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STABILITY OF MULTIPLE MARKETS: THE HICKS CONDITIONS

By LLOYD A. METZLER

I. INTRODUCTION

ONE of the most important achievements of modern economics is the integration of dynamic theory with the method of comparative statics. Until recently, dynamics and statics were two separate and distinct fields of inquiry, with virtually no connection.

The separation of the two types of analysis is exemplified by the differences between traditional price theory and traditional business-cycle theory. The former dealt with equilibrium prices and quantities, and described how these equilibria were altered by changes in tastes, costs, and methods of production. The behavior of prices when supply is not equal to demand was seldom discussed. In contrast traditional business-cycle theory described cumulative processes of expansion and contraction, placing much emphasis upon the time sequence by which a rise in prices or incomes leads to a further rise, or a fall to a further fall. While oscillations of the system about some normal level were analyzed, the nature of this norm was not clearly established. Nor was the cumulative process itself related to the equilibrium of the system. Dynamic economics, as exemplified by business-cycle theory, thus had little in common with comparative statics, the method of the classical economists.

One of the first steps toward an integration of statics and dynamics was taken by Professor Hicks in his *Value and Capital*.[1] Hicks insisted that comparative statics has no meaning unless the economic system is dynamically stable. In Chapter V, he said:

> The laws of change of the price-system, like the laws of change of individual demand, have to be derived from stability conditions. We first examine what changes are necessary in order that a given equilibrium system should be stable; then we make an assumption of regularity, that positions in the neighborhood of the equilibrium position will be stable also; and thence we deduce rules about the way in which the price-system will react to changes in tastes and resources.

The important innovation in Hicks's analysis is thus the derivation of properties of the equilibrium system from the conditions of stability of a corresponding dynamic system. He argues that, since only stable sys-

[1] J. R. Hicks, *Value and Capital*, London, 1939.

tems tend to approach equilibrium when disturbed, static analysis really implies stability; and the conditions of stability provide important information about the static system.

While the usefulness of dynamic economics in describing displacements of a static system was clearly demonstrated by Professor Hicks, his method remained incomplete, as Professor Samuelson pointed out, because, in the most general case, it lacked an explicit dynamic system.[2] Starting with the case of a single commodity, Hicks made the assumption that price tends to fall whenever supply exceeds demand and to rise when demand exceeds supply. From this assumption, it follows that a market involving only one commodity is stable provided a rise in price above equilibrium creates an excess supply, while a fall in price below equilibrium creates an excess demand. For under such conditions a displacement of price from equilibrium tends to be self-corrective; it sets in motion forces which restore equilibrium.

Hicks attributes these conclusions with respect to a single-commodity market to Walras and Marshall; he regards his own discussion in *Value and Capital* as simply a restatement of the Walras-Marshall conclusions. He then attempts to generalize the Walras-Marshall results to a market system involving a large number of commodities. Unfortunately, the generalization is attempted without the aid of an explicit dynamic system; instead of discussing the stability of a true dynamic system involving a large number of commodities, Hicks simply extends the Walras-Marshall conclusions to the case involving several commodities. In other words, he assumes, but does not prove, that an individual market within the system is stable, as in the single-commodity case, provided a reduction in price below equilibrium creates an excess demand for that particular commodity, while an increase in price creates an excess supply. Hicks realizes, of course, that in systems comprising several commodities the excess demand for and supply of a particular commodity cannot be discussed without reference to the prices of the other commodities. He therefore presents two definitions of stability, distinguishing one from the other according to the behavior of the other prices in the system. First, a market is defined as *imperfectly stable* if a fall in the price of a particular commodity creates an excess demand for that commodity, *after all other prices have adjusted themselves so that supply is again equal to demand in all markets except the one in question*. It is then tacitly assumed that the whole system of markets is dynamically stable if each market taken by itself is imper-

[2] See Paul A. Samuelson, "The Stability of Equilibrium: Comparative Statics and Dynamics," Econometrica, Vol. 9, April, 1941, pp. 97–120, esp. pp. 111–112.

fectly stable according to this definition.[3] Second, a market is defined as *perfectly stable* if a fall in price below equilibrium creates an excess demand *after any given subset of prices in other markets is adjusted so that supply again equals demand, with all remaining prices held constant*. And again it is assumed that if each single market is stable in this sense, the entire system will be dynamically stable.

Superficially, this seems plausible. In reality, however, the one-thing-at-a-time method cannot always be applied to a multiple-market system in the manner proposed by Hicks. It cannot be assumed, as in the Hicks analysis, that when the price of one commodity is out of equilibrium the prices of all other commodities are either unchanged or are instantaneously adjusted to their new equilibria. For this reason, the Hicks stability conditions cannot be accepted unless it is shown that they are related to the stability of a true dynamic system. The errors of the Hicks method were first demonstrated by Samuelson in his pioneer article on the significance of dynamics to static analysis.[4] It was there shown that imperfect stability, in the Hicks sense, is neither a necessary nor a sufficient condition for true dynamic stability. An example was given of a dynamic system which was unstable despite the fact that it was imperfectly stable in the Hicks sense. Another example was given of a system which was dynamically stable even though it was neither perfectly nor imperfectly stable according to Professor Hicks's definitions. In a later note, Professor Samuelson demonstrated that even perfect stability is insufficient to insure true dynamic stability under all circumstances.[5]

From the Samuelson examples, one might infer that Hicksian stability is only remotely connected with true dynamic stability. For some problems this is correct. But for others, the Hicks conditions are highly useful despite their lack of generality. Their usefulness in economics is due to two circumstances. In the first place, they provide a set of stability conditions which are independent of the speed of response of individual prices to discrepancies between supply and demand. In the following section, it will be shown that the Hicks conditions of perfect stability are necessary if stability is to be independent of such price responsiveness. Second, and more important, in a certain class of market systems Hicksian perfect stability is both necessary and sufficient for true dynamic stability. In particular, if all commodities are gross substitutes, the conditions of true dynamic stability are identical

[3] J. R. Hicks, *Value and Capital*, p. 67.

[4] P. A. Samuelson, *op. cit.*, pp. 111–112. This article presents the first complete synthesis of dynamics and comparative statics.

[5] P. A. Samuelson, "The Relation Between Hicksian Stability and True Dynamic Stability," Econometrica, Vol. 12, July–October, 1944, pp. 256–257.

with the Hicks conditions of perfect stability.[6] This proposition is demonstrated in III below.

II. STABILITY AND RELATIVE PRICE FLEXIBILITY

Consider a market for n commodities, and suppose that the demand and supply for the ith of these are $D_i(p_1, p_2, \cdots, p_n)$ and $S_i(p_1, p_2, \cdots, p_n)$, respectively, where $p_1, p_2, \cdots, p_n$ are the prices of the n commodities. For any given set of prices, the excess demand for the ith commodity is

$$x_i = D_i(p_1, p_2, \cdots, p_n) - S_i(p_1, p_2, \cdots, p_n).$$

In the neighborhood of equilibrium, the change in this excess demand with respect to a change in p_i, *after all other prices have adjusted themselves so that supply is equal to demand in the other markets*, is

$$\frac{dx_i}{dp_i} = \frac{\Delta}{\Delta_{ii}} \tag{1}$$

where $\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$, and where $a_{ij} \equiv \dfrac{\partial(D_i - S_i)}{\partial p_j} \equiv \dfrac{\partial x_i}{\partial p_j}$.

Since a rise in price must reduce excess demand when the system is imperfectly stable, imperfect stability in the Hicks sense implies that the minors, Δ_{ii}, of Δ all have signs opposite to the sign of Δ. Likewise, perfect stability means that for any mth-order principal minor, ${}^m\Delta$, of Δ, $(m<n)$, the cofactors ${}^m\Delta_{ii}$ have signs opposite the sign of ${}^m\Delta$. Thus, if the system is perfectly stable in the Hicks sense, the principal minors of Δ will alternate in sign, as follows:[7]

$$a_{ii} < 0, \quad \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} < 0, \text{ etc.},$$

so that the sign of Δ must be the same as the sign of $(-1)^n$.

The difficulty with these stability conditions, as Samuelson has shown, is that they are not derived from any explicit dynamic system. When a true dynamic system is introduced, it becomes apparent that

[6] The jth commodity in a group is defined as a gross substitute for the ith commodity if an increase in the price of the jth commodity creates excess demand for the ith commodity.

[7] J. R. Hicks, *op. cit.*, p. 315.

stability depends not only upon the slopes, a_{ij}, of the excess-demand functions, but also upon relative speeds of adjustment in the individual markets.[8] Suppose, for example, that the rate of change of price in each market is directly proportional to the difference between demand and supply in that market. Then for the ith commodity, we have

$$\frac{dp_i}{dt} = \kappa_i[D_i(p_1, p_2, \cdots, p_n) - S_i(p_i, p_2, \cdots, p_n)]. \tag{2}$$

The factor κ_i, in (2), is the "speed of adjustment" in the ith market. It measures the speed with which the price responds to a given discrepancy between demand and supply. For small deviations from equilibrium, (2) may be written[9]

$$\frac{dp_i}{dt} = \kappa_i a_{i1}(p_1 - p_1{}^0) + \kappa_i a_{i2}(p_2 - p_2{}^0) + \cdots + \kappa_i a_{in}(p_n - p_n{}^0) \qquad (i = 1, 2, \cdots, n), \tag{3}$$

where the superscripts indicate equilibrium prices. The stability of this system obviously depends both upon the speeds of adjustment, κ_i, and upon the slopes, a_{ij}, of the excess-demand functions.

The Hicks conditions are stated entirely in terms of the slopes a_{ij}. This means that the Hicks conditions can be true stability conditions only in systems in which stability is independent of the relative speeds of adjustment in the separate markets. In general, this will not be the case. Consider, for example, the following dynamic system:

$$\begin{aligned} \frac{dp_1}{dt} &= -\kappa_1(p_1 - p_1{}^0) - \kappa_1(p_2 - p_2{}^0), \\ \frac{dp_2}{dt} &= 2\kappa_2(p_1 - p_1{}^0) + \kappa_2(p_2 - p_2{}^0). \end{aligned} \tag{4}$$

This system possesses neither perfect nor imperfect stability in the

[8] This was first pointed out by Professor Oscar Lange, in *Price Flexibility and Employment*, Bloomington, Ind., 1944, pp. 94–99.

[9] The use of linear approximations in dynamic analysis is frequently criticized, particularly by those who have little appreciation of the difficulties presented by more complex systems. I believe the case for linear systems is much stronger than is commonly supposed. Most of the statistical investigations of such important functions as the propensity to consume and the propensity to import fail to show any significant departure from linearity. In business-cycle studies, the use of linear systems therefore has empirical support. In any case, Samuelson has shown that stability of linear approximations is a necessary condition, if not a sufficient one, for stability of more complicated dynamic systems (P. A. Samuelson, "The Stability of Equilibrium: Linear and Nonlinear Systems," ECONOMETRICA, Vol. 10, January, 1942, pp. 1–25).

Hicks sense. Actually, it may be shown that the system is stable or unstable in the true dynamic sense according as κ_1, the speed of adjustment in the first market, is greater or less than κ_2, the speed of adjustment in the second market.[10]

Since Hicks does not mention these speeds of adjustment explicitly, his analysis may be interpreted as an attempt to develop stability conditions which are independent of relative speeds of adjustment in individual markets. Indeed, the method of exposition which he uses clearly implies such independence. In discussing the effects of a change in the price of the *i*th commodity upon the excess demand for that commodity, he assumes that prices of all other commodities are adjusted to their new equilibria while no further change occurs—initially at least—in the price of the *i*th commodity. In other words, he assumes that the speed of adjustment in the *i*th market is very small relative to the speeds of adjustment in the other markets of the system. But when he considers the stability of any other market in the system, such as the *j*th market for example, he likewise assumes that the speed of adjustment in this market is small relative to the other markets. These assumptions are obviously inconsistent; the speed of adjustment in the *i*th market cannot be small relative to the speed in the *j*th market at the same time that the speed of adjustment in *j* is small relative to the speed in *i*. In reality, Hicks is considering a different dynamic system corresponding to the excess demand of each separate market. For this reason, the stability conditions which he derives cannot be applicable to the entire system of markets unless true dynamic stability is independent of relative speeds of adjustment in the separate markets.

More than this can be said, however. While the Hicks conditions may not be identical with true dynamic-stability conditions, it is easily demonstrated that *perfect stability, in the Hicks sense, is necessary for true stability if the market system is to be stable for all possible sets of speeds of adjustment.* To prove this proposition, consider the characteristic equation corresponding to the *n*-commodity system, (3). This market system is stable, for any initial prices, provided the roots of the following equation have no positive or zero real parts:

$$(5)\qquad \begin{vmatrix} \kappa_1 a_{11}-\lambda & \kappa_1 a_{12} & \cdots & \kappa_1 a_{1n} \\ \kappa_2 a_{21} & \kappa_2 a_{22}-\lambda & \cdots & \kappa_2 a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \kappa_n a_{n1} & \kappa_n a_{n2} & \cdots & \kappa_n a_{nn}-\lambda \end{vmatrix} \equiv \left| \kappa_i a_{ij} - \delta_{ij}\lambda \right| = 0.$$

[10] The characteristic equation of (4) is $\lambda^2+(\kappa_1-\kappa_2)\lambda+\kappa_1\kappa_2=0$. The real parts of the solutions of this equation are positive or negative according as κ_2 is greater than or less than κ_1.

This obviously requires that $A \equiv |\kappa_i a_{ij}|$ shall have the same sign as $(-1)^n$. For when λ becomes sufficiently large, the sign of $|\kappa_i a_{ij} - \delta_{ij}\lambda|$ is the same as the sign of $(-1)^n$, as may be seen by expanding (5). Hence if A, the determinant obtained by putting $\lambda = 0$ in (5), has a sign opposite to $(-1)^n$, $|\kappa_i a_{ij} - \delta_{ij}\lambda|$ must be equal to zero for some positive real value of λ. In other words, equation (5) must have a positive real root, and the system (3) must be unstable. Likewise, if $A = 0$, then $\lambda = 0$ is one root of (5), and the system is on the border line between stability and instability. Thus the system (3) cannot be stable unless its determinant, A, has the same sign as $(-1)^n$. But A is the same as the determinant of the Hicks static equations, except for multiplication by the positive speeds of adjustment $\kappa_1\kappa_2 \cdots \kappa_n$. Hence, sign $A =$ sign $(-1)^n$ is the last of the Hicks conditions of perfect stability.

Consider now what happens to the stability of (3) when the speeds of adjustment are changed. If the market system is to be stable for any set of speeds of adjustment, it must be stable when some of the κ_i are quite small, relative to others in the set. In other words, the system must be stable even when any subgroup of prices is completely inflexible. Suppose, for example, that the speeds of adjustment κ_{m+1}, $\kappa_{m+2}, \cdots, \kappa_n$ are all put equal to zero. The system (3) then becomes an mth-order system in the variables $p_1, p_2, \cdots, p_m$, and the determinant corresponding to the subset is

$$\begin{vmatrix} \kappa_1 a_{11} & \kappa_1 a_{12} & \cdots & \kappa_1 a_{1m} \\ \kappa_2 a_{21} & \kappa_2 a_{22} & \cdots & \kappa_2 a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \kappa_m a_{m1} & \kappa_m a_{m2} & \cdots & \kappa_m a_{mm} \end{vmatrix}.$$

From the preceding analysis, it is clear that stability requires that this determinant shall have the sign of $(-1)^m$. But apart from a positive factor $\kappa_1\kappa_2 \cdots \kappa_m$, the determinant is simply an mth-order principal minor of the determinant of the Hicks static equations. Since the argument holds for any value of m from 1 to n, it follows immediately that stability of a multiple-market system cannot be independent of relative speeds of adjustment in the separate markets unless the Hicks conditions of perfect stability are satisfied.[11]

What is the economic significance of this result? Why should anyone be interested in stability conditions which do not depend upon speeds

[11] Cf. Oscar Lange, *op. cit.*, p. 96. The proof just given of the necessity of the Hicks conditions is closely related to the Lange concept of *partial* stability of any given order. It is equivalent to the statement that stability cannot be independent of relative speeds of adjustment unless the system possesses partial stability of all orders.

of adjustment in individual markets? In describing a given market system, an economist is always confronted with a given set of relative speeds. Why not determine the stability conditions for this system simply in terms of these given speeds of adjustment? In other words, why should the economist be concerned about the effects of changes in relative speeds of adjustment?

Two answers may be given to these questions. First, the extent to which the stability of a group of markets depends upon speeds of adjustment is a question of considerable interest. It is important to know, for example, whether the inflexibility of certain prices is a stabilizing factor or whether the markets would be stable even if all prices were responsive to discrepancies between supply and demand. If the Hicks conditions of perfect stability are not satisfied, stability of the system clearly depends upon a relative inflexibility of certain prices. Second, and more important, the conditions which govern price responsiveness are much more obscure than are the static supply and demand conditions in individual markets. To a large extent, speeds of adjustment are determined by institutional factors such as the willingness or ability of buyers and sellers to hold or to reduce inventories. Static conditions of supply and demand, on the other hand, depend largely upon maximum-profits conditions of producers and consumers. For this reason, economists are usually more confident of their knowledge of supply and demand conditions than of their knowledge of such dynamic factors as speeds of adjustment. If possible, it is therefore desirable to describe market systems in terms which are independent of speeds of adjustment.

Nevertheless, an uncritical acceptance of the Hicks conditions may lead to serious errors, particularly if there is a high degree of complementarity between certain goods. While the Hicks conditions of perfect stability are *necessary* if stability is to be independent of relative speeds of adjustment, they are not always *sufficient*. In other words, a market may be unstable, for certain speeds of adjustment, even though it possesses perfect stability according to the Hicks criteria. Consider, for example, the following system:

$$
\begin{aligned}
\frac{dp_1}{dt} &= -\kappa_1(p_1 - p_1^0) - 0.8\kappa_1(p_2 - p_2^0), \\
\frac{dp_2}{dt} &= -\kappa_2(p_2 - p_2^0) - 0.8\kappa_2(p_3 - p_3^0), \\
\frac{dp_3}{dt} &= -10\kappa_3(p_1 - p_1^0) - \kappa_3(p_2 - p_2^0) - \kappa_3(p_3 - p_3^0).
\end{aligned} \tag{6}
$$

This system possesses perfect Hicksian stability, and yet it may be either stable or unstable dynamically, the outcome depending upon relative speeds of adjustment in the individual markets. If the three speeds of adjustment in the first, second, and third markets respectively are proportional to the three numbers 2, 2, 1, the system is stable. On the other hand, if the speeds of adjustment are proportional to 1, 1, 2, it may easily be shown that the system is unstable dynamically despite the fact that it satisfies the Hicks conditions for perfect stability.[12]

The preceding analysis has demonstrated two propositions. First, the Hicks conditions for perfect stability are necessary if a market system is to be stable for any set of relative speeds of adjustment. Second, the Hicks conditions are not always sufficient; some market systems may be stable with one set of relative speeds, and unstable with another, even though the static system is perfectly stable (in the Hicks sense) in both cases. If this were all that could be said, the usefulness of the Hicks conditions would be seriously limited, for true dynamic stability conditions would not be directly related to the Hicks conditions. In the following section, however, it will be shown that, for at least one class of market systems, the Hicks conditions of perfect stability are identical with the true dynamic conditions. This fact gives the Hicks conditions an important application, particularly in the field of income analysis.

III. SUBSTITUTION AND STABILITY CONDITIONS

In the present section, the following proposition will be proved: *If all commodities in a market system are gross substitutes, the conditions for true dynamic stability are identical with the Hicks conditions for perfect stability.*

The fact that all commodities are gross substitutes means that a rise in the price of the ith commodity, all other prices remaining unchanged,

[12] With the first set of speeds of adjustment (2, 2, 1), the characteristic equation of (6) is

$$\lambda^3+5\lambda^2+6.4\lambda+26.4=0.$$

By Descartes' rule of signs this equation has no positive or zero real roots. Moreover, by application of the Routh stability conditions, it may be shown that the above equation has no complex roots with nonnegative real parts. The system is therefore stable. (See E. J. Routh, *Advanced Rigid Dynamics*, London, 1905, pp. 297–301.) With the second set of speeds of adjustment (1, 1, 2), the characteristic equation of (6) is

$$\lambda^3+4\lambda^2+2.6\lambda+13.2=0.$$

While the equation has no nonnegative real roots, it fails to satisfy the Routh conditions and must therefore have complex roots with positive real parts.

A discussion of the Routh conditions may be found in R. A. Frazer, W. J. Duncan, and A. R. Collar, *Elementary Matrices*, Cambridge, England, 1938, p. 154. Readers unfamiliar with Routh's work on dynamics may easily verify the conclusions just presented by solving the two numerical equations.

reduces the excess demand for that commodity and increases the excess demands for all other commodities. Thus $a_{ii}<0$ and $a_{ij}>0$ for any i and any j. Consider again the dynamic system (3), repeated below for convenience:

$$\frac{dp_i}{dt} = \kappa_i \sum_{j=1}^{n} a_{ij}(p_j - p_j^0) \qquad (i = 1, 2, \cdots, n). \tag{7}$$

The speeds of adjustment, κ_i, in this system are determined partly by the units in which time is measured. But stability of the system is obviously independent of the time unit. Without loss of generality, we may therefore choose our time unit in such a way that $\kappa_i a_{ii}$ is less than unity in absolute value for all i. It is then possible to prove our theorem by consideration of the following related system of difference equations:

$$\begin{aligned} y_1(t) &= (\kappa_1 a_{11}+1)y_1(t-1)+\kappa_1 a_{12}y_2(t-1)+\cdots+\kappa_1 a_{1n}y_n(t-1),\\ y_2(t) &= \kappa_2 a_{21}y_1(t-1)+(\kappa_2 a_{22}+1)y_2(t-1)+\cdots+\kappa_2 a_{2n}y_n(t-1),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ y_n(t) &= \kappa_n a_{n1}y_1(t-1)+\kappa_n a_{n2}y_2(t-1)+\cdots+(k_n a_{nn}+1)y_n(t-1). \end{aligned} \tag{8}$$

The proof will be accomplished in two stages. First, it will be shown that the differential equations (7) are stable or unstable according as the difference equations (8) are stable or unstable. Second, it will be shown that the Hicks conditions of perfect stability are necessary and sufficient for stability of the difference equations (8).

The characteristic equation of the difference equations (8) is

$$\begin{vmatrix} \kappa_1 a_{11}+1-\rho & \kappa_1 a_{12} & \cdots & \kappa_1 a_{1n} \\ \kappa_2 a_{21} & \kappa_2 a_{22}+1-\rho & \cdots & \kappa_2 a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \kappa_n a_{n1} & \kappa_n a_{n2} & \cdots & \kappa_n a_{nn}+1-\rho \end{vmatrix} = 0. \tag{9}$$

If we substitute in (9), $\lambda=\rho-1$, we obtain the equation (5), which is the characteristic equation of the differential system (7). Thus the roots of (9) are greater by unity than the roots of (5), the characteristic equation of (7).

Now stability of the difference system (8) requires that the moduli of all the roots of (9) be less than unity (i.e., that all roots lie within the unit circle of the complex plane), whereas stability of (7) requires that the real parts of the roots of (5) be negative. If all the roots of (9) lie within the unit circle of the complex plane, it is obvious that the real parts of the roots of (5) will all be negative. Hence, stability of the difference equations (8) is sufficient to insure stability of the differential equations (7). But is it also necessary? Conceivably, one of the roots of (9) could lie outside the unit circle (if its value were $2\sqrt{-1}$, for example), and yet all the roots of (5) could have negative real parts. In this case, the differential equations would be stable even though the corresponding difference equations were unstable. But this is impossible in the present case, for it may be shown that *under the restrictions placed upon a_{ij} and a_{ii}, the root of* (9) *with the largest modulus is a positive real root.* From this proposition, it follows that if one of the complex roots of (9) has a modulus which is equal to or greater than unity, there must also be a positive real root equal to or greater than unity. And in this case (5) will have a root with a zero or positive real part, and the differential equations (7) will be unstable.

To prove that the root of (9) with the largest modulus is a positive real root, consider the behavior of the difference equations (8). Suppose that initially $y_1, y_2, \cdots, y_n$ are all positive, and that no further disturbances occur from outside the system. Then, since all the coefficients on the right of (8) are positive, $y_k(t)$ remains positive throughout. That is, (8) shows that $y_k(t) \geqq 0$ whenever $y_1(t-1), y_2(t-1), \cdots, y_n(t-1)$ are all greater than or equal to zero. Now the general solution of these difference equations, for y_k, is

$$y_k(t) = \sum r_i{}^t[g_i(t) \cos (\theta_i t) + h_i(t) \sin (\theta_i t)], \tag{10}$$

where r_i is the modulus of the ith distinct root of (9), and $g_i(t)$ and $h_i(t)$ are polynomials of degree one less than the multiplicity of the ith root, with coefficients dependent upon initial conditions. The solution, (10), includes real as well as complex roots. For positive real roots, $\theta_i = 0$, while for negative real roots, $\theta_i = \pi$. Suppose that r_1 is the largest modulus, and rewrite (10) as follows:

$$\begin{aligned} y_k(t) = r_1{}^t \Bigg[& g_1(t) \cos (\theta_1 t) + h_1(t) \sin (\theta_1 t) \\ & + \sum_{i \neq 1} \left(\frac{r_i}{r_1}\right)^t \{g_i(t) \cos (\theta_i t) + h_i(t) \sin (\theta_i t)\} \Bigg]. \end{aligned} \tag{11}$$

Since r_i/r_1 is less than unity for all i included in the summation,

$$\sum_{i \neq 1} \left(\frac{r_i}{r_1}\right)^t \{g_i(t) \cos (\theta_i t) + h_i(t) \sin (\theta_i t)\}$$

can be made as small as we please by taking t sufficiently large. Beyond a certain time interval, the sign of $y_k(t)$ will therefore be the same as the sign of $g_1(t) \cos (\theta_1 t) + h_1(t) \sin (\theta_1 t)$ for at least some values of t. This latter expression, in turn, is dominated, for sufficiently large t, by the parts of $g_1(t)$ and $h_1(t)$ with the highest power of t. If the dominant root (or dominant pair of conjugate complex roots) is repeated ν times, this leading expression will take the form $t^{\nu-1}[A \cos(\theta_1 t) + B \sin(\theta_1 t)] \equiv t^{\nu-1} C \cos(\theta_1 t + \phi_1)$. Thus, if t becomes sufficiently large, $y_k(t)$ will have the same sign as $C \cos(\theta_1 t + \phi_1)$ for at least some values of t. But unless θ_1 is zero, $\cos(\theta_1 t + \phi_1)$ assumes both positive and negative values for different integral values of t, and $y_k(t)$ therefore becomes negative for certain intervals of t regardless of whether C is positive or negative. Since this is clearly impossible when all the y_k's are initially positive, it follows that θ_1, the amplitude of the root with the largest modulus, must be zero, and this root must therefore be a positive real root.[13]

This completes the first part of our proof. Before presenting the second part, the preceding results may be summarized briefly. First, if the difference equations (8) are stable, all the roots of (9) must lie within the unit circle of the complex plane. In this case, no roots of (5), the characteristic equation of our differential system, can have positive or zero real parts, since the roots of the latter are less than those of (9) by exactly unity. It follows that stability of the difference equations (8) is sufficient to insure stability of the corresponding differential equations (7). Second, if the difference equations (8) are unstable, (9) must have at least one positive real root greater than unity, since the root of (9) with the largest modulus is a positive real root. But if (9) has a positive real root greater than unity, (5) will have a real root greater than zero, and the differential equations (7) will therefore be unstable. Thus, if the difference equations are unstable, the corresponding differential equations will likewise be unstable.

It has now been demonstrated that the differential system (7) is stable or unstable whenever the corresponding difference equations (8)

[13] This proof assumes that the coefficient, C, is not zero. While this may not be true for certain initial conditions, it is easily shown that C cannot be identically zero for all possible positive values of $y_1(0), y_2(0), \cdots, y_n(0)$. The coefficients such as C are linear functions of the initial values, which means that C can be zero only when the initial conditions satisfy an equation of the form $\alpha_1 y_1(0) + \alpha_2 y_2(0) + \cdots + \alpha_n y_n(0) = 0$. This equation defines a plane in n-dimensional space; any combination of $y_1(0), y_2(0), \cdots, y_n(0)$ which lies on this plane yields a zero value for C, while all other combinations yield nonzero values. Now it is obvious intuitively that no plane in n-dimensional space can include all combinations of the n variables which lie in the positive hyperquadrant. In other words, sets of positive values of $y_1(0), y_2(0), \cdots, y_n(0)$ may always be found outside the plane. Since the proof given in the text is valid for any combination of positive initial conditions, the values of $y_1(0), y_2(0), \cdots, y_n(0)$ may always be selected so as to make C nonzero. I am indebted to Professor Samuelson for pointing this out.

are stable or unstable. To complete the proof that the Hicks conditions of perfect stability are necessary and sufficient for stability of both the differential and the difference equations, it remains only to prove that they are necessary and sufficient for stability of the difference equations (8).

Consider again the characteristic equation (9). In its expanded form, this equation may be written

$$(12)\quad (\rho - 1)^n - \sum \kappa_i a_{ii}(\rho - 1)^{n-1} + \sum \kappa_i\kappa_j \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} (\rho - 1)^{n-2} + \cdots + (-1)^n \kappa_1\kappa_2 \cdots \kappa_n \Delta = 0,$$

where $\Delta \equiv |a_{ij}|$ as before.

We have seen that the difference equations (8) are stable provided this equation has no positive real roots equal to or greater than unity. Or, what amounts to the same thing, the system will be stable if the equation

$$(13)\quad \lambda^n - \sum \kappa_i a_{ii}\lambda^{n-1} + \sum \kappa_i\kappa_j \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \lambda^{n-2} + \cdots + (-1)^n \kappa_1\kappa_2 \cdots \kappa_n \Delta = 0$$

has no positive or zero real roots. Since the κ_i are all positive, we know by Descartes' rule of signs that (13) can have no positive or zero real roots if the determinants,

$$a_{ii}, \quad \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}, \quad \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}, \quad \text{etc.}$$

are alternately negative and positive. But these are the Hicks conditions of perfect stability. We have thus proved that perfect stability in the Hicks sense is sufficient for stability of both the difference equations (9) and the differential equations (7).

To prove that the Hicks conditions are also necessary, consider a system of m difference equations comprising a subset of the system (8):

$$(14)\quad \begin{aligned} \mathfrak{p}_1(t) &= (\kappa_1 a_{11}+1)\mathfrak{p}_1(t-1) + \kappa_1 a_{12}\mathfrak{p}_2(t-1) + \cdots + \kappa_1 a_{1m}\mathfrak{p}_m(t-1), \\ \mathfrak{p}_2(t) &= \kappa_2 a_{21}\mathfrak{p}_1(t-1) + (\kappa_2 a_{22}+1)\mathfrak{p}_2(t-1) + \cdots + \kappa_2 a_{2m}\mathfrak{p}_m(t-1), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathfrak{p}_m(t) &= \kappa_m a_{m1}\mathfrak{p}_1(t-1) + \kappa_m a_{m2}\mathfrak{p}_2(t-1) + \cdots + (\kappa_m a_{mm}+1)\mathfrak{p}_m(t-1). \end{aligned}$$

It can easily be shown that the complete system (8) is not stable unless this subsystem is also stable. To demonstrate this, we need only suppose, again, that all y_i of the complete system and all $\hat{y}_i$ of the subsystem are initially positive, and that each y_i of the complete system is initially greater than the corresponding $\hat{y}_i$ of the subsystem. Then at any time, t, $y_i(t) \geqq \hat{y}_i(t-1)$; i.e., for the initial conditions postulated, $y_i(t)$ of the complete system remains permanently greater than or equal to $\hat{y}_i(t)$ of the subset. The proof is a simple proof by induction. If $y_j(t-1) \geqq \hat{y}_j(t-1)$ holds for all j from 1 to m, then $y_i(t)$ is a sum of n positive or zero terms, of which m terms are greater than or equal to the corresponding terms in the summation for $\hat{y}_i(t)$. Hence, $y_i(t) \geqq \hat{y}_i(t)$ for all i from 1 to m.

If this is true, then clearly the complete system is not stable unless the subset is also stable. And a necessary condition for stability of the subset, as we have seen earlier, is that the mth-order determinant,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix},$$

shall have the same sign as $(-1)^m$. This must be true for any subset and for any value of m from 1 to n. Since the complete system is not stable unless all such subsets are stable, it follows that all mth-order principal minors of Δ must have the sign of $(-1)^m$. Thus, perfect stability in the Hicks sense is necessary for true dynamic stability of both the difference equations (8) and the differential equations (7).

IV. CONCLUSIONS

The fundamental assumption in the foregoing analysis is that the price of a particular commodity tends to rise whenever demand exceeds supply and to fall whenever supply exceeds demand. This assumption is accepted by Hicks as the basis for his conclusion that a single-commodity market is not stable unless demand and supply conditions are such that a rise in price above equilibrium creates an excess supply, while a fall in price below equilibrium creates an excess demand. Although the Hicks conclusion is valid for a system involving only one commodity, it cannot be generalized, in the manner attempted by Hicks, to a market system involving n commodities. In other words, it is inadmissible to assume, as Professor Hicks does in his discussion of *imperfect* stability, that a system of markets is necessarily stable if a fall in the price of a particular product leads to an excess demand for that product after all other prices have been adjusted so that supply is again equal to demand for each of the other commodities. Nor is it correct to assume, as in the Hicks discussion of *perfect* stability, that a market system is dynamically stable whenever a fall in the price of a given commodity leads to excess demand for that product after any given subset of prices of other commodities is adjusted so that supply again equals demand for these other commodities, with all remaining prices held constant. The Hicks discussion of perfect and imperfect stability tacitly assumes that whenever the price of a particular product deviates from equilibrium the prices of all remaining products either remain unchanged or are adjusted instantaneously to their new levels of equilibrium. Thus, the concepts of perfect and imperfect stability, developed by Professor Hicks, make no allowance for the fact that all prices may be out of equilibrium at the same time.

The error in Hicks' analysis of multiple-market systems is attributable to the fact that he generalizes the *conclusion* which was derived for a single-commodity market rather than the *method of analysis* for such a market. The correct generalization of the one-commodity case is a dynamic system which expresses the rate of change of the price of each commodity as a function of the discrepancy between demand for and supply of that commodity. The true conditions of stability are to be derived from this dynamic system, rather than from the properties of the corresponding static system. Samuelson has shown that these true stability conditions may differ from the Hicks conditions of perfect and imperfect stability.

Nevertheless, the foregoing analysis has demonstrated that, despite the error in the Hicks method, the Hicks conditions of perfect stability are applicable to certain economic problems. In the first place, a market system must be perfectly stable, in the Hicks sense, if the true dynamic system is to be stable for all possible speeds of price adjustment in individual markets. This means that in a market system which does not possess perfect Hicksian stability, true dynamic stability depends partly upon the fact that certain prices are relatively unresponsive to discrepancies between demand and supply; if these inflexible prices became more responsive, such a system might become unstable. A dynamic system can be stable for all possible speeds of adjustment only if it is perfectly stable in the Hicks sense.

Second, the analysis has shown that for at least one type of multiple-market system the Hicks conditions of perfect stability are identical with the true dynamic conditions. In particular, if all commodities are gross substitutes, the Hicks conditions are both necessary and sufficient for stability of the dynamic system. In most price problems, this conclusion may not be significant, for almost all markets have some degree of complementarity. In other problems, however, the Hicks conditions are highly useful. The study of income movements between countries, for example, requires the analysis of differential and difference equations which conform to the conditions described in III above. In other words, in a large part of income analysis, the Hicks conditions of perfect stability are both necessary and sufficient for true dynamic stability. These problems will be discussed in a later paper.

Since the Hicks conditions of perfect stability are now known to be necessary and sufficient for true stability of at least one class of markets, it is natural to speculate about the usefulness of these conditions for other classes of markets as well. The analysis presented above does not preclude the possibility that the Hicks conditions may be identical with the true dynamic conditions for certain classes of markets in which some goods are complementary. Indeed, Samuelson has previously demonstrated one such case; he has shown that if the static system is symmetrical, in the sense that $a_{ij}=a_{ji}$, and if the speed of adjustment is the same in all markets, the Hicks conditions of perfect stability are both necessary and sufficient, regardless of whether a_{ij} is positive or negative.[14] Further investigation may reveal other cases of a similar nature. In any event, an investigation which relates the true stability conditions to the minors of the static system will be highly useful, whether or not the final results are in accord with the Hicks conditions.

Board of Governors of the Federal Reserve System

[14] Paul A. Samuelson, "The Stability of Equilibrium: Comparative Statics and Dynamics," ECONOMETRICA, Vol. 9, April, 1941, p. 111.