Problem Set 4

Problem 1

Let Θ be a random variable with pdf $w(\theta)$ and let the conditional pdf of Θ given the observation $\vec{Y}$ be $w(\theta \mid \vec{Y})$. Show that the minimum mean absolute error (MMAE) estimate $\hat{\theta}_{\text{MMAE}}(\vec{y})$ is the conditional median (as given in lecture notes) of $w(\theta \mid \vec{Y})$. (**Hint:** For a non-negative random variable X, $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\ dx$.)

Problem 2 - (Poor, Ch. 4, Pr. 6)

You don't need to solve for $\hat{\theta}_{\text{MMAE}}$ in closed form.

Problem 3 - (Poor, Ch. 4, Pr. 8)

Problem 4

(a) Consider the joint probability density $p(x,z) = e^{-z}$ for $0 \le x \le z$ and $p(x,z) = 0$ otherwise. Find the pair (x,z) of values that maximize this density. Find the marginal density $p(z)$ and find the value of z that maximizes this.

(b) Let $p(x,y,z)$ be $y^2 e^{-yz}$ for $0 \le x \le z$ and $1 \le y \le 2$ and be 0 otherwise. Conditional on an observation $Y = y$, find the joint MAP estimate of X, Z. Find $p(z|y)$, the conditional pdf of Z, given $Y = y$, and find the MAP estimate of Z conditional on $Y = y$.

Problem 5 (Poor, Ch. 4, Pr. 11)

Replace "What happens when..." with "What happens when (i) $n \to \infty$ assuming $|\alpha| < 1$, (ii) $q^2 \to \infty$, (iii) $q^2 \to 0$".

In many problems of this sort it is *extremely* important to consider the support of the various pdfs. I suggest using the "indicator function" notation

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

for set A. For example, a unit-exponential r.v. Y has pdf $f_Y(y) = e^{-y} I_{[0,\infty)}(y)$.

Problem 6

(a) Let $X, Z_1, Z_2, \ldots, Z_n$ be independent zero-mean Gaussian random variables with variances $\sigma_X^2, \sigma_{Z_1}^2, \ldots, \sigma_{Z_n}^2$ respectively. Let $Y_j = h_j X + Z_j$ for $j \ge 1$ and let $\mathbf{Y} = [Y_1 \cdots Y_n]^T$. Show that the MMSE estimate of X conditional on $\mathbf{Y} = \mathbf{y} = [y_1, \cdots, y_n]^T$, is given by

$$\hat{X}(\mathbf{y}) = \sum_{j=1}^{n} g_j y_j; \text{ where } \quad g_j = \frac{h_j/\sigma_{Z_j}^2}{(1/\sigma_X^2) + \sum_{i=1}^{n} h_i^2/\sigma_{Z_i}^2}.$$

Hint: Let the row vector $\mathbf{g}^T$ be $K_{X\mathbf{Y}} K_{\mathbf{Y}}^{-1}$ and multiply $\mathbf{g}^T$ by $K_{\mathbf{Y}}$ to solve for $\mathbf{g}^T$.

(b) Let $\varepsilon \triangleq \hat{X}(\mathbf{Y}) - X$ and show that

$$\frac{1}{\sigma_\varepsilon^2} = \frac{1}{\sigma_X^2} + \sum_{i=1}^{n} \frac{h_i^2}{\sigma_{Z_i}^2}. \tag{1}$$

(c) Show that

$$\hat{X}(\mathbf{y}) = \sigma_\varepsilon^2 \sum_{j=1}^{n} \frac{h_j y_j}{\sigma_{Z_j}^2} \tag{2}$$

is valid.

(d) Show that the expression in (1) is equivalent to the iterative expression

$$\frac{1}{\sigma_{\varepsilon_n}^2} = \frac{1}{\sigma_{\varepsilon_{n-1}}^2} + \frac{h_i^2}{\sigma_{Z_i}^2}.$$

(e) Show that the expression in (2) is equivalent to the iterative expression

$$\hat{X}(\mathbf{y}_1^n) = \hat{X}(\mathbf{y}_1^{n-1}) + \frac{h_n \sigma_{\varepsilon_{n-1}}^2 [y_n - h_n \hat{X}(\mathbf{y}_1^{n-1})]}{h_n^2 \sigma_{\varepsilon_{n-1}}^2 + \sigma_{Z_n}^2}.$$