theory of potential games (Monderer and Shapley 1996). Consider a game with payoffs $V_i(x_i, \mathbf{x}_{-i})$. A function $\varphi(x_i, \mathbf{x}_{-i})$ is a *potential function* for this game if and only if for all x_i and x_i' and all $\mathbf{x}_{-i}$

$$\varphi(x_i, \mathbf{x}_{-i}) - \varphi(x_i', \mathbf{x}_{-i}) = V_i(x_i, \mathbf{x}_{-i}) - V_i(x_i', \mathbf{x}_{-i}) \quad \text{for all } i.$$

[9] Given a graph $\mathbf{M}$ and a scalar q such that $\mathbf{I} - q\mathbf{M}$ is invertible, the vector of Bonacich centralities $\mathbf{z}(q, \mathbf{M})$ is defined by $\mathbf{z}(q, \mathbf{M}) = (\mathbf{I} - q\mathbf{M})^{-1}\mathbf{M}\mathbf{1}$ (Bonacich 1987). The centrality measure for agent i, $z_i(q, \mathbf{M})$, can be understood as a weighted sum of the paths in $\mathbf{M}$ that start with i.

[10] When $|\lambda_{\min}(\mathbf{G})| = 1/\delta$, multiple equilibria are possible and the equilibrium set is strongly structured. Equilibria form a convex set, and all equilibria yield the same aggregate actions. In addition, one equilibrium is the limit of the unique equilibrium for $|\lambda_{\min}(\mathbf{G})| < 1/\delta$ as δ tends to $1/|\lambda_{\min}(\mathbf{G})|$ from below.

A potential function, if one exists, mirrors each agent's payoff function. For any i, changing actions from x_i to x_i', changes the value of the potential by exactly the same amount as i's payoffs.

If one game in our class is a potential game, we can use its potential to analyze the equilibria for all games in our class, since the games all have the same equilibria. Monderer and Shapley (1996) show for (continuous, twice-differentiable) payoffs V_i, there exists a potential function if and only if $\partial^2 V_i(\mathbf{x})/\partial x_i \partial x_j = \partial^2 V_j(\mathbf{x})/\partial x_j \partial x_i$ for all $i \neq j$. The game with quadratic payoffs $\tilde{U}_i$ has a potential function, since $\partial^2 \tilde{U}_i(\mathbf{x})/\partial x_i \partial x_j = \partial^2 \tilde{U}_j(\mathbf{x})/\partial x_j \partial x_i = -\delta g_{ij}$, and it is

$$\varphi(\mathbf{x}; \delta, \mathbf{G}) = \mathbf{x}^T \mathbf{1} - \frac{1}{2} \mathbf{x}^T (\mathbf{I} + \delta \mathbf{G}) \mathbf{x}.$$

Notice the matrix $(\mathbf{I} + \delta\mathbf{G})$ in the second term. It relates to Proposition 1's matrix formulation of best replies, and it will be central to our analysis.

Consider now maximizing the potential function:

$$\text{(P)} \qquad \max_{\mathbf{x}} \ \varphi(\mathbf{x}; \delta, \mathbf{G}) \quad \text{s.t.} \quad \forall i, x_i \geq 0.$$

Actions $\mathbf{x}$ are then a Nash equilibrium if and only if $\mathbf{x}$ satisfies the Kuhn-Tucker conditions of problem (P). The first order conditions for (P) mimic agents' individual best responses. Each agent chooses his action in the game *as if* he wants to maximize the potential, given other agents' actions.[11] We have:

LEMMA 1: *The set of Nash equilibria for a given* $\mathbf{G}$ *and* δ *corresponds to the set of maxima and saddle points of the potential function* $\varphi(\mathbf{x}; \delta, \mathbf{G})$ *on* $\mathbb{R}^n_+$.

A. *Unique Equilibrium and the Lowest Eigenvalue of the Network*

Since each agent acts as if he is maximizing the potential, there is a unique equilibrium when there is only one solution to the first order conditions of problem (P). With $x_i \geq 0$ for all i, a sufficient condition for a unique solution is a strictly concave potential function. Examining the matrix of second derivatives, the potential function is strictly concave if and only if $(\mathbf{I} + \delta\mathbf{G})$ is positive definite, which occurs if and only if $|\lambda_{\min}(\mathbf{G})| < 1/\delta$. Thus, we find:

PROPOSITION 2: *If* $|\lambda_{\min}(\mathbf{G})| < 1/\delta$, *there is a unique Nash equilibrium.*

For multiple equilibria to emerge, agents' actions must be sufficiently amplified, in possibly different directions. When $|\lambda_{\min}|$ is low, this does not occur.

Proposition 2 provides the best known condition for unique equilibria in these games.[12] It is stronger than Ballester, Calvó-Armengol, and Zenou's (2006) result

[11] Individual i's payoff is strictly concave in x_i, and $\varphi(x_i, \mathbf{x}_{-i})$ is strictly concave in each x_i, so for any $\mathbf{x}_{-i}$ a single x_i satisfies agent i's best response and the ith Kuhn-Tucker condition.

[12] The result is a network case of Neyman's (1997) finding that games with a strictly concave potential and compact convex action spaces have a unique correlated equilibrium. Since these games also satisfy the condition

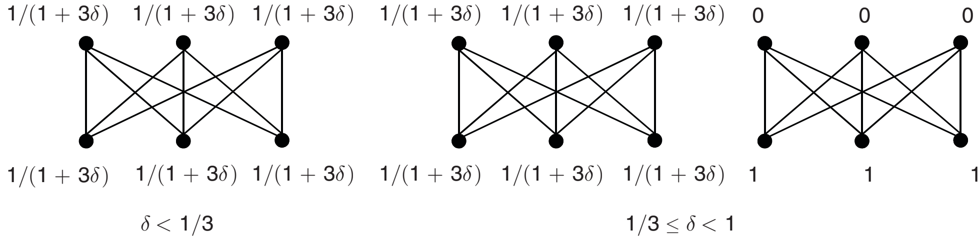

FIGURE 3. UNIQUE VERSUS MULTIPLE EQUILIBRIA IN REGULAR GRAPHS

that if $\lambda_{\max}(\mathbf{C} - \mathbf{G}) < 1/\delta - 1$ there is a unique equilibrium and all agents are active; our condition is also stronger than a classic uniqueness condition, contracting best replies, which here is $\lambda_{\max}(\mathbf{G}) < 1/\delta$.[13]

The condition $|\lambda_{\min}(\mathbf{G})| < 1/\delta$ is both necessary and sufficient for uniqueness in an important class of graphs—*regular graphs*, where all agents have the same number of neighbors. We establish this result by using the following fact.

PROPOSITION 3: *For any δ and any graph $\mathbf{G}$, if $|\lambda_{\min}(\mathbf{G})| > 1/\delta$, there exists at least one Nash equilibrium with inactive agents.*

Proposition 3 follows from the observation that the vector that globally maximizes a nonconcave quadratic function is non-interior. Thus for $|\lambda_{\min}(\mathbf{G})| > 1/\delta$, while the potential is not concave and multiple equilibria are possible, at least one equilibrium involves inactive agents.

Hence, for any graph $\mathbf{G}$ where there is an interior Nash equilibrium $\mathbf{x}^*(\delta)$ for any $\delta \in [0,1]$, this is the unique equilibrium for $|\lambda_{\min}(\mathbf{G})| < 1/\delta$. For $|\lambda_{\min}(\mathbf{G})| > 1/\delta$ there is at least one other equilibrium. In a regular graph where each agent has k neighbors, $x_i = 1/(1 + \delta k)$ for all i is a Nash equilibrium for any $\delta \in [0,1]$. Therefore,

COROLLARY 1: *For any δ and any graph $\mathbf{G}$ where each agent has k neighbors, there is a unique Nash equilibrium if and only if $|\lambda_{\min}(\mathbf{G})| < 1/\delta$.*

The following example illustrates.

Example 3 (Unique versus Multiple Equilibria in Regular Graphs): For any regular graph where each agent has k neighbors, $x_i = 1/(1 + k\delta)$ is a Nash equilibrium for any δ. Consider the network in Figure 3, which is called a complete bipartite graph since the agents are divided into two sets, with links to all agents in the other set. Here $k = 3$, and $\lambda_{\min}(\mathbf{G}) = -3$. For any δ, $x_i = 1/(1 + 3\delta)$ is a Nash equilibrium. By Proposition 2, for $\delta < 1/3$, this must be the unique equilibrium.

of "diagonal strict concavity," the result can also be viewed as a network application of Rosen's (1965) uniqueness result.

[13] For any $\mathbf{G}$, $|\lambda_{\min}(\mathbf{G})| \leq \lambda_{\max}(\mathbf{G})$, and this inequality is strict when no component of $\mathbf{G}$ is bipartite. See Theorem 0.13 in Cvetković, Doob, and Sachs (1979). For any $\mathbf{G}$, $|\lambda_{\min}(\mathbf{G})| \leq 1 + \lambda_{\max}(\mathbf{C} - \mathbf{G})$, and this inequality is strict, for example, when any agent has no more than $n/2$ neighbors. The Appendix provides details.

By Proposition 3, for $1/3 < \delta$ there is at least one equilibrium with inactive agents, so there are additional equilibria. Figure 3 shows the three equilibria when $1/3 < \delta$: in one equilibrium each agent plays $1/(1 + 3\delta)$ and in two equilibria agents in one set play 1, and agents in the other set play 0.

B. *Stable Equilibria*

This section refines the set of Nash equilibria by invoking the notion of stability. We consider the equilibria for which small changes in players' actions do not lead to widely divergent play. We find that the lowest eigenvalue again provides the key: An equilibrium $\mathbf{x}$ with active agents A is stable if and only if $|\lambda_{\min}(\mathbf{G}_A)|$ is sufficiently small. When $|\lambda_{\min}(\mathbf{G})| < 1/\delta$, the unique equilibrium is stable. When $|\lambda_{\min}(\mathbf{G})| > 1/\delta$, among the possibly multiple equilibria, all stable equilibria involve inactive agents.

We study a classic notion of stability which is a continuous version of textbook Nash tâtonnement.[14] Starting with a Nash equilibrium, change agents' play by a little bit. Agents then best respond to the new vector. The question is whether these best replies lead back to the original vector or not. Consider the following system of differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}; \delta, \mathbf{G}) - x_1 \\ &\vdots \\ \dot{x}_n &= f_n(\mathbf{x}; \delta, \mathbf{G}) - x_n, \end{aligned}$$

where $f_i(\mathbf{x}; \delta, \mathbf{G})$ is agent i's best response (2). By construction, a vector $\mathbf{x}$ is a stationary state of this system if and only if it is a Nash equilibrium. We say a Nash equilibrium $\mathbf{x}$ is *asymptotically stable* when this system converges back to $\mathbf{x}$ following any small enough perturbation. Formally, following Weibull (1995, Definition 6.5, p. 243), introduce $B(\mathbf{x}, \varepsilon) = \{\mathbf{y} \in \mathbb{R}^n_+ : ||\mathbf{y} - \mathbf{x}|| < \varepsilon\}$ and $\boldsymbol{\xi}(t, \mathbf{y})$, the value at time t of the unique solution to the system of differential equations that starts at $\mathbf{y}$. An equilibrium $\mathbf{x}$ is *asymptotically stable* if $\forall \varepsilon > 0, \exists \eta > 0 : \forall \mathbf{y} \in B(\mathbf{x}, \eta), \forall t \geq 0$, $\boldsymbol{\xi}(t, \mathbf{y}) \in B(\mathbf{x}, \varepsilon)$ and if $\exists \varepsilon > 0 : \forall \mathbf{y} \in B(\mathbf{x}, \varepsilon)$, $\lim_{t \to \infty} \boldsymbol{\xi}(t, \mathbf{y}) = \mathbf{x}$.

C. *Lowest Eigenvalue and Stable Equilibria*

To identify asymptotically stable equilibria, we again use the potential function. Starting from an equilibrium which is a *strict maximum* of the potential,[15] and modifying actions slightly, individual adjustments will lead back to the equilibrium. Locally there is no way to increase the potential, which reflects all agents' individual best replies. In contrast, if the equilibrium is not a strict maximum, but, say, a saddle

[14] See, e.g., Fisher (1961). We also characterize equilibria that are stable for discrete Nash tâtonnement, as in Bramoullé and Kranton (2007). The equilibria which are stable given discrete tâtonnement (almost always) implies stability given continuous best reply dynamics. But the reverse is not true. (Proofs are available upon request.)

[15] We say that a maximum is strict when it is unique in some open neighborhood.

point, then modifying agents' actions slightly, there will be a direction in which the potential is increasing and individual reactions lead away from the equilibrium. Hence the equilibrium is not stable. This relationship between the potential function and stability is formalized in the following lemma.

LEMMA 2: *The set of stable equilibria for a given* $\mathbf{G}$ *and* δ *is equal to the strict maxima of the potential function* $\varphi(\mathbf{x}; \delta, \mathbf{G})$ *on* $\mathbb{R}^n_+$. *A stable equilibrium exists for any* $\mathbf{G}$ *and almost any* δ.

While the set of Nash equilibria is equivalent to all the saddle points and maxima of the potential function on $\mathbb{R}^n_+$, stability eliminates the maxima that are not strict as well as the saddle points.

Whether an equilibrium $\mathbf{x}$ is stable, then, depends on the curvature of the potential around $\mathbf{x}_A$ which is captured in $|\lambda_{\min}(\mathbf{G}_A)|$. To see this, consider a Nash equilibrium $\mathbf{x}$ with active agents A, such that all inactive agents are strictly inactive.[16] Now perturb agents' play slightly by adding a vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ such that actions meet the constraint $\mathbf{x} + \boldsymbol{\varepsilon} \geq \mathbf{0}$. When $|\lambda_{\min}(\mathbf{G}_A)|$ is small relative to δ, the potential function is strictly concave in $\mathbf{x}_A$. The best replies converge back to $\mathbf{x}$; the agents are linked so that the network absorbs the impact of changes in play. When $|\lambda_{\min}(\mathbf{G}_A)|$ is large relative to δ, the potential function is not concave in $\mathbf{x}_A$. Some small perturbation $\boldsymbol{\varepsilon}$ can lead to large changes in best replies, the change reverberates through the network, and the equilibrium is not stable.

We obtain a necessary and sufficient condition for stability:

PROPOSITION 4: *Consider a graph* $\mathbf{G}$ *and a Nash equilibrium* $\mathbf{x}$ *with active agents* A *and strictly inactive agents.* $\mathbf{x}$ *is stable if and only if* $|\lambda_{\min}(\mathbf{G}_A)| < 1/\delta$.

The following example illustrates the lowest eigenvalue and equilibrium stability. We compare two networks, which are identical in the number of agents, the number of links, and the number of links per agent. The only difference is how these links are placed, so one network has a smaller lowest eigenvalue than the other. (We show why in the following section.)

Example 4 (Stable Equilibria and the Lowest Eigenvalue): Contrast the graphs in the left and right panels of Figure 4. In both graphs, each agent has three neighbors, so $x_i = 1/(1 + 3\delta)$ for all i is a Nash equilibrium in both graphs for any $\delta \in [0, 1]$. For the complete bipartite graph, on the left, $\lambda_{\min}(\mathbf{G}) = -3$; this equilibrium is stable for $\delta < 1/3$. For the "prism" graph, on the right, $\lambda_{\min}(\mathbf{G}) = -2$. The interior equilibrium is stable for $\delta < 1/2$. For intuition, compare agents' best replies after perturbing agents' play in the bipartite graph to perturbing play in the prism graph. Suppose agent 1 plays $1/(1 + 3\delta) + \varepsilon$. In the bipartite graph, all agents connected to 1 (agents 4, 5, and 6) are all on the other side of the network and will adjust their

[16] Agent i is strictly inactive if $\delta \sum_j g_{ij} x_j > 1$. A strictly inactive agent remains inactive following a small perturbation of others' play. We show in the Appendix that for any graph $\mathbf{G}$ and for almost any value of δ, inactive agents in all equilibria are all strictly inactive. We provide in the Appendix a characterization of stable equilibria when inactive agents are not necessarily strictly inactive.

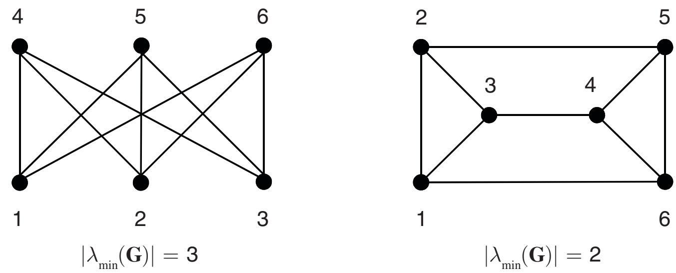

FIGURE 4. LOWEST EIGENVALUE AND STABLE EQUILIBRIA

play by the full amount. In the prism graph, agents 2 and 3 who are linked to 1 are also linked to each other and "share" the adjustment. The interior equilibrium is stable for a higher δ than in the complete bipartite graph.

There is another way to see why the lowest eigenvalue is primary for stability. To check for the stability of an equilibrium $\mathbf{x}$, we ultimately check the perturbation ε that gives the largest change in agents' best replies. This *maximal perturbation* derives from the eigenvectors associated with the lowest eigenvalue; i.e., $\mathbf{G}_A \varepsilon = \lambda_{\min} \varepsilon$. Since actions are strategic substitutes, it is the negative eigenvalues that matter, and the lowest eigenvalue gives the largest magnitude of the change. If this perturbation does not lead the system away from the equilibrium, then no other perturbation will.

We can easily show that unique equilibria are always stable; the unique equilibrium corresponds to a strict global maximum of the potential. This holds, in particular, for $|\lambda_{\min}(\mathbf{G})| < 1/\delta$,

COROLLARY 2: *A unique Nash equilibrium is stable.*

For $|\lambda_{\min}(\mathbf{G})| > 1/\delta$, the stability condition selects among the possibly multiple equilibria. We find that all stable equilibria are asymmetric and involve inactive agents. Thus, the common tendency in the literature to restrict attention to symmetric and interior Nash equilibria is ill-advised; these equilibria are not necessarily robust to small perturbations in agents' play.

PROPOSITION 5: *For $|\lambda_{\min}(\mathbf{G})| > 1/\delta$, all stable equilibria involve at least one inactive agent.*

We can further characterize stable equilibria: stable equilibria are minimal in terms of sets of active agents. For each stable equilibrium with active agents A, there is no other equilibrium with active agents A' where A' is a subset of A. That is, stable equilibria involve the greatest concentration of active agents (in set terms).

PROPOSITION 6: *Consider a stable equilibrium $\mathbf{x}$ with active agents A. There is no other equilibrium $\mathbf{x}'$ with active agents A' such that $A' \subset A$.*

One intuition again involves the lowest eigenvalue and the notion of *maximal perturbation*. For any two equilibria $\mathbf{x}$ and $\mathbf{x}'$ with sets of active agents such that $A' \subset A$, the maximal perturbation for A' has lower impact. We can see this relationship by comparing the lowest eigenvalues: for $A' \subset A$, $|\lambda_{\min}(\mathbf{G}_{A'})| \leq |\lambda_{\min}(\mathbf{G}_{A})|$.[17]

III. The Lowest Eigenvalue and Network Structure

Here we study how the lowest eigenvalue of a graph relates to network structure. There is only a small amount of recent mathematics on the relationship between the link structure and the lowest eigenvalue,[18] and it is not a common measure in other fields such as sociology or physics.

Overall, $|\lambda_{\min}(\mathbf{G})|$ tends to be larger when the network is more "two-sided," so that agents can be subdivided into two sets with few links within the sets but many links between them. We can gain much intuition from looking at the networks in Figure 4. For the bipartite graph $\lambda_{\min}(\mathbf{G}) = -3$, and for the prism graph $\lambda_{\min}(\mathbf{G}) = -2$. Each has six agents and nine links, and each is a regular graph. But in the bipartite graph there are two distinct sets of agents, with links between the sets but not within each set.

To see the importance of links between versus within two sets, consider a main characterization of the lowest eigenvalue:

$$\lambda_{\min}(\mathbf{G}) = \min_{||\boldsymbol{\varepsilon}||=1} \boldsymbol{\varepsilon}^t \mathbf{G} \boldsymbol{\varepsilon},$$

where the minimum is taken over all vectors with norm equal to one and is reached for an eigenvector associated with the lowest eigenvalue, i.e., a maximal perturbation. With this characterization, for any $\mathbf{G}$ we can divide agents in two sets depending on whether their actions are positive or negative in $\boldsymbol{\varepsilon}$. Define sets R and S as $R = \{i : \varepsilon_i \geq 0\}$ and $S = \{i : \varepsilon_i < 0\}$. The previous equality is then:

$$\lambda_{\min}(\mathbf{G}) = \sum_{i,j \in R} \varepsilon_i \varepsilon_j g_{ij} + \sum_{i,j \in S} \varepsilon_i \varepsilon_j g_{ij} + 2 \sum_{i \in R, j \in S} \varepsilon_i \varepsilon_j g_{ij}.$$

The first term captures the links within R, and the second term captures the links within S. Both are positive, since ε_i and ε_j have the same sign when i and j are in the same set. The third term captures the links between R and S, and it is negative since ε_i and ε_j then have opposite signs. This characterization shows why the eigenvalue captures substitutabilities. When the third term is larger in magnitude, the eigenvalue is more negative, and the more agents' actions rebound in opposite directions.

This formula gives a way to transform any network $\mathbf{G}$ into a network $\mathbf{G}'$ such that $|\lambda_{\min}(\mathbf{G}')| \geq |\lambda_{\min}(\mathbf{G})|$. We cut any number of links within R or S, and/or add any number of links between R or S. Precisely,

PROPOSITION 7: *For any graph* $\mathbf{G}$, *let* $\boldsymbol{\varepsilon}$ *be an eigenvector for* $\lambda_{\min}(\mathbf{G})$ *and let* $R = \{i : \varepsilon_i \geq 0\}$ *and* $S = \{i : \varepsilon_i < 0\}$. *Construct* $\mathbf{G}'$ *by removing links within R and/or S and/or by adding links between R and S, in any way. Then,* $|\lambda_{\min}(\mathbf{G}')| \geq |\lambda_{\min}(\mathbf{G})|$.

[17] This follows from the interlacing eigenvalue theorem, e.g., Horn and Johnson (1985, p.185).

[18] See Desai and Rao (1994); Alon and Sudakov (2000); Ye, Fan, and Liang (2009); Bell et al. (2008a, 2008b); and Trevisan (2009).

For example, in Figure 4 in the prism graph $R = \{1, 4, 5\}$, and $S = \{2, 3, 6\}$. Cutting the link between agents 2 and 3 yields a network with lowest eigenvalue approximately equal to -2.22. In the complete bipartite graph, set R is one side of the network and S is the other side; there are no further links to cut or add. It is the network with the largest possible $|\lambda_{\min}(\mathbf{G})|$ for six agents.

With these observations we can answer several questions concerning the structure of graphs and the lowest eigenvalue.

First, given a population of n agents and an unlimited number of links, what is the graph $\mathbf{G}$ with the largest $|\lambda_{\min}(\mathbf{G})|$ of all possible graphs? It is a complete bipartite graph with sides as equal as possible.[19] The main intuition lies in Proposition 7 which easily extends to prove this result. This problem was solved about thirty years ago, see e.g., Constantine (1985).

Second, given n agents and a limited number of links, what is the graph $\mathbf{G}$ with the largest $|\lambda_{\min}(\mathbf{G})|$? This is a more difficult question and only has been recently answered. Bell et al. (2008a, 2008b) show that dividing agents into two sets is critical for the graphs that solve this problem, and the structure of links between or within has a specific nested split shape.

Third, can we precisely relate the lowest eigenvalue to the "two-sidedness," or "bipartiteness," of the graph? We can do so by deriving a tight lower bound on $|\lambda_{\min}(\mathbf{G})|$. For any $\mathbf{G}$, consider any partition of the population in two sets P and Q and denote by $|\mathbf{G}_P|$ the number of links within P and by $|\mathbf{G}_{PQ}|$ the number of links between P and Q. The lower bound is proportional to the difference between the numbers of links between and within:[20]

$$|\lambda_{\min}(\mathbf{G})| \;\geq\; \frac{2}{n}(|\mathbf{G}_{PQ}| \;-\; |\mathbf{G}_P| \;-\; |\mathbf{G}_Q|).$$

Finally, how does the lowest eigenvalue change when links are added to a network? Adding links in a systematic way, can we see a pattern? The answer is yes and no. For the empty graph $\lambda_{\min}(\mathbf{G} = \mathbf{0}) = 0$, and for a complete graph $|\lambda_{\min}(\mathbf{C})| = 1$; starting from an empty graph and adding links, $|\lambda_{\min}|$ must increase and then eventually decrease. But there is no precise, general relationship between the number of links and the lowest eigenvalue. As we see in the following example, both the ascent and the descent can be non-monotonic.

Example 5 (Lowest Eigenvalue and Increasing Social and Geographic Interactions): Consider adding links to *local interaction graphs*, which are commonly used in economics to represent geography and social space. Agents are arranged in a circle and each has k neighbors, with $k/2$ to the left and $k/2$ to the right, for even $k = \{2, 4, 6, \ldots, n-1\}$ neighbors. Such networks provide a systematic way to capture increasing social and geographic interactions (see, for example, Ellison 1993, Rauch and Casella 2003, and Dixit 2003). Consider how the lowest eigenvalue changes as k increases. For $n = 19$ agents, Figure 5 shows the local interaction graph for $k = 4$, and Figure 6 plots $|\lambda_{\min}(\mathbf{L}_k)|$ for local interaction graphs

[19] We thank Noga Alon for initially providing us an intuitive and useful proof of this result.

[20] Equality holds for complete bipartite graphs with same size sides.

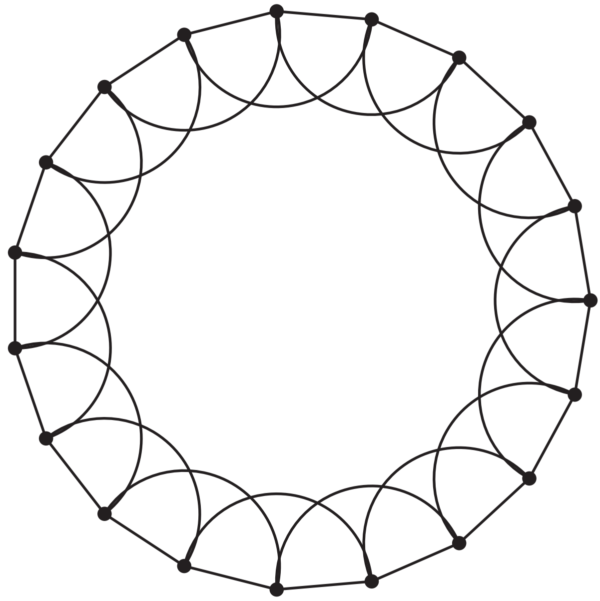

Figure 5. Local Interaction Network with $n = 19$ and $k = 4$

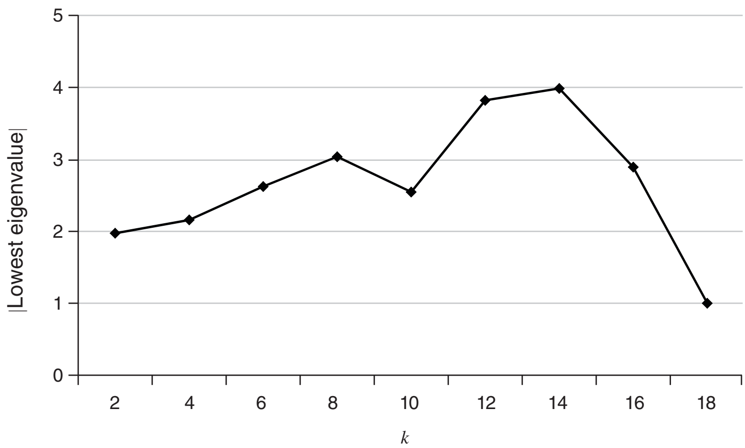

Figure 6. Lowest Eigenvalue for Local Interaction Networks, $n = 19$

$\mathbf{L}_k$, $k = \{2, 4, 6, \ldots, 18\}$. We see that $|\lambda_{\min}(\mathbf{L}_k)|$ is non-monotonic in k, but roughly inverse U-shaped, and reaches its highest value at $k = 14$.

IV. Comparative Statics

In this section we ask how changes in the graph or the payoff impact parameter δ affect equilibrium play. Strategic substitutes pose a challenge for comparative statics, especially in networks. Consider a new link between an agent i and j. The new link lowers the incentives of the newly connected agents and thus increases the incentives of their neighbors, and so on. The direct effects and the indirect effects can pull in opposite directions. Multiplicity of equilibria further complicates the analysis.

We use the potential function to overcome these difficulties and study changes in the aggregate equilibrium play $\sum_{i=1}^{n} x_i(\delta, \mathbf{G})$. We obtain clear results for differences in the *highest* equilibrium level of aggregate play. For a given δ and $\mathbf{G}$, let $\mathbf{x}^*(\delta, \mathbf{G})$ be an equilibrium with the highest aggregate play. This equilibrium is also a global maximum of the potential, since the potential function, evaluated at an equilibrium, is exactly proportional to aggregate play in that equilibrium.[21] Now consider adding a link to the graph. We find that the overall decreases dominate the increases. Aggregate play in the highest equilibrium falls. Essentially, an increase in $\mathbf{G}$ lowers the value of the potential. Since each agent acts as if he is maximizing the potential, the maximum they can reach is lower. The same is true for an increase in the payoff impact parameter δ. [22]

PROPOSITION 8: *Consider a δ and $\mathbf{G}$ and a highest-aggregate-play equilibrium $\mathbf{x}^*(\delta, \mathbf{G})$. Consider a δ' and $\mathbf{G}'$ where $\delta' \geq \delta$ and $\mathbf{G}$ is a subgraph of $\mathbf{G}'$ and any equilibrium vector $\mathbf{x}(\delta', \mathbf{G}')$. Then*

$$\sum_{i=1}^{n} x_i(\delta', \mathbf{G}') \leq \sum_{i=1}^{n} x_i^*(\delta, \mathbf{G}).$$

We derive a local version of this result in the Appendix. The aggregate outcome in *any* stable equilibrium falls following a small enough increase in δ or $\mathbf{G}$.

We can easily build examples of networks where individual play or the lowest level of aggregate play in equilibrium is non-monotonic for changes in δ or $\mathbf{G}$. This confirms the richness of comparative statics in network games of strategic substitutes and hence the interest of our approach.

V. Applications

A. *Information Creation, R&D, and Local versus Global Spillovers*

Information and innovation is one example of a privately provided public good that is consumed by those connected by social or geographic links. When making a decision—about which car or computer to buy, which crop to plant, which drug to use—people often investigate their options. People also glean this information from friends, colleagues, and neighbors. Scientific research similarly involve experiments and spread of results.[23]

[21] That is, for a given δ and $\mathbf{G}$ and an equilibrium $\mathbf{x}(\delta, \mathbf{G})$, $\varphi(\mathbf{x}; \delta, \mathbf{G}) = \frac{1}{2}\sum_{i=1}^{n} x_i(\delta, \mathbf{G})$. To see why, note that $\mathbf{x}^T(\mathbf{I} + \delta\mathbf{G})\mathbf{x} = \mathbf{x}_A^T(\mathbf{I} + \delta\mathbf{G}_A)\mathbf{x}_A$ if the set of active agents is A. By Proposition 1, $(\mathbf{I} + \delta\mathbf{G}_A)\mathbf{x}_A = \mathbf{1}$ and since $\mathbf{x}_A^T\mathbf{1} = \mathbf{x}^T\mathbf{1}$, then $\varphi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{1}$.

[22] Previous comparative statics results in the literature are special cases, especially Theorem 2 in Ballester, Calvó-Armengol, and Zenou (2006) and Bramoullé and Kranton's (2007) example for the circle. Bloch and Zenginobuz (2007) show spillovers can lead to lower public good provision.

[23] Economists have measured information creation and learning/appropriation in specific environments. For examples see Adams (2002), Coe and Helpman (1995), and Foster and Rosenzweig (1996).

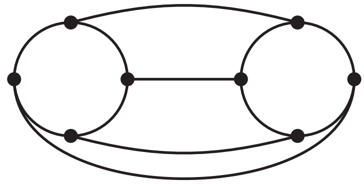

Figure 7. Networks with More Distant versus More Local Links

R&D is a case in point. A large body of empirical work investigates "spillovers,"[24] and the related divergence between private costs and the public benefit.[25] Themes in this literature are the personal nature of information sharing, where employees of similar firms talk with another and exchange information, and industrial clusters where firms (again, often through their employees) benefit from each others' experience and learning.[26]

The following analysis can capture how communication and interaction patterns affect innovation.[27] Consider a stylized model, where x_i is the information created by agent i, and let i's payoff be $\hat{U}_i(x_i, \mathbf{x}_{-i}) = b(x_i + \delta \sum_j g_{ij} x_j) - cx_i$, as earlier in the paper, where $b(\cdot)$ is the benefit of innovation, c is the marginal cost, and δ gives the substitutability between own and neighbors' innovation.[28]

Network models can capture different patterns of social interactions and geography, and the lowest eigenvalue gives a predictive measure of innovation. For example, below we start with a local interaction graph and add links, in different ways, to see the effects of greater local versus global spillovers.[29] In general, starting from a given network, adding links between the sets R and S identified from a lowest eigenvector, rather than further links within R or S, will create a new social structure in which innovation is more concentrated.

Example 6 (R&D and Local versus Global Spillovers): Consider two separated localities each with four agents (i.e., firms), where the network of relations in each locality is a $k = 2$ local interaction graph. Consider then the networks in Figure 7, each of which adds four links. On the left panel, each agent has a link to the other locality, representing access to distant sources of information; $|\lambda_{\min}(\mathbf{G}_a)| = 3$. On the right panel, two agents in each community have distant ties, and the other two have more local neighbors; $|\lambda_{\min}(\mathbf{G}_b)| \approx 2.24$. For $\delta < 1/3$, in both networks, innovation

[24] Classic works include Griliches (1992) and Jaffe (1986). For a review, see Audretsch and Feldman (2004).

[25] See, for example, Jaffe (1996) and Hall, Mairesse and Mohnen (2009).

[26] For comprehensive literature review see Horan (2012). Classic studies include Audretsch and Feldman (1996); Jaffe, Trajtenberg and Henderson (1993); Saxenian (1994); and von Hippel (1998).

[27] This model considers innovations that are substitutes, and agents that do not suffer a loss when a connected agent gains new technology. Such a model would capture smaller firms with little market power, which populate many datasets. A richer model would capture positive effects of spillovers and negative competitive effects. Recent empirical work (Bloom, Schankerman, and Van Reenen 2013) studies these possibilities and finds the net effects of spillovers are positive, supporting our model as a reduced form.

[28] Previous theoretical work studies $\delta = 1$ and highlights the polarized equilibria where only a few people provide information (Bramoullé and Kranton 2007 and Galeotti and Goyal 2010). With a general δ, we can study information flows that are not perfect and innovations that are not perfectly substitutable.

[29] Recent studies consider, for example, the movement of employees back and forth between localities (Saxenian 2007) and links between firms in clusters to global firms (Braunerhjelm and Feldman 2006).

emerges in the local communities; the unique stable outcome is $x_i = 1/(1 + 3\delta)$. For $1/3 \leq \delta \leq 1/2.24$, the stable outcomes diverge. On the left, the only stable outcome is concentrated innovation; on the right the stable outcomes involve diffuse and local innovation. Global spillovers, then, can lead to more concentrated innovation patterns.

These results are predictive in settings beyond R&D. Consider, for instance, the spread of innovations in rural communities. Agricultural yields are very low in large parts of the developing world, and large-scale diffusion of simple innovations such as fertilizers, irrigation, or high-yield crops could improve the livelihood and welfare of many (Sachs 2005). In their study of the Green Revolution in India, Foster and Rosenzweig (1996) find evidence of substitution and free-riding in farmers' experimentations with the new crops. Networks and space condition the spread and substitutability of information acquired in an agricultural context (Conley and Udry 2010). The results here indicate that strengthening local links will lead to more diffuse innovation.

B. *Crime and Social Networks*

While economists traditionally have focused on incentives and punishment (Becker 1968), social networks are also fundamental to criminal activity. Social interactions help explain the high variance of crime rates across time and space (Glaeser, Sacerdote, and Scheinkman 1996). Bayer, Hjalmarsson, and Pozen (2009) find evidence of peer effects in juvenile correction facilities.

This section studies a stylized model of crime and shows how incentives and social structure can jointly determine crime patterns. Building on Calvó-Armengol and Zenou (2004) and Ballester, Calvó-Armengol, and Zenou (2010), consider a population N of n individuals, and let x_i denote i's crime level. Agent i's returns from x_i are higher when total crime levels are lower, capturing the possibility that criminals compete for victims or territory. Agent i's costs are lower when i's friends engage in more crime, capturing peer effects. Together i's payoffs are

$$V_i(x_i, \mathbf{x}_{-i}; \alpha, \varphi, \mathbf{G}) = x_i\left(1 - \alpha \sum_{j \in N} x_j\right) - cx_i\left(1 - \varphi \sum_{j \in N} g_{ij} x_j\right),$$

where $\alpha \geq 0$ is the impact of total crime, $0 \leq c < 1$ is a cost parameter, and $\varphi \geq 0$ such that $\varphi \leq \alpha$ is the impact of friends' crime.[30] Maximizing subject to $x_i \geq 0$ yields the best reply:

$$x_i = \max\left(0, x_0 - \sum_j h_{ij} x_j\right),$$

where $x_0 = (1 - c)/(2\alpha)$ is individually optimal when no other agent engages in crime (i.e., $\mathbf{x}_{-i} = 0$), and $h_{ij} = \frac{1}{2}(1 - \frac{c\varphi}{\alpha} g_{ij})$. The matrix $\mathbf{H} = \left(\frac{1}{2}\mathbf{C} - \frac{c\varphi}{\alpha}\mathbf{G}\right)$ is key to our results, and the index $|\lambda_{\min}(\mathbf{H})|$ shapes the equilibrium set.[31]

[30] Under these assumptions we show in the Appendix that in equilibrium, $1 - \alpha\sum_j x_j \geq 0$ and $1 - \varphi\sum_j g_{ij} x_j \geq 0$.

[31] We study the full equilibrium set for these payoffs. Calvó-Armengol and Zenou (2004) and Ballester, Calvó-Armengol, and Zenou (2010) study a similar payoff function but with a discontinuity at $x_i = 0$, where agents earn strictly more from non-criminal activity than if their criminal activity is vanishingly small. They focus on small network effects, and the multiple equilibria that arise due to this discontinuity.

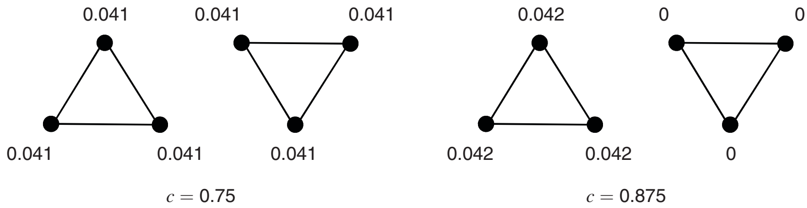

FIGURE 8. CRIME IN A NETWORK WITH CLIQUES

Studying the friendships, $\mathbf{G}$, and costs, c, as they feed into $|\lambda_{\min}(\mathbf{H})|$ yields predictions as to levels and patterns of crime.

Consider, first, outcomes for different friendship networks. A network $\mathbf{G}'$ that adds any links to $\mathbf{G}$ can involve higher total crime. By Proposition 8, the highest level of aggregate action is higher when elements h_{ij} are smaller. Adding a friendship link g_{ij} will decrease h_{ij} and hence increase crime. A network $\mathbf{G}'$ that adds particular links to $\mathbf{G}$ would lead to concentrated crime, along the lines of Example 6 in the R&D application. Here, adding "local" links rather than "global" links (identified from sets R and S for a lowest eigenvector for $\mathbf{G}$) would increase $|\lambda_{\min}(\mathbf{H})|$.

Consider next higher cost, c. Higher c has two opposing effects on crime levels. It directly decreases x_0—the classic incentive effect. But higher c also enhances peer effects, and thus can counteract the first effect. Total crime can fall, but individual crime levels for some agents can rise. Furthermore, gangs can arise for high values of c. Higher punishment can concentrate criminal activity within a subset of the population. As c increases, $|\lambda_{\min}(\mathbf{H})|$ tends to increase. For $c = 0$, the friendship links do not appear in $\mathbf{H}$. There is a single equilibrium, since $|\lambda_{\min}(\mathbf{H})| = \frac{1}{2} < 1$. Everyone engages in crime, and $x_i = 2x_0/(n+1)$ for all i. If c rises enough, and peer effects are large (φ sufficiently high), $|\lambda_{\min}(\mathbf{H})| > 1$ as soon as $|\lambda_{\min}(\mathbf{C} - \mathbf{G})| > 2$. In this case, when $\mathbf{G}$ contains a *clique* Q which is somewhat isolated, there is an equilibrium where each agent in Q is a criminal while each agent outside of Q does not engage in criminal activity.[32] Example 7 illustrates these outcomes.

Example 7 (Cost of Crime: Crime Levels and Gangs): Consider the friendship links $\mathbf{G}$ depicted in Figure 7: six agents are in two disconnected cliques. Let $\varphi = 0.6$ and $\alpha = 1$. Consider $c = 0.75$. With these values, $|\lambda_{\min}(\mathbf{H})| = 0.95 < 1$, hence there is unique and stable equilibrium. All agents engage in the same level of crime, $x_i \approx 0.041$. With a higher cost of crime, $c = 0.875$, $|\lambda_{\min}(\mathbf{H})| = 1.025 > 1$. An interior equilibrium where $x_i \approx 0.021$, exists, but it is neither unique nor stable. Stable equilibria now involve agents in one clique playing $x_Q \approx 0.042$, and agents in the other clique refraining from criminal activity. Crime is concentrated among fewer agents, overall crime level falls, but the individuals who engage in crime have higher crime levels.

These results give at least two insights into crime patterns. First, identical individuals, with the same observable characteristics and the same number of friendships, can have different outcomes. The division of the population into criminals and

[32] A *clique* is a set of agents such that any two agents in the set are linked. We show in the Appendix that the profile described is an equilibrium if every agent not in Q is connected to no more than $n_Q - 3$ agents in Q where n_Q is the size of the clique.

non-criminals can be an equilibrium phenomenon that depends on the entire network structure. Second, gangs are predicted by high cost of crime and dense, isolated friendship networks. Greater punishment can lead to higher levels of criminal activity perpetrated by individuals in detached subgroups of the population.

C. *Econometrics of Peer Effects and Spatial Interactions*

Much empirical work studies how individual outcomes are influenced by others' outcomes. "Peer effects" occur, for example, when a teenager's choices depend on the choices of his friends.[33] Spatial models consider geographic proximity and outcomes such as growth rates, technology adoption, and household demand.[34] The standard way to estimate such effects is to posit a system of simultaneous linear equations which is equivalent to our system of best-replies. Our analysis may then help applied researchers develop proper methods to measure such effects.

Consider some individual outcome y_i that does not take negative values, such as smoking rates or grades. To estimate social or spatial interactions, applied researchers often consider the following structural linear regression:

$$(S) \qquad y_i = \mathbf{X}_i\boldsymbol{\beta} - \delta \sum_j g_{ij} y_j + \varepsilon_i,$$

where the g_{ij}s are spatial links or friendships, δ is the interaction parameter, while X_i is a vector of individual covariates with associated parameters $\boldsymbol{\beta}$. Estimating parameters $\boldsymbol{\beta}$ and δ raises two difficulties. First, outcomes are simultaneously determined and hence the peers' outcome $\sum_j g_{ij} y_j$ on the right-hand side is endogenous. This is an aspect of the reflection problem (Manski 1993). Second, nonnegativity constraints $y_i \geq 0$ have to be taken into account.[35]

Applied economists are well aware of the first problem but typically neglect the second. Standard practice is to estimate the reduced-form of (S), which is well-defined when $(\mathbf{I} + \delta\mathbf{G})$ is invertible and given by $\mathbf{y}^* = (\mathbf{I} + \delta\mathbf{G})^{-1}(\mathbf{X}\beta + \varepsilon)$. However, this procedure will often produce negative values of y_i^* and hence will misestimate the model.[36] In principle, this can be simply addressed by considering the censored version of (S) $y_i = \max(0, \mathbf{X}_i\boldsymbol{\beta} - \delta\sum_j g_{ij} y_j + \varepsilon_i)$. In that case, the econometric system of simultaneous equations is equivalent to our system of best-replies (1) with heterogenous targets $\overline{x}_i = \mathbf{X}_i\boldsymbol{\beta} + \varepsilon_i$.[37]

Our analysis then contributes to the empirics of peer and spatial effects by making the link to the econometrics of games and multiple equilibria.[38] Finding the reduced-form $\mathbf{y}(\mathbf{X}, \boldsymbol{\beta}, \delta, \varepsilon)$ is equivalent to determining the Nash equilibria of the

[33] There is a large empirical literature on peer effects; see for example Duflo and Saez (2002), Gaviria and Raphael (2001), Kawaguchi (2004), Sacerdote (2001), and Trogdon, Nonnemaker, and Pais (2008). Economists have recently started to study peer effects in social networks, see Bramoullé, Djebbari, and Fortin (2009), and Blume et al. (2010).

[34] See, for example, Case (1991). For a review of spatial models and applications, see Anselin (2010).

[35] The outcome may also have a natural upper bound.

[36] In addition, $\mathbf{y}^*$ is necessarily unstable if $|\delta|$ is high enough. Our stability results imply that $\mathbf{y}^*$ is unstable if $\delta > 0$ and $|\lambda_{\min}(\mathbf{G})| > 1/\delta$ or if $\delta < 0$ and $\lambda_{\max}(\mathbf{G}) > 1/|\delta|$.

[37] We describe in Section VI below how our results extend to heterogeneous targets, to an upper bound and also to the "linear-in-means" formulations often used in applied studies of social or spatial interactions.

[38] Researchers are starting to make these connections for specific cases. See, for example, Card and Giuliano (2013).

corresponding game. Thus, model (S) should be estimated with appropriate bounds on $\mathbf{y}$ and attention needs to be paid to the multiple solutions that can arise. A classical way to address multiplicity is to introduce an equilibrium selection. For instance, Soetevent and Kooreman (2007) assume that all Nash equilibria are equally likely. The likelihood of observing outcome $\mathbf{y}$ would then be written as follows. Let S^* denote the set of Nash equilibria, which depends on $\mathbf{X}, \boldsymbol{\beta}, \delta$, and ε; $|S^*|$ their number; F the cdf of the error term; and $1_{S^*}(\mathbf{y})$ the indicator function equal to 1 if $\mathbf{y}$ is an equilibrium and 0 otherwise. Then,

$$L(\mathbf{y}|\mathbf{x}, \boldsymbol{\beta}, \delta) \;=\; \int \frac{1}{|S^*|} 1_{S^*}(\mathbf{y}) dF(\varepsilon).$$

To maximize the likelihood and estimate β and δ, the researcher needs an efficient algorithm to compute the number of Nash equilibria. Proposition 1 and Lemma 1 are the bases of such algorithms. Alternatively, it can be assumed that only stable equilibrium arise and all stable equilibria are equally likely. In this case, the likelihood function would consider only the subset of equilibria that are stable, and Proposition 4 would be central to empirical implementation. Bajari, Hong, and Ryan (2010) have proposed a parametric selection mechanism, which relies on an algorithm able to compute all Nash equilibria. In the Appendix we discuss how to build such an algorithm for these models.

VI. General Model

Our analysis of unique and stable equilibria generalizes and applies to many more games. The base model involves three assumptions (i) discrete and non-directed links, $g_{ij} = g_{ji} \in \{0, 1\}$, (ii) positive and homogeneous payoff impact, $\delta \in [0, 1]$, and (iii) homogeneous targets, $\bar{x}_i = \bar{x}$ for all i. In this section, we generalize the analysis.[39]

A. *Substitutes*, *Complements*, *and Individual Targets*

Consider a general specification which allows for any combination of strategic substitutes and complements and for individual target levels of play. Heterogeneous targets capture at least two well-known games, written in a network form: (a) a linear-demand Cournot model with differentiated products where firms have different marginal costs, and (b) strategic private provision of public goods à la Bergstrom, Blume, and Varian (1986) where consumers have Cobb-Douglas preferences and different incomes. Consider any weighted network $g_{ij} = g_{ji} \in \mathbb{R}$, any payoff parameter $\delta > 0$, and individual targets $\bar{x}_i$, where $\overline{\mathbf{x}} \equiv (\bar{x}_1, \ldots \bar{x}_n)$. The potential here is $\varphi(\mathbf{x}) = \mathbf{x}^T\overline{\mathbf{x}} - \frac{1}{2}\mathbf{x}^T(\mathbf{I} + \delta\mathbf{G})\mathbf{x}$. The curvature of the potential function and the Hessian $\nabla^2\varphi = -(\mathbf{I} + \delta\mathbf{G})$ are not affected by $\overline{\mathbf{x}}$. A new issue is that strategic complements can lead to ever-increasing play and nonexistence. The lowest

[39] Proofs are provided in the Appendix, which also indicates which comparative statics results extend to different settings. In the working paper version of this paper, we also show how our analysis can be applied, locally, to any potential game with nonlinear best-replies, under appropriate regularity conditions (Bramoullé, Kranton, and D'Amours 2010).

eigenvalue, we find, controls such amplification in this general model. An equilibrium exists and is unique if $|\lambda_{\min}(\mathbf{G})| < 1/\delta$.[40] Thus, Proposition 2 applies in this setup and our uniqueness condition ensures existence.[41] Moreover, this condition is less restrictive than the condition obtained in Ballester, Calvó-Armengol, and Zenou (2006) for a mix of substitutes and complements.[42]

The results just described hold when the payoff impact δ is sufficiently small and hence neighbors' actions are relatively unimportant. To study larger δ, and to capture realistically many applications, we place a natural upper bound l on actions (such as 24 hours in a day).[43] For $x_i \in [0, l]$, the best-reply is $f_i(\mathbf{x};\delta,\mathbf{G}) = \min\Big(l, \max\Big(0, \bar{x}_i - \delta\sum_{j=1}^{n} g_{ij}x_j\Big)\Big)$. The bound places an exogenous limit on the extent of amplification and guarantees existence of Nash equilibria, by standard results, for any δ and $\mathbf{G}$. Our analysis extends with the additional constraints on problem (P). Proposition 2 applies directly. Propositions 3, 4, and 5 extend by simply replacing active agents by agents who play an interior action ($0 < x_i < l$) and inactive agents by agents who play either 0 or l.

B. *Heterogeneous Payoff Impacts and Directed Networks*

Suppose δ_i is heterogeneous so that each agent i is impacted differently by neighbors' actions. For example, agents are affected by the average of their neighbors' actions, i.e., $\delta_i = 1/k_i$, as in peer effects and investment games (see e.g., Glaeser and Scheinkman 2003 and Angeletos and Pavan 2007). The best reply is then

$$f_i(\mathbf{x};\delta,\mathbf{G}) \;=\; \max\left(0, \bar{x}_i - \delta_i\left(\sum_{j=1}^{n} g_{ij}x_j\right)\right).$$

The game with modified quadratic payoffs $\tilde{U}_i$ now does not have a potential, since $\delta_i g_{ij} \neq \delta_j g_{ji}$.

We derive conditions for unique and stable equilibria as follows. By appropriate rescaling of the payoffs, obtain a "weighted potential," in the terminology of Monderer and Shapley (1996).[44] Introduce the network $\tilde{\mathbf{G}}$ such that $\tilde{g}_{ij} = \sqrt{\delta_i}\sqrt{\delta_j}g_{ij}$. The weighted potential is then strictly concave if and only if $|\lambda_{\min}(\tilde{\mathbf{G}})| < 1$. Using $\tilde{\mathbf{G}}$, all the results for unique, stable, and inactive-agent equilibria—Propositions 2, 3, 4, and 5—hold.

[40] Previous studies of pure complements (here all $g_{ij} \leq 0$) find that the highest eigenvalue determines when an interior equilibrium exists. Corbo, Calvó-Armengol, and Parkes (2007) show that a unique interior equilibrium exists if $\lambda_{\max}(-\mathbf{G}) < 1/\delta$ while existence fails to hold when $\lambda_{\max}(-\mathbf{G}) \geq 1/\delta$. Since $\lambda_{\max}(-\mathbf{G}) = -\lambda_{\min}(\mathbf{G})$, this is consistent with our analysis of the general case. Also, with any departure from pure complements, the relevant matrix has negative entries. The Perron-Frobenius Theorem and derivative results (e.g., the highest eigenvalue increases when the network expands) do not apply.

[41] When $|\lambda_{\min}(\mathbf{G})| < 1/\delta$, the potential is still a strictly concave quadratic function. Therefore, it must have a global maximum on $[0, +\infty[^n$ which constitutes the unique solution to the first-order conditions of problem (P).

[42] Ballester, Calvó-Armengol, and Zenou (2006, Theorem 1) shows: if $g_{ij} > 0$ for some pair (i,j), a unique interior equilibrium exists if $\lambda_{\max}(g_{\max}\mathbf{C} - \mathbf{G}) < 1/\delta - g_{\max}$ where $g_{\max} = \max_{i,j} g_{ij}$.

[43] Belhaj, Bramoullé, and Deroïan (2012) apply the theory of supermodular games to study pure complements when $x_i \in [0, l]$.

[44] Define $\tilde{\Pi}_i = \tilde{U}_i/\delta_i$. The game with payoffs $\tilde{\Pi}_i$ has the same best-replies and equilibria as the game with payoffs $\tilde{U}_i$. We can easily see that $\partial^2\tilde{\Pi}_i(\mathbf{x})/\partial x_i \partial x_j = \partial^2\tilde{\Pi}_i(\mathbf{x})/\partial x_j \partial x_i$ for all $i \neq j$. So this modified game has a potential function: $\varphi(\mathbf{x}) = \sum_i \Big(x_i\bar{x}_i - \frac{1}{2}x_i^2\Big)/\delta_i - \frac{1}{2}\sum_j g_{ij}x_ix_j$.

This rescaling procedure further indicates a class of directed graphs $\left(g_{ij} \neq g_{ji}\right)$ where our results apply.[45] Consider a directed graph $\mathbf{H}$ and payoff impacts γ_i such that for each i and j there are scalars α_i and α_j with the property $\alpha_i h_{ij} = \alpha_j h_{ji}$. Define a graph $\mathbf{H}'$ where for all i and j, $h'_{ij} \equiv \alpha_i h_{ij} = \alpha_j h_{ji} \equiv h'_{ji}$. The analysis of a linear-best response game with graph $\mathbf{H}$ is then equivalent to that of a symmetric graph $\mathbf{H}'$ and idiosyncratic payoff impacts $\delta_i = \gamma_i / \alpha_i$.

VII. Conclusion

This paper studies a large class of strategic games on networks. In games with linear best replies, we find the lowest eigenvalue is key to equilibrium outcomes. This simple network measure captures the extent to which the network amplifies the substitutabilities of agents' actions. When this amplification is small, there is a unique equilibrium, as agents' actions do not rebound in different directions. But when it is large, there are possibly many equilibria. In this case, however, we can refine the equilibrium set, again relying on a lowest eigenvalue. Equilibria are stable when the network of links connecting active agents does not overamplify agents' play. The lowest eigenvalue is new to research on economic and social networks, and we show that networks that are more "two-sided" have greater rebound of actions.

The analysis and results can provide predictions in specific applications and give guidance for empirical and experimental work. With small numbers of actors, the equilibrium set can be reasonably estimated for different graph structures and hypothesized payoff impact levels. For any given network, researchers can also calculate the lowest eigenvalue and use it either to test predictions or to calibrate their models. The analysis would indicate likely outcomes in geographic, social, or market settings, such as differentiated-product oligopoly, crime, innovation, and experimentation with new technologies.

More generally, this paper shows the importance of studying network relations. In a standard model, each agent is assumed to have the same direct effect on everyone else's payoffs. But people and firms do not interact equally and directly with every other actor. A firm can operate in different geographic markets; a researcher can exchange information with colleagues in different disciplines; a teenager can have different friends at school than at after-school activities. Agents have direct effect on some agents, but indirect effects—through the network—on others. This paper shows the force of these indirect, network, effects and show how the network can shape individual and aggregate outcomes.

Appendix

PROOF OF PROPOSITION 1:

For any Nash equilibrium $\mathbf{x}$ with active agents A: if $i \in A$, then $x_i > 0$ and $x_i = 1 - \delta \sum_{j \in N} g_{ij} x_j = 1 - \delta \sum_{j \in A} g_{ij} x_j$. If $i \notin A$, then $x_i = 0$ and $1 - \delta \sum_{j \in A} g_{ij} x_j \leq 0$. Using the matrix notation yields the result.

[45] We can also extend Proposition 2 to any directed graph $\mathbf{G}$ using Rosen's (1965) results: If $|\lambda_{\min}((\mathbf{G} + \mathbf{G}^T)/2)| < 1/\delta$ the game defined on $\mathbf{G}$ has a unique Nash equilibrium.

PROOF OF STATEMENTS IN FOOTNOTE 8:

The number of equilibria is finite for any $\mathbf{G}$ *and almost any* δ. Let $N = \{1, \ldots, n\}$. If $|\mathbf{I} + \delta \mathbf{G}_A| \neq 0$, there is at most one equilibrium with active agents A. $|\mathbf{I} + \delta \mathbf{G}_A| = 0$ if and only if $-1/\delta$ is an eigenvalue of $\mathbf{G}_A$. Therefore, for any δ except maybe a finite number of values, $\forall A$, $|\mathbf{I} + \delta \mathbf{G}_A| \neq 0$ and the number of equilibria is lower than the number of subsets of N.

An algorithm to compute all equilibria. Given a graph $\mathbf{G}$, consider a subset $S \subset N$. If $|\mathbf{I} + \delta \mathbf{G}_S| \neq 0$, compute the profile $\mathbf{x}_S = (\mathbf{I} + \delta \mathbf{G}_S)^{-1}\mathbf{1}$ and set $\mathbf{x}_{N-S} = \mathbf{0}$. Check whether $\mathbf{x}_S \geq \mathbf{0}$ and $\delta \mathbf{G}_{N-S,S}\mathbf{x}_S \geq \mathbf{1}$. If these two sets of linear inequalities are satisfied, $\mathbf{x}$ is an equilibrium. If either condition fails, it is not. Repeating this procedure for every subset of N yields all the equilibria for $\mathbf{G}$ and almost any δ. Since there are 2^n subsets of N, this algorithm runs in exponential time. Note that since the number of equilibria may be exponential, any alternative algorithm must also run in exponential time. Computer scientists have been searching for "partial enumeration methods" to reduce computing time in such problems (Murty and Yu 1988). We have derived several results (available upon request) that help identify the set of active agents and thus could be used to speed computation.

PROOF OF LEMMA 1:

The Kuhn-Tucker condition of problem (P) for agent i is: $x_i > 0 \Rightarrow \partial\varphi/\partial x_i = 0$ and $x_i = 0 \Rightarrow \partial\varphi/\partial x_i \leq 0$. Since $\partial\varphi/\partial x_i = \partial\tilde{U}_i/\partial x_i$, this is equivalent to the best-reply condition for agent i in the game with quadratic payoffs.

PROOF OF PROPOSITION 2:

By the theory of convex optimization, if φ is strictly concave, problem (P) has a unique solution characterized by the Kuhn-Tucker conditions. Since $\nabla^2\varphi = -(\mathbf{I} + \delta\mathbf{G})$, φ is strictly concave if and only if $\mathbf{I} + \delta\mathbf{G}$ is positive definite, which is equivalent to $1 + \delta\lambda_{\min}(\mathbf{G}) > 0$.

PROOF OF STATEMENT IN FOOTNOTE 13:

Let $\underline{\delta}_{BCAZ}(\mathbf{G}) \equiv 1/(1 + \lambda_{\max}(\mathbf{C}-\mathbf{G}))$, $k_{\min}(\mathbf{H})$ be the smallest number of neighbors any agent has in graph $\mathbf{H}$, and $k_{\max}(\mathbf{H})$ be the largest number of neighbors any agent has in graph $\mathbf{H}$. Suppose that $-1/\lambda_{\min}(\mathbf{G}) < \underline{\delta}_{BCAZ}(\mathbf{G})$ and consider δ such that $-1/\lambda_{\min}(\mathbf{G}) < \delta < \underline{\delta}_{BCAZ}(\mathbf{G})$. Ballester, Calvó-Armengol, and Zenou (2006) show there is then a unique equilibrium for δ and $\mathbf{G}$ and this equilibrium is interior. Next, in the Proof of Proposition 3, we show that for $-1/\lambda_{\min}(\mathbf{G}) < \delta$ there exists an equilibrium for δ and $\mathbf{G}$ which is a corner. This is a contradiction, hence $-1/\lambda_{\min}(\mathbf{G}) \geq \underline{\delta}_{BCAZ}(\mathbf{G})$. Next, observe that for any graph $\mathbf{H}$, $k_{\min}(\mathbf{H}) \leq \lambda_{\max}(\mathbf{H}) \leq k_{\max}(\mathbf{H})$ (Cvetković, Doob, and Sachs 1979). This implies that $\lambda_{\max}(\mathbf{C} - \mathbf{G}) \geq k_{\min}(\mathbf{C}-\mathbf{G}) = n - 1 - k_{\max}(\mathbf{G})$. Hence, $1/(1 + \lambda_{\max}(\mathbf{C}-\mathbf{G})) \leq 1/(n - k_{\max}(\mathbf{G}))$. Then, $-\lambda_{\min}(\mathbf{G}) \leq \lambda_{\max}(\mathbf{G}) \leq k_{\max}(\mathbf{G})$, hence $-1/\lambda_{\min}(\mathbf{G}) \geq 1/k_{\max}(\mathbf{G})$ and $-1/\lambda_{\min}(\mathbf{G}) > \underline{\delta}_{BCAZ}(\mathbf{G})$ if $k_{\max}(\mathbf{G}) < n/2$.

PROOF OF PROPOSITION 3:

We use three facts. First, for any δ and $\mathbf{G}$, there exists at least one vector, denoted $\mathbf{x}^{**}(\delta, \mathbf{G})$, that globally maximizes the potential $\varphi(\mathbf{x}; \delta, \mathbf{G})$. This is true because problem (P) is equivalent to maximizing $\varphi(\mathbf{x})$ over $[0,1]^n$, hence a maximum exists because the domain is compact and $\varphi(\mathbf{x})$ is continuous. Second, by Lemma 1 any maximum of the potential is a Nash equilibrium, so $\mathbf{x}^{**}(\delta,\mathbf{G})$ is a Nash equilibrium. Third, if $|\lambda_{\min}(\mathbf{G})| > 1/\delta$, the potential is a non-concave quadratic function. Then, there must be a direction along which the potential increases without bound. Hence, the vector $\mathbf{x}^{**}(\delta,\mathbf{G})$ must not be interior, and for this equilibrium $A \neq N$.

PROOF OF COROLLARY 1:

No further argument is necessary beyond what is stated in the text.

PROOF OF LEMMA 2:

Introduce $h_i(\mathbf{x}) = f_i(\mathbf{x}) - x_i$ and $H(\mathbf{x}) = \big(h_1(\mathbf{x}),\dots,h_n(\mathbf{x})\big)$. We first show that the potential increases along the trajectories of the system. Here, $\frac{d}{dt}\varphi\big(\mathbf{x}(t)\big) = \sum_i \frac{\partial\varphi}{\partial x_i}\dot{x}_i = \nabla\varphi(\mathbf{x}) \cdot H(\mathbf{x})$.

LEMMA A1: $\forall \mathbf{x} \in \mathbb{R}^n_+$, $\nabla\varphi(\mathbf{x}) \cdot H(\mathbf{x}) \geq 0$ *and* $\nabla\varphi(\mathbf{x}) \cdot H(\mathbf{x}) = 0$ *if and only if* $\mathbf{x}$ *is a Nash equilibrium.*

PROOF:

We have: $\frac{\partial\varphi}{\partial x_i} = 1 - \delta\sum_j g_{ij}x_j - x_i$ and $h_i(\mathbf{x}) = 1 - \delta\sum_j g_{ij}x_j - x_i$ if $\delta\sum_j g_{ij}x_j \leq 1$ and $-x_i$ if $\delta\sum_j g_{ij}x_j > 1$. Then, $\frac{\partial\varphi}{\partial x_i}h_i = (1 - \delta\sum_j g_{ij}x_j - x_i)^2$ in the first case and $x_i(\delta\sum_j g_{ij}x_j - 1 + x_i)$ in the second case. Therefore, $\nabla\varphi(\mathbf{x}) \cdot H(\mathbf{x}) \geq 0$. Equality occurs if and only if $x_i = 1 - \delta\sum_j g_{ij}x_j$ if $\delta\sum_j g_{ij}x_j \leq 1$, and $x_i = 0$ if $\delta\sum_j g_{ij}x_j > 1$.

Next, consider some profile $\mathbf{x}$. There are three possibilities.

(i) $\mathbf{x}$ is not a maximum of the potential. There is then a sequence $\{\mathbf{x}^m\}_{m=1}^{\infty}$ converging to $\mathbf{x}$ such that $\varphi(\mathbf{x}^m) > \varphi(\mathbf{x})$ for all m. Since φ cannot decrease along trajectories, starting at $\mathbf{x}^m$ the system cannot converge back to $\mathbf{x}$. Hence $\mathbf{x}$ is not asymptotically stable.

(ii) $\mathbf{x}$ is a maximum but is not a strict maximum of the potential. There is then a sequence $\{\mathbf{x}^m\}_{m=1}^{\infty}$ of maxima converging to $\mathbf{x}$ such that $\mathbf{x}^m \neq \mathbf{x}$ and $\varphi(\mathbf{x}^m) = \varphi(\mathbf{x})$. Since $\mathbf{x}^m$ is an equilibrium, it is a steady state of the system of differential equations. Thus, starting at $\mathbf{x}^m$ the system does not converge back to $\mathbf{x}$, and $\mathbf{x}$ is not asymptotically stable.

(iii) $\mathbf{x}$ is a strict maximum of the potential. If there exists an $\varepsilon > 0$ such that $\mathbf{x}$ is the unique equilibrium on $B_O(\mathbf{x},\varepsilon) \cap \mathbb{R}^n_+$, then we can apply Theorem 6.4 in Weibull (1995). Precisely, the Lyapunov function is equal to $\varphi(\mathbf{x}) - \varphi(\mathbf{y})$ for any $\mathbf{y} \in B_O(\mathbf{x},\varepsilon) \cap \mathbb{R}^n_+$. Hence $\mathbf{x}$ is asymptotically stable. Thus, we only need to show that $\mathbf{x}$ is locally the unique equilibrium. Suppose not. There then exists a sequence of equilibria $\{\mathbf{x}^m\}_{m=1}^{\infty}$ converging to $\mathbf{x}$ such that $\mathbf{x}^m$ is not a potential maximum. Without loss of generality, we can assume that

$\forall m$, $\mathbf{x}^m$ have the same active agents A since there is a finite number of subsets of N. Hence $(\mathbf{I} + \delta\mathbf{G}_A)\mathbf{x}_A^m = 1$ and $\delta\mathbf{G}_{N-A,A}^m\mathbf{x}_A \geq \mathbf{1}$. Taking the limit shows that $(\mathbf{I} + \delta\mathbf{G}_A)\mathbf{x}_A = 1$ and $\delta\mathbf{G}_{N-A,A}\mathbf{x}_A \geq \mathbf{1}$, and hence that any active agent of $\mathbf{x}$ is in A. Therefore, we can write $\mathbf{x}^m = \mathbf{x} + \mathbf{y}^m$ with $\mathbf{y}^m$ such that $\mathbf{y}_{N-A}^m = \mathbf{0}$ and $(\mathbf{I} + \delta\mathbf{G}_A)\mathbf{y}_A^m = 0$. Next, compute $\varphi(\mathbf{x}^m)$ in two different ways. First, $\varphi(\mathbf{x}^m) = \frac{1}{2}(\mathbf{x}^m)^T\mathbf{1} = \varphi(\mathbf{x}) + \frac{1}{2}(\mathbf{y}^m)^T\mathbf{1}$ since $\mathbf{x}^m$ and $\mathbf{x}$ are equilibria. Second, by definition, $\varphi(\mathbf{x}^m) = \varphi(\mathbf{x} + \mathbf{y}^m) = \mathbf{x}^T\mathbf{1} + (\mathbf{y}^m)^T\mathbf{1} - \frac{1}{2}(\mathbf{x} + \mathbf{y}^m)^T(\mathbf{I} + \delta\mathbf{G}) \times(\mathbf{x} + \mathbf{y}^m)$. The last expression is equal to $(\mathbf{x}_A + \mathbf{y}_A^m)^T(\mathbf{I} + \delta\mathbf{G}_A)(\mathbf{x}_A + \mathbf{y}_A^m) = \mathbf{x}_A^T(\mathbf{I} + \delta\mathbf{G}_A)\mathbf{x}_A$. Thus, $\varphi(\mathbf{x}^m) = \varphi(\mathbf{x}) + (\mathbf{y}^m)^T\mathbf{1}$. This shows that $(\mathbf{y}^m)^T\mathbf{1} = \mathbf{0}$. Hence $\varphi(\mathbf{x}^m) = \varphi(\mathbf{x})$ which contradicts the fact that $\mathbf{x}$ is a locally strict maximum.

We know that for any $\mathbf{G}$ and almost any δ, the number of equilibria is finite. In that case, the global maximum of the potential is necessarily strict and hence is a stable equilibrium.

PROOF OF PROPOSITION 4 AND CHARACTERIZATION OF STABLE EQUILIBRIA:

We analyze the second-order conditions of problem (P). Consider an equilibrium $\mathbf{x}$ with active agents A and inactive agents who are not strictly inactive K, such that $A \cup K = \{i : x_i + \delta\sum_j g_{ij}x_j = 1\}$ and $K = \{i : x_i = 0 \text{ and } \delta\sum_j g_{ij}x_j = 1\}$. The application of Theorems 3.4 and 3.6 in Lee, Tam, and Yen (2005) to our setting yields the following result.

LEMMA A2: *Second-order conditions of problem* (*P*).

$\mathbf{x}$ *is a maximum of* φ *on* $\mathbb{R}_+^n$	*if and only if*	$\forall \varepsilon \in \mathbb{R}^{A\cup K}$ *such that* $\varepsilon_K \geq \mathbf{0}$, $\varepsilon^T(\mathbf{I} + \delta\mathbf{G}_{A\cup K})\varepsilon \geq 0$
$\mathbf{x}$ *is a strict maximum of* φ *on* $\mathbb{R}_+^n$	*if and only if*	$\forall \varepsilon \in \mathbb{R}^{A\cup K} - \{\mathbf{0}\}$ *such that* $\varepsilon_K \geq \mathbf{0}$, $\varepsilon^T(\mathbf{I} + \delta\mathbf{G}_{A\cup K})\varepsilon > 0$.

We derive necessary conditions. Observe that when $\varepsilon_K = \mathbf{0}$, $\varepsilon^T(\mathbf{I} + \delta\mathbf{G}_{A\cup K})\varepsilon = \varepsilon_A^T(\mathbf{I} + \delta\mathbf{G}_A)\varepsilon_A$. Therefore, if $\mathbf{x}$ is a maximum then $\mathbf{I} + \delta\mathbf{G}_A$ is positive semi-definite while if $\mathbf{x}$ is a strict maximum, $\mathbf{I} + \delta\mathbf{G}_A$ is positive definite. In contrast, observe that if $\mathbf{x}$ is a minimum of φ, then $\mathbf{I} + \delta\mathbf{G}_A$ must be negative semi-definite which is impossible. Therefore, any Nash equilibrium which is not a maximum of the potential is a saddle point. When $K = \varnothing$, the second-order conditions take a simple form: $\mathbf{x}$ is a maximum if and only if $\mathbf{I} + \delta\mathbf{G}_A$ is positive semi-definite and it is a strict maximum if and only if $\mathbf{I} + \delta\mathbf{G}_A$ is positive definite. This proves Proposition 4.

PROOF OF STATEMENTS IN FOOTNOTE 16:

Inactive agents are strictly inactive in all equilibria for any $\mathbf{G}$ *and almost any* δ. Consider a graph $\mathbf{G}$ such that $|\mathbf{I} + \delta\mathbf{G}_A| \neq 0$. Define $P_i(\delta) = |\mathbf{I} + \delta\mathbf{G}|[(\mathbf{I} + \delta\mathbf{G})^{-1}\mathbf{1}]_i$. Consider the classic relation between a matrix's inverse and its cofactors (e.g., Horn and Johnson 1985, p.20). Let m_{ij} be the determinant of the submatrix of $\mathbf{I} + \delta\mathbf{G}$

obtained by removing the ith row and jth column. Then, $P_i(\delta) = \sum_{j=1}^{n}(-1)^{i+j}m_{ji}$ hence P_i is a polynomial of degree less than or equal to $n-1$ in δ. In addition, $P_i(0) = 1$ so P_i cannot have more than $n-1$ zeros. Thus, for almost any δ, the profile $\mathbf{y} = (\mathbf{I} + \delta\mathbf{G})^{-1}\mathbf{1}$ is well-defined and satisfies $\forall i, y_i \neq 0$. Next, for any equilibrium $\mathbf{x}$, $\mathbf{x}_{A\cup K} = (\mathbf{I} + \delta\mathbf{G}_{A\cup K})^{-1}\mathbf{1}$ if $|\mathbf{I} + \delta\mathbf{G}_{A\cup K}| \neq 0$, hence for almost any $\delta, \forall i \in A \cup K, x_i > 0$ and $K = \varnothing$.

PROOF OF COROLLARY 2:

A global maximum of the potential on $\mathbb{R}_+^n$ always exists and is an equilibrium by Lemma 1. The unique equilibrium is the global maximum of the potential. Since it is strict, by Lemma 2 it must be stable.

PROOF OF PROPOSITION 5:

If an equilibrium $\mathbf{x}$ is stable, then $|\lambda_{\min}(\mathbf{G}_A)| \leq 1/\delta$. Thus, if $|\lambda_{\min}(\mathbf{G})| > 1/\delta$, we necessarily have $A \neq N$.

PROOF OF PROPOSITION 6:

Consider a stable equilibrium $\mathbf{x}$ with active agents A. For any $\mathbf{y} \in \mathbb{R}^A$, define $\varphi_A(\mathbf{y}; \delta, \mathbf{G}) = \varphi(\hat{\mathbf{y}}; \delta, \mathbf{G})$ where $\hat{\mathbf{y}}$ is such that $\hat{\mathbf{y}}_A = \mathbf{y}$ and $\hat{\mathbf{y}}_{N-A} = \mathbf{0}$ and let (P_A) be the problem: $\max_{\mathbf{y}}\varphi_A(\mathbf{y})$ under the constraints that $\forall i \in A, y_i \geq 0$. Since $\mathbf{x}$ is a strict maximum of φ, $\mathbf{I} + \delta\mathbf{G}_A$ is positive definite. Therefore, φ_A is strictly concave and $\mathbf{x}_A$ is the unique solution to the Kuhn-Tucker conditions of (P_A). If $\mathbf{x}'$ is another equilibrium with active agents $A' \subset A$, $\mathbf{x}'_A$ must solve the Kuhn-Tucker conditions of (P_A), which is impossible.

PROOF OF PROPOSITION 7:

Consider $\boldsymbol{\varepsilon}$ such that $\lambda_{\min}(\mathbf{G}) = \boldsymbol{\varepsilon}^t\mathbf{G}\boldsymbol{\varepsilon}$ and let $\mathbf{G}'$ denote the new graph. Observe that $\varepsilon_i\varepsilon_j \geq 0$ when $i,j \in R$ or $i,j \in S$ while $\varepsilon_i\varepsilon_j \leq 0$ if $i \in R, j \in S$. This means that $\boldsymbol{\varepsilon}^t\mathbf{G}'\boldsymbol{\varepsilon} \leq \boldsymbol{\varepsilon}^t\mathbf{G}\boldsymbol{\varepsilon}$. Since $\lambda_{\min}(\mathbf{G}') \leq \boldsymbol{\varepsilon}^t\mathbf{G}'\boldsymbol{\varepsilon}$, the result follows.

PROOF OF STATEMENTS IN SECTION III:

The graph with largest $|\lambda_{\min}|$ *is a complete bipartite graph with sides as equal as possible.* Proposition 6 implies that a graph with largest $|\lambda_{\min}|$ is complete bipartite. Then, in a complete bipartite graph with sides of sizes p and $n-p$, we have $\lambda_{\min} = -\sqrt{p(n-p)}$ (Cvetković, Doob, and Sachs 1979), which is lowest when $p = n/2$ if n is even and $(n-1)/2$ if n is odd.

A tight lower bound on $|\lambda_{\min}(\mathbf{G})|$. Define $\boldsymbol{\varepsilon}$ such that $\varepsilon_i = +\varepsilon$ if $i \in P$ and $\varepsilon_i = -\varepsilon$ if $i \in Q$. We have: $\boldsymbol{\varepsilon}^t\mathbf{G}\boldsymbol{\varepsilon} = \varepsilon^2(2|G_P| + 2|G_Q| - 2|G_{PQ}|)$. In addition, $||\boldsymbol{\varepsilon}|| = \sqrt{n}\varepsilon$ so choosing $\varepsilon = 1/\sqrt{n}$ yields $||\boldsymbol{\varepsilon}|| = 1$ and $\lambda_{\min}(\mathbf{G}) \leq \frac{2}{n}(|\mathbf{G}_P| + |\mathbf{G}_Q| - |\mathbf{G}_{PQ}|)$. When n is even and $\mathbf{G}$ is a complete bipartite graph with equal sizes, we obtain $|\mathbf{G}_P| = |\mathbf{G}_Q| = 0$ and $|\mathbf{G}_{PQ}| = \left(\frac{n}{2}\right)^2$. The lower bound is then equal to $\frac{n}{2} = \lambda_{\min}$.

PROOF OF PROPOSITION 8:

Notice first that for any profile $\mathbf{x}$, $\varphi(\mathbf{x}; \delta', \mathbf{G}') \leq \varphi(\mathbf{x}; \delta, \mathbf{G})$. Therefore, we have: $\varphi(\mathbf{x}(\delta', \mathbf{G}'); \delta', \mathbf{G}') \leq \varphi(\mathbf{x}^*(\delta', \mathbf{G}'); \delta', \mathbf{G}') \leq \varphi(\mathbf{x}^*(\delta', \mathbf{G}'); \delta, \mathbf{G}) \leq \varphi(\mathbf{x}^*(\delta, \mathbf{G}); \delta, \mathbf{G})$ where the first inequality holds because $\mathbf{x}^*(\delta', \mathbf{G}')$ is a global maximum of $\varphi(\mathbf{x}; \delta', \mathbf{G}')$ and the third inequality holds because $\mathbf{x}^*(\delta, \mathbf{G})$ is a global maximum of the potential $\varphi(\mathbf{x}; \delta, \mathbf{G})$.

Local Comparative Statics

PROPOSITION A1:

For δ and $\mathbf{G}$, consider a stable equilibrium $\mathbf{x}(\delta, \mathbf{G})$ with active agents A. Now consider for δ' and $\mathbf{G}'$ where $\delta' \geq \delta$ and $\mathbf{G}$ is a subgraph of $\mathbf{G}'$ and consider any equilibrium $\mathbf{x}(\delta', \mathbf{G}')$ with active agents A' such that $A' \subset A$. Then, $\sum_{i=1}^n x_i' \leq \sum_{i=1}^n x_i$.

PROOF:

Consider a stable equilibrium $\mathbf{x}$ with active agents A. Define φ_A as above. Then, φ_A is strictly concave and $\mathbf{x}_A$ is a global maximum of φ_A on $\mathbb{R}_+^A$. Since $A' \subset A$, $\frac{1}{2}\sum_{i \in N} x_i' = \varphi_A(\mathbf{x}_A'; \delta', \mathbf{G}') \leq \varphi_A(\mathbf{x}_A'; \delta, \mathbf{G}) \leq \varphi_A(\mathbf{x}_A; \delta, \mathbf{G}) = \frac{1}{2}\sum_{i \in N} x_i$. The first inequality comes from the monotonicity of the potential. The second inequality holds because $\mathbf{x}_A$ is a global maximum of φ_A.

The conditions of Proposition A1 hold when the new parameters are "close enough" so that no new agents are active. The existence of such a close-by equilibrium when $\mathbf{G} = \mathbf{G}'$ is guaranteed for almost any δ if the increase in δ is small.

PROOF OF STATEMENT IN FOOTNOTE 30:

Consider an equilibrium $\mathbf{x}$ and suppose that $1 - \alpha\sum_j x_j < 0$. For any i such that $x_i > 0$, $\alpha x_i = 1 - \alpha\sum_j x_j - c(1 - \varphi\sum_j g_{ij}x_j)$. Summing up over active agents yields: $\alpha\sum_j x_j = n_A(1 - \alpha\sum_j x_j) - c\sum_{i \in A}(1 - \varphi\sum_j g_{ij}x_j)$. Then, $\varphi\sum_j g_{ij}x_j \leq \alpha\sum_j x_j$ and $1 - \varphi\sum_j g_{ij}x_j \geq 1 - \alpha\sum_j x_j$ so $-c\sum_{i \in A}(1 - \varphi\sum_j g_{ij}x_j) \leq -cn_A(1 - \alpha\sum_j x_j)$. Therefore, $\alpha\sum_j x_j \leq n_A(1 - c)(1 - \alpha\sum_j x_j) < 0$ which is impossible.

PROOF OF STATEMENT IN FOOTNOTE 32:

Observe that $\lambda_{\min}(\mathbf{H})$ is continuous in $\mathbf{H}$. Therefore, $\exists \varepsilon > 0$ such that $|\lambda_{\min}(\mathbf{H})| > 1$ if $|\lambda_{\min}(\mathbf{C} - \mathbf{G})| > 2$ and $c\varphi/\alpha \geq 1 - \varepsilon$. Next, consider a profile such that everyone in the clique plays $x > 0$ while everyone outside of the clique plays 0. Equilibrium conditions for clique insiders amount to $x = 2x_0/(n_C + 1 - (n_C - 1)c\varphi/\alpha)$. The equilibrium condition for clique outsider i is satisfied if and only if $n_Q - 1 - n_{iQ} \geq \alpha/(c\varphi)$ where n_{iQ} is i's number of clique friends. If ε is small enough, this is equivalent to $n_{iQ} \leq n_Q - 3$.

Proofs for Section VI

Substitutes, Complements, and Individual Targets: Suppose that agents have Cobb-Douglas utilities and allocate income, w_i, between private good consumption, q_i, and public good provision, x_i. Let p be the relative price of the public good. Benefits from the public good flow to neighbors, weighted by $\beta \in [0, 1]$. Individual i then maximizes $U_i = q_i^\alpha(x_i + \beta\sum_i g_{ij}x_j)^{1-\alpha}$ subject to the budget constraint $q_i + px_i \leq w_i$. Simple computations yield the correspondence with $\overline{x}_i = \frac{1-\alpha}{p}w_i$ and $\delta = \alpha\beta$.

Consider δ^* such that a unique interior equilibrium exists and varies continuously with δ for any $\delta < \delta^*$. Suppose that $1/|\lambda_{\min}(\mathbf{G})| < \delta^*$. If $\delta < 1/|\lambda_{\min}(\mathbf{G})|$, the equilibrium is the global maximum of the potential on $[0,+\infty[^n$. If $\delta > 1/|\lambda_{\min}(\mathbf{G})|$, the potential is a non-concave quadratic function and hence there is a direction where it increases without bounds. Its global maximum on $[0,+\infty[^n$ either has some inactive agent or does not exist. The first case is a contradiction. In the second case, by continuity, the equilibrium must diverge when δ tends to $1/|\lambda_{\min}(\mathbf{G})|$ from below, which is also a contradiction. Thus, $\delta^* \leq 1/|\lambda_{\min}(\mathbf{G})|$.

Lemma 1 extends: A profile $\mathbf{x}$ is an equilibrium if and only if it solves the Kuhn-Tucker conditions of the problem: $\max\varphi(\mathbf{x})$ subject to $\mathbf{x} \in [0,l]^n$. These conditions are now: $x_i = 0 \Rightarrow \partial\varphi/\partial x_i \leq 0;\ \ 0 < x_i < l \Rightarrow \partial\varphi/\partial x_i = 0$ and $x_i = l \Rightarrow \partial\varphi/\partial x_i \geq 0$. Hence, there is unique equilibrium if the potential is strictly concave, and Proposition 2 holds. Similarly, Lemma 2 extends and the second-order conditions tell us that an equilibrium is stable if and only if $|\lambda_{\min}(\mathbf{G}_A)| < 1/\delta$ where $A = \{i : 0 < x_i < l\}$ and when actions at the boundary are strict: $x_i = 0 \Rightarrow \delta\sum_j g_{ij}x_j > \overline{x}_i$ and $x_i = l \Rightarrow l - \delta\sum_j g_{ij}x_j < \overline{x}_i$. So Propositions 3, 4, and 5 extend. In addition, under substitutes $(g_{ij} \geq 0)$ we see that $\varphi(\mathbf{x}) = \frac{1}{2}\sum_i \overline{x}_i x_i$ for any equilibrium $\mathbf{x}$. Therefore, the highest level of weighted aggregate action $\sum_i \overline{x}_i x_i$ in equilibrium decreases weakly following an increase in δ or $\mathbf{G}$ or a decrease in any $\overline{x}_j$.

Heterogenous Payoff Impacts: Let $\mathbf{D}$ be the diagonal matrix such that $d_{ii} = 1/\delta_i$. Then, the Hessian of the weighted potential is $-(\mathbf{D} + \mathbf{G})$. For any $\mathbf{x} \in \mathrm{R}^n$, define $\mathbf{y}$ such that $y_i = x_i/\sqrt{\delta_i}$. We see that $\mathbf{x}^T(\mathbf{D} + \mathbf{G})\mathbf{x} = \mathbf{y}^T(\mathbf{I} + \tilde{\mathbf{G}})\mathbf{y}$. Therefore, φ is strictly concave if and only if $\mathbf{I} + \tilde{\mathbf{G}}$ is positive definite. Next, observe that $\varphi(x) = \frac{1}{2}\sum_i x_i\overline{x}_i/\delta_i$ for any equilibrium $\mathbf{x}$. Suppose first that δ_i does not depend on $\mathbf{G}$. $\forall i, \overline{x}_i x_i - \frac{1}{2}x_i^2 \geq 0$, hence $\varphi(\mathbf{x}; \delta', \mathbf{G}) \leq \varphi(\mathbf{x}; \delta, \mathbf{G})$ for any $\mathbf{x}$ if $\delta' \geq \delta$. Then an increase in δ_i, in $\mathbf{G}$ or a decrease in $\overline{x}_i$ leads to a decrease in the highest level of weighted aggregate action $\sum_i x_i\overline{x}_i/\delta_i$ in equilibrium. Suppose next that $\delta_i = \delta/k_i$. Now adding links to $\mathbf{G}$ leads to an increase in the highest equilibrium level of $\sum_i k_i\overline{x}_i x_i$. To see why, consider i and j such that $g_{ij} = 0$ and $\mathbf{G}' = \mathbf{G} + ij$. We have: $\varphi(\mathbf{x}; \delta, \mathbf{G}') - \varphi(\mathbf{x}; \delta, \mathbf{G}) = \overline{x}_i x_i + \overline{x}_j x_j - \frac{1}{2}x_i^2 - \frac{1}{2}x_j^2 - \delta x_i x_j \geq \overline{x}_i x_i + \overline{x}_j x_j - \frac{1}{2}x_i^2 - \frac{1}{2}x_j^2 - x_i x_j \geq 0$ if $x_i \leq \overline{x}_i$ and $x_j \leq \overline{x}_j$. Thus, $\varphi(\mathbf{x}; \delta, \mathbf{G}') \geq \varphi(\mathbf{x}; \delta, \mathbf{G})$ for any $\mathbf{x}$.

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