# Artificial Intelligence: Solutions to Exercise 4

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1. Transform the following formulas to conjunctive normal form.

   (a) $a \Leftrightarrow b$

   (b) $a \wedge b \Leftrightarrow a \vee b$

   (c) $a \wedge (a \Leftrightarrow b) \Rightarrow b$

   Solution:

   (a) We use $p \Rightarrow q \equiv q \vee \neg p$. Thus $a \Rightarrow b \equiv b \vee \neg a$ and $b \Rightarrow a \equiv a \vee \neg b$. Thus $a \Leftrightarrow b \equiv (a \vee \neg b) \wedge (b \vee \neg a)$.

   (b) One can follow the algorithm from the lecture. here, it is faster tus use a truth table to see that $(a \wedge b \Leftrightarrow a \vee b) \equiv a \vee b$, which is in conjunctive normal form.
   Alternativel, we follow the algorithm in the lecture

      i. Remove implications

      $$\begin{aligned} a \wedge b \Leftrightarrow a \vee b &\equiv (a \wedge b \Rightarrow a \vee b) \wedge (a \vee b \Rightarrow a \wedge b) \\ &\equiv ((a \vee b) \vee \neg(a \wedge b)) \wedge ((a \wedge b) \vee \neg(a \vee b)) \end{aligned}$$

      ii. Use De Morgan's laws to move negations in front of each symbol

      $$\begin{aligned} &((a \vee b) \vee \neg(a \wedge b)) \wedge ((a \wedge b) \vee \neg(a \vee b)) \\ &\equiv \underbrace{((a \vee b) \vee (\neg a \vee \neg b))}_{=p} \wedge \underbrace{((a \wedge b) \vee (\neg a \wedge \neg b))}_{=q} \end{aligned}$$

      iii. We use the distributive laws for the two formulas $p$ and $q$. Actually, $p$ is already a clause, but we will try to further simplify

      $$\begin{aligned} p &\equiv (a \vee b) \vee (\neg a \vee \neg b) \equiv ((a \vee b) \vee \neg a) \vee ((a \vee b) \vee \neg b) \\ &\equiv a \vee \neg a \vee b \vee a \vee b \vee \neg b \\ &\equiv a \vee b \\ q &\equiv (a \wedge b) \vee (\neg a \wedge \neg b) \equiv ((a \wedge b) \vee \neg a) \wedge ((a \wedge b) \vee \neg b) \\ &\equiv ((a \vee \neg a) \wedge (b \vee \neg a)) \wedge ((a \vee \neg b) \wedge (b \wedge \neg b)) \\ &\equiv (b \vee \neg a) \wedge (a \vee \neg b) \end{aligned}$$

      Now, since $p$ is a clause and $q$ is already in conjunctive normal form we find

      $$a \wedge b \Leftrightarrow a \vee b \equiv p \wedge q \equiv (a \vee b) \wedge (b \vee \neg a) \wedge (a \vee \neg b)$$

(c) $a \wedge (a \Leftrightarrow b) \Rightarrow b \equiv T$

2. Explain the difference between the following expressions

$$p \Rightarrow q$$
$$p \models q$$
$$p \vdash q$$

Solution: Todo

3. The resolution rule

$$\frac{\begin{array}{l} A \vee B \\ \neg A \vee C \end{array}}{B \vee C}$$

is an inference rule and not an equation. Check, whether $(A \vee B) \wedge (\neg A \wedge C) \equiv (B \vee C)$ is correct?

Solution: Already the second row of the following truth table shows that the formula is incorrect.

| $A$ | $B$ | $C$ | $\neg A$ | $A \vee B$ | $\neg A \vee C$ | $(A \vee B) \wedge (\neg A \vee C)$ | $B \vee C$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T | T |
| T | T | F | F | T | F | F | T |
| $\vdots$ | $\vdots$ | ... | ... | ... | ... | ... | $\vdots$ |

4. Show by resolution that

(a) the formula $(a \vee b) \wedge (\neg b \vee c) \Rightarrow (a \vee c)$ is a tautology.

(b) the formula $\neg(\neg \text{gasintank} \wedge (\text{gasintank} \vee \neg \text{carstarts}) \Rightarrow \neg \text{carstarts})$ is unsatisfiable.

Solution:

(a) We look for complementary literals and find $b$ and $\neg b$. Then

$$\frac{\begin{array}{l} a \vee b \\ \neg b \vee c \end{array}}{a \vee c}$$

Now

$$\frac{\begin{array}{l} a \vee c \\ \neg(a \vee c) \end{array}}{\emptyset}$$

(b) Here $\neg$gasintank and gasintank are complementary literals and the resolution

$$\frac{\begin{array}{l} \text{gasintank} \vee \neg \text{carstarts} \\ \neg \text{gasintank} \end{array}}{\neg \text{carstarts}}$$

Thus (¬gasintank ∧ (gasintank ∨ ¬carstarts) ⊢ ¬carstarts. Negating yields carstarts ⇒ ¬carstarts, which is unsatisfiable.

5. Solve the following case with the help of a resolution proof: "If the criminal had an accomplice, then he came in a car. The criminal had no accomplice and did not have the key, or he had the key and an accomplice. The criminal had the key. Did the criminal come in a car or not?"

   Solution: We read from the text the following formulas (as usualy connected by conjunctions ∧) and assumed to be true.

$$\begin{aligned}
&accomplice \Rightarrow car\\
&(\neg accomplice \wedge \neg key) \vee (accomplice \wedge key)\\
&key
\end{aligned}$$

   We form a conjunctive normal form for the first

$$accomplice \Rightarrow car \equiv car \vee \neg accomplice$$

   and for the second formula using the distributive law

$$\begin{aligned}
&(\neg accomplice \wedge \neg key) \vee (accomplice \wedge key)\\
&\equiv (\neg accomplice \vee (accomplice \wedge key)) \wedge (\neg key \vee (accomplice \wedge key))\\
&\equiv ((\neg accomplice \vee accomplice) \wedge (\neg accomplice \vee key))\\
&\qquad \wedge ((\neg key \vee accomplice) \wedge (\neg key \wedge key)).
\end{aligned}$$

   With $a \vee \neg a = T$ and $c \wedge T = c$ we find

$$\begin{aligned}
&(\neg accomplice \wedge \neg key) \vee (accomplice \wedge key)\\
&\equiv T \wedge (\neg accomplice \vee key)) \wedge ((\neg key \vee accomplice) \wedge T)\\
&\equiv (\neg accomplice \vee key) \wedge (\neg key \vee accomplice)
\end{aligned}$$

   This is a conjuction of two clauses, which can be used in the axioms. We find

$$\begin{aligned}
&car \vee \neg accomplice\\
&\neg accomplice \vee key\\
&\neg key \vee accomplice\\
&key.
\end{aligned}$$

   For a proof by contradiction we add the negation of the query *car*

$$\begin{aligned}
&car \vee \neg accomplice\\
&\neg accomplice \vee key\\
&\neg key \vee accomplice\\
&key\\
&\neg car
\end{aligned}$$

We use resolution

$$\frac{\begin{array}{l} key \\ \neg key \vee accomplice \end{array}}{accomplice}$$

and using the resolvent we have

$$\frac{\begin{array}{l} accomplice \\ \neg accomplice \vee car \end{array}}{car}$$

The last resolution

$$\frac{\begin{array}{l} car \\ \neg car \end{array}}{\emptyset}$$

proves the contradiction. Thus, the criminal came in a car.