STABILITY IN MATHEMATICAL GAME THEORY

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Abstract. Game theory – the study of strategic decision making in interactive environments – has numerous applications in economics, political science, biology, computer science, and public policy. This paper addresses games where individuals act in isolation (decision problems), interact strategically and independently (strategic games), and coordinate their strategic efforts (cooperative games). Along the way, approaches aimed at stability are emphasized, and particular attention is given to the concepts of Nash equilibrium and the core.

Contents

1. A Note about Stability

The main (though not exclusive) focus of this paper is stability. In many ways, stability is the game theoretic analog to chemical equilibrium. Just as chemical equilibrium is the state in which both reactants and products have no further tendency to change with time, a set of actions is deemed stable when no player – or, in the case of cooperative games, set of players – would change their action choice given the opportunity. Stability defines both Nash equilibrium in strategic games and the core in cooperative ones, and it remains closely related to several other convenient properties (i.e. dominance, rationality, and balance) as well, as the coming pages will establish in greater detail.

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2. Preferences and Utility

Since game theory studies how people make decisions in interactive environments, we begin by considering what a decision problem involves. In a decision problem, a decision maker faces options (out of a set A). Furthermore, the decision maker has preferences over the elements of A, here modeled by the binary relation $\succeq \subset A \times A$, or a preference relation. For each $a, b \in A$, $a \succeq b$ can be read as "the decision maker prefers a to b or is indifferent between a and b."

We assume that $\succeq$ is:

(1) complete, that is, $\forall\ a, b \in A$, a preference relationship exists between a,b ($a \succeq b$ or $b \succeq a$ or both)
(2) transitive, i.e., $\forall\ a, b, c \in A$, if $a \succeq b$ and $b \succeq c$, then $a \succeq c$.

We call a complete and transitive binary relation over a set A a weak preference over A. And so, it follows that:

Definition 2.1. A *decision problem* is a pair $(A, \succeq)$, where A is a set of options and $\succeq$ represents a weak preference over A.

One way to think about weak preference is as the union of strict preference ($\succ$) and indifference ($\sim$). All three of these binary relations (as with all binary relations) can be characterized as reflexive, symmetric, asymmetric, and/or antisymmetric.

Proposition 2.2. *Let $(A, \succeq)$ be a decision problem. Then*

i) The strict preference, $\succ$, is asymmetric and transitive.
ii) The indifference, $\sim$, is reflexive, symmetric, and transitive (i.e., an equivalence relation).

These results follow directly from definitions, but they provide us with some useful properties of preferences.

Just as inequalities can be used to order real numbers, the weak preference relation is a way of ordering preferences. And so, it is sometimes possible to map preferences associated with objects to numerical levels of preference strength. A utility function is just such a map, as the definition below elucidates.

Definition 2.3. Let $(A, \succeq)$ be a decision problem. A *utility function* representing $\succeq$ is a function $u : A \to \mathbb{R}$ satisfying for each pair $a, b \in A$ that $a \succeq b$ if and only if $u(a) \geq u(b)$.

The utility function could have alternately been defined in terms of strict preference and equivalence – where $a, b \in A$, $a \succ b$ if and only if $u(a) > u(b)$ and $a \sim b$ if and only if $u(a) = u(b)$ – without changing the meaning or ramifications.

Utility functions can be a great way of accurately capturing the decision maker's preferences and analyzing trends within them; however, not every decision problem can be represented by a utility function, as the example below illustrates.

Example 2.4. Let $(\mathbb{R}^n, \succeq)$ be a decision problem where for each pair $x, y \in \mathbb{R}^n$, $x \succeq y$ if and only if $x = y$ or there is $i \in 1, 2, ..., n$ such that $\forall\ j < i$, $x_j = y_j$ and $x_i > y_i$ (this is called the lexicographic order). Although one can create a utility function for trivial subsets of the lexicographic order (i.e., where $n = 1$ only), whenever $n \geq 2$, no utility function can represent the preference relation. To see why this is true, suppose that u is a utility function representing $\succeq$. For each $x \in$

$\mathbb{R}, (x, ..., x, 1) \succ (x, ..., x, 0)$. Therefore it follows that $u(x, ..., x, 1) > u(x, ..., x, 0)$ and we can find an $f(x) \in \mathbb{Q}$[1] such that $u(x, ..., x, 1) > f(x) > u(x, ..., x, 0)$. Thus, $f(x)$ is a function from $\mathbb{R}$ to $\mathbb{Q}$, but it cannot be injective since $\mathbb{Q}$ is countable while $\mathbb{R}$ is not. Injectivity is a requisite property of utility functions: if a utility function were not injective, then at least two different preferences would be associated with the same utility level. This would violate the condition that $u(a) = u(b)$ implies $a \sim b$, since there would be at least one situation where $u(a) = u(b)$ but $a \succ b$ or $a \prec b$.

This raises questions regarding what conditions are necessary so that a decision problem can be represented by a utility function. The following theorem will begin to provide an answer.

Theorem 2.5. *Let $(A, \succeq)$ be a decision problem. If A is a countable set, then there is a utility function u representing $\succeq$.*

Proof. Since A is countable, it can be denoted as $\{a_1, a_2, ...\}$. For each pair $i, j \in \mathbb{N}$, we can define a pointwise function:

$$f(x) := \begin{cases} 1 \text{ if } a_i, a_j \in A \text{ and } a_i \succ a_j \\ 0 \text{ otherwise} \end{cases}$$

Using this pointwise function, we can define a utility function where, for each $a_i \in A$, $u(a_i) := \sum_{j=1}^{\infty} \frac{1}{2^j} f(x)$. Since this converges (by ratio test), u must be well-defined. Furthermore, because $\forall\ a, b \in A$, $a \succeq b$ if and only if $u(a) \geq u(b)$, the utility function represents $\succeq$. □

This proposition is handy, as it gives us an easy way to check whether a decision problem $(A, \succeq)$ can be represented by a utility function: simply check whether the set A is countable, and representability follows. However, an even stronger statement can be made regarding when a decision problem can be represented by a utility function and, moreover, when a utility function represents a decision problem. The conditions it outlines (as seen in the theorem below) will be those used to determine representability for the remainder of the paper.

Definition 2.6. Let $(A, \succeq)$ be a decision problem. A set $B \subset A$ is *order dense in* A if, for each pair $a_1, a_2 \in A$ with $a_2 \succeq a_1$, there exists $b \in B$ such that $a_2 \succeq b \succeq a_1$.

Definition 2.7. Let $(A, \succeq)$ be a decision problem and let $a_1, a_2 \in A$ with $a_2 \succeq a_1$. Then, (a_1, a_2) is a *gap* if, $\forall\ b \in A$, either $b \succeq a_2$ or $a_1 \succeq b$. If (a_1, a_2) is a gap, then a_1 and a_2 are *gap extremes*. Let A^* be the set of gap extremes of A.

Theorem 2.8. *Let $(A, \succeq)$ be a decision problem and assume that $\succeq$ is antisymmetric. Then, $\succeq$ can be represented by a utility function if and only if there is a countable set $B \subset A$ that is order dense in A.*

Proof. Please refer to Appendix A. □

While the technical nature of the representability theorem proof makes it better suited for the appendix, the following lemma (used in the proof of the representability theorem) merits mention here.

[1]We know that $f(x) \in \mathbb{Q}$ because utility levels must be rational numbers.

Lemma 2.9. *Let $(A, \succeq)$ be a decision problem and assume that $\succeq$ is antisymmetric.*

i) If there is a countable set $B \subset A$ that is order dense in A, then A^* is countable.

ii) If there is a utility function representing $\succeq$, then A^* is countable.

Proof. i) Let $B \subset A$ be a countable set order dense in A, and define A_1^* as the set of superior gap extremes and A_2^* as the set of inferior gap extremes. Since $\succeq$ is antisymmetric and B is order dense in A, there is $b \in B$ such that $a_1 = b$ or $a_2 = b$ (or both a_1 and a_2 are equal to elements of the set B). Therefore, by matching every superior gap extreme not in B with its inferior gap extreme (which must be in B by order density), we can create a bijection between $A_1^* \backslash B$ and a subset of B (hereafter referred to as C). Because $A_1^* \backslash B$ has the same number of elements as C (a subset of B), $A_1^* \backslash B$ must be countable. By the same reasoning, $A_2^* \backslash B$ must also be countable. We also know that $(A^* \cap B)$ is countable because it involves an intersection with a countable set. Thus, $(A_1^* \setminus B) \cup (A_2^* \setminus B) \cup (A^* \cap B)$ is countable, as any finite union of countable sets is countable. Incidentally, $A = (A_1^* \setminus B) \cup (A_2^* \setminus B) \cup (A^* \cap B)$, so A is a countable set.

ii) Let u be a utility function that represents $\succeq$. For each gap in (a_1, a_2), there must be a $q \in \mathbb{Q}$ such that $u(a_2) > q > u(a_1)$, since the utility function represents $\succeq$. Define a set $\hat{Q}$ such that it contains one q for each gap (a_1, a_2). Then there exists a bijection between $\hat{Q}$ the set A_1^* (as defined in part i), as both are of cardinality equal to the number of gaps in set A. Because $\hat{Q}$ is a subset of of $\mathbb{Q}$ and thus countable, A_1^* too must be countable. By the same reasoning, A_2^* is countable as well. Thus, $A^* = A_1^* \cup A_2^*$ must be a countable set. □

Notice that, so far, we have only discussed preferences relative to each other. Indeed, when discussing possible outcomes in a decision-making problem, the numbers associated with outcomes are generally only important relative to each other. This is because people act based on preferences, and their actions change only when the order of their preferences changes. For this reason, oftentimes the only information we care to extract from utility functions is ordinal. Put differently, for any $a_1, a_2 \in A$, the magnitude of $u(a_1)$ and $u(a_2)$ usually doesn't matter to us, although $a_1 \succeq a_2$ if and only if $u(a_1) \geq u(a_2)$. And so, multiple utility functions can represent the same preference schemes. A standard for when different utility functions represent the same preferences will be addressed in the upcoming proposition, but first the notion of strictly increasing must be formalized.

Definition 2.10. A function $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if, for every pair $x, y \in \Omega$, $x > y$ if and only if $f(x) > f(y)$.

Proposition 2.11. *Let $(A, \succeq)$ be a decision problem and assume that u is a utility function representing $\succeq$. Then $\bar{u}$ is another function representing $\succeq$ if and only if there is a strictly increasing function $f : u(A) \subset \mathbb{R} \to \mathbb{R}$ such that, for each $a \in A$, $\bar{u}(a) = f(u(a))$.*

This proposition provides us with the means to change utility functions without altering the ordinal information they convey, or alternately to determine whether different utility functions represent the same preferences. However, it is important to note that while cardinal preferences don't change game outcomes, they can still convey valuable information regarding the strengths of preferences. For this reason, utility functions are distinctive from their monotonic transformations (that is, from the set of $\bar{u}$ described above).

3. Strategic Games and Nash Equilibrium

While decision problems model how individuals makes decisions in isolation, strategic games examine interactive decision making. Strategic games are static models involving players who make decisions simultaneously and independently (although these decisions usually impact the other players as well). Here, we assume that players have complete information; that is, they know what the outcome will be for each possible action of each player. If it helps, one can think of players as able to communicate with each other before the game but unable to make binding agreements amongst one another.

This model also takes a common assumption of economics: that individuals are rational, or trying to maximize their own utility. As was the case with decisions in isolation, individuals have preferences over the possible outcomes (which we now associate with utility levels).

Notation 3.1. We denote the set of players of a given game by $N := \{1, 2, ..., n\}$.

Definition 3.2. An n-player *strategic game* with set of players N is a pair $G := (A, u)$ whose elements represent the following:

Sets of strategies: For each $i \in N, A_i$ is the nonempty set of strategies of player i and $A := (A_1, A_2, ..., A_n)$ is the set of possible strategies.

Payoff functions: For each $i \in N, u_i : A \to \mathbb{R}$ is the payoff function of player i and $u := (u_1, u_2, ..., u_n)$. Hence, u_i assigns, to each possible strategy $a \in A$, the payoff that player i gets if a is played.

Remark 3.3. Although a pair (A, u) can be used to represent a strategic game concisely and precisely, all of the following elements are implicitly involved in a strategic game:

- $\{A_i\}_{i \in N}$, the strategy sets of the players
- R, the set of possible outcomes
- A function $f : A \to R$, an assignment to each possible strategy a of its corresponding outcome
- $\{\succeq_i\}_{i \in N}$, the preferences of the players over outcomes in R
- $\{U_i\}_{i \in N}$, the utility functions of the players, which represent their preferences on R

For this reason, a strategic game can be viewed as a simplification in which, for each $i \in N$ and each $a \in A$, $u_i(a) = U_i(f(a))$.

Notation 3.4. Two-player strategic games can be modeled in tables of the following form, which present utility payoffs as a function of selected actions. By convention, we represent tables such that player 1 chooses between rows and player 2 chooses between columns.

	a_1	a_2
a_1	$(u_1(a_1, a_1), u_2(a_1, a_1))$	$(u_1(a_1, a_2), u_2(a_1, a_2))$
a_2	$(u_1(a_2, a_1), u_2(a_2, a_1))$	$(u_1(a_2, a_2), u_2(a_2, a_2))$

Example 3.5. Battle of the Sexes (or Bach or Stravinsky)

In this classic game, two individuals want to spend time together but have different preferences as to where to go. Let's say that Player 1 prefers the opera, whereas Player 2 would rather see her brother's football game. In table form:

	O	F
O	(2,1)	(0,0)
F	(0,0)	(1,2)

The game has two solutions, depending on how it is conceived. The first assumes that the two individuals can converse. If this is the case, then whichever individual is able to convey that "no matter what, I will be attending X event" will hold the upper hand. If Player 2 convinces Player 1 that she will be attending the football game, she in effect eliminates column 1 from the range of possible outcomes. This leaves Player 1 to choose a row, but since one column has been eliminated, he is effectively left to choose between the opera, given Player 2 will not be there ((O,F) = (0,0)) and the football game, given that Player 2 will be there ((F,F) = (1,2)). Under the assumption that he is utility-maximizing, we can be sure that he will choose the football game, which results in the outcome of (F,F), where Player 1 receives a payoff of 1 and Player 2 of 2.

If neither individual can convince the other as to where he or she will be, a second optimal solution remains: Player 1 picks the opera (the first row) with probability 0.5 and Player 2 picks the football game (the second column) with $p = 0.5$. This would suggest that the payoffs for Players 1 and 2 become $(0.5 \cdot 0.5 \cdot 2) + (0.5 \cdot 0.5 \cdot 0) + (0.5 \cdot 0.5 \cdot 0) + (0.5 \cdot 0.5 \cdot 1) = 0.75$.

The second solution to Battle of the Sexes brings up an interesting possibility that is present in some games – abstaining from choosing a single action in favor of choosing a probability distribution of actions. An action choice consisting of a probability distribution of actions is called a mixed strategy, and it is an element of the mixed extension, defined below.

Notation 3.6. Let $G = (A, u)$ be a finite game. Then ΔA represents the set of probability distributions over A. More technically:

$$\Delta A = \{x \in [0,1]^{|A|} : |\{a \in A : x(a) > 0\}| < \infty \text{ and } \textstyle\sum_{a \in A} x(a) = 1\}.$$

Definition 3.7. Let $G = (A, u)$ be a finite game. The *mixed extension* of G is the strategic game $E(G) := (S, u)$, whose elements are the following:

Set of (mixed) strategies: For each $i \in N$, $S_i := \Delta A_i$ and $S := \prod_{i \in N} S_i$. For each $s \in S$ and each $a \in A$, let $s(a) := s_1(a_1) \cdot ... \cdot s_n(a_n)$.

Payoff functions: For each $s \in S$, $u_i(s) := \sum_{a \in A} u_i(s)s(a)$ and $u := (u_1, u_2, ..., u_n)$.

Notation 3.8. Given a game $G = (A, u)$ and a possible strategy $a \in A$, let $(a_{-i}, \hat{a}_i)$ denote the profile $(a_1, ..., a_{i-1}, \hat{a_i}, a_{i+1}, ..., a_n)$.

Definitions 3.9. Let G be a finite game and let $E(G)$ be its mixed extension. Let $s_i, \bar{s}_i$ be two strategies of player i in $E(G)$. Finally, let S_i be the set of all possible strategies. Then $\bar{s}_i$ *dominates* s_i if:

i) For each $\hat{s}_i \in S_i$, $u_i(\hat{s}_{-i}, \bar{s}_i) \geq u_i(\hat{s}_{-i}, s_i)$, and

ii) There is $\tilde{s}_i \in S_i$ such that $u_i(\tilde{s}_{-i}, \bar{s}_i) > u_i(\tilde{s}_{-i}, s_i)$.

Furthermore, $\bar{s}_i$ is called a *dominant strategy* if it dominates all $s_i \in S_i$.

In the first solution to Battle of the Sexes, Player 2 influences the outcome of the game by limiting Player 1's options to one where her ideal outcome is his ideal outcome as well. We can also parse this phenomenon in terms of dominant strategies. Neither player has a dominant strategy in the Battle of the Sexes game when both players have two possible actions (here, opera or football game) and there are four possible outcomes ((O,O), etc.). If one player commits to an action, though, that individual eliminates two possible outcomes (either a row or a column, depending on who acted), and a dominant strategy emerges for the other player: to make the same choice as the previous actor.

Definition 3.10. Let G be a finite game and let $E(G)$ be its mixed extension. G is *undominated* if and only if no dominant strategy exists.

We observe that Battle of the Sexes is undominated in its complete form but contains subsets with dominant (and dominated) strategies. Next, we will consider one of the most famous games in game theory, a game that not only has a dominant strategy but whose dominant strategy leads to a very interesting result.

Example 3.11. Prisoner's Dilemma
Two suspects – each guilty of both a major crime and a minor crime – are being held in separate cells. They are both known to be guilty of the minor crime, but convicting one of the major crime depends on the other's testimony. The cops offer each prisoner a deal: defect on your accomplice, and charges will not be pressed against you for either crime, though your accomplice will be prosecuted for both. Thus, if one confesses but his accomplice stays quiet, the accomplice faces 15 years in prison while the confessor goes home. If both stay quiet, they just get prosecuted for the minor crime (1 year); if both defect, they face 10 years in prison each.

We assume that each individual's utility depends only on how many years he himself spends in prison. Because we only care about ordinal preferences in this example, the exact numbers are of little significance; for simplicity's sake, we will equate utility levels with the negative of the number of years spent in prison. The game can thus be represented by the following table:

	Q	D
Q	(-1,-1)	(-15,0)
D	(0,-15)	(-10,-10)

For either player, regardless of what the other player does, he is better off defecting (since $0 > -1$ and $-10 > -15$). Hence, defecting constitutes the dominant strategy for both players. Interestingly though, both players would both have been better off had they remained quiet (with a utility payoff of -1 each, as opposed to -10 each). But since neither can guarantee that the other will remain quiet under this formulation of the game, they cannot collaborate towards the outcome (Q, Q).

Clearly, although (D, D) is the outcome if both prisoners pursue their dominant strategies, it is far from ideal for either party. But would either change their action after the fact if they could? Let's assume that the game has resulted in (D, D): without loss of generality, we can ask whether Player 1 would change his choice from (D, D) to (Q, D), given the opportunity. Since this would shift his utility from -10 to the even less palatable payoff of -15, Player 1 is unlikely to deviate, even if he is granted that liberty. Likewise, Player 2 would only be hurt by changing his choice from (D, D) to (D, Q). In this sense, the outcome (D, D) is stable; this notion of stability is exactly what a Nash equilibrium expresses.

Definition 3.12. Let $G = (A, u)$ be a strategic game. A *Nash equilibrium* of G is a possible strategy $a^* \in A$ such that, for each $i \in N$ and each $\hat{a} \in A$,

$$u(a^*) \geq u(a^*_{-i}, \hat{a}).$$

In other words, a Nash equilibrium is an outcome such that no utility-maximizing player would change his or her action, given that all the other players' actions are final. If we take a look at Prisoner's Dilemma again, we notice that only (D, D) is a Nash equilibrium. This makes sense, as neither player stands to gain by deviating at (D, D); however, every other action pair will incline one player or the other to deviate, and this process will continue until the action pair (D, D) is arrived upon.

Battle of the Sexes has two Nash equilibria: (O,O) and (F,F). However, the number of Nash equilibria can vary greatly from game to game, as evidenced by the following example.

Example 3.13. Matching Pennies

Two people have one coin each. Simultaneously, they decide which side of their coin to show (heads or tails). If they both show the same side, Player 1 gets a payoff of 1 (Player 2 gets a payoff of -1); if they show different sides of their coins, Player 2 gets a payoff of 1 (Player 1 gets a payoff of -1). The game can thus be modeled by the table below:

	H	T
H	(1,-1)	(-1,1)
T	(-1,1)	(1,-1)

Remark 3.14. In Matching Pennies, regardless of the outcome, $u_1(a_i) + u_2(a_i) = 0$. Games of this form (that is, where $\forall\ i \in N, a_i \in A, u_1(a_i) + ... + u_n(a_i) = 0$) are known as zero-sum games.

We can see that, as formulated, there is no Nash equilibrium to the Matching Pennies game. This is because, for every possible outcome, one of the players would stand to gain from shifting their choice to their other action option. If this occurs, then the other person stands to gain from shifting their choice to their other action option, and so on. Thus, the game as formulated is never stable in the sense that someone will always want to change their action choice.

However, if one allows mixed strategies, then Matching Pennies does have a Nash equilibrium: one where the strategy of each player is to choose randomly and without bias between heads and tails (in other words, to assign a probability of $\frac{1}{2}$ to each action strategy a_1 and a_2). Before discussing how we can ensure that this is the unique Nash equilibrium, we must introduce the concept of best response.

Definition 3.15. Let $G = (A, u)$ be a strategic game such that, for each $i \in N$,

(1) there is $m_i \in N$ such that A_i is a nonempty and compact subset of $\mathbb{R}^{m_i}$
(2) u_i is continuous

Then, for each $i \in N$, i's *best reply correspondence*, $BR_i : A_{-i} \to A_i$, is defined, for each $a_{-i} \in A_{-i}$, by:

$$BR_i(a_{-i}) := \{a_i \in A_i : u_i(a_{-i}, a_i) = \max_{\tilde{a}_i \in A_i} u_i(a_{-i}, \tilde{a}_i)\}.$$

Let $BR : A \to A$ be defined, for each $a \in A$, as $BR(a) := \prod_{i \in N} BR_i(a_{-i})$.

To prove that the randomization strategy is a Nash equilibrium, we must show that no one gains from deviation. Without loss of generality, let us say that Player 1 changes the assignment of probabilities such that he chooses a_1 with probability $\frac{1}{2} + \delta$ where $0 < \delta < \frac{1}{2}$. Then the best response of Player 2 would be to choose a_2 all of the time.

In this scenario, their respective expected payoffs would be -2δ and 2δ respectively. Player 1 can maximize his payoff by setting $\delta = 0$ (that is, assign a probability of $\frac{1}{2}$ to each action strategy). Of course, if Player 2 is still consistently choosing the action a_2, then Player 1 would want to choose a_2 as well with probability of 1. In this case, Player 1 would receive an expected payoff of 1 and Player 2 a payoff of -1. Thus, Player 2 would be best off choosing randomly and without bias between heads and tails as well; this makes intuitive sense, since the game is symmetric.

Contrasting the two versions of Matching Pennies (where mixed strategies aren't or are allowed) illustrates how game conditions can affect the existence and number of Nash equilibria. This elicits the question: under what conditions can we be certain that a Nash equilibrium exists, or, by the same token, doesn't exist? Nash's Theorem answers this question, but before we can prove it, we must first address two other theorems and define several concepts upon which they depend.

Definitions 3.16. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. A *correspondence* F from X to Y is a map $F : X \to 2^Y$. Furthermore, a correspondence F is *nonempty-valued*, *closed-valued*, or *convex-valued*, if for each $x \in X, F(x)$ is, respectively, a nonempty, closed, or convex subset of Y.

Definition 3.17. A function $f : X \to Y$ is *continuous* if, for each sequence $\{x_k\} \subset X$ converging to $\tilde{x} \in X$ and each open set $Y^* \subset Y$ such that $f(\tilde{x}) \in Y^*$, there is $k_0 \in \mathbb{N}$ such that, for each $k \geq k_0$, $f(x_k) \in Y^*$.

Definition 3.18. A correspondence F is *upper hemicontinuous* if, for each sequence $\{x_k\} \subset X$ converging to $\bar{x} \in X$ and every open set $Y^* \subset Y$ such that $F(\bar{x}) \subset Y^*$, there is a $k_0 \in \mathbb{N}$ such that, for each $k \geq k_0$, $F(x_k) \in Y^*$.

Definition 3.19. Let $m \in \mathbb{N}$ and $A \subset \mathbb{R}^m$ be a convex set. A function $f : A \to \mathbb{R}$ is *quasi-concave* if, for each $r \in \mathbb{R}$, the set $\{a \in A : f(a) \geq r\}$ is convex or, equivalently, if, for each $a, b \in A$ and each $\alpha \in [0, 1]$, $f(\alpha a + (1 - \alpha)b) \geq min\{f(a), f(b)\}$.

Proposition 3.20. *Let $G = (A, u)$ be a strategic game such that, for each $i \in N$,*

(1) A_i is a nonempty and compact subset of $\mathbb{R}^{m_i}$,
(2) u_i is continuous, and
(3) for each a_{-i}, $u_i(a_i, \cdot)$ is quasi-concave on A_i.

Then, for each $i \in N, BR_i$ is an upper hemicontinuous, nonempty-valued, closed-valued, and convex valued correspondence. Therefore, BR also satisfies the previous properties.

Proof. Please refer to Appendix B. □

Theorem 3.21. *(Nash's Theorem) Let $G = (A, u)$ be a strategic game such that, for each $i \in N$,*

(1) A_i is a nonempty, convex, and compact subset of $\mathbb{R}^{m_i}$,
(2) u_i is continuous, and
(3) for each a_{-i}, $u_i(a_i, \cdot)$ is quasi-concave on A_i.

Then, the game G has at least one Nash equilibrium.

Remark 3.22. Simply stated, Nash's theorem tells us that every finite game has a mixed strategy equilibrium. (A strategic game $G := (A, u)$ is finite if, for each $i \in N, |A_i| < \infty$.)

John Nash – the namesake of the equilibrium and the theorem, as well as the protagonist of *A Beautiful Mind* – proved Nash's Theorem using the Kakutani fixed-point theorem, a generalization of Brouwer's fixed-point theorem. We will proceed in the same manner.

Theorem 3.23. *(Kakutani fixed-point theorem) Let $X \subset \mathbb{R}^n$ be a nonempty, convex, and compact set. Let $F : X \to X$ be an upper hemicontinuous, nonempty-valued, closed-valued, and convex-valued correspondence. Then, there is $\tilde{x} \in X$ such that $\tilde{x} \in F(\tilde{x})$, i.e., F has a fixed-point.*

Remark 3.24. There are several different ways to prove the Kakutani fixed-point theorem; the classic approach involves deriving it from results of the Sperner lemma. However, because this proof is rather technical and ultimately outside the scope of this paper, we will omit it and take Kakutani fixed-point theorem as given. The interested reader is encouraged to consult *An Introductory Course on Mathematical Game Theory* [1] for more on the Kakutani fixed-point theorem.

Proof. (Nash's Theorem) If a is a fixed point of the correspondence $BR : A \to A$, then a is a Nash equilibrium of G. Proposition 3.19 (combined with our assumption that A_i is convex) gives us that the correspondence $BR : A \to A$ satisfies all the conditions specified in Kakutani fixed-point theorem. Hence, BR has a fixed point, and this fixed point is the Nash equilibrium. $\square$

4. Cooperative Games and the Core

So far, we have only examined games where players act individually. In this section, we will consider cooperative games, or games where players can work together to arrive at collaborative outcomes. A group of collaborating players is called a coalition.

4.1. Non-Transferable Utility Games.

Non-transferable utility games constitute the most general class of cooperative game. As their name suggests, these games are characterized by the property that utility cannot be transferred across players. In NTU games, allocations may or may not be enforced by binding agreements.

Notation 4.1. As before, the set of players continues to be denoted by $N := \{1, ..., n\}$; however, we now also consider subsets $S \subset N$. For each $S \subset N$, we refer to S as a coalition, where $|S|$ denotes the number of players in S. Furthermore, the coalition N - that is, the set of all players - is often called the *grand coalition*.

Notation 4.2. Let x, y be vectors in $\mathbb{R}^{|S|}$. We say that $y \leq x$ if and only if $y_i \leq x_i$ for each $i \in N$, where $N = |S|$.

Definitions 4.3. Given $S \subset N$ and a set $A \subset \mathbb{R}^{|S|}$, we say that A is *comprehensive* if for each $x, y \in \mathbb{R}^{|S|}$ such that $x \in A$ and $y \leq x$, we have that $y \in A$. Furthermore, the *comprehensive hull* of a set A is the smallest comprehensive set containing A.

Definition 4.4. An n-player *nontransferable utility game* (NTU-game) is a pair (N, V) where N is the set of players and V is a function that assigns, to each coalition $S \subset N$, a set $V(S) \subset \mathbb{R}^{|S|}$. By convention, $V(\varnothing) := \{0\}$. Moreover, for each $S \subset N, S \neq \varnothing$:

(1) $V(S)$ is a nonempty and closed subset of $\mathbb{R}^{|S|}$.

(2) $V(S)$ is comprehensive. Moreover, for each $i \in N$, $V(\{i\}) \neq \mathbb{R}$, i.e., there is $v_i \in \mathbb{R}$ such that $V(\{i\}) = (-\infty, v_i]$.
(3) The set $V(S) \cap \{y \in \mathbb{R}^{|S|} : \text{for each } i \in S, y_i \geq v_i\}$ is bounded.

Remark 4.5. Although NTU-games are represented simply by a pair (N, V), all of the following elements are involved:

For each $S \subset N$,

- $R^{|S|}$, is the set of outcomes the players in coalition S can obtain by themselves,
- $\{\succeq_i^S\}_{i \in S}$, the preferences of the players in S over the outcomes in $R^{|S|}$, and
- $\{U_i^S\}_{i \in S}$, the utility functions of the players, which represent their preferences on $R^{|S|}$.

For this reason, an NTU-game can be viewed as a simplification where, for each $S \subset N$ and each $x \in V(S)$, an outcome $r \in R^{|S|}$ exists such that, for every $i \in S$, $x_i = U_i^S(r)$.

Remark 4.6. The requirement that $V(S)$ be comprehensive simply ensures that players in a coalition S could throw away utility if they want to (although, as presumed utility-maximizing individuals, they would not want to do so).

Definitions 4.7. Let (N, V) be an NTU-game. We call the vectors in $\mathbb{R}^N$ *allocations.* An allocation $x \in \mathbb{R}^N$ is *feasible* if there is a partition of $\{S_1, ..., S_k\}$ of N satisfying that, for every $j \in \{1, ..., k\}$, there is $y \in V(S_j)$ such that, for each $i \in S_j, y_i = x_i$.

Example 4.8. The banker game

Consider the NTU game given by:

$$\begin{aligned}
v(\{i\}) &= \{x_i : x_i \leq 0\}, i \in \{1,2,3\} \\
v(\{1,2\}) &= \{(x_1, x_2) : x_1 + 4x_2 \leq 1000 \text{ and } x_1 \leq 1000\} \\
v(\{1,3\}) &= \{(x_1, x_3) : x_1 \leq 0 \text{ and } x_3 \leq 0\} \\
v(\{2,3\}) &= \{(x_2, x_3) : x_2 \leq 0 \text{ and } x_3 \leq 0\} \\
v(N) &= \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 1000\}
\end{aligned}$$

Assuming that 1 unit of utility = \$1, we can interpret the game in the following way. No player can get anything on his or her own; however, Player 1 can get \$1000 with the help of Player 2. Player 1 can reward Player 2 for his assistance by sending him money, but the money Player 1 sends Player 2 will be lost with probability 0.75. Player 3 (the banker) can ensure safe transactions between players 1 and 2. So the question becomes: how much should Player 1 pay Player 2 for his help in getting the money and Player 3 for her help as an intermediary?

When it comes to NTU-games, the main goal of previous analysis has been to determine appropriate rules for selecting feasible allocations. Such rules are referred to as solutions, which can strive towards different ideals, for example, stability, equity, or fairness. If a solution selects a single allocation per game, the solution can be called an allocation rule.

4.2. **Bargaining Problems.** Bargaining games constitute a special class of NTU-games. In a bargaining problem, players have to agree on which allocation should be chosen from the set F. If they cannot agree, the default allocation d is selected.

Definitions 4.9. An n-player *bargaining problem* with set of players N is a pair (F, d) whose elements are the following:

Feasible set: F, the comprehensive hull of a compact and convex subset of $\mathbb{R}^N$
Disagreement point: d, a specific allocation of F (see Remark 4.11).

Furthermore, B^N is the set of possible n-player bargaining problems. Moreover, given a bargaining problem $(F, d) \in B^N$, we define the compact set $F_d := \{x \in F : x \geq d\}$.

Definition 4.10. An *allocation rule* for n-player bargaining problems is a map $\phi : B^N \to \mathbb{R}^N$ such that, for each $(F, d) \in B^N$, $\phi(F, d) \in F_d$.

Remark 4.11. An n-player bargaining problem (F, d) can be seen as an NTU-game (N, V), where $V(N) := F$ and for each nonempty coalition $S \neq N$, $V(S) := \{y \in \mathbb{R}^{|S|} : \text{for each } i \in S, y_i \leq d_i\}$.

Remark 4.12. As mentioned above, the disagreement point delivers the utilities where no agreement is reached. It is assumed that there exists $x \in F$ such that $x > d$; otherwise, no agreement could be achieved, and a coalition would not form.

Notation 4.13. Given $(F, d) \in B^N$, let $g^d : \mathbb{R}^N \to \mathbb{R}$ be defined for each $x \in \mathbb{R}^N$, by $g^d(x) := \prod_{i \in N}(x_i - d_i)$, i.e., if $x > d$, $g^d(x)$ represents the product of the gains of the players at x with respect to their utilities at the disagreement point.

Proposition 4.14. *Let $(F, d) \in B^N$. Then, there is a unique $z \in F_d$ that maximizes the function g^d over the set F_d.*

Proof. First, we know that g^d has at least one maximum in F_d since g_d is continuous and F_d is compact.

Suppose that there are more than one $z \in F_d$ that maximizes g^d; more specifically, supposed that there exist $z, \hat{z} \in F_d$ such that $z \neq \hat{z}$ and:

$$\max_{x \in F_d} g^d(x) = g^d(z) = g^d(\hat{z}).$$

For each $i \in N$, $z_i > d_i$ and $\hat{z} > d_i$. Because F_d is convex, $\bar{z} := \frac{z}{2} + \frac{\hat{z}}{2} \in F_d$. If $g^d(\bar{z})$ were to be greater than $g^d(z)$, that would contradict the fact that z is a maximum; we will now show that this is in fact the case. We know that:

$$\log(g^d(\bar{z})) = \textstyle\sum_{i \in N} \log(\bar{z}_i - d_i) = \sum_{i \in N} \log(\frac{z_i - d_i}{2} + \frac{\hat{z} - d_i}{2}).$$

By the strict concavity of the logarithmic functions, the above equality is strictly larger than:

$$\textstyle\sum_{i \in N}(\frac{1}{2}\log(z_i - d_i) + \frac{1}{2}\log(\hat{z}_i - d_i)) = \frac{1}{2}\log(g^d(z)) + \frac{1}{2}\log(g^d(\hat{z}) = \log(g^d(z)).$$

Hence, $g^d(\bar{z}) > g^d(z)$, a contradiction. $\square$

Definition 4.15. The *Nash solution (NA)*, is defined, for each bargaining problem $(F, d) \in B^N$, as $NA(F, d) := z$, where z in turn is defined such that

$$g^d(z) = \max_{x \in F(d)} g^d(x) = \max_{x \in F(d)} \prod_{i \in N}(x_i - d_i).$$

Proposition 4.12 makes sure that the Nash solution is a well-defined allocation rule. As a result, given a bargaining problem $(F, d) \in B^N$, a unique allocation in F_d will maximize the product of the gains of the players with respect to the disagreement point. Next, we will proceed to characterize the Nash solution by several convenient properties. But before we can do so, we must consider an auxiliary lemma.

Lemma 4.16. *Let $(F, d) \in B^N$ and let $z := NA(F, d)$. For each $x \in \mathbb{R}^N$, let $h(x) := \sum_{i \in N} \prod_{j \neq i} (z_j - d_j) x_i$. Then for each $x \in F$, $h(x) \leq h(z)$.*

Proof. Please refer to Appendix C. □

Now we are ready to define several properties of some solutions, which we will soon establish as characteristics of the Nash solution.

Definitions 4.17. Let ϕ be an allocation rule.

Pareto Efficiency (EFF): The allocation rule satisfies EFF if, for each $(F, d) \in B^N$, $\phi(F, d)$ is a Pareto efficient allocation. An allocation $\phi(F, d)$ is Pareto efficient if for every $\phi_0(F, d)$ such that $\phi_0(F, d) > \phi(F, d)$ for any individual i, there exists an individual j for whom $\phi_0(F, d) < \phi(F, d)$.

Symmetry (SYM): Let π denote a permutation of elements of N and, given $x \in \mathbb{R}^N$, let x^π be defined, for each $i \in N$, by $x_i^\pi := x_{\pi(i)}$. A bargaining problem $(F, d) \in B^N$ is symmetric if, for each permutation π of the elements of N, we have that:

(1) $d^\pi = d$, and
(2) for each $x \in F$, $x^\pi \in F$.

We have ϕ satisfying SYM if, for each symmetric bargaining problem $(F, d) \in B^N$, we have that for each pair $i, j \in N$, $\phi_i(F, d) = \phi_j(F, d)$.

Covariance with positive affine transformations (CAT): $f^A : \mathbb{R}^N \to \mathbb{R}^N$ is a positive affine transformation if, for each $i \in N$, there are $a_i, b_i \in \mathbb{R}$, with $a_i > 0$, such that, for each $x \in \mathbb{R}^N$, $f_i^A(x) = a_i x_i + b_i$. We have ϕ satisfying CAT if, for each $(F, d) \in B^N$ and each positive affine transformation f^A:

$$\phi(f^A(F), f^A(d)) = f^A(\phi(F, d)).$$

Independence of irrelevant alternatives (IIA): We have ϕ satisfying IIA if, for each pair of problems $(F, d), (\hat{F}, d) \in B^N$ where $\hat{F} \subset F$, $\phi(F, d) \in \hat{F}$ implies that $\phi(\hat{F}, d) = \phi(F, d)$.

Remark 4.18. EFF basically ensures that no one can be made better off without making someone else worse off. SYM, in turn, gives us that the outcome of the game is not biased towards certain players. CAT tells us that the choice of utility representations should not affect the allocation rule. Lastly, IIA says that if the feasible set is diminished and the proposal of the original allocation rule is still feasible in the new version, then the allocation rule must make the same proposal in the new problem.

Theorem 4.19. *The Nash solution is the unique allocation rule for n-player bargaining problems that satisfies EFF, SYM, CAT, and IIA.*

Proof. To check that NA satisfies EFF, SYM, CAT, and IIA, we let ϕ be an allocation rule for n-player bargaining problem that satisfies the four properties and let $(F, d) \in B^N$. Furthermore, let $z := NA(F, d)$. We just need to show that $\phi(F, d) = z$.

Let $U := \{x \in \mathbb{R}^N : h(x) \leq h(z)\}$, where h is defined as in Lemma 4.14, by which we know that $F \subset U$. Since we have by assumption that ϕ is CAT, let f^A be the positive affine transformation which associate, to each $x \in \mathbb{R}^N$, the vector $(f_1^A(x), ..., f_n^A(x))$ where, for each $i \in N$, $f_i^A(x) := \frac{1}{z_i - d_i} x_i - \frac{d_i}{z_i - d_i}$.

Now, we can compute $f^A(U)$ as follows:

$$\begin{aligned} f^A(U) &= \{y \in \mathbb{R}^N : (f^A)^{-1}(y) \in U\} \\ &= \{y \in \mathbb{R}^N : h((f^A)^{-1}(y)) \leq h(z)\} \\ &= \{y \in \mathbb{R}^N : h((z_1 - d_1) + d_1, ..., (z_n - d_n)y_n + d_n) \leq h(z)\} \\ &= \{y \in \mathbb{R}^N : \sum_{i \in N} \prod_{j \neq i} (z_j - d_j)((z_i - d_i)y_i + d_i) \leq \sum_{i \in N} \prod_{j \neq i} (z_j - d_j) z_i\} \end{aligned}$$

Skipping a few steps of (very straightforward) algebra, we arrive at:

$$\begin{aligned} f^A(U) &= \{y \in \mathbb{R}^N : \sum_{i \in N} \prod_{j \in N} (z_j - d_j) y_i \leq \sum_{i \in N} \prod_{j \in N} (z_j - d_j)\} \\ &= \{y \in \mathbb{R}^N : \prod_{j \in N} (z_j - d_j) \sum_{i \in N} y_i \leq \prod_{j \in N} (z_j - d_j) \sum_{i \in N} 1\} \\ &= \{y \in \mathbb{R}^N : \sum_{i \in N} y_i \leq n\}. \end{aligned}$$

Thus, $f^A(d) = (0, ..., 0)$ and $(f^A(U), f^A(d))$ is symmetric. Because ϕ satisfies EFF and SYM by assumption, $\phi(f^A(U), f^A(d)) = (1, ..., 1)$. As ϕ also satisfies CAT by assumption, $\phi(U, d) = (f^A)^{-1}((1, ..., 1)) = z$. Since $z \in F$, $F \subset U$, and we are given that ϕ satisfies IIA, $\phi(F, d) = z$. □

Not only does NA constitute the sole solution to an allocation rule satisfying EFF, SYM, CAT, IIA, but furthermore, none of these properties are unnecessary, as the proposition below affirms.

Proposition 4.20. *None of the axioms used in the characterization of the Nash solution given in Theorem 4.17 are superfluous.*

Proof. It suffices to show that, for each possible combination of three of the four properties, a solution that is not the Nash solution can be found.

Remove EFF: For each bargaining problem (F, d), the allocation rule ϕ defined as $\phi(F, d) := d$ satisfies SYM, CAT, and IIA but is not the Nash solution.

Remove SYM: Let the allocation rule ϕ be defined for an arbitrary bargaining problem (F, d) as follows:

$$\phi_i(F, d) := \begin{cases} \max_{x \in F_d} x_i \text{ for } i = 1 \\ \max_{x \in (F_d^i)} := \{x \in F_d : \forall j < i,\ x_j = \phi_j(F, d)\}\ \forall i > 1 \end{cases}$$

This allocation is known as "serial dictatorships" since there is an ordering of players and each player choose the allocations he prefers among the remaining options in turn. It satisfies EFF, CAT, and IIA but is not the Nash solution.

Remove CAT: Let the allocation rule ϕ be defined for an arbitrary bargaining problem (F, d) as follows:

$$\phi(F, d) := d + \bar{t}(1, ..., 1), \text{where } t := \max\{t \in \mathbb{R} : d + t(1, ..., 1) \in F_d\}.$$

This allocation is known as the "egalitarian solution"; it satisfies EFF, SYM, and IIA but is not the Nash solution.

Remove IIA: The "Kalai-Smorodinsky solution" (which will be defined formally below) satisfies EFF, SYM, and CAT but is not the Nash solution. □

Example 4.21. Consider the two-player bargaining problem (F, d) where $d = 0$ and F is the comprehensive hull of the set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 2\}$. What is the Nash solution? Because we know that NA always satisfies EFF and SYM, we can easily deduce that $NA(F, d) = (1, 1)$.

Now consider the problem $(\tilde{F}, d)$, where $\tilde{F} = F \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 1\}$. What is the new Nash solution? Since NA always satisfies IIA, $\tilde{F} \subset F$, and $NA(F, d) \in \tilde{F}$, we know that $NA(\tilde{F}, d) = (1, 1)$ and the Nash solution is unchanged.

Definition 4.22. The *utopia point* of a bargaining problem $(F, d) \in B^N$ is given by the vector $b(F, d) \in \mathbb{R}^N$ where, for each $i \in N$,

$$b_i(F, d) := \max_{x \in F_d} x_i.$$

Remark 4.23. For each $i \in N$, we can see that $b_i(F, d)$ represents the largest utility that player i can get in F_d. Thus, the utopia point of a bargaining problem is the point at which every player i gets their maximal utility payoff in F_d.

Definition 4.24. For each $(F, d) \in B^N$ let $\bar{t}$ be defined as $\bar{t} := \max\{t \in \mathbb{R} : d + t(b(F, d) - d) \in F_d\}$. The *Kalai-Smorodinsky solution* (KS), is defined by

$$KS(F, d) := d + \bar{t}(b(F, d) - d).$$

Remark 4.25. In the above definition, because F_d is compact, we can be sure that $\bar{t}$ is well-defined.

Definition 4.26. Let $(F, d), (\hat{F}, d) \in B^N$ be a pair of bargaining problems such that $\hat{F}_d \subset F_d$. Let $i \in N$ be such that, for each $j \neq i$, $b_j(\hat{F}, d) = b_j(F, d)$. If ϕ is an allocation rule for n-player bargaining problems that satisfies *individual monotonicity* (IM), then $\phi_i(\hat{F}, d) \leq \phi_i(F, d)$.

Remark 4.27. One possible interpretation of IM is that, if utility possibilities increase (decrease) in such a way that they can only affect one individual, then that individual should be made better off (worse off).

Example 4.28. We return once more to our bargaining problem example, this time with the Kalai-Smorodinsky in mind. We observe $b(F, d) = (\sqrt{2}, \sqrt{2})$ and $\bar{t} = 1/\sqrt{2}$. Thus, $KS_i(F, d) = 0 + 1/\sqrt{2}(\sqrt{2} - 0)$ where $i = \{1, 2\}$, or $KS(F, d) = (1, 1)$.

Taking $(\tilde{F}, d)$ as before, $b(F, d) = (\sqrt{2}, 1)$ and $\bar{t} = 1/\sqrt{3}$. Thus, $KS(\tilde{F}, d) = (\sqrt{2/3}, \sqrt{1/3})$. Unlike with the Nash solution, the KS solution shifts when the maximum potential payoff of Player 2 is diminished (as it is when proceeding from F to $\tilde{F}$). This is because KS satisfies IM, not IIA.

Notation 4.29. We denote the convex hull of set A as conv$\{A\}$. The convex hull of set A is, incidentally, defined as follows:

$$\operatorname{conv}\{A\} := \{\textstyle\sum_{i=1}^k \alpha_i x_i : k \in \mathbb{N},\ x_i \in A,\ \alpha_i \in \mathbb{R},\ \alpha_i \geq 0, \text{ and } \sum_{i=1}^k \alpha_1 = 1\}.$$

Theorem 4.30. *The Kalai-Smorodinsky solution is the unique allocation rule for two-player bargaining problems that satisfies EFF, SYM, CAT, and IM.*

Proof. First, we check that KS satisfies EFF, SYM, CAT, and IM. Because for each $i \in N$, $b_i(F,d)$ represents the largest utility each player i can get in F_d, because $\bar{t}$ is the largest possible $t \in \mathbb{R}$ that satisfies the constraints laid forth by (F,d), and because d is fixed, KS must be EFF. Furthermore, because for every $(F,d) \in B^N$, $KS_i(F,d) = KS_j(F,d)$ for each $i,j \in N$, KS is SYM. KS is CAT because of the following equality (where f^A defined as in definition of CAT):

$$KS(f^A(F), f^A(d)) = f^A(d) + \bar{t}(b(f^A(F), f^A(d)) - f^A(d)) = f^A(d + \bar{t}(b(F,d) - d)) = f^A(KS(F,d))$$

Finally, we claim KS is IM. Let $\hat{F}_d \subset F_d$. Then it follows that $b(\hat{F},d) \leq b(F,d)$. Thus, if for each $j \neq i$, $b_j(\hat{F},d) = (F,d)$, then $\phi_i(\hat{F},d) \leq \phi_i(F,d)$, since $b_i(\hat{F},d) \leq b_i(F,d)$.

Now, we let $(F,d) \in B^2$ and let ϕ be an allocation rule for two-player bargaining problems that satisfies the four properties. We need to show that $\phi(F,d) = KS(F,d)$. Because ϕ and KS satisfy CAT, we can assume without loss of generality that $b(F,d) = (1,1)$ and $d = (0,0)$. By the definition of KS, it must lie in the segment joining (0,0) and (1,1) and, hence $KS_1(F,d) = KS_2(F,d)$ (that is to say, both players recieve the same payoff). Define $\hat{F}$ by:

$$\hat{F} := \{x \in \mathbb{R}^2 : \exists\, y \in \text{conv}\{(0,0),(0,1),(1,0),KS(F,d)\} \text{ with } x \leq y\}.$$

We observe that, by this definition, $\hat{F} \subset F$ and $b(\hat{F},d) = b(F,d)$. Thus, since ϕ satisfies IM, we have that $\phi(\hat{F},d) \leq \phi(F,d)$. Because $\hat{F}$ is symmetric, $\phi(\hat{F},d) = KS(F,d)$ by SYM and EFF of ϕ. And so we have $KS(F,d) \leq \phi(F,d)$. Finally, since $KS(F,d)$ is Pareto efficient, we have $KS(F,d) = \phi(F,d)$. □

And so, EFF, SYM, CAT, and IM can only be satisfied by an allocation of a two-person bargaining problem if that allocation is the Kalai-Smorodinsky solution. Next, we consider these same three of these four properties (EFF, SYM, and IM) in the context of bargaining problems involving more than two people; here, we find another interesting result.

Theorem 4.31. *Let $n > 2$. Then, there is no solution for n-player bargaining problems satisfying EFF, SYM, and IM.*

Proof. Let $n > 2$ and suppose that ϕ is a solution for n-player bargaining problems satisfying EFF, SYM, and IM. Let $d = (0,...,0)$ and let $\hat{F}$ be defined as follows:

$$\hat{F} := \{x \in \mathbb{R}^N : \exists\, y \in \text{conv}\{(0,1,1,...,1),(1,0,1,...,1)\} \text{ with } x \leq y\}.$$

By EFF, $\phi(\hat{F},d)$ belongs to the segment joining $(0,1,1,...,1)$ and $(1,0,1,...,1)$ and hence $\phi_3(\hat{F},d) = 1$. Let

$$F = \{x \in \mathbb{R}^N : \textstyle\sum_{i=1}^N x_i \leq n-1 \text{ and}, \forall i \in N, x_i \leq 1\}.$$

Because ϕ satisfies EFF and SYM, for each $i \in N$, $\phi_i(F,d) = \frac{n-1}{n}$. However, $\hat{F} \subset F$ and $b(\hat{F},d) = b(F,d) = (1,...,1)$. And so, by IM, $\phi(\hat{F},d) \leq \phi(F,d)$. But this contradicts $\phi(\hat{F},d) = 1 > \frac{n-1}{n} = \phi(F,d)$. □

Due to their convenient properties, the Nash and Kalai-Smorodinsky solutions are by far the best-studied solutions to bargaining problems.

4.3. **Transferable Utility Games.** In many ways, transferable utility games are similar to NTU games. Different coalitions can be formed among the players in N, and these coalitions can choose among a range of allocations. Depending on the rules of the particular TU game (as with NTU games), binding agreements may or may not be made.

That being said, TU games depart from the NTU game framework in one very important way: in a TU game, utility can be transferred freely and without penalty among players. More formally, given a coalition S and an allocation $x \in V(S) \subset \mathbb{R}^{|S|}$, all the allocations that can be obtained from x by tranfers of utility among the players in S belong to $V(S)$. For this reason, $V(S)$ can be characterized by a single number:

$$\max_{x \in V(S)} \sum_{i \in S} x_i.$$

This number is incidentally called the worth of coalition S.

Definition 4.32. A *TU-game* is a pair (N, v), where N is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function of the game (that is, a map of the set of all possible coalitions of players to a set of payments). By convention, $v(\varnothing) := 0$.

Remark 4.33. TU games imply the existence of a numéraire good, or a standard against which value can be calculated to measure the relative worth of different goods and services (e.g., money).

Notation 4.34. When possible, we denote the game (N, v) by v. Also, we notate $v(\{i\})$ and $v(\{i\}\{j\})$ as $v(i)$ and $v(ij)$, respectively.

Definition 4.35. Let $(N, v) \in G^N$ and let $S \subset N$. The *restriction of* (N, v) *to the coalition* S is the TU game (S, v_s) where, for each $T \subset S$, $v_s(T) := v(T)$.

Example 4.36. Divide a million

A multimillionaire dies and leaves \$1 million to his three nephews on the condition that a majority among them agree on how to divide the sum; if they cannot agree, then the million dollars will be burned. We can model this situation as the TU game (N, v) where $N = \{1, 2, 3\}$, $v(1) = v(2) = v(3) = 0$, and $v(12) = v(13) = v(23) = v(N) = 1$.

Example 4.37. The glove game

In this game, three individuals own one glove each. The price of a pair of gloves is \$1, and gloves are only bought in left-right pairs. Let's say Player 1 has a left glove and Players 2 and 3 each have a right glove. Then the situation can be modeled by a TU game (N, v) where $N = \{1, 2, 3\}$, $v(1) = v(2) = v(3) = v(23) = 0$, and $v(12) = v(13) = v(N) = 1$.

Example 4.38. The visiting professor

In this example, three American political science departments, from universities in Chicago (group 1), Boston (group 2), and Washington DC (group 3), plan to invite a visiting professor from France next year. To minimize the costs, they coordinate the courses so the professor makes a tour visiting Chicago, Boston, and DC. Now, the three departments want to allocate the cost of the tour among themselves. For this reason, they have estimated the travel cost (in dollars) of the visit for all the possible coalitions of groups: $c(1) = 1500$, $c(2) = 1600$, $c(3) = 1900$, $c(12) = 2900$, $c(13) = 3000$, $c(23) = 1600$, and $c(N) = 3000$. Let $N =$

$\{1, 2, 3\}$. While (N, c) is a TU game (more specifically, a cost game), it represents disutility; thus, for consistency, we will examine the associated saving game (N, v) where, for each $S \subset N$, v is given by

$$v(S) = \sum_{i \in S} c(i) - c(S).$$

Thus, $v(1) = v(2) = v(3) = 0$, $v(12) = 200$, $v(13) = 400$, $v(23) = 1900$, and $v(N) = 2000$.

Among TU games, those that observe some or all of the following convenient properties are of particular importance.

Definitions 4.39. A TU game $v \in G^N$ is *superadditive* if, for each pair $S, T \subset N$ where $S \cap T = \varnothing$,

$$v(S \cup T) \geq v(S) + v(T).$$

A TU game $v \in G^N$ is *weakly superadditive* if, for each player $i \in N$ and each coalition $S \subset N \backslash \{i\}$,

$$v(S) + v(i) \leq v(S \cup \{i\}).$$

A TU game $v \in G^N$ is *additive* if, for each player $i \in N$ and each coalition $S \subset N \backslash \{i\}$, $v(S) + v(i) = v(S \cup \{i\})$. In particular, for each $S \subset N$,

$$v(S) = \sum_{i \in S} v(\{i\}).$$

Definitions 4.40. A TU game is *monotonic* if, for each pair $S, T \subset N$ with $S \subset T$, we have $v(S) \leq v(T)$.

A TU game $v \in G^N$ is *zero-normalized* if, for each player $i \in N$, $v(i) = 0$. Furthermore, the *zero-normalization of* v is the zero-normalized game w defined, for each $S \subset N$ by $w(S) := v(S) - \sum_{i \in S} v(i)$.

A TU game $v \in G^N$ is *zero-monotonic* if its zero-normalization is a monotonic game.

The main goal of TU game theory is to define solutions which are desirable. This desirability is most often a function of the stability or fairness of the allocation. Though approaches targeted at fairness will not be addressed in this paper, the interested reader is encouraged to consult *An Introduction to Mathematical Game Theory* [1], which examines a variety of approaches to fairness (Shapley value, the nucleolus, and the τ-value). Our focus will turn instead to the core, by far the most widely respected approach to capture stability in cooperation theory.

4.4. **The Core and Related Concepts.** Before we can detail the core as a notion of stability, we must first define several terms upon which the concept of the core depends.

Definition 4.41. Let $v \in G^N$ and let $x \in \mathbb{R}^N$ be an allocation. Then, x is *efficient* if $\sum_{i \in N} x_i = v(N)$.

Remark 4.42. For superadditive games, efficiency means that the total benefit from cooperation is shared among all of the players.

Definition 4.43. The allocation x is *individually rational* if, for each $i \in N$, $x_i \geq v(i)$.

Individual rationality is a logical provision because it ensures that no player gets less than what he can get by himself. As a utility maximizing agent, a player would not join a coalition unless this were the case.

Definition 4.44. The *set of imputations* of a TU game, $I(v)$, consists of all the efficient and individually rational allocations. As such, $I(v)$ defined as

$$I(v) := \{x \in \mathbb{R}^N : \textstyle\sum_{i \in N} x_i = v(N) \text{ and, for each } i \in N,\ x_i \geq v(i)\}.$$

Remark 4.45. The set of imputations of a superadditive game is always empty.

Because individual rationality is a provision of the set of imputations, no player has an incentive to block an allocation of $I(v)$. However, coalitions may still stand to gain from blocking the allocation.

Definition 4.46. Let $v \in G^N$. An allocation $x \in \mathbb{R}^N$ is *coalitionally rational* if, for each $S \subset N$, $\sum_{i \in S} x_i \geq v(S)$.

Definition 4.47. Let $v \in G^N$. The *core* of v, $C(v)$, is defined by

$$C(v) := \{x \in I(v) : \text{for each } S \subset N,\ \textstyle\sum_{i \in S} x_i \geq v(S)\}.$$

The elements of $C(v)$ are called *core allocations*.

Remark 4.48. Core allocations are also called strong Nash equilibria. The parallel is apt: just as a Nash equilibrium is an outcome such that no utility-maximizing player would change his or her action, a core allocation represents an outcome where no coalition of utility-maximizing players would change its action.

Thus, since the definition of the core contains a provision for coalitional rationality, it is stable in the sense that no coalition has an incentive to block a core allocation. Next, we will observe that the core is stable in another way as well.

Definition 4.49. Let $v \in G^N$. Let $S \subset N$, $S \neq \varnothing$, and let $x, y \in I(v)$. We say that y *dominates* x *through* S if

(1) for each $i \in S$, $y_i > x_i$ and
(2) $\sum_{i \in S} y_i \leq v(S)$.

We say y *dominates* x if there is a nonempty coalition $S \subset N$ such that y dominates x through S. Furthermore, x is an *undominated* imputation of v if there is no $y \in I(v)$ such that y dominates x.

Proposition 4.50. *Let $v \in G^N$. Then, if $x \in C(v)$, x is undominated.*

Proof. Let $x \in C(v)$ and suppose that there is $y \in I(v)$ and $S \subset N$, $S \neq \varnothing$, such that y dominates x through S. By the definition of the core, we know that $\sum_{i \in S} x_i \geq v(S)$. Since y dominates x through S, we have that $\sum_{i \in S} y_i \leq v(S)$. Thus, since $\sum_{i \in S} y_i > \sum_{i \in S} x_i$, we have that $v(S) \geq \sum_{i \in S} y_i > \sum_{i \in S} x_i \geq v(S)$, a contradiction. $\square$

We can now analyze the stability of previously considered TU games.

Notation 4.51. Let $v \in G^N$. Then $v = \{(v_1, v_2, ..., v_N)\}$ represents the allocation where each player $i \in N$ receives a payoff of v_i.

Example 4.52. Divide a million (continued from page 17)

The core of the game is empty, as, for each allocation, at least one each individual stands to gain from blocking that allocation. The fact that the core is empty indicates that the bargaining situation modeled by the game is highly unstable.

Example 4.53. The glove game (continued from page 17)

The core of the glove game consists of a single allocation: $\{(1,0,0)\}$. Thus, the player who has the unique glove (here, the only left glove) receives all the benefits. One interpretation of this is that the price of the right gloves gets driven to zero because each player in possession of a right glove is willing to sell it for any amount greater than zero (since they receive a payoff of zero if they don't sell it). Thus, each tries to undercut the other's price, and both players' prices decrease until they reach their mutual point of indifference, zero.

Example 4.54. The visiting professor (continued from page 17)

The core of the savings game v associated with the visiting professor allocation problem is the nonempty set $\{x \in I(v) : x_1 + x_2 \geq 200,\ x_1 + x_3 \geq 400, \text{ and } x_2 + x_3 \geq 1900\}$.

Definition 4.55. A TU game $v \in G^N$ is a *simple game* if

(1) it is monotonic,
(2) for each $S \subset N$, $v(S) \in \{0, 1\}$, and
(3) $v(N) = 1$.

Notation 4.56. We denote by S^N the class of simple games with n players.

Definition 4.57. Let $v \in S^N$ be a simple game. A coalition $W \subset N$ is a *winning coalition* if $v(W) = 1$.

Remark 4.58. To characterize a simple game v, it is enough to specify the collection W of its winning coalitions.

Definition 4.59. Let $v \in S^N$. Then, a player $i \in N$ is a *veto player* in v if $v(N\backslash\{i\}) = 0$.

Proposition 4.60. *Let $v \in S^N$ be a simple game. Then, $C(v) \neq \varnothing$ if and only if there is at least one veto player in v. Moreover, if $C(v) \neq \varnothing$, then*

$$C(v) = \{x \in I(v) : \textit{for each nonveto player } i \in N,\ x_i = 0\}.$$

Proof. Let $v \in S^N$, $x \in C(v)$, and A be the set of veto players.

(Forward direction) Suppose that $A = \varnothing$. Then, for each $i \in N$, $v(N\backslash\{i\}) = 1$ and, hence,

$$0 = v(N) - v(N\backslash\{i\}) \geq \sum_{j \in N} x_j - \sum_{j \in N\backslash\{i\}} x_j = x_i \geq 0.$$

So $x_i = 0$ for each $i \in N$. Thus, $\sum_{i \in N} x_i = 0$, which contradicts the efficiency of x.

(Backward direction) Suppose that $A \neq \varnothing$. Furthermore, denote the elements of A as $\{a_1, ..., a_k\}$ where $1 \leq k \leq N$. We claim that there is an $x \in C(v)$ such that

$$x := \begin{cases} \sum_i x_i = 1 \text{ where player } i \in \{a_1, ..., a_k\} \\ x_j = 0 \text{ if player } j \in N\backslash\{a_1, ..., a_k\}. \end{cases}$$

We see that x thus defined lies in the set of imputations, since $\sum_{i \in N} = 1 = v(N)$ and $x_i \geq v(i)$ for each $i \in N$. We also observe that for any coalition including all of the veto players, $\sum_{i \in S} x_i = 1 = v(A)$; likewise, for any coalition that doesn't include all of the veto players, $\sum_{i \in S} x_i = 0 = v(S)$. Thus, for each $S \subset N$, $\sum_{i \in S} x_i \geq v(S)$, so $x \in C(v)$.

(The "Moreover" statement) Suppose there exists $x_0 \in S^N$ such that $x_0 \in C(v)$ but is not of the form x. If it is not of the form x because a player $i \in \{a_1, ..., a_k\}$ is excluded from the veto player's coalition, then $\sum_{i\in N} x_i = 0 < 1 = v(N)$ and efficiency is violated. If it is not of the form x because a player $j \in N \backslash \{a_1, ..., a_k\}$ is included in the veto player's coalition and player j receives $x_j = 0$, then this is equivalent to player j not being a member of the coalition. (Player j would not receive $x_j > 0$ because such a strategy is dominated.) Thus x_0 must be of the form x if it lies in $C(v)$. □

Now, we turn our attention to a fourth and final standard of stability which the core meets: balance.

Definition 4.61. A family of coalitions $F \subset 2^N \backslash \{\varnothing\}$ is *balanced* if there are positive real numbers $\{\alpha_S : S \in F\}$ such that, for each $i \in N$,

$$\sum_{\substack{s\in F \\ i\in S}} \alpha_S = 1.$$

The numbers $\{\alpha_S : S \in F\}$ are called *balancing coefficients*.

Definition 4.62. A TU game $v \in G^N$ is *balanced* if, for each balanced family F, with balancing coefficients $\{\alpha_S : S \in F\}$, $\sum_{S\in F} \alpha_S v(S) \leq v(N)$. A TU game is *totally balanced* if, for each $S \subset N$, the TU game (S, v_s) is balanced.

One possible interpretation of balance is that, given a balanced family of coalitions F, the coefficient α_S associated with a given $S \in F$ represents the time the players in S are allocating to S. The balancing condition then requires that any given player has a set unit of time to allocate among its different coalitions in F. Thus, the balancing condition represents the impossibility of the players to allocate their time among the different coalitions in a manner that yields an aggregate payoff greater than $v(N)$.

This interpretation suggests that, in order for a TU game to be balanced, the worth of the coalitions different from N must be relatively small as compared to the grand coalition's worth. That observation proves to be the intuition behind the Bondareva-Shapley theorem, a useful result that lays out a necessary and sufficient condition for a game to have a nonempty core.

Theorem 4.63. *(Bondareva-Shapley theorem)*
Let $v \in G^N$. Then, $C(v) \neq \varnothing$ if and only if v is balanced.

Proof. (Forward direction) Let $v \in G^N$ be such that $C(v) \neq \varnothing$. Let $x \in C(v)$ and let F be a balanced family with balancing coefficients $\{\alpha_S : S \in F\}$. Then

$$\sum_{S\in F} \alpha_S(v_S) \leq \sum_{S\in F} \sum_{i\in S} \alpha_S x_i = \sum_{i\in N} (x_i \sum_{\substack{s\in F \\ i\in S}} \alpha_S) = \sum_{i\in N} x_i = v(N).$$

Thus, v is balanced.

(Backward direction) In the backward direction, we assume v is balanced and use a linear programming problem and its dual as well as the duality theorem to establish that there exists at least one element $\bar{x} \in C(v)$. However, because linear programming has not been addressed in this paper and the proof is rather technical, we will omit it. For the complete proof, as well as an alternative proof (which uses the minimax theorem and results of noncooperative game theory), the interested reader is encouraged to consult *An Introductory Course on Mathematical Game Theory* [1]. □

And so, we end by observing that the core is stable in four senses: it is individually rational, coalitionally rational, a set of undominated strategies, and balanced.

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Appendix A

Theorem A.1. *Let $(A, \succeq)$ be a decision problem and assume that $\succeq$ is antisymmetric. Then, $\succeq$ can be represented by a utility function if and only if there is a countable set $B \subset A$ that is order dense in A.*

Proof. (Forward direction) Let $(A, \succeq)$ be a decision problem where $\succeq$ is antisymmetric and assume that there is a utility function u representing $\succeq$ in A. Define $\bar{\mathbb{Q}}^2$ to be the following subset of $\mathbb{Q}^2$:

$$\bar{\mathbb{Q}}^2 := \{(q_1, q_2) \in \mathbb{Q}^2 : \text{there is } a \in A \text{ such that } q_2 > u(a) > q_1\}$$

Furthermore, define a function $g : \bar{\mathbb{Q}}^2 \to A$ to be such that, for every $(q_1, q_2) \in \bar{\mathbb{Q}}^2$, $q_2 > u(g(q_1, q_2)) > q_1$. Because the rationals are countable (provable by Cantor's Diagonalization Theorem), $\bar{B} := g(\bar{\mathbb{Q}}^2)$ is countable; moreover, by Lemma 2.10, so is the set $B := A^* \cup \bar{B}$. Is B order dense in A? Let $a_1, a_2 \in A$ be such that $a_2 \succ a_1$. If (a_1, a_2) is a gap, then, since A^* is the set of gap extremes and $A^* \subset B$, one can take $b = a_1$ or $b = a_2$ and observe that B is order dense in A. If (a_1, a_2) is not a gap, then there are $q_1, q_2 \in \mathbb{Q}$ and $\bar{a} \in A$ such that $a_2 \succ \bar{a} \succ a_1$ and $u(a_2) > q_2 > u(\bar{a}) > q_1 > u(a_1)$. Thus, $(q_1, q_2) \in \bar{\mathbb{Q}}^2$ and there is $b \in B$ such that $q_2 > u(b) > q_1$, so $a_2 \succ b \succ a_1$. And so, B is order dense in A in both cases.

(Backward direction) Let $B \subset A$ be a countable set order dense in A. We call a the first element in A if there is not $\bar{a} \in A$ such that $\bar{a} \succeq a$ (where $\bar{a} \neq a$); likewise, we call a the last element in A if there is not $\bar{a} \in A$ such that $\bar{a} \preceq a$. While the first and last elements of A might not exist, if they do, they are unique, since the set is ordered. So we can define $\bar{B}$ to be the set containing B and the first and last elements of A, if they exist. Since B is countable by Lemma 2.10, so is $\bar{B}$, and hence $B^* := \bar{B} \cup A^*$ as well. Furthermore, Theorem 2.5 tells us that there is a utility function $\bar{u}$ representing $\succeq$ in B^*. For each $a \in A$, define $u(a)$ as follows

$$u(a) := \sup\{\bar{u}(b) : b \in B^*, a \succeq b\}.$$

We claim that u is well-defined. To show that this is true, let $a \in A$. If $a \in B^*$, then $u(a) = \bar{u}(a)$; if $a \notin B^*$, then there exist $a_1, a_2 \in A$ such that $a_2 \succ a \succ a_1$. Moreover, since B is order dense in A and $a \notin B^*$, there must be $b_1, b_2 \in B$ such that $a_2 \succeq b_2 \succ a \succ b_1 \succeq a_1$. Thus the set $\{\bar{u}(b) : b \in B^*, a \succeq b\}$ is nonempty, as at the very least $\bar{u}(b_1)$ is in it, and bounded above by $\bar{u}(b_2)$. And so, $\{\bar{u}(b) : b \in B^*, a \succeq b\}$ has a supremum for each $a \in A$ and $u(a)$ is thus well defined.

Now all that remains is to check whether u is a utility function representing $\succeq$ in A. Let $a_1, a_2 \in A$ such that $a_2 \succ a_1$. If there were to be $b_1, b_2 \in B^*$ such that $a_2 \succeq b_2 \succ b_1 \succeq a_1$, then we could conclude that $u(a_2) \geq \bar{u}(b_2) > \bar{u}(b_1) \geq u(a_1)$. This means that $u(a_2) > u(a_1)$ and also implies that, for each pair $a_1, a_2 \in A$ such that $u(a_2) > u(a_1)$, we have $a_2 \succ a_1$. Thus, u would be a utility function.

We claim that there are $b_1, b_2 \in B^*$ such that $a_2 \succeq b_2 \succ b_1 \succeq a_1$. If $a_1, a_2 \in B^*$, then $b_1 = a$ and $b_2 = a$ gives us the desired result (i.e., that $a_2 \succeq b_2 \succ b_1 \succeq a_1$). If $a_1 \notin B^*$, because B is order dense in A, there is $b_2 \in B$ (and thus in B^*) such that $a_2 \succeq b_2 \succ a_1$. Because (a_1, b_2) is not a gap (since $a_1 \in B^*$, it cannot be a gap), there must be $\tilde{a} \in A$ such that $b_2 \succ \tilde{a} \succ a_1$. Therefore, by density of B in A, there is $b_1 \in B$ such that $\tilde{a} \succeq b_1 \succ a_1$. Applying the same logic to the case where $a_2 \notin B^*$, we derive the desired result again (i.e., that $a_2 \succeq b_2 \succ b_1 \succeq a_1$). ∎

Appendix B

Proposition B.1. *Let $G = (A, u)$ be a strategic game such that, for each $i \in N$,*

(1) A_i is a nonempty and compact subset of $\mathbb{R}^{m_i}$,
(2) u_i is continuous, and
(3) for each a_{-i}, $u_i(a_i, \cdot)$ is quasi-concave on A_i.

Then, for each $i \in N, BR_i$ is an upper hemicontinuous, nonempty-valued, closed-valued, and convex valued correspondence. Therefore, BR also satisfies the previous properties.

Proof. Let $i \in N$.

- *Nonempty-valued*: Since u_i is given to be continuous and defined on A_i, a compact set, it must have a maximum contained in A_i. Since BR_i is defined as these very maxima, BR_i cannot be empty.
- *Closed-valued*: We know that a continuous image of a compact space is compact. Therefore, because u_i is continuous and the set of strategies is compact, BR_i is compact. Furthermore, since all the elements of BR_i are in $\mathbb{R}$, BR_i must be closed by Heine-Borel.
- *Convex-valued*: Let $a_{-i} \in A_{-i}$ and $\tilde{a_i} \in BR_i(a_{-i})$. Let $r := u_i(a_{-i}, a_i) \geq r$. Convex-valuedness is implied by the quasi-concavity of u_i.
- *Upper hemicontinuous*: Suppose BR_i is not upper hemicontinuous. Thus, there is a sequence $a_k \subset A_{-i}$ converging to $\tilde{a} \in A_{-i}$ and an open set $B^* \in A_i$ with $BR_i(\tilde{a}) \subset B^*$, satisfying that, for each $k_0 \in \mathbb{N}$, there is $k \geq k_0$ such that $BR_i(a_k) \not\subset B^*$. This implies that there exists a sequence $\{\tilde{a}^m\} \subset A_i$, such that, for each $m \in \mathbb{N}$, $\tilde{a}^m \in BR_i(a^m) \backslash B^*$. Since A_i is compact and since compactness in $\mathbb{R}$ implies sequential compactness in $\mathbb{R}$, $\{\tilde{a}^m\}$ has a convergent subsequence. Without loss of generality, assume that $\{\tilde{a}^m\}$ itself converges and let $\hat{a}$ be its limit. As B^* is an open set, $A_i \backslash B^*$ must is closed. Hence, $\hat{a} \in A_i \backslash B^*$ (because closed sets contain their limit points) and, therefore, $\hat{a} \notin BR_i(\tilde{a})$ and we have a contradiction.

$\square$

Appendix C

Lemma C.1. *Let* $(F, d) \in B^N$ *and let* $z := NA(F, d)$. *For each* $x \in \mathbb{R}^N$, *let* $h(x) := \sum_{i \in N} \prod_{j \neq i} (z_j - d_j) x_i$. *Then for each* $x \in F$, $h(x) \leq h(z)$.

Proof. Suppose not; that is, suppose there is $x \in F$ such that $h(x) > h(z)$. For each $\epsilon \in (0, 1)$, let $x^\epsilon := \epsilon x + (1 - \epsilon) z$. Since F is convex, $x^\epsilon \in F$. Because $z \in F_d$ and $z > d$, then, for sufficiently small ϵ, $x^\epsilon \in F_d$. Furthermore,

$$\begin{aligned} g^d(x^\epsilon) &= \prod_{i \in N} (\epsilon x_i + (1 - \epsilon) z_i - d_i) \\ &= \prod_{i \in N} (z_i - d_i + \epsilon(x_i - z_i)). \end{aligned}$$

Moreover, by factoring, we can arrive at:

$$\begin{aligned} g^d(x^\epsilon) &= \prod_{i \in N} (z_i - d_i) + \epsilon \sum_{i \in N} \prod_{j \neq i} (z_j - z_j) + \sum_{i=2}^{n} \epsilon^i f_i(x, z, d) \\ &= g^d(z) + \epsilon(h(x) - h(z)) + \sum_{i=2}^{n} \epsilon^i f_i(x, z, d), \end{aligned}$$

where for each $i \in N$, $f_i(x, z, d)$ is a function depending on x, z, and d alone. Hence, because $h(x) > h(z)$, we have that $g^d(x^\epsilon) > g^d(z)$ for ϵ sufficiently small, a contradiction of

$$z = \max_{x \in F_d} g^d(x).$$

□