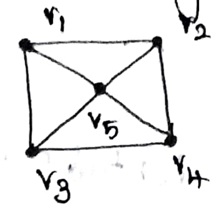
2/7/19

Graph :-

A Graph $G = (V, E)$ where V is a finite non Empty set of vertices and E is a finite set of Edges where the Edges connects the vertices.

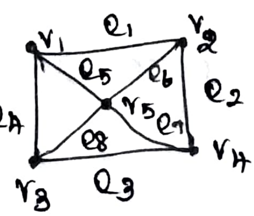
Eg:- G:



$V = \{ v_1, v_2, v_3, v_4, v_5 \}$

$E = \{ (v_1, v_2)\ (v_1, v_5)\ (v_1, v_3)\ (v_2, v_4)\ (v_2, v_5), (v_3, v_4), (v_3, v_5), (v_4, v_5) \}$

G :



$V = \{ v_1, v_2, v_3, v_4, v_5 \}$

$E = \{ e_1, e_2, e_3\ e_4, e_5, e_6, e_7, e_8 \}$

Adjacent vertices and Adjacent Edges :-

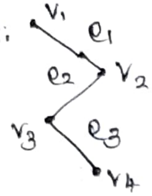
Adjacent vertices :-

An Edge connects two vertices and the end points of an edge are called adjacent vertices.

Adjacent Edges :-

If two or More edges meet at a common vertex those edges are called adjacent Edges.

Eg:- G:



Adjacent vertices $= \{(v_1, v_2)\ (v_2, v_3)\ (v_3, v_4)\}$

Adjacent Edges $= \{(e_1, e_2)\ (e_2, e_3)\}$

Isolated vertex and Pendant vertex:-

Isolated vertex:-

A vertex is called an isolated vertex if no edge is incident on it (or)

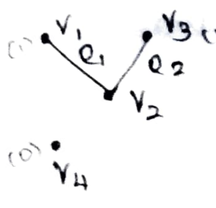
A vertex with degree "zero" is called isolated vertex.

Pendant vertex:-

A vertex is called a Pendant vertex if only one edge is incident on it (or)

A vertex with degree "one" is called Pendant vertex.

Eg:- G:



(0) Isolated vertex :- v_4

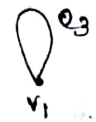
(1) Pendant vertex :- v_1, v_3

Loop and parallel Edges:-

Loop:-

A loop is an edge whose End vertices are the same.

Eg:-

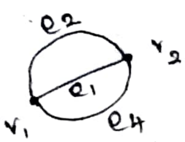


$e_3 : (v_1, v_1)$

parallel Edges:- / Multiple Edges:-

If two end vertices are connected by more than one Edge, those edges are called parallel (or) Multiple Edges.

Eg:-



$e_1 : (v_1, v_2)$

$e_2 : (v_1, v_2)$

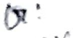
$e_4 : (v_1, v_2)$

e_1, e_2, e_4 are parallel Edges.

Trivial Graph:-

A Graph G is said to be trivial if it has one vertex and no Edges.

Eg:- G:



$V = \{v_1\}$

$E = \{\phi\}$

Null graph :-

* A graph G is said to be a null graph if it has a finite no. of vertices with no Edges.

Eg:- G:

$V = \{v_1, v_2, v_3, v_4\}$

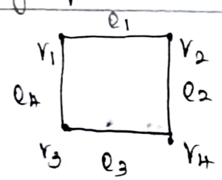
$E = \phi$

* A Null graph is a finite collection of isolated vertices.

Simple graph :-

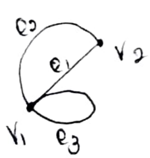
A graph G is said to be simple if it has no loops and no parallel Edges.

Draw a graph with 4 vertices :-

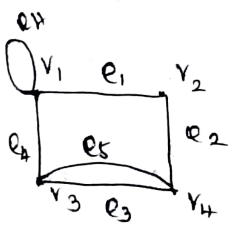


Simple Graph

Not simple Graph :-



G1:



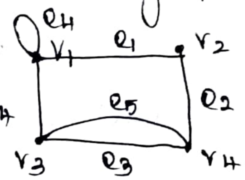
It has loop and parallel Edges - so it is called not simple graph.

LOOP :- $G_1, e_4 = \{v_1, v_1\}$ and parallel $e_5 = \{v_3, v_4\}$ and $e_3 = \{v_3, v_4\}$

General or Multigraph :-

A graph with loops and parallel edges is called general (or) Multigraph.

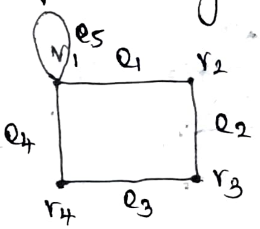
Eg:- G1:



Pseudograph :-

A graph G is called a pseudograph if it has either loop (or) parallel Edges but not the both.

Eg:- G1:

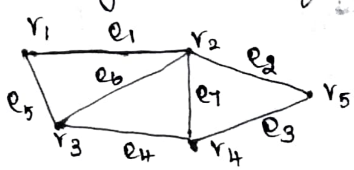


Finite and Infinite graph :-

A graph G is said to be a finite graph if its vertex set and Edge set are finite.

Eg:- G1:

A finite graph with 5 vertices & 7 Edges



A graph G is said to be infinite if its vertex set (or) Edge set (or) both are infinite

G : $V = \{v_1, v_2, v_3 \ldots\}$

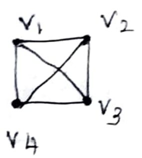
$E = \{e_1, e_2, e_3 \ldots\}$

such graph does not exist.

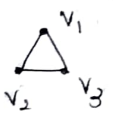
5/7/19 Complete graph (or) K_n graph :-

A simple graph G is said to be complete, if every vertex 'v' of G is adjacent to every other vertices of G.

Eg:- K_4 :-



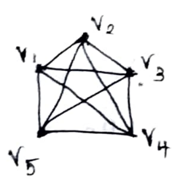
K_3 :-



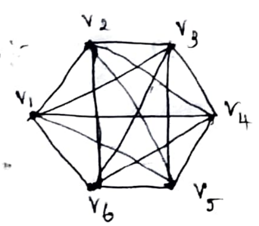
K_2 :-

K_1 :- •

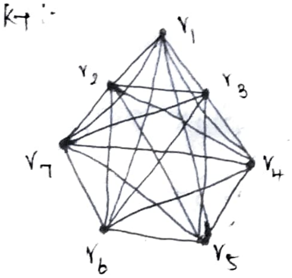
K_5 :-



K_6 :-



K_7 :-



Undirected, directed and mixed graph :-

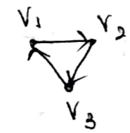
A graph G = (V, E) where V is finite non-Empty set of vertices and E is finite set of undirected edges which connects vertices is called undirected graph.

Eg :-

Note :- In undirected graph, Every Edge (i, j) = (j, i) must.

A graph G = (V, E) where V is finite non-Empty set of vertices and E is finite set of directed edges which connects vertices is called directed graph.

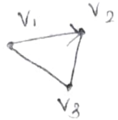
Eg :-



Note : In directed graph, Every Edge $(i, j) \neq (j, i)$ (or) need not to be edge $(i, j) = (j, i)$

A graph is said to be a mixed graph if it has both undirected and directed edges.

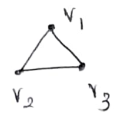
Eg:-

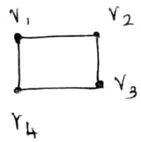


Regular graph:-

A simple graph G is said to be regular if all the vertices have the same degree.

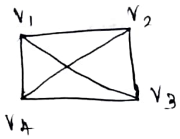
Eg:-





Note:- All complete graph are regular

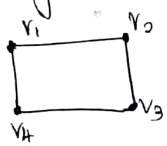
Eg:-



degree = 3

But not all the regular graph are complete.

Eg:-



degree - 2

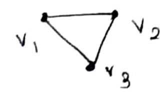
cyclic graph:-(or) C_n:

A graph is said to be cyclic if all its vertices have degree 2.

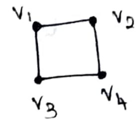
Eg:- C_1:-

C_2:-

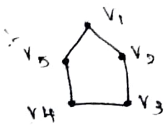
C_3:-



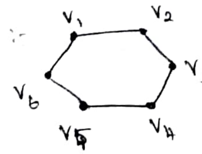
C_4:-



C_5:-



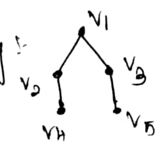
C_6:-



Acyclic graph:-

A graph with no cycle is called Acyclic graph.

Eg:-

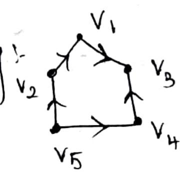


Note:- Acyclic graph is also called "Tree"

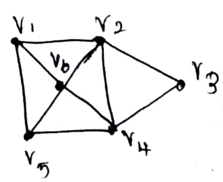
Directed Acyclic graph (DAG):

A directed graph with no cycle is called DAG.

Eg:-



Minimum and maximum degree of a graph:-



$d(v_1) = 3$ $d(v_5) = 3$

$d(v_2) = 4$ $d(v_6) = 4$

$d(v_3) = 2$

$d(v_4) = 4$

max degree of Graph:

$\Delta(G) = 4$

min degree of graph:

$\delta(G) = 2$

1) Fundamental theorem (or) Handshaking theorem:

Statement:

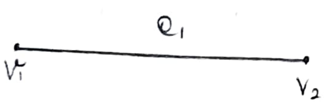
In graph G, the sum of the degrees of the vertices is equal to twice the no. of Edges.

(i.e) $\sum_{i=1}^{n} d(v_i) = 2e$

Proof:

Let G be a graph with 'n' vertices and 'e' edges.

As Each Edge constitute two degrees, one for one vertex and the one degree for the other vertex an edge connects two vertices.



e_1 gives 1 degree for v_1

e_1 gives 1 degree for v_2

Therefore, sum of a degrees of all the vertices of G is equal to two times the no. of edges.

(i.e) $\therefore \sum_{i=1}^{n} d(v_i) = 2e$

$\therefore$ Hence Proved.

2) A No. of vertices with odd degrees in a graph G is always Even.

Proof:

Let G be a graph with 'n' vertices and 'e' edges.

In G, there can be 'x' no. of odd degree and 'y' no. of even degree vertices where,

(i.e) $\therefore x+y=n$

(i.e) $\sum_{i=1}^{n} d(v_i) = \underbrace{\sum_{j=1}^{x} d(v_j)}_{\text{odd}} + \underbrace{\sum_{k=1}^{y} d(v_k)}_{\text{even}}$

We know that,

$\sum_{i=1}^{n} d(v_i) = 2e$

$\underbrace{\sum_{j=1}^{x} d(v_j)}_{\text{odd}} + \underbrace{\sum_{k=1}^{y} d(v_k)}_{\text{even}} = 2*e$

$$\sum_{i=1}^{x} d(v_i) = 2*e - \sum_{k=1}^{y} d(v_k)$$

odd — Even — even

Statement

As the sum of odd ~~number~~ degree vertices results an even number. We can conclude, the no. of odd degree vertices should be Even.

Then only it results the Σ ~~has~~ as even.

1) How many Edges are there in a graph with 10 vertices, Each of which degree 6.

Solu:

$\Sigma d(v_i) = 2e = 10+10+10+10+10+10 = 2e$

$60 = 2e$

$e = \frac{60}{2}$

$\therefore e = 30$

2) Can simple graph G exist with 5 vertices with each vertex with degree 5.

Solu:

$\Sigma d(v_i) = 2e$

$25 = 2e$

$e = \frac{25}{2}$

$\therefore e = 12.5$

$\therefore$ Such graph cannot Exist.

3) For the following degree sequence 1, 1, 2, 3 verify whether there exist graph (or) Not.

Solu:

$\Sigma d(v_i) = 2e$

$1+1+2+3 = 2e$

$7 = 2e$

$\therefore e = \frac{7}{2} = 3.5$

$\therefore$ Graph does not exist.

4) Verify whether there exist graphs with 28 Edges and 12 vertices in the following cases:

a) Degree of vertices is either 3 (or) 8.

b) Degree of vertices is either 3 (or) 4.

c) Degree of vertices is either 3 (or) 6.

a) Solu:

Let 'k' be the no. of vertices with degree 3

$3k + (12-k)8 = 2\times 28$

$3k + 96 - 8k = 56$

$96 - 5k = 56$

$96 - 56 = 5k$

$40 = 5k$

$\therefore k = \frac{40}{5} = 8$

b) Solu:

Let 'k' be the no. of vertices with degree 3.

$$3k + (12-k)4 = 2 * 28$$

$$3k + 48 - 4k = 56$$

$$48 - k = 56$$

$$48 - 56 = k$$

$$\therefore k = -8$$

$\therefore$ graph does not exist.

c) Solu:

Let 'k' be the no. of vertices with degree 3.

$$3k + (12-k)6 = 2*28$$

$$3k + 72 - 6k = 56$$

$$72 - 3k = 56$$

$$72 - 56 = 3k$$

$$16 = 3k$$

$$k = \frac{16}{3}$$

$$\therefore k = 5.33$$

$\therefore$ graph does not exist.

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1) The maximum degree of any vertex in a simple graph 'G' with 'n' vertices is 'n-1'.

Proof:

A graph is said to be simple if and only if (iff) it has no loops and no parallel edges.

In a simple graph, every vertex will be adjacent to every other vertices in the graph only when it is complete.

Thus, in a graph G with n vertices, the vertex 'v' can be adjacent to atmost all the remaining 'n-1' vertices of G, when it is complete.

Thus, a vertex v can have a maximum degree of 'n-1'.

2) The maximum no. of edges in a simple graph G with n vertices is $\frac{n(n-1)}{2}$ (or)

Show that a complete graph G with 'n' vertices has $\frac{n(n-1)}{2}$ edges.

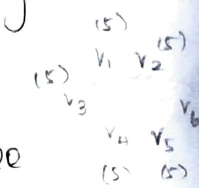
Proof:

A simple graph is said to be complete if every vertex 'v' of G is adjacent to every other vertices of G.

Eg:- In a complete graph 'G' with 'n' vertices, each vertex will have a degree of 'n-1'.

We know that, sum of degrees of all vertices in G $\Big\} = 2e$

$$\sum_{i=1}^{n} d(v_i) = 2e$$



$$(n-1)+(n-1)+(n-1)+\cdots+n \text{ times} = 2e$$

$$n(n-1) = 2e$$

$\boxed{n=6}$

$n-1$

$$\therefore e = \frac{n(n-1)}{2}$$

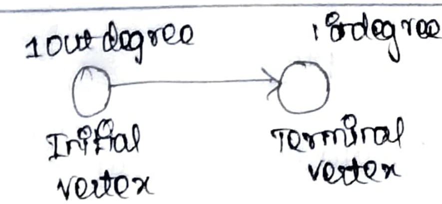
Thus, a complete graph with n vertices will have a maximum of $\frac{n(n-1)}{2}$ edges.

3) In a directed graph 'G', show that $\Sigma deg_G^{+} = \Sigma deg_G^{-} = e$

Proof:

In a directed graph, every edge has initial and terminal vertex.

Thus, a directed edge constitutes one outdegree to the initial vertex and one indegree to the terminal vertex.



Thus, in a above graph G, no. of indegree equal to no. of outdegree equal to e.

$0+1=1$

Thus, in a directed graph G, with n vertices sum of the indegree = sum of the outdegree = e

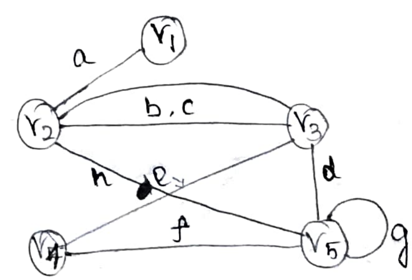
Thus, $\Sigma deg_G^{+} = \Sigma deg_G^{-} = e$

sum of outdegree sum of indegree

∴ Hence proved.

4) Let $G=(V,E)$ where $V=\{v_1, v_2, v_3, v_4, v_5\}$ and the edges are given by $a \to (v_1,v_2)$, $b \to (v_2,v_3)$, $c \to (v_2,v_3)$, $d \to (v_3,v_5)$, $e \to (v_3,v_4)$, $f \to (v_4,v_5)$, $g \to (v_5,v_5)$, $h \to (v_5,v_2)$. Draw G.

Sol:



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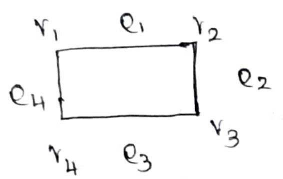
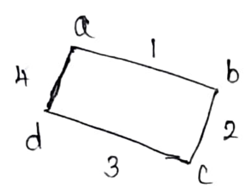
Isomorphism of Graphs :-

Two graphs G_1 and G_2 are said to be isomorphic if there exists a one to one (1-1) correspondance between their vertices set V_1 and V_2, their edge sets E_1 and E_2 such that the incidence relation is preserved.

Eg :-



G_1



G_2

In G_1 :-

vertex	degree	Edgeset
v_1	2	$\{e_1, e_4\}$
v_2	2	$\{e_1, e_2\}$
v_3	2	$\{e_2, e_3\}$
v_4	2	$\{e_3, e_4\}$

In G_2 :-

vertex	degree	Edgeset
a	2	$\{1, 4\}$
b	2	$\{1, 2\}$
c	2	$\{2, 3\}$
d	2	$\{3, 4\}$

Incidence relation :-

$v_1 \to a$ $\quad e_1 \to 1$

$v_2 \to b$ $\quad e_2 \to 2$

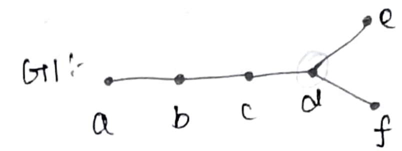
$v_3 \to c$ $\quad e_3 \to 3$

$v_4 \to d$ $\quad e_4 \to 4$

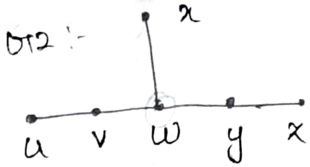
$\therefore$ Hence G_1 and G_2 are isomorphic.

1) Check whether the following two graphs are isomorphic (or) Not :-

G_1 :-



G_2 :-



In G_1 :-

vertex	degree
a	1
b	2
c	2
d	3
e	1
f	1

In G_2 :-

vertex	degree
u	1
v	2
w	3
x	1
y	2
z	1

In G_1, vertex d (degree-3) is incident with vertices c (degree-2), e (degree-1) & f (degree-1).

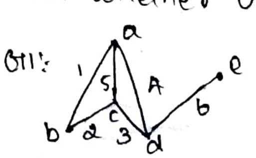
In G_2, vertex w (degree-3) is incident with vertices v (degree-2), y (degree-2) & x (degree-1).

Hence, the incidence relation is not preserved.

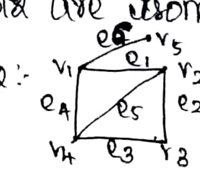
$\therefore$ G_1 and G_2 are not isomorphic.

2) Check whether G_1 and G_2 are isomorphic :

G_1 :



G_2 :-



In G_1: In G_2:

vertex	degree	Edgeset	vertex	degree	Edgeset
a	3	{1,4,5}	v_1	3	{e_1, e_4, e_6}
b	2	{1,2}	v_2	3	{e_1, e_2, e_5}
c	3	{2,3,5}	v_3	2	{e_2, e_3}
d	3	{3,4,6}	v_4	3	{e_3, e_4, e_5}
e	~~2~~ 1	{6}	v_5	1	{e_6}

Incidence relation:

In G_1, vertex d (degree-3) is incident with vertices a (degree-3), c (degree-3) & e (degree-1).

In G_2, vertex v_1 (degree-3) is incident with vertices v_2 (degree-3), v_4 (degree-3) & v_5 (degree-1).

$\therefore$ The incident relation between G_1 and G_2 shall be

$d \to v_1$	$1 \to e_2$
$e \to v_5$	$2 \to e_3$
$a \to v_2$	$3 \to e_4$
$c \to v_4$	$4 \to e_1$
$b \to v_3$	$5 \to e_5$
	$6 \to e_6$

$\therefore$ Hence G_1 and G_2 are isomorphic.

Sub-graphs:

Let G be graph with vertex set $V(G)$ and Edge set $E(G)$.

Let H be a graph with the vertex set $V(H)$ and Edge set $E(H)$.

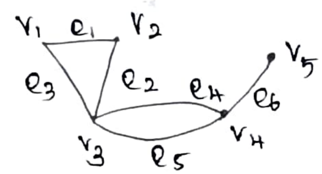
Then H is said to be a sub-graph of G.

iff $V(H) \subseteq V(G)$

and

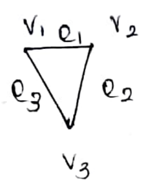
$E(H) \subseteq E(G)$

G:-

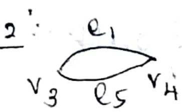


Subgraphs of G:

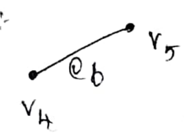
H1:



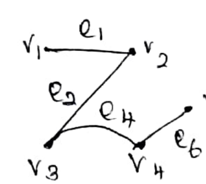
H2:



H3:



H4:-



Note:-

* Every graph is its own sub-graph.
* A sub-graph of a sub-graph of G is also a

sub-graph of G.

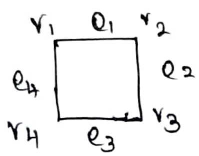
* A single vertex in a graph G is a sub-graph of G.

* A single edge together with its own end vertices is also a subgraph of G.

15-7-19 Path, walk and circuit :-

walk :-

A walk in a graph is a finite sequence of vertices and edges alternatively.



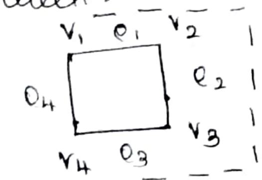
$\therefore v_1 e_1, v_2, e_2, v_3 e_3, v_4 e_4.$

Two types of walk/path :-

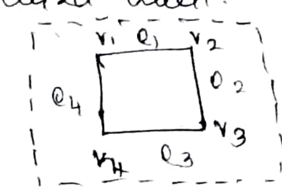
1) open-walk :- A walk is said to be open if the initial and terminal vertices are different.

2) closed-walk :- A walk is said to be closed if the initial and terminal vertices are same.

Open walk/path :-



closed walk/path :-

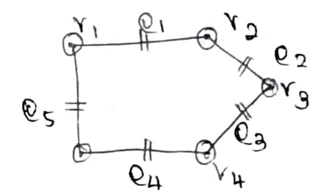


Path :-

A route taking from one vertices to other vertices such that no vertices reappear.

Length of the path :-

Eg:



$v_1 \to v_5$

Length of the path from

$v_1 \to v_5 = e_5 \Rightarrow 1$

$e_1 - e_2 - e_3 - e_4 \Rightarrow 4$

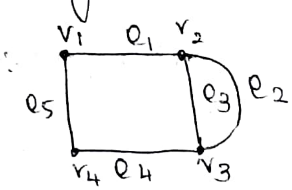
Simple path :-

All vertices are distinct in a path.

Trail :-

All the edges are distinct in a path.

Eg:

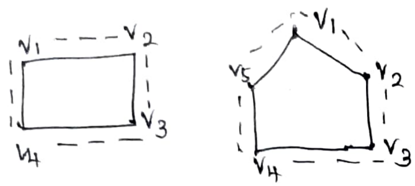


$e_1 - e_5 - e_4 - e_2 - e_3$

Circuit:-

A closed path in which all the vertices are distinct except initial and terminal vertices.

Eg:-



Euler Graph → covers Edges:-

Euler Circuit:- complete

* A circuit which visits Every edge once, where starting and Ending point are at the same Vertex.

* Euler Circuit is possible only if Every vertices have even ~~decrease~~ degree

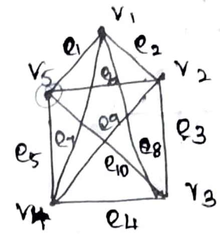
Euler path:- Not complete

* An Euler path visits Every edges only once.

* Euler path is possible but Euler Circuit is not possible only if exactly 2 vertices have odd degrees.

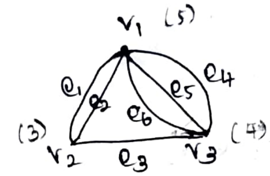
* No Euler path is possible if more than 2 vertices have odd degree.

Eg:-



Euler Circuit:- $e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_9 - e_7 - e_8 - e_{10} - e_1$

Eg:-

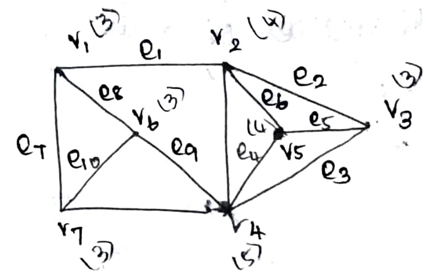


∴ Euler circuit is not possible.

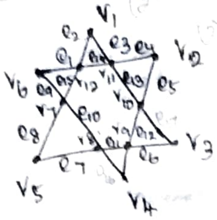
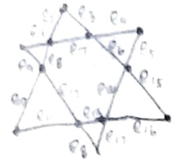
Euler path: $e_1 - e_2 - e_3 - e_6 - e_5 - e_4$

∴ Euler Circuit is not possible because starting and Ending vertices are not same.

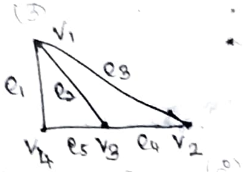
Eg:-



∴ No Euler path and no Euler Circuit because, it has more than 2 vertices have odd degrees.

Eg:-  

∴ It has no Euler circuit and Euler path.

Eg:- 

∴ Euler path is possible but Euler circuit is not possible.

Theorem:

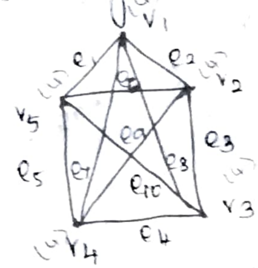
For given connected graph G is an Euler graph if all vertices of G is of even degree.

Proof:

Suppose G is an Euler graph then, G contains an closed walk covering all the edges.

In tracking the closed ~~path~~/walk, every time we walk meets a vertices v, it goes through two new edges incident on v with one entered and other exited.

Every vertex in G is also of Even degrees. This process is repeated untill we obtain a closed walk covering an all the edges in G.

Eg:- 

Euler circuit :- $e_1-e_2-e_3-e_4-e_5-e_6-e_9-e_7-e_8-e_{10}-e_1$

Euler path :- $e_1-e_2-e_3-e_4-e_5-e_6-e_9-e_7-e_8-e_{10}$

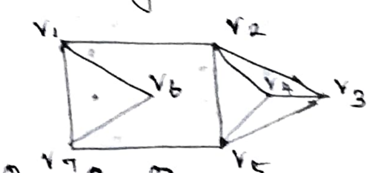
Hamiltonian Graph :→ covers vertices.

Hamiltonian circuit :-

An hamiltonian circuit visits every vertices once, where the starting and ending vertices are same.

Hamiltonian Path :-

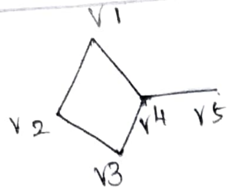
It visits every vertices only once.

Eg:- 

Hamiltonian circuit:
$V_1-V_2-V_3-V_4-V_5-V_7-V_6-V_1$

Hamiltonian Path:
$V_1-V_2-V_3-V_4-V_5-V_7-V_6$

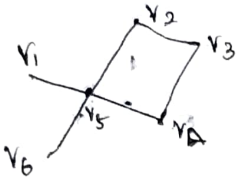
Eg:



and it has no circuit.

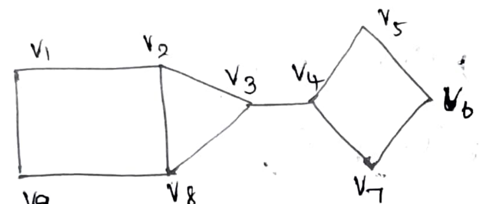
Path: $V_1 - V_2 - V_3 - V_4 - V_5$

Eg:



∴ It has no path and no circuit.

Eg:



Path: $V_1 - V_9 - V_8 - V_2 - V_3 - V_4 - V_5 - V_6 - V_7$

∴ It has no circuit because there is a repetition of vertices

16/4/19 Trees:-

A tree is a connected graph without any circuit.

Eg:

G_1 G_2 G_3 G_4 G_5

G_1 G_2

Here G_1, G_2 are not a tree because it forms a circuit.

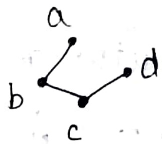
Note:-

* The trees always has n vertices and n-1 edges.

Theorem:-

If In a graph G, there is one and only one path between every pair of vertices, then G is a tree. G is connected.

Proof:



* Graph G has no circuit,
* Let a and b be any two vertices in the circuit.
* There exist two path between a and d which is a contradiction.
* Therefore, G has no circuit.
* Hence, in a graph G has no circuit which is proved as a tree.

Theorem:-

A connected graph G is a tree if and only if adding an edge between any 2 vertices in G create exactly one circuit.

Proof:-

* Let G be a connected graph.
* An Addition of an edge between any two

vertices in G creates one circuit.

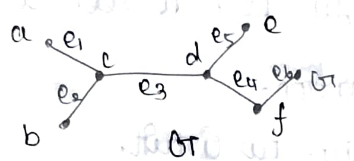
* So without this new edge a graph G is acyclic.

* Thus G is a connected acyclic graph and so it is a tree.

Minimally connected graph:

A connected graph is said to be minimum if the removal of any one edge from the graph G makes an disconnected graph.

Eg:



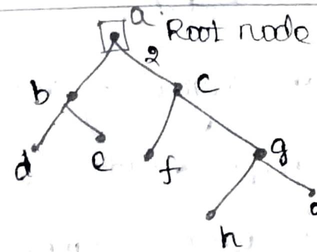
If the edge e4 is removed, the graph is disconnected.

18/7/19 **Rooted and Binary trees:-**

A tree in which one vertex is distinguished from all other vertex is called Rooted tree.

A binary tree is used for decision making and sometimes called an decision tree.

Eg:



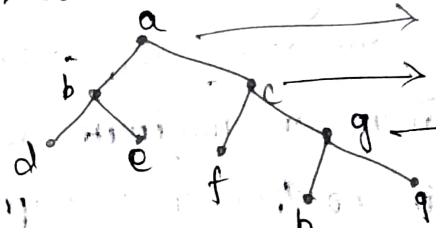
Ancestor node of b and c = a

Descendant node of a = b, c

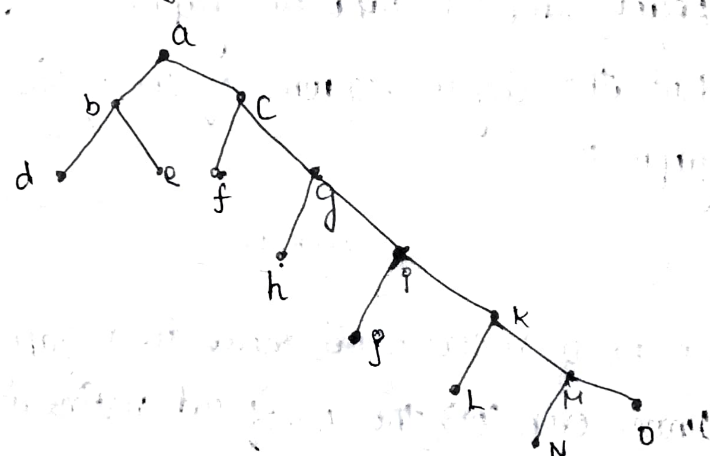
Note:

A binary tree is defined as a tree in which there is exactly one vertex of degree 2, and each of the remaining vertex of degree 1, 3

Levels of tree:- → n-1



1) Draw a binary tree with 15 vertex



Part-A

1) Define graph and degree of vertex.

2) Define directed graph & undirected graph.

3) For the following degree sequence 1,1,2,3 verify whether there exist a graph or not.

4) Can there be a graph consisting of the vertices v_1, v_2, v_3, v_4 with $deg(v_1) = 2$, $deg(v_2) = 3$, $deg(v_3) = 2$, $deg(v_4) = 2$.

5) Define Bipartite graph.

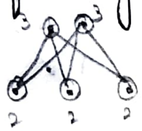
6) Define graph and acyclic graph with ex.

7) Define maximum and minimum degrees of a graph. (1.89)

8) How many edges are there in a graph with 10 vertices each of degree six?

9) Define complete Bipartite graph. (1.11)

10) Find the degree sequence of the following graph $K_{2,3}$

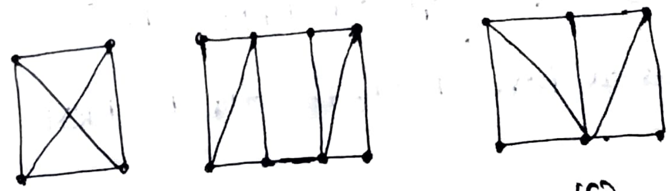


3,3,2,2,2

Part-B

11) The no. of vertices of odd degree in a graph is always even (or) the no. of odd vertices is always even.

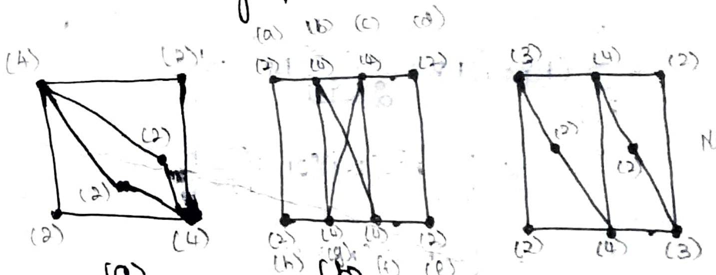
12) Show that the following graphs are Hamiltonian but not Euler.



(a) (b) (c)

13) If G is a simple graph with n vertices and k-components, then it can have ~~atleast~~ atmost $(n-k)(n-k+1)/2$ edges.

14) Find which of the following graph are Euler graph and Hamilton graphs.



(a) (b) (c)

No circuit

Euler graph — Every vertices have even degree

Euler graph

path: a-b-f-e-d-c-g-h

circuit: a-b-f-e-d-c-g-h-a

Hamilton graph — Exactly 2 vertices have odd degree, so Euler graph is not possible

13) proof:

Let the no. of vertices in each of the k components of a graph G be $n_1, n_2, \dots n_k$.

Thus we have, $n_1 + n_2 + \dots + n_k = n$, $n_k \geq 1$

The proof of the theorem depends on an algebraic inequality

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)$$

Now the maximum no. of edges in the ith component of G (which is a simple connected graph) is $\frac{1}{2} n_i(n_i - 1)$. Therefore, the maximum no. of edges in G is

$$\frac{1}{2} \sum_{i=1}^{k} (n_i - 1) n_i = \frac{1}{2} \left(\sum_{i=1}^{k} n_i^2 \right) - \frac{n}{2}$$

$$\leq \frac{1}{2} \left[n^2 - (k-1)(2n-k) \right] - \frac{n}{2}$$

$$= \frac{1}{2} (n-k)(n-k+1)$$

$\leq \frac{1}{2}[n^2-(k-1)(2n-k)$

26/7/19

RECURRENCE RELATION

Definition:

[It is a relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one (or) more of the preceding terms $a_0, a_1, a_2, \dots a_{n-1}$ for all $n \geq 0$.]

Linear Recurrence Relation:

A recurrence relation is said to be a linear if the relation can be expressed by a linear function of a fixed number of preceding terms.

Eg: 1, 2, 4, 8, 16, 32, ...

$a_n = 2a_{n-1}, n \geq 1 \Rightarrow a_1 = 2a_0$

[∴ 1 preceding terms]

0, 1, 1, 2, 3, 5, 8, 13, 21, ...

$F_N = F_{N-1} + F_{N-2}, n \geq 2$

$F_2 = F_1 + F_0$

[∴ 2 preceding terms]

Otherwise, the relation is non-linear recurrence relation.

Homogeneous and Non-Homogeneous recurrence relation:

1) The recurrence relation can be classified as homogeneous ~~(or) Non-homogeneous~~ if the term

sequence $a_0 = a_1 = a_2 = \dots = 0$ satisfies the relation is said to be Homogeneous. Otherwise, the relation is called non-Homogeneous.

(*) If a_n can be expressed in terms of a_{n-1}, a_{n-2} ...... a_{n-k}, then the relation is said to be K-order relation.

First order recurrence relation:

i) Find the general term of the sequences:

a) 2, 10, 50, 250, ...

b) 6, −18, 54, −162, ...

c) 7, 14/5, 28/25, 56/125, ...

Soln:

a) Here, $a_0 = 2$

$a_1 = 5 * 2 = 10 \Rightarrow 5a_0$

$a_2 = 5 * 10 = 50 \Rightarrow 5a_1 \Rightarrow 5.5a_0 \Rightarrow 5^2 a_0$

$a_3 = 5 * 50 = 250 \Rightarrow 5a_2 \Rightarrow 5.5.5.a_0 \Rightarrow 5^3 a_0$

⋮

$\boxed{a_n = 5a_{n-1}}$ $n \geq 1$, $a_0 = 2$

The solution is,

$a_n = 5^n . a_0$

$a_n = 5^n . (2)$

$\boxed{\therefore a_n = 2(5)^n, n \geq 0}$

b) 6, −18, 54, −162, ...

Here, $a_0 = 6$

$a_1 = -18 \Rightarrow (-3)a_0$

$a_2 = 54 \Rightarrow -3a_1 \Rightarrow -3(-3)a_0 \Rightarrow 9 \times 6 = 54$

$a_3 = -162 \Rightarrow (-3)a_2$

⋮

$\boxed{a_n = (-3)a_{n-1}, n \geq 1}$ $a_0 = 6$

The solution is,

$a_n = -3^n a_0$

$a_n = (-3)^n 6$, $n \geq 0$

$\boxed{\therefore a_n = 6(-3)^n}$ $n \geq 0$

c) $7, \frac{14}{5}, \frac{28}{25}, \frac{56}{125}, \dots$

Here, $a_0 = 7$

$a_1 = \frac{14}{5} = \frac{2a_0}{5}$

$a_2 = \frac{28}{25} = \frac{2}{5} a_1 = \frac{2}{5} \times \frac{2}{5}(7) = \frac{4 \times 7}{25} = \frac{28}{25}$

$a_3 = \frac{56}{125} = \frac{2}{5} a_2 = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \times 7$

$= \frac{8 \times 7}{125} = \frac{56}{125}$

The solution is,

$a_n = \left(\frac{2}{5}\right)^n a_0$, $n \geq 0$

$= \left(\frac{2}{5}\right)^n (7) = 7\left(\frac{2}{5}\right)^n$ $\boxed{a_n = 7\left(\frac{2}{5}\right)^n}$

$\boxed{\therefore a_n = \left(\frac{2}{5}\right)^n a_0, n \geq 0, a_0 = 7}$

2) Find the general solution for the following recurrence relation:

a) $a_{n+1} - 1.5a_n = 0$

b) $a_{n+1} = 3a_n,\ n \geq 0,\ a_0 = 5$

c) $3a_{n+1} - 4a_n = 0,\ n \geq 0,\ a_1 = 5$

d) $a_n = 7a_{n-1},\ n \geq 1,\ a_2 = 98$

Soln:

a) $a_{n+1} = 1.5a_n$

$n=0,\quad a_1 = 1.5a_0$

$n=1,\quad a_2 = 1.5a_1 = (1.5)^2 a_0$

$n=2,\quad a_3 = 1.5a_2 = (1.5)^3 a_0$

$\vdots$

The general solution is,

$\boxed{a_n = (1.5)^n a_0,\ n \geq 0}$

b) $a_{n+1} = 3a_n,\ n \geq 0,\ a_0 = 5$

$n=0 \Rightarrow a_1 = 3a_0 = 3(5)$

$n=1 \Rightarrow a_2 = 3a_1 = 3.3(5) = 3^2(5)$

$n=2 \Rightarrow a_3 = 3a_2 = 3.(3^2)5 = (3)^3.5$

$\vdots$

The general solution is,

$\boxed{a_n = (3)^n.5,\ n \geq 0}$

c) $3a_{n+1} - 4a_n = 0,\ n \geq 0,\ a_1 = 5$

$3a_{n+1} = 4a_n$

$a_{n+1} = \frac{4}{3}a_n$

$n=0,\ a_1 = \frac{4}{3}a_0$

$a_1 = \frac{4}{3}(a_0)$

$5 = \frac{4}{3}a_0 \Rightarrow \frac{15}{4}$ $\quad \boxed{\therefore a_0 = \frac{15}{4}}$

$a_1 = \frac{4}{3}\left(\frac{15}{4}\right)$

$n=1,\ a_2 = \frac{4}{3}a_1$

$= \frac{4}{3} \times \frac{4}{3} \times \frac{15}{4} = \left(\frac{4}{3}\right)^2\left(\frac{15}{4}\right)$

$n=2,\ a_3 = \frac{4}{3}a_2$

$= \frac{4}{3} \times \left(\frac{4}{3}\right)^2\left(\frac{15}{4}\right) = \left(\frac{4}{3}\right)^3\left(\frac{15}{4}\right)$

$\vdots$

The solution is,

$\boxed{a_n = \left(\frac{4}{3}\right)^n\left(\frac{15}{4}\right),\ n \geq 0}$

d) $a_n = 7a_{n-1},\ n \geq 1,\ a_2 = 98$

$a_n = 7a_{n-1}$

$n=1,\ a_1 = 7a_0$

$a_0 = \frac{a_1}{7} \Rightarrow \boxed{a_1 = 7a_0}$ $\quad \therefore \boxed{a_1 = 14}$

$n=2$, $a_2 = 7a_1$

$a_2 = 7\times 7a_0 = (7)^2 a_0$

$n=3$, $a_3 = 7a_2$

where $a_3 = (7)^3 a_0$

$98 = 49\, a_0$

$\frac{98}{49} = a_0$

$\therefore \boxed{a_0 = 2}$

$a_3 = 7a_2$

$= (7)^3 \times 2$

$\vdots$

The general solution is,

$a_n = (7)^n 2 \;, n \geq 0$

$\boxed{a_n = 2.(7)^n \;, n \geq 0}$

$n=2$

$a_2 = 7a_1$

$98 = 7a_1$

$a_1 = \frac{98}{7}$

$\boxed{a_1 = 14}$

$n=1$,

$a_1 = 7a_0$

$14 = 7a_0$

$\boxed{a_0 = 2}$

$\therefore a_n = 7a_{n-1}, n \geq 0, a_0 = 2$

$a_n = 2.(7)^n, n \geq 0$

3) Solve the following recurrence relation;

(i) If $a_n, n \geq 0$ is the solution of a recurrence relation $a_{n+1} - da_n = 0$, $a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{2401}$ what is d?

(ii) Find a_{12} if $a_{n+1}^2 = 5a_n^2, n \geq 0, a_0 = 2$

(iii) The no. of bacteria in a culture is 1000 and this increases 250% every two hours. Use recurrence relation to determine no. of bacteria present after one day.

Soln:-

(i) $a_{n+1} - da_n = 0$

$a_{n+1} = da_n, n \geq 0$

The general solution is,

$\boxed{a_n = d^n a_0, n \geq 0}$

when, $n = 3$,

$a_3 = d^3 a_0$

$\frac{153}{49} = d^3 a_0 \rightarrow (A)$

when, $n = 5$,

$a_5 = d^5 a_0$

$\frac{1377}{2401} = d^5 a_0 \rightarrow (B)$

$\frac{(B)}{(A)} \Rightarrow \frac{d^5 a_0}{d^3 a_0} = \frac{1377}{\cancel{2401}_{49}} \times \frac{\cancel{49}}{153}$

$= \frac{\cancel{1377}\,9}{49 \times \cancel{153}}$

$= \frac{9}{49}$

$d^2 = \frac{9}{49}$

$\boxed{d = \frac{3}{7}}$

$$\begin{array}{r} 49 \\ 49 \overline{)\,2401} \\ 196 \\ \hline 441 \\ 441 \\ \hline 0 \end{array}$$

$\boxed{\therefore a_n = \left(\frac{3}{7}\right)^n a_0, n \geq 0}$

(ix) Let, $b_n = a_n^2$

when $n=0$,

$b_0 = a_0^2$

$b_0 = (2)^2 = 4$

$b_n = a_n^2$

Let, $n = n+1$

$b_{n+1} = a_{n+1}^2$

$b_{n+1} = a_{n+1}^2$

$= 5a_n^2$

$b_{n+1} = 5b_n$

$b_{n+1} - 5b_n = 0, \quad n \geq 0, \; b_0 = 4$

$\boxed{\therefore b_n = (5)^n b_0}$, $n \geq 0$, $b_0 = 4$.

$b_n = 4(5)^n, \; n \geq 0$

$a_n^2 = 4(5)^n$

$a_n = \sqrt{4(5)^n}$

$\boxed{a_n = 2(5)^{n/2}}$

(15625)

$a_{12} = 2(5)^{12/2}$

$= 2(5)^6$

$= 2(15625)$

$\boxed{a_{12} = 31250}$

(viii) Given :-

$a_0 = 1000$

Let a_n be the no. of bacteria after 24 hours

Rate of increase is $a_n * \frac{250}{100}$

$= 2.5a_n$

after 2 hrs :- $a_1 = a_0 + 2.5a_0$

$= a_0(1+2.5) = 3.5a_0$

after 4 hrs :- $a_2 = a_1 + 2.5a_1$

$= a_1(1+2.5) = 3.5a_1$

after 24 hrs :- $a_{12} = a_{11} + 2.5a_{11}$

$= a_{11}(1+2.5) = 3.5a_{11}$

$\therefore$ The recurrence relation is,

$a_{n+1} = 3.5a_n$

$a_{n+1} = 3.5a_n$

$\therefore$ The general solution is,

$a_n = (3.5)^n a_0$

$a_n = 1000(3.5)^n$

$a_n = (3.5)^n a_0$

$a_n = (1000)(3.5)^n$

$a_{12} =$

When $n = 12$

$a_{12} = 1000(3.5)^{12}$

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4) A person invest Rs. 10,000/- at 10.5% interest per year compounded monthly. Find and solve the recurrence

relation for the value of Interest after n months what is the value for the end of 1 yr. How long it takes for double the investment.

Sol:

let a_0 be the initial investment = 10000

Interest rate = $\frac{10.5}{100}$ yearly

∴ the monthly interest rate

$= \frac{10.5}{100 \times 12}$

$= 0.00875$

At the End of first month,

the investment becomes,

$a_1 = a_0 + (0.00875)a_0$

$a_1 = 1.00875\, a_0$

At the End of 2nd month,

the investment becomes,

$a_2 = a_1 + (0.00875)a_1$

$a_2 = 1.00875\, a_1$

∴ At the End of 'n' months,

the investment becomes,

$\boxed{a_n = 1.00875\, a_{n-1}}$

The general solution is,

$\boxed{a_n = (1.00875)^n a_0}$

At the end of the 1st year,

$n = 12$

$a_{12} = (1.00875)^{12}(10,000)$

To double how long it takes,

$a_n = 2a_0$

$2a_0 = (1.00875)^n a_0$

$2 = (1.00875)^n$

Take log on both sides,

$\log 2 = n \log(1.00875)$

$n = \frac{\log 2}{\log 1.00875}$

$n = \frac{0.30102995}{0.003783547}$

$\frac{0.301029995}{0.003783547}$

$n = 79.6$ months

∴ It takes 80 months to double the investment.

5) Suppose an investment interest rate is compounded quarterly. Find the time taken for an investment of Rs. 500/- to double, Assuming an interest rate of 8%

annually.

Solu:

Initial Investment, $a_0 = 500$

Rate of interest $= \frac{8}{100}$ yearly.

Rate of interest for a Quater $= \frac{8^2}{100 \times 4}$

$= 0.02$ per Quarter.

At the End of 1st Quarter, the investment becomes, $a_1 = a_0 + 0.02\, a_0$

$a_1 = 1.02\, a_0$

At the End of 2nd Quarter, the investment becomes, $a_2 = a_1 + 0.02\, a_1$

$a_2 = 1.02\, a_1$

$\therefore$ The general recurrence relation is,

$\boxed{a_n = 1.02\, a_{n-1}}$

$\therefore$ The general solution is,

$\boxed{a_n = (1.02)^n a_0}$

To find the period to double the investment,

$a_n = 2\, a_0$

$a_n = 2(500)$

$a_n = 1000$

$a_n = (1.02)^n a_0$

$2a_0 = (1.02)^n a_0$

$2 = (1.02)^n$

Take log on both sides.

$\log 2 = n \log(1.02)$

$n = \frac{\log 2}{\log 1.02}$

$n = \frac{0.30102}{0.00860}$

$\boxed{n = 35}$ quarters

$35/4 = 8.75$
$= 9$

$\therefore$ It takes 9 years to double the investment

b) A bank pays annual interest on deposits compounding the interest once in 3 months. If a deposit doubles in 6 years and 6 months. What is the % of interest paid by the bank.

Solu:

Let a_0 be the initial investment.

Rate of interest per year be 'x' percentage $= \frac{x}{100}$

Quarterly interest rate $= \frac{x}{400}$

At the End of 1^{st} Quarter,

the Investment becomes,

$$a_1 = a_0 + \left(\frac{x}{400}\right) a_0$$

$$a_1 = \left(1 + \frac{x}{400}\right) a_0$$

At the End of 2^{nd} Quarter,

the Investment becomes,

$$a_2 = a_1 + \left(\frac{x}{400}\right) a_1$$

$$a_2 = \left(1 + \frac{x}{400}\right) a_1$$

∴ The general recurrence relation is, (after n Quarters),

$$a_n = \left(1 + \frac{x}{400}\right) a_{n-1}$$

∴ The general solution is,

$$a_n = \left(1 + \frac{x}{400}\right)^n a_0$$

↳ compounded Quarterly

In 6 years and 6 months,

1 yr - 12 months

6 yrs - 72 months + 6 months

$= \frac{78}{3}$ months

= 26 Quarters

$$2\% = \left(1 + \frac{x}{400}\right)^{26} \%$$

Taking log on both sides,

$$\log_e 2 = 26 \log_e \left(1 + \frac{x}{400}\right)$$

$$\log_e \left(1 + \frac{x}{400}\right) = \frac{\log_e 2}{26}$$

$$= 0.0266$$

$$\log_e \left(1 + \frac{x}{400}\right) = 0.0266$$

$$1 + \frac{x}{400} = e^{0.0266}$$

$$1 + \frac{x}{400} = 1.0269$$

$$\frac{x}{400} = 0.0269$$

$$x = 10.76$$

$$\boxed{x = 10.8}$$

Rate of Interest = 10.8% Quarterly

3a) 1.19 Second Order Linear Homogeneous Recurrence Relation

The general form is,

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0, \; n \geq 2$$

To find characteristic Equation,

Put $a_n = c r^n$

$C_n(cr^n) + C_{n-1}(cr^{n-1}) + C_{n-2}(cr^{n-2}) = 0, \; n \geq 2$

where $c \neq 0, r \neq 0$

order $= 2$, i.e. $n = 2$

$C_n r^2 + C_{n-1} r^1 + C_{n-1} r = 0, \; c \neq 0, r \neq 0$

This quadratic equ is to be solved for two roots r_1 and r_2.

Case-1:- r_1 and r_2 are real and distinct.

The general solution is,

$$a_n = A(r_1)^n + B(r_2)^n$$

Case-2:- r_1 and r_2 are real and equal.

The general solution is,

$$a_n = (A+Bn)\, r^n$$

Case-3:- r_1 and r_2 are complex. general solution is

$$a_n = r^n (A \cos n\theta + B \sin n\theta)$$

where,

$$r_1 = P + iq, \quad r_2 = P - iq$$

$$r = \sqrt{P^2 + q^2}$$

$$\theta = \tan^{-1}\left(\frac{q}{P}\right)$$

① Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$, $n \geq 2$ and $a_0 = -1$, $a_1 = 8$

Solu:

Given:- $a_n + a_{n-1} - 6a_{n-2} = 0$

The characteristic Equ is,

$$r^2 + r - 6 = 0 \rightarrow ①$$

Solving Equ ①

$$(r+3)(r-2) = 0$$

$$r_1 = -3, \; r_2 = 2$$

$\therefore$ The roots are real & distinct.

The general solution is,

$$a_n = A(r_1)^n + B(r_2)^n$$

$$a_n = A(-3)^n + B(2)^n$$

Given, $a_0 = -1$

$n = 0$

$$a_0 = A(-3)^0 + B(2)^0$$

$$-1 = A + B \rightarrow ②$$

$a_1 = 8$

$n = 1$

$$a_1 = A(-3)^1 + B(+2)^1$$

$$8 = -3A + 2B \rightarrow ③$$

Solving ② & ③

$② \times ② \Rightarrow -2 = 2A + 2B$

$③ \Rightarrow 8 = -3A + 2B$

$-10 = 5A$

$\boxed{A = -2}$

Put $A = -2$ in ①

$-1 = -2 + B$

$\boxed{B = 1}$

The general solution is,

$a_n = (-2)(-3)^n + 1.(2)^n, \; n \geq 0$

2) Solve $a_n = 5a_{n-1} + 6a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 3$

Sol:

Given:

$a_n = 5a_{n-1} + 6a_{n-2}$

$a_n - 5a_{n-1} - 6a_{n-2} = 0$

The characteristic Equ is,

$r^2 - 5r - 6 = 0$

$(r-6)(r+1) = 0$

$r_1 = 6, \; r_2 = -1$

∴ the roots are real & distinct

The general solution is,

$a_n = A(r_1)^n + B(r_2)^n$

$a_n = A(6)^n + B(-1)^n$

Given, $a_0 = 1$

$n = 0$

$a_0 = A(6)^0 + B(-1)^0$

$1 = A + B \rightarrow ①$

$a_1 = 3$, $n = 1$

$a_1 = A(6) + B(-1)$

$3 = 6A - B \rightarrow ②$

Solving equ ① & ②

$A + \cancel{B} = 1$

$6A - \cancel{B} = 3$

$7A = 4$

$\boxed{A = 4/7}$

Sub $A = 4/7$ in ①

$\frac{4}{7} + B = 1$

$B = 1 - \frac{4}{7}$

$B = \frac{7-4}{7} = \frac{3}{7}$

$\boxed{B = \frac{3}{7}}$

The general solution is,

$a_n = \left(\frac{4}{7}\right)(6)^n + \left(\frac{3}{7}\right)(-1)^n, \; n \geq 0$

3) Solve $2a_{n+2} = 11a_{n+1} - 5a_n$, $n \geq 0$, $a_0 = 2$, $a_1 = -8$

Solu:

Given: $2a_{n+2} = 11a_{n+1} - 5a_n$

rewrite the given Equ:

$2a_{n+2} - 11a_{n+1} + 5a_n = 0,\ n \geq 0$

The characteristic Equ is,

$2r^2 - 11r + 5 = 0$

$2r^2 - 10r - r + 5 = 0$

$2r(r-5) - 1(r-5) = 0$

$(2r-1)(r-5) = 0$

$r_1 = 1/2,\ r_2 = 5$ (or) $r_1 = 5,\ r_2 = 1/2$

$\therefore$ The roots are real & distinct.

The general solution is,

$a_n = A(r_1)^n + B(r_2)^n$

$\therefore a_n = A(5)^n + B\left(\frac{1}{2}\right)^n$

Given, $a_0 = 2$

$a_0 = A(5)^0 + B\left(\frac{1}{2}\right)^0$

$2 = A + B \rightarrow ①$

$a_1 = -8$

$a_1 = A(5) + B(1/2)$

$-8 = 5A + B/2 \rightarrow ②$

$① \times 5 \rightarrow 10 = \cancel{5A} + 5B$

$2 \rightarrow -8 = \cancel{5A} + B/2$

$18 = 5B - \frac{B}{2}$

$18 = \frac{10B - B}{2}$

$18 = \frac{9B}{2}$

$\frac{18 \times 2}{9} = B$

$\boxed{B = 4}$

Sub $B = 4$ in ①

$A + 4 = 2$

$A = 2 - 4 = -2$

$\boxed{A = -2}$

The general solution is,

$a_n = (-2)(5)^n + (4)\left(\frac{1}{2}\right)^n,\ n \geq 0$

4) Solve $a_n - 6a_{n-1} + 9a_{n-2} = 0,\ n \geq 2,\ a_0 = 5,\ a_1 = 12$

Sol:

Given: $a_n - 6a_{n-1} + 9a_{n-2} = 0$

The characteristic Equ is,

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r_1 = 3,\ r_2 = 3$

$\therefore$ The roots are real and Equal.

The general solution is,

$a_n = (A + Bn)r^n$

$a_n = (A + Bn)(3)^n$

Given, $a_0 = 5,\ n = 0$

$a_0 = (A + B(0))(3)^0$

$\boxed{5 = A}$

$a_1 = 12,\ n = 1$

$a_1 = (A + B(1))3$

$12 = (A + B)3$

$12 = 3A + 3B \rightarrow ①$

Sub $A=5$ in (1)

$12 = 3(5) + 3B$

$12 = 15 + 3B$

$12 - 15 = 3B$

$-3 = 3B$

$\boxed{B = -1}$

The general solution is,

$a_n = (5-n)(3)^n \Rightarrow (5+(-1)n)(3)^n$

$\boxed{\therefore a_n = (5-n)(3)^n}$, $n \geq 0$.

5) Solve $a_n = 2(a_{n-1} - a_{n-2})$, $n \geq 2$, $a_0 = 1$, $a_1 = 2$

Soln:

Given: $a_n = 2(a_{n-1} - a_{n-2})$

$a_n = 2a_{n-1} - 2a_{n-2}$

rewrite the equ,

$a_n - 2a_{n-1} + 2a_{n-2} = 0$, $n \geq 2$

The characteristic equ.

$r^2 - 2r + 2 = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad a=1$, $b=-2$, $c=2$

$r = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2}$

$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm i2}{2}$

$= \frac{2(1 \pm i)}{2} = 1 \pm i$

$r_1 = 1+i$, $r_2 = 1-i$

$[r_1 = p + iq]$ $\quad [r_2 = p - iq]$

$\therefore$ The roots are complex.

$r_1 \Rightarrow p = 1, q = 1$

$r_2 \Rightarrow p = 1, q = -1$

$\boxed{r = \sqrt{p^2 + q^2}}$

$r = \sqrt{1+1}$

$\therefore r = \sqrt{2}$

$\boxed{\theta = \tan^{-1}\left(\frac{q}{p}\right)}$

$\theta = \tan^{-1}\left(\frac{+1}{1}\right) = \tan^{-1}(+1)$

$\therefore \theta = 45^\circ = \pi/4$

	0 0°	$\pi/6$ 30°	$\pi/4$ 45°	$\pi/3$ 60°	$\pi/2$ 90°
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

The general solution is,

$\therefore a_n = r^n(A \cos n\theta + B \sin n\theta)$

$\therefore a_n = (\sqrt{2})^n\left(A \cos \frac{n\pi}{4} + B \sin \frac{n\pi}{4}\right)$

Given, $a_0 = 1$

$n = 0$,

$a_0 = (\sqrt{2})^0 (A\cos 0^\circ + B\sin 0^\circ)$

$\boxed{A=1} \rightarrow (1)$

$a_1 = 2,\ n=1$

$a_1 = \sqrt{2}\left(A\cos\frac{\pi}{4} + B\sin\frac{\pi}{4}\right)$

$2 = \sqrt{2}\left(\frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}}\right)$

$2 = A+B \rightarrow (2)$

Put $A=1$ in (2)

$\boxed{B=1}$

∴ The general solution is,

$$a_n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right),\ n\geq 0$$

---

7) Solve $F_{n+2} = F_{n+1} + F_n$, (or) $F_n = F_{n-1} + F_{n-2}$, $n\geq 2$, $n\geq 0$. Given $F_0 = 0$, $F_1 = 1$

Soln:

Given :- $F_{n+2} = F_{n+1} + F_n$

rewrite the equ,

$F_{n+2} - F_{n+1} - F_n = 0$

The characteristic equ is,

$r^2 - r - 1 = 0$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$r = \frac{1 \pm \sqrt{(1)^2 - 4(-1)}}{2}$$

$$r = \frac{1 \pm \sqrt{1+4}}{2}$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

$r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$

$r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$

∴ The roots are real and distinct

The general solution is,

$a_n = A(r_1)^n + B(r_2)^n$

$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

when $n=0$, $F_0 = 0$

$F_0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0$

$0 = A+B \rightarrow (1)$

$n=1$, $F_1 = 1$

$F_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$

$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \rightarrow (2)$

equ (1) $\times \left(\frac{1+\sqrt{5}}{2}\right) \Rightarrow 0 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1+\sqrt{5}}{2}\right)$

Solving.

$$A\left(\frac{1+\sqrt5}{2}\right)+B\left(\frac{1+\sqrt5}{2}\right)=0$$

$$A\left(\frac{1+\sqrt5}{2}\right)+B\left(\frac{1-\sqrt5}{2}\right)=1$$

$$(-)\qquad(-)\qquad(-)$$

$$B\left(\frac{1+\sqrt5}{2}\right)-B\left(\frac{1-\sqrt5}{2}\right)=-1$$

$$B(1+\sqrt5)-B(1-\sqrt5)=-2$$

$$B+\sqrt5B-B+\sqrt5B=-2$$

$$2(\sqrt5B)=-2$$

$$\sqrt5B=-1$$

$$\boxed{B=\frac{-1}{\sqrt5}}$$

Sub B in ①

$$A+B=0$$

$$A-\frac{1}{\sqrt5}=0$$

$$\boxed{A=\frac{1}{\sqrt5}}$$

The general solution is,

$$F_n=\frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^n-\frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^n,\ n\geq0.$$

7) Solve $2a_{n+3}=a_{n+2}+2a_{n+1}-a_n$, $n\geq0$, $a_0=0$, $a_1=1$, $a_2=2$.

Soln :-

Given :- $2a_{n+3}=a_{n+2}+2a_{n+1}-a_n$.

$$2a_{n+3}-a_{n+2}-2a_{n+1}+a_n=0$$

The characteristic equation is,

$$2r^3-r^2-2r+1=0$$

	2	-1	-2	1
1	0	2	1	-1
	2	1	-1	0

$$2r^2+r-1=0$$

$$2r^2+2r-r-1=0$$

$$2r(r+1)-1(r+1)=0 \Rightarrow (2r-1)(r+1)=0$$

$$(r-1)(2r-1)(r+1)=0$$

$$r_1=1,\ r_2=1/2,\ r_3=-1.$$

∴ The roots are real and distinct.

The general solution is,

$$a_n=A(r_1)^n+B(r_2)^n+C(r_3)^n$$

$$a_n=A(1)^n+B(1/2)^n+C(-1)^n$$

Given $a_0=0$

$$a_0=A(1)^0+B(1/2)^0+C(-1)^0$$

$$a_0=A+B+C=0\rightarrow ①$$

then $a_1=1$

$$a_1=A(1)+B(1/2)+C(-1)$$

$$1=A+B(1/2)-C\rightarrow ②$$

equ ① + equ ②

$A+B+C=0$

$A+B/2-C=1$

$2A+\left(B+\frac{B}{2}\right)=1$

$2A+\left(\frac{2B+B}{2}\right)=1 \Rightarrow 2A+\left(\frac{3B}{2}\right)=1 \Rightarrow 4A+3B=2 \rightarrow (4)$

then $a_2=2$

$a_2=A(1)^2+B(1/2)^2+C(-1)^2$

$2=A+B/4+C \rightarrow (3)$

Solve (2) and (3)

$A+B/2-C=1$

$A+B/4+C=2$

$2A+\left(\frac{B}{2}+\frac{B}{4}\right)=3 = 2A+\left(\frac{2B+B}{4}\right)=3 \Rightarrow 8A+3B=12 \rightarrow (5)$

Solve equ (4) and (5)

$4A+3B=2$

$8A+3B=12$

$\cancel{+4A} = +10$

$A=\frac{10}{4}$

$\boxed{A=5/2}$

Sub $A=5$ in (5)

$2(-5)+3B=12$

$-10+3B=12$

$3B=12+10$

$3B=22$

$\cancel{B=22/3}$

Sub $A=5/2$ in (5)

$8(5/2)+3B=12$

$\frac{40}{2}+3B=12$

$3B=12-\frac{40}{2}$

$3B=12-20$

$\boxed{B=-8/3}$

Sub A, B in (1)

$\frac{5}{2}-\frac{8}{3}+C=0$

$\frac{15-16}{6}+C=0 \Rightarrow \boxed{C=1/6}$

The general solution is

$a_n=\left(\frac{5}{2}\right)(1)^n+\left(\frac{-8}{3}\right)(1/2)^n+(1/6)(-1)^n$

3) Find the recurrence relation for the sequence $8, 24/7, 72/49, 216/343 \ldots$

Soln:

Here, $a_0=8$

$a_1=\frac{24}{7}=\frac{3a_0}{7}$

$a_2=\frac{72}{49}=\frac{3}{7}a_1=\frac{3\times3\times8}{7\times7}=\frac{72}{49}$

$a_3=\frac{216}{343}=\frac{3}{7}[a_2]=\frac{3}{7}\times\frac{3}{7}\times\frac{3}{7}\times8$

$=\frac{27\times8}{49\times7}=\frac{216}{343}$

$\therefore a_1=\frac{3a_0}{7}$

$a_2=\frac{3}{7}a_1$

$a_3=\frac{3}{7}a_2$

The solution is,

$a_n=\left(\frac{3}{7}\right)^n a_0, \; n\geq0$

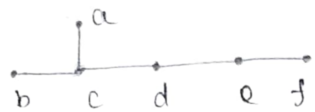
$\boxed{a_n=\left(\frac{3}{7}\right)^n a_0, \; n\geq0, \; a_0=8}$ (or)

$\boxed{a_n=8\left(\frac{3}{7}\right)^n, \; n\geq0}$

5) Find center, radius and eccentricity of the following:

Soln:

let :



Eccentricity $E(a) = 2$ $E(e) = 3$

$E(b) = 2$ $E(f) = 4$

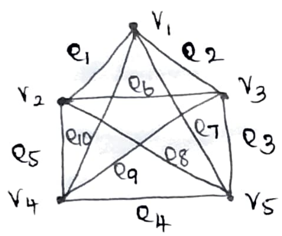
$E(c) = 1$

$E(d) = 2$

centre $= c$

radius $= 1$ [eccentricity of a centre]

6) Simple graph with 5 vertices and 10 Edges:-



5/8/19 **Non-Homogeneous recurrence Relation:-**

The general solution shall be

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$a_n^{(h)}$ - general solution of the associated homogeneous recurrence solution is obtained by the solution $f(n)=0$

$a_n^{(P)}$ - particular solution for the given non-homogeneous recurrence solution / relation.

First order Non-Homogeneous recurrence relation:

$$a_n + C_{n-1} a_{n-1} = K r^n$$

where K - constant, $n \in Z^*$

there are 2 cases :-

case-1:-

If r^n is not a solution of the associated homogeneous recurrence relation,

$$a_n + C_{n-1} a_{n-1} = 0$$

then $a_n^{(P)} = A r^n$, A is a constant.

case-2:-

If r^n is a solution of the associated homogeneous recurrence relation,

$$a_n + C_{n-1} + a_{n-1} = 0$$

then $a_n^{(P)} = B . n . r^n$; B is a constant.

General form:-

I^{st} order:-

$$a_n + C_{n-1} a_{n-1} = f(n), n \geq 1$$

II^{nd} order:-

$$a_n + C_{n-1} a_{n-1} + a_{n-2} = f(n), n \geq 2$$

1) Solve $a_n - a_{n-1} = f(n)$, $n \geq 1$.

Solu:-

The given recurrence relation is a Ist order Non-homogeneous recurrence relation.

$$a_n = a_{n-1} + f(n)$$

when,

$n=1$, $a_1 = a_0 + f(1)$

$n=2$, $a_2 = a_1 + f(2)$

$= a_0 + f(1) + f(2)$

$n=3$, $a_3 = a_2 + f(3)$

$= a_0 + f(1) + f(2) + f(3)$

$n=4$, $a_4 = a_3 + f(4)$

$= a_0 + f(1) + f(2) + f(3) + f(4)$

when,

$n=n$, $a_n = a_0 + f(1) + f(2) + f(3) + f(4) + \dots + f(n)$

$$\boxed{\therefore a_n = a_0 + \sum_{i=1}^{n} f(i)}$$

2) Solve :- $a_n - a_{n-1} = 3n^2$, $n \geq 1$, $a_0 = 7$.

Solu:-

The given recurrence relation is Ist order non-homogeneous recurrence relation.

The general solution is,

$$a_n = a_0 + \sum_{i=1}^{n} f(i)$$

Given, $f(n) = 3n^2$

The general solution is,

$$a_n = 7 + \sum_{i=1}^{n} (3i^2)$$

$$= 7 + 3 \sum_{i=1}^{n} i^2$$

$$= 7 + 3 \left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$\therefore a_n = 7 + \frac{n(n+1)(2n+1)}{2}$$

3) Solve :- $a_n - 3a_{n-1} = 5(7^n)$, $n \geq 1$, $a_0 = 2$.

Solu:

The given recurrence relation is Ist order Non-homogeneous recurrence relation.

The general solution is,

$$a_n = a_n^{(h)} + a_n^{(p)}$$

To find $a_n^{(h)}$:-

Put $f(n) = 0$

$a_n - 3a_{n-1} = 0$

$a_n = 3a_{n-1}$

$a_n = c(3)^n$ is the solution.

To find $a_n^{(p)}$.

$f(n) = 5.(7^n)$

It is of the form $A.(7^n)$

Put $a_n = A.7^n$ into the given recurrence relation.

$A.7^n - 3.A.7^{n-1} = 5(7)^n$

$\div$ both sides by 7^{n-1}

$A.7 - 3A = 5.7^{n-n+1}$

$7A - 3A = 5(7)$

$4A = 35$

$\boxed{\therefore A = \frac{35}{4}}$

$\therefore a_n^{(p)} = \left(\frac{35}{4}\right)7^n$

$\therefore a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = C(3^n) + \frac{35}{4}.7^n$

Given, $a_0 = 2$

$\therefore n = 0$,

$a_0 = C(3^0) + \frac{35}{4}.7^0$

$2 = C + \frac{35}{4}$

$C = 2 - \frac{35}{4}$

$\boxed{\therefore C = \frac{-27}{4}}$

$\therefore a_n = \left(\frac{-27}{4}\right)3^n + \frac{35}{4}.7^n$

$= \left(-\frac{3^3}{4}\right)3^n + \left(\frac{5.7}{4}\right)7^n$

$\boxed{\therefore a_n = \left(\frac{-1}{4}\right)3^{n+3} + \left(\frac{5}{4}\right)7^{n+1}}$

A) Solve: $a_n - 3a_{n-1} = 5(3^n)$, $n \geq 1$, $a_0 = 2$

Solu:-

The given recurrence relation is 1st order non-homogeneous recurrence relation.

The general solution is,

$a_n = a_n^{(h)} + a_n^{(p)}$

To find $a_n^{(h)}$:-

Put $f(n) = 0$

$a_n - 3a_{n-1} = 0$

$a_n = 3a_{n-1}$

$\therefore a_n = C(3)^n$ is the solution.

To find $a_n^{(p)}$:-

$f(n) = 5.(3^n)$

It is of the form $A.(3^n)$

Put $a_n = A.3^n$ into the given recurrence relation.

$A.3^n - 3.A.3^{n-1} = 5.(3)^n$

$\div$ both sides by 3^{n-1}

$A.3^{n-n+1} - 3A = 5.3$

$3A - 3A = 5(3)$

Since 3^n is a solution for the relation

Put $a_n = B.n.3^n$

$B.n.3^n - 3.B(n-1)3^{n-1} = 5.3^n$

$\div$ by 3^n

$B.n - B(n-1) = 5$

$nB - nB + B = 5$

$\boxed{B = 5}$

$\boxed{\therefore a_n^{(P)} = 5.n.3^n}$

$\therefore a_n = a_n^{(h)} + a_n^{(P)}$

$a_n = c(3)^n + 5(n)3^n$

Given $a_0 = 2$

$n=0, \quad a_0 = c(3)^0 + 5(0)3^0$

$2 = c(1)$

$\boxed{c = 2}$

$\boxed{\therefore a_n = 2(3)^n + 5(n)3^n}$ $\boxed{a_n = (2+5n)3^n}$

Second order Non-Homogeneous recurrence relation:-

1) Solve: $a_{n+2} - 4a_{n+1} + 3a_n = 200, n \geq 0, a_0 = 3000$

$a_1 = 3300$.

Soln:

The given Equation is a 2nd order non-Homogeneous recurrence relation.

The general solution is,

$a_n = a_n^{(h)} + a_n^{(P)}$

To find $a_n^{(h)}$:

Put $f(n) = 0$

$a_{n+2} - 4a_{n+1} + 3a_n = 0$

-4

3

$-3 \quad -1$

$r^2 - 4r + 3 = 0$

$(r-3)(r-1) = 0$

$r_1 = 1, r_2 = 3$

$\therefore$ The general solution is,

~~$a_n = A(3)^n + B(1)^n$~~

$\therefore a_n = A(3^n) + B(1)^n$

To find $a_n^{(P)}$:

Here $f(n) = -200$

It is of the form, $A.n$

Put $a_n = A.n$

$A(n+2) - 4A(n+1) + 3A(n) = -200$

$An + 2A - 4An - 4A + 3An = -200$

$-2A = -200$

$\boxed{A = 100}$

$\therefore a_n^{(p)} = 100n$

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n = c_1(3^n) + c_2(1) + 100n$

Given, $a_0 = 3000$

$n = 0$

$a_0 = c_1(3^0) + c_2(1) + 100(0)$

$3000 = c_1 + c_2 \rightarrow (1)$

Given, $a_1 = 3300$

$n = 1,$

$a_1 = c_1(3^1) + c_2(1) + 100(1)$

$3300 = 3c_1 + c_2 + 100$

$3c_1 + c_2 = 3200 \rightarrow (2)$

Solve (1) and (2)

$c_1 + \cancel{c_2} = 3000$

$3c_1 + \cancel{c_2} = 3200$

$-2c_1 = -200$

$\boxed{c_1 = 100}$

Sub c_1 in (1)

$100 + c_2 = 3000$

$c_2 = 3000 - 100$

$\boxed{c_2 = 2900}$

$\boxed{\therefore a_n = 100(3^n) + 2900 + 100n}$

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2) Solve $f_n = 3f_{n-1} + 10f_{n-2} + 7.5^n$, $n \geq 2$, $f_0 = 4$, $f_1 = 3$

Soln:

Given, $f_n - 3f_{n-1} - 10f_{n-2} = 7.5^n$

The given eqn is second order non-homogeneous recurrence relation with $f_n = 7.5^n$

The general solution is,

$a_n = a_n^{(h)} + a_n^{(p)}$.

To find $a_n^{(h)}$:

Put $f(n) = 0$

$f_n - 3f_{n-1} - 10f_{n-2} = 0$

$r^2 - 3r - 10 = 0$

$(r-5)(r+2) = 0$

$r_1 = 5,\ r_2 = -2$

$\therefore$ The roots are real and distinct.

$\therefore a_n^{(h)} = c_1(5)^n + c_2(-2)^n$

To find $a_n^{(p)}$:

Here $f(n) = 7.5^n$

It is of the form is $A.(5^n)$

Put $a_n = A.5^n$

$A.5^n - 3A.5^{n-1} - 10A.5^{n-2} = 7.5^n$

$\div$ both sides by 5^{n-1}

$A.5 - 3A - 10A.\frac{1}{5} = 7.5$

Since 5^n is a solution for the relation.

Put $f(n) = B.n.5^n$

$B.n.5^n - 3B(n-1).5^{n-1} - 10B(n-2)5^{n-2} = 7.5^n$

÷ both sides by 5^n

$B.n - 3B(n-1).\frac{1}{5} - 10.B.(n-2).\frac{1}{5^2} = 7$

$Bn - \frac{3Bn}{5} + \frac{3B}{5} - \frac{2Bn}{5} + \frac{4B}{5} = 7$

$Bn\left(1 - \frac{3}{5} - \frac{2}{5}\right) + \frac{7B}{5} = 7$

$0 + \frac{7B}{5} = 7$

$\boxed{B = 5}$

$\therefore a_n^{(P)} = 5.n.5^n$

$\boxed{\therefore a_n^{(P)} = n.5^{n+1}}$

$\therefore a_n = C_1(5)^n + C_2(-2)^n + n.5^{n+1}$

Given : $f_0 = 4, f_1 = 3$

$n=0, \quad a_0 = C_1 + C_2$

$\boxed{C_1 + C_2 = 4} \rightarrow (1)$

$n=1, \quad a_1 = 5C_1 - 2C_2 + 5^2$

$3 = 5C_1 - 2C_2 + 25$

$\boxed{5C_1 - 2C_2 = -22} \rightarrow (2)$

$(1) \times (-2)$

$\Rightarrow -2C_1 - 2C_2 = -8$

$(2) \Rightarrow 5C_1 - 2C_2 = -22$

$(-) \quad (+) \quad (+)$

$-7C_1 = 14$

$\boxed{C_1 = -2}$

Put $C_1 = -2$ in (1)

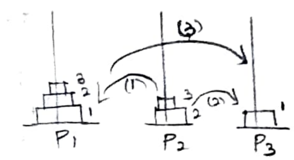
$-2 + C_2 = 4$

$\boxed{C_2 = 6}$

$\boxed{\therefore a_n = (-2)5^n + 6(-2)^n + n.5^{n+1}}$

3) (7m) (8m) Identify the recurrence relation for towers of Hanoi problem and hence solve it.

Solu:



Let a_n be the no. of moves it takes to transfer n disks from P_1 to P_3

For $(n+1)$ disks, we do the following.

(i) To transfer the top n disks from P_1 to P_2 as per the directions it takes a_n steps. (a_n)

(ii) Transfer the largest disks from P_1 to P_3. It takes one step. (1)

$n=0, a_0=0$
$n=1, a_1=1$

(iii) Transfer n disks from P_2 to largest disk on P_3 as per the directions. It takes a_n steps. (a_n)

The equation is 1st order Non-Homogeneous recurrence relation of the form,

$$a_{n+1} = a_n + 1 + a_n$$

$$a_{n+1} = 2a_n + 1$$

$$\boxed{a_{n+1} - 2a_n = 1}$$

where $f(n) = 1$

The general solution is,

$$a_n = a_n^{(h)} + a_n^{(P)}$$

To find $a_n^{(h)}$:

Put $f(n) = 0$

$$a_{n+1} - 2a_n = 0 \quad \rightarrow r - 2 = 0 \quad \boxed{r = 2}$$

$$\therefore a_n^{(h)} = C(2)^n$$

To find $a_n^{(P)}$:

$$a_{n+1} - 2a_n = 1$$

It is of the form $f(n) = A$.

$$a_n = A$$

$$A - 2A = 1$$

$$-A = 1$$

$$\boxed{A = -1}$$

$$\therefore a_n^{(P)} = (-1)$$

$$\therefore a_n = a_n^{(h)} + a_n^{(P)}$$

$$\therefore a_n = C(2)^n - 1$$

$\cancel{a_{n+1} = 2a_n + 1}$ $\quad a_0 = 0$

[$\because a_0 = 0$, it take 0 moves to transfer 0 disk, $n = 0$]

$$\Rightarrow a_0 = C(2)^0 - 1$$

$$0 = C - 1$$

$$\boxed{C = 1}$$

[$\because a_1 = 1$, it takes 1 moves to transfer 1 disk, $n = 1$]

$$\boxed{\therefore a_n = 2^n - 1}$$

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UNIT-IV PERMUTATIONS & COMBINATIONS

Permutations:-

An Ordered arrangement of objects from a set of distinct objects.

Notations: nP_r (or) ^nP_r where $n = $ no. of objects. $n \geq r$

$r = $ no. of times.

The general formula is,

$$nP_r = \frac{n!}{(n-r)!}$$

The general rules is,

$$nP_0 = 1, \quad nP_1 = n \qquad [0! = 1]$$

$$nP_n = n! \qquad [n! = (n-1)!\,n]$$

1) Evaluate:- (i) $P(4,3)$ (ii) $P(7,2)$ (iii) $P(12,3)$

(iv) $P(2, 4) \rightarrow$ Not Possible $n < r$

Sol:- (i) $P(4,3) = 4P_3$

$$= \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4! = 4 \times 3 \times 2 \times 1 = 24$$

(ii) $P(7,2) = {}^7P_2$

$= \frac{7!}{(7-2)!} = \frac{7!}{5!} = \frac{7\times6\times\cancel{5!}}{\cancel{5!}} = 42$

(iii) $P(12,3) = {}^{12}P_3$

$= \frac{12!}{(12-3)!} = \frac{12!}{9!} = \frac{12\times11\times10\times9!}{9!}$

$= 1320$

2) Find r if (i) ${}^6P_r = 360$ (ii) $P(13,r) = 156$

Solu:

(i) Given : ${}^6P_r = 360$

$360 = 6\times5\times4\times3$

$\boxed{r=4}$ (or)

$\left[\because {}^6P_4 = \frac{6!}{(6-4)!} = \frac{6\times5\times4\times3\times\cancel{2!}}{\cancel{2!}}\right]$

$\boxed{r=4}$

$$\begin{array}{r|l} 6 & 360 \\ \hline 5 & 60 \\ \hline 4 & 12 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

(ii) $P(13,r) = 156$:

$156 = 13\times12$

$\boxed{r=2}$

$$\begin{array}{r|l} 13 & 156 \\ \hline 12 & 12 \\ \hline & 1 \end{array}$$

3) Find n : (i) $P(n,2) = 72$ (ii) $P(n,4) = 42\, P(n,3)$

(iii) $P(n,2) = 90$, (iv) $2P(n,2) + 50 = P(2n,2)$

Solu:

(i) $P(n,2) = 72$

$n(n-1) = 72$

$n^2 - n = 72$

$n^2 - n - 72 = 0$

$(n-9)(n+8) = 0$

$n = 9, -8$

$\boxed{\therefore n=9}$

(ii) $P(n,4) = 42\, P(n,3)$

$[n(n-1)\cancel{(n-2)}(n-3)] = 42[n(n-1)\cancel{(n-2)}]$

$(n-3) = 42$

$n = 42+3$

$\boxed{n=45}$

(iii) $P(n,2) = 90$

$n(n-1) = 90$

$n^2 - n = 90$

$n^2 - n - 90 = 0$

$(n-10)(n+9) = 0$

$n = 10, -9$

$\boxed{\therefore n=10}$

(iv) $2P(n-2) + 50 = P(2n,2)$

$2[n(n-1)] + 50 = 2n(2n-1)$

$2[n(n-1)] + 50 = 2n(2n-1)$

$2n(n-1) + 50 = 2n(2n-1)$

$2n^2 - 2n + 50 = 4n^2 - 2n$

$50 = 2n^2$

$n^2 = \frac{50}{2}$

$n^2 = 25,\ n = \pm5$

$\boxed{\therefore n=5}$

4) For non-negative n and r if $n+1 \geq r$, Prove that $P(n+1, r) = (n+1/n+1-r) P(n, r)$

Solu:

Given:

non-negative n and r if $n+1 \geq r$

To prove: $P(n+1, r) = \frac{(n+1)!}{(n+1-r)!}$ $\left[\because P(n,r) = \frac{n!}{(n-r)!}\right]$

$= \frac{(n+1)}{(n+1-r)} P(n,r)$

$\therefore P(n+1, r) = \frac{(n+1)}{(n+1-r)} P(n,r)$

$\left[\because (n+1)! = (n+1)\, n! \qquad n! = n(n-1)!\right]$

$(n+1-r)! = (n+1-r)(n+1-r-1)!$

$= \frac{(n+1)\, n!}{(n+1-r)(n+1-r-1)!} = \frac{(n+1)}{(n+1-r)} \cdot \frac{n!}{(n-r)!}$

$(n+1-r-1) = (n-r)!$

$= \frac{(n+1)}{(n+1-r)} P(n,r)$

$\therefore$ Hence Proved.

Fundamental principle of counting:

1) Addition Principle:

If an Event E can Occur in m ways and another Event F can occur in n ways and suppose both cannot occur together, then E or F can occur in m+n ways (mutually Exclusive).

2) Multiplication Principle:

Suppose an Event E can Occur in m ways and associated with each occurrence of E, Another Event F can occur in n ways then the total no. of occurrence of two Events can be in order of mn ways (mutually Inclusive).

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1) A hostel library has 120 books on maths, 90 books on physics, 10 books on computer science & 10 books on chemistry. A student wishes to choose one of these books for study. Find the no. of ways he can choose a book.

Solu:

Borrowing a book out of the four choices is a mutually exclusive event. By rule of some

$120+90+10+10 = 230$ ways

2) A person has 10 shirts and 7 ties Find the no. of ways of choosing a shirt and a tie.

Solu:

choosing a shirt and a tie is a mutually Inclusive event. By rule of product the no. of way of choosing a start and the is 10×7 = 70 ways.

3) How many ways can we get a sum of 7 or 11 when two distinguishable dice are thrown.

Solu:-

No. of ways for getting a sum of 7 =

(1,6) (2,5) (3,4) (4,3) (5,2) (6,1)

= 6 ways.

No. of ways for getting a sum of 11 =

(6,5) (5,6) = 2 ways.

∴ The no. of way for getting a sum of 7 or 11

= 6+2

= 8 ways

∴ This is a mutually exclusive event.

4) How many four digit number divisible by 5 can be formed using the digits (3,7,1,5,6)

a) Repetition of digits is permitted.

b) Repetition of digits not permitted.

Solu:-

A no. is said to be divisible by 5, if its unit position has 5.

a) Repetition of digits permitted:-

No's available - 37156

1000's D_4	100's D_3	10's D_2	units D_1
5 ways	5 ways	5 ways	$\boxed{5}$ 1 way

* choosing unit as 5 there is only one way.

* choosing a number for 10's → all 5 no's are allowed

∴ Repetition Permitted = 5 ways.

* choosing no's for 100's → all 5 no's available = 5 ways.

* choosing a no. for 1000's → all 5 no's available = 5 ways

∴ No. of ways = 5×5×5×1

= 125

b) Repetition of digits not permitted:- [No repetition of no.]

1000's D_4	100's D_3	10's D_2	units D_1
2 ways	3 ways	4 ways	$\boxed{5}$ 1 way

* Units has to be 5
* No. of ways = 1 way.
* choosing 10's only 4 no's available = 4 ways (1 no excluded)
* choosing a no. for 100's, no. of ways = 3 (2 no's excluded)
* choosing a no. for 1000's, no. of ways = 2 (3 no's excluded)

∴ No. of ways = 2*3*4*1

= 24.

5) There are 6 candidates for classical, 3 for mathematical and 2 for natural science, scholarship.
(i) how many ways these scholarships be awarded.
(ii) how many ways one of these scholarships be awarded.

Sol:
(i) It is mutually inclusive.
∴ The no. of ways $= 6 * 3 * 2$
$= 36$

(ii) It is mutually exclusive
∴ The no. of ways. $= 6+3+2$
$= 11$

6) How many positive integers 'n' can be formed using digits 3, 4, 4, 5, 5, 6, 7, If we want to exceed 5,000,000

Sol:
3, 4, 4, 5, 5, 6, 7 > 50,00,000

∴ $D_7 \underbrace{D_6 D_5 D_4 D_3 D_2 D_1}_{\text{6 digits can be anything}}$

D_7 has to be 5, 6, 7

(i) If $D_7 = 5$
remaining we have:- 3, 4, 4, 5, 6, 7
2 No. of 4
1 No. of 3, 5, 6, 7

No. of ways $= \frac{6!}{2!1!1!1!1!} = 6 \times 5 \times 4 \times 3 = 360$

(ii) If $D_7 = 6$
remaining we have :- 3, 4, 4, 5, 5, 7
2 No. of 4, 5
1 No. of 3, 7

No. of ways $= \frac{6!}{2!2!1!1!} = 6 \times 5 \times 3 \times 2 = 180$

(iii) If $D_7 = 7$
remaining we have :- 3, 4, 4, 5, 5, 6
2 No. of 4, 5
1 No. of 3, 6

No. of ways $= \frac{6!}{2!2!1!1!} = 180$

∴ Total No. of ways $= 360 + 180 + 180 = 720$

7) Find the No. of permutations of letters of word MASSASAUGA
(i) In how many of this all 4 A's are together?
(ii) How many of them begin's with S?

Sol:
MASSASAUGA → 10 letter word

In this, 4 No. of A
3 No. of S
1 No. of M, U, G

No. of permutation $= \frac{10!}{4!3!1!1!1!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1}$
$= 25200$ ways

(i) All 4 A's are together [AAA] → treat this as 1 letter

remaining MSSSUGI

3 No. of S

1 No. of MUG

$$\text{No. of permutation} = \frac{7!}{3!1!1!1!}$$

$= 7 \times 6 \times 5 \times 4 = 840$ ways.

(ii) Begin with S :-

remaining MASSAAUGTA

4 No. of A

2 No. of S

1 No. of MUG

$$\text{No. of permutation} = \frac{9!}{4!2!1!1!1!}$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4!\,2!}$$

$= 9 \times 8 \times 7 \times 3 \times 5$

$= 7560$ ways.

8) How many arrangement are there for all the letters in the word SOCIOLOGICAL? In how many of these arrangements. a) A & G are adjacent b) all the vowels are adjacent.

Solu: SOCIOLOGICAL → No. of letters = 12

3 No. of O

2 No. of C, L, I

1 No. of S, G, A

$$\text{No. of arrangements} = \frac{12!}{3!2!2!2!1!1!1!}$$

$$= \frac{12 \times 10 \times 11 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!\,2 \times 2 \times 2}$$

$= 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 3 \times 5$

$= 9979200$

a) A and G are adjacent :-

∴ A and G can be treated as 1 letter.

3 No. of O

2 No. of C, L, I

1 No. of S

$$\text{No. of arrangements} = \frac{11!}{3!2!2!2!1!}$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 3!}{3!\,2 \times 2 \times 2 \times 1}$$

$= 11 \times 10 \times 9 \times 8 \times 7 \times 3 \times 5$

No. of arrangements $= 831600$

∴ Letters A and G can be arranged in 2 ways, A&G, G&A

No. of ways $= 831600 + 831600$

$= 1663200$ ways.

b) All the vowels are adjacent :-

vowel = a, e, i, o, u

In the given word, we have [A I O]

[balance Sum:- Pg No: 95]

13/8/19

1) It is required to seat 6 men and 5 women in a row so that women occupy even places. How many arrangements are possible.

Solu:-

women

2 3 4 5 6 7 8 9 10 11

No. of men = 6

No. of women = 5

Total no. of seats required = 6+5 = 11

Out of which; 2, 4, 6, 8, 10 will be allocated to women.

5 seats 5 women:-

No. of arrangement $= 5P_5$

$= 5!$

$= 120$

6 seats 6 men:-

No. of arrangements $= 6P_6$

$= 6!$

$= 720$

$\therefore$ Total No. of arrangements:-

$= 120 \times 720$

$= 86400$

2) How many permutations of a, b, c, d, e, f, g:- PLA

(i) End with a

(ii) Begin with c

(iii) Begin with c and End with a

(iv) c and a occupies End places.

{a, b, c, d, e, f}

Solu:-

No. of letters = 7 = 6

(i) End with a:-

No. of End places = 1, It has to be 'a'

No. of letters = 1, It can be arranged in $1P_1$ way = 1

Remaining positions = 6 = 5

No. of letters available = 6, = 5

No. of arrangements $= 6P_6$ $5P_5$

$= 6!$ $5!$

$= 720$ $= 120$

$\therefore$ Total arrangements $: 720 \times 1$ $= 120 \times 1$

$= 720$ $= 120$

(ii) Begin with c:-

No. of beginning position = 1, It has to be 'c'.

No. of letter = 1, It can be arranged in $1P_1 = 1$

Remaining position = 6 = 5

No. of available letters = 6 = 5

No. of arrangements $= 6P_6$ $= 5P_5$

$= 6!$ $= 5!$

$= 720$ $= 120$

$\therefore$ Total arrangement $= 720 \times 1$

$= 720.$

(iii) Begin with c and End with a:

Beginning place = 1 Position, it has to be 'c'.

No. of letter = 1

No. of arrangement $= {}^{1}P_{1}$

$= 1$

End place = 1 position, it has to be 'a'

No. of letter = 1

No. of arrangements $= {}^{1}P_{1}$

$= 1$

Remaining places = 5, = 4

No. of letters available = 5 = 4

No. of arrangements $= {}^{5}P_{5} = {}^{4}P_{4}$

$= 5! = 4!$

$= 120 = 24$

$\therefore$ Total arrangements $= 1 \times 120 \times 1 = 1 \times 24 \times 1$

$= 120 = 24$

(iv) c and a occupies End places:

last two places must be 'a' and 'c'

No. of positions = 2

No. of letters = 2

No. of arrangements $= {}^{2}P_{2}$

$= 2! = 2$

Remaining position = 5 = 4

No. of letters available = 5 = 4

No. of arrangements $= {}^{5}P_{5} = {}^{4}P_{4}$

$= 5! = 4!$

$= 120 = 24$

$\therefore$ Total No. of arrangements $= 2 \times 120 = 2 \times 24$

$= 240 = 48$

Combinations:

It is the selections of objects without regard to order. It is denoted by,

$${}^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$$

1) Find (i) $C(10,4)$

(ii) $\binom{12}{7}$

Solu:

(i) $C(10,4)$

$= {}^{10}C_{4}$

$= \frac{10!}{(10-4)!\,4!}$

$= \frac{10!}{6!\,4!} = \frac{10 \times 9^{3} \times \cancel{8}^{4} \times 7 \times \cancel{6!}}{\cancel{6!} \times 1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} = 210$

(ii) $\binom{12}{7} = {}^{12}C_{7} = \frac{12!}{5!\,7!} = \frac{12 \times 11 \times \cancel{10}^{2} \times \cancel{9}^{3} \times \cancel{8}^{2} \times \cancel{7!}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1 \times \cancel{7!}}$

$= 792$

2) Find the no. of arrangements of all letters in the word "TALLAHASSEE". How many of these arrangements have no adjacent A's.

Solu:-

No. of letters = 11

Out of which, there are

3 = A's,

2 = L, S, E

1 = T, H

$\therefore$ No. of arrangements

$$= \frac{11!}{3! \times 2! \times 2! \times 2! \times 1! \times 1!}$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3! \times 2 \times 2 \times 2 \times 1}$$

990
840
39600
792
831600

= 831600 ways

If we disregard A's, we have 8 letters

they can be arranged in

$$= \frac{8!}{2! \times 2! \times 2! \times 1! \times 1!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2 \times 2 \times 2!}$$

280
18
2240
280
5040

= 5040 ways

The possible ways A's can be placed so that they are not adjacent is given below:-

↑ T ↑ L ↑ L ↑ H ↑ S ↑ S ↑ E ↑ E ↑

No. of positions = 9

No. of letters (A) = 3

In this 9 places, the 3 A's can be placed in $9C_3$ ways

$$= \frac{9!}{(9-3)!\,3!} = \frac{9!}{6!\,3!}$$

$$= \frac{9 \times 8 \times 7 \times 6!}{6! \times 3 \times 2 \times 1}$$

= 84 ways

5040
84
20160
40320
423360

$\therefore$ Total No. of arrangements, so that no A's are adjacent = 5040 × 84

= 423360 ways

balance sum 8) b)

make them together as 1 letter, remaining,

2 No. of C, L

1 No. of S, G

No. of arrangements $= \frac{7!}{2!\,2!\,1!\,1!}$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3}{2}$$

= 1260 ways

9) How many 3 letter word can be form out of the

word TRIANGLE?

Solu

TRIANGLE → No. of letters - 8

∴ No. of 3 letter words that can be formed

$= {}^8P_3$

$= \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8\times7\times6\times5!}{5!}$

$= 8\times42 = 336$ ways.

10) How many 4 letter word with or without meaning can be formed of the letters of the word LOGARITHMS?

Solu

No. of letter in LOGARITHMS = 10

∴ No. of 4 letter words that can be formed

$= {}^{10}P_4$

$= \frac{10!}{(10-4)!} = \frac{10\times9\times8\times7\times6!}{6!}$

$= 10\times9\times8\times7$

$= 90\times56 = 5040$ ways.

16/8/19 3) How many arrangements of letters in the word MISSISSIPPI will have no consecutive Ss.

Solu:

No. of letters = 11

There are 4 S and I, 2 → P, 1 → M

∴ No. of arrangements $= \frac{11!}{4!\,4!\,2!\,1!}$

$= \frac{11\times10\times9\times8\times7\times6\times5\times4!}{4!\times4\times3\times2\times1\times2\times1}$

$= 34650$.

If we disregard S. we have 7 letters, they can be arranged in

$= \frac{7!}{4!\,2!\,1!}$

$= \frac{7\times6\times5\times4!}{4!\times2\times1}$

$= 105$.

↑ M ↑ I ↑ I ↑ I ↑ P ↑ P ↑ I ↑

There are 8 possible positions for placing 'S' so that they will not be consecutive.

∴ No. of arrangements can be

$= 105 * {}^8C_4$

$= 105 * \frac{8!}{4!\,4!}$

$= 105 * \frac{8\times7\times6\times5\times4!}{4!\times4\times3\times2\times1}$

$= 105 * 70$

$= 7350$ ways

Binomial Theorem:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r \cdot y^{n-r}$$

1) Determine the co-efficients of x^5y^2 in the expansion of $(x+y)^7$

Solu:

$$(x+y)^7 = \sum_{r=0}^{7} \binom{7}{r} x^r y^{7-r}$$

co-efficient of x^5y^2, $r=5$

$\therefore$ coefficient of $x^5y^2 = 7C_5 \cdot x^5 \cdot y^{7-5}$

$$= \frac{7!}{5!\,2!} x^5y^2$$

$$= \frac{7\times 6\times 5!}{5!\,2\times 1}$$

$$= 21$$

2) Find the co-efficients of a^5b^2 in the expansion of $(2a-3b)^7$

Solu:

$$(2a-3b)^7 = \sum_{r=0}^{7} \binom{7}{r} (2a)^r (-3b)^{7-r}$$

co-efficient of a^5b^2, $r=5$

$\therefore$ coefficient of $a^5b^2 = 7C_5 (2a)^5 (3b)^{7-5}$

$$= \frac{7\times 6\times 5!}{5!\,2!} (32)a^5(9)b^2$$

189
32
378
567
6048

$= (21\times 32\times 9)a^5b^2$

$= 6048$

3) Find the co-efficients of x^9y^3 in $(2x-3y)^{12}$

Solu:

$$(2x-3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} (2x)^r (-3y)^{12-r}$$

32×4
128

co-efficients of x^9y^3, $r=9$

$\therefore$ co-efficients of $x^9y^3 = 12C_9 (2x)^9 (-3y)^{12-9}$

20×11
20
20
220

$$= \frac{12\times 10\times 11\times 9!}{9!\,3!} (512)x^9(-27)y^3$$

$= (2\times 10\times 11)(512)x^9(-27)y^3$

$= (220\times 512\times -27)x^9y^3$

$= ~~3082860~~ -3041280$

4) A committee of 12 is to be selected from 10 men and 10 women. In how many ways the selection can be carried out if

a) No restriction.

b) there must be 6 men & 6 women.

c) there must be even no. of women.

d) there must be more women than men.

e) there must be atleast 8 men.

Solu:

No. of men = 10
No. of women = 10 $\therefore$ Total = 10+10 = 20

comittee size = 12

a) No restriction:-

12 has to be selected from 20:-

$$20C_{12} = \frac{20\times19\times18\times17\times16\times15\times14\times13\times12!}{8!\,12!}$$

$8\times7\times6\times5\times4\times3\times2$

$= 19\times3\times17\times2\times15\times13$

$= 19\times17\times30\times13$

$= 125970$

b) 6 men and 6 women:-

$$6\text{ men} \rightarrow 10C_6 = \frac{10!}{4!\,6!} = \frac{10\times9\times8\times7\times6!}{6!\times4\times3\times2\times1}$$

$= 210$

$6\text{ women} \rightarrow 10C_6 = 210$

$\therefore$ 6 men & 6 women $= 210 * 210$

$= 44100$

c) Even no. of women:- (2,4,6,8,10)

$$\sum_{q=1}^{5} \underset{\text{Men}}{10C_{12-2q}} \cdot \underset{\text{women}}{10C_{2q}}$$

$$= \underset{M}{10C_{10}} \cdot \underset{W}{10C_2} + \underset{M}{10C_8} \cdot \underset{W}{10C_4} + \underset{M}{10C_6} \cdot \underset{W}{10C_6} + \underset{M}{10C_4} \cdot \underset{W}{10C_8} + \underset{M}{10C_2} \cdot \underset{W}{10C_{10}}$$

d) More women than men:-

$$\sum_{q=2}^{5} \underset{\text{women}}{10C_{12-q}} \cdot \underset{\text{Men}}{10C_q}$$

	W	M
q=2,	10	2
q=3,	9	3
q=4,	8	4
q=5,	7	5

$$= 10C_{10} \cdot 10C_2 + 10C_9 \cdot 10C_3 + 10C_8 \cdot 10C_4 + 10C_7 \cdot 10C_5$$

e) Atleast 8 men:-

$$\sum_{q=8}^{10} \underset{\text{Men}}{10C_q}\ \underset{\text{women}}{10C_{12-q}}$$

	M	W
q=8,	8	4
q=9,	9	3
q=10	10	2

$$= 10C_8 \cdot 10C_4 + 10C_9 \cdot 10C_3 + 10C_{10} \cdot 10C_2$$

5) How many bytes contain
(i) exactly 2 one's
(ii) exactly 4 one's
(iii) exactly 6 one's
(iv) Atleast 6 one's.

Soln:

No. of bytes = 8.

(i) exactly 2 one's:-

$$8C_2 = \frac{8!}{6!\,2!} = \frac{8\times7\times6!}{6!\times2}$$

$= 28$

(ii) exactly 4 one's:-

$$8C_4 = \frac{8!}{4!\,4!} = \frac{8\times7\times6\times5\times4!}{4!\times4\times3\times2}$$

$= 2\times7\times5 = 70$

(iii) exactly 6 one's:-

$$8C_6 = \frac{8!}{2!\,6!} = \frac{8\times7\times6!}{6!\times2} = 28.$$

(ii) Atleast 6 one's:

$$\sum_{i=6}^{8} 8C_i$$

$= 8C_6 + 8C_7 + 8C_8$

$= 28 + 8 + 1$

$= 37$

6) A student is to answer 7 out of 10 question on exam. In how many ways he can make this selection

(i) No restrictions. - 120

(ii) He must answer 1^{st} 2 questions. - 56

(iii) He must answer atleast 4 out of 1^{st} 6 questions. - 100

Solu:

No. of questions = 10

(i) No restrictions:

Answer 7 out of 10:

$$10C_7 = \frac{10!}{3!\,7!}$$

$$= \frac{10 \times 9 \times 8 \times 7!}{3 \times 2 \times 7!}$$

$= 120$

(ii) Answer 1^{st} 2 questions:

Answer 1^{st} two questions:

$$8C_5 = \frac{8!}{3!\,5!}$$

$$= \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 5!}$$

$= 56$

(iii) Answer atleast 4 out of 1^{st} 6 questions:

Answer atleast 4 out of 1^{st} 6 questions:

$= 6C_4 \cdot 4C_3 + 6C_5 \cdot 4C_2 + 6C_6 \cdot 4C_1$

$= (15 \times 4) + (6 \times 6) + (1 \times 4)$

$= (60) + (36) + (4)$

$= 100$

7) A gym teacher of a school must arrange 4 volleyball team of 9 girls each from 36 fresh women girls in her physical Education class. How many ways she can select the 4 teams A, B, C, D.

Solu:

No. of girls = 36

Each team = 9 girls:

Team A:

$$36C_9 = \frac{36!}{27!\,9!}$$

Team B:

$$27C9 = \frac{27!}{18!9!}$$

Team C:

$$18C9 = \frac{18!}{9!9!}$$

Team D:

$$9C9 = \frac{9!}{9!0!}$$

$$= \frac{36!}{27!9!} * \frac{27!}{18!9!} * \frac{18!}{9!9!} * \frac{9!}{9!9!}$$

$$= \frac{36!}{9!*9!*9!*9!} \text{ ways}$$

Principle of Inclusion and Exclusion:

* If every Element of a set A is also an element of set B, we say A is included in B, and we write, $A \subseteq B$.

* A set A is called a proper set of B if $A \subseteq B$ and $A \neq B$.

* n(A) is called cardinal No. of set A

$$\boxed{\begin{aligned} &n(A\cup B) = n(A)+n(B)-n(A\cap B) \\ &n(A\cup B\cup C) = n(A)+n(B)+n(C)-n(A\cap B)-n(A\cap C)- \\ &\quad n(B\cap C)+n(A\cap B\cap C)\end{aligned}}$$

10) Solve the recurrence relation : $a_n = 7a_{n+1}$, $n \geq 1$, $a_2 = 98$

Solu:

$a_n = 7a_{n+1}$

$n=1,\ a_1 = 7a_2$ $\qquad n=2,\ a_2 = 7a_3$

$n=3,\ a_3 = 7a_4 = 7(a_2)_2$

$n=4,\ a_4 = 7a_5 = 7(a_2)_3 \ldots$

The general solution is,

$a_n = 7(a_2)_{n+1}$

$\therefore a_n = 7(98)_{n+1}$, $n \geq 1$.

$7a_{n+1} - a_n = 0$

$a_n = (7)^n a_0$, $n \geq 1$

$n=2$, $a_2 = (7)^2 a_0$

$98 = (7)^2 a_0$

$\frac{98}{49} = \boxed{a_0 = 2}$

$\boxed{a_n = 2\times(7)^n,\ n \geq 1}$

9) Solve the recurrence relation : $a_n = n.a_{n-1}$, $n \geq 1$, $a_1 = 1$

Solu:

$a_n = n.a_{n-1}$

$n=1,\ a_1 = 1a_0$

$n=2,\ a_2 = 2.a_1 = 2\times 1a_0 = 2a_0 = 2\times 1\ a_0$

$n=3,\ a_3 = 3a_2 = 3(2a_0) = 6a_0 = 3\times 2\times 1\ a_0$

$n=4,\ a_4 = 4a_3 = 4(6a_0) = 24a_0 = 4\times 3\times 2\times 1\ a_0$

The general solution is,

$a_n = (n!)a_0$, $n \geq 1$

8) Find 'n' $P(n,3) = 3P(n,2)$

Solu:

$[\cancel{n(n-1)}(n-2)] = 3[\cancel{n(n-1)}]$

$n-2 = 3$

$n = 5$

$\therefore \boxed{n=5}$

7) Solve the recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = 0$, $a_0 = 5, a_1 = 1, n \geq 2$.

Solu:

Given $a_n - 6a_{n-1} + 9a_{n-2} = 0$

The characteristic Equation is,

$$r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0$$

$$r_1 = 3, \ r_2 = 3$$

$\therefore$ The roots are real and ~~distinct~~ Equal

The general solution is,

$$a_n = (A+Bn)r^n$$

$$a_n = (A+Bn)(3)^n$$

Given: $a_0 = 5$, $a_1 = 1$, $n=0$

$$a_0 = (A+B(0))3^0$$

$$\boxed{5 = A}$$

$a_1 = 1$, $n = 1$

$$a_1 = (A+B)3$$

$$a_1 = 3A + 3B$$

$$3A + 3B = 1$$

$$3(5) + 3B = 1$$

$$15 + 3B = 1$$

$$3B = -14$$

$$\boxed{B = \frac{-14}{3}}$$

$\therefore$ The general solution is,

$$\therefore a_n = \left(5 + \left(-\frac{14}{3}\right)n\right)(3)^n, \quad n \geq 0.$$

6) How many permutation of size 3 can one produce with letters m, r, a, f and t.

Solu:

permutations of size 3 out of 5 letters

$$= 5P_3$$

$$= \frac{5!}{(5-2)!} = \frac{5 \times 4 \times 3 \times 2!}{2!} = 60$$

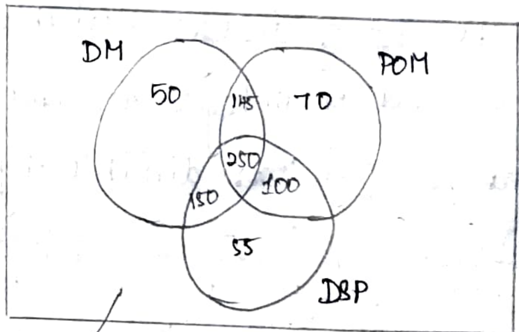
1) A survey conducted among 1000 students, 595 like DM, 565 like POM, 550 like DSP, 395 like DM and POM, 350 like POM and DSP, 400 like DSP and DM, and 250 like all the 3.

1) Find the no. of students who like atleast one of the subject?

2) Find the no. of students who like POM and don't like DM and DSP?

3) How many don't like all the 3 subjects?

Solu:



(i) 815 - Add all the Numbers.

(ii) 70 - Only POM.

(iii) 185.

2) How many positive integers not exceeding 500 are divisible by 7 or 11.

Solu:

Let A be the set of numbers divisible by 7.

Let B be the set of numbers divisible by 11.

then, $|A| = \left|\frac{500}{7}\right| = 71$

$|B| = \left|\frac{500}{11}\right| = 45$

$|A \cap B| = \left|\frac{500}{7 \times 11}\right| = 6$

$\therefore$ No. of integers divisible by 7 or 11 is

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 71 + 45 - 6$

$= 110.$

3) Determine the no. of positive integers between 1 and 250 that are not divisible by 2, 3 or 5.

Solu:

Let A be the set of integers divisible by 2.

Let B be the set of integers divisible by 3.

Let C be the set of integers divisible by 5.

$|A| = \left|\frac{250}{2}\right| = 125$, $|B| = \left|\frac{250}{3}\right| = 83$, $|C| = \left|\frac{250}{5}\right| = 50$

$|A \cap B| = \left|\frac{250}{2 \times 3}\right| = 41$, $|B \cap C| = \left|\frac{250}{3 \times 5}\right| = \left|\frac{250}{15}\right| = 14$,

$|A \cap C| = \left|\frac{250}{2 \times 5}\right| = \left|\frac{250}{10}\right| = 25$

$|A \cap B \cap C| = \left|\frac{250}{2 \times 3 \times 5}\right| = \left|\frac{250}{30}\right| = 8$

$\therefore$ No. of integers that are divisible by 2, 3 or 5 is

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 125 + 83 + 50 - 41 - 14 - 25 + 8$

$= 184.$

$\therefore$ No. of positive integers not divisible by 2, 3 or 5

$= 250 - 184$

$= 66.$

4) Determine the no. of positive integer n, $1 < n < 2000$, that are a) Not divisible by 2 or 3 or 5.

b) Not divisible by 2, 3, 5 or 7.

Solu:

Let A be the set of positive integers divisible by 2.

Let B be the set of positive integers divisible by 3.

Let C be the set of positive integers divisible by 5.

Let D be the set of positive integers divisible by 7.

$|A| = \left|\frac{2000}{2}\right| = 1000$ $|B| = \left|\frac{2000}{3}\right| = 666$

$|C| = \left|\frac{2000}{5}\right| = 400$ $|D| = \left|\frac{2000}{7}\right| = 283.$

$|A \cap B| = \left|\frac{2000}{6}\right| = 333$ $|A \cap C| = \left|\frac{2000}{10}\right| = 200$

$|A\cap D| = \left\lfloor \frac{2000}{14} \right\rfloor = 144$ $\quad |B\cap C| = \left\lfloor \frac{2000}{15} \right\rfloor = 133$

$|B\cap D| = \left\lfloor \frac{2000}{21} \right\rfloor = 95$ $\quad |C\cap D| = \left\lfloor \frac{2000}{35} \right\rfloor = 57$

$|A\cap B\cap D| = \left\lfloor \frac{2000}{42} \right\rfloor = \left\lfloor \frac{2000}{42} \right\rfloor = 47.61$

$|B\cap C\cap D| = \left\lfloor \frac{2000}{3\times5\times7} \right\rfloor = 19$

$|A\cap B\cap C\cap D| = \left\lfloor \frac{2000}{210} \right\rfloor = 9.52 = 9$ $\quad |A\cap B\cap C| = \left\lfloor \frac{2000}{2\times3\times5} \right\rfloor = 66$

∴ No. of integers that are divisible by 2,3,5,7 is

$[\,|A\cup B\cup C\cup D| = |A|+|B|+|C|+|D| - |A\cap B| - |A\cap C| - |A\cap D| - |B\cap C| - |B\cap D| - |C\cap D| + |A\cap B\cap C\cap D|.$

$= 1000+666+400+283-334-200-144-133-95-57+9$

$= 1395\,]$ ✗ → Ans: 867

a) Not divisible by 2,3,5:

$= 1000+666+400-334-200-133+66$

$|A\cap B\cap C| = \left\lfloor \frac{2000}{30} \right\rfloor = 66 \quad = 1466$

$= 2000 - 1466 = 534$

b) Not divisible by 2,3,5,7: – Ans on (115)

Ans:

* $|A\cup B\cup C\cup D| = |A|+|B|+|C|+|D| - |A\cap B| - |A\cap C| - |A\cap D| - |B\cap C| - |B\cap D| - |C\cap D| + |A\cap B\cap C| + |B\cap C\cap D| - |A\cap B\cap C\cap D|$

$= 1000+666+400+285-333-200-142-133-95-57+66+19-9 = 867$

$|A\cup B\cup C\cup D|$

balance sum .115

4) Determine the co-efficient of a) xyz^2 in $(x+y+z)^4$ b) xyz^2 in $(w+x+y+z)^4$ c) xyz^2 in $(2x-y-z)^4$ d) xyz^{+1} in $(x-2y+3z^{-1})^4$ e) $w^3x^2yz^2$ in $(2w-x+3y-2z)^8$

Soln:

a) xyz^2 in $(x+y+z)^4$:

$4C_{1,1,2} = \frac{4!}{1!1!2!} = \frac{4!}{2!} = \frac{24}{2} = 12$

b) xyz^2 in $(w+x+y+z)^4$:

$4C_{1,1,2} = \frac{4!}{1!1!2!} = \frac{4!}{2!} = \frac{24}{2} = 12$

c) xyz^2 in $(2x-y-z)^4$:

$4C_{1,1,2} * (2)^1(-1)^1(-1)^2$

$= \frac{4!}{1!1!2!} * 2*(-1)*(1)$

$= \boxed{\frac{4!}{2!}} * 12 * (2) * (-1)$

$= -24$

d) xyz^2 in $(x-2y+3z-1)^4$:

$= 4C_{1,1,2} * (1)^1(-2)^1(3)^2$

$= \frac{4!}{1!1!2!} * 1*-2*9 = +12*(-18) = -216$

e) $w^3x^2yz^2$ in $(2w-x+3y-2z)^8$

$= 8C(3,2,1,2 * (2)^3(-1)^2(3)^1(-2)^2$

$= \frac{8!}{3!2!1!2!} * 8*1*3*4$

$= \frac{8\times7\times6\times5\times4\times3!}{3!\ 2\times2} * 96$

$= 161280.$

5) Determine the sum of all the co-efficients in the expansion of

a) $(x+y)^3$ b) $(x+y)^{10}$ c) $(x+y+z)^{10}$ d) $(w+x+y+z)^5$

e) $(2x+y+3z)^3$

Soln:

$(x+y)^n = \sum_{r=0}^{n} nC_{n-r}\, x^r y^n$

a) $(x+y)^3$:-

$(x+y)^3 = \sum_{r=0}^{3} 3C_{3-r}\, x^r y^{3-r}$

$= 3C_3 x^0 y^3 + 3C_2 x^1 y^2 + 3C_1 x^2 y^1 + 3C_0 x^3 y^0$

$= 1.y^3 + 3xy^2 + 3x^2y + 1x^3$

$= 1+3+3+1$

Another Method:

$(2^3) = 8.$

b) $(x+y)^{10}$:-

$(x+y)^{10} = \sum_{r=0}^{10} 10C_{10-r}\, x^r y^{10-r}$

$= 10C_{10} x^0 y^{10} + 10C_9 x^1 y^9 + 10C_8 x^2 y^8 + 10C_7 x^3 y^7 +$

$10C_6 x^4 y^6 + 10C_5 x^5 y^5 + 10C_4 x^6 y^4 + 10C_3 x^7 y^3 +$

$10C_2 x^8 y^2 + 10C_1 x^9 y^1 + 10C_0 x^{10} y^0$

$= 1y^{10} + 10xy^9 + 45x^2y^8 + 120x^3y^7 + 210x^4y^6 + 252x^5y^5 +$

$210x^6y^4 + 120x^7y^3 + 45x^8y^2 + 10x^9y + 1x^{10}$

$= (1+10+45+120+210+252+210+120+45+10+1)$

$= 1024.$

Another Method:

$(x+y)^{10} = (1+1)^{10} = (2)^{10} = 1024.$

c) $(x+y+z)^{10}$:-

$(x+y+z)^{10} = (3)^{10} = 59049$

d) $(w+x+y+z)^5$:-

$(w+x+y+z)^5 = \sum_{r=0}^{5} 5C_{5-r}\, x^r y^{5-r}$

$= 5C_5 x^0 y^5 + 5C_4 x^1 y^4 + 5C_3 x^2 y^3 + 5C_2 x^3 y^2 + 5C_1 x^4 y^1 +$

$5C_0 x^5 y^0$

$= 1y^5 + 5xy^4 + 10x^2y^3 + 10x^3y^2 + 5x^4y + 1x^5$

$= 1+5+10+10+5+1 =$

Another Method:

$(w+x+y+z)^5 = (4)^5 = 1024.$

e) $(2x+y+3z)^3$

$(2x+y+3z)^3 = (2+1+3)^3 = (6)^3 = 216$

Derangements:

A derangement of objects in a finite set S is a permutation arrangement of elements so that no element appear in its original position.

The Number of derangement is given by

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

1) Find the derangement of 1,2,3,4 :-

Solu:

$n=4$

$$D_4 = 4! \left(\frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!}\right) + \left(\frac{(-1)^4}{4!}\right)$$

$$= 4! \left[1 + (-1) + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)$$

$$= 24\left(\frac{12-4+1}{24}\right)$$

$$D_4 = 9$$

[∴ 1234

{2,1,4,3   4,1,2,3

2,3,4,1   4,3,1,2

2,4,1,3   4,3,2,1 }]

3,1,4,2

3,4,1,2

3,4,2,1

2) Find the derangement of 1,2,3 :-

Solu: $n=3$

$$D_3 = 3! \left[\frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!}\right]$$

$$= 6\left[1-1+1/2-1/6\right]$$

$$D_3 = 6\left[\frac{3-1}{6}\right] = 2$$

[∴ Derangement of 1,2,3 :-

2,3,1

3,1,2]

3) Find the derangement of 1,2,3,4,5,6 :

Solu:

$n=6$

$$D_6 = 6! \left[\frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \frac{(-1)^5}{5!} + \frac{(-1)^6}{6!}\right]$$

$$= 720\left[1-1+1/2-1/6+1/24-1/120+1/720\right]$$

$$= 720\left[\frac{360-120+30-6+1}{720}\right]$$

$$D_6 = 265$$

balance sum of (4)

* $|A\cup B\cup C| = |A|+|B|+|C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|.$

$= 1000+666+400-333-200-133+66$

$\therefore |A\cup B\cup C| = 1466$

a) No. of positive integer Not divisible by 2,3,5:

= 2000 − 1466

= 534.

b) No. of positive integer Not divisible by 2,3,5,7:

= 2000 − 867

= 1133.

22/8/19

4) 5 gentlemen A, B, C, D, F attend a party were before joining the party they leave their overcoat in a cloak room. After the party, the overcoats got mixed up and are returned to the gentleman in a random manner. Find the no. of ways can the coats be returned to them such that no gentlemen will get his own coat. Also find the probability that none receives his own coat.

Solu:

Given, $n = 5$

None of the 5 gentlemen gets his own coat is called derangements

$$D_5 = 5!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$$

$$= 120\left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right]$$

$$= \cancel{120}\left[\frac{60 - 20 + 5 - 1}{\cancel{120}}\right] = 44$$

Probability that none receives his own coat

= No. of permutations in which none gets his own coat / No. of all permutations of the coat

$$= D_5 / 5!$$

$$= \frac{44}{120} = \frac{44}{120} = \frac{11}{30}$$

$= 0.3666 = 0.367$

5) How many derangements of {1,2,3,4,5,6} begin with 1,2,3 in some order and end with 1,2,3 in some order.

Solu:

Given: {1,2,3,4,5,6}

Begin with 1,2,3:

1) 231 564

2) 312 645

$\left.\begin{array}{l} * \; 231 \; 564 \\ * \; 231 \; 645 \\ * \; 312 \; 564 \\ * \; 312 \; 645 \end{array}\right\} = 4$ derangements

End with 1,2,3:

1) 456 123

2) 465 123

3) 564 123

* 123
* 132
* 213
* 231
* 312
* 321

4) 5 4 6   123

5) 6 4 5   123

6) 6 5 4   123

Similarly for 132, 213, 231, 312, 321

∴ No. of derangements = 36

∴ The last 3 position can be =

{123, 132, 213, 231, 312, 321} = 6

∴ The first 3 position can be =

{456, 465, 564, 546, 645, 654} = 6

∴ No. of derangements = 6×6 = 36

26/8/19 Arrangements with forbidden positions:

* A board is a subset of squares of an n by n [n×n] chessboard.

* Given: a board B, let $r_k(B)$ denote the no. of ways to place K rooks on B, so that no two rooks are in the same row or column.

* Such rooks are said to be non-attacking. The rook polynomial of B is $r_B(K) = r_0(B) + r_1(B)x + r_2(B)x^2 + r_3(B)x^3 + \dots + r_n(B)x^n$ for a board B $r_0(B) = 1$ and $r_1(B)$ = No. of squares in B.

Note:-

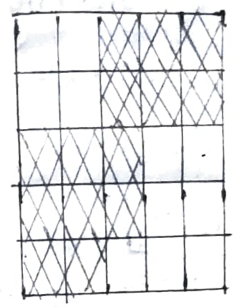
1) The rook polynomial of n by n [n×n] board is

$$\sum_{K=0}^{n} K!\,[nC_K]^2 . x^K$$

2) In general, let B be a board. Suppose a squares in B can be partitioned into pairwise disjoint sub boards $B_1, B_2, \dots B_n$ then

$$\therefore r_B(x) = r_{B_1}(x) . r_{B_2}(x) . r_{B_3}(x) \dots r_{B_n}(x)$$

1) Find the rooks polynomial for the given chess board where the shaded portions are not the part of the chess board.



Sol:

There are 11 unshaded squares which are the part of the chess board B.

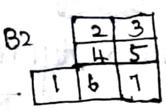
B consist of two sub Boards B1 and B2 where B1 is 2×2 sub boards and B2 consists of 7 squares

B1:-

1	2
3	4

2×2

B2



$\therefore r_B(x) = r_{B_1}(x) . r_{B_2}(x)$

Also, $r_B(x) = \sum_{K=0}^{2} K!\,[2C_K]^2 x^K$

$= 0!\,[2C_0]^2 x^0 + 1!\,[2C_1]^2 x^1 + 2!\,[2C_2]x^2$

$\therefore r_{B_1}(x) = 1 + 4x + 2x^2$

$r_{B2}(x) = r_0(B_2) + r_1(B_2)x + r_2(B_2)x^2 + r_3(B_2)x^3$

where, $r_0(B_2) = 1$

$r_1(B_2) = 7$

$r_2(B_2)$ = No. of 2 rooks can be placed that are not attacking each other.

$= \{(1,2)\ (1,3)\ (1,4)\ (1,5)\ (2,5)\ (2,7)\ (3,4)\ (3,6)\ (4,7)\ (5,6)\} = 10$

$r_3(B_2)$ = No. of 3 rooks can be placed that are not attacking each other.

$= \{(1,2,5), (1,3,4)\} = 2$

$\therefore r_{B2}(x) = 1+7x+10x^2+2x^3$

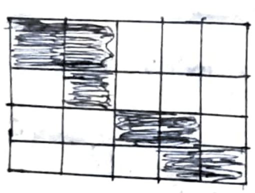
$\boxed{\therefore r_B(x) = r_{B1}(x) \cdot r_{B2}(x)}$

$= (1+4x+2x^2)(1+7x+10x^2+2x^3)$

$= 1+7x+10x^2+2x^3+4x+28x^2+40x^2+8x^4+$
$2x^2+14x^3+20x^4+4x^5$

$\boxed{\therefore r_B(x) = 1+11x+40x^2+56x^3+28x^4+4x^5}$

2) A board consist of a shaded part of the figure. Find the rooks polynomial.



Soln:- Here the Board B consist of 7 shaded squares.

The Board B has 2 sub boards B1 and B2 where B1 has 3 shaded squares and B2 has 4 shaded squares.

B1:-  B2:- 

$r_B(x) = r_{B1}(x) \cdot r_{B2}(x)$

To find $r_{B1}(x)$:-

$\therefore r_{B1}(x) = r_0(B_1) + r_1(B_1)x + r_2(B_2)x^2$

$r_0(B_1) = 1$

$r_1(B_1) = 3$

$r_2(B_1) = \{(1,3)\} = 1$

$\boxed{\therefore r_{B1}(x) = 1+3x+1x^2}$

To find $r_{B2}(x)$:-

$\therefore r_{B2}(x) = r_0(B_2) + r_1(B_2)x + r_2(B_2)x^2 + r_3(B_2)x^3$

$r_0(B_2) = 1$

$r_1(B_2) = 4$

$r_2(B_2) = \{(1,3)\ (1,4)\ (2,4)\} = 3$

$r_3(B_2) = 0$

$\boxed{\therefore r_{B2}(x) = 1+4x+3x^2}$

$\therefore r_B(x) = r_{B1}(x) \cdot r_{B2}(x)$

$= (1+3x+x^2)(1+4x+3x^2)$

$= 1+4x+3x^2+3x+12x^2+9x^3+x^2+4x^3+3x^4$

$\boxed{\therefore r_B(x) = 1+7x+16x^2+13x^3+3x^4}$

3/9/19

1) write a note on arrangements with forbidden positions:

* Suppose m objects are to be arranged in n places where $n \geq m$.
* Suppose that there are constraints under which some objects cannot occupy certain places, such places are called the forbidden positions for the objects.
* Eg:- we may need to match applicants to jobs, where some of the applicants cannot hold certain jobs.

2) To find the no. of arrangements with forbidden positions we use rook R polynomial.

* The No. of ways of carrying out the arrangements with forbidden position is given by

$$\overline{N} = S_0 - S_1 + S_2 - S_3 \ldots\ldots (-1)^n S_n$$

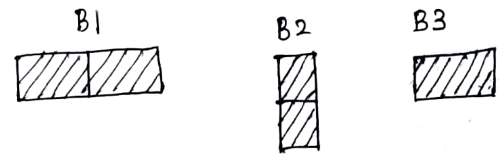
where $S_0 = n!$ and

$S_K = (n-K)! * (r_K)$ where R_K = coefficient of x^K in the rook's polynomial.

3) An apple, a banana, a mango and an orange have to be distributed to 4 boys B1, B2, B3, B4. The boys B1 and B2 do not wish to have apple, the boy B3 does not want banana or mango and the boy B4 refuses orange. How many ways the distribution can be made so that no boy is dissatisfied.

sdu:

	B1	B2	B3	B4
apple				
Banana				
mango				
orange				



$$r_B(x) = r_{B1}(x) * r_{B2}(x) * r_{B3}(x)$$

To find $r_{B1}(x)$:-

$$r_{B1}(x) = r_0(B_1) + x \cdot r_1(B_1)$$

$$= 1 + x \cdot 2$$

$$\therefore r_{B1}(x) = 1 + 2x$$

To find $r_{B2}(x)$:-

$$r_{B2}(x) = r_0(B_2) + x \cdot r_1(B_2)$$

$$= 1 + 2x$$

To find $r_{B3}(x)$:-

$$r_{B3}(x) = r_0(B_3) + x\, r_1(B_3)$$

$$= 1 + x$$

$r_B(x) = (1+2x)(1+2x)(1+x)$

$= (1+2x+2x+4x^2)(1+x)$

$= (1+4x+4x^2)(1+x)$

$= 1+4x+4x^2+x+4x^2+4x^3$

$\boxed{\therefore r_B(x) = 1+5x+8x^2+4x^3}$

$\bar{N} = S_0 - S_1 + S_2 - S_3$

$S_0 = n! = 4!$ ($\because$ no. of fruits $= 4$)

$= 24$

$S_1 = (n-k)!\, r_k(x)$

$= (4-1)!\, r_1(x)$

$= 3! * 5 = 30$

$S_2 = (4-2)!\, r_2(x)$

$= 2!\, 8 = 16$

$S_3 = (4-3)!\, r_3(x)$

$= 1!\, 4$

$= 4$

$\bar{N} = 24 - 30 + 16 - 4 = 6$ $\boxed{\therefore \bar{N} = 6}$

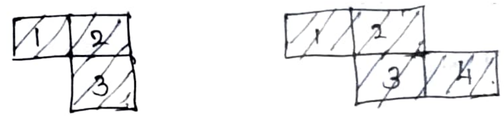
$\therefore$ There are 6 No. of ways of distribution of fruits so that No boy is dissatisfied.

2) 4 officers F1, F2, F3, F4 ~~arrive~~ arrive late for a dinner party. Find the only 1 chair at each of the 5 tables t_1, t_2, t_3, t_4, t_5 is vacant. F1 will not sit at t_1 or t_2, F2 will not sit at t_2, F3 will not sit at t_3 (or) t_4 and F4 will not sit at t_4 or t_5. Find the No. of ways that they can occupy the seats.

Soln:

	t_1	t_2	t_3	t_4	t_5
F1					
F2					
F3					
F4					

B1 B2



$r_B(x) = r_{B1}(x) * r_{B2}(x)$

To find $r_{B1}(x)$:

$r_{B1}(x) = r_0(B_1) + x \cdot r_1(B_1) + x^2 r_2(B_1)$

$= 1 + 3x + x^2(1)$

$\therefore r_{B1}(x) = 1 + 3x + x^2$

$r_{B2}(x) = r_0(B_2) + x \cdot r_1(B_2) + x^2 r_2(B_2)$

$\therefore r_{B2}(x) = 1 + 4x + 3x^2$

$r_B(x) = (1+3x+x^2)(1+4x+3x^2)$

$= 1+4x+3x^2+3x+12x^2+9x^3+x^2+4x^3+3x^4$

$\therefore r_B(x) = 1+7x+16x^2+13x^3+3x^4$

To find the no. of ways of sitting,

$$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4$$

$$n = 5$$

$$S_0 = n! = 5! = 120$$

$$S_1 = (5-1)!\, r_1(x)$$

$$= 24 * 7 = 168$$

$$S_2 = (5-2)!\, r_2(x)$$

$$= 6(16) = 96$$

$$S_3 = (5-3)!\, r_3(x)$$

$$= 2(13) = 26$$

$$S_4 = (5-4)!\, r_4(x)$$

$$= 1(3) = 3$$

$$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4$$

$$= 120 - 168 + 96 - 26 + 3$$

$$= 219 - 194$$

$$= 25$$

∴ There are 25 no. of ways the officers can be seated.

19/9

Note:

* A simple graph consisting of one circuit with $n \geq 3$ is two-chromatic, if n is even and 3 chromatic if n is odd.

3-chromatic
n is odd

2-chromatic
n is even

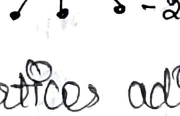
3-chromatic
n is odd

2-chromatic
n is even

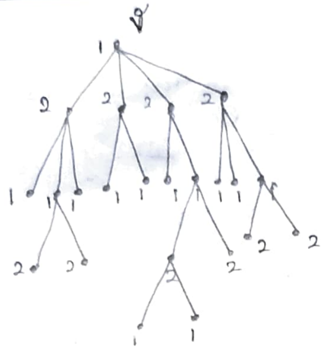
Prove that Every tree with 2 or more vertices is 2-chromatic.

Proof:

* consider a rooted tree T with root v
paint v with colour 1



* paint all the vertices adjacent to v with colour 2.
* Next, paint all the vertices adjacent to this vertices with colour 1.



* Now we observe that, all vertices with odd distance from v have colour 2 and vertices at even distance from v have colour 1.
* Along any path in T, the vertices are of alternative colours. Since there is one path between any two vertices in a tree and no two adjacent vertices have the same colour.
* Thus, any tree T with two or more vertices is 2-chromatic.

2) Prove that a graph G with atleast one edge is 2-chromatic if and only if it has no circuits of odd length.

Proof:

* Let G be a 2-chromatic graph with atleast one edge.

* If G has a circuit of odd length, we would need atleast 3 colours for that circuit.
* But G is 2-chromatic, hence G contains no circuit of odd length.
* Conversely,

Let G be a connected graph with no circuits of odd length.

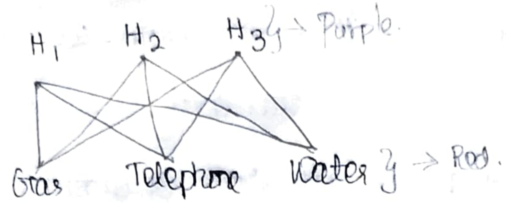
* Consider a spanning tree T of G where T can be properly coloured with 2 colours.
* Now, add edges one by one since G has no circuits of odd length, the end vertices of every chord being replaced are differently coloured in T.
* Thus, G is properly coloured with 2 colours (i.e) G is 2-chromatic.

20/9/19

3) Define bipartite graph and prove that a graph G is 2-chromatic iff G is bipartite.

Proof:

A graph G is bipartite if its vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in G joints a vertex in V_1 with a vertex in V_2.



$V = \{H_1, H_2, H_3, G, T, W\}$

$V_1 = \{H_1, H_2, H_3\}$

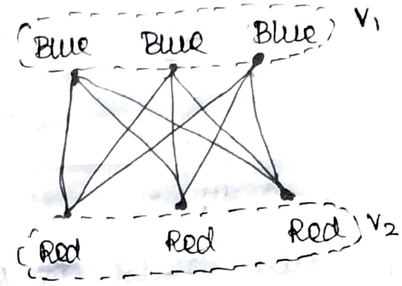
$V_2 = \{G, T, W\}$

Proof:

* Let G be a 2-chromatic graph then the vertex set V of G can be decomposed into 2 independent subsets V_1 and V_2 such that a set of vertices in V_1 can be painted with one colour and a set of vertices of V_2 shall be painted with another colour.
* An edge of G connects a ~~vedge~~ vertex of V_1 and vertex of V_2.
* Therefore G is bi-partite.
* Conversely,

Let G be a bi-partite graph, a vertex set V of G can be partitioned into 2 independent sets V_1 and V_2, we can use colour C_1 to paint the vertices of V_1 and colour C_2 to paint the vertices of V_2

* Therefore G is 2-chromatic.



2-chromatic

4) Define chromatic polynomial and derive an expression for it.

Sol:

* The No. of different ways of colouring of a graph with n vertices that can be obtained using lamda (λ) colours (or) fewer colours can be expressed as a polynomial $P_n(\lambda)$ of G.
* Let G be a graph with n vertices and let C_i be the different ways of properly colouring G using exactly 'i' different colours.
* Since 'i' colours can be choosen out of λ colours in λC_i different ways.
* There are $C_i \lambda C_i$ different ways of properly colouring

G using exactly 'i' colours out of 'λ' colours.

(ie) $P_n(\lambda) = \sum_{i=1}^{n} c_i \, {}^{\lambda}C_i$

$$= c_1 \frac{\lambda}{1!} + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \dots + c_n \frac{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)}{n!}$$

* If G is a graph with atleast one edge then G requires atleast 2 colours for proper colouring of G

$$\boxed{\therefore c_1 = 0}$$

* If G has n vertices then it can be properly coloured with n colours in n factorial ways.

$$\boxed{\therefore c_n = n!}$$

5) Find a chromatic polynomial for a complete graph K_n (OR) prove that the graph of n vertices is a complete graph iff its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)$

Proof:

Note:

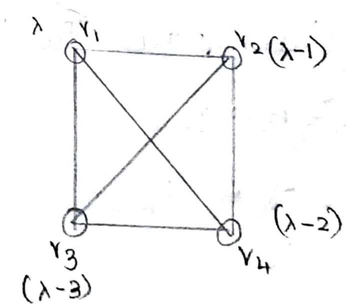
* [Every vertex is adjacent to every other vertices - complete graph].

* Let G be a complete graph with n vertices and let λ be a no. of colours.
* Here the 1st vertex of G can be coloured in λ ways.
* 2nd vertex of G can be coloured in $(\lambda-1)$ ways.
* 3rd vertex of G can be coloured in $(\lambda-2)$ ways

⋮

* nth vertex of G can be coloured in $(\lambda-(n-1))$ ways.
* As given the graph is complete, every vertex is adjacent to every other vertices, no two vertex shall have the same colour.
* Therefore, a complete graph G can be coloured in $\lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)$ ways.
* Thus, chromatic polynomial of K_n is

$$\boxed{\therefore P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\dots(\lambda-n+1)}$$



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5) write a note on four colour problem:

The four colour problem is an example for proper colouring of regions in a planar graph.

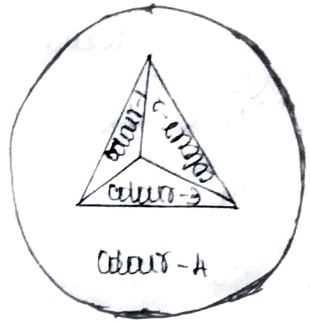
The 4-colour problem states that every planar graph can be coloured with 4 colours in such a way that 2-adjacent regions get different colours.

The proper colouring of region is also called map colouring because in atlas different countries are coloured such that countries with common boundaries are shown in different colours.

It has been shown that all maps containing less than 40 regions can be properly coloured with 4 colours.

The 4-colour problem states that "Every planar graph is 4-colourable"

Eg: The complete graph K_4 is 4-colourable



*) State and prove five colour theorem:

Statement:

The vertices of every planar graph can be properly coloured with 5-colours.

proof:

The above statement can be proved using mathematical induction on the no. of vertices of the graph.

consider a planar graph G with n vertices

for $n=1$, the graph can be coloured with any one of the 5-colours.

Now, let G be a connected planar graph, then atleast one vertex is of degree less than (or) equal to 5.

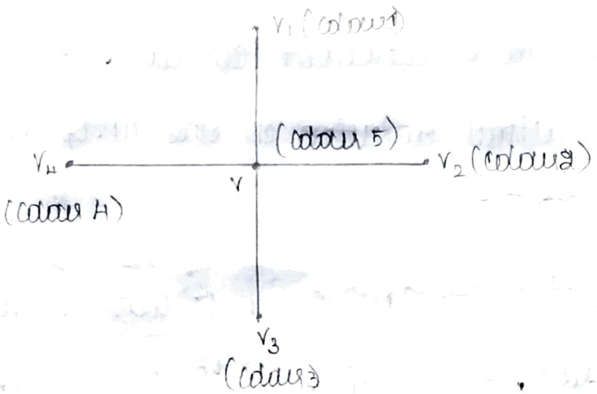
Let G' be the sub graph that is obtained from

G after deleting the vertex v (i.e) G' contains (n-1) vertices obtained from G by deleting vertex v and all edges incident on v.

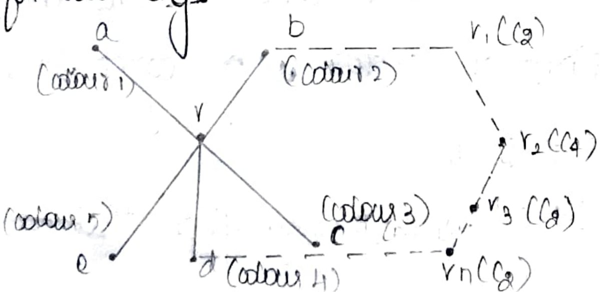
Assume that the theorem is true for G' (i.e) G' can be coloured with 5-colours

case-(i): If $\deg(v) \le 4$, then $G' = (G - v)$ can be properly coloured.

Hence if $\deg(v) \le 4$, there is no difficulty.



case-(ii): If $\deg(v) = 5$, then all the 5 colours are used for colouring.



Suppose there is a path in G' between vertices b and d coloured alternatively. b by c_2, v_1 by c_4, v_2 by c_2 and so on

Then the similar path between c and e coloured alternatively with colours c_3 and c_5 cannot exist.

Hence, there is no path of alternative colours c_3 and c_5 through vertices c and e.

Had we assume that there was no path between b and d painted alternatively with colours c_2 and c_4, we would have used c_2 (or) c_4 instead of c_5.

Hence the theorem

8) prove that if a,b are two non-adjacent vertices in a graph G and G' is a graph obtained by adding an edge between a and b, G'' is a simple graph obtained from G by fusing the vertices a&b together and replacing the sets of parallel edges with simple edges, then $P_n(\lambda)$ of $G = P_n(\lambda)$ of G'

$+ P_{n-1}(\lambda)$ of G''

Proof :-

* The No. of ways of properly colouring of Graph is of two ways:

a) vertices a and b coloured with same colours

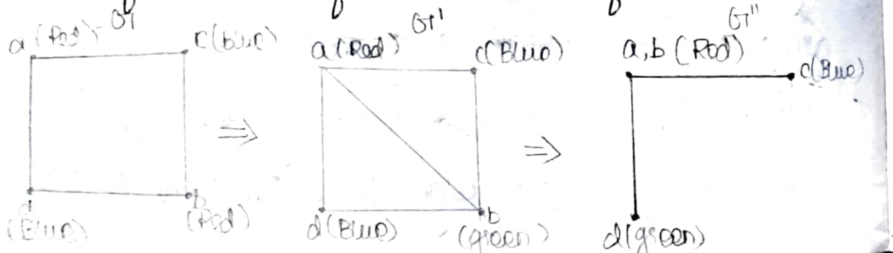
b) vertices a and b are of different colours.

* The No. of ways of properly colouring of G such that a, b are of different colours = No. of ways of proper colouring of G'.

* The No. of ways of proper colouring of G such that a, b are of same colour = No. of ways of proper colouring of G''

* Therefore:

$P_n(\lambda)$ of G = $P_n(\lambda)$ of G' + $P_{n-1}(\lambda)$ of G''



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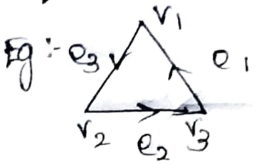
1) Define Directed graph (or) Di-graph and Explain the different types of Di-graph.

A graph G is said to be a directed graph if every edge of G is directed.

Eg :-

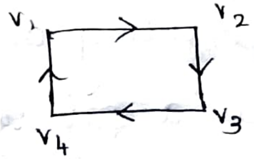


Types of Di-graph :-

1) Simple (or) strict Di-graph :-

A Di-graph with no self loops and parallel edges is called simple (or) strict Digraph.

Eg :-

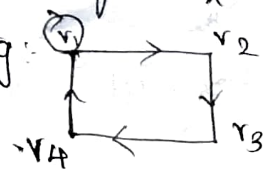


2) Oriented graph :-

A Di-graph without parallel edges is called oriented Di-graph.

A simple (or) strict Di-graph itself is also a oriented Di-graph.

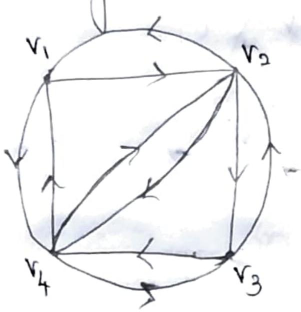
Eg :



3) Symmetric Di-graph:

In a Di-graph, if for every edge (a, b) there is also an edge (b, a), is called symmetric Di-graph.

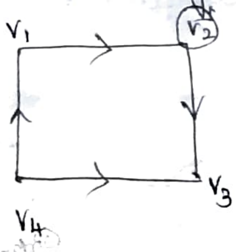
Eg:
V1, V2, V3, V4



4) Asymmetric Di-graph:

It is a Di-graph having atmost one directed edge between a pair of vertices with or without self loop is called Assymmetric (or) Anti-symmetric Di-graph.

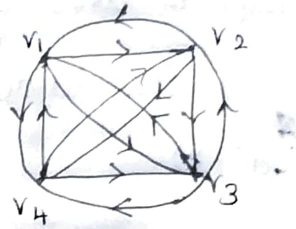
Eg:-



5) Complete Di-graph:

It is a Di-graph in which there is exactly one edge directed for every vertex to every other vertex.

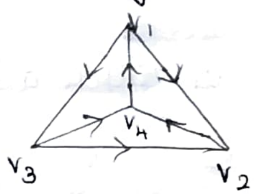
Eg:-



6) Complete Asymmetric Di-graph:

It is a Digraph with exactly one edge directed between every pair of vertices.

Eg:-



Note:

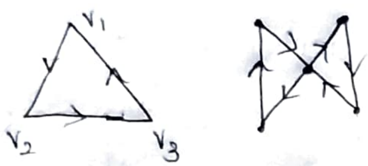
* A complete symmetric Di-graph with n vertices will have $n(n-1)$ Edges
* A complete A-symmetric Di-graph with n vertices will have $n(n-1)/2$ Edges.
* A complete A-symmetric Di-graph is also called as tournament (or) complete tournament.

7) Iso-graph (or) Balanced Di-graph (or) pseudo symmetric Di-graph:

It is a Di-graph in which for every vertex v, the indegree equals the out-degree (i.e)

$d^{+}(v) = d^{-}(v)$

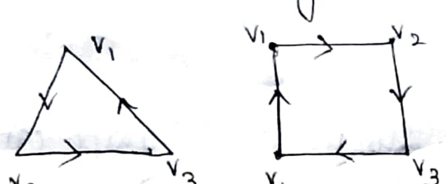
Eg:



8) Regular Di-graph:

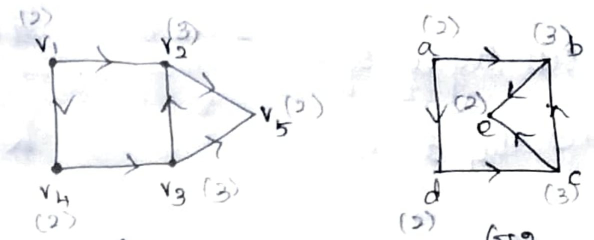
It is a Di-graph in which all the vertices will have the same indegree and out-degree.

Eg:



9) Isomorphism of the Di-graph:

Two Di-graphs are isomorphic if their corresponding undirected graph are isomorphic and the directions of the corresponding edges must also agree



G1 G2

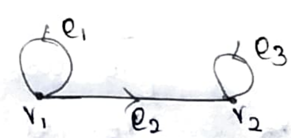
Here, G1 and G2 are isomorphic.

2) Define self loop, indegree, outdegree of vertex, isolated vertex, pendant vertex, parallel edges.

Self-loop:

An edge for which the initial and terminal vertices are same is called self loop.

Eg:



Here e_1, e_3 are self loops.

Indegree:

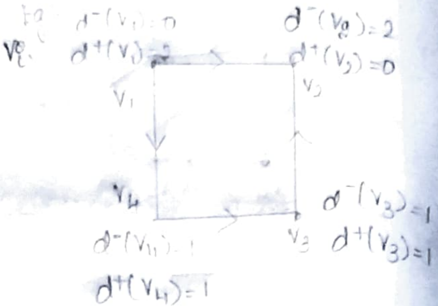
The No. of edges incident to the vertex v_i is called indegree of v_i.

Denoted by $d^{-}(v_i)$

Outdegree:

The No. of edges incident out of the vertex v_i is called out-degree of v_i.

Denoted by $d^+(v_i)$



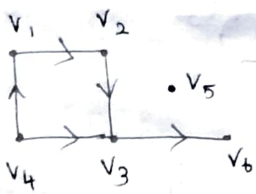
Isolated vertex:-

In a Di-graph, a vertex for which the indegree and out degree are both zero is called an Isolated vertex.

Pendant Vertex:-

In a Di-graph, a vertex whose degree (sum of indegree + outdegree) is one is called pendant vertex.

Eg:



[v_5 = Isolated
v_6 = pentant]

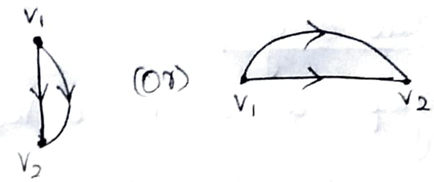
$d^-(v_5) = d^+(v_5) = 0$

$d^-(v_6) + d^+(v_6) = 1$

Parallel Edges:-

Two Edges are said to be parallel if they are mapped on to the same ordered pair of vertices.

Eg:-



3) Describe the types of relations and their Di-graphs:-

If $X=\{x_1, x_2, \ldots x_n\}$ is a set of objects, a binary relation R exists between the pair of vertices (v_i, v_j).

In this case, we write $v_i R v_j$.

Types of relations:-

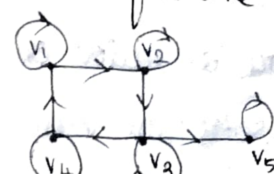
a) Reflexive relation:

A relation R on set X that satisfies $x_i R x_i$ for every/all [$\forall$] $x_i \in X$ is called a Reflexive relation.

(or)

A Di-graph in which every vertex has a self loop is called reflexive di-graph.

Eg:-



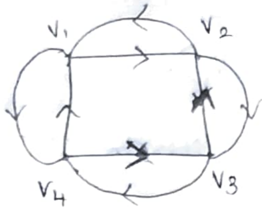
Irreflexive di-graph:

A di-graph in which no vertex has a self loop is called irreflexive di-graph.

Symmetric Digraph:

A relation R on a set X is symmetric if $x_i R x_j = x_j R x_i$

Eg:

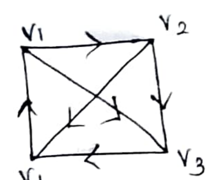


Transitive Relation:

A relation R on a set X is transitive if $x_i R x_j$ and $x_j R x_k$

$\Rightarrow x_i R x_k$

Eg:



Equivalence Relation:

A binary relation is called an Equivalence relation if it is reflexive, symmetric and

transitive.

Eg: