Detecting Stochastic Signals in Noise

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ECE 7001: Stochastic Processes, Detection,
and Estimation

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Outline

1. Detecting Stochastic Signals in Noise
2. Performance Evaluation of Signal Detectors
3. Sequential Detection

Detecting Stochastic Signals in Noise

- Consider $H_1 : \vec{Y} = \vec{S} + \vec{N}$, $H_0 : \vec{Y} = \vec{N}$, where $\vec{N} \sim \mathcal{N}(\vec{0}, \sigma^2 I)$, $\vec{S} \sim \mathcal{N}(\vec{0}, \Sigma_s)$, $\vec{N}$ and $\vec{S}$ are independent

$$
\begin{aligned}
\text{under} \quad H_1 : & \vec{Y} \sim \mathcal{N}(\vec{0}, \Sigma_s + \sigma^2 I) \\
LLR(\vec{y}) &= \frac{1}{2} \vec{y}^T (\sigma^{-2} I - (\sigma^2 I + \Sigma_s)^{-1}) \vec{y} + \text{constant} \\
\sigma^{-2} I &= \sigma^{-2} (\sigma^2 I + \Sigma_s)(\sigma^2 I + \Sigma_s)^{-1} \\
&= (I + \sigma^{-2} \Sigma_s)(\sigma^2 I + \Sigma_s)^{-1} \\
\Rightarrow LLR(\vec{y}) &= \frac{1}{2} \vec{y}^T \underbrace{(\sigma^{-2} \Sigma_s (\sigma^2 I + \Sigma_s)^{-1})}_{R} \vec{y} + \text{constant}
\end{aligned}
$$

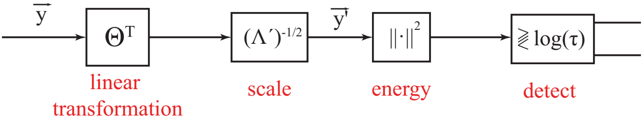
LRT: $\vec{y}^T R \vec{y} \gtrless \log \tau \ \rightarrow$ quadratic detector

$$
\begin{aligned}
R &= \sigma^{-2} Q \Lambda_s Q^T (\sigma^2 Q Q^T + Q \Lambda_s Q^T)^{-1} \\
&= \sigma^{-2} Q \Lambda_s Q^T (Q(\sigma^2 \vec{I} + \Lambda_s) Q^T)^{-1} \\
&= \sigma^{-2} Q \Lambda_s Q^T Q (\sigma^2 \vec{I} + \Lambda_s)^{-1} Q^T \\
&= Q \Lambda_s (\sigma^4 \vec{I} + \sigma^2 \Lambda_s)^{-1} Q^T \\
&= Q \Lambda' Q^T
\end{aligned}
$$

detector:

whitening $\rightarrow$ define $\vec{y'} \triangleq (\Lambda')^{1/2} Q^T \vec{y} \Rightarrow \vec{y} = Q(\Lambda')^{-1/2} \vec{y'}$

$\Rightarrow \vec{y}^T R \vec{y} = (\vec{y'})^T (\Lambda')^{-1/2} Q^T Q \Lambda' Q^T Q (\Lambda')^{-1/2} \vec{y'} = (\vec{y'})^T (\vec{y'}) = ||\vec{y'}||^2$



- When the signal has iid samples, $\Sigma_s = \sigma_s^2 \vec{I} \Rightarrow R = \frac{\sigma_s}{\sigma^2(\sigma^2+\sigma_s^2)}\vec{I}$
  $\vec{y}^T R \vec{y} = \alpha ||\vec{y}||^2 \ \Rightarrow \ LRT : ||\vec{y}||^2 \gtreqless \log \tau'$ (energy detector)

Performance Evaluation of Signal Detectors

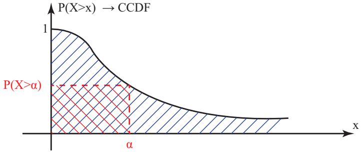
Motivation: In non-Gaussian scenarios, it may be very difficult to evaluate P_F and P_M. Here we derive bounds on them.

Chernoff bound:

Markov inequality: Let X be a non-negative valued r.v. with $E[X] < \infty$. Then, for all $\alpha > 0$,

$$P(X > \alpha) \leq \frac{E[X]}{\alpha}$$

Proof:



$E[X] \rightarrow$ Blue area $\geq$

Red area, for all $\alpha > 0$

The Chernoff Bound: Let $Z = \exp(rX)$ for some r.v. X and $r \in \mathbb{R}$. We know $Z \geq 0$ w.p. 1 and let $E[Z] = E[\exp(rX)] \triangleq g_X(r) = \int_{-\infty}^{\infty} e^{rx} p(x) dx$ exist for α range $[r_-, r_+]$ of values. Applying the Markov inequality,

$$P(\exp(rX) > \alpha) = P(e^{rX} > e^{r\beta}) \leq \frac{E[\exp(rX)]}{e^{r\beta}}$$

- $$\left.\begin{aligned} P(X > \beta) \leq g_X(r) e^{-r\beta}, \quad r > 0 \\ P(X < \beta) \leq g_X(r) e^{-r\beta}, \quad r < 0 \end{aligned}\right\} \quad \text{for all } r \in [r_-, r_+]$$
- The tightest bound can be found by minimizing $g_X(r)e^{-r\beta}$ over the values of $r \in [r_-, 0)$ or $r \in (0, r_+]$

$$P(X > \beta) \leq \arg\min_{r \in (0, r_+]} \quad g_X(r) e^{-r\beta}$$

$$P(X < \beta) \leq \arg\min_{r \in [r_-, 0)} \quad g_X(r) e^{-r\beta}$$

Application to signal detection:

- $$\begin{aligned} P_F = P(LLR(\vec{Y}) > \tau | H_0) &\leq E[\exp(rLLR(\vec{Y})) \big| H_0] \, e^{-r\tau}, \quad r > 0 \\ &= g_0(r) e^{-r\tau} \\ &\triangleq \exp(\gamma_0(r)) \\ &= \exp(\gamma_0(r) - r\tau) \end{aligned}$$

We defined $\gamma_0(r) \triangleq \log g_0(r)$,

$$\begin{aligned}
\gamma_0(0) &= 0 \\
\gamma_0'(0) &= E[LLR(\vec{Y})\big|H_0] \\
\gamma_0''(0) &= var(LLR(\vec{Y})\big|H_0) \\
&\vdots \rightarrow \text{ not the central moments}
\end{aligned}$$

$$(\text{Also,} \quad \gamma_0''(r) \geq 0 \quad \text{for all} \quad r \in [r^-, r^+])$$

- Similarly

$$\begin{aligned}
1 - P_D = P_M &= P(LLR(\vec{Y}) < \tau\big|H_1) \\
&\leq E[\exp(rLLR(\vec{Y}))\big|H_1]e^{-r\tau}, \quad r < 0 \\
&\triangleq \exp(\gamma_1(r) - r\tau)
\end{aligned}$$

tightest bounds:

$$P_F \leq \min_{r>0} \quad \exp(\gamma_0(r) - r\tau), \quad r > 0 \text{ (i)}$$

$$P_M \leq \min_{r<0} \quad \exp(\gamma_1(r) - r\tau), \quad r < 0 \text{ (ii)}$$

- $\gamma(\cdot)$ is a convex function. If $\gamma(\cdot)$ is differentiable,

$$\text{(i)} \quad \gamma_0'(r_0) - \tau = 0 \quad (r_0 \triangleq \min_{r>0} \quad \exp(\gamma_0(r) - r\tau))$$

$$\text{(ii)} \quad \gamma_1'(r_1) = \tau \quad (r_1 \triangleq \min_{r<0} \quad \exp(\gamma_1(r) - r\tau))$$

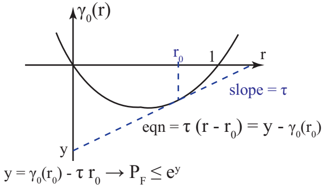
- More on r_0, r_1 :

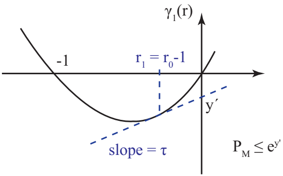
$$
\begin{aligned}
g_1(r) = E[\exp(rLLR(\vec{y}))\big|H_1] &= \int \exp(r\log L(\vec{Y}))\, p(\vec{y}\big|H_1)\, d\vec{y} \\
&= \int [L(\vec{y})]^r\, p(\vec{y}\big|H_1)\, d\vec{y} \\
&= \int [L(\vec{y})]^r\, \frac{p(\vec{y}\big|H_1)}{p(\vec{y}\big|H_0)} p(\vec{y}\big|H_0)\, d\vec{y} \\
&= \int [L(\vec{y})]^{r+1}\, p(\vec{y}\big|H_0)\, d\vec{y} \\
&= E[\exp((r+1)LLR(\vec{Y}))\big|H_0] = g_0(r+1)
\end{aligned}
$$

$$\Rightarrow\ \gamma_0(r+1) = \gamma_1(r)\ \Rightarrow\ r_0 = r_1 + 1$$

$$\Rightarrow\ \gamma_0(1) = \gamma_1(0) = 0, \quad \gamma_0(0) = \gamma_1(-1) = 0$$

- Combine everything: $(E[LLR(\vec{Y})\big|H_0] < 0)$





- Therefore, for the detection rules of the form: $LLR(\vec{y}) \gtreqless \tau$,

1.
   - $P_F \leq \exp(\gamma_0(r_0) - r_0\tau) \rightarrow$ tightest bound
   - $P_F \leq e^{-\tau} \quad (r = 1$ is also valid in Chernoff bound)
   - $P_F \leq \exp(\min_{r>0} (\gamma_0(r) - r\tau))$
   - $P_F \leq \exp(\min_{r>0} \gamma_0(r)) \rightarrow$ tightest bound for ML detection

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- $P_M \leq \exp(\gamma_1(r_1) - r_1\tau) \rightarrow$ tightest bound
- $P_M \leq e^{\tau}$
- $P_M \leq \exp(\min_{r<0} \gamma_1(r))$

IID Signals

$$LLR(\vec{y}) = \sum_{k=1}^{n} LLR(y_k) \Rightarrow g_i(r) = \prod_{k=1}^{n} g_{i,k}(r) = [g_{i,k}(r)]^n$$

$$\Rightarrow \gamma_i(r) = n\gamma_{i,1}(r)$$

- $$\begin{aligned} P_F &= P(LLR(\vec{Y}) > \tau | H_0) \\ &= P\left(\sum_{k=1}^{n} LLR(Y_i) > \tau | H_0\right) \\ &\leq \min_{r>0} \exp(n\gamma_{0,1}(r) - r\tau) \\ &\leq \exp(\min_{r>0} n\gamma_{0,1}(r)) \end{aligned}$$

- If we want $P_F \leq e^{-\alpha}$ $\Rightarrow$ regardless of τ we can achieve this by choosing $n \approx \frac{-\alpha}{\min \ \gamma_{0,1}(r)}$ (as long as $\tau > 0$)

Sequential Detection

- Non-sequential detection: fix n, observe $Y_1, ..., Y_n$ and make the decision. Suppose we have $\Pi_0, \Pi_1, \vec{Y} = Y_1, Y_2, ..., Y_n$.
- Let $LLR(\vec{Y}) = \beta$

$$\Rightarrow \log \frac{p(\vec{y}\big|H_1)}{p(\vec{y}\big|H_0)} = \beta \Rightarrow \frac{\Pi_1 \, p(\vec{y}\big|H_1)}{\Pi_0 \, p(\vec{y}\big|H_0)} = e^{\beta} \frac{\Pi_1}{\Pi_0} \triangleq \beta'$$

- Divide both the numerator and denominator by $P(\vec{y})$

$$\frac{P(H_1\big|\vec{Y})}{P(H_0\big|\vec{Y})} = \beta' \Rightarrow P(H_1\big|\vec{Y}) = \frac{\beta'}{1+\beta'} = \frac{1}{1+e^{\beta}\frac{\Pi_0}{\Pi_1}}$$

$$P(H_0\big|\vec{Y}) = 1 - P(H_1\big|\vec{Y})$$

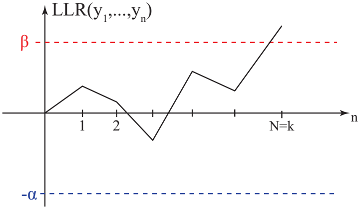
$$= \frac{1}{1+\beta'} = \frac{1}{1+e^{\beta}\frac{\Pi_1}{\Pi_0}}$$

- $P(\text{false alarm } |\vec{Y} = \vec{y}) = \frac{\beta'}{1+\beta'}$
- $P(\text{misdetection } |\vec{Y} = \vec{y}) = \frac{1}{1+\beta'}$

When we make the decision, we know the probability of error.

Sequential test: Keep taking observations until $LLR(\vec{Y})$ crosses either $\beta > 0$ or $-\alpha < 0$. Let N be the first time $LLR(Y_1, Y_2, ..., Y_n, ...)$ crosses either $-\alpha$ or β. Here N is called a "stopping rule". Then,

$$\delta(y_1, ..., y_N) = \begin{cases} 1, & LLR(y_1, ..., y_N) > \beta \\ 0, & LLR(y_1, ..., y_N) < -\alpha \end{cases}$$



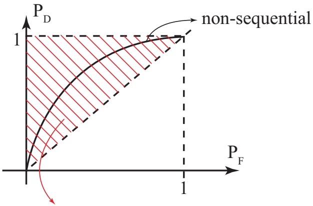
Notes: Independent observations

1. performance comparison: Conditional or $N = k$,

$$r > 0 \to P_F = \exp(k\gamma_{0,1}(r) - r\beta),$$

$$r < 0 \to P_M = \exp(k\gamma_{1,1}(r) - r\alpha)$$

other bonds: $r = 1 \Rightarrow P_F \leq e^{-\beta}$, $r = -1 \Rightarrow P_M \leq e^{-\alpha}$



sequential test can achieve any point in
the shaded area by adjusting (α,β) together

2. Do we require $E[N] \gg 1$ to achieve the above performance?

Let $\alpha, \beta \gg 1 \Rightarrow LLR(Y_1, ..., Y_N|H_0) = \sum_{k=1}^{N} LLR(Y_k|H_0) \approx$
$$\begin{cases} \beta, & w.p.\ e^{-\beta} \\ -\alpha, & w.p.\ 1 - e^{-\beta} \end{cases}$$
If $\beta, \alpha \gg 1$ compared to the overshoot / undershoot

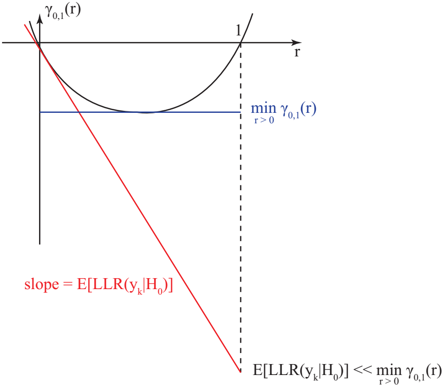
Wald's equality:

$$E[N \big| H = H_0] = \frac{E[LLR(Y_1, ..., Y_N) \big| H_0]}{LLR(Y_k) \big| H_0}$$
$$\approx \frac{\beta e^{-\beta} - \alpha(1 - e^{-\beta})}{E[LLR(Y_k) \big| H_0]}$$
$$\approx \frac{-\alpha}{E[LLR(Y_k) \big| H_0]}$$

Compare this with non-sequential test:

to achieve $P_F \leq e^{-\alpha}$,

$$n \approx \frac{-\alpha}{\min_{r>0} \gamma_{0,1}(r)} \quad \text{observation is sufficient}$$



$\Rightarrow$ observations with sequential detection $\gg$ # observations with non-sequential detection for the same P_F