3. Solving cubic equations

Now let us move on to the solution of cubic equations. Like a quadratic, a cubic should always be re-arranged into its standard form, in this case

$$ax^3 + bx^2 + cx + d = 0$$

The equation

$$x^2 + 4x - 1 = \frac{6}{x}$$

is a cubic, though it is not written in the standard form. We need to multiply through by x, giving us

$$x^3 + 4x^2 - x = 6$$

and then we subtract 6 from both sides, giving us

$$x^3 + 4x^2 - x - 6 = 0$$

This is now in the standard form

When solving cubics it helps if you know one root to start with.

Example

Suppose we wish to solve

$$x^3 - 5x^2 - 2x + 24 = 0$$

given that $x = -2$ is a solution.

There is a theorem called the **Factor Theorem** which we do not prove here. It states that if $x = -2$ is a solution of this equation, then $x + 2$ is a factor of this whole expression. This means that $x^3 - 5x^2 - 2x + 24 = 0$ can be written in the form

$$(x + 2)(x^2 + ax + b) = 0$$

where a and b are numbers.

Our task now is to find a and b, and we do this by a process called **synthetic division**. This involves looking at the coefficients of the original cubic equation, which are 1, -5, -2 and 24. These are written down in the first row of a table, the starting layout for which is

$$\begin{array}{cccc|c} 1 & -5 & -2 & 24 & x = -2 \\ & & & & \\ \hline 1 & & & & \end{array}$$

Notice that to the right of the vertical line we write down the known root $x = -2$. We have left a blank line which will be filled in shortly. In the first position on the bottom row we have brought down the number 1 from the first row.

The next step is to multiply the number 1, just brought down, by the known root, -2, and write the result, -2, in the blank row in the position shown.

$$\begin{array}{cccc|c} 1 & -5 & -2 & 24 & x = -2 \\ & -2 & & & \\ \hline 1 & & & & \end{array}$$

The numbers in the second column are then added, $-5 + -2 = -7$, and the result written in the bottom row as shown.

$$\begin{array}{rrrr|l} 1 & -5 & -2 & 24 & x=-2 \\ & -2 & & & \\ \hline 1 & -7 & & & \end{array}$$

Then, the number just written down, -7, is multiplied by the known root, -2, and we write the result, 14, in the blank row in the position shown.

$$\begin{array}{rrrr|l} 1 & -5 & -2 & 24 & x=-2 \\ & -2 & 14 & & \\ \hline 1 & -7 & & & \end{array}$$

Then the numbers in this column are added:

$$\begin{array}{rrrr|l} 1 & -5 & -2 & 24 & x=-2 \\ & -2 & 14 & & \\ \hline 1 & -7 & 12 & & \end{array}$$

The process continues:

$$\begin{array}{rrrr|l} 1 & -5 & -2 & 24 & x=-2 \\ & -2 & 14 & -24 & \\ \hline 1 & -7 & 12 & 0 & \end{array}$$

Note that the final number in the bottom row (obtained by adding 24 and -24) is zero. This is confirmation that $x = -2$ is a root of the original cubic. If this value turns out to be non-zero then we do not have a root.

At this stage the coefficients in the quadratic that we are looking for are the first three numbers in the bottom row. So the quadratic is

$$x^2 - 7x + 12$$

So we have reduced our cubic to

$$(x+2)(x^2 - 7x + 12) \quad = \quad 0$$

The quadratic term can be factorised to give

$$(x+2)(x-3)(x-4) \quad = \quad 0$$

giving us the solutions $x = -2,\ 3 \text{ or } 4$

In the previous Example we were given one of the roots. If a root is not known it's always worth trying a few simple values.