A THEOREM ON EXPECTATIONS AND THE STABILITY OF EQUILIBRIUM

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The relationship between extrapolative expectations and dynamic stability is studied in the context of a multiple market system. Metzler's theorem on the Hicksian stability conditions is extended to include a simple expectations function. It is shown that a stable dynamic system can absorb the effects of some extrapolation of price movements and remain stable.

THE CONDITIONS under which a Walrasian system of multiple markets will be stable have been investigated by a number of authors under the implicit assumption of static expectations, [3], [5], [6], [7]. It often has been assumed that expectations based upon an extrapolation of current rates of change (rather than upon the assumption that the future would be like the present) would prevent the system from converging onto its equilibrium position at all.[2] Since interesting results have been scarce and difficult to achieve even in the case of static expectations, it is not surprising that little has been done with the relationship between extrapolative expectations and dynamic stability. In this paper, we shall introduce, under rather restrictive assumptions, a type of extrapolative expectations and we shall test their effects on the stability of a dynamic system.

Excess demands in a multiple market system are usually taken to be functions of the current prices of all goods. Ideally, it would be desirable also to include expected prices for all future time periods and for all individuals and all assets as arguments of the excess demand functions. A theory of such formal generality, however, would be necessarily devoid of much content. Abstractions and simplifying assumptions are necessary. There are many possible expectations functions by which people might relate current and expected prices, and there is a variety of ways to represent plausibly the type of extrapolative expectations which we wish to describe. Our choice was made largely on the grounds of mathematical simplicity. Though our assumptions are quite special, we believe that the results have more general significance in that they show that a stable dynamic system can have "room" for some extrapolative expectations and that the critical amount of extrapolation, from the point of view of stability, is related to the inertia of the system.

First of all, we assume that for any price, the expectations of all individuals can be represented adequately by those of a "representative individual." This device, traditional in economic theory, can be justified by the argument that whatever amount the market demands, there is some price expectation which, if held by everybody, would result in the same demand. Secondly, we assume away "cross effects" of price expectations on excess demand, i.e., only the expected price for the ith good significantly affects excess demand for that good. Though

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[2] See Hicks, [3], p. 225.

obviously not realistic, this assumption is a natural first approximation to reality.

It is difficult to justify the frequently made assumption that expectations for all future time periods can be represented by one "expected price." For given current prices and given expected prices one month hence, demand can still vary with expectations for subsequent months. Reactions to price changes which are thought to be one-way movements will differ from reactions to changes which are expected to be reversed. This difficulty can be avoided partially by the introduction of the distinction between induced and autonomous expectations of price change. The difference between any expected price and its corresponding current price can be thought of as divisible into two components. One part is, from the point of view of a short-run dynamic model, autonomous. It is the result of knowledge about price histories and causal factors relating to particular prices. A new invention may set up the expectation that a certain price will fall when the invention is applied to production. This sort of expectation can be taken as exogenous. On the other hand, part of the expected price change is induced by actual changes in current prices. These expectations are definitely endogenous to the dynamic system and dynamic stability can be affected by them. The induced component of expected price changes for all future periods is much more likely to have the same sign than is the autonomous part. In the analysis that follows, we assume that induced expectations can be represented by one "expected price" for each good and that autonomous expected price changes are small. We shall also assume that during the time interval under consideration, changes in the asset structure, as such, can be neglected.

Letting P_i represent the current price of the ith good and P_i', the expected price, excess demand functions for the n goods can be written

$$X_i = X_i(P_1, \cdots, P_n, P_i') \qquad (i = 1, \cdots, n). \tag{1}$$

In equilibrium,

$$X_i = 0 \qquad (i = 1, \cdots, n) \tag{2}$$

and

$$P_i' = P_i \qquad (i = 1, \cdots, n). \tag{3}$$

The latter conditions are necessary for equilibrium, for if expected prices do not equal present prices, the latter cannot be in equilibrium (except possibly for an instant). If the expectations are correct, prices will change, whence they were not in equilibrium. If the expectations are incorrect, then current behavior is based upon false expectations and hence it will be modified as the expectations fail to be verified. Equations (1), (2) and (3) are sufficient to determine the equilibrium values of all the variables.[3]

[3] It is easy enough to construct examples in which the equations (1), (2) and (3) have either no meaningful solution or several. In the latter case, the "local" results of this note can be taken to apply in the neighborhood of any such equilibrium. We do not consider the problem of stating sufficient conditions that at least one equilibrium point actually exists. But see Arrow and Debreu [**1**].

If all prices are flexible, their dynamic behavior can be approximated by equations of the form

$$\dot{P}_i = K_i X_i \qquad (i = 1, \cdots, n) \tag{4}$$

where the K_i are positive constants. We shall assume that induced changes in expected prices are governed by the relationship

$$P'_i = P_i + \eta_i \dot{P}_i \qquad (i = 1, \cdots, n). \tag{5}$$

This is a simple variant of Metzler's "coefficient of expectations," [5]. When $\eta_i = 0$, expectations are static and current prices are expected to persist. When $\eta_i > 0$, expected prices are an extrapolation of the trend in current prices and expectations may be described as "extrapolative." When $\eta_i < 0$, expectations in that market might be described as conservative in that expected prices lag behind current prices.

By Taylor's theorem, we can approximate (1) in the neighborhood of equilibrium by the linear expression

$$X_i = \sum_j a_{ij}(P_j - P_j^0) + b_i(P'_i - P_i^0) \qquad (i = 1, \cdots, n) \tag{6}$$

where $a_{ij} = \partial X_i/\partial P_j$, and $b_i = \partial X_i/\partial P'_i$, both evaluated in the neighborhood of equilibrium. Substituting (5) and (6) into (4), we obtain the linear system

$$\dot{P}_i = K_i \sum_j a_{ij}(P_j - P_j^0) + K_i b_i (P_i - P_i^0) + K_i b_i \eta_i \dot{P}_i \qquad (i = 1, \cdots, n). \tag{7}$$

If all $\eta_i = 0$, the roots of the system with static expectations are the characteristic roots of the matrix

$$A = \begin{vmatrix} K_1(a_{11} + b_1) & \cdots & K_1 a_{1n} \\ \cdots & \cdots & \cdots \\ K_n a_{n1} & \cdots & K_n(a_{nn} + b_n) \end{vmatrix}.$$

Metzler [5] has shown that if all goods are gross substitutes, i.e., all of the off-diagonal elements are positive, necessary, and sufficient conditions for stability are that the principal minors of the matrix alternate in sign, with sign $(-1)^n$ where n is the order of the minor.[4] Metzler's result can be extended to include non-static expectations.

THEOREM: *If the parameters of the system with static expectations fulfil the conditions for the Metzler theorem, then a necessary and sufficient condition for the stability of the system* (7) *with any expectations is that* $1/K_i > b_i\eta_i$ *for all markets.*

The stability conditions on expectations require that the reciprocal of the price reaction coefficients, i.e., the coefficients of insensitivity of prices to excess demand, be greater than the destabilizing force of the extrapolative expectations. Only in the limiting case of infinite K_i will $\eta_i \leqq 0$ be necessary for stability.

[4] This condition on the minors is known as the Hicks condition. See the mathematical appendix of [2]. The condition is also necessary and sufficient when the matrix is symmetric regardless of the signs of the elements.

In other cases, the effects of the expectations may be damped by the friction and inertia of the system and the system may asymptotically converge to equilibrium.

For any real square matrix A, let $\phi(A)$ be the largest of the real parts of the characteristic roots. A is said to be *stable* if and only if $\phi(A) < 0$.

The characteristic equation of (7) can be written

$$\det | DA - \lambda I | = 0 \tag{8}$$

where D is a diagonal matrix with elements $d_i = (1 - K_i b_i \eta_i)^{-1}$. Given that A is stable, we shall prove that $R(\lambda) < 0$ if and only if the elements of D are positive.

Sufficiency. If all of the elements of D are positive, the signs of the elements and the signs of the minors of DA will be the same as those of the corresponding elements and minors of A since each of the latter will have been multiplied by a positive factor. Hence, DA will also fulfil the conditions of the Metzler theorem and it will be stable.

Necessity. In order to prove that the stability of DA implies that $d_i > 0$, we must first prove three lemmas.

LEMMA 1: *If no off-diagonal element of A is negative, then $\phi(A)$ is a characteristic root of A and it does not decrease when any off-diagonal element of A increases.*

PROOF: Choose s so that $s + a_{ii} > 0$ for all i. Then $sI + A$ is a positive matrix whose characteristic roots are s greater than those of A. If λ is a characteristic root of A, $\lambda + s$ is a characteristic root of $sI + A$. $sI + A$ has a real characteristic root, λ_o, such that $| \lambda + s | \leqq \lambda_o$.[5] Since $R(z) \leqq | z |$ for any complex number, $R(\lambda + s) = R(\lambda) + s \leqq \lambda_o$ or $R(\lambda) \leqq \lambda_o - s$. Since $\lambda_o - s$ is a characteristic root of A and a real number, clearly, $\phi(A) = \lambda_o - s$.

The second part of the Lemma is a well known property of λ_o, and hence of $\lambda_o - s$ if the diagonal elements are constant so that s need not vary.[6]

LEMMA 2: *If A is a nonnegative matrix and if $Ax \geqq \lambda x$ for some real λ and for some $x \geq 0$ ($\geq$ indicating that not all $x = 0$), then $\lambda \leqq \phi(A)$.*

PROOF: Wielandt[7] has shown that for any nonnegative indecomposable matrix

$$\lambda_o = \max_{x \geqq 0} \min_i \left(\sum_{j=1}^{n} a_{ij} x_j / x_i \right)$$

where λ_o is defined as it is in the proof of Lemma 1. For positive A, clearly $\lambda_o = \phi(A)$. Then, for the particular λ and x in the hypothesis

$$\phi(A) = \lambda_o \geqq \min_i \left(\sum_{j=1}^{n} a_{ij} x_j / x_i \right) \geqq \lambda. \tag{8}$$

[5] See Debreu and Herstein [2], Theorem I.

[6] See *ibid*.

[7] Wielandt [8], equation (4). Though the Wielandt result is stated only for nonnegative indecomposable matrices, the validity of the lemma for all nonnegative matrices can be shown from the present proof by a trivial limiting argument.

LEMMA 3: *Let A be a stable matrix with only nonnegative off-diagonal elements and D a nonsingular diagonal matrix. Then $\phi(DA) \neq 0$.*

PROOF: Choose s so that $B = sI + A$ is a nonnegative matrix. Suppose $\phi(DA) = 0$. Then DA has a pure imaginary characteristic root, λ, and a characteristic vector, $x \neq 0$, such that $DAx = \lambda x$. Multiplying by D^{-1} and adding sx to both sides, we obtain the system of equations

$$Bx = (sI + D^{-1})x. \tag{9}$$

Consider the absolute values of both sides of each equation. The absolute value of the sum on the left-hand side is less than or equal to the sum of the absolute values. On the right-hand side, $sI + \lambda D^{-1}$ is a diagonal matrix, so that the absolute value of each is the absolute value of a single non-zero term. Therefore, letting the elements of B be b_{ij},

$$\sum_j b_{ij} | x_j | \geqq | s + \lambda/d_i | \cdot | x_i |. \tag{10}$$

Let $\mu = \min_i | s + \lambda/d_i |$ and $y_i = | x_i |$. Then (10) implies

$$By \geqq \mu y, \qquad y \geq 0. \tag{11}$$

However, by Lemma 2,

$$\mu \leqq \phi(B) = s + \phi(A). \tag{12}$$

Since λ is a pure imaginary number and d_i is real, it is easy to see that $| s + \lambda/d_i | \geqq s$ for each i whence $\mu \geqq s$. But this and (12) imply $\phi(A) \geqq 0$ which is a contradiction since A was assumed stable.

THEOREM[8]: *If A has all negative diagonal elements and no negative off-diagonal elements, D is a diagonal matrix, and both A and DA are stable, then the diagonal elements of D are positive.*

PROOF: If D were singular, then DA would be singular and have 0 as a charteristic root, so that $\phi(DA) \geqq 0$, contrary to hypothesis. Therefore

$$D \text{ is nonsingular and } d_i \neq 0. \tag{13}$$

Let the variable matrix $A(t)$ be defined for $0 \leqq t \leqq 1$ as

$$a_{ij}(t) = a_{ij} \quad \text{for} \quad i \leqq j,$$
$$a_{ij}(t) = (1 - t)a_{ij} \quad \text{for} \quad i > j.$$

Then

$$A(t) \text{ has nonnegative off-diagonal elements for } 0 \leqq t < 1, \tag{14}$$

and $A(0) = A$.

As t increases, some elements decrease while none increase. Thus (14) and

[8] The A matrix of this theorem is more general than that of the Metzler theorem which must have positive off-diagonal elements. The extension of the Metzler theorem follows as a special case.

Lemma 1 imply that $\phi[A(t)]$ is a non-increasing function of t for $0 \leqq t < 1$. Hence, $\phi[A(t)] \leqq \phi[A(0)] = \phi(A) < 0$, the latter inequality being true by hypothesis. Therefore,

$$(15) \qquad A(t) \text{ is a stable matrix for } 0 \leqq t < 1.$$

From (13), (14) and (15) and Lemma 3, $\phi[DA(t)] \neq 0$ for $0 \leqq t < 1$. By hypothesis, $DA = DA(0)$ is stable, so that $\phi[DA(0)] < 0$. Since $\phi[DA(t)]$ is a continuous function of t, it follows that $\phi[DA(t)] < 0$ for $0 \leqq t < 1$, and hence, by continuity,

$$(16) \qquad \phi[DA(1)] \leqq 0.$$

But $A(1)$, and hence $DA(1)$, has only zeroes below the diagonal so that its characteristic roots are precisely its diagonal elements, $d_i a_{ii}$. (16) implies, then, that $d_i a_{ii} \leqq 0$ for all i. Since $a_{ii} < 0$, $d_i \geqq 0$ and (13) implies that $d_i > 0$. Therefore, the stability of DA implies that $d_i > 0$ whence the condition $1/K_i > b_i \eta_i$ is necessary as well as sufficient for stability.

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REFERENCES

[1] Arrow, K. J., and G. Debreu: "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, Vol. 22, No. 3, July, 1954, pp. 265–290.

[2] Debreu, G., and I. N. Herstein: "Nonnegative Square Matrices," *Econometrica*, Vol. 21, No. 4, October 1953, pp. 597–607.

[3] Hicks, J. R.: *Value and Capital*, Second Edition, Oxford University Press, London 1946.

[4] Metzler, L. A.: "The Nature and Stability of Inventory Cycles," *Review of Economic Statistics*, Vol. XXIII, August, 1941.

[5] ———: "Stability of Multiple Markets: The Hicks Conditions," *Econometrica*, Vol. 13, No. 4, October 1945, pp. 277–292.

[6] Samuelson, P. A.: "The Stability of Equilibrium: Comparative Statics and Dynamics," *Econometrica*, Vol. 9, No. 2, April 1941, pp. 97–120.

[7] ———: "The Relation Between Hicksian Stability and True Dynamic Stability," *Econometrica*, Vol. 12, Nos. 3 & 4, July–October, 1944, pp. 256–257.

[8] Wielandt, H.: "Unzerlegbare, nicht negative Matrizen," *Mathematische Zeitschrift*, Vol. 52, March 1950, pp. 642–648.