Advanced Classical Physics, Autumn 2015

Carlo R. Contaldi

October 16, 2015

Preface

These are the lecture notes for the third-year Advanced Classical Physics course in the 2015-16 academic year at Imperial College London. They are based on the notes which I inherited from the previous lecturers including Professor Angus MacKinnon, and Professor Arttu Rajantie.

The notes are designed to be self-contained, but there are also some excellent textbooks, which I want to recommend as supplementary reading. The core textbooks are

- Classical Mechanics (5th Edition), Kibble & Berkshire (Imperial College Press 2004),
- Classical Mechanics (3rd Edition), Goldstein, Poole & Safko (Addison-Wesley 2002),

and these books may also be useful:

- Classical Mechanics (2nd Edition), McCall (Wiley 2011),
- Classical Mechanics, Gregory (Cambridge University Press 2006),
- Mechanics (3rd Edition), Landau & Lifshitz (Elsevier 1976),
- Classical Electrodynamics (3rd Edition), Jackson (Wiley 1999),
- The Classical Theory of Fields, Landau & Lifshitz (Elsevier 1975).

The course assumes Mechanics, Relativity and Electromagnetism as background knowledge. Being a theoretical course, it also makes heavy use of most aspects of the compulsory mathematics courses. Mathematical Methods is also useful, but it is not a formal prerequisite, and all necessary concepts are introduced as part of this course, although in a less general way.

The course syllabus was redefined in 2014 by introducing material on Dynamical Systems from the older Dynamical Systems and Chaos course that has been discontinued.

Office Hours — Huxley 505

- Tuesdays 11–12am All term.
- Fridays 10–11am All term.

Rapid Feedback

- Nine sessions.
- Problem sheets handed out Monday/Tuesday lecture.
- Question marked as RF will be marked by assistants.
- Hand in RF work by following Monday 2pm at UG Office.
- Pick up marked work from 12pm Wednesday at UG Office.
- RF Session, LT2, 1-2pm on Fridays (from 23 October)

Aims and Objectives

Aims

To cover the topics of

Rotation

Centrifugal and Coriolis forces, Inertia tensor, Principal axes of inertia, Gyroscopes.

Lagrangian and Hamiltonian Mechanics

Calculus of variations, Action integral, Principle of least action, Generalised co-ordinates and momenta, Normal modes, Lagrangian for charged particle motion in electromagnetic field, Hamiltonian, Hamilton's equations. Canonical transformations. Poisson Brackets. Continuous symmetries.

Dynamical Systems

First and second order dynamical systems. Phase space picture and stability analysis. Action angle variables. Integrability of systems.

Electrodynamics and Relativity

4–vectors, Lorentz transformations for electromagnetic fields, Magnetic field required by relativity, 4–vector potential, Maxwell's equations in 4–vector form.

Contents

Chapter 1

Rotating Frames

1.1 Angular Velocity[1]

In order to describe rotation, you need to know its speed and the direction of the rotation axis. The speed is a scalar quantity and it is given by the angular frequency $\omega = 2\pi/T$, where T is the period of rotation, and the axis is a direction in space, so it can be represented by a unit vector $\hat{\mathbf{n}}$.

It is natural and useful to combine these to an angular velocity vector $\boldsymbol{\omega} = \omega\hat{\mathbf{n}}$. The sign of the vector $\boldsymbol{\omega}$ is determined by the *right-hand-rule*: If you imagine gripping the axis of rotation with the fingers of your right hand, your thumb will point to the direction of $\boldsymbol{\omega}$.

The angular velocity vector is called an *axial* vector (or sometimes a *pseudo-vector*), which means that is has different symmetry properties from a normal (*polar*) vector. Consider the effect of reflection in a plane containing the vector, e.g. a vector in the $\hat{\mathbf{z}}$ direction reflected in the $(y-z)$ plane. A *polar* vector is unchanged under such an operation, whereas an *axial* vector changes sign, as the direction of rotation is reversed.

For example, for the rotation of the earth (against the background of the stars), the angular velocity ω takes the value

$$\omega = \frac{2\pi}{86164\mathrm{s}} = 7.292 \times 10^{-5}\,\mathrm{s}^{-1}\,. \tag{1.1.1}$$

The angular momentum vector $\boldsymbol{\omega}$ points up at the North Pole.

Consider now a point $\mathbf{r}$ on the the rotating body, e.g. Blackett Lab. at latitude $51.5°\,\mathrm{N}$. This point is moving tangentially eastwards with a speed $v = \omega r \sin\theta$, where r is the distance from the origin (assumed to be on the axis of rotation (e.g. the centre of the Earth) and θ is the angle between the vectors $\mathbf{r}$ and ω (i.e., for Blackett, $\theta = 90° - 51.5° = 38.5°$). Hence the velocity of the point $\mathbf{r}$ may be written

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{r}\,. \tag{1.1.2}$$

We note here that geographers tend to measure latitude from the equator whereas the angle θ in spherical polar co-ordinates is defined from the pole. Thus the geographical designation $51.5°\,\mathrm{N}$ corresponds to $\theta = 38.5°$.

[1] Kibble & Berkshire, chapter 5

1.2 Transformation of Vectors

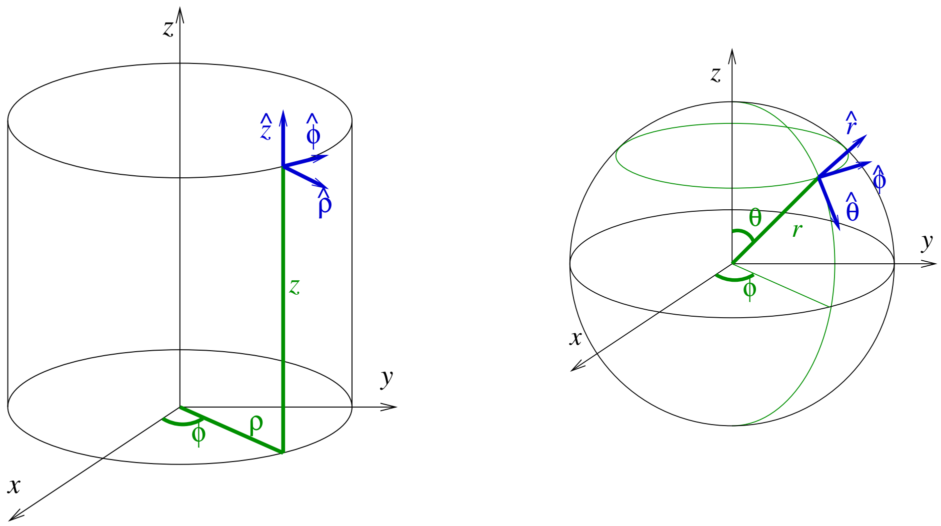

Figure 1.1: Cylindrical (left) and spherical (right) coordinate systems

Actually the result (1.1.2) is valid for any vector fixed in the rotating body, not just for position vectors. So, in general we may write

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{A}\,. \tag{1.2.1}$$

In particular we can consider the case of a set of orthogonal unit vectors, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\mathbf{k}}$ fixed in the body, chosen in accordance with the right-hand rule (index finger points in the direction of $\hat{\boldsymbol{\imath}}$, middle finger in the direction of $\hat{\boldsymbol{\jmath}}$ and thumb in the direction of $\hat{\mathbf{k}}$), which we can use as a basis of a rotating coordinate system. It is often convenient to choose $\hat{\boldsymbol{\imath}}$ to point radially away from the rotation axis and $\hat{\mathbf{k}}$ to be in the direction $\boldsymbol{\omega}$, so that $\hat{\boldsymbol{\jmath}}$ points in the direction of motion. This forms the basis of *cylindrical coordinates* (ρ, ϕ, z) and are often denoted by $\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\phi}}$, $\hat{\mathbf{k}} = \hat{\mathbf{z}}$.

On the other hand, for motion on the surface for a rotating sphere such as the Earth, a convenient choice is to take $\hat{\boldsymbol{\imath}}$ pointing south, $\hat{\boldsymbol{\jmath}}$ pointing east and $\hat{\mathbf{k}}$ pointing up (i.e. away from the centre). These form the basis of *spherical coordinates* (r, θ, ϕ), with $\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\phi}}$, $\hat{\mathbf{k}} = \hat{\mathbf{r}}$.

We have to be careful to distinguish between the point of view of an observer on the rotating object and one in a fixed (*inertial*) frame of reference observing the situation from outside. The basis vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\mathbf{k}}$ are fixed from the point of view of the rotating observer to the inertial observer. We shall adopt the convention of using subscripts I and R to denote quantities in the inertial and rotating frames respectively (N.B. Kibble & Berkshire use a different convention).

Given a set of basis vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\mathbf{k}}$, we can write any vector $\mathbf{A}$ as

$$\mathbf{A} = A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\mathbf{k}}\,. \tag{1.2.2}$$

We want to write down an expression which relates the rates of change of this vector in the two frames. We first note that a scalar quantity cannot depend on the choice of frame and that A_x, A_y and A_z may each be considered as such scalars. Hence

$$\left.\frac{\mathrm{d}A_x}{\mathrm{d}t}\right|_{\mathrm{I}} = \left.\frac{\mathrm{d}A_x}{\mathrm{d}t}\right|_{\mathrm{R}} \qquad \text{etc.} \tag{1.2.3}$$

so that the differences between the vector $\mathbf{A}$ in the 2 frames must be solely related to the difference in the basis vectors. Hence

$$\begin{aligned}
\left.\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right|_{\mathrm{I}} &= \left(\frac{\mathrm{d}A_x}{\mathrm{d}t}\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}A_y}{\mathrm{d}t}\hat{\boldsymbol{\jmath}} + \frac{\mathrm{d}A_z}{\mathrm{d}t}\hat{\mathbf{k}}\right) + \left(A_x\frac{\mathrm{d}\hat{\boldsymbol{\imath}}}{\mathrm{d}t} + A_y\frac{\mathrm{d}\hat{\boldsymbol{\jmath}}}{\mathrm{d}t} + A_z\frac{\mathrm{d}\hat{\mathbf{k}}}{\mathrm{d}t}\right) \\
&= \left(\frac{\mathrm{d}A_x}{\mathrm{d}t}\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}A_y}{\mathrm{d}t}\hat{\boldsymbol{\jmath}} + \frac{\mathrm{d}A_z}{\mathrm{d}t}\hat{\mathbf{k}}\right) + \Big(A_x(\boldsymbol{\omega}\times\hat{\boldsymbol{\imath}}) + A_y(\boldsymbol{\omega}\times\hat{\boldsymbol{\jmath}}) + A_z(\boldsymbol{\omega}\times\hat{\mathbf{k}})\Big)\,.
\end{aligned} \tag{1.2.4}$$

The expression in the first brackets on the right-hand-side is precisely the time derivative measured by the rotating observer, so this is the relation we wanted: It relates the time derivatives measured by inertial and rotating observers. We can write it compactly as in the compact form

$$\left.\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right|_{\mathrm{I}} = \left.\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right|_{\mathrm{R}} + \boldsymbol{\omega}\times\mathbf{A}\,. \tag{1.2.5}$$

1.3 Equation of Motion

Applying Eq. (1.2.5) to the position of the particle $\mathbf{r}$, its velocity may be written as

$$\mathbf{v}_{\mathrm{I}} = \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{I}} = \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{R}} + \boldsymbol{\omega}\times\mathbf{r} = \mathbf{v}_{\mathrm{R}} + \boldsymbol{\omega}\times\mathbf{r}. \tag{1.3.1}$$

Now consider an object subject to a force $\mathbf{F}$. Newton's second law applies in the inertial frame, so we have

$$m\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{I}} = \mathbf{F}\,. \tag{1.3.2}$$

To write this equation in the rotating frame, we differentiate the velocity $\mathbf{v}_{\mathrm{I}}$ using Eq. (1.2.5) again,

$$\begin{aligned}
\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{I}} &= \left.\frac{\mathrm{d}\mathbf{v}_{\mathrm{I}}}{\mathrm{d}t}\right|_{\mathrm{I}} \\
&= \left.\frac{\mathrm{d}\mathbf{v}_{\mathrm{R}}}{\mathrm{d}t}\right|_{\mathrm{I}} + \boldsymbol{\omega}\times\left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{I}} \\
&= \left.\frac{\mathrm{d}\mathbf{v}_{\mathrm{R}}}{\mathrm{d}t}\right|_{\mathrm{R}} + \boldsymbol{\omega}\times\mathbf{v}_{\mathrm{R}} + \boldsymbol{\omega}\times(\mathbf{v}_{\mathrm{R}} + \boldsymbol{\omega}\times\mathbf{r}) \\
&= \left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{R}} + 2\boldsymbol{\omega}\times\left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{R}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\,.
\end{aligned} \tag{1.3.3}$$

Alternatively we can write this very concisely as

$$\begin{aligned}
\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{I}} &= \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{\mathrm{R}} + \boldsymbol{\omega}\times\right)^2 \mathbf{r} \qquad (1.3.4)\\
&= \left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{R}} + 2\boldsymbol{\omega}\times \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{R}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\ .
\end{aligned}$$

We now rearrange (1.3.3) and combine it with (1.3.2) to obtain an equation of motion for the particle in the rotating frame

$$m\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{R}} = \mathbf{F} - 2m\boldsymbol{\omega}\times\mathbf{v}_{\mathrm{R}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\ . \qquad (1.3.5)$$

The second term on the right-hand-side of (1.3.5) is the *Coriolis force* while the final term, pointing away from the axis, is the *centrifugal* force. They are called *fictitious forces*, because they do not represent real physical interactions but appear only because of the choice of the coordinate system. Fictitious forces are always proportional to the mass of the particle, so that the corresponding acceleration is independent of mass. Of course, this is also true for gravity, and in fact general relativity describes gravity as a fictitious force.

1.4 Coriolis Force

Equation (1.3.5) shows that, due to the Coriolis force, an object moving at velocity $\mathbf{v}$ in the rotating frame, experiences apparent acceleration

$$\mathbf{a}_{\mathrm{Cor}} = -2\boldsymbol{\omega}\times\mathbf{v}. \qquad (1.4.1)$$

For example, imagine a car travelling north along Queen's Gate at $50\,\mathrm{km/h}$. To calculate the Coriolis acceleration it experiences, let us choose a set of orthogonal basis vectors that rotate with the Earth, for example $\hat{\boldsymbol{\imath}}$ pointing east, $\hat{\boldsymbol{\jmath}}$ north and $\hat{\mathbf{k}}$ up. With this choice the angular velocity vector has components

$$\boldsymbol{\omega} = (0, \omega\sin\theta, \omega\cos\theta)\ , \qquad (1.4.2)$$

and the car has velocity $\mathbf{v} = v\hat{\boldsymbol{\jmath}}$. The Coriolis acceleration is, therefore,

$$\begin{aligned}
\mathbf{a}_{\mathrm{Cor}} &= -2\boldsymbol{\omega}\times\mathbf{v} = -2\times\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & \omega\sin\theta & \omega\cos\theta \\ 0 & v & 0 \end{vmatrix} = 2\omega v\cos\theta\,\hat{\boldsymbol{\imath}}\\
&\approx 2\times(7.292\times10^{-5}\,\mathrm{s}^{-1})\times\left(\frac{50\times10^{3}\,\mathrm{m}}{3600\,\mathrm{s}}\right)\times\cos(38.5^\circ)\,\hat{\boldsymbol{\imath}}\\
&\approx 1.5\,\mathrm{mm/s}^2\ \text{eastwards}\,, \qquad (1.4.3)
\end{aligned}$$

which is equivalent to a velocity change of $\approx 9\,\mathrm{cm/s}$ after $1\,\mathrm{min}$. In this context it's not a big effect and can safely be ignored. There are other contexts in which it is anything but negligible, however.

Figure 1.2: Gaspard-Gustave de Coriolis (1792–1843) (left), Léon Foucault (1819–68) (centre) and Sir Joseph Larmor (1857–1942) (right)

1.5 Centrifugal Force

According to Eq. (1.3.5), the acceleration due to the centrifugal force is

$$\mathbf{a}_{\rm cf} = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,. \tag{1.5.1}$$

. To calculate this, it is useful to note the general identity for the triple cross product,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}. \tag{1.5.2}$$

Using the same coordinates as in Section 1.4, with $\mathbf{r} = r\hat{\mathbf{k}}$, we find

$$\begin{aligned}\mathbf{a}_{\rm cf} &= (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{r} - (\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega} = \omega^2 r\left(0, -\sin\theta\cos\theta, \sin^2\theta\right) \\ &= \omega^2 r\sin\theta(0, -\cos\theta, \sin\theta).\end{aligned} \tag{1.5.3}$$

The acceleration points away from the rotation axis, and for example in London, it has the strength

$$a_{\rm cf} = \omega^2 r\sin\theta \approx (7.3\times 10^{-5}\,{\rm s}^{-1})^2 \times 6.4\times 10^6\,{\rm m}\times\sin(38.5^\circ) \approx 0.02\,{\rm m/s}^2. \tag{1.5.4}$$

Because this force is independent of velocity, we cannot distinguish it locally from the gravitational force, and therefore it acts essentially as a small correction to it, changing not only the apparent strength but also the direction of the gravitational force. In other situations the centrifugal force can be very important, especially when rotation speeds are high.

It is useful to consider the special case of a particle at rest in the rotating frame, so that the left-hand-side of Eq. (1.3.5) vanishes and $\mathbf{v}_{\rm R} = 0$ as well. In that case, Eq. (1.3.5) implies that there has to be a real physical force

$$\mathbf{F} = m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,. \tag{1.5.5}$$

This is known as the *centripetal force*, and it is the total net force acting on the particle to keep it fixed in the rotating frame. Because it is a real force, it has to be due to some type of physical interaction, such as gravitational or electrostatic force, a support force provided by, e.g., a rope, or usually a combination of several different physical forces.

1.6 Examples

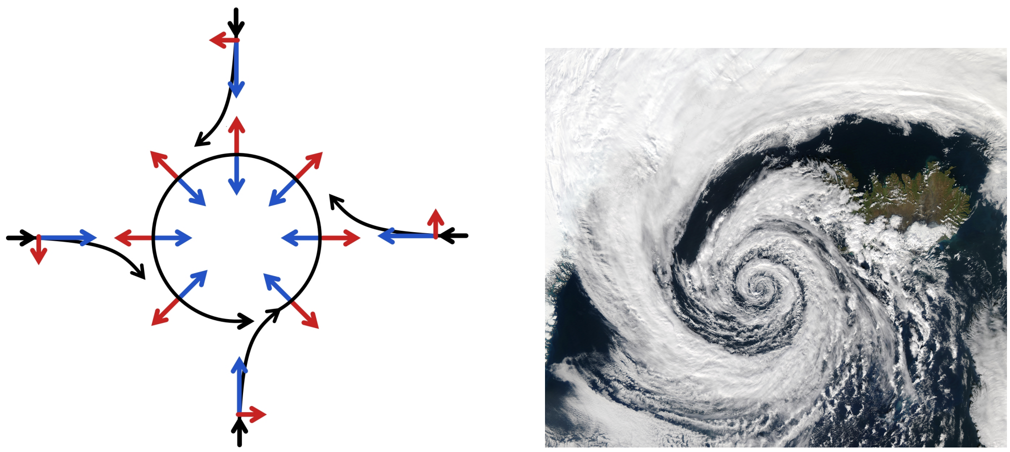

Figure 1.3: Left: schematic representation of flow around a low-pressure area in the Northern hemisphere. The pressure-gradient force is represented by blue arrows, the Coriolis acceleration (always perpendicular to the velocity) by red arrows. Right: this low pressure system over Iceland spins counter-clockwise due to balance between the Coriolis force and the pressure gradient force.

1.6.1 Weather

Probably the most important effect attributed to the Coriolis effect is in meteorology: Winds don't flow from areas of high pressure to those of low pressure but instead tend to flow round the minima and maxima of pressure, giving rise to cyclones and anticyclones respectively. A simple way to understand this is to consider a simple uniform pressure gradient in the presence of a Coriolis force, giving an equation of motion such as

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{1}{\rho}\boldsymbol{\nabla}p - 2\boldsymbol{\omega}\times\mathbf{v}\,. \tag{1.6.1}$$

The general solution of such a problem is complicated. However, if we confine ourselves to 2–dimensions and ignore the component of $\boldsymbol{\omega}$ parallel to the surface, just as we did in our discussion of the Foucault pendulum, we can always find a solution of (1.6.1) with a constant velocity such that the 2 terms on the right cancel. In such a solution $\boldsymbol{\nabla}p$ must be perpendicular to $\mathbf{v}$, as a cross product is always perpendicular to both vectors. Hence (1.6.1) has a solution in which the velocity is perpendicular to the pressure gradient. In such a system the wind always follows the isobars (lines of constant pressure), a pressure minimum is not easily filled and a cyclone (or anticyclone) is stable. This is illustrated in Fig. 1.3a.

1.6.2 Foucault's Pendulum

Consider a pendulum which is free to move in any direction and is sufficiently long and heavy that it will swing freely for several hours. Ignoring the vertical component both of the pendulum's motion and of the Coriolis force, the equations of motion for the bob (in the coordinate system described above) are

$$\begin{aligned} \ddot{x} &= -\frac{g}{\ell}x + 2\omega\cos\theta\dot{y}\,, &(1.6.2)\\ \ddot{y} &= -\frac{g}{\ell}y - 2\omega\cos\theta\dot{x}\,, &(1.6.3)\end{aligned}$$

or, using the complex number trick from Section 1.6.3 with $\tilde{r} = x + \mathrm{i}y$,

$$\frac{\mathrm{d}^2\tilde{r}}{\mathrm{d}t^2} + 2\mathrm{i}\Omega\frac{\mathrm{d}\tilde{r}}{\mathrm{d}t} + \omega_0^2\tilde{r} = 0\,, \tag{1.6.4}$$

where $\Omega = \omega\cos\theta$ and $\omega_0^2 = g/\ell$. Using standard methods for second order differential equations we obtain the general solution

$$\tilde{r} = A\mathrm{e}^{-\mathrm{i}(\Omega-\omega_1)t} + B\mathrm{e}^{-\mathrm{i}(\Omega+\omega_1)t}\,, \tag{1.6.5}$$

where $\omega_1^2 = \omega_0^2 + \Omega^2$. In particular, if the pendulum is released from the origin with velocity $(v_0, 0)$, we have $A = -B = v_0/2\mathrm{i}\omega_1$, so that[2]

$$\tilde{r} = \frac{v_0}{\omega_1}\mathrm{e}^{-\mathrm{i}\Omega t}\sin\omega_1 t, \tag{1.6.6}$$

which means in terms of the original variables x and y,

$$\begin{aligned} x &= \frac{v_0}{\omega_1}\cos\Omega t\sin\omega_1 t\,,\\ y &= -\frac{v_0}{\omega_1}\sin\Omega t\sin\omega_1 t\,. &(1.6.7)\end{aligned}$$

We can also write this in terms of polar co-ordinates, (ρ, ϕ), as

$$\rho = \frac{v_0}{\omega_1}\sin\omega_1 t \qquad \phi = -\Omega t \quad .$$

As $\Omega \ll \omega_0$, the period of oscillation is much less than a day, the result is easy to interpret: the pendulum swings with a basic angular frequency ω_1 ($\approx \omega_0$) but the plane of oscillation rotates with angular frequency Ω. At the pole, $\theta = 0$ the plane of the pendulum apparently rotates once a day. In other words, the plane doesn't rotate at all but the Earth rotates under it once a day. On the other hand, at the equator, $\theta = \frac{\pi}{2}$ and $\Omega = 0$ the plane of the pendulum is stable. In South Kensington

$$\Omega = \omega\cos\theta = (7.292\times10^{-5}\,\mathrm{s}^{-1})\times\cos(38.5^\circ) \Rightarrow T = 30.58\,\mathrm{hr}\,. \tag{1.6.8}$$

Note that this is the time for a complete rotation of the plane of the pendulum through 360°. However, after it has rotated through 180° it would be hard to tell the difference between that and the starting position.

A working Foucault pendulum may be seen in the Science Museum.

[2] N.B. There is an error in the example given in K & B p 117.

1.6.3 Particle in Magnetic Field

The equation of motion for a charged particle in a magnetic field takes the form

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = q\mathbf{v}\times\mathbf{B}, \tag{1.6.9}$$

which we can write as

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\left(\frac{q}{m}\mathbf{B}\right)\times\mathbf{v} = \boldsymbol{\omega}\times\mathbf{v}. \tag{1.6.10}$$

If we identify $\boldsymbol{\omega}$ with $-q\mathbf{B}/m$, we note that this has same form as Eq. (1.2.5) for a velocity which is constant in the rotating frame. Hence we should expect the motion of a particle in a magnetic field to be similar to motion of a free particle in a rotating frame with $\boldsymbol{\omega} = -q\mathbf{B}/m$, namely circular motion with angular frequency ω. In this case this is called the *cyclotron* motion and $\omega = qB/m$ is the cyclotron frequency.

We can check this by solving the equation of motion. We first choose a coordinate system such that $\mathbf{B}$ is in the positive $\hat{\mathbf{k}}$ direction i.e. $\boldsymbol{\omega}$ is in the negative $\hat{\mathbf{k}}$ direction. Then, since $\boldsymbol{\omega}\times\hat{\mathbf{k}} = 0$, the z component of $\mathbf{v}$ is constant, and the x and y components satisfy

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \omega v_y, \quad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\omega v_x. \tag{1.6.11}$$

Define a complex variable $\tilde{v} = v_x + \mathrm{i}v_y$, these can be written as a single equation

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}t} = -\mathrm{i}\omega\tilde{v}, \tag{1.6.12}$$

which is easy to solve,

$$\tilde{v} = \tilde{v}_0\exp(-\mathrm{i}\omega t)\,. \tag{1.6.13}$$

Finally we obtain v_x and v_y by taking the real and imaginary parts of $\tilde{v}$. The solution is circular movement in a clockwise direction around the z axis (aligned with $\mathbf{B}$. If there is no initial component of the velocity in the z direction i.e. $v_z = 0$ then the motion is a circle in a fixed plane. If the initial $v_z \neq 0$ then the motion is a helical path around the z-axis with constant v_z.

We can take this analogy further by generalising Eq. (1.6.10) to the full Lorentz force, including an electric field $\mathbf{E}$ to obtain

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{v}\times\mathbf{B} + q\mathbf{E}\,. \tag{1.6.14}$$

where we immediately see that $\frac{q}{m}\mathbf{E}$ is analogous to the rate of change of the velocity in the rotating frame. We can re-write the equation in the canonical form by defining $\boldsymbol{\omega} = -q\mathbf{B}/m$ as before such that

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \boldsymbol{\omega}\times\mathbf{v} + \frac{\omega}{B}\mathbf{E}\,, \tag{1.6.15}$$

with $\omega = |\boldsymbol{\omega}|$ and $B = |\mathbf{B}|$.

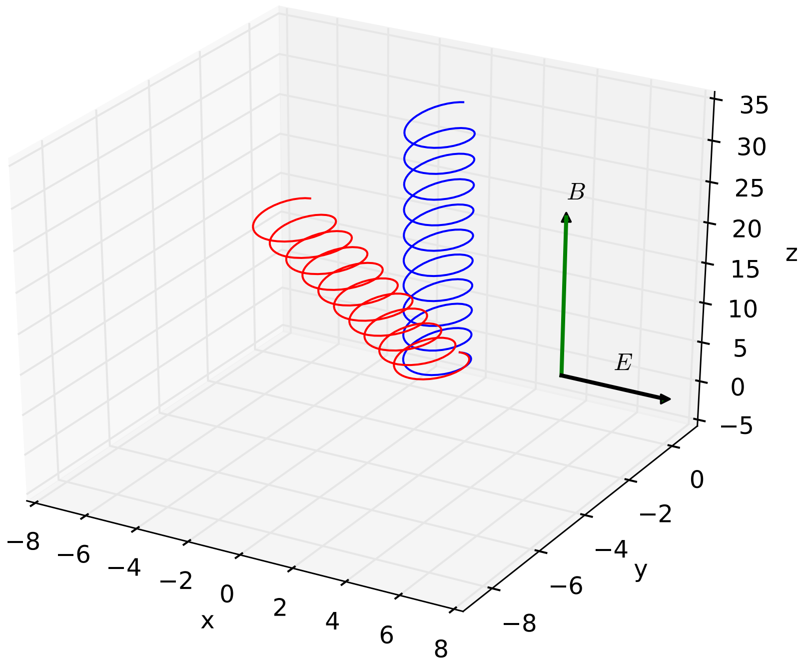

Figure 1.4: Charged particle motion for magnetic field aligned along the z axis. The initial v_z and v_x are non-zero and the particle starts at the origin. The red solution is with for the full lorentz case with and electric field aligned in the x direction. The drift of the helical motion in this case is along the direction perpendicular to both B and E.

If we consider now the simple case in which $\mathbf{E}$ is in the x–direction and $\mathbf{B}$ in the z–direction, we can write the x and y components of Eq. (1.6.15) in the form

$$\begin{aligned} \frac{\mathrm{d}v_x}{\mathrm{d}t} &= \omega v_y + \frac{\omega}{B}E \\ \frac{\mathrm{d}v_y}{\mathrm{d}t} &= -\omega v_x \,. \end{aligned} \tag{1.6.16}$$

Using the simple transformation $v_y' = v_y + (E/B)$, Eq. (1.6.16) becomes

$$\begin{aligned} \frac{\mathrm{d}v_x}{\mathrm{d}t} &= \omega v_y' \\ \frac{\mathrm{d}v_y'}{\mathrm{d}t} &= -\omega v_x \,, \end{aligned} \tag{1.6.17}$$

which is the same as Eq. (1.6.10). Hence the complete solution is a circular motion with an additional drift with speed E/B in the y–direction, perpendicular to both $\mathbf{E}$ and $\mathbf{B}$.

More generally, whenever we find that the time derivative of any vector is of the form (1.2.5), we can immediately tell that it rotates with the corresponding angular velocity $\boldsymbol{\omega}$.

1.6.4 Larmor Effect

It is sometimes useful to consider a rotating frame, not because the system is itself rotating, but because it helps to simplify the mathematics. In this sense it is similar to choosing an appropriate coordinate system.

Consider a particle of charge q moving around a fixed point charge $-q'$ in a uniform magnetic field $\mathbf{B}$. The equation of motion is

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -\frac{k}{r^2}\hat{\mathbf{r}} + q\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{B}\,, \tag{1.6.18}$$

where $k = qq'/4\pi\epsilon_0$.

Rewriting (1.6.18) in a rotating frame, we obtain

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{R}} + 2\boldsymbol{\omega} \times \left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\frac{k}{mr^2}\hat{\mathbf{r}} + \frac{q}{m}\left(\left.\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|_{\mathrm{R}} + \boldsymbol{\omega} \times \mathbf{r}\right) \times \mathbf{B}\,. \tag{1.6.19}$$

If we choose $\boldsymbol{\omega} = -\left(q/2m\right)\mathbf{B}$, the terms in the velocity fall out and we are left with

$$\left.\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right|_{\mathrm{R}} = -\frac{k}{mr^2}\hat{\mathbf{r}} + \left(\frac{q}{2m}\right)^2 \mathbf{B} \times (\mathbf{B} \times \mathbf{r})\ . \tag{1.6.20}$$

In a weak magnetic field we may ignore terms in B^2, such as the last term in (1.6.20), so that we are left with an expression which is identical to that of the system without the magnetic field. What does this mean?

The solution of (1.6.20) is an ellipse (as shown by Newton, Kepler, etc.). Hence, the solution in the inertial frame must be the same ellipse but rotating with the angular frequency $\boldsymbol{\omega} = -\left(q/2m\right)\mathbf{B}$. If this is smaller than the period of the ellipse then the effect is that the major axis of the ellipse slowly rotates. Such a behaviour is known as *precession.* We shall see other examples of this later.

Note that there are some confusing diagrams on the internet and in textbooks which purport to illustrate this precession. There are two special cases to consider: firstly when the magnetic field is perpendicular to the plane of the ellipse. In this case the major axis of the ellipse rotates about the focus, while remaining in the same plane. In the second special case the magnetic field is in the plane of the ellipse. Here the ellipse rotates about an axis through the focus and perpendicular to the major axis. For a circular orbit, the first case the orbit would remain circular, but with slightly different periods clockwise and anti-clockwise, whereas the second would behave like a plate rotating while standing on its edge.

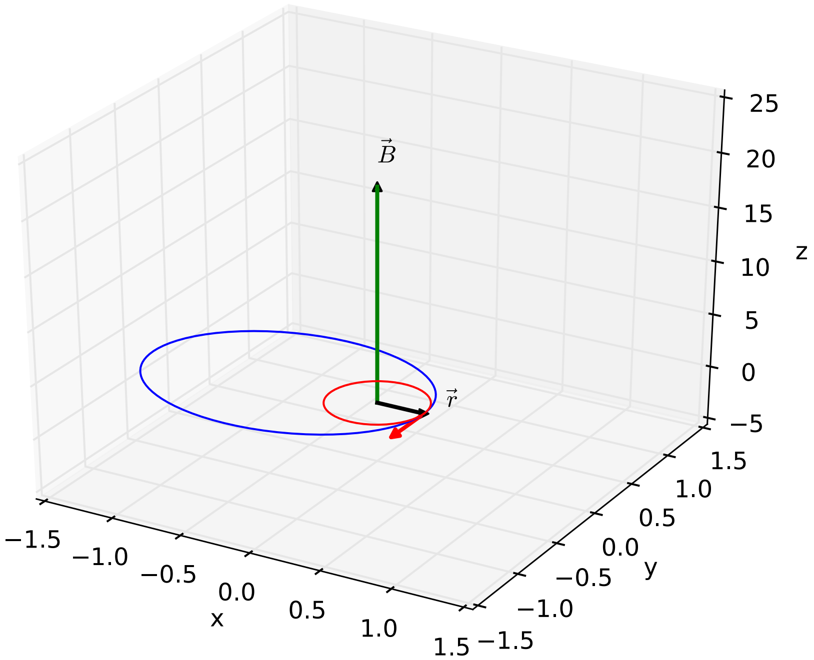

Figure 1.5: The Larmor effect with $\mathbf{B}$ aligned perpendicular to the elliptical orbit in the rotating frame. In the inertial frame the ellipse rotates in the same plane as shown by the red orbit taken by the "perihelion" point at $\mathbf{r}$. The perihelion accelerates in the direction $-\mathbf{B} \times \mathbf{r}$ shown by the red arrow.

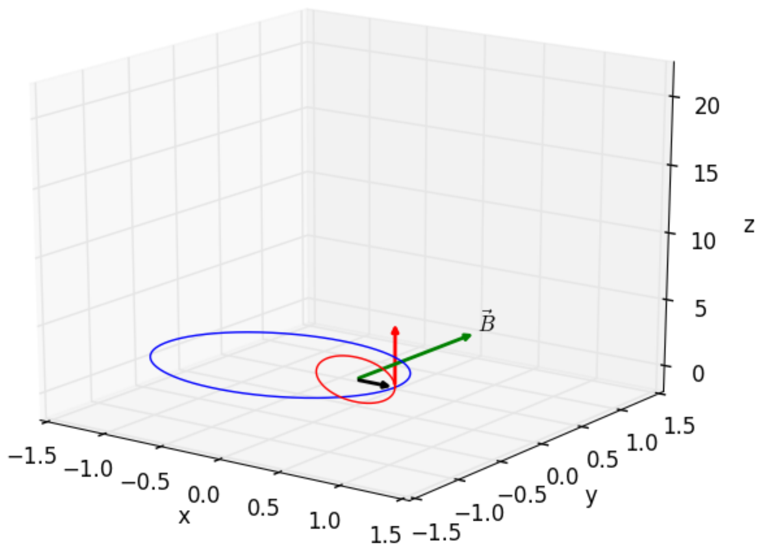

Figure 1.6: Same but for the magnetic field lying in the plane of the elliptical orbit.