AE341 Modeling and Analysis Lab
Exercise 4

Dynamic Analysis of the A-frame mechanism

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(Dated: October 24, 2021)

In this exercise, we intend to perform a dynamic Analysis of the A-frame structure that is used for the recovery of the Dragon capsule. In the previous exercise, we had performed the kinematic analysis for the structure, and found the poses(position, orientation), velocities and accelerations for all the links of the mechanism. We will use the obtained results to first determine the accelerations of the center of gravity of each of the links, and then, using the equations of motion, determine the actuating torque required for the corresponding motion of the structure. We will also obtain the reaction forces developed at the joints connecting the links.

I. INTRODUCTION

In the preceding exercise, we focused on the kinematics of the A-frame structure, while neglecting the forces that caused the motion of the links.This exercise addresses the problem of determining these forces. This type of analysis, which deals with the determination of forces causing the motion of mechanisms and machines, is termed as kinetics or dynamic analysis. Dynamic analysis is generally of two types - **Forward Dynamics**, and **Inverse Dynamics**. In case of the former, we know the actuating forces and moments, the poses and the velocities(along with the inertia parameters) of all the links in the mechanism, and the purpose is to determine the accelerations of the links, and the reaction forces developed at the joints while in case of the latter, we are aware of the poses, velocities and the accelerations(along with the inertia parameters) of all the links in the mechanism, and our purpose is to determine the actuating forces and torques that cause the corresponding motion of the mechanism, along with the reaction forces at the joints. In this exercise, we will perform the Inverse Dynamics analysis for the A-frame structure, as we already the poses, velocities and accelerations for the links from kinematic analysis performed in exercise 3.

II. THEORY

The first step of Inverse dynamic analysis of a mechanism is to make a free-body diagram for all the movable links of the mechanism, in order to get a clear idea of all the forces and torques acting (both external forces as well as the reaction forces at the joints). The free body diagrams for the links on one side of the structure are clearly depicted in figures 2, 3 and 4. The FBDs would be the same for the links on the other half as well(due to symmetry of the structure and symmetrical distribution of the external forces and actuating torque)

All the joints of the mechanism are revolute joints, which means that there will only be reaction forces between the links and no reaction torques, and that the reaction forces will have unknown x- and y-components.

The next step would involve mentioning all the known parameters of the mechanism, followed by the equations of motion for all the links. Since the mechanism motion is planar only, there will be 3 equations of motion, namely, two force equations (one each for x- and y-directions), and a moment equation (for rotation about the z-axis). The final step involved would be to solve for all the unknowns. We will obtain linear equations such that the number of unknowns matches the number of equations. So, solving for the variables would simply involve writing the equations down in the matrix form and the use of a commonly available commercial package which can easily solve matrix equations.

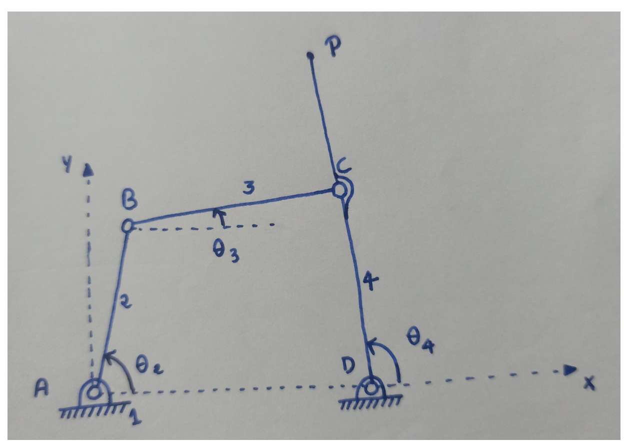

FIG. 1. Schematic representation for the A-frame structure, side view

‘ The masses of the links have been assumed based on their lengths and cross sections($\rho = 7950\ Kg/m^2$).

Following are the formulae used for determining the mass moment of inertia for the movable links.

$$I_2 = (m_2/12) * AB^2$$
$$I_3 = (m_3/12) * BC^2$$
$$I_4 = (m_4/12) * DP^2$$

II.1. Equations of Motion

The equations of motions for links 2, 3 and 4 are as mentioned in this section. The moment equations have been considered about the center of gravity for each of the links, while the accelerations in the force equations are the accelerations corresponding to the center of gravity for each of the links. In total, there are 9 equations, and 9 unknowns. In the appendix A, we have explained how the system of 9 equations has been solved simultaneously for the 9 unknowns.

Link 2

This is the link on which we are considering the actuating torque(τ_a) to act on. Since it is a 1 DoF mechanism, there can only be one actuating force/torque. We have considered the link to be of uniform cross section(0.2 $\times$ 0.2 m^2), with the center of gravity G_2 lying at the midpoint of the axis joining the points A and B. The force balance and moment balance equations for link-2 are written from fig. 2. ‘

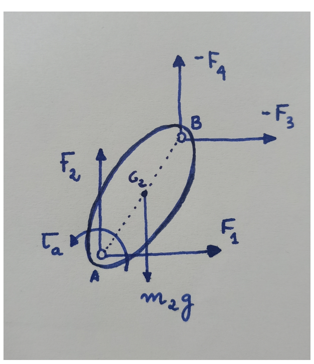

FIG. 2. Free body diagram for the link AB

$$\begin{aligned} m_2\, \ddot{x}_2 &= F_1 - F_3 \\ m_2\, \ddot{y}_2 &= F_2 - F_4 - m_2 g \\ I_2\, \ddot{\theta}_2 &= \tau_A - F_1(y_A - y_2) + F_2(x_A - x_2) \\ &\quad + F_3(y_B - y_2) - F_4(x_B - x_2) \end{aligned}$$

Link 3

These are the equations for link 3. We have considered the link to be of uniform cross section(0.15 $\times$ 0.15 m^2), with the center of gravity G_3 lying at the midpoint of the axis joining the points B and C. The force balance and moment balance equations for link-3 are written from fig. 3.

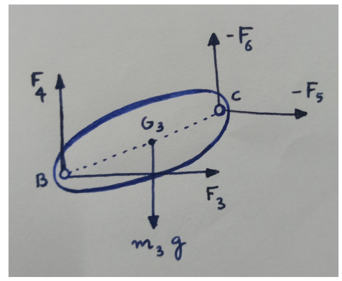

FIG. 3. Free body diagram for the link BC

$$\begin{aligned} m_3\, \ddot{x}_3 &= F_3 - F_5 \\ m_3\, \ddot{y}_3 &= F_6 - F_4 - m_3 g \\ I_3\, \ddot{\theta}_3 &= -F_3(y_B - y_3) + F_4(x_B - x_3) \\ &\quad + F_5(y_C - y_3) - F_6(x_C - x_3) \end{aligned}$$

Link 4

These are the equations for link 4. We have considered the link to be of uniform cross section (0.25 $\times$ 0.25 m^2), with the center of gravity G_4 lying at the midpoint of the axis joining the points P and D. The force balance and moment balance equations for link-4 are written from fig. 4.

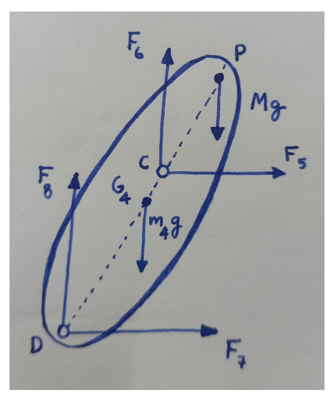

FIG. 4. Free body diagram for the link PCD

$$\begin{aligned} m_4\, \ddot{x}_4 &= F_5 + F_7 \\ m_4\, \ddot{y}_4 &= F_6 + F_8 - m_4 g - T \\ I_4\, \ddot{\theta}_4 &= -F_5(y_C - y_4) + F_6(x_C - x_4) \\ &\quad - F_7(y_D - y_4) + F_8(x_D - x_4) - T(x_P - x_4) \end{aligned}$$

II.2. Auxiliary equations

The system of equations require the position and the accelerations of the centre of gravity of the links, which are to be computed separately using the results of the kinematic analysis. We are considering the origin of the co-ordinate system to be located at the point A, and so, we will try to write the positions of all the COMs(center of masses) with respect to the point A. The corresponding equations are given below:

$$\overrightarrow{\boldsymbol{AG_3}} = \overrightarrow{\boldsymbol{AB}} + \overrightarrow{\boldsymbol{BG_3}}$$
$$\overrightarrow{\boldsymbol{AG_4}} = \overrightarrow{\boldsymbol{AD}} + \overrightarrow{\boldsymbol{DG_4}}$$

Link-2

$$\begin{aligned}
x_2 &= \left(\frac{AB}{2}\right) \cos\theta_2 \\
\ddot{x}_2 &= -\frac{AB}{2}\left(\ddot{\theta}_2 \sin\theta_2 + (\dot{\theta}_2)^2 \cos\theta_2\right) \\
y_2 &= \left(\frac{AB}{2}\right) \sin\theta_2 \\
\ddot{y}_2 &= \frac{AB}{2}\left(\ddot{\theta}_2 \cos\theta_2 - \left(\dot{\theta}_2\right)^2 \sin\theta_2\right)
\end{aligned}$$

Link-3

$$\begin{aligned}
x_3 &= AB \cos\theta_2 + \left(\frac{BC}{2}\right) \cos\theta_3 \\
\ddot{x}_3 &= -AB\left(\ddot{\theta}_2 \sin\theta_2 + \left(\dot{\theta}_2\right)^2 \cos\theta_2\right) \\
&\quad - \frac{BC}{2}\left(\ddot{\theta}_3 \sin\theta_3 + \left(\dot{\theta}_3\right)^2 \cos\theta_3\right) \\
y_3 &= AB \sin\theta_2 + \left(\frac{BC}{2}\right) \sin\theta_3 \\
\ddot{y}_3 &= AB\left(\ddot{\theta}_2 \cos\theta_2 - \left(\dot{\theta}_2\right)^2 \sin\theta_2\right) \\
&\quad + \frac{BC}{2}\left(\ddot{\theta}_3 \cos\theta_3 - \left(\dot{\theta}_3\right)^2 \sin\theta_3\right)
\end{aligned}$$

Link-4

$$\begin{aligned}
x_4 &= DA + \left(\frac{DA}{2}\right) \cos\theta_4 \\
\ddot{x}_4 &= -\frac{DP}{2}\left(\ddot{\theta}_4 \sin\theta_4 + \left(\dot{\theta}_4\right)^2 \cos\theta_4\right) \\
y_4 &= \left(\frac{DP}{2}\right) \sin\theta_4 \\
\ddot{y}_4 &= \frac{DP}{2}\left(\ddot{\theta}_4 \cos\theta_4 - \left(\dot{\theta}_4\right)^2 \sin\theta_4\right)
\end{aligned}$$

Also, the locations of the joints are given by:

$$\begin{aligned}
x_A &= 0, & y_A &= 0 \\
x_B &= AB \cos\theta_2 & y_B &= AB \sin\theta_2 \\
x_C &= DA + CD \cos\theta_4 & y_C &= CD \sin\theta_4 \\
x_D &= DA & y_D &= 0 \\
x_P &= DA + DP \cos\theta_4 & y_P &= DP \sin\theta_4
\end{aligned}$$

III. DATA PRESENTATION

In this section, we have presented the plots representing the variation of the reaction forces and the actuating torque with time. We have considered two cases - constant angular acceleration, and constant angular velocity for the input link AB, and then performed the kinematic and inverse dynamic analysis for both these cases and plotted the forces and actuating torque corresponding to the cases separately. The analysis is done in Julia.(appendix section B subsections 1 and 2)

III.1. Case 1: Constant angular acceleration

We assume θ_2 to be of the form:

$$\theta_2 = \zeta_0 + \beta_0\, t^2$$

where:

$$\zeta_0 = \frac{\pi}{6}, \quad \beta_0 = \frac{\pi}{90 \times 45}$$

Therefore, we get:

$$\dot{\theta}_2 = 2\,\beta_0\, t$$
$$\ddot{\theta}_2 = 2\,\beta_0$$

We use the above equations to perform kinematic analysis and pass on the kinematic relations to the dynamic analysis to solve the system of equations and the plots for the reaction forces and actuating torque as functions of time has been drawn.(in figs. 8,9,10,11,12,13,14,15 and 16)

Following are the plots for the angular positions, velocities and accelerations for each of the three movable links are given below, corresponding to the input mentioned above, followed by the plots for the forces and torque.

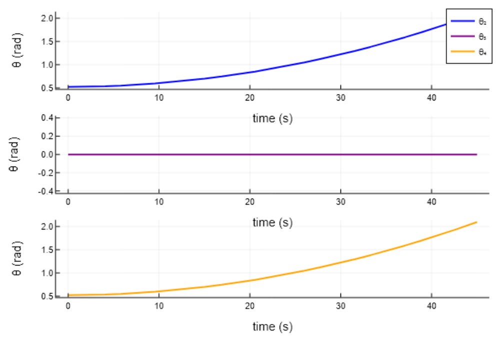

FIG. 5. Plot for θ vs time for the three links - Case 1

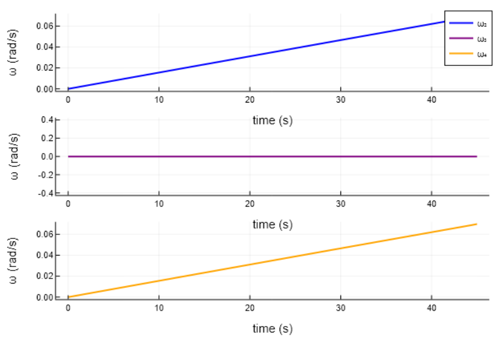

FIG. 6. Plot for $\dot{\theta}$ vs time for the three links - Case 1

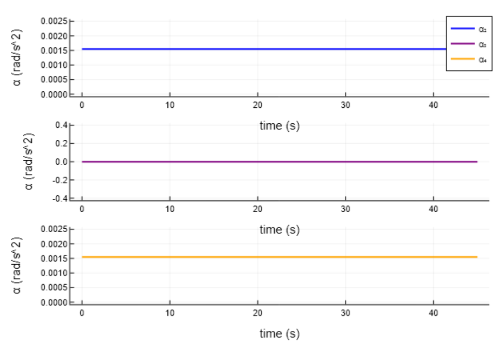

FIG. 7. Plot for $\ddot{\theta}$ vs time for the three links - Case 1

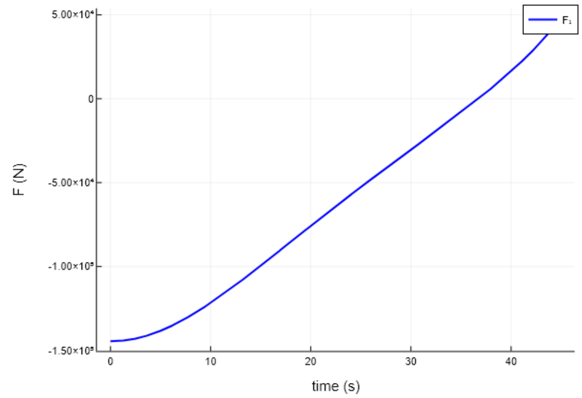

FIG. 8. Plot for F_1 vs time - Case 1

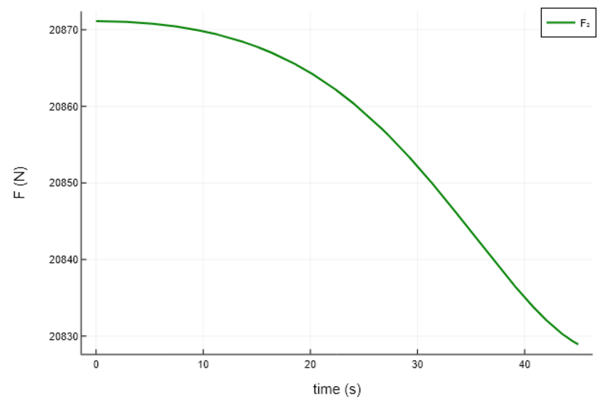

FIG. 9. Plot for F_2 vs time - Case 1

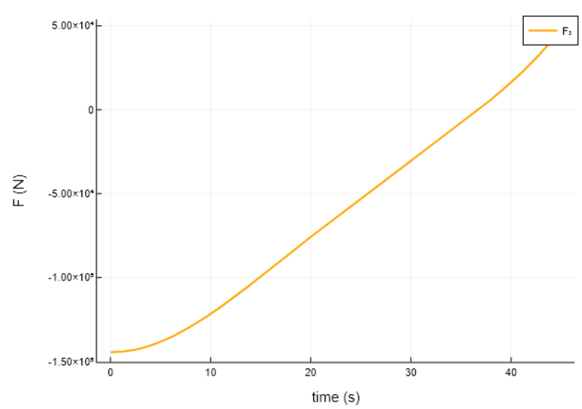

FIG. 10. Plot for F_3 vs time - Case 1

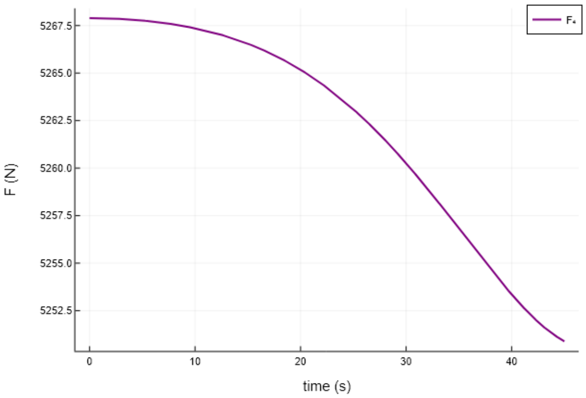

FIG. 11. Plot for F_4 vs time - Case 1

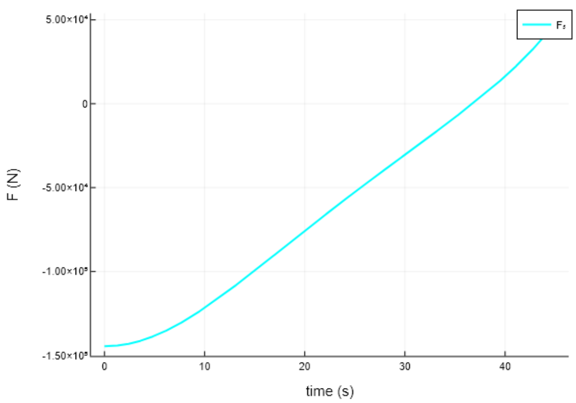

FIG. 12. Plot for F_5 vs time - Case 1

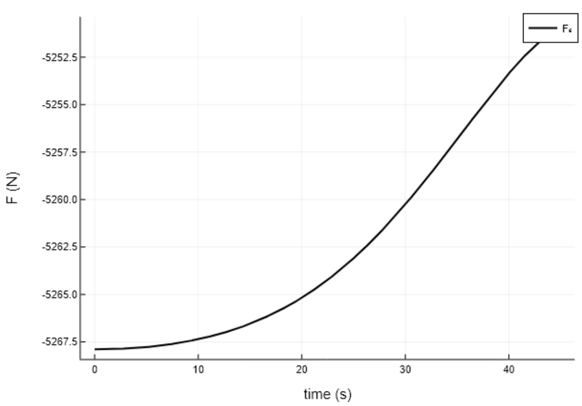

FIG. 13. Plot for F_6 vs time - Case 1

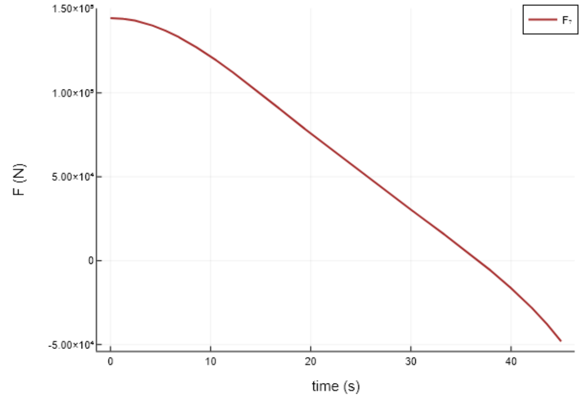

FIG. 14. Plot for F_7 vs time - Case 1

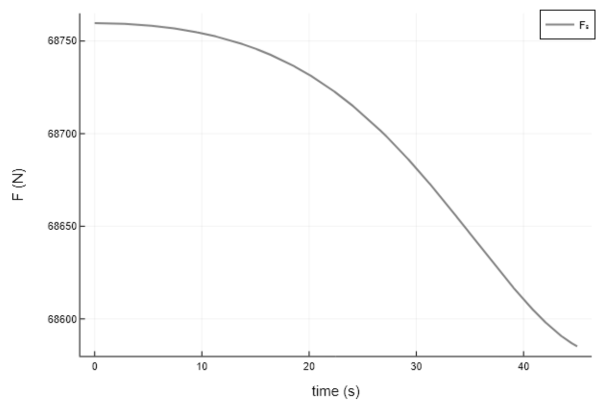

FIG. 15. Plot for F_8 vs time - Case 1

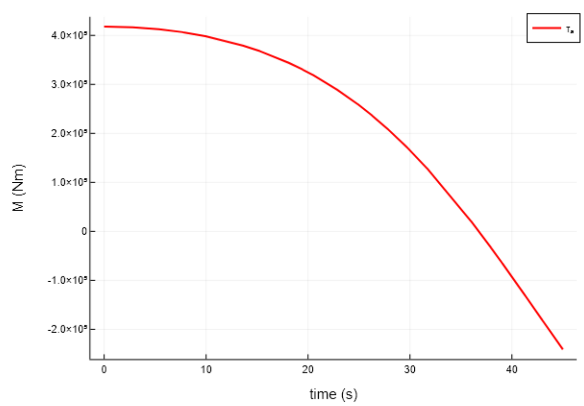

FIG. 16. Plot for τ_A vs time - Case 1

III.2. Case 2: Constant angular velocity

We assume θ_2 to be of the form:

$$\theta_2 = \zeta_0 + \beta_0\, t$$

where:

$$\zeta_0 = \frac{\pi}{6}, \quad \beta_0 = \frac{\pi}{90}$$

Therefore, we get:

$$\dot{\theta}_2 = \beta_0$$
$$\ddot{\theta}_2 = 0$$

We use the above equations to perform kinematic analysis and pass on the kinematic relations to the dynamic analysis to solve the system of equations and the plots for the reaction forces and actuating torque as functions of time has been drawn.(in figs. 20,21,22,23,24,25,26,27 and 28) Following are the plots for the angular positions, velocities and accelerations for each of the three movable links are given below, corresponding to the input mentioned above, followed by the plots for the forces and torque.

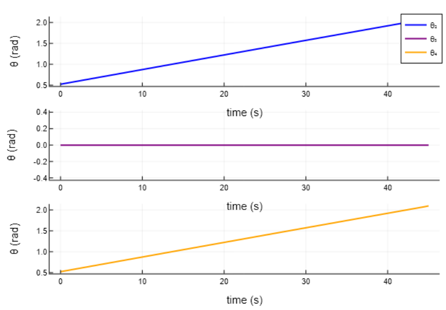

FIG. 17. Plot for θ vs time for the three links - Case 2

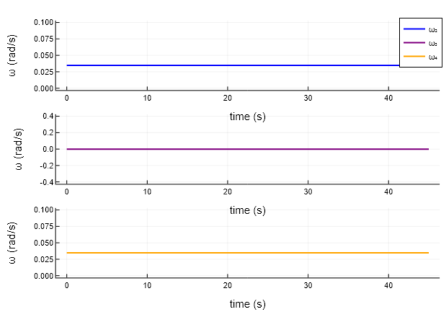

FIG. 18. Plot for $\dot{\theta}$ vs time for the three links - Case 2

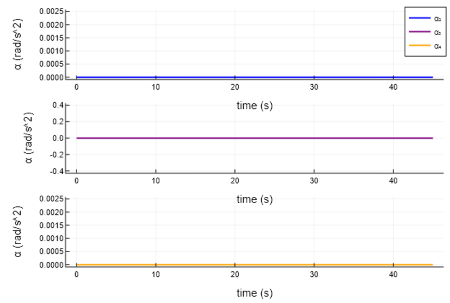

FIG. 19. Plot for $\ddot{\theta}$ vs time for the three links - Case 2

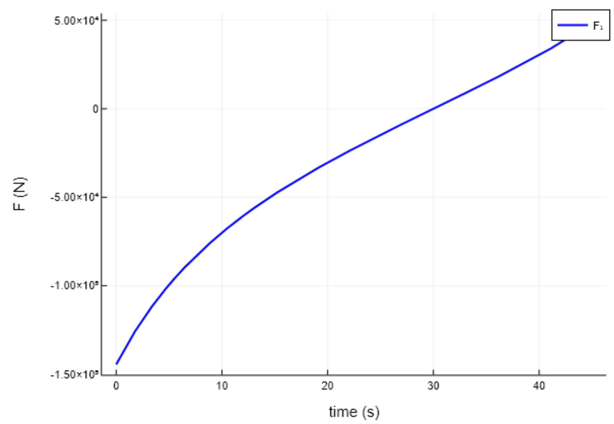

FIG. 20. Plot for F_1 vs time - Case 2

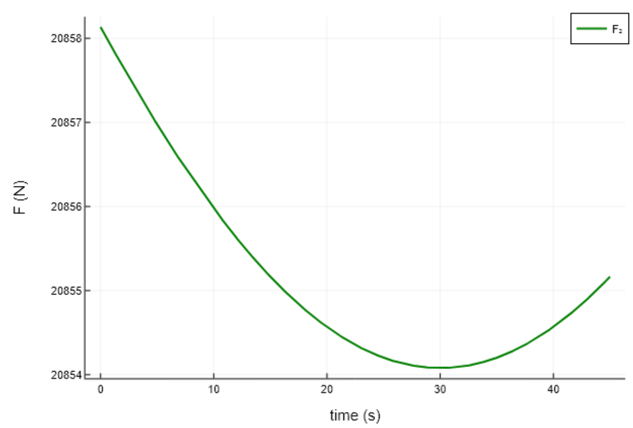

FIG. 21. Plot for F_2 vs time - Case 2

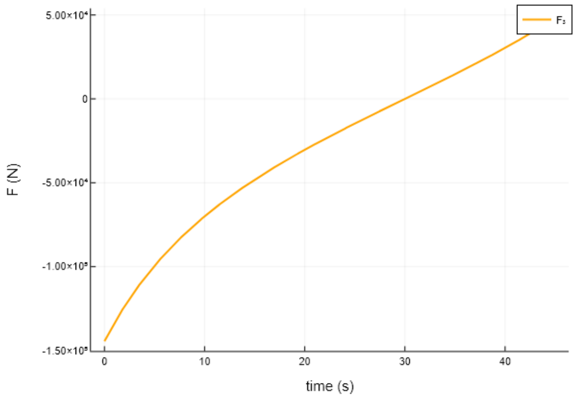

FIG. 22. Plot for F_3 vs time - Case 2

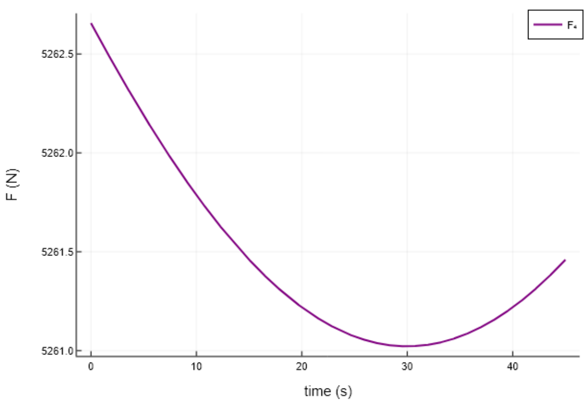

FIG. 23. Plot for F_4 vs time - Case 2

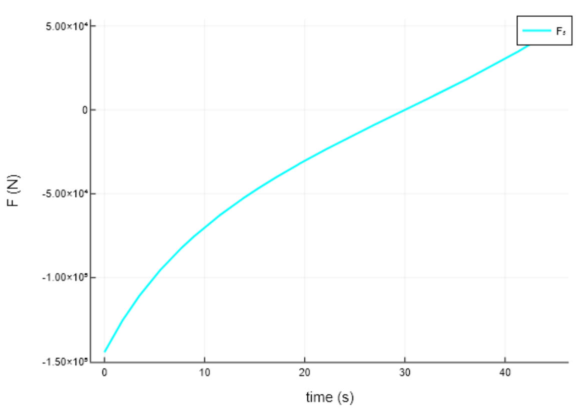

FIG. 24. Plot for F_5 vs time - Case 2

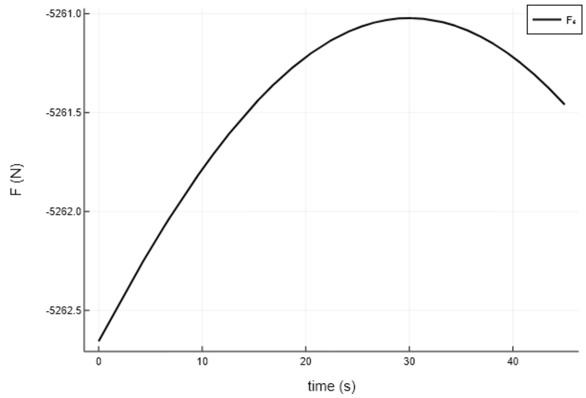

FIG. 25. Plot for F_6 vs time - Case 2

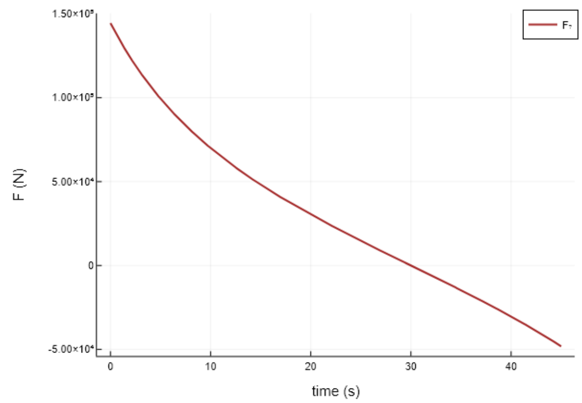

FIG. 26. Plot for F_7 vs time - Case 2

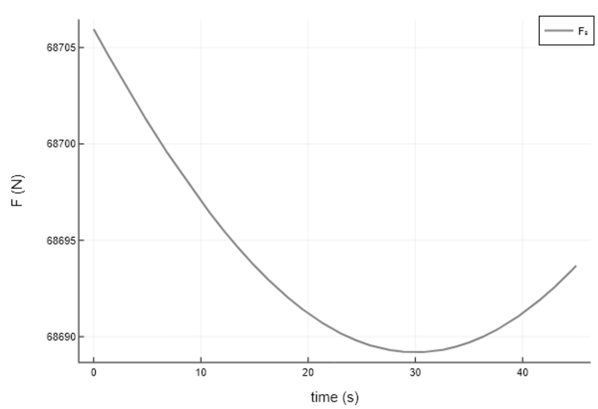

FIG. 27. Plot for F_8 vs time - Case 2

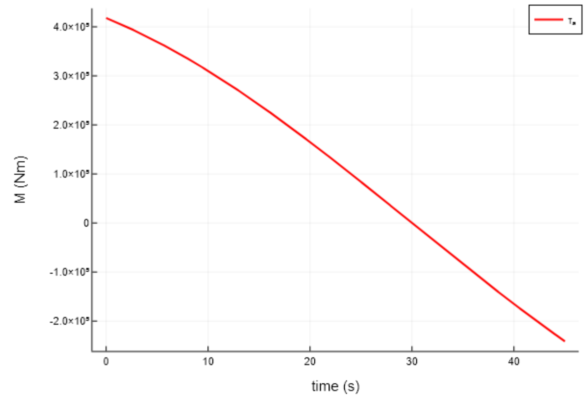

FIG. 28. Plot for $F\tau_A$ vs time - Case 2

IV. RESULTS AND DISCUSSIONS

The obtained plots displayed in the preceding section clearly present the variation of the reaction forces, along with the required actuating torque as the Dragon capsule is lifted out of the ocean and placed onto the Dragon's nest on the GO Navigator's deck. We are going to discuss the obtained plots for both the cases considered.

IV.1. Case 1 - Constant angular acceleration

1. In this case, we observe that the maximum actuating torque is required at t=0, since it is at that instant that the torque due to the gravitational forces is the highest as the component of the radial distance from the hinges A and D perpendicular to the gravitational forces is the highest. As the angles θ_2 and θ_4 increase, the perpendicular distance decreases.

2. As mentioned in the previous point, the distance from the hinge perpendicular to the gravitational forces keeps on decreasing as the two angles θ_2 and θ_4 increase, which means that the required actuating torque also keeps on decreasing monotonically.

3. When the angles θ_2 and θ_4 cross $\pi/2$, the direction of torque due to the gravitational forces reverses and becomes anticlockwise, which means that the actuating torque (that was previously anticlockwise in direction) will now change direction and act in the clockwise sense. At $\theta_2 = \pi/2$, the actuating torque is barely positive (while the horizontal reaction forces are also negligible) in order to provide just the required constant angular acceleration (the torques due to the gravitational forces are zero). The actuating torque and the horizontal reaction forces are zero (at t=36.755 seconds) just beyond the instant at which $\theta_2 = \pi/2$, at which instant the required constant acceleration is provided completely by the anticlockwise torque caused by the gravitational forces.

IV.2. Case 2 - Constant angular velocity

1. Similar to the previous case, the maximum actuating torque is required at t=0 (because torque due to gravity is maximum then and it has to be balanced by the actuating torque.)

2. At t=30 s, θ_2 reaches 90^o and at that instant, actuating torque required becomes zero as torques due to gravity on the links become zero and the angular acceleration is zero.

V. CONCLUSION

In this exercise, we performed the dynamic inverse analysis for four bar mechanism that enables the A-frame crane to lift the Dragon crew module from the ocean for two cases-constant angular acceleration and constant angular velocity. We obtain the actuating torque profiles for both the specified cases based on which the hydraulic actuator can be designed.

APPENDIX

Appendix A: matrix form of the equations

The equations used in the inverse dynamic analysis are represented in a matrix form as:

$$AX = B$$

Where:

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ (y_A - y_2) & (x_A - x_2) & (y_B - y_2) & -(x_B - x_2) & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -(y_B - y_3) & (x_B - x_3) & (y_C - y_3) & -(x_C - x_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -(y_C - y_4) & (x_C - x_4) & -(y_D - y_4) & (x_D - x_4) & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ \tau_A \end{bmatrix}$$

$$B = \begin{bmatrix} m_2\ \ddot{x}_2 \\ m_2\ \ddot{y}_2 + m_2\ g \\ I_2\ \ddot{\theta}_2 \\ m_3\ \ddot{x}_3\ m_3\ \ddot{y}_3 + m_3\ g \\ I_3\ \ddot{\theta}_3 \\ m_4\ \ddot{x}_4 \\ m_4\ \ddot{y}_4 + m_4\ g + T \\ I_4\ \ddot{\theta}_4 + T(x_p - x_4) \end{bmatrix}$$

The coefficient matrix is inverted and multiplied with the constant vector to compute the solution vector at each time step.

$$A^I = inv(A)$$
$$X = BA^I$$

Appendix B: Scripts used for modelling

1. Julia code for performing kinematic analysis of four-bar mechanism

```
"""
Julia script for kinematics analysis of four-bar mechanism.
Contains sections for position, velocity, and acceleration analyses.
"""

using Plots
plotlyjs()

function ex3()

# fixed parameters
AB=5
BC=6
CD=5
DA=6
DP=10
θ₁=π

#time limit
t=0:0.01:45

# position analysis

#independent coordinate
ζ₀=π/6 # angle subtended by the A-frame [rad]
#case 1
# β₀=dζ=90*π/(180*45*45)
# θ₂=ζ=ζ₀.+β₀.*t.^2
#case 2
β₀=dζ=90*π/(180*45)
θ₂=ζ=ζ₀.+β₀.*t

x=-AB.*cos.(θ₂).-DA.*cos.(θ₁)
y=-AB.*sin.(θ₂).-DA.*sin.(θ₁)

z=(x.^2 .+ y.^2 .+ CD^2 .- BC^2)./(2 .* CD)

θ₄=acos.(-z./sqrt.(x.^2 .+ y.^2)) .+ atan.(y,x)

y_C=CD.*sin.(θ₄)
y_B=AB.*sin.(θ₂)

x_C=CD.*cos.(θ₄).+DA
x_B=AB.*cos.(θ₂)

θ₃=atan.(y_C .- y_B,x_C .- x_B)

#plotting
p1=plot(t,θ₂,linewidth=2,xaxis="time (s)",yaxis="θ (rad)",
    label="θ₂",color="blue")
p2=plot(t,θ₃,linewidth=2,xaxis="time (s)",yaxis="θ (rad)",
    label="θ₃",color="purple",ylim=[-0.4,0.4])
p3=plot(t,θ₄,linewidth=2,xaxis="time (s)",yaxis="θ (rad)",
    label="θ₄",color="orange")
display(plot(p1,p2,p3,layout=(3,1)))
savefig("./plots/ex3-pb1.png")

#trajectory of point of suspension
xₚ=DA .+ DP.*cos.(θ₄)
yₚ=DP.*sin.(θ₄)
display(plot(xₚ,yₚ,label="trajectory",xaxis="x (m)",yaxis="y (m)",
    linewidth=2,xlim=[0,15],xticks=0:1:15,ylim=[0,15],yticks=0:1:15))
savefig("./plots/ex3-trajectory.png")


#velocity analysis

#independent coordinate
# case 1
# θ̇₂=dθ₂=2 .*β₀.*t
# case 2
θ̇₂=dθ₂=β₀.*ones(length(t))

X=ones(2,length(t))

```

```
#solving the equations
for i in 1:length(t)
    A=[BC.*sin.(θ₃[i])  -CD.*sin.(θ₄[i]); BC.*cos.(θ₃[i])  -CD.*cos.(θ₄[i])]
    # X=[θ̇₃;θ̇₄]
    B=[-AB.*θ̇₂[i].*sin.(θ₂[i]); -AB.*θ̇₂[i].*cos.(θ₂[i])]

    Aᴵ=inv(A)
    @assert size(Aᴵ,2)==size(B,1) "dimension mismatch"
    X[:,i]=Aᴵ*B
end

(θ̇₃,θ̇₄)=[X[i,:] for i in 1:size(X,1)]

#plotting
p4=plot(t,θ̇₂,linewidth=2,xaxis="time (s)",yaxis="ω (rad/s)",
    label="ω₂",color="blue",ylim=[0,0.1])
p5=plot(t,θ̇₃,linewidth=2,xaxis="time (s)",yaxis="ω (rad/s)",
    label="ω₃",color="purple",ylim=[-0.4,0.4])
p6=plot(t,θ̇₄,linewidth=2,xaxis="time (s)",yaxis="ω (rad/s)",
    label="ω₄",color="orange",ylim=[0,0.1])
display(plot(p4,p5,p6,layout=(3,1)))
savefig("./plots/ex3-pb2.png")


#acceleration analysis

#independent coordinate
#case 1
# θ̈₂=dθ̇₂=2*β₀.*ones(length(t))
#case 2
θ̈₂=dθ̇₂=zeros(length(t))

Y=ones(2,length(t))

#solving the equations
for i in 1:length(t)
    C=[BC*cos(θ₃[i])  -CD*cos(θ₄[i]); BC*sin.(θ₃[i])  -CD*sin.(θ₄[i])]
    # Y=[θ̈₃;θ̈₄]
    D=[(-AB*θ̈₂[i]*cos.(θ₂[i])+AB*(θ̇₂[i]^2)*sin(θ₂[i])+BC*(θ̇₃[i]^2)*sin.(θ₃[i])
            -CD*(θ̇₄[i]^2)*sin(θ₄[i]));
        (-AB*θ̈₂[i]*sin(θ₂[i])-AB*(θ̇₂[i]^2)*cos(θ₂[i])-BC*(θ̇₃[i]^2)*cos.(θ₃[i])
            +CD*(θ̇₄[i]^2)*cos(θ₄[i]))]

            Cᴵ=inv(C)
    @assert size(Cᴵ,2)==size(D,1) "dimension mismatch"
    Y[:,i]=Cᴵ*D
end

(θ̈₃,θ̈₄)=[Y[i,:] for i in 1:size(Y,1)]

#plotting
p7=plot(t,θ̈₂,linewidth=2,xaxis="time (s)",yaxis="α (rad/s^2)",
    label="α₂",color="blue",ylim=[0,0.0025])
p8=plot(t,θ̈₃,linewidth=2,xaxis="time (s)",yaxis="α (rad/s^2)",
    label="α₃",color="purple",ylim=[-0.4,0.4])
p9=plot(t,θ̈₄,linewidth=2,xaxis="time (s)",yaxis="α (rad/s^2)",
    label="α₄",color="orange",ylim=[0,0.0025])
display(plot(p7,p8,p9,layout=(3,1)))
savefig("./plots/ex3-pb3.png")

K=(θ₂,θ₃,θ₄,θ̇₂,θ̇₃,θ̇₄,θ̈₂,θ̈₃,θ̈₄)

return K

end
```

2. Julia code for performing dynamic analysis of four-bar mechanism

```
"""
Julia script for dynamic analysis of four-bar mechanism.
Includes the kinematic analysis file and computes the reaction
forces and actuating torque required as a function of time.
"""

using Plots
plotlyjs()

include("ex3.jl") #file for kinematic analysis

K=ex3() #kinematic analysis

#extracting the kinematic state vectors
(θ₂,θ₃,θ₄,θ̇₂,θ̇₃,θ̇₄,θ̈₂,θ̈₃,θ̈₄)=K

#fixed parameters
#lengths and angles
AB=5
BC=6
CD=5
DA=6
DP=10
θ₁=π

#masses
m=3000
m₂=AB*0.2*0.2*7950
m₃=BC*0.15*0.15*7950
m₄=DP*0.25*0.25*7950
g=9.81

#tension force due to the capsule
T=m*g/2

#moment of inertia
I₂=(m₂/12)*AB^2
I₃=(m₃/12)*BC^2
I₄=(m₄/12)*DP^2

#time limit
t=0:0.01:45

#poses and accelerations of centres of gravity of the links
#link 2
x_2=(AB/2).*cos.(θ₂)
ẍ₂=-(AB/2).*θ̈₂.*sin.(θ₂).-(AB/2).*(θ̇₂.^2).*cos.(θ₂)
y_2=(AB/2).*sin.(θ₂)
ÿ₂=(AB/2).*(θ̈₂.*cos.(θ₂).-(θ̇₂.^2).*sin.(θ₂))

#link 3
x_3=AB.*cos.(θ₂).+(BC/2)*cos.(θ₃)
ẍ₃=-AB.*(θ̈₂.*sin.(θ₂).+(θ̇₂.^2).*cos.(θ₂)) .- (BC/2).*(θ̈₃.*sin.(θ₃).+(θ̇₃.^2).*cos.(θ₃))
y_3=AB.*sin.(θ₂).+(BC/2)*sin.(θ₃)
ÿ₃=AB.*(θ̈₂.*cos.(θ₂).-(θ̇₂.^2).*sin.(θ₂)) .+ (BC/2).*(θ̈₃.*cos.(θ₃).-(θ̇₃.^2).*sin.(θ₃))

#link 4
x_4=DA.+(DP/2).*cos.(θ₄)
ẍ₄=-(DP/2).*(θ̈₄.*sin.(θ₄).+(θ̇₄.^2).*cos.(θ₄))
y_4=(DP/2).*sin.(θ₄)
ÿ₄=(DP/2).*(θ̈₄.*cos.(θ₄).-(θ̇₄.^2).*sin.(θ₄))

#locations of the joints
x_A=zeros(size(x_2))
y_A=zeros(size(y_2))
x_B=AB.*cos.(θ₂)
y_B=AB.*sin.(θ₂)
x_C=DA.+CD.*cos.(θ₄)
y_C=CD.*sin.(θ₄)
x_D=DA.*ones(size(x_4))
y_D=y_A=zeros(size(y_4))
x_P=DA.+DP.*cos.(θ₄)
y_P=DP.*sin.(θ₄)

#solution matrix
X=zeros(9,length(t))
```

```

#solving the equations
for i in 1:length(t)
    R1=[1 0 -1 0 0 0 0 0 0]
    R2=[0 1 0 -1 0 0 0 0 0]
    R3=[-(y_A[i]-y_2[i]) (x_A[i]-x_2[i]) (y_B[i]-y_2[i]) -(x_B[i]-x_2[i]) 0 0 0
     0 1]
    R4=[0 0 1 0 -1 0 0 0 0]
    R5=[0 0 0 1 0 -1 0 0 0]
    R6=[0 0 -(y_B[i]-y_3[i]) (x_B[i]-x_3[i]) (y_C[i]-y_3[i]) -(x_C[i]-x_3[i]) 0
     0 0]
    R7=[0 0 0 0 1 0 1 0 0]
    R8=[0 0 0 0 0 1 0 1 0]
    R9=[0 0 0 0 -(y_C[i]-y_4[i]) (x_C[i]-x_4[i]) -(y_D[i]-y_4[i]) (x_D[i]-x_4[i]
     ) 0]
    A=vcat(R1,R2,R3,R4,R5,R6,R7,R8,R9)
    B=[m2*ẍ2[i],(m2*ÿ2[i]+m2*g),I2*θ̈2[i],m3*ẍ3[i],(m3*ÿ3[i]+m3*g),I3*θ̈3[i],m4*ẍ4
     [i],(m4*ÿ4[i]+m4*g+T),I4*θ̈4[i]+T*(x_P[i]-x_4[i])]

    AI=inv(A)
    @assert size(AI,2)==size(B,1) "dimension mismatch"
    X[:,i]=AI*B
end

#extracting the solution vectors
(F1,F2,F3,F4,F5,F6,F7,F8,τa)=[X[i,:] for i in 1:size(X,1)]

#plotting
display(plot(t,F1,linewidth=2,xaxis="time (s)",yaxis="F (N)",label="F1",color="
      blue"))
savefig("./plots/ex4-41.png")
display(plot(t,F2,linewidth=2,xaxis="time (s)",yaxis="F (N)",label="F2",color="
      green"))
savefig("./plots/ex4-42.png")
display(plot(t,F3,linewidth=2,xaxis="time (s)",yaxis="F (N)",label="F3",color="
      orange"))
savefig("./plots/ex4-43.png")
display(plot(t,F4,linewidth=2,xaxis="time (s)",yaxis="F (N)",label="F4",color="
      purple"))
savefig("./plots/ex4-44.png")
display(plot(t,F5,linewidth=2,xaxis="time (s)",yaxis="F (N)",label="F5",color="
      cyan"))
savefig("./plots/ex4-45.png")
display(plot(t,F6,linewidth=2,xaxis="time (s)",yaxis="F (N)",label="F6",color="
      black"))
savefig("./plots/ex4-46.png")
display(plot(t,F7,linewidth=2,xaxis="time (s)",yaxis="F (N)",label="F7",color="
      brown"))
savefig("./plots/ex4-47.png")
display(plot(t,F8,linewidth=2,xaxis="time (s)",yaxis="F (N)",label="F8",color="
      grey"))
savefig("./plots/ex4-48.png")
display(plot(t,τa,linewidth=2,xaxis="time (s)",yaxis="M (Nm)",label="τa",color="
      red"))
savefig("./plots/ex4-49.png")
```