Chapter 4

Hamiltonian Mechanics

4.1 Hamilton's Equations[1]

In the Lagrangian formulation of mechanics, the state of the system is descibed by a set of N generalised coordinates q_i and and their time derivatives $\dot{q}_i$. For brevity we shall write it in the form $L(q, \dot{q})$ and let the q and $\dot{q}$ stand for the whole set.

In Eq. (3.2.11) we defined the generalised momentum $p_i(q, \dot{q}) = \partial L/\partial \dot{q}_i$. The basic idea of the Hamiltonian formulation of mechanics is to use coordinates and momenta (q, p) rather than coordinates and velocities $(q, \dot{q})$ to parameterise the system. The space of momenta and coordinates is known as the *phase space*, so the Hamiltonian describes the evolution in phase space. Although the two formulations are mathematically and physically equivalent, the Hamiltonian formulation provides new insight especially when moving to quantum mechanics which is usually described in the Hamiltonian formulation.

To move from velocities to momenta, one has to invert Eq. (3.2.11) to find the velocities in terms of coordinates and momenta,

$$\dot{q}_i = \dot{q}_i(q, p). \tag{4.1.1}$$

Having done this, we define the *Hamiltonian function* H by Legendre transformation

$$H(q, p) = \sum_{i=1}^{N} p_i \dot{q}_i(q, p) - L\left(q, \dot{q}(q, p)\right). \tag{4.1.2}$$

Note that H is a function of coordinates q_i and momenta p_i, whereas the Lagrangian L is a function of coordinates q_i and velocities $\dot{q}_i$. This is analogous to transformations between different thermodynamic potentials in statistical physics. We will see that we can obtain the equations of motion of the system from the Hamiltonian, and therefore it determines the dynamics of the system.

As a simple example, consider a particle in a one-dimensional potential $V(x)$. The Lagrangian is

$$L = \tfrac{1}{2} m\dot{x}^2 - V(x). \tag{4.1.3}$$

[1] Kibble & Berkshire, chapter 12

In this case, the generalised momentum p is just the usual momentum

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \tag{4.1.4}$$

which we can easily invert to find $\dot{x} = p/m$. Therefore the Hamiltonian is

$$H(x,p) = p\dot{x} - L(x,\dot{x}) = \frac{p^2}{m} - \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} + V(x). \tag{4.1.5}$$

This Hamiltonian is, of course, familiar from quantum mechanics.

In order to see how the equations of motion arise from the Hamiltonian, let us calculate its derivatives with respect to momenta and coordinates. Using Eqs. (4.1.2) and (3.2.11), we find

$$\frac{\partial H}{\partial p_j} = \dot{q}_j + \sum_i p_i \frac{\partial \dot{q}_i}{\partial p_j} - \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial p_j} = \dot{q}_j, \tag{4.1.6}$$

and similarly for the derivative with respect to the coordinate q_j,

$$\frac{\partial H}{\partial q_j} = \sum_i p_i \frac{\partial \dot{q}_i}{\partial q_j} - \frac{\partial L}{\partial q_j} - \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial q_j} = -\frac{\partial L}{\partial q_j} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = -\dot{p}_j, \tag{4.1.7}$$

where we used the Euler-Lagrange equation (3.2.1) in the last step. These results are known as *Hamilton's equations*,

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}. \end{aligned} \tag{4.1.8}$$

Together, they determine the evolution of coordinates and momenta, and they are therefore the equations of motion in the Hamiltonian formulation.

We can also calculate the time derivative of the Hamiltonian,

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial t} + \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial H}{\partial p_i}\dot{p}_i \\ &= \frac{\partial H}{\partial t} + \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} \right) = \frac{\partial H}{\partial t}. \end{aligned} \tag{4.1.9}$$

In particular this means that if the Hamiltonian has no explicit time dependence (i.e. $\partial H/\partial t = 0$), then the Hamiltonian is conserved.

To understand better the physical meaning of the Hamiltonian, let us assume that the system is *natural* (see Section 3.5.1), so that the kinetic energy can be written as

$$T = \frac{1}{2}\sum_{ij} a_{ij}\dot{q}_i\dot{q}_j. \tag{4.1.10}$$

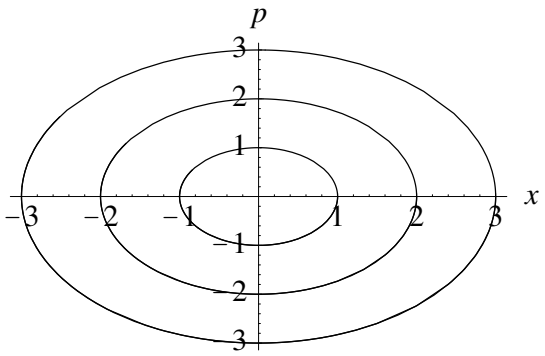

Figure 4.1: Phase space trajectories of the harmonic oscillator.

The Lagrangian is $L = T - V$, and therefore the momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \sum_j a_{ij}\dot{q}_j. \tag{4.1.11}$$

Using this, we can write the Hamiltonian as

$$H = \sum_i p_i\dot{q}_i - L = \sum_{ij} a_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{ij} a_{ij}\dot{q}_i\dot{q}_j + V(q) = \frac{1}{2}\sum_{ij} a_{ij}\dot{q}_i\dot{q}_j + V(q) = T + V. \tag{4.1.12}$$

This demonstrates that the Hamiltonian is nothing but the total energy of the system, and which also explains why it is conserved.

As a simple example, let us consider a particle in a harmonic potential. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2. \tag{4.1.13}$$

Hamilton's equations are

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = \frac{p}{m}, \\ \dot{p} &= -\frac{\partial H}{\partial x} = -kx. \end{aligned} \tag{4.1.14}$$

The evolution of the system can be desribed by a trajectory in phase space (x, p). In the harmonic case, the solutions are ellipses (see Fig. 4.1). Note that because Hamilton's equations are first-order, the phase space trajectories cannot cross, but they can have fixed points at which the derivatives vanish. One can often deduce the qualitative properties of the solutions by simply finding the fixed points and considering the behaviour near them. In the Harmonic case, there is one fixed point at $x = p = 0$.

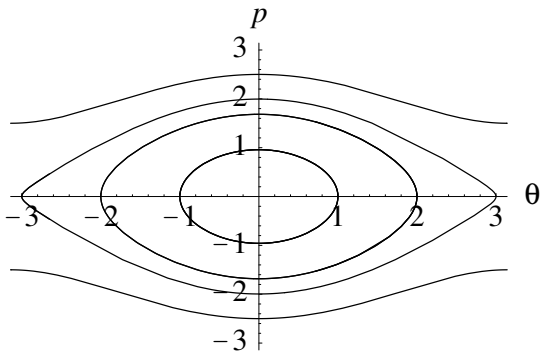

Figure 4.2: Phase space trajectories of a non-linear pendulum.

A less trivial example is a non-linear simple pendulum. The Lagrangian is

$$L = \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta. \tag{4.1.15}$$

The generalised momentum is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \tag{4.1.16}$$

from which we find

$$\dot{\theta} = \frac{p_\theta}{mR^2}. \tag{4.1.17}$$

Therefore the Hamiltonian is

$$H = p_\theta\dot{\theta} - L = \frac{p_\theta^2}{mR^2} - \frac{1}{2}\frac{p_\theta^2}{mR^2} - mgR\cos\theta = \frac{1}{2}\frac{p_\theta^2}{mR^2} - mgR\cos\theta. \tag{4.1.18}$$

Hamilton's equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgR\sin\theta. \tag{4.1.19}$$

These equations have fixed points at $p_\theta = 0$ and $\theta = n\pi$ where n is an arbitrary integer. Even integers correspond to the minimum-energy state in which the pendulum is pointing down, and the odd integers are unstable states in which the pendulum points exactly up. By considering the qualitative motion we can draw a phase space diagram of the trajectories (Fig. 4.2) without having to actually solve the equations.

4.2 Poisson Brackets

Suppose now that we want to study the behaviour of some other quantity which depends on p and q such that its value changes as p and q change under control of the Hamiltonian. Let $F(q, p, t)$

be such a quantity. The *total* time derivative of F is

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}t} &= \frac{\partial F}{\partial t} + \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i\right) \\ &= \frac{\partial F}{\partial t} + \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \\ &= \frac{\partial F}{\partial t} + \{F, H\}\end{aligned} \tag{4.2.1}$$

where the quantity

$$\{F, H\} = \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{4.2.2}$$

is known as a *Poisson bracket.* Compare this with the equation for an operator in quantum mechanics (in the Heisenberg picture)

$$\frac{\mathrm{d}\hat{F}}{\mathrm{d}t} = \frac{\partial \hat{F}}{\partial t} + \frac{1}{\mathrm{i}\hbar}[\hat{F}, \hat{H}]\,, \tag{4.2.3}$$

where $[\hat{F}, \hat{H}] = \hat{F}\hat{H} - \hat{H}\hat{F}$ is the *commutator* of $\hat{F}$ and $\hat{H}$.

Let us note some basic properties of Poisson brackets. They are antisymmetric, $\{F, H\} = -\{H, F\}$. If F has no explicit time–dependence, $\frac{\partial F}{\partial t} = 0$, and the Poisson bracket $\{F, H\}$ is zero, then F is independent of time; F represents a conserved quantity. The Poisson brackets for coordinates and momenta are simply

$$\{q_i, p_j\} = \delta_{ij} \qquad \{q_i, q_j\} = 0 \qquad \{p_i, p_j\} = 0\,, \tag{4.2.4}$$

which resemble the canonical commutation relations in quantum mechanics. We can also write Hamilton's equations in terms of Poisson brackets as

$$\dot{p}_i = \{p_i, H\} \qquad \dot{q}_i = \{q_i, H\}\ . \tag{4.2.5}$$

4.3 Symmetries and Conservation Laws

We have already noted in Sec. 3.2 a relationship between symmetry and conservation laws. If the Lagrangian does not depend on coordinate q_i, then the corresponding momentum p_i is conserved. In terms of the Hamiltonian, we have the corresponding result that if H does not depend on q_i, then

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0, \tag{4.3.1}$$

so p_i is conserved.

So far, this result is only useful if we have chosen the coordinate system to reflect the symmetry. For example, if we have a two-dimensional problem with rotation symmetry, and we express

it in polar coordinates (r, θ), the Hamiltonian does not depend on the polar angle θ, and we find that the corresponding momentum is conserved.

However, if we had chosen to use Cartesian coordinates (x, y), we could not use the above result. In this case, the Hamiltonian would be

$$H = \frac{p_x^2 + p_y^2}{2m} + V\left(\sqrt{x^2 + y^2}\right). \tag{4.3.2}$$

In order to treat a situation like this, we can consider an infinitesimal rotation. When we rotate the system by an infinitesimal angle $\delta\theta$, the coordinates change as

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + \delta x \\ y + \delta y \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} -y\delta\theta \\ x\delta\theta \end{pmatrix}, \tag{4.3.3}$$

and correspondingly the momenta change as

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} \to \begin{pmatrix} p_x + \delta p_x \\ p_y + \delta p_y \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} \delta p_x \\ \delta p_y \end{pmatrix} = \begin{pmatrix} -p_y\delta\theta \\ p_x\delta\theta \end{pmatrix}. \tag{4.3.4}$$

If we find that the Hamiltonian does not change under such a rotation, we say that the Hamiltonian is *invariant*.

More generally, consider a transformation generated by some function $G(q, p)$ under which the coordinates and momenta change as

$$\delta q_i = \frac{\partial G}{\partial p_i}\delta\lambda, \quad \delta p_i = -\frac{\partial G}{\partial q_i}\delta\lambda, \tag{4.3.5}$$

where $\delta\lambda$ is an infinitesimal parameter. For the above rotation, the generator is

$$G = xp_y - yp_x = L_z, \tag{4.3.6}$$

i.e., the z component of the angular momentum.

Under this transformation, the Hamiltonian changes as

$$\begin{aligned} \frac{dH}{d\lambda} &= \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \lambda} + \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \lambda} \right) \\ &= \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial G}{\partial q_i} \right) = \{H, G\}. \end{aligned} \tag{4.3.7}$$

This means that the Hamiltonian is invariant under the transformation if $\{H, G\} = 0$. But if this is the case, then G is conserved because then

$$\frac{dG}{dt} = \{G, H\} = 0. \tag{4.3.8}$$

Therefore, we have found a more general form of Noether's theorem: If the Hamiltonian is invariant under a continuous transformation, then the generator G of the transformation is a conserved charge.

Familiar examples of this are rotations, for which the conserved charge is the angular momentum $\mathbf{L}$ as we saw above and translations, for which the conserved charge is momentum $\mathbf{p}$. To see the latter, consider $G = p_x$. Then we have

$$\delta x = \frac{\partial G}{\partial p_x}\delta\lambda = \delta\lambda, \quad \delta p_x = -\frac{\partial G}{\partial x}\delta\lambda = 0, \tag{4.3.9}$$

so this is, indeed, the generator of translations.

Conservation of energy (or, equivalently, the Hamiltonian H) can also be understood in this way. Choosing $G = H$ as the generator, the coordinates and momenta transform as

$$\begin{aligned} \delta q_i &= \frac{\partial H}{\partial p_i}\delta\lambda = \dot{q}_i\delta\lambda, \\ \delta p_i &= -\frac{\partial H}{\partial q_i}\delta\lambda = \dot{p}_i\delta\lambda. \end{aligned} \tag{4.3.10}$$

This means just time translation $t \to t + \delta\lambda$, so energy conservation is a direct consequence of time translation invariance.

These are deep results because they mean that the conservation of angular momentum, momentum and energy, which we derived earlier using the equations of motion, and actually very general results and are valid whenever the Hamiltonian has the corresponding symmetries, irrespective of it precise form.

4.4 Canonical Transformations

In addition to the coordinate transformations considered in the previous section, the Hamiltonian formulation allows even more freedom in how to parameterise the system. To see this, note that the time evolution of a function $F(q, p; t)$ depends only on the Poisson bracket (4.2.2)

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}. \tag{4.4.1}$$

Therefore, if one finds another set of coordinates Q and momenta P that gives the same Poisson brackets, i.e.,

$$\{f, g\}_{Q,P} = \{f, g\}_{q,p} \tag{4.4.2}$$

for every pair of functions f and g, then they will be equally valid variables to parameterise the system. A transformation

$$\begin{aligned} q &\to Q(q, p), \\ p &\to P(p, q), \end{aligned} \tag{4.4.3}$$

that takes the original coordinates q and momenta p to the new ones satisfying Eq. (4.4.2) is known a *canonical transformation*. In fact, it turns out that it is enough to check that the Poisson brackets (4.2.4) for coordinates and momenta are unchanged under the transformation, i.e.,

$$\{Q_i, Q_j\}_{q,p} = \{P_i, P_j\}_{q,p} = 0, \quad \{Q_i, P_j\}_{q,p} = \delta_{ij}. \tag{4.4.4}$$

Any set of variables that satisfy these conditions are called *canonical conjugates*.

As an example, consider the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 + cpx, \tag{4.4.5}$$

where c is a constant. The last term, which is linear in p, makes this Hamiltonian unusual, but we can use a canonical transformation to write it in a more standard form. To see how to do that, let us first rearrange the terms to write the Hamiltonian as

$$H = \frac{(p + cmx)^2}{2m} + \frac{1}{2}(k - c^2m)x^2. \tag{4.4.6}$$

This suggests a transformation

$$P = p + cmx, \quad Q = x. \tag{4.4.7}$$

It is easy to check that they satisfy the conditions (4.4.4),

$$\{Q, Q\} = \{P, P\} = 0, \quad \{Q, P\} = \frac{\partial Q}{\partial x}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial x} = 1 \times 1 - 0 = 1. \tag{4.4.8}$$

Therefore Q and P and canonical conjugates, and (4.4.7) is a canonical transformation. In terms of Q and P, the Hamiltonian is simply

$$H = \frac{P^2}{2m} + \frac{1}{2}(k - c^2m)Q^2, \tag{4.4.9}$$

which is just a standard harmonic oscillator. This illustrates how one can sometimes use canonical transformations to turn the Hamiltonian into a familiar form, for which the solution is already known. For example, one can always remove a linear term in p by the transformation

$$Q = q, \quad P = p + f(q). \tag{4.4.10}$$

It is important to understand that canonical transformations generally mix coordinates and momenta, and therefore the distinction between them essentially disappears in the Hamiltonian formulation. Because of this, it is interesting to see what a canonical transformation means in the Lagrangian formulation, which uses only coordinates to describe the system. In general, the transformation changes the Lagrangian, so we denote the new Lagrangian by L'. The Hamiltonian is unchanged, so we have

$$H = p\dot{q} - L = P\dot{Q} - L', \tag{4.4.11}$$

from which we find

$$L' = P\dot{Q} - p\dot{q} + L. \tag{4.4.12}$$

Considering, for example, the transformation (4.4.10), this is

$$L' = L + f(q)\dot{q}. \tag{4.4.13}$$

The extra term is just the total time derivative of the integral $F(q) = \int^q f(q')dq'$,

$$L' = L + \frac{dF}{dt}. \tag{4.4.14}$$

Such a total derivative changes the action by a constant, and therefore does not affect the dynamics.