# Chapter 2

# Rigid Bodies

## 2.1 Many-Body Systems[1]

Let us consider a system consisting $N$ particles, which we label by an integer $a = 1, \ldots, N$ (or other letters $b, c, \ldots$ at the start of the alphabet). We denote their positions by $\mathbf{r}_a$ and masses by $m_a$.

The net force acting on each particle, is the sum of the forces due to each other particle in the system as well as any external forces. Therefore Newton's second law for particle $a$ has the form

$$m_a \ddot{\mathbf{r}}_a = \sum_{b \neq a} \mathbf{F}_{ab} + \mathbf{F}_a^{\text{ext}}, \tag{2.1.1}$$

where $\mathbf{F}_{ab}$ is the force on particle $a$ due to particle $b$, and $\mathbf{F}_a^{\text{ext}}$ is the external force acting on particle $a$. Note that the external force is generally dependent on the particle's position, velocity etc. and is therefore different for each particle, which is why it has the index $a$.

Instead of trying to solve the motion of each particle, let us first look at the motion of the system as a whole. For that purpose, it is useful to define the total mass

$$M = \sum_a m_a, \tag{2.1.2}$$

and the centre or mass

$$\mathbf{R} = \frac{1}{M} \sum_a m_a \mathbf{r}_a. \tag{2.1.3}$$

We also define the total momentum of the system as the sum of the momenta of the individual particles,

$$\mathbf{P} = \sum_a \mathbf{p}_a = \sum_a m_a \dot{\mathbf{r}}_a = M\dot{\mathbf{R}}. \tag{2.1.4}$$

Differentiating this with respect to time gives

$$\dot{\mathbf{P}} = \sum_a m_a \ddot{\mathbf{r}}_a = \sum_{ab} \mathbf{F}_{ab} + \sum_a \mathbf{F}_a^{\text{ext}} = \sum_a \mathbf{F}_a^{\text{ext}}, \tag{2.1.5}$$

[1] Kibble & Berkshire, chapter 9

where the sum of the inter-particle forces $\mathbf{F}_{ab}$ vanishes because of Newton's third law $\mathbf{F}_{ba} = -\mathbf{F}_{ab}$. Therefore the rate of change of the total momentum is given by the total external force. In particular, the total momentum $\mathbf{P}$ is conserved in isolated systems, i.e., when there are no external forces.

Similarly, we define the total angular momentum $\mathbf{L}$ as the sum of angular momenta $\mathbf{l}_a = m_a\mathbf{r}_a \times \dot{\mathbf{r}}_a$ of the individual particles,

$$\mathbf{L} = \sum_a \mathbf{l}_a = \sum_a m_a \mathbf{r}_a \times \dot{\mathbf{r}}_a \,. \tag{2.1.6}$$

(N.B. Kibble & Berkshire use $\mathbf{J}$ for the angular momentum. We shall stick to the more conventional $\mathbf{L}$ here.) Its rate of change is given by

$$\begin{aligned}\dot{\mathbf{L}} &= \sum_a m_a \dot{\mathbf{r}}_a \times \dot{\mathbf{r}}_a + \sum_a m_a \mathbf{r}_a \times \ddot{\mathbf{r}}_a = \sum_a m_a \mathbf{r}_a \times \ddot{\mathbf{r}}_a \\ &= \sum_a \mathbf{r}_a \times \left( \sum_b \mathbf{F}_{ab} + \mathbf{F}_a^{\rm ext} \right). \end{aligned} \tag{2.1.7}$$

Using Newton's third law, we can write this as

$$\dot{\mathbf{L}} = \frac{1}{2} \sum_{ab} (\mathbf{r}_a - \mathbf{r}_b) \times \mathbf{F}_{ab} + \sum_a \mathbf{r}_a \times \mathbf{F}_a^{\rm ext}. \tag{2.1.8}$$

In general, the first term on the right-hand-side is non-zero, but it vanishes if we assume that the inter-particle forces are *central*, which means that the force $\mathbf{F}_{ab}$ between particles $a$ and $b$ is in the direction of their separation vector $(\mathbf{r}_a - \mathbf{r}_b)$. In that case we have

$$\dot{\mathbf{L}} = \sum_a \mathbf{r}_a \times \mathbf{F}_a^{\rm ext} \equiv \boldsymbol{\tau}, \tag{2.1.9}$$

where the right-hand-side is known as the *torque*. The torque is zero and the angular momentum is conserved if the system is isolated or if the external forces all point to the origin (i.e., $\mathbf{F}_a^{\rm ext} || \mathbf{r}_a$).

Note that there are some forces that are not central, such as the electromagnetic force between moving charges, and in that case $\mathbf{L}$ is not conserved. (In fact, the electromagnetic field can carry angular momentum, and when it is included the total angular momentum is still conserved.)

It is often useful to separate the coordinates $\mathbf{r}_a$ into centre of mass and relative contributions

$$\mathbf{r}_a = \mathbf{R} + \mathbf{r}_a^* \tag{2.1.10}$$

such that, by definition,

$$\sum_a m_a \mathbf{r}_a^* = 0\,. \tag{2.1.11}$$

Substituting this into Eq. (2.1.6) gives

$$\begin{aligned}
\mathbf{L} &= \sum_a m_a \left(\mathbf{R} + \mathbf{r}_a^*\right) \times \left(\dot{\mathbf{R}} + \dot{\mathbf{r}}_a^*\right) \\
&= \left(\sum_a m_a\right) \mathbf{R} \times \dot{\mathbf{R}} + \left(\sum_a m_a \mathbf{r}_a^*\right) \times \dot{\mathbf{R}} + \mathbf{R} \times \left(\sum_a m_a \dot{\mathbf{r}}_a^*\right) + \sum_a m_a \mathbf{r}_a^* \times \dot{\mathbf{r}}_a^* \\
&= M\mathbf{R} \times \dot{\mathbf{R}} + \mathbf{L}^* \qquad \text{where} \qquad \mathbf{L}^* = \sum_a m_a \mathbf{r}_a^* \times \dot{\mathbf{r}}_a^* \,, \qquad (2.1.12)
\end{aligned}$$

as the other terms are zero due to Eq. (2.1.11). The angular momentum can be separated into a centre of mass part, $M\mathbf{R} \times \dot{\mathbf{R}}$, and the angular momentum about the centre of mass, $\mathbf{L}^*$.

The rate of change of the relative angular momentum $\mathbf{L}^*$ may be written as the sum of the moments of the particles about the centre of mass due to external forces alone

$$\begin{aligned}
\dot{\mathbf{L}}^* &= \dot{\mathbf{L}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(M\mathbf{R} \times \dot{\mathbf{R}}\right) = \sum_a \mathbf{r}_a \times \mathbf{F}_a^{\mathrm{ext}} - M\mathbf{R} \times \ddot{\mathbf{R}} \\
&= \sum_a \left(\mathbf{r}_a - \mathbf{R}\right) \times \mathbf{F}_a^{\mathrm{ext}} = \sum_a \mathbf{r}_a^* \times \mathbf{F}_a^{\mathrm{ext}}. \qquad (2.1.13)
\end{aligned}$$

This means that we can often study the centre-of-mass motion and the relative motion separately from each other. In particular, if the external forces are position-independent, the relative angular momentum evolves independently of the centre-of-mass motion.

Likewise, the total kinetic energy separates into the kinetic energy of the centre of mass and the kinetic energy relative to the centre of mass,

$$T = \frac{1}{2}\sum_a m_a \dot{\mathbf{r}}_a^2 = \frac{1}{2}\sum_a m_a \left(\dot{\mathbf{R}} + \dot{\mathbf{r}}_a^*\right) \cdot \left(\dot{\mathbf{R}} + \dot{\mathbf{r}}_a^*\right) = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_a m_a \dot{\mathbf{r}}_a^{*2}. \qquad (2.1.14)$$

The above results apply generally to many-body systems, but for the rest of this Section we will focus on a special class of them known as *rigid bodies*. These are many-body systems in which all distances $|\mathbf{r}_a - \mathbf{r}_b|$ between particles are fixed. The whole system can still move and rotate. In reality, a rigid body a mathematical idealisation because it requires infinitely strong forces between particles, but in many cases it is a very good approximation.

## 2.2 Rotation about a Fixed Axis

We first consider the case in which the body is free to rotate about a fixed axis. Cylindrical polar coordinates $(\rho, \phi, z)$ are ideally suited for this situation (see Fig. 1.1). If we choose the $z$ axis as the rotation axis, then for every particle $a$ the coordinate $z_a$ and $\rho_a$ are fixed and only the angular coordinate $\phi_a$ changes as $\dot{\phi}_a = \omega$. Then we can write the $z$ component of the angular momentum (2.1.6) as

$$L_z = \sum_a m_a \rho_a \left(\rho_a \dot{\phi}\right) = \sum_a m_a \rho_a^2 \omega = I\omega \,, \qquad (2.2.1)$$

where $\rho\dot{\phi}$ is the tangential velocity and $I = \sum_a m_a \rho_a^2$ is the *moment of inertia* about the axis. As $I$ is obviously constant we can write its rate of change as

$$\dot{L}_z = I\dot{\omega} = \sum_a \rho_a F_{a\phi}, \tag{2.2.2}$$

where $F_{a\phi}$ is the component of the external force $\mathbf{F}_a^{\text{ext}}$ in the $\hat{\phi}$ direction.

Similarly we can write the kinetic energy in terms of $I$ and $\omega$ as

$$T = \sum_a \tfrac{1}{2} m_a \left(\rho_a \dot{\phi}_a\right)^2 = \tfrac{1}{2} I\omega^2 \,. \tag{2.2.3}$$

Note the similarity of these expressions to the corresponding linear ones where $m \mapsto I$ and $v \mapsto \omega$

$$p = mv \qquad \leftrightarrow \qquad L = I\omega \tag{2.2.4}$$

$$T = \tfrac{1}{2} m v^2 \qquad \leftrightarrow \qquad T = \tfrac{1}{2} I\omega^2 \,. \tag{2.2.5}$$

Of course, there is no reason why the axis should be through the centre of mass, and for example in a pendulum it is not. If we define the origin to be on the axis then we can define $\mathbf{R}$ as the distance of the centre of mass from that axis. In general the axis would be free to move, so in order for it to remain fixed, there must be a support force $\mathbf{Q}$ that prevents it from moving. From brevity, we refer to it as the “force at axis”. Denoting the sum of all other external forces by $\mathbf{F}$, we can write Newton’s second law as

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{Q} + \mathbf{F} \,. \tag{2.2.6}$$

Using $\dot{\mathbf{R}} = \boldsymbol{\omega} \times \mathbf{R}$ we can write

$$\ddot{\mathbf{R}} = \dot{\boldsymbol{\omega}} \times \mathbf{R} + \boldsymbol{\omega} \times \dot{\mathbf{R}} = \dot{\boldsymbol{\omega}} \times \mathbf{R} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) \,. \tag{2.2.7}$$

The first of these is the tangential acceleration and the second one is the centripetal force which keeps the centre of mass on a circular trajectory. From these equations we can determine the support force $\mathbf{Q}$ required to keep the axis fixed.

### 2.2.1 Compound Pendulum

As an example, let us consider a *compound pendulum*, which is a rigid body attached to a pivot and subject to a gravitational force. We take the $z$–axis to be the axis of rotation, which is now horizontal, and $\hat{\mathbf{x}}$ to be pointing downwards. Then the pendulum is subject to an external gravitational force $\mathbf{F} = (Mg, 0, 0)$ acting through its centre of mass. In terms of the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$, we can write this as

$$\mathbf{F} = Mg\cos\phi\hat{\boldsymbol{\rho}} - Mg\sin\phi\hat{\boldsymbol{\phi}}. \tag{2.2.8}$$

Thus the equation of motion (2.2.2) is

$$I\ddot{\phi} = -MgR\sin\phi, \tag{2.2.9}$$

and the energy conservation equation is

$$E = T + V = \tfrac{1}{2}I\dot{\phi}^2 - MgR\cos\phi = \text{constant}. \tag{2.2.10}$$

For small amplitudes, $\phi \ll 1$, Eq. (2.2.9) reduces to the equation for a simple harmonic oscillator,

$$\ddot{\phi} = -\frac{MgR}{I}\phi, \tag{2.2.11}$$

with period $\Theta = 2\pi\sqrt{I/MgR}$.

Rewriting Eq. (2.2.7) in polar coordinates and noting that only $\phi$ actually changes we can calculate the net force on the system and hence the support force $\mathbf{Q}$ at the axis,

$$\begin{aligned}
\dot{\mathbf{P}} &= M\ddot{\mathbf{R}} = MR\ddot{\phi}\,\hat{\boldsymbol{\phi}} - MR\dot{\phi}^2\,\hat{\boldsymbol{\rho}} && (2.2.12)\\
\Rightarrow \mathbf{Q} &= \dot{\mathbf{P}} - \mathbf{F} = \left(-Mg\cos\phi - MR\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + \left(Mg\sin\phi + MR\ddot{\phi}\right)\hat{\boldsymbol{\phi}} && (2.2.13)\\
&= -\left[Mg\cos\phi\left(1 + \frac{2MR^2}{I}\right) + \frac{2MR}{I}E\right]\hat{\boldsymbol{\rho}} + Mg\sin\phi\left(1 - \frac{MR^2}{I}\right)\hat{\boldsymbol{\phi}}\,, && (2.2.14)
\end{aligned}$$

where, in the final step, we have substituted from Eqs. (2.2.10) and (2.2.9) to eliminate $\dot{\phi}$ and $\ddot{\phi}$. Note that, in contrast with a simple pendulum (for which $I = MR^2$), the force $\mathbf{Q}$ is not in the radial direction $\hat{\boldsymbol{\rho}}$.

### 2.2.2 Centre of Percussion

As a further example, consider a compound pendulum which is initially at rest. An external force $\mathbf{F}$ in the angular direction $\hat{\boldsymbol{\phi}}$ is then applied at distance $d$ from the pivot point for a short period of time. We want to calculate the force $\mathbf{Q}$ needed to keep the axis fixed.

To make this more concrete, you can think of the pendulum as a tennis racket which you are holding in your hand, so that your hand acts as the pivot point. A ball hits the racket at distance $d$ from your hand and exerts a force $\mathbf{F}$ on the racket. We want to calculate the support force $\mathbf{Q}$ which your hand has to provide to remain stationary, or equivalently the impact you will feel with your hand.

Because initially the racket is not rotating, $\dot{\phi} = 0$, and therefore Eqs. (2.2.6) and (2.2.7) become

$$\begin{aligned}
\mathbf{Q} + \mathbf{F} &= \dot{\mathbf{P}} = M\ddot{\mathbf{R}} = MR\ddot{\phi}\hat{\boldsymbol{\phi}},\\
I\ddot{\phi} &= dF_\phi, && (2.2.15)
\end{aligned}$$

from which we find

$$Q_\phi = MR\ddot{\phi} - F_\phi = \frac{MRd}{I}F_\phi - F_\phi = \left(\frac{MRd}{I} - 1\right)F_\phi. \tag{2.2.16}$$

If the distance at which the ball hits the racket is $d = I/MR$, the linear and rotational motion balance each other and the pivot point does not feel any impact. This point is known as the *centre of percussion*. In sport it is also called the "sweet spot", because you hit the ball but feel no impact with your hand.

## 2.3 Inertia Tensor

In general, the angular momentum vector

$$\mathbf{L} = \sum_a m_a \mathbf{r}_a \times \dot{\mathbf{r}}_a = \sum_a m_a \mathbf{r}_a \times (\omega \times \mathbf{r}_a) \tag{2.3.1}$$

is not parallel to the angular velocity $\omega$.

Let us use the Cartesian coordinates and write the position vector as

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \tag{2.3.2}$$

For simplicity, we first assume that $\omega$ is in the $z$ direction, so that

$$\boldsymbol{\omega} = \omega \hat{\mathbf{k}} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} . \tag{2.3.3}$$

Then we have

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) = (\mathbf{r} \cdot \mathbf{r})\boldsymbol{\omega} - (\mathbf{r} \cdot \boldsymbol{\omega})\mathbf{r} = (x^2 + y^2 + z^2)\boldsymbol{\omega} - z\omega\mathbf{r} = \begin{pmatrix} -xz\omega \\ -yz\omega \\ (x^2 + y^2)\omega \end{pmatrix} . \tag{2.3.4}$$

Using this in Eq. (2.3.1), we find the components of the angular momentum vector $\mathbf{L}$,

$$\begin{aligned} L_x &= -\sum_a m_a x_a z_a \omega, \\ L_y &= -\sum_a m_a y_a z_a \omega, \\ L_z &= \sum_a m_a \left(x_a^2 + y_a^2\right) \omega \, . \end{aligned} \tag{2.3.5}$$

We can summarise these by writing

$$L_x = I_{xz}\omega, \qquad L_y = I_{yz}\omega, \qquad L_z = I_{zz}\omega, \tag{2.3.6}$$

where

$$I_{xz} = -\sum_a m_a x_a z_a, \qquad I_{yz} = -\sum_a m_a y_a z_a, \qquad I_{zz} = \sum_a m_a \left(x_a^2 + y_a^2\right) . \tag{2.3.7}$$

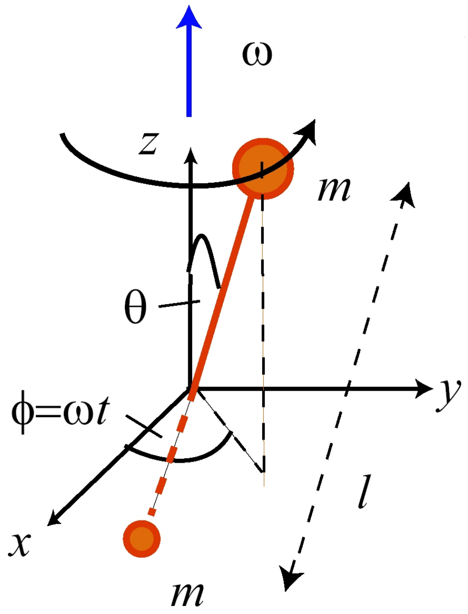


Figure 2.1:

$I_{zz}$ is the moment of inertia as previously defined. $I_{xz}$ and $I_{yz}$ are sometimes known as *products of inertia.*

As an example of a simple system for which the angular momentum is not parallel the angular velocity, consider a rigid rod with equal masses on either end (a dumbbell) inclined at an angle $\theta$ to the axis of rotation. If the masses are at $\pm\mathbf{r}$ then the total angular momentum is

$$\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}} + m(-\mathbf{r}) \times (-\dot{\mathbf{r}}) = 2m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \ , \tag{2.3.8}$$

which is clearly in a direction perpendicular to $\mathbf{r}$. Because it is also perpendicular to the vector $(\boldsymbol{\omega} \times \mathbf{r})$, it has to lie on the plane spanned by $\boldsymbol{\omega}$ and $\mathbf{r}$, and therefore it is rotating around the axis $\boldsymbol{\omega}$ together with the rod.

Using Eq. (2.3.5), we can write the components of the angular momentum as

$$\mathbf{L} = \begin{pmatrix} -2mxz\omega \\ -2myz\omega \\ 2m(x^2+y^2)\omega \end{pmatrix} = 2m\rho\omega \begin{pmatrix} -z\cos\phi \\ -z\sin\phi \\ \rho \end{pmatrix} . \tag{2.3.9}$$

For completeness, let us write down the angular momentum vector $\mathbf{L}$ for a general angular velocity $\boldsymbol{\omega}$. We have

$$\begin{aligned}\mathbf{L} &= \sum_a \left[(\mathbf{r}_a \cdot \mathbf{r}_a)\boldsymbol{\omega} - (\mathbf{r}_a \cdot \boldsymbol{\omega})\mathbf{r}_a\right] \\ &= \sum_a m_a \left[(x_a^2 + y_a^2 + z_a^2)\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} - (x_a\omega_x + y_a\omega_y + z_a\omega_z)\begin{pmatrix}x_a\\ y_a\\ z_a\end{pmatrix}\right] \\ &= \sum_a m_a \begin{pmatrix}(y_a^2 + z_a^2)\omega_x - x_a y_a \omega_y - x_a z_a \omega_z \\ -x_a y_a \omega_x + (x_a^2 + z_a^2)\omega_y - y_a z_a \omega_z \\ -x_a z_a \omega_x - y_a z_a \omega_y + (x_a^2 + y_a^2)\omega_z\end{pmatrix}. \end{aligned} \tag{2.3.10}$$

Using linear algebra, we can write this as a product of a matrix and a vector

$$\mathbf{L} = \sum_a m_a \begin{pmatrix} y_a^2 + z_a^2 & -x_a y_a & -x_a z_a \\ -x_a y_a & x_a^2 + z_a^2 & -y_a z_a \\ -x_a z_a & -y_a z_a & x_a^2 + y_a^2 \end{pmatrix} \begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}, \tag{2.3.11}$$

or more concisely

$$\mathbf{L} = \bar{\bar{\mathbf{I}}} \cdot \boldsymbol{\omega}, \tag{2.3.12}$$

where the three-by-three matrix

$$\bar{\bar{\mathbf{I}}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \sum_a m_a \begin{pmatrix} y_a^2 + z_a^2 & -x_a y_a & -x_a z_a \\ -x_a y_a & x_a^2 + z_a^2 & -y_a z_a \\ -x_a z_a & -y_a z_a & x_a^2 + y_a^2 \end{pmatrix} \tag{2.3.13}$$

is known as the *inertia tensor*. In general, a tensor is a geometric object that describes a linear relation between two or more vectors. In this case, the inertia tensor describes the linear relation between $\boldsymbol{\omega}$ and $\mathbf{L}$, and can be represented by a three-by-three matrix. Just like the components of a vector, the elements of the matrix $\bar{\bar{\mathbf{I}}}$ change under rotations. For more details, see Appendix A.9 in Kibble&Berkshire.

Finally, it is often convenient to work in the component notation. Labelling the coordinates $x$, $y$ and $z$ by $i, j \in \{1, 2, 3\}$, we can write the components of the inertia tensor in a compact form as

$$I_{ij} = \sum_a m_a \left(\mathbf{r}_a^2 \delta_{ij} - r_{ai} r_{aj}\right), \tag{2.3.14}$$

where $\delta_{ij}$ is the Kronecker delta (that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$), and $r_{ai}$ is the $i$th component of the position vector $\mathbf{r}_a$. In the component notation, Eq. (2.3.12) becomes

$$L_i = \sum_j I_{ij}\omega_j. \tag{2.3.15}$$

## 2.4 Principal Axes of Inertia

We can make use of our knowledge of the properties of matrices to understand the meaning of the inertia tensor $\bar{\bar{\mathbf{I}}}$. We note that $\bar{\bar{\mathbf{I}}}$ is symmetric, $I_{xy} = I_{yx}$, so that the eigenvalues of $\bar{\bar{\mathbf{I}}}$ are real. We denote these eigenvalues by $I_1$, $I_2$ and $I_3$ and call them the *principal moments of inertia*. The corresponding eigenvectors, which we denote by $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ are called the *principal axes of inertia*. They are orthogonal to each other, and we choose them to be unit vectors. By definition, the eigenvectors satisfy

$$\bar{\bar{\mathbf{I}}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i, \tag{2.4.1}$$

where again $i \in \{1, 2, 3\}$. This also means that if the angular velocity $\boldsymbol{\omega}$ is parallel to a principal axis, then the angular momentum $\mathbf{L}$ is parallel to it.

It is convenient to work in a coordinate system based on the principal axes, and write

$$\boldsymbol{\omega} = \sum_i \omega_i \hat{\mathbf{e}}_i. \tag{2.4.2}$$

The angular momentum is then

$$\mathbf{L} = \sum_i I_i \omega_i \hat{\mathbf{e}}_i. \tag{2.4.3}$$

It is important to note that the principal axes rotate with the body. They therefore represent a rotating frame of reference (see Chapter 1).

Using the identity $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, the kinetic energy can be expressed as

$$\begin{aligned} T &= \sum_a \tfrac{1}{2} m_a \dot{\mathbf{r}}_a \cdot \dot{\mathbf{r}}_a = \sum_a \tfrac{1}{2} m_a (\boldsymbol{\omega} \times \mathbf{r}_a) \cdot (\boldsymbol{\omega} \times \mathbf{r}_a) = \sum_a \tfrac{1}{2} m_a \boldsymbol{\omega} \cdot [\mathbf{r}_a \times (\boldsymbol{\omega} \times \mathbf{r}_a)] \\ &= \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2} \boldsymbol{\omega} \cdot \bar{\bar{\mathbf{I}}} \cdot \boldsymbol{\omega} = \frac{1}{2} \sum_i I_i \omega_i^2. \end{aligned} \tag{2.4.4}$$

The principal axes can always be found by diagonalising the inertia tensor $\bar{\bar{\mathbf{I}}}$, but calculations become easier if one already knows their directions because then one can choose them as the coordinate axes. It is therefore useful to know that any symmetry axis is always a principal axis, and than the direction normal to any symmetry plane is also a principal axis.

If two of the principal moments of inertia are equal, say $I_1 = I_2$, we say that the body is a *symmetric body*. In this case, any linear combination of the $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$, so any two orthogonal directions on the plane spanned by $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ can be chosen as the principal axis. Note that although a system with an axis of cylindrical symmetry, e.g. a cylinder or a cone, would certainly be a symmetric body in this sense, it is not necessary. In fact any system with a more than 2-fold rotational symmetry would suffice, e.g. a triangular prism, or the two principal moments could be equal just by chance in spite of the body have no geometrical symmetry. In the case of a symmetric body, Eq. (2.4.3) becomes

$$\mathbf{L} = I_1 (\omega_1 \hat{\mathbf{e}}_1 + \omega_2 \hat{\mathbf{e}}_2) + I_3 \omega_3 \hat{\mathbf{e}}_3 \,. \tag{2.4.5}$$

If all 3 moments of inertia are equal, we say the body is *totally symmetric*. Again, this can happen either by symmetry, as in a sphere, cube, regular tetrahedron or any of the five regular solids, or by coincidence. In the case of a totally symmetric body, we have $\mathbf{L} = I\boldsymbol{\omega}$ and $\mathbf{L}$ is always in the same direction as $\boldsymbol{\omega}$. In that case the choice of the 3 principal axes is completely arbitrary, as long as they are mutually perpendicular.

## 2.5 Calculation of Moments of Inertia

### 2.5.1 Shift of Origin

It is often useful to be able to relate the moments of inertia about different pivots, e.g. when a body is pivoted around a point other than its centre of mass. We write the position vector $\mathbf{r}_a$ has the sum of the centre-of-mass position $\mathbf{R}$ and the position relative to the centre-of-mass $\mathbf{r}^*_a$, i.e., $\mathbf{r}_a = \mathbf{R} + \mathbf{r}^*_a$. Then, by definition,

$$\sum_a m_a \mathbf{r}^*_a = 0. \tag{2.5.1}$$

Therefore we can write the components of the inertia tensor (2.3.14) as

$$\begin{aligned} I_{ij} &= \sum_a m_a \left[ \left(\mathbf{R}^2 + \mathbf{r}^*_a\right)^2 \delta_{ij} - (R_i + r^*_{ai})(R_j + r^*_{aj}) \right] \\ &= \sum_a m_a \left[ \mathbf{R}^2 \delta_{ij} + (\mathbf{r}^*_a)^2 \delta_{ij} - R_i R_j - r^*_{ai} r^*_{aj} \right] = M \left( \mathbf{R}^2 \delta_{ij} - R_i R_j \right) + I^*_{ij}, \end{aligned} \tag{2.5.2}$$

where

$$I^*_{ij} = \sum_a m_a \left[ (\mathbf{r}^*_a)^2 \delta_{ij} - r^*_{ai} r^*_{aj} \right]. \tag{2.5.3}$$

If we know the inertia tensor with respect to the centre of mass $\overline{\overline{\mathbf{I}}}^*$, we can use these relations to easily calculate with respect to any origin we want. This is known as the *parallel axes theorem.* Note that the principal axes about a general point are not necessarily parallel to those about the centre of mass, unless the point itself lies on one of the principal axes.

### 2.5.2 Continuous Solid

Generally we have a continuous solid rather than a group of point particles. In this case the sums become integrals and the masses, $m_a$ become densities, $\rho(\mathbf{r})$, so that we have

$$I_{ij} = \int \rho(\mathbf{r}) \left( r^2 \delta_{ij} - r_i r_j \right) \, \mathrm{d}^3 r. \tag{2.5.4}$$

### 2.5.3 Routh's Rule

Let us now imagine that the coordinate axes have been chosen to agree with the principal axes. We can then see from Eq. (2.5.4) that we can split the principal moments of inertia such that

$$I_1^* = K_y + K_z, \qquad I_2^* = K_x + K_z, \qquad I_3^* = K_x + K_y, \tag{2.5.5}$$

where

$$K_i = \int_V \rho r_i^2 \, \mathrm{d}^3 r. \tag{2.5.6}$$

It is now useful to ask how the principal moments change if we rescale (i.e. stretch or squeeze) the body in directions along the principal axes. To do this in practice, let us first consider the original body $\tilde{V}$ assuming that we know the constants $\tilde{K}_i$ defined by by Eq. (2.5.6). The rescaled body $V$ is obtained by rescaling the coordinates as $r_i = a_i \tilde{r}_i$ for each $i \in \{1,2,3\}$. The constants $K_i$ in Eq.(2.5.6) change to

$$K_i = \int_V \rho {r_i}^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = a_1 a_2 a_3 a_i^2 \int_{\tilde{V}} \rho \tilde{r}_i^2 \, \mathrm{d}\tilde{x} \, \mathrm{d}\tilde{y} \, \mathrm{d}\tilde{z} = a_1 a_2 a_3 a_i^2 \tilde{K}_i. \tag{2.5.7}$$

We can also note that the total mass of the body $V$ is

$$M = \int_V \rho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = a_1 a_2 a_3 \int_{\tilde{V}} \rho \mathrm{d}\tilde{x} \, \mathrm{d}\tilde{y} \, \mathrm{d}\tilde{z} = a_1 a_2 a_3 \tilde{M}, \tag{2.5.8}$$

where $\tilde{M}$ is the mass of the original body. Hence $K_i \propto a_i^2 M$, and we can write

$$K_i = \lambda_i a_i^2 M, \tag{2.5.9}$$

where $\lambda_z$ is a dimensionless number, which is the same for all bodies of the same general type. Hence we have *Routh's rule* which states that

$$\begin{aligned} I_1^* &= M\left(\lambda_y a_y^2 + \lambda_z a_z^2\right), \\ I_2^* &= M\left(\lambda_x a_x^2 + \lambda_z a_z^2\right), \\ I_3^* &= M\left(\lambda_x a_x^2 + \lambda_y a_y^2\right). \end{aligned} \tag{2.5.10}$$

By checking the standard bodies we obtain the following values for the coefficients: $\lambda = \frac{1}{3}$ for 'rectangular' axes, $\lambda = \frac{1}{4}$ for 'elliptical' axes and $\lambda = \frac{1}{5}$ for 'ellipsoidal' ones. This covers most special cases. For example, a sphere is an ellipsoid with $a_x = a_y = a_z = a$ and each principal moment of inertia is $\frac{2}{5}Ma^2$, whereas a cube is a parallelepiped with $a_x = a_y = a_z = a$ and $I = \frac{2}{3}Ma^2$.

For a cylinder we have $\lambda_x$ and $\lambda_y$ elliptical and $\lambda_z$ rectangular. This nomenclature can be confusing as it refers to the symmetry of the corresponding integrals and not to symmetry about the axes. For a cylinder with $a_x = a_y \neq a_z$ we have

$$I_1^* = I_2^* = M\left(\tfrac{1}{4}a_x^2 + \tfrac{1}{3}a_z^2\right) \qquad I_3^* = M\left(\tfrac{1}{4}a_x^2 + \tfrac{1}{4}a_x^2\right) = \tfrac{1}{2}Ma_x^2, \tag{2.5.11}$$

and therefore a flat circular plate, i.e. a cylinder with $a_z = 0$, has

$$I_1^* = I_2^* = \tfrac{1}{4}Ma_x^2 \qquad I_3^* = \tfrac{1}{2}Ma_x^2. \tag{2.5.12}$$

Conversely, a thin rod is a cylinder with $a_x = a_y = 0$ and

$$I_1^* = I_2^* = \tfrac{1}{3}Ma_z^2 \qquad I_3^* = 0. \tag{2.5.13}$$

## 2.6 Effect of Small Force

Suppose a body is rotating about a principal axis such that $\boldsymbol{\omega} = \omega\hat{\mathbf{e}}_3$ and $\mathbf{L} = I_3\omega\hat{\mathbf{e}}_3$. Then

$$\dot{\mathbf{L}} = I_3\dot{\boldsymbol{\omega}} = 0\,, \tag{2.6.1}$$

the axis will remain fixed in space and the angular velocity will be constant. Note that this would not be true if $\boldsymbol{\omega}$ were not a principal axis.

Suppose now that the axis is fixed at the origin and a small force $\mathbf{F}$ is applied to the axis at point $\mathbf{r}$. Then the equation of motion becomes

$$\dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F}\,. \tag{2.6.2}$$

The body will acquire a small component of angular velocity perpendicular to its axis. However, if the force is small, this will be small compared with the angular velocity of rotation about the axis. We may then neglect the angular momentum components normal to the axis and again write

$$\dot{\mathbf{L}} = I_3\dot{\boldsymbol{\omega}} = \mathbf{r} \times \mathbf{F}\,. \tag{2.6.3}$$

Since $\mathbf{r} \times \mathbf{F}$ is perpendicular to $\boldsymbol{\omega}$ ($\mathbf{r}$ is parallel to $\boldsymbol{\omega}$) the magnitude of $\omega$ does not change ($\mathrm{d}\omega^2/\mathrm{d}t = 2\boldsymbol{\omega}\cdot\dot{\boldsymbol{\omega}} = 0$). Its direction does change, however, in the direction of $\mathbf{r} \times \mathbf{F}$ and hence perpendicular to the applied force $\mathbf{F}$.

Figure 2.2:

As an example, consider a child's spinning top. In general, the rotation axis is not exactly vertical. We consider the point as which the top touches the ground as the pivot point, and use it as our origin. There is a gravitational force $\mathbf{F} = -Mg\hat{\mathbf{k}}$, acting at the centre of mass at position

$\mathbf{R} = R\hat{\mathbf{e}}_3$. Eq. (2.6.3) gives

$$\begin{aligned} I_3\omega\frac{\mathrm{d}\hat{\mathbf{e}}_3}{\mathrm{d}t} &= -MgR\,\hat{\mathbf{e}}_3 \times \hat{\mathbf{k}} \\ \Rightarrow \frac{\mathrm{d}\hat{\mathbf{e}}_3}{\mathrm{d}t} &= \left(\frac{MgR}{I_3\omega}\right)\hat{\mathbf{k}} \times \hat{\mathbf{e}}_3. \end{aligned} \tag{2.6.4}$$

This has the same form as Eq. (1.2.1), i.e.,

$$\frac{\mathrm{d}\hat{\mathbf{e}}_3}{\mathrm{d}t} = \boldsymbol{\Omega} \times \hat{\mathbf{e}}_3, \tag{2.6.5}$$

Which means that the principal axis $\hat{\mathbf{e}}_3$ rotates around the vertical direction $\hat{\mathbf{k}}$ with angular velocity

$$\boldsymbol{\Omega} = \frac{MgR}{I_3\omega}\hat{\mathbf{k}}\,. \tag{2.6.6}$$

The analysis is only valid when $\Omega \ll \omega$ or when $MgR \ll I_3\omega^2$; the potential energy associated with the tilt is much smaller than the kinetic energy of the rotation. The system is very similar to Larmor precession (see section 1.6.4). The expression for $\Omega$ tells us a great deal about this system. Note that $\Omega$ is inversely proportional to both the moment of inertia $I_3$ and the angular frequency $\omega$. This implies that to minimise the precession and hence to improve the stability of the system we have to choose both to be large: we require a fat rapidly spinning body.

This is the basis of the gyroscope: the high stability of such a rapidly rotating body makes it ideal for use in navigation, especially near the poles where a compass is almost useless. It can also be used, e.g. , to provide an "artificial horizon" when flying blind, either in cloud or at night.

## 2.7 Rotation about a Principal Axis

As the principal axes are fixed in the body we are really dealing with a rotating frame. We here return to the notation used in Section 1 to distinguish between the inertial and rotating frames. The rate of change of the angular momentum in the inertial frame is

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{\mathrm{I}} = \sum_a \mathbf{r}_a \times \mathbf{F}_a = \boldsymbol{\tau}\,. \tag{2.7.1}$$

Eq. (1.2.5) relates this to the rate of change measured in the rotating frame,

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{\mathrm{I}} = \left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{\mathrm{R}} + \boldsymbol{\omega} \times \mathbf{L}\,. \tag{2.7.2}$$

On the other hand, because in the rotating frame the principal axes and principal moments are fixed, we have

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{\mathrm{R}} = I_1\dot{\omega}_1\hat{\mathbf{e}}_1 + I_2\dot{\omega}_2\hat{\mathbf{e}}_2 + I_3\dot{\omega}_3\hat{\mathbf{e}}_3\,, \tag{2.7.3}$$

and, therefore,

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{\mathrm{R}} + \boldsymbol{\omega}\times\mathbf{L} = \boldsymbol{\tau}. \tag{2.7.4}$$

Calculating the cross product

$$\begin{aligned}\boldsymbol{\omega}\times\mathbf{L} &= \begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \omega_1 & \omega_2 & \omega_3\\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3\end{vmatrix}\\ &= (I_3-I_2)\omega_2\omega_3\hat{\mathbf{e}}_1 + (I_1-I_3)\omega_1\omega_3\hat{\mathbf{e}}_2 + (I_2-I_1)\omega_1\omega_2\hat{\mathbf{e}}_3,\end{aligned} \tag{2.7.5}$$

we find the Euler equations

$$\begin{aligned}I_1\dot{\omega}_1 + (I_3-I_2)\,\omega_2\omega_3 &= \tau_1\,,\\ I_2\dot{\omega}_2 + (I_1-I_3)\,\omega_3\omega_1 &= \tau_2\,,\\ I_3\dot{\omega}_3 + (I_2-I_1)\,\omega_1\omega_2 &= \tau_3\,.\end{aligned} \tag{2.7.6}$$

In principle these equations could be solved to give $\boldsymbol{\omega}(t)$. In practice, however, we often don't have the force expressed in a useful form to do this and, in any case, it is easier to solve this system using Lagrangian methods (see chapter 3).

For the moment we concentrate on studying the stability of the motion in the absence of external forces ($\boldsymbol{\tau} = 0$). Suppose that the object is rotating about the principal axis $\hat{\mathbf{e}}_3$ and that $\omega_1 = \omega_2 = 0$ then it is obvious from Eq. (2.7.6) that the object will continue indefinitely to rotate about $\hat{\mathbf{e}}_3$. On the other hand let us suppose that the motion deviates slightly from this such that $\omega_1$ and $\omega_2$ are much smaller than $\omega_3$. We may therefore ignore any terms which are quadratic in $\omega_1$ and $\omega_2$ so that, from the third line in Eq. (2.7.6), we have $\dot{\omega}_3 = 0$ and $\omega_3$ is constant.

We look for solutions of the form[2]

$$\omega_1 = a_1\mathrm{e}^{\gamma t} \qquad \omega_2 = a_2\mathrm{e}^{\gamma t} \tag{2.7.7}$$

where $a_1, a_2$ and $\gamma$ are constants. Substituting this into Eq. (2.7.6) gives

$$I_1\gamma a_1 + (I_3-I_2)\,\omega_3 a_2 = 0 \tag{2.7.8}$$

$$I_2\gamma a_2 + (I_1-I_3)\,\omega_3 a_1 = 0\,, \tag{2.7.9}$$

which is a $2\times 2$ eigenvalue problem with a solution

$$\gamma^2 = \frac{(I_3-I_2)\,(I_1-I_3)}{I_1 I_2}\omega_3^2\,. \tag{2.7.10}$$

We note that $\omega_3^2/I_1I_2$ is always positive. Hence, if $I_3$ is the smallest or the largest of the 3 moments of inertia $\gamma^2$ is negative. In that case $\gamma$ is imaginary and the motion is oscillatory. Hence its amplitude does not change, and we say that the rotation is stable.

[2]Those doing computational physics will note the similarity between this analysis and the stability analysis considered there.

However, if $I_3$ is the middle of the three moments then $\gamma^2$ is positive and $\gamma$ is real. There are two independent solutions with opposite signs of $\gamma$, and in general the solution is a linear combination of them. However, at late times ($t \gg 1/\gamma$) the solution with a positive exponent dominates. Hence $\omega_1$ and $\omega_2$ tend to grow exponentially and the motion about $\hat{e}_3$ is unstable: any small deviation from rotation about $\hat{e}_3$ will tend to grow.

You can test this by trying to spin an appropriately dimensioned object, such as a book or a tennis racket. It is much easier to spin it around the axis with the smallest or the largest moment of inertia, but not the middle one.

## 2.8 Euler's Angles

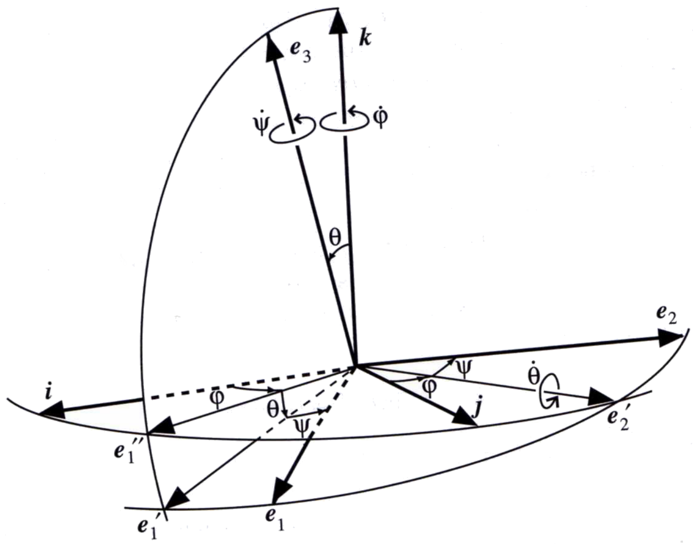


Figure 2.3:

In order to describe the orientation of a solid body we require 3 angles. The conventional way to do this is to define angles $(\phi, \theta, \psi)$ these is known as *Euler's Angles,*, which are illustrated in Fig. 2.3. Note however that there are several different conventions for Euler's Angles. We shall stick to the one used by Kibble & Berkshire, known as the $y$–convention. The meaning of the angles is, essentially, that $\phi$ and $\theta$ are the usual spherical coordinates expressing the direction of the principal axis $\hat{e}_3$, and $\psi$ expresses the orientation of the object about this axis.

Let us construct the angles in detail. We can obviously express the orientation of the body by giving the orientations of the three principal axes, i.e., by a triplet of orthogonal unit vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$. To show that we can parameterise these with the three Euler angles, let us start with the orientation $(\hat{\imath}, \hat{\jmath}, \hat{k})$, which means that the principal axes are aligned with the axes of our original Cartesian coordinate system. As illustrated in Fig. 2.3, we then carry out three steps:

- We first rotate by $\phi$ about the $\hat{\mathbf{k}}$ axis. The changes the directions of the first two principal axes, and we denote the new directions by $\hat{\mathbf{e}}_1''$ and $\hat{\mathbf{e}}_2'$. Thus, the orientation of the principal axes changes as $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\mathbf{k}}) \to (\hat{\mathbf{e}}_1'', \hat{\mathbf{e}}_2', \hat{\mathbf{k}})$.
- Secondly we rotate by $\theta$ about the second principal axis $\hat{\mathbf{e}}_2'$. This changes the directions of the first and third principal axes to $\hat{\mathbf{e}}_1'$ and $\hat{\mathbf{e}}_3$, so the orientation of the body changes as $(\hat{\mathbf{e}}_1'', \hat{\mathbf{e}}_2', \hat{\mathbf{k}}) \to (\hat{\mathbf{e}}_1', \hat{\mathbf{e}}_2', \hat{\mathbf{e}}_3)$.
- Finally we rotate by $\psi$ about the third principal axis $\hat{\mathbf{e}}_3$, to bring the first principal axis to direction $\hat{\mathbf{e}}_1$ and the second principal axis to $\hat{\mathbf{e}}_2$, i.e., $(\hat{\mathbf{e}}_1', \hat{\mathbf{e}}_2', \hat{\mathbf{e}}_3) \to (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$.

Using these three rotations we can reach any orientation $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ we want, and therefore the orientation of the body is fully parameterised by the three Euler angles.

Because the three angles $(\phi, \theta, \psi)$ correspond to rotations about the axes $\hat{\mathbf{k}}$, $\hat{\mathbf{e}}_2'$ and $\hat{\mathbf{e}}_3$, respectively. Note that these axes are not mutually perpendicular. We can, nevertheless, use them to express the angular velocity $\boldsymbol{\omega}$ in terms of Euler angles as

$$\boldsymbol{\omega} = \dot{\phi}\hat{\mathbf{k}} + \dot{\theta}\hat{\mathbf{e}}_2' + \dot{\psi}\hat{\mathbf{e}}_3 \,. \tag{2.8.1}$$

For a symmetric system such as a gyroscope we can choose $\hat{\mathbf{e}}_3$ as the symmetry axis and, as $I_1 = I_2$, any two mutually perpendicular axes as the other two. In this case the most convenient are $\hat{\mathbf{e}}_1'$ and $\hat{\mathbf{e}}_2'$ as two of the axes are already used in Eq. (2.8.1). We can therefore use that $\hat{\mathbf{k}} = -\sin\theta\,\hat{\mathbf{e}}_1' + \cos\theta\,\hat{\mathbf{e}}_3$ to obtain

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\hat{\mathbf{e}}_1' + \dot{\theta}\hat{\mathbf{e}}_2' + \left(\dot{\psi} + \dot{\phi}\cos\theta\right)\hat{\mathbf{e}}_3 \,, \tag{2.8.2}$$

where the unit vectors are mutually perpendicular and, for a symmetric body, principal axes.

Using Eq. (2.8.2), we can express the angular momentum and kinetic energy as

$$\mathbf{L} = -I_1\dot{\phi}\sin\theta\,\hat{\mathbf{e}}_1' + I_1\dot{\theta}\hat{\mathbf{e}}_2' + I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\hat{\mathbf{e}}_3 \tag{2.8.3}$$

$$T = \tfrac{1}{2}I_1\dot{\phi}^2\sin^2\theta + \tfrac{1}{2}I_1\dot{\theta}^2 + \tfrac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 . \tag{2.8.4}$$

To find equations of motion we could either translate this into Cartesian coordinates, $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\mathbf{k}}$, or try to write the equations in terms of the Euler angles. Either way is difficult. It is much easier to use Lagrangian methods (see Chapter 3).

In the meantime we can consider the free motion, with no forces. In this case $\mathbf{L}$ is a constant. We therefore choose the vector $\hat{\mathbf{k}}$ to be in the direction of $\mathbf{L}$ such that

$$\mathbf{L} = L\hat{\mathbf{k}} = -L\sin\theta\,\hat{\mathbf{e}}_1' + L\cos\theta\hat{\mathbf{e}}_3 \,. \tag{2.8.5}$$

This must be equal to Eq. (2.8.3) so that by equating components we can write

$$I_1\dot{\phi}\sin\theta = L\sin\theta \tag{2.8.6}$$

$$I_1\dot{\theta} = 0 \tag{2.8.7}$$

$$I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right) = L\cos\theta \tag{2.8.8}$$

From Eq. (2.8.7) we deduce that $\theta$ is constant. As long as $\sin\theta \neq 0$, Eq. (2.8.6) implies that that $\dot{\phi}$ is constant, too,

$$\dot{\phi} = \frac{L}{I_1}, \tag{2.8.9}$$

and hence, from Eq. (2.8.8), we find that $\dot{\psi}$ is also a constant,

$$\dot{\psi} = L\cos\theta\left(\frac{1}{I_3} - \frac{1}{I_1}\right). \tag{2.8.10}$$

We conclude therefore that the axis $\hat{\mathbf{e}}_3$ rotates around $\mathbf{L}$ at a constant rate $\dot{\phi}$ and at an angle $\theta$ to it. In addition the body spins about the axis $\hat{\mathbf{e}}_3$ at a constant rate $\dot{\psi}$. The angular velocity vector $\boldsymbol{\omega}$ deduced from Eq. (2.8.2) is

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\hat{\mathbf{e}}_1' + \left(\dot{\psi} + \dot{\phi}\cos\theta\right)\hat{\mathbf{e}}_3 \tag{2.8.11}$$

which describes a cone around the direction of $\mathbf{L}$.

Note that this appears very similar to precession (see Section 2.6): $\psi$ is the angle of rotation of the gyroscope around its axis, $\theta$ is the angle between the gyroscope axis and the angular momentum and $\phi$ is the angle which describes the precession around this direction. However, here we are describing free rotation with no external forces involved.