# Artificial Intelligence: Exercise 3

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Week 3

1. Given are the following formulas.
   (a) $\neg(a \vee b) \Leftrightarrow \neg a \vee \neg b$
   (b) $(a \Rightarrow b) \Leftrightarrow (\neg b \Rightarrow \neg a)$
   (c) $((a \Rightarrow b) \wedge (b \Rightarrow a)) \Leftrightarrow (a \Leftrightarrow b)$
   (d) $(a \vee b) \wedge (\neg b \vee c) \Rightarrow (a \vee c)$

   Which of the formulas are tautologies? Which of the formulas in (a)-(d) are semantically equivalent?

2. Which of the following formulars are satisfiable, unsatisfiable or logically valid?
   (a) $a \Rightarrow a$
   (b) $a \wedge b \Rightarrow b$
   (c) $a \vee b \wedge \neg a$

3. Describe the following situations as formulas in propositional logic.
   (a) If the graphics card of the computer is defect, the screen output is not readable.
   (b) In a plane there is a valve which can either be shut, open, half open. Describe the fact that two of the three errors 'Open valve blocked.', 'Half open valve blocked' and 'Shut valve blocked' occur simultaneously.

4. Consider a parking lot with four sites named A,B,C and D and the following propositions.

$$a = \text{"Site A is vacant."} \tag{1}$$
$$b = \text{"Site B is vacant."} \tag{2}$$
$$c = \text{"Site C is vacant."} \tag{3}$$
$$d = \text{"Site D is vacant."} \tag{4}$$
$$z = \text{"There are two vacant sites."} \tag{5}$$
$$o = \text{"The parking lot is completely occupied."} \tag{6}$$

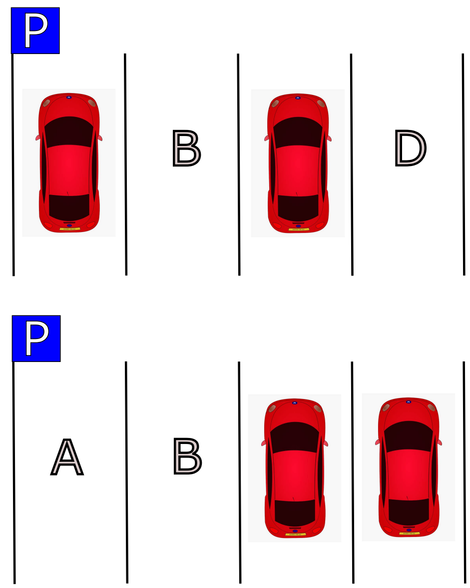


Figure 1: Two different occupation configurations of the parking lot with four sites.

Using these atomic formulas as the signature, provide a formula $q$ which has exactly two models. The formula should represent the two occupations in Figure 1.

**Solution:**

1. (a) The formula, lets call it $p$ with $p = \neg(a \vee b) \Leftrightarrow \neg a \vee \neg b$ can be analyzed by the truth table.

| $a$ | $b$ | $a \wedge b$ | $\neg a$ | $\neg b$ | $\neg(a \wedge b)$ | $\neg a \wedge \neg b$ | $\neg(a \vee b) \Leftrightarrow \neg a \wedge \neg b$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F | T |
| T | F | F | F | T | T | F | F |
| F | T | F | T | F | T | F | F |
| F | F | F | T | T | T | T | T |

The model of the formular is $\mathcal{M}(p) = \{(I(a) = T, I(b) = T),\ (I(a) = F, I(b) = F)\}$. Thus, the model is not equal to all possible interpretations or $\mathcal{M}(\neg p) \neq \emptyset$, i.e. the model is satisfiable and falsifiable. It is not a tautology.

(b) The formula $q = (a \Rightarrow b) \Leftrightarrow (\neg b \Rightarrow \neg a)$ can be analyzed by the truth table.

| $a$ | $b$ | $\neg a$ | $\neg b$ | $a \Rightarrow b$ | $\neg b \Rightarrow \neg a$ | $(a \Rightarrow b) \Leftrightarrow (\neg b \Rightarrow \neg a)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

Here, the model contains all interpretations, or $\mathcal{M}(\neg q) \neq \emptyset$, i.e. the model is satisfiable and not falsifiable. It is a tautology. The formula $q$ is not equivalent to $p$ from (a).

(c) The formula $r = ((a \Rightarrow b) \wedge (b \Rightarrow a)) \Leftrightarrow (a \Leftrightarrow b)$ can be analyzed by the truth table.

| $a$ | $b$ | $a \Rightarrow b$ | $b \Rightarrow a$ | $a \Leftrightarrow b$ | $(a \Rightarrow b) \wedge (b \Rightarrow a)$ | $((a \Rightarrow b) \wedge (b \Rightarrow a)) \Leftrightarrow (a \Leftrightarrow b)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | F | F | T |
| F | F | T | T | T | T | T |

Here, the model $\mathcal{M}(r)$ includes all interpretations and thus $r$ is a tautology and thus satisfiable and not falsifiable. We can write $r \equiv T$. The formula $r$ is equivalent to $q$ from (b), but not to $p$ from (a).

(d) This formula $s$ can be analyzed by the truth table

| $a$ | $b$ | $c$ | $a \vee b$ | $\neg b \vee c$ | $(a \vee b) \wedge (\neg b \vee c)$ | $a \vee c$ | $(a \vee b) \wedge (\neg b \vee c) \Rightarrow (a \vee c)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | T | T |
| T | F | T | T | T | T | T | T |
| F | T | T | T | T | T | T | T |
| T | F | F | T | T | T | T | T |
| F | T | F | T | F | F | F | T |
| F | F | T | F | T | F | T | T |
| F | F | F | F | T | F | F | T |

Here, the model $\mathcal{M}(s)$ includes all interpretations and thus $s$ is a tautology, i.e. $s = T$. We can write $s \equiv T$. The formula $r$ is not equivalent to $p$ . It is equivalent to $r \equiv T$ from (c) and to $q$ from (b). i.e. $q \equiv r \equiv s = T$.

2. (a) The formula $a \Rightarrow a$ is a tautology and therfore its is staisfiable, bot not unsatifiable. The formula is also logically valid, because this is the same as saying its a tautology.

(b) We check the truth table

| $a$ | $b$ | $a \wedge b$ | $a \wedge b \Rightarrow b$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

and find that $a \wedge b \Rightarrow b$ is a tautology (logically valid). Hence, it is satisfiable and not unsatisfiable.

(c) We check the truth table

| $a$ | $b$ | $\neg a$ | $a \vee b$ | $a \vee b \wedge \neg a$ |
|---|---|---|---|---|
| T | T | F | T | F |
| T | F | F | T | F |
| F | T | T | T | T |
| F | F | T | F | F |

and find that $a \wedge b \Rightarrow b$ is satisfiable and not unsatisfiable. It is, however, not a tautology.

3. (a) We define two variables

- $a =$ 'The graphics card of the computer works flawlessly.'
- $b =$ 'The screen output is readable.'.

For propsition

$p =$ 'If the graphics card of the computer is defect, the screen output is not readable.'

we want the following truth table:

| a | b | p | Explanation |
|---|---|---|---|
| T | T | T | Both the craphics card is ok and the screen is readable. |
| T | F | T | The screen output can be unreadable for there reasons, even if $a = T$. |
| F | T | F | If the screen is readble, the graphics card can not be defect. |
| F | F | T | Both the graphics card is defect and the screen is unreadable. |

The formula $p = b \Rightarrow a \equiv a \vee \neg b$ describes exactly the situation:

| $a$ | $b$ | $\neg b$ | $b \Rightarrow a \equiv a \vee \neg b$ |
|---|---|---|---|
| T | T | F | T |
| T | F | T | T |
| F | T | F | F |
| F | F | T | T |

(b) We introduce the three atomic variables

- $o =$ 'The valve is open.',
- $s =$ 'The valve is shut.',
- $b =$ 'The valve is blocked.'.

Now, the half open valve can be describes as

$$\neg o \wedge \neg s = \text{'The valve is half open.'}$$

The three errors are described logically as

$$o \wedge b = \text{'Open valve is blocked'} = f$$
$$s \wedge b = \text{'Shut valve is blocked'} = g$$
$$(\neg o \wedge \neg s) \wedge b = \text{'Half open valve is blocked'} = h$$

If two of these tree errors occur sumultaneously we describe it as

$$((o \wedge b) \wedge (s \wedge b)) \wedge ()$$

The proposition that two of th three errors occur simultaneously is described as

$$\begin{aligned}(f \wedge g) \vee (f \wedge h) \vee (g \wedge h) = &((o \wedge b) \wedge (s \wedge b)) \\ &\vee ((o \wedge b) \wedge ((\neg o \wedge \neg s) \wedge b)) \\ &\vee ((s \wedge b) \wedge ((\neg o \wedge \neg s) \wedge b)) \, .\end{aligned}$$

Remark: This is of course only possible if there are two valves with the same possible states! A single valve can not be open or half open at the same time. If we want to describe a single valve and we want to express that is is either open, half open or blockef we need the formula

$$(o \wedge s) \vee (o \wedge (\neg o \wedge \neg s)) \vee (s \wedge (\neg o \wedge \neg s)) = F$$

4. $p = (\neg c \wedge b) \wedge (\neg a \vee \neg d) \wedge z$ To check this, it is enough to analyze the interpretation with $z = T$, i.e. where two sites are vacant. There are $\binom{4}{2} = 6$ interprations for $a, b, c, c$ with two times $F$:

| $a$ | $b$ | $c$ | $d$ | $z$ | $\neg c \wedge b$ | $\neg a \vee \neg d$ | $(\neg c \wedge b) \wedge (\neg a \vee \neg d)$ | $p = (\neg c \wedge b) \wedge (\neg a \vee \neg d) \wedge z$ |
|---|---|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T | T | T |
| F | T | T | F | T | F | T | F | F |
| F | F | T | T | T | F | T | F | F |
| T | F | F | T | T | F | F | F | F |
| T | F | T | F | T | F | T | F | F |
| F | T | F | T | T | T | T | T | T |

The last and the first row correspond to the occupations in Figure 1.