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On eigenvalue bounds of two classes of two-by-two block indefinite matrices

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ABSTRACT

Based on Theorem 2.1 of [Z.-Z. Bai, M.K. Ng, Z.-Q. Wang, Constraint preconditioners for symmetric indefinite matrices, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 410–433] we present the eigenvalue bounds of two classes of two-by-two block indefinite matrices. Our results extend the existing ones about two-by-two block nonsingular and symmetric indefinite matrices with symmetric positive definite (1, 1) block. Numerical examples confirm that our estimated eigenvalue bounds are sharp and effective in practice.

1. Introduction

Given matrices $A \in \mathbb{R}^{p\times p}, E \in \mathbb{R}^{p\times (n-p)}, F \in \mathbb{R}^{(n-p)\times p}$ and $C \in \mathbb{R}^{(n-p)\times (n-p)}$, we consider the structured system of linear equations

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} A & E \\ F & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{1.1}$$

Here we are interested in two classes of linear systems:

- **Case (i):** In (1.1) A is nonsingular and symmetric indefinite, C is symmetric, E is of full rank, $F = E^T$, and the Schur complement $S = C - E^T A^{-1} E$ is nonsingular.
- **Case (ii):** In (1.1) A is symmetric positive definite, and the Schur complement $\widetilde{S} = C - FA^{-1}E$ is nonsingular and symmetric.

These hypotheses imply that matrices $\mathcal{A}$ in Cases (i) and (ii) are both nonsingular; see [2].

Linear systems of Cases (i) and (ii) arise frequently in many scientific and engineering applications, including constrained optimization, mixed finite element approximation of elliptic partial differential equations, circuit analysis, weighted and equality constrained least squares estimation, and so forth; see [1,3,6–9,11–13,15]. In recent years, a large amount of work has been done on evaluating the eigenvalue bounds of the coefficient matrices of linear systems of the form (1.1). The analysis of this behavior has attracted many authors' attentions; see [1,3,4,9,14].

The aim of this paper is to estimate the eigenvalue bounds of the coefficient matrices $\mathcal{A}$ in Cases (i) and (ii), respectively. Based on Theorem 2.1 of [5] we present the eigenvalue bounds of two classes of two-by-two block indefinite matrices. The examples we adopt in the paper are about constraint quadratic programming problems. The first example shows the sharpness of the eigenvalue bounds estimated by our new Theorem 2.1. Also, the last example illustrates that the eigenvalue bounds given by our new Theorem 3.1 can sharply estimate the real parts of exact eigenvalue intervals. Therefore, these two numerical examples both confirm the effectiveness of our new results in practical applications.

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The rest of this paper is organized as follows. In Section 2 we give eigenvalue bounds of the coefficient matrix $\mathcal{A}$ in Case (i). Eigenvalue bounds of the coefficient matrix $\mathcal{A}$ in Case (ii) are presented in Section 3. Finally, in Section 4 we give numerical examples to illustrate the sharpness of our estimated eigenvalue bounds.

2. Eigenvalue bounds for $\mathcal{A}$ for Case (i)

In this section we derive eigenvalue bounds for the matrix $\mathcal{A}$ in Case (i). Since the matrix $\mathcal{A}$ in Case (i) is symmetric, its eigenvalues are all real. We are able to obtain accurate bounds for its negative and positive eigenvalues as follows. Throughout this paper we denote $sp(\cdot)$ by the spectrum of a matrix.

Theorem 2.1. *Let $\mathcal{A} \in \mathbb{R}^{n\times n}$ be a nonsingular matrix of the block form in Case (i). Assume that*

$$\begin{aligned}
&sp(A) \subseteq [\delta_1, \Delta_1] \cup [-\delta_2, -\Delta_2](\delta_1, \Delta_1, \delta_2, \Delta_2 > 0),\\
&sp(E^T A^{-1} E) \subseteq [-\Omega_1, -\omega_1] \cup [\omega_2, \Omega_2](\Omega_1, \Omega_2, \omega_1, \omega_2 > 0),\\
&sp(S) \subseteq [-\Theta, -\theta] \cup [\gamma, \Gamma](\Theta, \theta, \gamma, \Gamma > 0).
\end{aligned}$$

Then $sp(\mathcal{A}) \subseteq \mathcal{I}_+ \cup \mathcal{I}_-$, with

$$\begin{aligned}
&\mathcal{I}_+ = \left[\vartheta, \frac{1}{2}\left(\tilde{\Omega} + \Delta + \Gamma + \sqrt{(\tilde{\Omega} + \Delta + \Gamma)^2 - 4\Delta\Gamma}\right)\right];\\
&\mathcal{I}_- = \left[\frac{1}{2}\left(\tilde{\omega} + \Delta - \tilde{\Theta} - \sqrt{(\tilde{\omega} + \Delta - \tilde{\Theta})^2 + 4\tilde{\Theta}\Delta}\right) - 2\Delta_2, \varrho\right],
\end{aligned}$$

where

$$\begin{aligned}
&\delta = \min\{\delta_1, \delta_2\}, \quad \Delta = \max\{\Delta_1, \Delta_2\}, \quad \tilde{\omega} = \max\{0, \omega_2\}, \quad \tilde{\Omega} = \Omega + \frac{2\|E\|_2^2}{\Delta_2}, \quad \tilde{\Theta} = \Theta + \frac{2\|E\|_2^2}{\Delta_2},\\
&\vartheta^{-1} = \frac{\delta + \gamma + \tilde{\Omega} + \sqrt{(\delta + \gamma + \tilde{\Omega})^2 - 4\delta\gamma}}{2\delta\gamma} + \frac{\sqrt{2}(2 + \sqrt{2})}{\delta} + \sqrt{2}\max\left\{\frac{1}{\theta}, \frac{1}{\gamma}\right\}\left(\frac{(2 + \sqrt{2})\tilde{\Omega} + 2\sqrt{\tilde{\Omega}\delta}}{\delta}\right),\\
&\varrho^{-1} = \frac{\delta - \theta + \tilde{\Omega} + \sqrt{(\delta - \theta + \tilde{\Omega})^2 + 4\delta\theta}}{-2\delta\theta} - \frac{\sqrt{2}(2 + \sqrt{2})}{\delta} - \sqrt{2}\max\left\{\frac{1}{\theta}, \frac{1}{\gamma}\right\}\left(\frac{(2 + \sqrt{2})\tilde{\Omega} + 2\sqrt{\tilde{\Omega}\delta}}{\delta}\right).
\end{aligned}$$

Proof. We let the spectral decomposition of A have the form

$$A = U^T \Sigma U,$$

where $U \in \mathbb{R}^{p\times p}$ is an orthogonal matrix and $\Sigma = diag(\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^{p\times p}$ is a diagonal matrix, with $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_p$ being nonzero eigenvalues of A. Without loss of generality, we assume that A has k $(k = 1, \ldots, p-1)$ negative eigenvalues. For the convenience of our discussion, we define matrices

$$\begin{aligned}
&A_1 = U^T \Sigma_1 U, \quad K_1 = U^T \tilde{\Sigma}_1 U, \quad K_2 = U^T \tilde{\Sigma}_2 U,\\
&D = \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad L = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{E^T A_1^{-1}} & \mathbf{I} \end{pmatrix}, \quad T = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix},\\
&\mathcal{A}_1 = \begin{pmatrix} K_1 & E \\ E^T & C \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} K_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
&\Sigma_1 = diag\left(\sqrt{|\lambda_1|}i, \ldots, \sqrt{|\lambda_k|}i, \sqrt{\lambda_{k+1}}, \ldots, \sqrt{\lambda_p}\right),\\
&\tilde{\Sigma}_1 = diag(|\lambda_1|, \ldots, |\lambda_k|, \lambda_{k+1}, \ldots, \lambda_p),\\
&\tilde{\Sigma}_2 = diag(\lambda_1 - |\lambda_1|, \ldots, \lambda_k - |\lambda_k|, 0, \ldots, 0).
\end{aligned}$$

Then it holds that

$$E^T K_1^{-1} E = E^T A^{-1} E + E^T U^T \tilde{\Sigma}_3 U E, \tag{2.1}$$

$$\mathcal{A} = DLTL^T D = \mathcal{A}_1 + \mathcal{A}_2, \tag{2.2}$$

where

$$\tilde{\Sigma}_3 = diag\left(\frac{1}{|\lambda_1|} - \frac{1}{\lambda_1}, \ldots, \frac{1}{|\lambda_k|} - \frac{1}{\lambda_k}, 0, \ldots, 0\right).$$

Evidently, we have

$$\begin{cases} \lambda_{\max}(\mathcal{A}) \leqslant \lambda_{\max}(\mathcal{A}_1) + \lambda_{\max}(\mathcal{A}_2), \\ \lambda_{\min}(\mathcal{A}) \geqslant \lambda_{\min}(\mathcal{A}_1) + \lambda_{\min}(\mathcal{A}_2). \end{cases} \tag{2.3}$$

In addition, we define

$$\begin{aligned} &A_2 = U^T \Xi U, \quad \widetilde{D} = \begin{pmatrix} A_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad H_1 = \widetilde{D}^{-1} L^{-T} T^{-1} L^{-1} \widetilde{D}^{-1}, \\ &H_2 = \widehat{M} L^{-T} T^{-1} L^{-1} \widehat{M} = \begin{pmatrix} \Pi^2 + \Pi A_1^{-1} E S_1^{-1} E^T A_1^{-1} \Pi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \\ &H_3 = \widetilde{D}^{-1} L^{-T} T^{-1} L^{-1} \widehat{M} = \begin{pmatrix} A_2^{-1} \Pi + A_2^{-1} A_1^{-1} E S_1^{-1} E^T A_1^{-1} \Pi & \mathbf{0} \\ -\mathbf{S_1^{-1} E^T A_1^{-1}} \Pi & \mathbf{0} \end{pmatrix}, \\ &H_4 = \widehat{M} L^{-T} T^{-1} L^{-1} \widetilde{D}^{-1} = H_3^T, \end{aligned}$$

where

$$\Xi = diag\left(\sqrt{|\lambda_1|}, \ldots, \sqrt{|\lambda_k|}, \sqrt{\lambda_{k+1}}, \ldots, \sqrt{\lambda_p}\right), \quad \widehat{M} = \begin{pmatrix} \Pi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

with

$$\Pi = (-i-1) U^T diag\left(\frac{1}{\sqrt{|\lambda_1|}}, \ldots, \frac{1}{\sqrt{|\lambda_k|}}, 0, \ldots, 0\right) U.$$

By (2.2) and some direct computations we obtain

$$\mathcal{A}^{-1} = D^{-1} L^{-T} T^{-1} L^{-1} D^{-1} = (\widetilde{D}^{-1} + \widehat{M}) L^{-T} T^{-1} L^{-1} (\widetilde{D}^{-1} + \widehat{M}) = H_1 + H_2 + H_3 + H_4,$$

which implies

$$\begin{cases} \lambda_{\max}(\mathcal{A}^{-1}) \leqslant \lambda_{\max}(H_1) + \lambda_{\max}(H_2) + \lambda_{\max}(H_3 + H_4), \\ \lambda_{\min}(\mathcal{A}^{-1}) \geqslant \lambda_{\min}(H_1) + \lambda_{\min}(H_2) + \lambda_{\min}(H_3 + H_4). \end{cases} \tag{2.4}$$

Based on the assumptions

$$sp(A) \subseteq [\delta_1, \Delta_1] \cup [-\delta_2, -\Delta_2], \quad sp(S) \subseteq [-\Theta, -\theta] \cup [\gamma, \Gamma]$$

and

$$sp(E^T A^{-1} E) \subseteq [-\Omega_1, -\omega_1] \cup [\omega_2, \Omega_2],$$

from (2.1) we can deduce

$$\begin{cases} sp(K_1) \subseteq [\delta, \Delta], \\ sp(E^T K_1^{-1} E) \subseteq [\tilde{\omega}, \tilde{\Omega}], \\ sp(C - E^T K_1^{-1} E) \subseteq [-\tilde{\Theta}, \Gamma], \end{cases} \tag{2.5}$$

where

$$\delta = \min\{\delta_1, \Delta_2\}, \quad \Delta = \max\{\Delta_1, \delta_2\}, \quad \tilde{\omega} = \max\{0, \omega_2\}$$

and

$$\tilde{\Omega} = \Omega + \frac{2\|E\|_2^2}{\Delta_2}, \quad \tilde{\Theta} = \Theta + \frac{2\|E\|_2^2}{\Delta_2}.$$

Then by the definitions of $\mathcal{A}_1$ and H_1 as well as (2.5) and the approach in [5] we have (see page 413–414 in [5])

$$\begin{cases} \lambda_{\max}(\mathcal{A}_1) \leqslant \frac{1}{2}\left(\tilde{\Omega} + \Delta + \Gamma + \sqrt{(\tilde{\Omega} + \Delta + \Gamma)^2 - 4\Delta\Gamma}\right), \\ \lambda_{\max}(H_1) \leqslant \frac{1}{2}\left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{\tilde{\Omega}}{\delta\gamma} + \sqrt{\left(\frac{1}{\delta} + \frac{1}{\gamma} + \frac{\tilde{\Omega}}{\delta\gamma}\right)^2 - \frac{4}{\delta\gamma}}\right), \end{cases} \tag{2.6}$$

(see page 415–416 in [5])

$$\lambda_{\min}(\mathcal{A}_1) \geqslant \frac{1}{2}\left(\tilde{\omega} + \Delta - \tilde{\Theta} - \sqrt{(\tilde{\omega} + \Delta - \tilde{\Theta})^2 + 4\tilde{\Theta}\Delta}\right) \tag{2.7}$$

and

$$\lambda^{-}_{\max}(H_1^{-1}) \leqslant \frac{1}{2}\left(\tilde{\Omega} + \delta - \theta - \sqrt{(\tilde{\Omega} + \delta - \theta)^2 + 4\theta\delta}\right),$$

i.e.,

$$\lambda_{\min}(H_1) = \frac{1}{\lambda^{-}_{\max}(H_1^{-1})} \geqslant -\frac{1}{2}\left(\frac{\tilde{\Omega}}{\delta\theta} + \frac{1}{\theta} - \frac{1}{\delta} + \sqrt{\left(\frac{\tilde{\Omega}}{\delta\theta} + \frac{1}{\theta} - \frac{1}{\delta}\right)^2 + \frac{4}{\delta\theta}}\right), \tag{2.8}$$

where $\lambda^{-}_{\max}(\cdot)$ denotes the largest negative eigenvalue of a matrix. In terms of the definition of the matrix $\mathcal{A}_2$, we can obtain

$$\lambda_{\max}(\mathcal{A}_2) \leqslant 0, \quad \lambda_{\min}(\mathcal{A}_2) \geqslant -2\delta_2.$$

By the definitions of matrices H_2, H_3 and H_4, some straightforward computations lead to

$$\begin{cases} \lambda_{\max}(H_3 + H_4) \leqslant 2\sigma_{\max}(H_3) \leqslant 2\sqrt{2}\left(\frac{1}{\delta} + \max\left\{\frac{1}{\theta}, \frac{1}{\gamma}\right\}\left(\frac{\tilde{\Omega}}{\delta} + \frac{\sqrt{\tilde{\Omega}}}{\sqrt{\delta}}\right)\right), \\ \lambda_{\min}(H_3 + H_4) \geqslant -2\sigma_{\max}(H_3) \geqslant -2\sqrt{2}\left(\frac{1}{\delta} + \max\left\{\frac{1}{\theta}, \frac{1}{\gamma}\right\}\left(\frac{\tilde{\Omega}}{\delta} + \frac{\sqrt{\tilde{\Omega}}}{\sqrt{\delta}}\right)\right), \\ \lambda_{\max}(H_2) \leqslant \sigma_{\max}(H_2) \leqslant \frac{2}{\delta}\left(1 + \tilde{\Omega}\max\left\{\frac{1}{\theta}, \frac{1}{\gamma}\right\}\right), \\ \lambda_{\min}(H_2) \geqslant -\sigma_{\max}(H_2) \geqslant -\frac{2}{\delta}\left(1 + \tilde{\Omega}\max\left\{\frac{1}{\theta}, \frac{1}{\gamma}\right\}\right). \end{cases} \tag{2.9}$$

Therefore, from (2.3), (2.4), (2.6)–(2.9) we have

$$\lambda_{\max}(\mathcal{A}) \leqslant \frac{1}{2}\left(\tilde{\Omega} + \Delta + \Gamma + \sqrt{(\tilde{\Omega} + \Delta + \Gamma)^2 - 4\Delta\Gamma}\right),$$

$$\lambda_{\max}(\mathcal{A}^{-1}) \leqslant \frac{\delta + \gamma + \tilde{\Omega} + \sqrt{(\delta + \gamma + \tilde{\Omega})^2 - 4\delta\gamma}}{2\delta\gamma} + \sqrt{2}\max\left\{\frac{1}{\theta}, \frac{1}{\gamma}\right\}\left(\frac{(2+\sqrt{2})\tilde{\Omega} + 2\sqrt{\tilde{\Omega}\delta}}{\delta}\right) + \frac{\sqrt{2}(2+\sqrt{2})}{\delta}$$

and

$$\lambda_{\min}(\mathcal{A}) \geqslant \frac{1}{2}\left(\tilde{\omega} + \Delta - \tilde{\Theta} - \sqrt{(\tilde{\omega} + \Delta - \tilde{\Theta})^2 + 4\tilde{\Theta}\Delta}\right) - 2\delta_2,$$

$$\lambda_{\min}(\mathcal{A}^{-1}) \geqslant \frac{\delta - \theta + \tilde{\Omega} + \sqrt{(\delta - \theta + \tilde{\Omega})^2 + 4\delta\theta}}{-2\delta\theta} - \sqrt{2}\max\left\{\frac{1}{\theta}, \frac{1}{\gamma}\right\}\left(\frac{(2+\sqrt{2})\tilde{\Omega} + 2\sqrt{\tilde{\Omega}\delta}}{\delta}\right) - \frac{\sqrt{2}(2+\sqrt{2})}{\delta}.$$

This shows $sp(\mathcal{A}) \subseteq \mathcal{I}_+ \cup \mathcal{I}_-$. $\square$

Remark 2.1. By Theorem 2.1 we can derive the spectral bounds of the matrix block $\mathcal{A}$ in order to know the eigenvalue distribution of the matrix block $\mathcal{A}$. Spectral information for the matrix blocks A and C, and the largest singular value of E can be practically available. From Theorem 2.1 we see that the eigenvalue bounds of the matrix $\mathcal{A}$ in Case (i) need spectral knowledge on the matrix $E^TA^{-1}E$ and the Schur complement S. Much work on eigenvalue bounds of Schur complement has been discussed, for example, we refer to [18,19]. In the following we will give some simple examples to estimate the eigenvalue bounds given in Theorem 2.1.

Example 2.1. Consider Case (i) with the matrix $\mathcal{A}$ being chosen as follows:

$$A = \begin{pmatrix} -10 & 0 \\ 0 & 10 \end{pmatrix}, \quad C = \begin{pmatrix} -0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad E = \begin{pmatrix} 0.1 & 0.05 \\ 0 & 0.1 \end{pmatrix}.$$

Then we can easily compute the eigenvalue bounds of the matrix $\mathcal{A}$ as shown in Table 2.1.

Table 2.1
The eigenvalue bounds for the matrix $\mathcal{A}$.

Case (i)	Example 2.1
$sp(A)$	$-10, 10$
$sp(E^TA^{-1}E)$	$-0.0011, 0.0009$
$sp(C - E^TA^{-1}E)$	$-0.0990, 0.0993$
$\mathcal{I}_+ \cup \mathcal{I}_-$	$[-14.2411, -0.0918] \cup [0.0920, 10.0042]$
$sp(\mathcal{A})$	$-10.0013, -0.0990, 0.0992, 10.001$

From Table 2.1 we see that the estimated eigenvalue bounds by making use of Theorem 2.1 well approximate to the exact values. Both negative and positive eigenvalue intervals are approximated sharply.

Remark 2.2. In [2] the author gave the eigenvalue bounds for the nonsingular saddle point matrices of Hermitian and indefinite $(1,1)$ and $(2,2)$ blocks without imposing the restrictions that the $(1,1)$ blocks are positive definite on the kernels of the $(2,1)$ blocks. The Case (i) in this paper is similar to that considered in [2]. However, we have used different method to derive the eigenvalue bounds of matrix $\mathcal{A}$.

Example 2.2. In the following we compute the eigenvalue bounds of the matrix given by Examples 3.1 and 3.2 in [2].

From Table 2.2 we see that the upper bound of negative eigenvalues and the lower bound of positive eigenvalues are sharper than the ones in [2].

3. Eigenvalue bounds for $\mathcal{A}$ for Case (ii)

In this section we derive eigenvalue bounds for the matrix $\mathcal{A}$ in Case (ii). We can give accurate bounds on the positive real part, the negative real part, and the imaginary part of its eigenvalues. Throughout this subsection we denote the positive real part of the spectrum of a matrix by $\mathcal{I}^{(Re)}(sp_+(\cdot))$, the negative real part by $\mathcal{I}^{(Re)}(sp_-(\cdot))$, and the imaginary part by $\mathcal{I}^{(Im)}(sp(\cdot))$. Firstly, we give the following useful lemma.

Lemma 3.1 [10]. *Let $W \in \mathbb{R}^{n\times n}$ and $\mathcal{H}(W), \mathcal{S}(W), \lambda(W)$ be the Hermitian part, the skew-Hermitian part, and the arbitrary eigenvalue of the matrix W, respectively. Then*

$$\lambda_{\min}(\mathcal{H}(W)) \leqslant \mathcal{I}^{(Re)}(\lambda(W)) \leqslant \lambda_{\max}(\mathcal{H}(W))$$

and

$$\lambda_{\min}(\mathcal{S}(W)) \leqslant \mathcal{I}^{(Im)}(\lambda(W)) \leqslant \lambda_{\max}(\mathcal{S}(W)).$$

Theorem 3.1. *Let $\mathcal{A} \in \mathbb{R}^{n\times n}$ be a nonsingular matrix of the block form in Case (ii). Assume that*

$$sp(A) \subseteq [\delta, \Delta](\delta, \Delta > 0), \quad sp(\widetilde{S}) \subseteq [-\tilde{\Theta}, -\tilde{\theta}] \cup [\tilde{\gamma}, \tilde{\Gamma}] \quad (\tilde{\Theta}, \tilde{\theta}, \tilde{\Gamma}, \tilde{\gamma} > 0),$$
$$sp\left[\frac{1}{4}(E^T + F)A^{-1}(E + F^T)\right] \subseteq [\omega, \Omega] \quad (\omega, \Omega > 0),$$
$$sp(S_1) = sp\left[\frac{1}{2}(C + C^T) - \frac{1}{4}(E^T + F)A^{-1}(E + F^T)\right] \subseteq [-\Theta, -\theta] \cup [\gamma, \Gamma] \quad (\Theta, \theta, \Gamma, \gamma \geqslant 0).$$

If

$$\min\left\{\frac{1}{\Phi_1^2 + \eta^2}, \frac{1}{\Phi_2^2 + \eta^2}\right\} > \frac{1}{4}(\|E - F^T\|_2 + \|C - C^T\|_2)^2,$$

Table 2.2
The eigenvalue bounds of matrix $\mathcal{A}$.

Case (i)	$(\alpha, \beta) = (2, 1)(\mu, m) = (2, 1)$	$(\alpha, \beta) = (\frac{1}{9}, \frac{1}{6})(\mu, m) = (4, 2)$
$\mathcal{I}_-$	$[-1.0510, -0.2468]$	$[-0.4082, -1.7881E - 7]$
$\mathcal{I}_+$	$[0.7306, 4.1090]$	$[8.6396E - 8, 0.3333]$
$\mathcal{I}_-^{[2]}$	$[-1.0452, -0.2429]]$	$[-0.4085, -1.7868E - 7]$
$\mathcal{I}_+^{[2]}$	$[0.7139, 4.1807]$	$[8.6333E - 8, 0.3336]$
$sp_-(\mathcal{A})$	$[-1.0000, -0.2537]$	$[-0.4083, -1.7870E - 7]$
$sp_+(\mathcal{A})$	$[0.7452, 4.0085]$	$[8.6380E - 8, 0.3334]$

then

$$\mathcal{I}^{(Im)}(sp(\mathcal{A}))\subseteq\left[-\frac{1}{2}\left(\|E-F^T\|_2+\|C-C^T\|_2\right),\frac{1}{2}(\|E-F^T\|_2+\|C-C^T\|_2)\right],$$
$$\mathcal{I}^{(Re)}(sp_+(\mathcal{A}))\subseteq\left[\left(\frac{1}{\Phi_1^2+\eta^2}-\frac{1}{4}\left(\|E-F^T\|_2+\|C-C^T\|_2\right)^2\right)^{\frac{1}{2}},\hat{\lambda}\right],$$
$$\mathcal{I}^{(Re)}(sp_-(\mathcal{A}))\subseteq\left[\tilde{\lambda},-\left(\frac{1}{\Phi_2^2+\eta^2}-\frac{1}{4}\left(\|E-F^T\|_2+\|C-C^T\|_2\right)^2\right)^{\frac{1}{2}}\right],$$

where

$$\eta=\delta^{-1}\tilde{\gamma}^{-1}\|E-F^T\|_2\left(\sqrt{\frac{\Omega}{\delta}}+1\right),$$
$$\hat{\lambda}=\frac{1}{2}\left(\Delta+\Gamma+\Omega+\sqrt{(\Delta+\Gamma+\Omega)^2-4\Delta\Gamma}\right),$$
$$\tilde{\lambda}=\frac{1}{2}\left(\omega+\Delta-\Theta-\sqrt{(\omega+\Delta-\Theta)^2+4\Theta\Delta}\right),$$
$$\Phi_1=\frac{1}{2}\left[\sqrt{\left(\frac{1}{\delta}+\frac{1}{\tilde{\gamma}}+\frac{\Omega}{\delta\tilde{\gamma}}\right)^2-\frac{4}{\delta\tilde{\gamma}}}+\frac{1}{\delta}+\frac{1}{\tilde{\gamma}}+\frac{\Omega}{\delta\tilde{\gamma}}\right]-\frac{1}{4\tilde{\theta}}\|E-F^T\|_2^2\delta^{-2},$$
$$\Phi_2=-\frac{1}{2}\left[\frac{1}{\tilde{\theta}}-\frac{1}{\delta}+\sqrt{\left(\frac{\Omega}{\delta\tilde{\theta}}+\frac{1}{\tilde{\theta}}-\frac{1}{\delta}\right)^2+\frac{4}{\delta\tilde{\theta}}}+\frac{\Omega}{\delta\tilde{\theta}}\right]-\frac{1}{4\tilde{\gamma}}\|E-F^T\|_2^2\delta^{-2}.$$

Proof. For the convenience of our discussion we define matrices

$$D=\begin{pmatrix}A^{\frac{1}{2}}&\mathbf{0}\\\mathbf{0}&I\end{pmatrix},\quad T=\begin{pmatrix}I&\mathbf{0}\\\mathbf{0}&\widetilde{S}\end{pmatrix},\quad M=\begin{pmatrix}I&A^{-\frac{1}{2}}E\\\mathbf{0}&I\end{pmatrix},\quad \widetilde{L}=\begin{pmatrix}I&\mathbf{0}\\FA^{-\frac{1}{2}}&I\end{pmatrix},$$

$$J=\begin{pmatrix}2I&-A^{-\frac{1}{2}}(F^T+E)\\\mathbf{0}&2I\end{pmatrix},\quad K=\begin{pmatrix}\mathbf{0}&A^{-\frac{1}{2}}(F^T-E)\\\mathbf{0}&\mathbf{0}\end{pmatrix}.$$

Then by direct computations we get

$$\mathcal{A}=D\widetilde{L}TMD$$

and

$$\mathcal{H}(\mathcal{A}^{-1})=\frac{1}{2}D^{-1}(M^{-1}T^{-1}\widetilde{L}^{-1}+\widetilde{L}^{-T}T^{-1}M^{-T})D^{-1}=\frac{1}{4}D^{-1}(JT^{-1}J^T-KT^{-1}K^T)D^{-1}. \tag{3.1}$$

By straightforward calculations we obtain

$$\begin{cases}\lambda_{\max}\left(-\frac{1}{4}D^{-1}KT^{-1}K^TD^{-1}\right)\leqslant\frac{1}{4}\tilde{\theta}^{-1}\|E-F^T\|_2\delta^{-2},\\ \lambda_{\min}\left(-\frac{1}{4}D^{-1}KT^{-1}K^TD^{-1}\right)\geqslant-\frac{1}{4}\tilde{\gamma}^{-1}\|E-F^T\|_2\delta^{-2}.\end{cases} \tag{3.2}$$

By applying the similar techniques to [5] (see pages 413–414 in [5]) we have

$$\lambda_{\max}\left(\frac{1}{4}D^{-1}JT^{-1}J^TD^{-1}\right)\leqslant\lambda_{\max}\left(\frac{1}{4}D^{-1}J\begin{pmatrix}I&\mathbf{0}\\\mathbf{0}&\frac{1}{\tilde{\gamma}}I\end{pmatrix}J^TD^{-1}\right)\leqslant\frac{1}{2}\left[\sqrt{\left(\frac{1}{\delta}+\frac{1}{\tilde{\gamma}}+\frac{\Omega}{\delta\tilde{\gamma}}\right)^2-\frac{4}{\delta\tilde{\gamma}}}+\frac{1}{\delta}+\frac{1}{\tilde{\gamma}}+\frac{\Omega}{\delta\tilde{\gamma}}\right]. \tag{3.3}$$

It then follows that

$$\lambda_{\min}\left(\frac{1}{4}D^{-1}JT^{-1}J^TD^{-1}\right)\geqslant\lambda_{\min}\left(\frac{1}{4}D^{-1}J\begin{pmatrix}I&\mathbf{0}\\\mathbf{0}&-\frac{1}{\tilde{\theta}}I\end{pmatrix}J^TD^{-1}\right)\geqslant-\frac{1}{2}\left[\sqrt{(\frac{\Omega}{\delta\tilde{\theta}}+\frac{1}{\tilde{\theta}}-\frac{1}{\delta})^2+\frac{4}{\delta\tilde{\theta}}}+\frac{\Omega}{\delta\tilde{\theta}}+\frac{1}{\tilde{\theta}}-\frac{1}{\delta}\right]. \tag{3.4}$$

By Lemma 3.1 and (3.1)–(3.4) we have

$$\mathcal{I}^{(Re)}(\lambda(\mathcal{A}^{-1})) \leqslant \lambda_{\max}(\mathcal{H}(\mathcal{A}^{-1})) \leqslant \frac{1}{2}\left(\sqrt{\left(\frac{1}{\delta}+\frac{1}{\tilde{\gamma}}+\frac{\Omega}{\delta\tilde{\gamma}}\right)^2-\frac{4}{\delta\tilde{\gamma}}}+\frac{1}{\delta}+\frac{1}{\tilde{\gamma}}+\frac{\Omega}{\delta\tilde{\gamma}}\right)-\frac{1}{4\tilde{\theta}}\|E-F^T\|_2^2\delta^{-2} \equiv \Phi_1 \tag{3.5}$$

and

$$\mathcal{I}^{(Re)}(\lambda(\mathcal{A}^{-1})) \geqslant \lambda_{\min}(\mathcal{H}(\mathcal{A})^{-1}) \geqslant -\frac{1}{2}\left(\frac{1}{\tilde{\theta}}-\frac{1}{\delta}+\sqrt{\left(\frac{\Omega}{\delta\tilde{\theta}}+\frac{1}{\tilde{\theta}}-\frac{1}{\delta}\right)^2+\frac{4}{\delta\tilde{\theta}}}+\frac{\Omega}{\delta\tilde{\theta}}\right)-\frac{1}{4\tilde{\gamma}}\|E-F^T\|_2^2\delta^{-2} \equiv \Phi_2. \tag{3.6}$$

We now consider the bounds for $\mathcal{I}m(sp(\mathcal{A}))$ and $\mathcal{I}m(\lambda(\mathcal{A}^{-1}))$. Since

$$\mathcal{S}(\mathcal{A}) = \frac{1}{2i}\begin{pmatrix} \mathbf{0} & \mathbf{E}-\mathbf{F}^{\mathbf{T}} \\ \mathbf{F}-\mathbf{E}^{\mathbf{T}} & \mathbf{C}-\mathbf{C}^{\mathbf{T}} \end{pmatrix},$$

by making use of Lemma 3.1 we obtain

$$\mathcal{I}^{(Im)}(sp(\mathcal{A})) \subseteq \left[-\frac{1}{2}\left(\|E-F^T\|_2+\|C-C^T\|_2\right), \frac{1}{2}\left(\|E-F^T\|_2+\|C-C^T\|_2\right)\right]. \tag{3.7}$$

Some direct computations lead to

$$\mathcal{S}(\mathcal{A}^{-1}) = \frac{1}{2i}\left(\mathcal{A}^{-1}-\mathcal{A}^{-T}\right) = \frac{1}{2i}\begin{pmatrix} Q_1 & Q_2 \\ -Q_2^T & \mathbf{0} \end{pmatrix},$$

where

$$Q_1 = A^{-1}\left(E\widetilde{S}^{-1}F - F^T\widetilde{S}^{-1}E^T\right)A^{-1} = \frac{1}{2}A^{-1}\left[(E-F^T)\widetilde{S}^{-1}(F+E^T)-(E+F^T)\widetilde{S}^{-1}(E^T-F)\right]A^{-1}$$

and

$$Q_2 = A^{-1}(F^T-E)\widetilde{S}^{-1}.$$

We can verify that

$$\sigma_{\max}(Q_1) \leqslant \sqrt{\Omega}\tilde{\gamma}^{-1}\delta^{-\frac{3}{2}}\|E-F^T\|_2, \quad \sigma_{\max}(Q_2) \leqslant \tilde{\gamma}^{-1}\delta^{-1}\|E-F^T\|_2,$$

where $\sigma_{\max}(\cdot)$ denotes the largest singular value of a matrix. From Lemma 3.1, after simple computations we have

$$\mathcal{I}^{(Im)}(\lambda(\mathcal{A}^{-1})) \subseteq [-\eta, \eta], \tag{3.8}$$

where

$$\eta = \tilde{\gamma}^{-1}\delta^{-1}\|E-F^T\|_2\left(\sqrt{\frac{\Omega}{\delta}}+1\right).$$

Based on the assumptions

$$sp(A) \subseteq [\delta, \Delta], \quad sp\left[\frac{(E^T+F)A^{-1}(E+F^T)}{4}\right] \subseteq [\omega, \Omega], \quad sp(S_1) \subseteq [-\Theta, -\theta] \cup [\gamma, \Gamma],$$

following the same approach as in [5], we can deduce (see page 413–414 in [5])

$$\mathcal{I}^{(Re)}(sp_+(\mathcal{A})) \leqslant \lambda_{\max}(\mathcal{H}(\mathcal{A})) = \lambda_{\max}\begin{pmatrix} A & \frac{F^T+E}{2} \\ \frac{E^T+F}{2} & \frac{C^T+C}{2} \end{pmatrix} \leqslant \frac{1}{2}\left(\Delta+\Gamma+\Omega+\sqrt{(\Delta+\Gamma+\Omega)^2-4\Delta\Gamma}\right), \tag{3.9}$$

(see page 415–416 in [5])

$$\mathcal{I}^{(Re)}(sp_-(\mathcal{A})) \geqslant \lambda_{\min}(\mathcal{H}(\mathcal{A})) = \lambda_{\min}\begin{pmatrix} A & \frac{F^T+E}{2} \\ \frac{E^T+F}{2} & \frac{C^T+C}{2} \end{pmatrix} \geqslant \frac{1}{2}\left(\omega+\Delta-\Theta-\sqrt{(\omega+\Delta-\Theta)^2+4\Theta\Delta}\right). \tag{3.10}$$

Let any eigenpair of the matrix $\mathcal{A}$ be (μ, x) and $(\tilde{\mu}, x)$ be the corresponding eigenpair of $\mathcal{A}^{-1}$. We will consider the case about the positive real eigenvalue bounds for $\mathcal{A}$, i.e., $\mathcal{I}^{(Re)}(\mu) > 0$. Then it holds that

$$\begin{cases} \mathcal{I}^{(Re)}(\tilde{\mu}) = \frac{\mathcal{I}^{(Re)}(\mu)}{\mathcal{I}^{(Re)2}(\mu)+\mathcal{I}^{(Im)2}(\mu)}, \\ (\mathcal{I}^{(Re)2}(\mu)+\mathcal{I}^{(Im)2}(\mu))(\mathcal{I}^{(Re)2}(\tilde{\mu})+\mathcal{I}^{(Im)2}(\tilde{\mu})) = 1. \end{cases} \tag{3.11}$$

From (3.11) we can easily obtain $\mathcal{I}^{(Re)}(\tilde{\mu}) > 0$. Combining (3.5), (3.7), (3.8) and (3.11) we have

$$\left(\mathcal{I}^{Re2}(\mu) + \frac{1}{4}\left(\|E - F^T\|_2 + \|C - C^T\|_2\right)^2\right)\left(\Phi_1^2 + \eta^2\right) \geqslant 1.$$

Straightforward computations imply that if

$$\frac{1}{\Phi_1^2 + \eta^2} > \frac{1}{4}\left(\|E - F^T\|_2 + \|C - C^T\|_2\right)^2,$$

then

$$\mathcal{I}^{(Re)}(\mu) \geqslant \left(\frac{1}{\Phi_1^2 + \eta^2} - \frac{1}{4}\left(\|E - F^T\|_2 + \|C - C^T\|_2\right)^2\right)^{\frac{1}{2}},$$

i.e.,

$$\mathcal{I}^{(Re)}(sp_+(\mathcal{A})) \geqslant \left(\frac{1}{\Phi_1^2 + \eta^2} - \frac{1}{4}\left(\|E - F^T\|_2 + \|C - C^T\|_2\right)^2\right)^{\frac{1}{2}},$$

which combines with (3.9) leads to the upper and lower bounds for $\mathcal{I}^{(Re)}(sp_+(\mathcal{A}))$.

We now discuss the case about the negative real eigenvalue bounds for $\mathcal{A}$, i.e., $\mathcal{I}^{(Re)}(\mu) < 0$. Then by (3.11) we have $\mathcal{I}^{(Re)}(\tilde{\mu}) < 0$. By (3.6), (3.7), (3.8) and (3.11) and simple computations we can derive if

$$\frac{1}{\Phi_2^2 + \eta^2} > \frac{1}{4}\left(\|E - F^T\|_2 + \|C - C^T\|_2\right)^2,$$

then

$$\mathcal{I}^{(Re)}(\mu) \leqslant -\left(\frac{1}{\Phi_2^2 + \eta^2} - \frac{1}{4}\left(\|E - F^T\|_2 + \|C - C^T\|_2\right)^2\right)^{\frac{1}{2}},$$

i.e.,

$$\mathcal{I}^{(Re)}(sp_-(\mathcal{A})) \leqslant -\left(\frac{1}{\Phi_2^2 + \eta^2} - \frac{1}{4}(\|E - F^T\|_2 + \|C - C^T\|_2)^2\right)^{\frac{1}{2}},$$

which, together with (3.10), immediately leads to the upper and lower bounds for $\mathcal{I}^{(Re)}(sp_-(\mathcal{A}))$. $\square$

Remark 3.1. Special case $F = E^T$ and $C^T = C$ in Theorem 3.1 is considered. We know that now $\Theta = \tilde{\Theta}, \theta = \tilde{\theta}, \gamma = \tilde{\gamma}, \Gamma = \tilde{\Gamma}, \eta = 0$. It is easy to check that

$$\frac{1}{\Phi_1^2 + \eta^2} = \frac{1}{\Phi_1^2} > 0 = \frac{1}{4}\left(\|E - F^T\|_2 + \|C - C^T\|_2\right)^2$$

and

$$\frac{1}{\Phi_2^2 + \eta^2} = \frac{1}{\Phi_2^2} > 0 = \frac{1}{4}\left(\|E - F^T\|_2 + \|C - C^T\|_2\right)^2,$$

are satisfied. Then by Theorem 3.1 we have

$$\begin{aligned}
&\mathcal{I}^{(Im)}(sp(\mathcal{A})) = \{0\},\\
&\mathcal{I}^{(Re)}(sp_+(\mathcal{A})) \in \left[\frac{1}{2}\left(\Omega + \delta + \gamma - \sqrt{(\Omega + \delta + \gamma)^2 - 4\delta\gamma}\right), \frac{1}{2}\left(\Omega + \Delta + \Gamma\sqrt{(\Omega + \Delta + \Gamma)^2 - 4\Delta\Gamma}\right)\right],\\
&\mathcal{I}^{(Re)}(sp_-(\mathcal{A})) \in \left[\frac{1}{2}\left(\omega + \Delta - \Theta - \sqrt{(\omega + \Delta - \Theta)^2 + 4\Theta\Delta}\right), \frac{1}{2}\left(\Omega + \delta - \theta - \sqrt{(\Omega + \delta - \theta)^2 + 4\delta\theta}\right)\right],
\end{aligned}$$

which are the results in Theorem 2.1 in [5]. This shows that Theorem 3.1 extends Theorem 2.1 in [5].

Remark 3.2. From Theorem 3.1 we can derive the spectral bounds of the matrix $\mathcal{A}$ in order to know its eigenvalue distribution. From Theorem 3.1 we realize that the eigenvalue bounds for the matrix $\mathcal{A}$ in Case (ii) require spectral information about the Schur complements S_1 and $\widetilde{S}$. Much work on eigenvalue bounds of Schur complement has been discussed, for example, we refer to [18,19]. In the following a simple example is given to illustrate the results in Theorem 3.1.

Example 3.1. Consider Case (ii) with the matrix $\mathcal{A}$ being chosen as follows:

$$A = \begin{pmatrix} 0.2 & -0.1 \\ -0.1 & 0.2 \end{pmatrix}, \quad E = \begin{pmatrix} -0.01 & -0.02 \\ 0.01 & 0.01 \end{pmatrix},$$

$$F = \begin{pmatrix} -0.01 & -0.01 \\ -0.01 & 0.02 \end{pmatrix}, \quad C = \begin{pmatrix} -0.1 & 0.02 \\ 0.01 & 0.1 \end{pmatrix}.$$

By computations we have

$$\omega = 0.0006, \quad \Omega = 0.0196, \quad \Theta = \theta = 0.1021, \quad \gamma = \Gamma = 0.0819, \quad \tilde{\theta} = \tilde{\Theta} = 0.1013, \quad \tilde{\gamma} = \tilde{\Gamma} = 0.0913.$$

It is easy to check that

$$\frac{1}{\Phi_1^2 + \eta^2} > \frac{1}{4}\left(\|E - F^T\|_2 + \|C - C^T\|_2\right)^2, \quad \frac{1}{\Phi_2^2 + \eta^2} > \frac{1}{4}\left(\|E - F^T\|_2 + \|C - C^T\|_2\right)^2,$$

in Theorem 3.1 are satisfied. By Theorem 3.1 we have the eigenvalue bounds for the matrix $\mathcal{A}$ shown in Table 3.1.

It follows from Table 3.1 that the positive real part and the negative real part of the eigenvalue intervals are approximately sharp.

4. Numerical Experiments

In this section we give two numerical examples about constraint quadratic programming problems to sharply estimate the eigenvalue bounds for the coefficient matrices $\mathcal{A}$ of the linear systems in Cases (i) and (ii), respectively. The first example is of the form

$$\min_{x \in \mathbb{R}^p} \left(g^T x + \frac{1}{2} x^T A x\right),$$

subject to $E^T x = 0$.

Given the data A and $E, C = (c_{i,j})_{n-p,n-p}$ is defined as

$$c_{ij} = \begin{cases} 0, & i = j, \ 1 \leqslant i \leqslant [\frac{(n-p)}{2}], \\ 1, & i = j, \ [\frac{(n-p)}{2}] + 1 \leqslant i \leqslant n - p, \\ 0, & otherwise. \end{cases}$$

Here the block matrix

$$\begin{pmatrix} A & E \\ E^T & C \end{pmatrix}$$

has the same structure as in Case (i). For more details about choices of the matrix blocks, we can refer to [16,17]. Our aim is to estimate the eigenvalue bounds of the block matrix mentioned above by Theorem 2.1.

For different choices of p and n the numerical results are listed in Tables 4.1, 4.2. From Table 4.1, Table 4.2 we see that the eigenvalue bounds given by Theorem 2.1 estimate the exact spectral intervals sharply.

Our second example is described as follows. The nonlinear programming problem of the form

$$\min_{x \in \mathbb{R}^p} f(x), \tag{4.1}$$

subject to $c(x) = 0$,

where $f : \mathbb{R}^p \to \mathbb{R}$ and $c : \mathbb{R}^p \to \mathbb{R}^{n-p}$ are twice-continuously differentiable. Given a suitable constraint qualification, an optimal solution x of (4.1), together with a Lagrange multiplier vector $v \in \mathbb{R}^{n-p}$, has to satisfy the first-order necessary optimality conditions associated with (4.1). These conditions may be written in the form

Table 3.1
The eigenvalue bounds for the matrix $\mathcal{A}$.

$\mathcal{I}^{(Re)}(sp_+(\mathcal{A}))$	$[0.0951, 0.3286]$
$\mathcal{I}^{(Re)}(sp_-(\mathcal{A}))$	$[-0.1432, -0.0843]$
$\mathcal{I}^{(Im)}(sp(\mathcal{A}))$	$[-0.1307, 0.1307]$
$sp(\mathcal{A})$	$-0.1009, 0.3023, 0.0993 + 0.0067i, 0.0993 - 0.0067i$

Table 4.1
The eigenvalue bounds for the matrix $\mathcal{A}$ for Case (i).

p, n	$p = 256, n = 480$	$p = 512, n = 960$
$\mathcal{I}(sp_-(\mathcal{A}))$	$[-50.5246, -0.0905]$	$[-3.2417, -0.0687]$
$\cup\mathcal{I}(sp_+(\mathcal{A}))$	$\cup[0.0389, 26.6648]$	$\cup[0.0416, 4.0435]$
$sp(\mathcal{A})$	$[-48.2071, -0.0929]$	$[-3.1607, -0.0809]$
	$\cup[0.0472, 26.5825]$	$\cup[0.0450, 3.7118]$

Table 4.2
The eigenvalue bounds for the matrix $\mathcal{A}$ for Case (i).

p, n	$p = 768, n = 1440$	$p = 1024, n = 1920$
$\mathcal{I}(sp_-(\mathcal{A}))$	$[-8.4075, -2.1350]$	$[-10.2491, -4.0648]$
$\cup\mathcal{I}(sp_+(\mathcal{A}))$	$\cup[0.0953, 3.3940]$	$\cup[0.9304, 13.5192]$
$sp(\mathcal{A})$	$[-7.4483, -2.2027]$	$[-10.1632, -5.3026]$
	$\cup[0.1061, 3.1739]$	$\cup[1.1907, 11.6884]$

Table 4.3
The eigenvalue bounds for the matrix $\mathcal{A}$ for Case (ii).

p, n	$p = 64, n = 120$	$p = 128, n = 240$
$\mathcal{I}^{(Re)}(sp_-(\mathcal{A}))$	$[-3.7341, -0.0035]$	$[-6.4700, -0.7146]$
$\cup\mathcal{I}^{(Re)}(sp_+(\mathcal{A}))$	$\cup[0.3675, 9.7380]$	$\cup[1.2561, 3.6490]$
$\mathcal{I}^{(Im)}(sp(\mathcal{A}))$	$[-1.9518, 1.9518]$	$[-7.1839, 7.1839]$
$\mathcal{I}^{(Re)}_{exact}(sp_-(\mathcal{A}))$	$[-3.4834, -0.0059]$	$[-6.3173, -1.1042]$
$\cup\mathcal{I}^{(Re)}_{exact}(sp_+(\mathcal{A}))$	$\cup[0.4410, 9.6936]$	$\cup[1.5402, 3.4734]$
$\mathcal{I}^{(Im)}(sp(\mathcal{A}))$	$[-0.8509, 1.2037]$	$[-3.4115, 7.0638]$

Table 4.4
The eigenvalue bounds for the matrix $\mathcal{A}$ for Case (ii).

p, n	$p = 256, n = 480$	$p = 512, n = 960$
$\mathcal{I}^{(Re)}(sp_-(\mathcal{A}))$	$[-4.9946, -0.0140]$	$[-17.8213, -2.2554]$
$\cup\mathcal{I}^{(Re)}(sp_+(\mathcal{A}))$	$\cup[0.0042, 12.3180]$	$\cup[0.1061, 1.8246]$
$\mathcal{I}^{(Im)}(sp(\mathcal{A}))$	$[-0.0764, 0.0764]$	$[-6.8504, 6.8504]$
$\mathcal{I}^{(Re)}_{exact}(sp_-(\mathcal{A}))$	$[-4.8507, -0.0366]$	$[-17.6140, -2.4611]$
$\cup\mathcal{I}^{(Re)}_{exact}(sp_+(\mathcal{A}))$	$\cup[0.0091, 11.7039]$	$\cup[0.1402, 0.9881]$
$\mathcal{I}^{(Im)}_{exact}(sp(\mathcal{A}))$	$[-0.0001, 0.0705]$	$[-3.5120, 6.4429]$

Table 4.5
The eigenvalue bounds for the matrix $\mathcal{A}$ for Case (ii).

p, n	$p = 1024, n = 1920$	$p = 2048, n = 3840$
$\mathcal{I}^{(Re)}(sp_-(\mathcal{A}))$	$[-11.8025, -0.5470]$	$[-42.7051, -3.4050]$
$\cup\mathcal{I}^{(Re)}(sp_+(\mathcal{A}))$	$\cup[2.1248, 10.0816]$	$\cup[14.8642, 21.1805]$
$\mathcal{I}^{(Im)}(sp(\mathcal{A}))$	$[-0.8041, 0.8041]$	$[-5.8270, 5.8270]$
$\mathcal{I}^{(Re)}_{exact}(sp_-(\mathcal{A}))$	$[-11.7638, -0.6011]$	$[-40.0781, -4.1125]$
$\cup\mathcal{I}^{(Re)}_{exact}(sp_+(\mathcal{A}))$	$\cup[2.1635, 9.8742]$	$\cup[14.9006, 20.8124]$
$\mathcal{I}^{(Im)}_{exact}(sp(\mathcal{A}))$	$[-0.1240, 0.6953]$	$[-4.9704, 3.7206]$

$$\begin{cases} g(x) - E(x)^T y = 0, \\ c(x) - s = 0, \\ YPe = 0, \quad y \geqslant 0, \ s \geqslant 0, \end{cases}$$

where $g(x) = \nabla f(x)$ and $E(x) = c'(x)$ and e is an $n - p-$ dimensional vector of ones. Here, we denote by upper-case letters Y and P, the diagonal matrices formed by y and s respectively. By Newton's method we get the linear equation and should consider the coefficient matrix of the equation as follows

$$\begin{pmatrix} A(x,y) & \mathbf{0} & -\mathbf{E}(\mathbf{x})^{\mathbf{T}} \\ \mathbf{E}(\mathbf{x}) & -\mathbf{I_0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} & \mathbf{P} \end{pmatrix}, \tag{4.2}$$

where I_0 stands for identity matrix and $A(x,y) = \nabla^2 f(x) - \sum_{i=1}^{n-p} y_i \nabla^2 c_i(x)$.

Here we choose matrices $A(x,y), E(x), Y = -EA^{-1}E^T$ and P to ensure that the block matrix (4.2) has the same structure as in case (ii) and conditions in Theorem 3.1 can be satisfied. For more details about choices of the matrix blocks, we can refer to [16,17]. We will estimate the eigenvalue bounds of the block matrix (4.2) by Theorem 3.1.

Here $\mathcal{I}_{exact}^{(Re)}(sp_+(\mathcal{A})), \mathcal{I}_{exact}^{(Re)}(sp_-(\mathcal{A}))$ and $\mathcal{I}_{exact}^{(Im)}(sp(\mathcal{A}))$ denote the exact values of the positive real part, the negative real part and imaginary part of the spectrum of matrix $\mathcal{A}$, respectively.

The numerical results are listed in Tables 4.3, 4.4, 4.5. It is easy from Tables 4.3, 4.3, 4.3 to see that the eigenvalue bounds given by Theorem 3.1 sharply estimate the real parts of exact eigenvalue intervals.

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