Chapter 5

Dynamical Systems

Dynamical systems with simple solutions have been examined so far by considering individual solutions to equations of motions with a restricted set of parameters and initial (or boundary) conditions.

In this section we will introduce a new method that allows us to gain a much wider understanding of *all* possible solutions of a system under different assumptions for either control parameters or initial conditions. The method involves examining the geometry of the *phase space* of the system and was first introduced by Poincaré.

These methods are particularly useful when the system being considered displays behaviour that is dependent, in a non-trivial way, on initial conditions or system parameters. The complicated behaviour these systems can display may not be understood, or indeed expected, by examining individual solutions.

5.1 Phase Portraits[1]

In the following we will describe the state of a dynamical system using a number of *continuous* functions $x_i(t)$ with $i = 1, 2, ..., N$ and t is the independent variable *eg.* time.

The functions could describe positions, momenta, velocities, densities etc. and the equations of motion for the system can be written in vector notations as

$$\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t)\,, \tag{5.1.1}$$

where the functions $\mathbf{F}$ contain any number of control parameters $\mathbf{c}$ describing how the generalised phase velocities $\dot{\mathbf{x}}$ react to the generalised state coordinates $\mathbf{x}$. The control parameters could describe system properties such as mass, spring constants, gravity etc..

You should be familiar with the fact that higher order systems can all be reduced to the first order form of Eq. (5.1.1) by including each derivative of the state variable as extra variables. For

[1]Kibble & Berkshire, chapter 13

example if the system is described by

$$\frac{d^3x}{dt^3} = G\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right) , \tag{5.1.2}$$

we can define extra variables $y \equiv dx/dt$ and $z = d^2/dt^2$ and consider the first order system described by three variables

$$\begin{aligned} \frac{dx}{dt} &= y\,, \\ \frac{dy}{dt} &= z\,, \\ \frac{dz}{dt} &= G(t, x, y, z)\,, \end{aligned} \tag{5.1.3}$$

and recover the form of Eq. (5.1.1) by grouping $\mathbf{x} = (x, y, z)$ and $\mathbf{F} = (y, z, G)$.

If the functions $\mathbf{F}$ do not depend on the independent variable t explicitly then the system is called *autonomous* and can be examined by considering the phase trajectories in an N-dimensional phase space. If $\mathbf{F}$ depends explicitly on t then the system is *non-autonomous* but can be reduced to autonomous form by adding an extra variable $x_{N+1} = t$ with equation of motion $dx_{N+1}/dt = 1$. In this case the phase portrait has an additional dimension. We will only consider autonomous systems here as they are easier to interpret geometrically, particularly if $N \leq 3$.

5.1.1 First order systems

As an example of a first order system we can look at the *logistic* equation, a differential equation describing the growth of a population x in a scenario of limited resources. The system is described by the single equation

$$\dot{x} = kx - \sigma x^2\,, \tag{5.1.4}$$

with k and σ positive control parameters. The system is separable and its solution is

$$x(t) = \frac{kx_0}{[\sigma x_0 + (k - \sigma x_0)\exp(-kt)]}\,, \tag{5.1.5}$$

for initial condition $x(0) = x_0$.

The solutions can be plotted in the (x, t) plane as shown in Fig. 5.1 where the arrows indicate the direction of the population growth or phase velocity.

The plot shows the existence of two stationary, or *critical*, points in the system where $\dot{x} = 0$. This happens when $x = 0$ and $x = k/\sigma$. These are the only points at which the phase velocity $\dot{x}$ can change sign.

The position of x_0 with respect to these two lines determines the global nature of the solutions *i.e.* the line $x = k/\sigma$ is an *attractor* of solutions and they all approach this value given enough time. The line $x = 0$ is a *repeller* and all solutions with $0 < x_0 < k/\sigma$ move away from it towards $x = k/\sigma$.

In fact this global picture can be understood by disregarding the independent variable t explicitly and plotting the phase portrait, in this case a *phase line*.

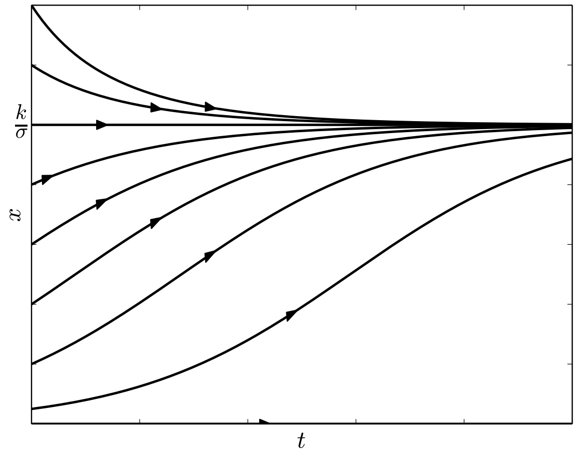


Figure 5.1:

5.1.2 Analysis of critical points

To build a phase portrait we need to find any critical points (or lines for higher order, see later) and then carry out a stability analysis to determine their nature.

For the first order, autonomous system, this can be done by considering small deviations from the critical point $x = x_0 + \xi$, where $\xi \ll 1$, and expanding to linear order in the deviation such that the system is now

$$\dot{x_0} + \dot{\xi} = F(x_0 + \xi) \approx F(x_0) + \xi \left.\frac{dF}{dx}\right|_{x=x_0} + \mathcal{O}(\xi^2)\,, \tag{5.1.6}$$

since, by definition, $\dot{x}_0 = F(x_0)$ we have

$$\dot{\xi} = \xi \left.\frac{dF}{dx}\right|_{x=x_0} + \mathcal{O}(\xi^2)\,, \tag{5.1.7}$$

which suggests solutions of the type $\xi(t) \sim \exp(\lambda t)$ with

$$\lambda = \left.\frac{dF}{dx}\right|_{x=x_0} . \tag{5.1.8}$$

This leads to three possible categories for the critical points:

- $dF/dx|_{x=x_0} \in \Re^- \rightarrow$ stable (attractor).
- $dF/dx|_{x=x_0} \in \Re^+ \rightarrow$ unstable (repeller).
- $dF/dx|_{x=x_0} \in \Im \rightarrow$ oscillatory (libration).

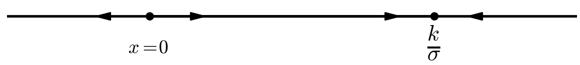


Figure 5.2:

Applying this to the logistic equation example we have

$$\left.\frac{dF}{dx}\right|_{x=k/\sigma} = -k\,, \tag{5.1.9}$$

and

$$\left.\frac{dF}{dx}\right|_{x=0} = k\,, \tag{5.1.10}$$

for the two critical points which implies that the $x = k/\sigma$ point is an attractor and the point at $x = 0$ is a repeller as we have already seen.

5.2 Second order systems

This analysis can be extended to second order systems where the phase portrait is two dimensional ($N = 2$). Here, after reducing to two first order equations, we have autonomous systems of the form

$$\begin{aligned} \dot{x} &= F(x,y)\,, \\ \dot{y} &= G(x,y)\,. \end{aligned} \tag{5.2.1}$$

The slope of a trajectory in the phase plane is given by

$$\frac{dy}{dx} = \frac{F}{G}\,, \tag{5.2.2}$$

everywhere except at critical points where $F = G = 0$. Trajectories can only intersect at critical points where degeneracies are allowed.

As before we consider small displacements from critical points (x_0, y_0) such that $x = x_0 + \xi$ and $y = y_0 + \eta$. Expanding to linear order we then have

$$\dot{\xi} = \xi \left.\frac{\partial F}{\partial x}\right|_{x=x_0,y=y_0} + \eta \left.\frac{\partial F}{\partial y}\right|_{x=x_0,y=y_0} + \mathcal{O}(\xi^2, \eta^2)\,,$$
$$\dot{\eta} = \xi \left.\frac{\partial G}{\partial x}\right|_{x=x_0,y=y_0} + \eta \left.\frac{\partial G}{\partial y}\right|_{x=x_0,y=y_0} + \mathcal{O}(\xi^2, \eta^2)\,, \tag{5.2.3}$$

or, in matrix form and dropping higher order terms,

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}\right|_{x=x_0,y=y_0} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \overline{\overline{\mathbf{M}}} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} \,. \tag{5.2.4}$$

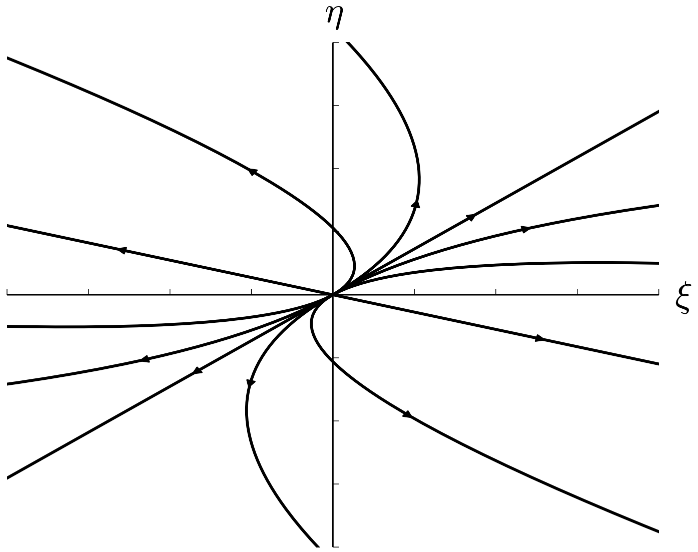
Once again, since the Jacobian $\overline{\overline{\mathbf{M}}}$ does not depend explicitly on the independent variable t we expect linear superpositions of solutions of the form $\exp(\lambda t)$ for ξ and η as general solutions with

$$\overline{\overline{\mathbf{M}}} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix} \,. \tag{5.2.5}$$

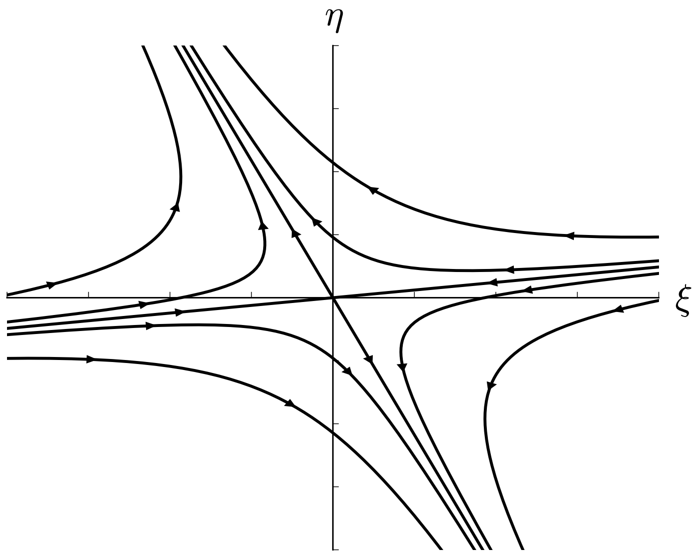
Thus the analysis of critical points is an eigen problem and we can identify the nature of all possible critical points through the properties of the eigen system.

The critical points can be categorised based on the eigenvalues λ_1 and λ_2 of matrix $\overline{\overline{\mathbf{M}}}$.

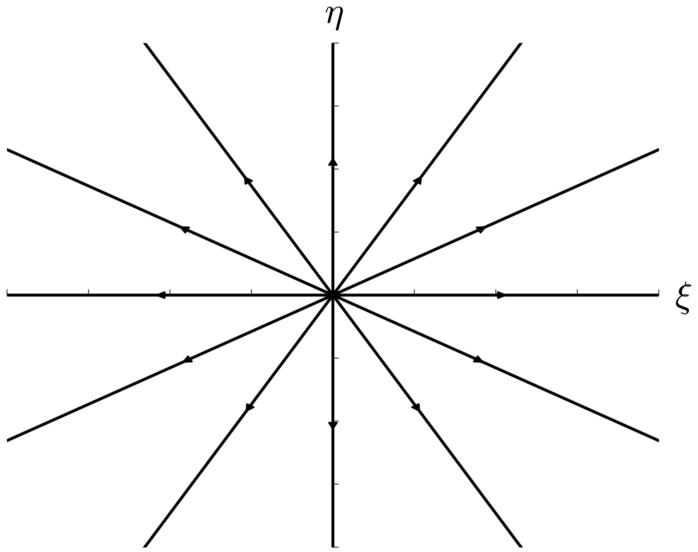
1. $\lambda_1 \neq \lambda_2$, real, same sign $\rightarrow$ improper, stable (negative), or unstable (positive), node.



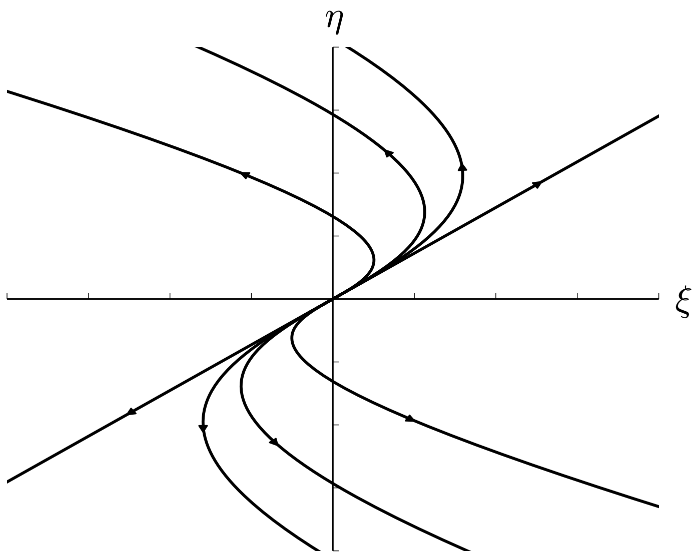
2. λ_1, λ_2, real, opposite sign $\rightarrow$ saddle point.



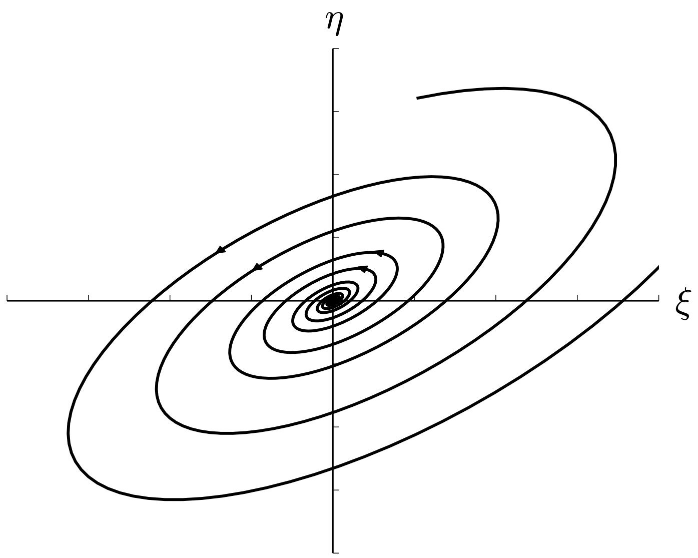
3. $\lambda_1 = \lambda_2 = \lambda$ and $\overline{\overline{\mathbf{M}}}$ can be diagonalized (two eigenvectors exist) $\rightarrow$ proper, stable (negative), unstable (positive), node.



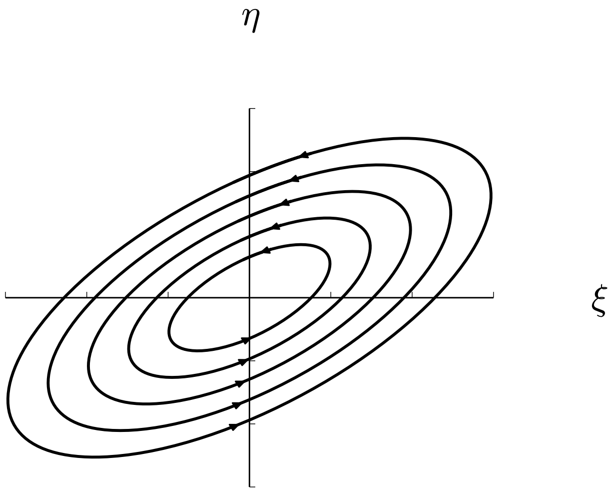
4. $\lambda_1 = \lambda_2 = \lambda$ and $\overline{\overline{\mathbf{M}}}$ cannot be diagonalized (only one eigenvectors exist) $\rightarrow$ improper or inflected, stable (negative), unstable (positive), node.



5. λ_1 and λ_2 are complex conjugates $\mu \pm i\nu$ with both real and imaginary components $\rightarrow$ spiral, stable (negative μ), unstable (positive μ), node.



6. λ_1 and λ_2 are pure imaginary conjugates $\pm i\nu \rightarrow$ stable centre.



7. If $|\overline{\overline{\mathbf{M}}}| = 0$ (singular matrix) $\rightarrow$ critical point is not isolated.

8. If one or two eigenvectors exist and are real they give the orientation of the asymptotes for the inflection and improper nodes respectively.

5.2.1 Simple pendulum

As an example consider the simple pendulum problem with a mass m attached to a rigid, massless rod of length l. The equation of motion for the angle of the rod with the vertical axis is

$$\ddot{\theta} = -\frac{g}{l}\sin\theta\,, \tag{5.2.6}$$

and the potential energy is

$$V(\theta) = mgl(1-\cos\theta)\,. \tag{5.2.7}$$

Looking at the equation of motion we can reduce it to two first order equations by defining the velocity $\sigma = \dot{\theta}$

$$\begin{aligned} \dot{\sigma} &= -\frac{g}{l}\sin\theta \equiv F(\sigma,\theta)\,, \\ \dot{\theta} &= \sigma \equiv G(\sigma,\theta)\,. \end{aligned} \tag{5.2.8}$$

This defines the matrix $\overline{\overline{\mathbf{M}}}$ as

$$\overline{\overline{\mathbf{M}}} \equiv= \begin{pmatrix} 0 & -\frac{g}{l}\cos\theta \\ 1 & 0 \end{pmatrix}\,, \tag{5.2.9}$$

whose eigenvalues are given by

$$\lambda_\pm = \pm i\sqrt{\frac{g}{l}\cos\theta}\,. \tag{5.2.10}$$

There is an infinite series of critical points at $\sigma = 0$, $\theta = n\pi$ with $n = -\infty, .., -1, 0, 1, ..., +\infty$. The eigen values at the three critical points with $\pm 2\pi$ of the origin are

$$\begin{aligned} \lambda_-(0) &= \lambda_+(0) = i\sqrt{\frac{g}{l}}\,, \\ \lambda_\pm(\pi) &= \pm\sqrt{\frac{g}{l}}\,, \\ \lambda_\pm(-\pi) &= \mp\sqrt{\frac{g}{l}}\,. \end{aligned} \tag{5.2.11}$$

We therefore have a stable centre at the origin and unstable saddles at π and $-\pi$. The phase portrait is shown in Fig. 5.3 for $g/l = 1$. In this case the eigenvectors of $\overline{\overline{\mathbf{M}}}$ are

$$\begin{pmatrix} -i \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} i \\ 1 \end{pmatrix}\,, \tag{5.2.12}$$

at $\theta = 0$ and

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,, \tag{5.2.13}$$

for the points at $\theta = \pm\pi$. This gives axes ($x = 1$, $y = 0$) and ($x = 0$, $y = 1$) for the centre at $\theta = 0$ and ($x = y$, $x = -y$) for the asymptotes at the saddle points.

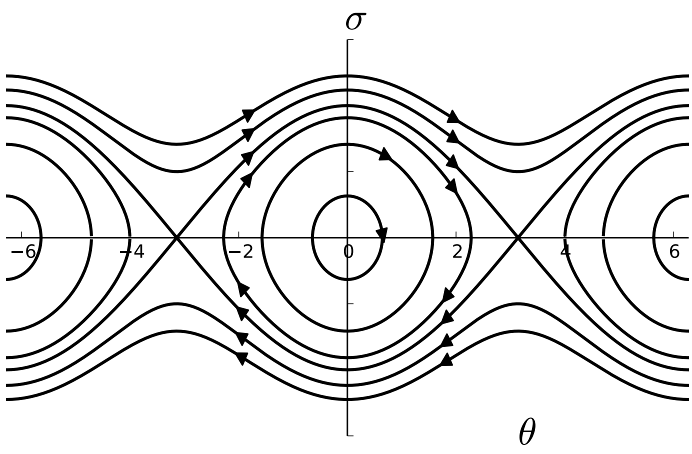


Figure 5.3:

5.2.2 Lotka-Volterra system

This is a model of competing species (prey and predators) with populations x, y. The system is given by two first order differential equations for the populations each involving a feedback and an interaction component

$$\begin{aligned} \dot{x} &= ax - bxy\,, \\ \dot{y} &= -cy + dxy\,, \end{aligned} \tag{5.2.14}$$

with a, b, c, and d are positive parameters quantifying the level of feedback and interaction. Feedback is positive for the prey but its interaction with predators results in a negative contribution to the population (*vice versa* for predators).

The system has two critical points

- (0,0) with

$$\begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}\,, \tag{5.2.15}$$

  with eigenvalues $\lambda = a$, $-c$ and eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{5.2.16}$$

  This is a saddle point with asymptotes aligned with the x and y axes.

- $(c/d,\, a/b)$ with

$$\begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{pmatrix}\,, \tag{5.2.17}$$

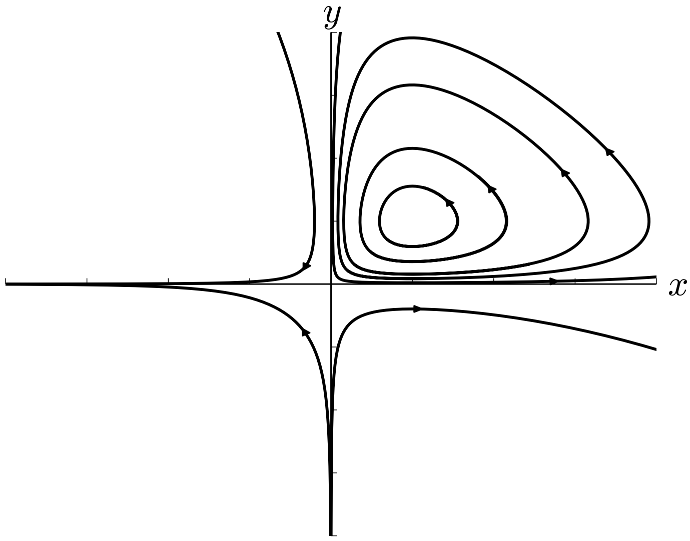


Figure 5.4:

with eigenvalues $\lambda = \pm i\sqrt{ac}$. This is a centre.

The phase portrait is shown in Fig. 5.4. Topologically we can see that the centre in the positive quadrant is a *global* centre for the positive quadrant - the prey/predator population undergoes a periodic oscillation. This can also be shown by obtaining solutions for the phase lines directly by integrating the separable equation

$$\frac{dx}{dy} = \frac{x(a - by)}{y(dx - c)}\,, \tag{5.2.18}$$

giving

$$dx - c\ln|x| + by - a\ln|y| = H(x, y)\,, \tag{5.2.19}$$

with H a constant. Thus each contour corresponds to a different value of H which is analogous to the conserved energy of the system.

The analogy with a Hamiltonian system can be made explicit by introducing new variables $x = \exp(p)$ and $y = \exp(q)$. Then the conserved quantity H is given by

$$H(p, q) = de^p - cp + be^q - aq\,, \tag{5.2.20}$$

and the system equations take on a Hamiltonian form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = -c + de^p\,, \\ \dot{p} &= -\frac{\partial H}{\partial q} = a - be^q\,. \end{aligned} \tag{5.2.21}$$

5.3 Liouville's theorem

Liouville's theorem concerns the evolution of phase space trajectories of Hamiltonian systems such as those examined in 5.2.2. The generalisation of such systems with n dimensions is to describe the instantaneous position in phase space $\mathbf{r}(t)$ using $2n$ coordinates (the phase space)

$$\mathbf{r}(t) = (q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)\,, \tag{5.3.1}$$

with Hamilton's equations describing the phase space velocity

$$\mathbf{v} = \dot{\mathbf{r}} = (\dot{q}_1, \dot{q}_2, ..., \dot{q}_n, \dot{p}_1, \dot{p}_2, ..., \dot{p}_n)\,. \tag{5.3.2}$$

Liouville's theorem states that, for a Hamiltonian system, a given set of initial solutions, forming an initial volume V in the phase space, evolves as an incompressible fluid in the $2n$ dimensional phase space. This means that the volume occupied by the set of solutions does not change in time. In analogy with the Euler fluid equations the incompressible fluid condition can be stated as the condition that the velocity is divergenceless

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0\,. \tag{5.3.3}$$

If we consider this condition in terms of the phase space coordinates

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial \dot{q}_1}{\partial q_1} + ... + \frac{\partial \dot{q}_n}{\partial q_n} + \frac{\partial \dot{p}_1}{\partial p_1} + ... + \frac{\partial \dot{p}_n}{\partial p_n}\,, \tag{5.3.4}$$

then we can see why (5.3.3) holds since for a Hamiltonian system we have

$$\frac{\partial \dot{q}_1}{\partial q_1} + \frac{\partial \dot{p}_1}{\partial p_1} = \frac{\partial H}{\partial q_1 p_1} - \frac{\partial H}{\partial p_1 q_1} = 0\,, \text{ etc.} \tag{5.3.5}$$

The solutions must remain on the same constant energy surface with $H(q,p) = E$ and this links energy conservation to the phase space volume. Notice that although the volume is conserved it can change shape and for most systems the set of solutions can 'twist and fold' to cover most of the phase space in a complicated structure given enough time. This is known behaviour is known as *ergodic*.

In systems where there are additional symmetries i.e. more conserved quantities, the set of solutions may be constrained to simpler surfaces. In this case the set of solutions may not spread out to cover all the available phase space. This is known as *non-ergodic* behaviour.

Liouville's theorem is part of a set of powerful theorems that relate the symmetries of Hamiltonian systems to the topology of manifolds covered by the allowed solutions in the phase space and, ultimately, the integrability of the systems[2].

5.4 Third order systems

First and second order systems can display critical points and lines. Systems higher than third order can display much more complex critical behaviour. However the presence of constants of the motion (conserved quantities) reduced the effective dimensionality of the problem, this makes the interpretation of the system easier.

[2] see Kibble & Berkshire Chapter 14

5.4.1 Rigid Body Rotation

As an example of a third order system that is constrained due to conservation laws we can consider the free rotation of a three dimensional, rigid body as done in Section 2.7. We can now look at the global behaviour of a freely rotating body by considering its entire phase space.

In terms of the angular momenta about the principle axes we can write Eq. 2.7.6 as

$$\begin{aligned}\dot{L}_1 + \frac{(I_3 - I_2)}{I_2 I_3} L_2 L_3 &= 0\,,\\ \dot{L}_2 + \frac{(I_1 - I_3)}{I_3 I_1} L_3 L_1 &= 0\,,\\ \dot{L}_3 + \frac{(I_2 - I_1)}{I_1 I_2} L_1 L_2 &= 0\,.\end{aligned} \tag{5.4.1}$$

Note that this system is non-linear in the momenta. However we can still carry out a stability analysis by considering perturbations about special points. the system can then be expanded to linear order in the perturbations to obtain a linear system of the type Eq. (5.2.4). The eigenvalues of the system can then be calculated to assess stability.

Let's assume we are rotating about the second principle axis initially such that $L_1 = L_3 = 0$ and consider small perturbations in each angular momentum. You will work out that the geometry of the this critical point to depends on eigenvalues

$$\lambda \propto \pm\sqrt{(I_3 - I_2)(I_2 - I_1)}\,, \tag{5.4.2}$$

such that for the case $I_1 < I_2 < I_3$ we has an unstable critical point. Considering the stability of all six critical points for this case we can sketch the entire phase space geometry as shown in Fig. 5.5. The stable critical points indicate a stable precession of the angular momentum around the stable principle axes. This is the same "tennis racquet" result we obtained in Section 2.7.

For the case $I_1 = I_2 < I_3$ we get a phase diagram as in Fig. Dyn:racquet1 with stable precession around the preferred principle axis. This case is similar to the spinning top case.

5.4.2 Lorentz System

This is a very well known third order system originally derived by Lorentz to describe convective flow in a two dimensional slab of fluid placed in a temperature gradient. The system is given by

$$\begin{aligned}\dot{x} &= \sigma(y - x)\,,\\ \dot{y} &= \rho x - y - xz\,,\\ \dot{z} &= -\beta z + xy\,,\end{aligned} \tag{5.4.3}$$

where (x, y, z) describe the convective state of the system[3] and σ, β, and ρ are positive control parameters. σ describes the ratio between diffusion of momentum and heat of the fluid (mechanical *vs* thermodynamic transport), β is the aspect ratio of the slab, and ρ is related to the temperature difference applied across the slab.

[3] x is the convective intensity, y is the temperature difference between ascending and descending parcels of fluid and z is the difference in the vertical temperature profile from a linear relationship.

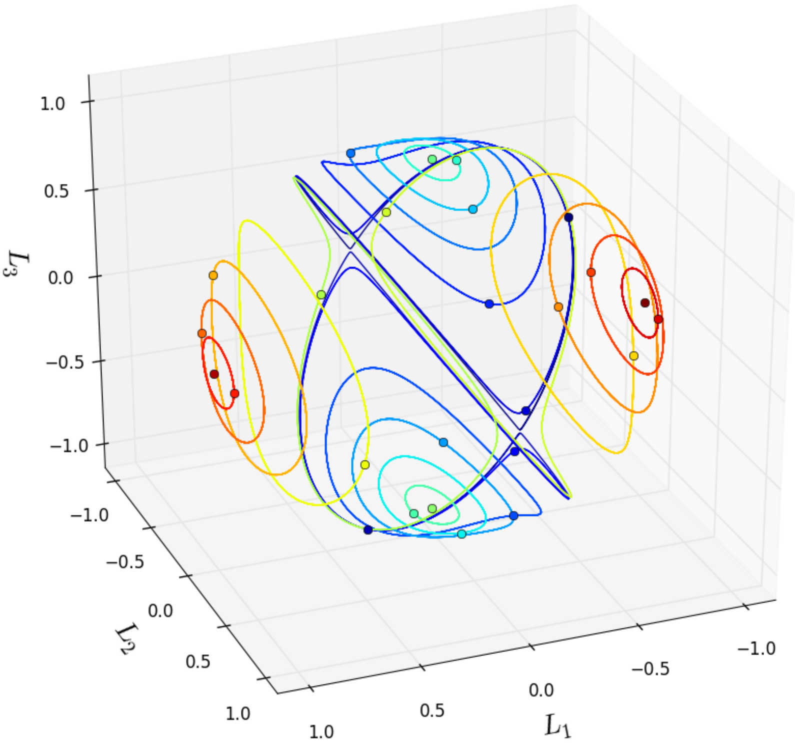


Figure 5.5:

It can be shown that for $\rho < 1$ there exists only one critical point in this system, at the origin (0,0,0). The stability of this point can be analysed in a similar manner as before by introducing small perturbations about the critical point and linearising the system. In this case we still have a third order system after doing so and so we will generally have three eigenvalues but the interpretation, in terms of the stability, is the same.

For $\rho < 1$ the origin is an attractor. This is identified with simple heat flow across the slab with no convective rolls. For $\rho > 1$ the origin is unstable and two more critical points appear at $x = y = \pm\sqrt{\beta(\rho - 1)}$ and $z = (\rho - 1)$.

For $1 < \rho < \rho_{\text{crit}}$ where

$$\rho_{\text{crit}} = \frac{\sigma(\sigma + \beta + 3)}{(\sigma - \beta - 1)}\,, \tag{5.4.4}$$

with $\sigma > \beta + 1$ the two new points are attractors and the system displays steady convective rolls.

For $\rho > \rho_{\text{crit}}$ all three points are unstable. This state is identified with strong convective turbulence. However it can be shown that there is still a *attraction basin* since

$$\boldsymbol{\nabla} \cdot \dot{\mathbf{x}} = -(\sigma + 1 + \beta) < 0\,. \tag{5.4.5}$$

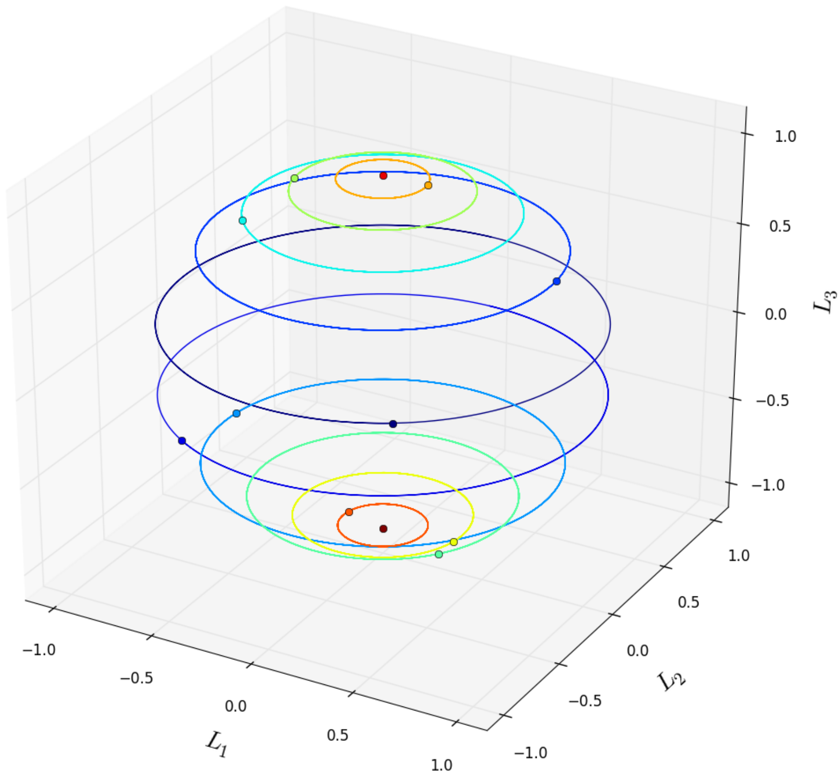


Figure 5.6:

The system is not Hamiltonian and the phase space volume is decreasing. In fact this is called a *strange attractor* as trajectories are attracted into a basin that has vanishing phase space volume and they fold onto this shape in very complex paths. This behaviour is typical of *chaotic* systems.

The structure in the basin (see Fig. 5.7) is hierarchical or *fractal* in nature. And the shape of the attractor depends strongly on the initial conditions of the system.

5.4.3 Sensitivity to initial conditions

The Lorentz system is an example of *chaotic* motion that is possible in three or more dimensions. This is despite the fact that the volume of phase space occupied by the solutions is bounded (e.g. approaching infinitesimal in the Lorentz case as time evolves).

In the case of only two effective dimensions chaotic motion cannot exist if the volume is bounded. If we consider the time evolution of the generalised n-dimensional distance

$$d = \sqrt{\Delta q_1^2 + \Delta q_2^2 + \ldots + \Delta q_n^2}\,, \tag{5.4.6}$$

between two solutions separated by $\Delta\mathbf{q}$ then it can be shown that d evolves at most linearly in

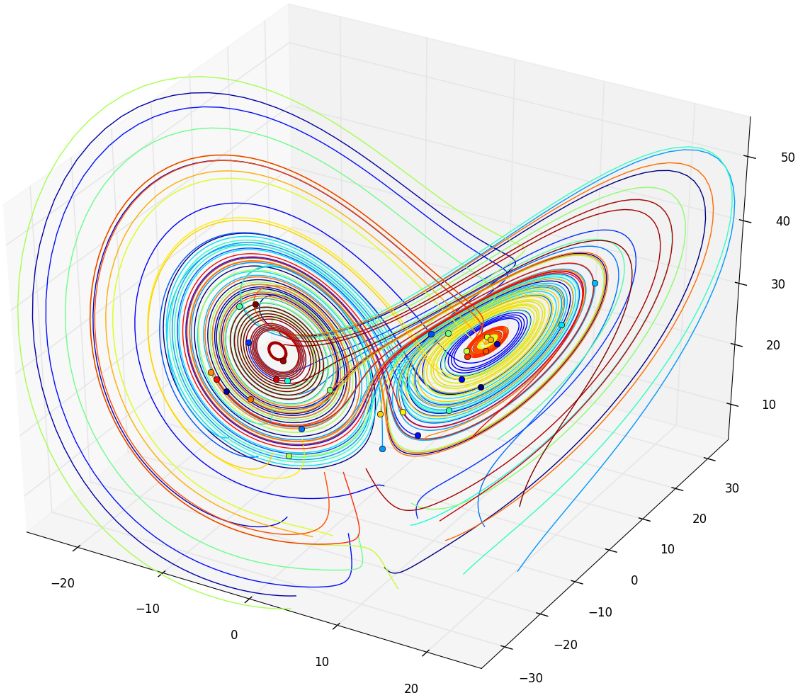


Figure 5.7:

two dimensions or less. For example in the case of a simple harmonic oscillator the distance is bounded in time and for a simple pendulum it grows approximately linearly.

For $n \geq 3$ the distance generally increases exponentially unless there are significant constraints in the motion (e.g. rigid body rotation with angular momentum conservation). The reason for this is that the phase space trajectories can fold and wrap around themselves without intersecting *whilst* often remaining in a bounded volume of phase space as per the Liouville theorem.

In general the distance in higher order systems will evolve as $d \sim \exp(\lambda t)$ where λ is the *Lyapunov exponent* of the system. In particular, for an n-dimensional system we have already seen how we can relate the evolution of a perturbation in each dimension to the eigenvalues of the Jacobian matrix $\overline{\overline{\mathrm{M}}}$. We can expand the distance on the othogonal basis provided by eigenvectors of the matrix $\overline{\overline{\mathrm{M}}}$ as

$$\mathbf{d} = \sum_{i=1}^{n} d_i \mathbf{e}_i \, , \tag{5.4.7}$$

with coefficients $d_i \sim \exp(\lambda_i t)$. The eigenvalues constitute the *Lyapunov spectrum* of the system and if *any* of them have a positive real component we will have exponential growth of $|\mathbf{d}|$ eventually - this is guaranteed if the system is autonomous.

In we can define a condition for a system to be chaotic by looking at the *maximal Lyapunov* *exponent for the system* i.e.

$$\lim_{t\to\infty} |\mathbf{d}| = \lim_{t\to\infty} \left(\sum_{i=1}^{n} d_i^2 \right)^{1/2} \sim d_j \,, \tag{5.4.8}$$

where j corresponds to the eigenvalue

$$\lambda_j = \max_{1\le i\le n} \mathbb{R}(\lambda_i) \,. \tag{5.4.9}$$

A chaotic system in $n \geq 3$ is then one that has a bounded cover in phase space *and* a positive definite maximal lyapunov exponent. The only way to avoid this is to have truly periodic trajectories in the n-dimensional phase space i.e. purely imaginary eigenvalues for $\overline{\overline{\mathbf{M}}}$.

Chaotic systems lead to the loss of practical predictability due to the sensitivity to initial conditions

5.5 Integrability

We have seen how the existence of symmetries in Hamiltonian systems lead to conserved quantities and a reduction in the effective degrees of freedom of the system. If there are enough symmetries in the system then the we are able to obtain a full solution, by integration, of the system. This is known as *integrability*.

For a system with $H(q_i, ..., q_N, p_i, ..., p_N, t)$ we need N independent, constant functions $F_i(q_i, ..., q_N, p_i, ..., p_N)$ with $i = 1, ..., N$, leading to N conserved quantities, in order to obtain a full solution via integration. A system with less than N constant functions is said to be *non-integrable*.

For an autonomous system with $\partial F_i/\partial t = 0$ this requires N Poisson bracket relations

$$\frac{dF_i}{dt} = -\{F_i, H\} = 0 \,, \tag{5.5.1}$$

for integrability.

Another condition for the F_i is that the functions are *in involution*

$$\{F_i, F_j\} = 0 \text{ for } i, j = 1, ..., N \,, \tag{5.5.2}$$

i.e. that the constraints are independent.

Each function F_i reduces the effective dimension of the phase space by one degree of freedom by constraining the trajectory onto a $2N-1$-dimensional surface where F_i is constant. Then N F_i lead to an effective reduction by N i.e. the possible phase space trajectories span an N-dimensional subspace or manifold $\mathbf{M}$ of the original $2N$-dimensional phase space.

The possible topology of the F_i constant surface is quite specific. The normal to the surface is given by the gradient of the constraints

$$\boldsymbol{\nabla} F_i = (\boldsymbol{\nabla}_q F_i, \boldsymbol{\nabla}_p F_i) \,, \tag{5.5.3}$$

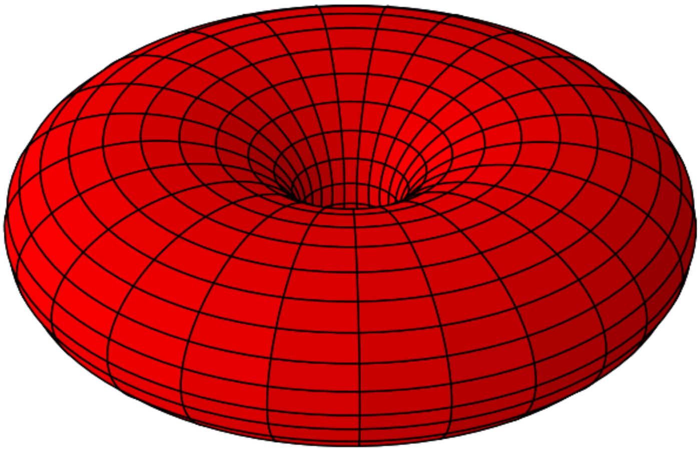


Figure 5.8:

and the condition (5.5.2) can be interpreted as a set of orthogonality conditions

$$\mathbf{v}_j \cdot \boldsymbol{\nabla} F_i = 0 \;\text{ for }\; i, j = 1, ..., N \,, \tag{5.5.4}$$

where

$$\mathbf{v}_j = (\boldsymbol{\nabla}_p F_i, -\boldsymbol{\nabla}_q F_i) \;. \tag{5.5.5}$$

This means that the set of vectors $\mathbf{v}_j$ are parallel to the surface $\mathbf{M}$ and can be used as an orhogonal coordinate system spanning the entire surface without singularities.

The topology of such a surface is an N-torus and different values of F_i give nested tori. It cannot be a sphere, for example, because all orthogonal coordinates on a sphere necessarily have a singularity - also known as the ‘hairy ball theorem’.

Phase trajectories of an integrable system are therefore constrained to lie on an N-torus embedded in the $2N$-dimensional phase space. The manifolds are known as *invariant tori* because a trajectory initial on a torus will stay on it forever.

5.5.1 Action/Angle Variables

The fact that an integrable system has trajectories constrained to an N-torus is a hint that there is a special set of generalised coordinates. This is just a statement that the N constraints F_i imply N conserved quantities which should lead to N ignorable coordinates.

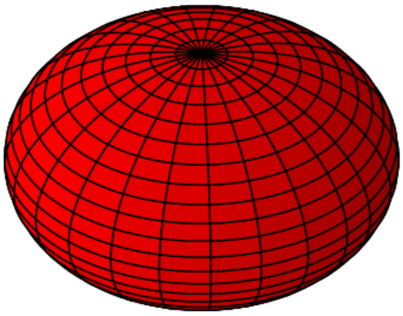


Figure 5.9:

An ignorable coordinate is one that does not appear explcitily in the Hamiltonian. As we have seen this leads to conservation in the momentum related to that coordinate. For example if q_α does not appear in the Hamiltonian then Hamilton's equation gives

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} = 0\,, \tag{5.5.6}$$

and hence p_α is conserved. In this case q_α is ignorable and the momentum is just assumed to be constant.

A system may not be written in terms of ingorable coordinates but if we know it is integrable we should be able to define a canonical transformation that enables us to write it in terms of N such coordinates. This is known as a transformation to Action/Angle variables I_i and ϕ_i that satisfy the following conditions;

1. The Hamiltonian form of the equations of motion are preserved with

$$\begin{aligned} H(q_i, p_i) \quad &\rightarrow \quad \widetilde{H}(I_i) \equiv H\,, \\ \dot{q}_i = \frac{\partial H}{\partial p_i} \quad &\rightarrow \quad \dot{\phi}_i = \frac{\partial \widetilde{H}}{\partial I_i} \equiv \omega_i(I_i)\,, \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad &\rightarrow \quad \dot{I}_i = -\frac{\partial \widetilde{H}}{\partial \phi_i} = 0\,. \end{aligned} \tag{5.5.7}$$

2. The new momenta I_i are functions of F_i.

3. The new coordinates ϕ_i are all ignorable.

Since I_i are all constants we have that the angle variables all evolve at uniform rate and their equation of motion can be integrated to obtain

$$\phi_i(t) = \omega_i t + \beta_i \,, \tag{5.5.8}$$

with β_i also a constant.

As an example consider the case with one coordinate q and momentum p. Assuming the canonical transformation is achieved by a generating function $W(q, I)$ such that

$$p = \frac{\partial W}{\partial q} \text{ and } \phi = \frac{\partial W}{\partial I} \,, \tag{5.5.9}$$

then we can define

$$W = \int_0^q p\, dq \,, \tag{5.5.10}$$

or

$$\phi = \frac{\partial}{\partial I} \int_0^q p\, dq \,. \tag{5.5.11}$$

The action variable I can be defined by considering the conserved area enclosed by the 1-torus in the phase space

$$I = \frac{1}{2\pi} \oint p\, dq \,. \tag{5.5.12}$$

The angle nature of ϕ is seen by transforming to polar coordinates to describe the phase space with radius $R = \sqrt{2I}$.

As an example consider an oscillator problem of the form

$$H(q, p) = \frac{p^2}{2m} + V(q) \,. \tag{5.5.13}$$

In this case the only constraint is that H is constant. This is sufficient to make the system integrable since we have a one coordinate systems with a two dimensional phase space. The 1-torus is a *loop* in the phase space.

We can solve for p to define the action variable i.e.

$$p = \pm\sqrt{2m(H - V(q)} \,, \tag{5.5.14}$$

and we can define the action variable as

$$I = \frac{1}{2\pi} \oint p\, dq = \frac{\sqrt{2m}}{2\pi} \left[\int_{q_1}^{q_2} \sqrt{H - V(q)} + \int_{q_2}^{q_1} \sqrt{H - V(q)} \right] \,. \tag{5.5.15}$$

Applying this to the harmonic oscillator case with $V(q) = 1/2kq^2$ the integral becomes the area of the ellipse at constant H. The ellipse has semi-axes of length

$$\begin{aligned} a &= \sqrt{\frac{2H}{k}} \text{ at } p = 0\,, \\ b &= \sqrt{m2H} \text{ at } q = 0\,, \end{aligned}$$

giving the area $A = \pi\, a\, b = 2\pi H\sqrt{m/k}$. Then

$$I = H\sqrt{\frac{m}{k}} \text{ and } H = \sqrt{\frac{k}{m}}I\,. \tag{5.5.16}$$

We can immediately find the frequency of the solution at this point without working out the exact form of the solution i.e.

$$\omega = \frac{\partial H}{\partial I} = \sqrt{\frac{k}{m}}\,. \tag{5.5.17}$$