Review of elements of Calculus (functions in one variable)

Mainly adapted from the lectures of prof Greg Kelly

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http://online.math.uh.edu/HoustonACT/

https://sites.google.com/site/gkellymath/home/calculus-powerpoints

Functions

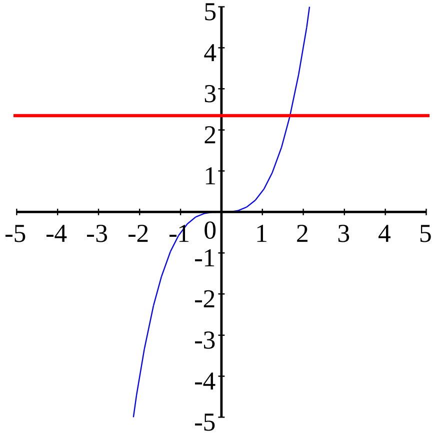
A relation is a <u>function</u> if:
for each x there is one and only one y.

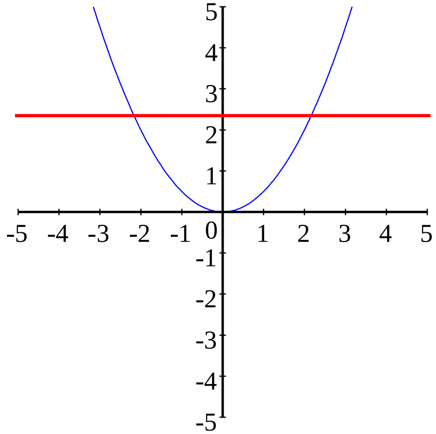
A relation is a <u>one-to-one</u> if also:
for each y there is one and only one x.

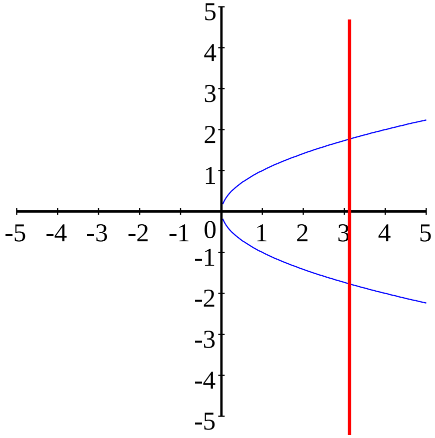
In other words, a function is one-to-one on domain D if:

$$f(a) \neq f(b) \text{ whenever } a \neq b$$

To be one-to-one, a function must pass the horizontal line test as well as the vertical line test.







$$y = \frac{1}{2}x^3$$

$$y = \frac{1}{2}x^2$$

$$x = y^2$$

one-to-one

<u>not</u> one-to-one

not a function

(also not one-to-one)

Inverse functions:

$$f(x) = \frac{1}{2}x + 1$$

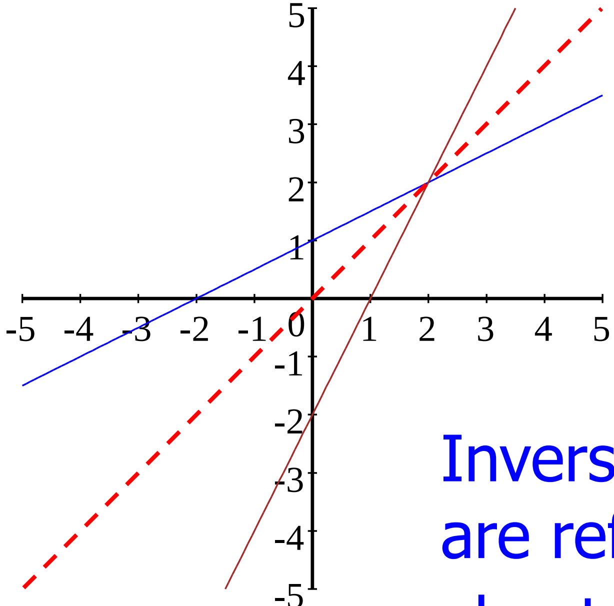
$$y = \frac{1}{2}x + 1$$

Solve for x:

$$y - 1 = \frac{1}{2}x$$

$$2y - 2 = x$$

$$x = 2y - 2$$



Inverse functions are reflections about $y = x$.

Switch x and y: $y = 2x - 2$ ➡ $f^{-1}(x) = 2x - 2$

Inverse of function $f(x) = a^x$

This is a one-to-one function, therefore it has an inverse.

The inverse is called a logarithm function.

Example: $16 = 2^4$ $4 = \log_2 16$ Two raised to what power is 16?

The most commonly used bases for logs are 10: $\log_{10} x = \log x$

and e: $\log_e x = \ln x$

$y = \ln x$ is called the natural log function.

$y = \log x$ is called the common log function.

Properties of Logarithms

$$\boxed{a^{\log_a x} = x \quad \log_a a^x = x} \quad (a > 0\,,\, a \neq 1\,, \quad x > 0)$$

Since logs and exponentiation are inverse functions, they "un-do" each other.

Product rule: $\log_a xy = \log_a x + \log_a y$

Quotient rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$

Power rule: $\log_a x^y = y \log_a x$

Change of base formula: $\log_a x = \frac{\ln x}{\ln a}$

Trigonometric Functions

Even and Odd Trig Functions:

- "Even" functions behave like polynomials with even exponents, in that when you change the sign of x, the y value doesn't change.

Cosine is an even function because: $\cos\left(-\theta\right)=\cos\left(\theta\right)$

Secant is also an even function, because it is the reciprocal of cosine.

Even functions are symmetric about the y - axis.

Even and Odd Trig Functions:

- "Odd" functions behave like polynomials with odd exponents, in that when you change the sign of x, the sign of the y value also changes.

Sine is an odd function because: $\sin(-\theta) = -\sin(\theta)$

Cosecant, tangent and cotangent are also odd, because their formulas contain the sine function.

Odd functions have origin symmetry.

Shifting, stretching, shrinking the graph of a function

Vertical stretch or shrink;
reflection about x-axis

$|a|>1$ is a stretch.

Vertical shift

Positive d moves up.

$$y = a\,f\left(b\left(x+c\right)\right)+d$$

Horizontal stretch or shrink;
reflection about y-axis

$|b|>1$ is a shrink.

Horizontal shift

Positive c moves left.

The horizontal changes happen in the opposite direction to what you might expect.

Amplitude and period in trigonometric functions

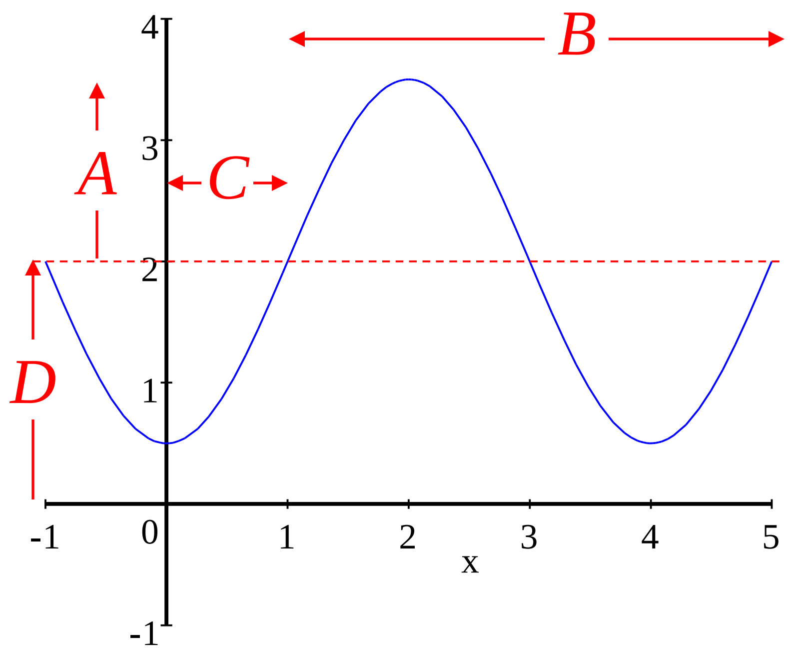
- $|A|$ is the **amplitude**.

Vertical shift

$$f(x) = A\sin\left[\frac{2\pi}{B}(x-C)\right]+D$$

$|B|$ is the **period**.

Horizontal shift

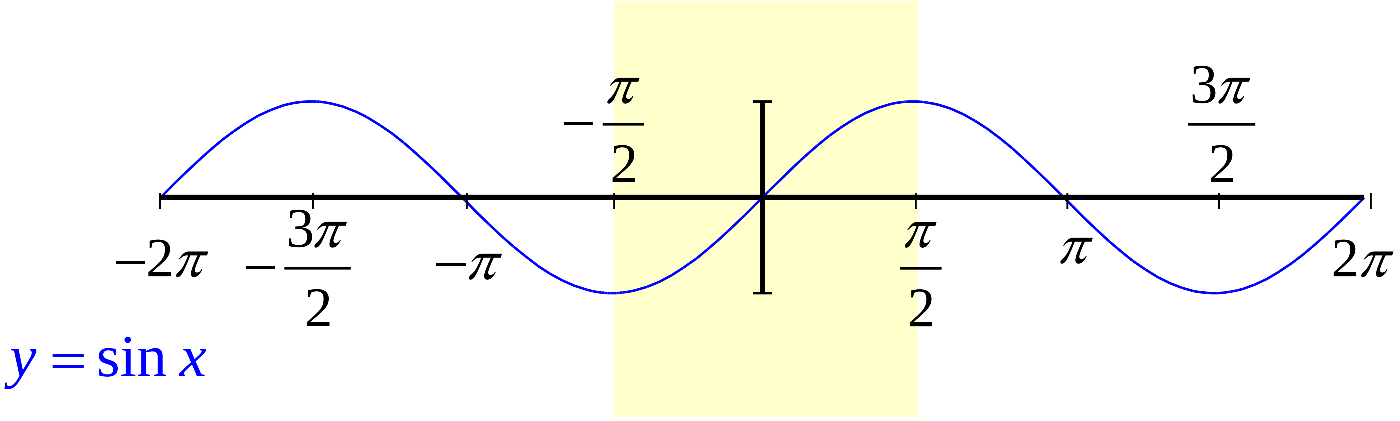


$$y = 1.5\sin\left[\frac{2\pi}{4}(x-1)\right]+2$$

Invertibility of trigonometric functions

Trig functions are not one-to-one.

However, the domain can be restricted for trig functions to make them one-to-one.

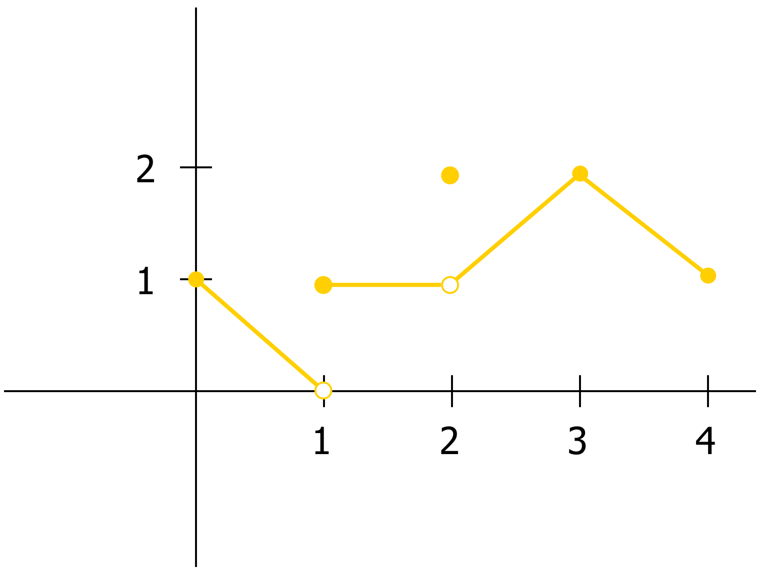


These restricted trig functions have inverses.

Continuity

Most of the techniques of calculus require that functions be <u>continuous</u>. A function is continuous if you can draw it in one motion without picking up your pencil.

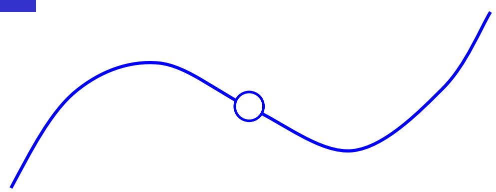
A function is continuous at a point if the limit is the same as the value of the function.

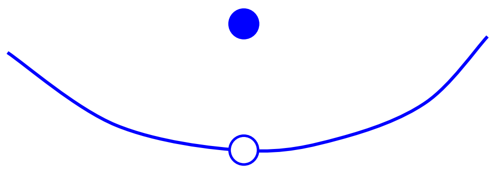


This function has <u>discontinuities</u> at x=1 and x=2.

It is continuous at x=0 and x=4, because the one-sided limits match the value of the function

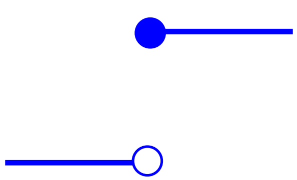
Removable Discontinuities:



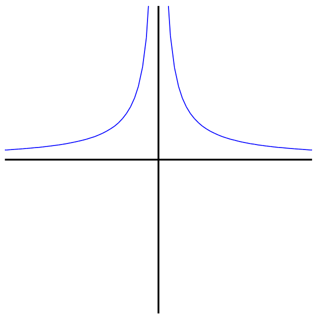


(You can fill the hole.)

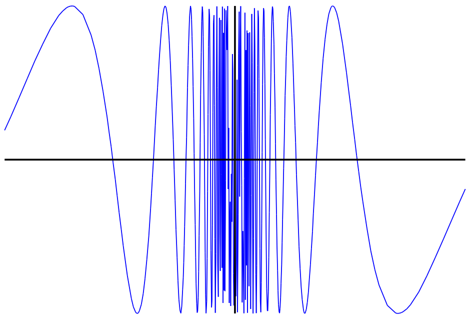
Essential Discontinuities:



jump



infinite



oscillating

Removing a discontinuity:

$f\left(x\right)=\frac{x^3-1}{x^2-1}$ has a discontinuity at $x=1$.

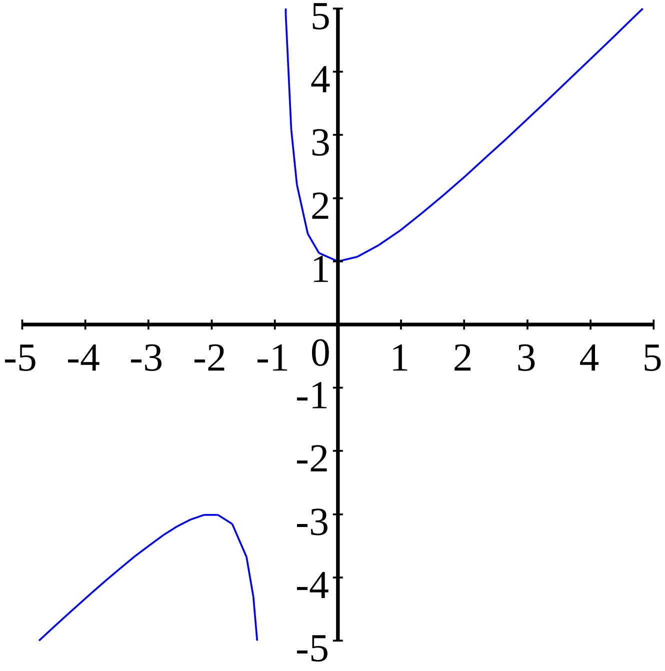
Write an extended function that is continuous at $x=1$.

$$\lim_{x\to 1}\frac{x^3-1}{x^2-1}=\lim_{x\to 1}\frac{\cancel{\left(x-1\right)}\left(x^2+x+1\right)}{\left(x+1\right)\cancel{\left(x-1\right)}}=\frac{1+1+1}{2}=\frac{3}{2}$$

$$f\left(x\right)=\begin{cases}\frac{x^3-1}{x^2-1}, & x\neq 1\\ \frac{3}{2}, & x=1\end{cases}$$

Note: There is another discontinuity at $x=-1$ that can not be removed.

Removing a discontinuity:



$$f(x)=\begin{cases} \dfrac{x^3-1}{x^2-1}, & x\neq 1 \\ \dfrac{3}{2}, & x=1 \end{cases}$$

Note: There is another discontinuity at $x=-1$ that can not be removed.

→

Continuous functions can be added, subtracted, multiplied, divided and multiplied by a constant, and the new function remains continuous.

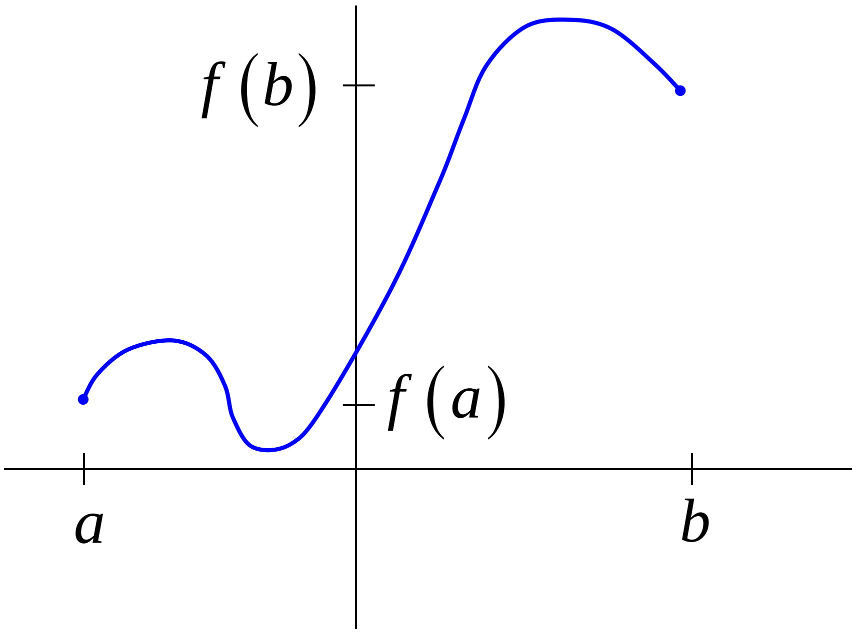
Also: Composites of continuous functions are continuous.

examples: $y = \sin\left(x^2\right)$ $y = \left|\cos x\right|$

→

Intermediate Value Theorem

If a function is continuous between a and b, then it takes on every value between $f(a)$ and $f(b)$.



Because the function is continuous, it must take on every y value between $f(a)$ and $f(b)$.

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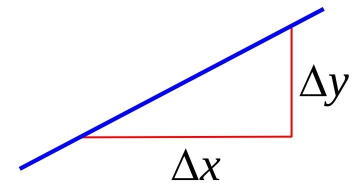
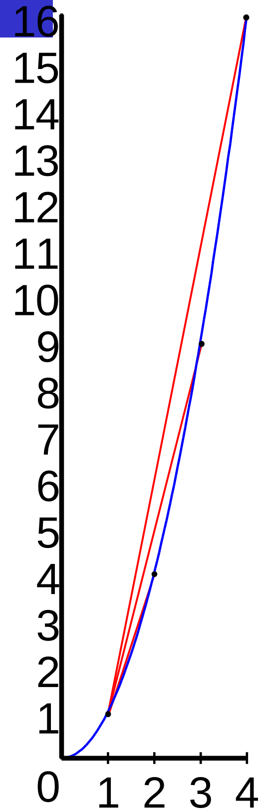
Rates of Change and Tangent Lines

Slope of a line

The slope of a line is given by: $m = \dfrac{\Delta y}{\Delta x}$





$$f(x) = x^2$$

The slope at (1,1) can be approximated by the slope of the secant through (4,16).

$$\frac{\Delta y}{\Delta x} = \frac{16-1}{4-1} = \frac{15}{3} = 5$$

We could get a better approximation if we move the point closer to (1,1). ie: (3,9)

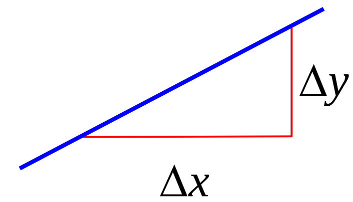
$$\frac{\Delta y}{\Delta x} = \frac{9-1}{3-1} = \frac{8}{2} = 4$$

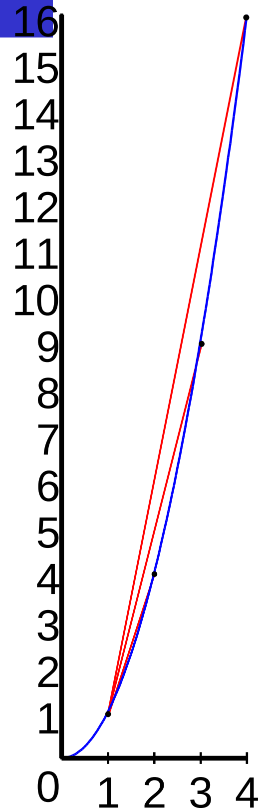
Even better would be the point (2,4).

$$\frac{\Delta y}{\Delta x} = \frac{4-1}{2-1} = \frac{3}{1} = 3$$

Slope of a line

The slope of a line is given by: $m = \dfrac{\Delta y}{\Delta x}$





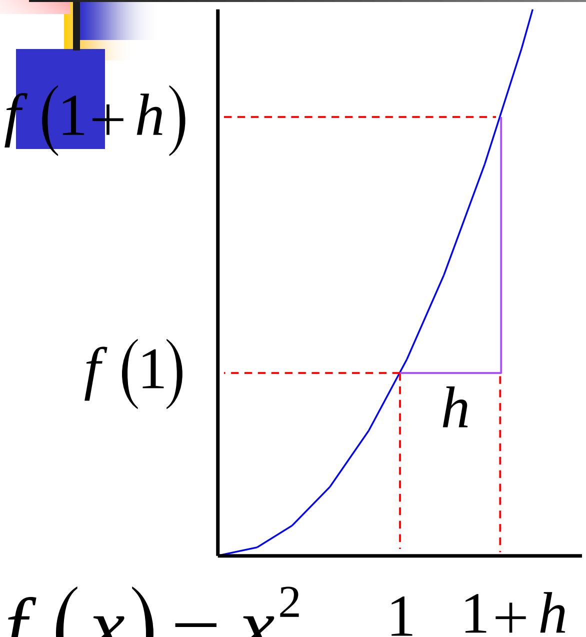
If we got really close to (1,1), say (1.1,1.21), the approximation would get better still

$$\frac{\Delta y}{\Delta x} = \frac{1.21-1}{1.1-1} = \frac{.21}{.1} = 2.1$$

How far can we go?

$$f\left(x\right) = x^2$$

Slope of a line



$$f(x) = x^2$$

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{f(1+h) - f(1)}{h}$$

$$\text{slope at } (1,1) = \lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}} = 2$$

The slope of the curve $y = f(x)$ at the point $P(a, f(a))$ is:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The slope of a curve at a point is the same as the slope of the tangent line at that point.

In the previous example, the <u>tangent line</u> could be found using $y - y_1 = m\left(x - x_1\right)$.

If you want the <u>normal line</u>, use the negative reciprocal of the slope. (in this case, $-\frac{1}{2}$)

(The normal line is perpendicular.)

Derivatives

f'(x^n)=nx

f'(nx)=n

Derivatives

$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ is called the derivative of f at a.

We write: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

"The derivative of f with respect to x is …"

There are many ways to write the derivative of $y = f(x)$

→

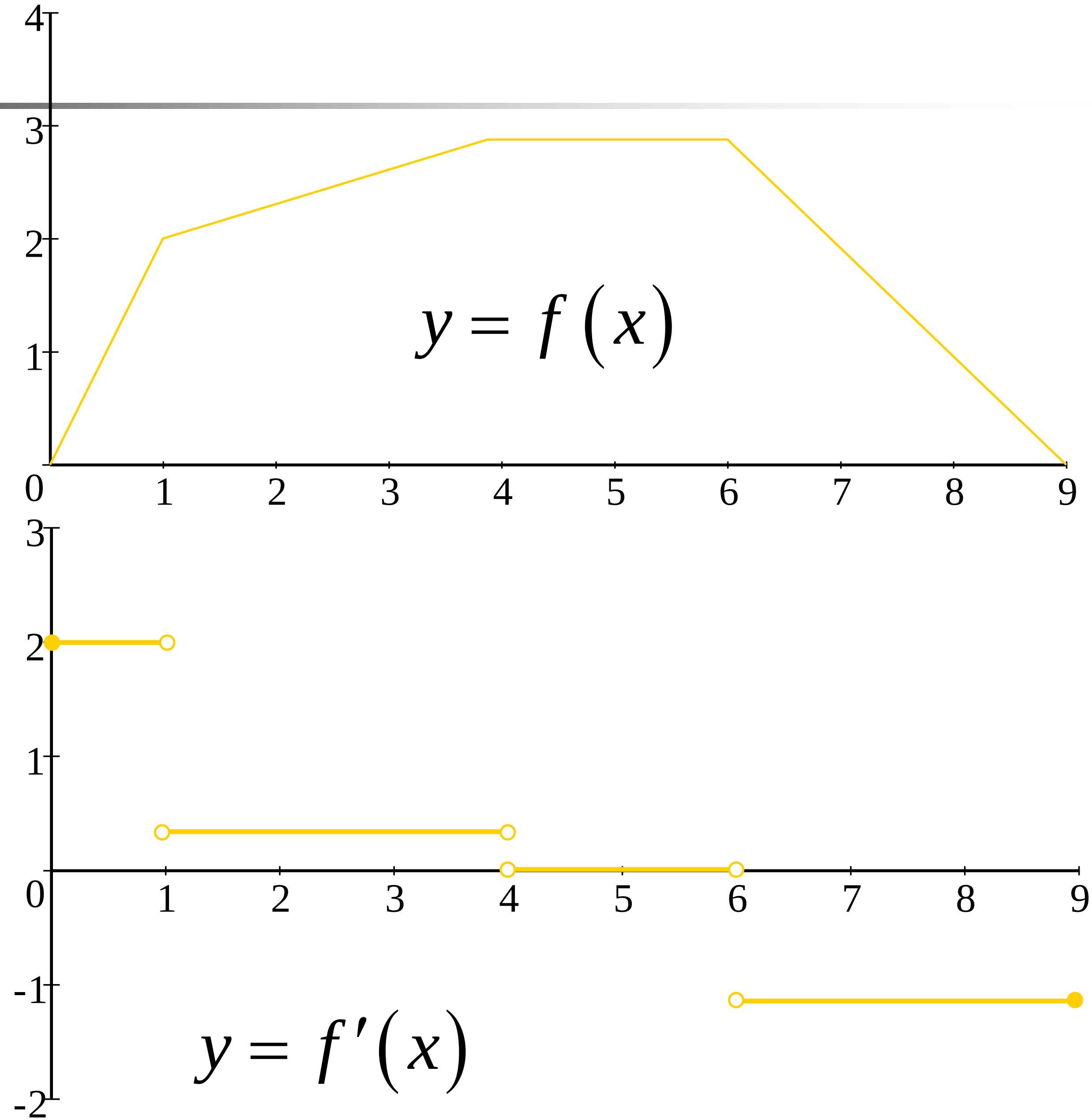
$f'(x)$ "f prime x" or "the derivative of f with respect to x"

y' "y prime"

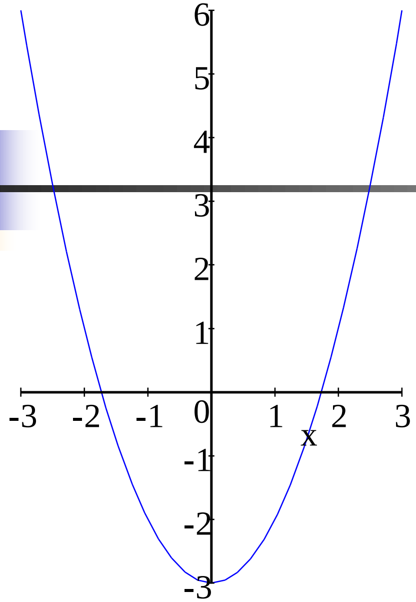
$\frac{dy}{dx}$ "the derivative of y with respect to x"

$\frac{df}{dx}$ "the derivative of f with respect to x"

$\frac{d}{dx} f(x)$ "the derivative of f of x"

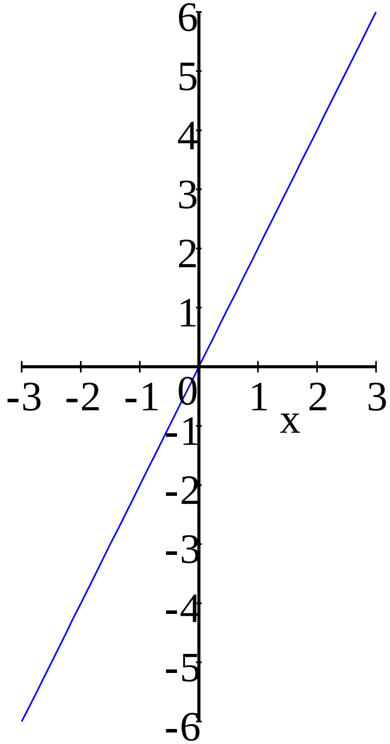


The derivative is the slope of the original function.



$$y = x^2 - 3$$

$$y' = \lim_{h \to 0} \frac{(x+h)^2 - 3 - (x^2 - 3)}{h}$$



$$y' = \lim_{h \to 0} \frac{\cancel{x^2} + 2x\cancel{h} + \cancel{h^2} - \cancel{x^2}}{\cancel{h}}$$

$$y' = \lim_{h \to 0} 2x + \underbrace{h}_{0}$$

$$y' = 2x$$

Differentiability

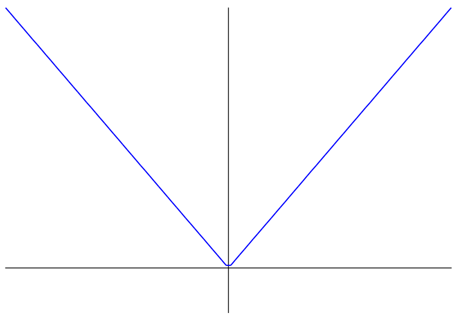
A function is differentiable if it has a derivative everywhere in its domain. It must be continuous and smooth. Functions on closed intervals must have one-sided derivatives defined at the end points.

Differentiability

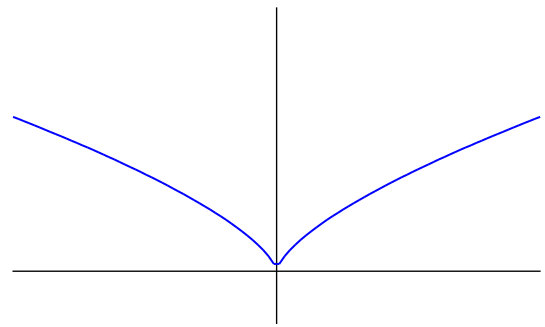
To be differentiable, a function must be <u>continuous</u> and <u>smooth</u>.

Derivatives will fail to exist at:

$f(x) = |x|$

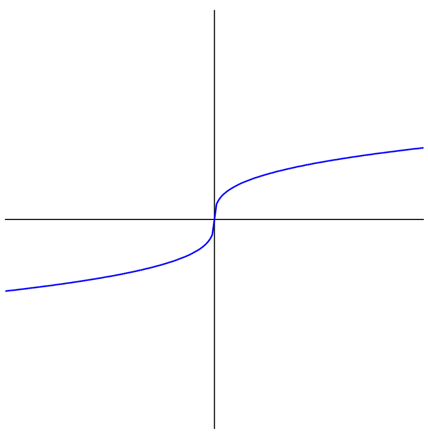


corner



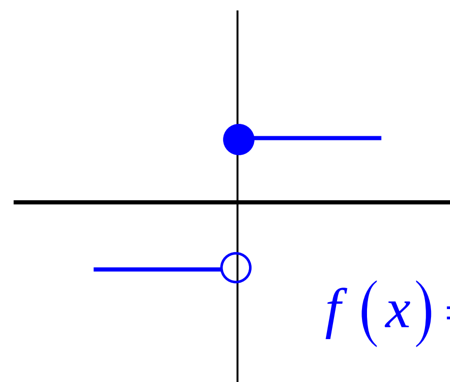
$f(x) = x^{\frac{2}{3}}$

cusp



$f(x) = \sqrt[3]{x}$

vertical tangent



$$f(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

discontinuity

Rules for Differentiation

If the derivative of a function is its slope, then for a constant function, the derivative must be zero.

$$\boxed{\frac{d}{dx}(c)=0}$$

example: $y=3$

$$y'=0$$

The derivative of a constant is zero.

Derivatives of monomials

$$\frac{d}{dx}x^2 = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{\left(\cancel{x^2} + 2x\cancel{h} + h^{\cancel{2}}\right) - \cancel{x^2}}{\cancel{h}} = 2x$$

$$\frac{d}{dx}x^3 = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0}\frac{\left(\cancel{x^3} + 3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\overset{2}{\cancel{3}}}\right) - \cancel{x^3}}{\cancel{h}} = 3x^2$$

$$\frac{d}{dx}x^4 = \lim_{h\to 0}\frac{\left(\cancel{x^4} + 4x^3\cancel{h} + 6x^2h^{\cancel{2}} + 4xh^{\overset{2}{\cancel{3}}} + h^{\overset{3}{\cancel{4}}}\right) - \cancel{x^4}}{\cancel{h}} = 4x^3$$

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

(Pascal's Triangle)

We observe a pattern: $2x \quad 3x^2 \quad 4x^3 \quad 5x^4 \quad 6x^5 \quad \ldots$

Derivatives of monomials

We observe a pattern: $2x \quad 3x^2 \quad 4x^3 \quad 5x^4 \quad 6x^5 \quad \ldots$

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}}$$

power rule

examples:

$$f(x) = x^4$$

$$f'(x) = 4x^3$$

$$y = x^8$$

$$y' = 8x^7$$

Constant multiple rule:

$$\frac{d}{dx}\left(cu\right)=c\frac{du}{dx}$$

examples:

$$\frac{d}{dx}cx^{n}=cnx^{n-1}$$

$$\frac{d}{dx}7x^{5}=7\cdot 5x^{4}=35x^{4}$$

When we used the difference quotient, we observed that since the limit had no effect on a constant coefficient, that the constant could be factored to the outside.

Sum and difference rules:

$$\boxed{\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}}$$

$$\boxed{\frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}}$$

$$y=x^4+12x$$

$$y'=4x^3+12$$

$$y=x^4-2x^2+2$$

$$\frac{dy}{dx}=4x^3-4x$$

Product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Notice that this is <u>not</u> just the product of two derivatives.

$$\frac{d}{dx}\left[\left(x^2+3\right)\left(2x^3+5x\right)\right] = \left(x^2+3\right)\left(6x^2+5\right) + \left(2x^3+5x\right)\left(2x\right)$$

$$\frac{d}{dx}\left(2x^5+5x^3+6x^3+15x\right)$$

$$\frac{d}{dx}\left(2x^5+11x^3+15x\right) \qquad 6x^4+5x^2+18x^2+15+4x^4+10x^2$$

$$10x^4+33x^2+15 \qquad 10x^4+33x^2+15$$

Quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Alternative:

F'(v\u)=f'(v)*u^-1+f'(u^-1)*v

$$\frac{d}{dx}\frac{2x^3+5x}{x^2+3} = \frac{\left(x^2+3\right)\left(6x^2+5\right)-\left(2x^3+5x\right)\left(2x\right)}{\left(x^2+3\right)^2}$$

Derivatives of trigonometric functions

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\sec x = \sec x \cdot \tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cdot \cot x$$

Derivatives of Exponential and Logarithmic Functions

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

Chain rule

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

If $f \circ g$ is the composite of $y = f(u)$ and $u = g(x)$, then:

$$(f \circ g)' = f'_{\text{at } u=g(x)} \cdot g'_{\text{at } x}$$

Example

$$y=\sin\left(x^2-4\right)$$

$$y'=\cos\left(x^2-4\right)\cdot\frac{d}{dx}\left(x^2-4\right)$$ Differentiate the outside function...

$$y'=\cos\left(x^2-4\right)\cdot 2x$$ ...then the inside function

At $x=2,\ y'=4$

→

Example

$$\frac{d}{dx}\cos^2(3x)$$

$$\frac{d}{dx}\left[\cos(3x)\right]^2$$

It looks like we need to use the chain rule again!

$$2\left[\cos(3x)\right]\cdot\frac{d}{dx}\cos(3x)$$

$$2\cos(3x)\cdot -\sin(3x)\cdot\frac{d}{dx}(3x)$$

The chain rule can be used more than once.

$$-2\cos(3x)\cdot\sin(3x)\cdot 3$$

(That's what makes the "chain" in the "chain rule"!)

$$-6\cos(3x)\sin(3x)$$

→

Higher Order Derivatives:

$$y' = \frac{dy}{dx}$$ is the first derivative of y with respect to x.

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$ is the <u>second</u> derivative.

(y double prime)

$$y''' = \frac{dy''}{dx}$$ is the third derivative.

$$y^{(4)} = \frac{d}{dx}y'''$$ is the fourth derivative.

Extreme Values of Functions

Global and Local extrema

Absolute extreme values are either maximum or minimum points on a curve.

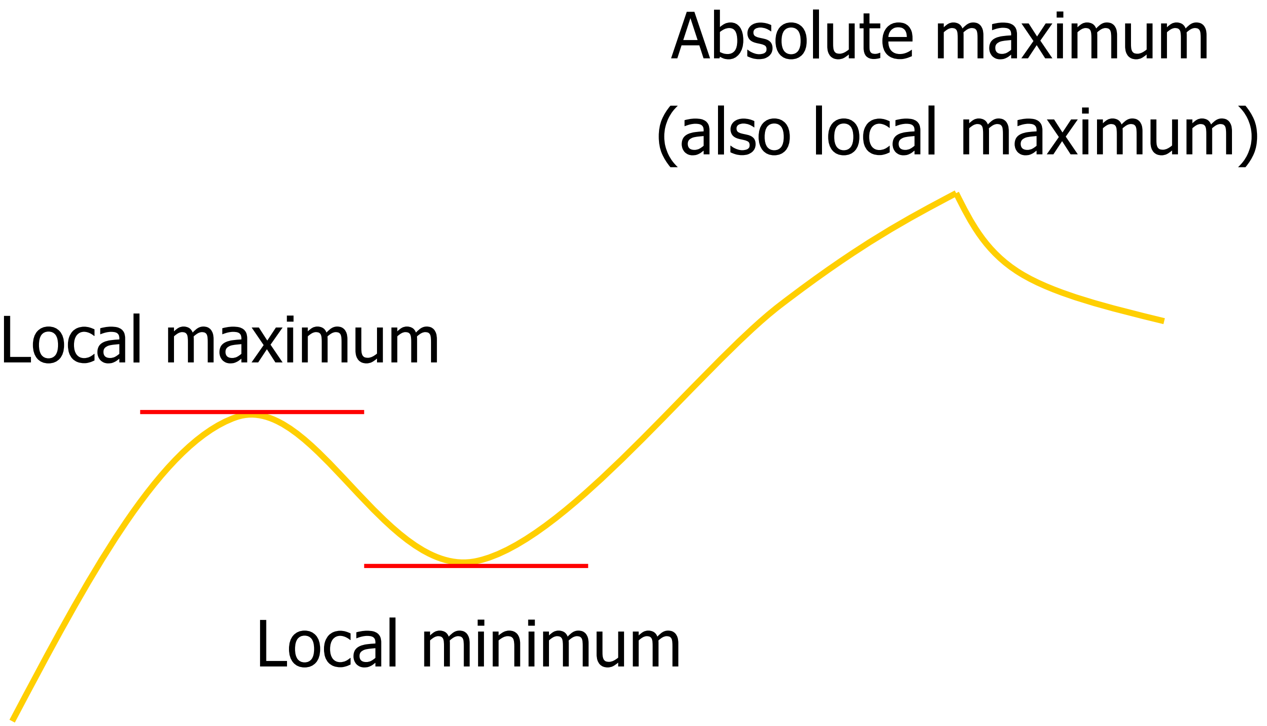
They are sometimes called global extremes.

They are also sometimes called absolute extrema. (*Extrema* is the plural of the Latin *extremum*.)

A local maximum is the maximum value within some open interval.

A local minimum is the minimum value within some open interval.

Global and Local extrema



Notice that local extremes in the interior of the function occur where f' is zero or f' is undefined.

→

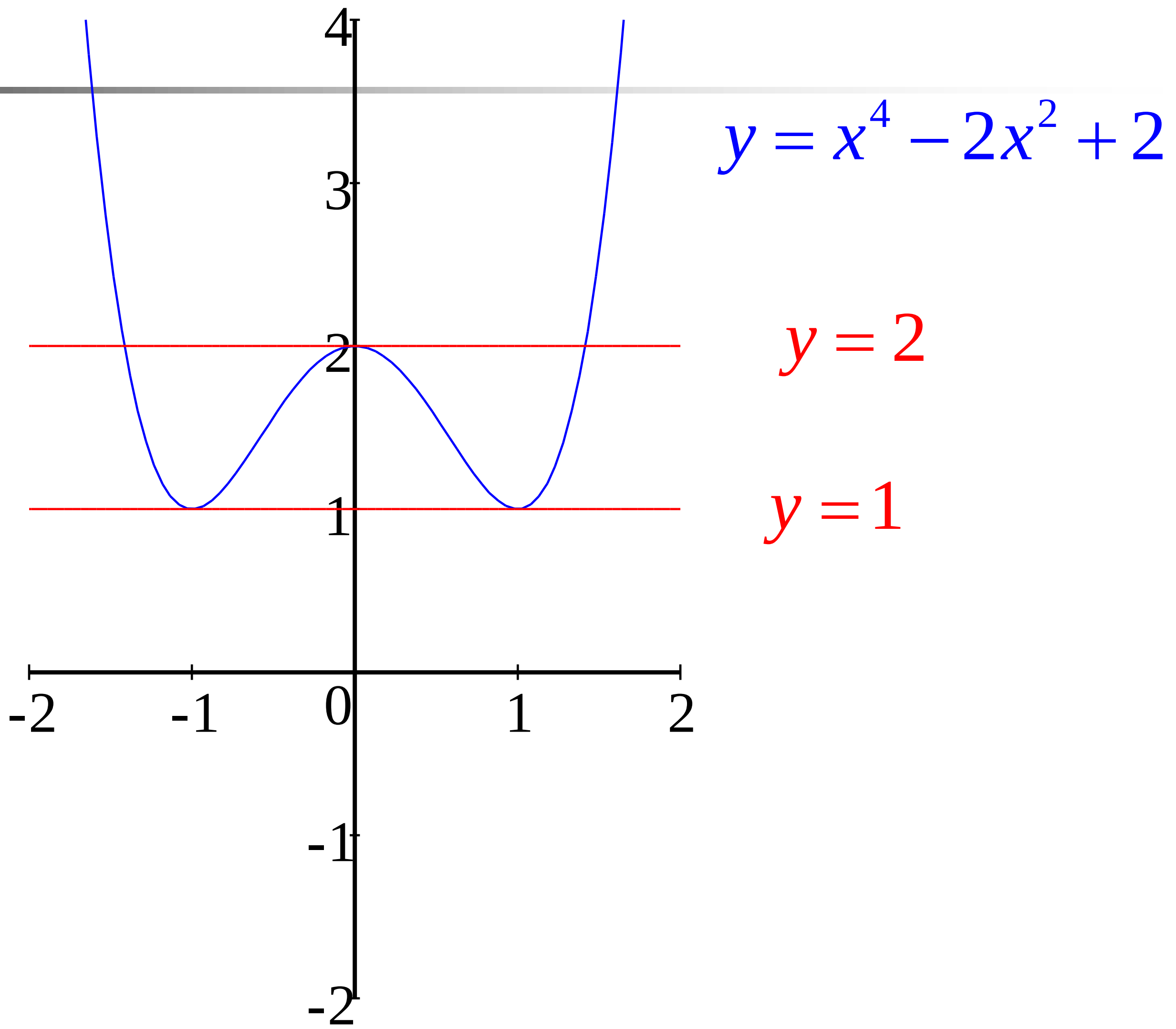
Local Extreme Values:

If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then

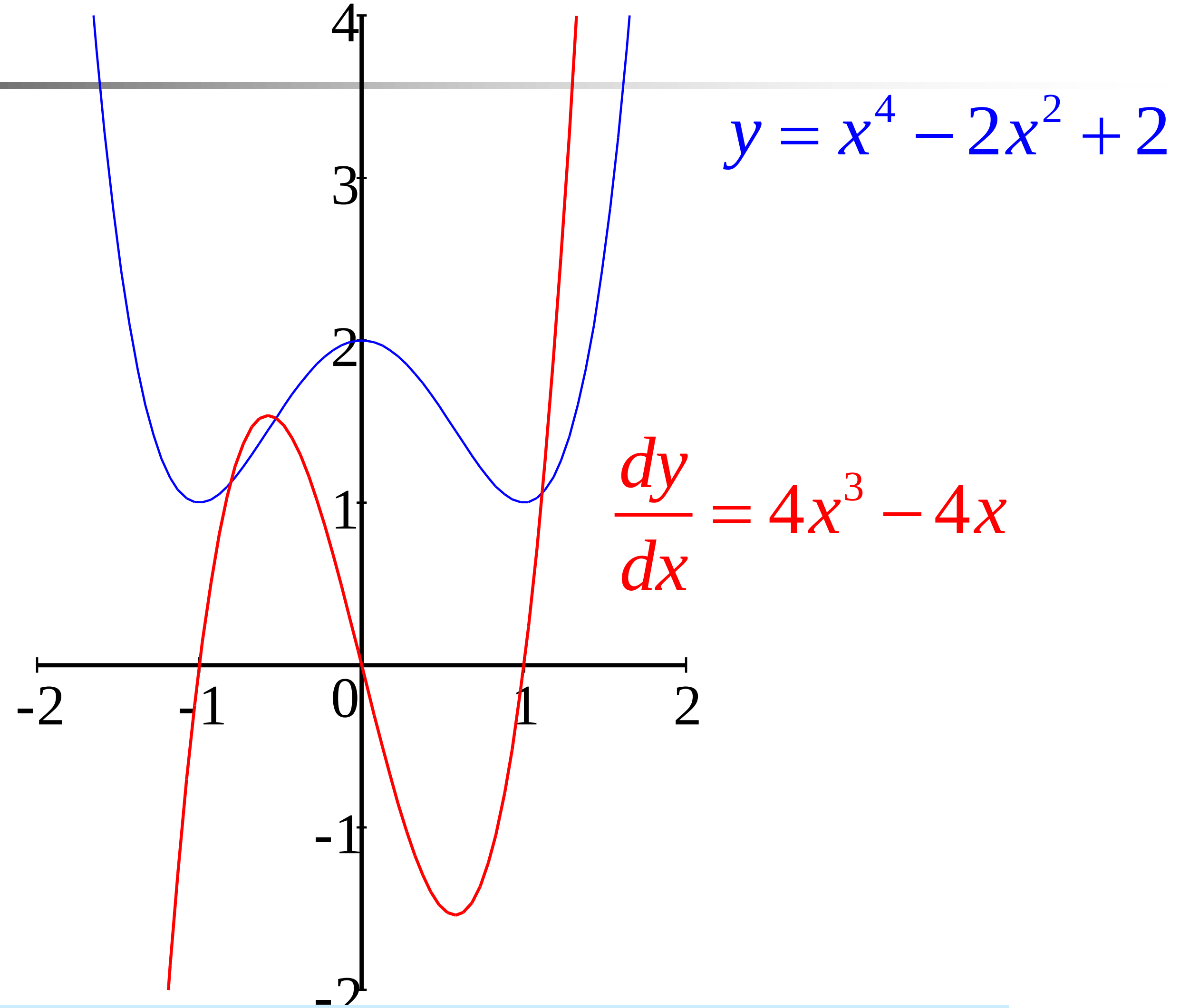
$$f'(c) = 0$$

First derivate to determine if it is a MAX or a MIN then the second derivate:
f''>0=MIN
f''<0=max

$\rightarrow$



$$\frac{dy}{dx} = 4x^3 - 4x$$



First derivative (slope) is zero at:

$$x = 0, -1, 1$$

Critical Point:

A point in the domain of a function f at which $f' = 0$ or f' does not exist is a **critical point** of f.

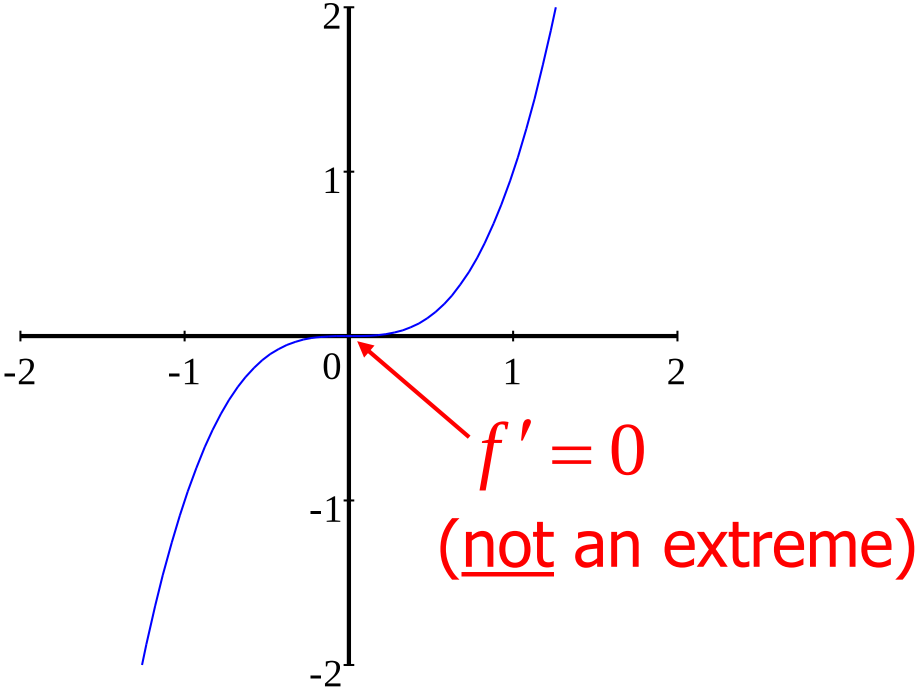
Note:
Maximum and minimum points in the interior of a differentiable function always occur at critical points,

BUT

critical points are not always maximum or minimum values.

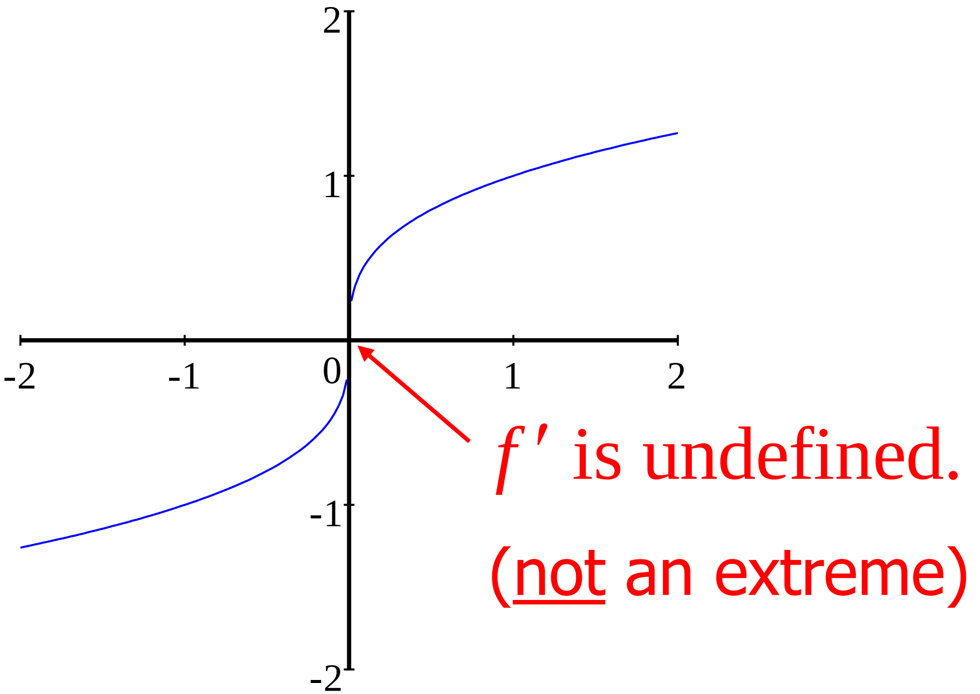
Critical points are <u>not always</u> extremes!

$$y = x^3$$



If f''=0 USUALLY is a flex

$y = x^{1/3}$



Finding absolute extrema

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

$$f(x) = x^{2/3}$$

$$f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$$

There are no values of x that will make the first derivative equal to zero.

$$f'(x) = \frac{2}{3\sqrt[3]{x}}$$

The first derivative is undefined at x=0, so (0,0) is a critical point.

Because the function is defined over a closed interval, we also must check the endpoints.

Finding absolute extrema

$$f(x) = x^{2/3} \quad D = [-2, 3]$$

At: $x = 0 \quad f(0) = 0$

To determine if this critical point is actually a maximum or minimum, we try points on either side, without passing other critical points.

$$f(-1) = 1 \qquad f(1) = 1$$

Since 0<1, this must be at least a local minimum, and possibly a global minimum.

At: $x = -2$ $f(-2) = (-2)^{\frac{2}{3}} \approx 1.5874$

At: $x = 3$ $f(3) = (3)^{\frac{2}{3}} \approx 2.08008$

Absolute minimum: $(0, 0)$

Absolute maximum: $(3, 2.08)$

Finding Maxima and Minima Analytically:

1. Find the derivative of the function, and determine where the derivative is zero or undefined. These are the critical points.

2. Find the value of the function at each critical point.

3. Find values or slopes for points between the critical points to determine if the critical points are maximums or minimums.

4. For closed intervals, check the end points as well.

Using Derivatives for Curve Sketching

Rules

First derivative:

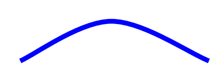
y' is positive → Curve is rising.

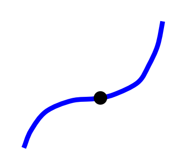
y' is negative → Curve is falling.

y' is zero → <u>Possible</u> local maximum or minimum.

Second derivative:

y'' is positive → Curve is concave up. 

y'' is negative → Curve is concave down. 

y'' is zero → <u>Possible</u> inflection point (where concavity changes). 

Example:

Graph $y = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

There are roots at $x = -1$ and $x = 2$.

$y' = 3x^2 - 6x$

Possible extreme at $x = 0,\ 2$.

We can use a chart to organize our thoughts.

Set $y' = 0$

First derivative test:

$0 = 3x^2 - 6x$

$0 = x^2 - 2x$

$0 = x(x-2)$

$x = 0,\ 2$

y'	$+$	0	$-$	0	$+$
		0		2	

$y'(1) = 3 \cdot 1^2 - 6 \cdot 1 = -3$ ➡ negative

$y'(-1) = 3(-1)^2 - 6(-1) = 9$ ➡ positive

$y'(3) = 3 \cdot 3^2 - 6 \cdot 3 = 9$ ➡ positive

$\rightarrow$

Example:

Graph $y = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

There are roots at $x = -1$ and $x = 2$.

$y' = 3x^2 - 6x$

Set $y' = 0$

$0 = 3x^2 - 6x$

$0 = x^2 - 2x$

$0 = x(x-2)$

$x = 0,\ 2$

<u>Possible</u> extreme at $x = 0,\ 2$.

First derivative test:

y': $+$ 0 $-$ 0 $+$

0 2

$\therefore$ maximum at $x = 0$

minimum at $x = 2$

$\rightarrow$

Example:

Graph $y = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

There are roots at $x = -1$ and $x = 2$.

$y' = 3x^2 - 6x$

Possible extreme at $x = 0,\ 2$.

Or you could use the second derivative test:

$y'' = 6x - 6$

$y''(0) = 6 \cdot 0 - 6 = -6$

Because the second derivative at $x = 0$ *is negative, the graph is concave down and therefore* $(0,4)$ *is a local maximum.*

$y''(2) = 6 \cdot 2 - 6 = 6$

Because the second derivative at $x = 2$ *is positive, the graph is concave up and therefore* $(2,0)$ *is a local minimum.*

Example:

Graph $y = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

We then look for inflection points by setting the second derivative equal to zero.

$$y'' = 6x - 6$$

$$0 = 6x - 6$$

$$6 = 6x$$

$$1 = x$$

<u>Possible</u> inflection point at $x = 1$.

y''	$-$	0	$+$
		1	

$y''(0) = 6 \cdot 0 - 6 = -6$ ➡ negative

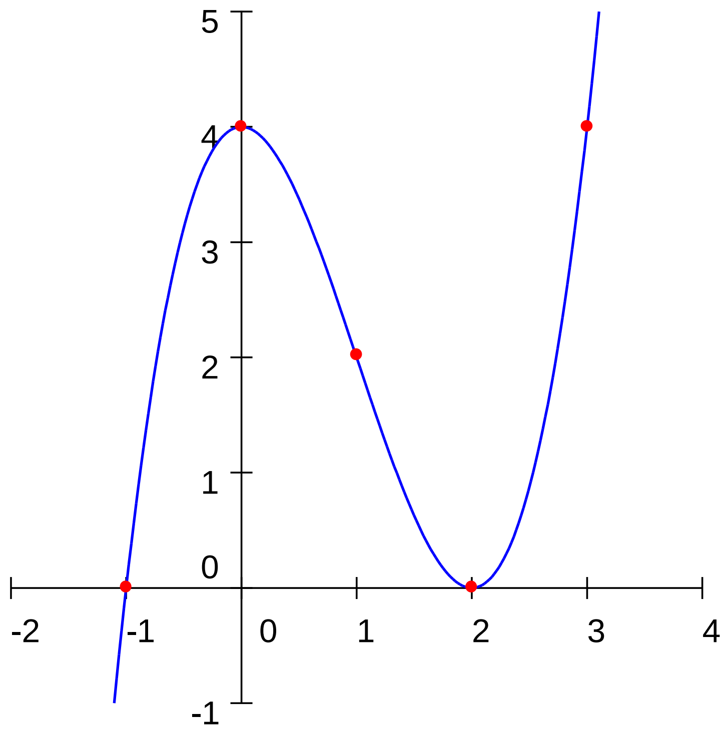
$y''(2) = 6 \cdot 2 - 6 = 6$ ➡ positive

There is an inflection point at $x = 1$ because the second derivative changes from negative to positive.

$\rightarrow$

Make a summary table:

x	y	y'	y''	
-1	0	9	-12	rising, concave down
0	4	0	-6	local max
1	2	-3	0	falling, inflection point
2	0	0	6	local min
3	4	9	12	rising, concave up



Taylor Series

Suppose we wanted to find a fourth degree polynomial of the form:

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

that approximates the behavior of $f(x) = \ln(x+1)$ at $x = 0$

If we make $P(0) = f(0)$, and the first, second, third and fourth derivatives the same, then we would have a pretty good approximation.

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$f(x) = \ln(x+1)$$

$$f(x) = \ln(x+1) \qquad P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$f(0) = \ln(1) = 0 \qquad P(0) = a_0 \rightarrow \boxed{a_0 = 0}$$

$$f'(x) = \frac{1}{1+x} \qquad P'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

$$f'(0) = \frac{1}{1} = 1 \qquad P'(0) = a_1 \rightarrow \boxed{a_1 = 1}$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad P''(x) = 2a_2 + 6a_3 x + 12a_4 x^2$$

$$f''(0) = -\frac{1}{1} = -1 \qquad P''(0) = 2a_2 \rightarrow \boxed{a_2 = -\frac{1}{2}}$$

Derive both the function and the polinome and put the point in th funcction and the derivate of the polinome should be the same as the result of the same grade derivate of the function

$$P(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4$$

$$f(x)=\ln(x+1)$$

$$f'''(x)=2\cdot\frac{1}{(1+x)^3}$$

$$P'''(x)=6a_3+24a_4x$$

$$f'''(0)=2$$

$$P'''(0)=6a_3 \rightarrow \boxed{a_3=\frac{2}{6}}$$

$$f^{(4)}(x)=-6\frac{1}{(1+x)^4}$$

$$P^{(4)}(x)=24a_4$$

$$f^{(4)}(0)=-6$$

$$P^{(4)}(0)=24a_4 \rightarrow \boxed{a_4=-\frac{6}{24}}$$

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

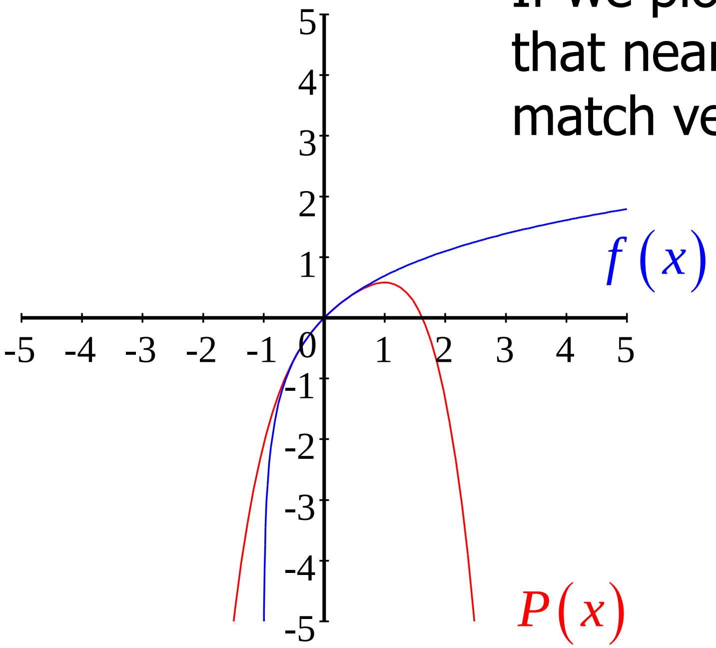
$$f(x) = \ln(x+1)$$

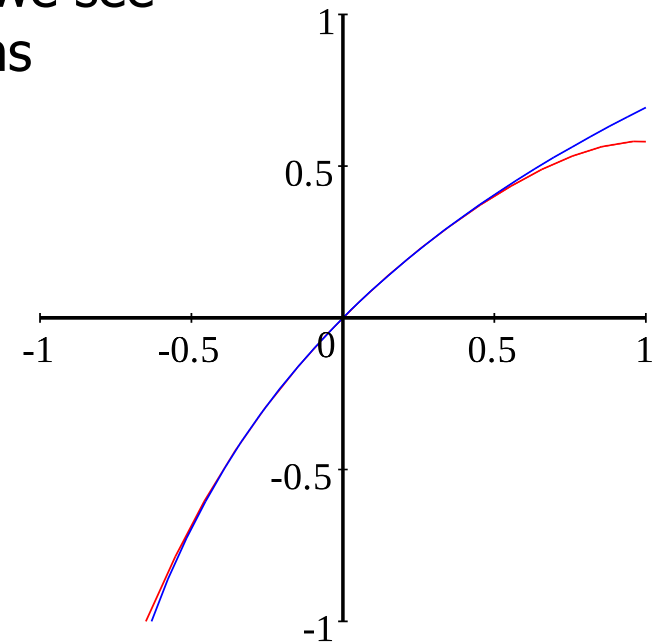
$$P(x) = 0 + 1x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4$$

$$P(x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

$$f(x) = \ln(x+1)$$

If we plot both functions, we see that near zero the functions match very well!





Maclaurin Series:

(generated by f at $x=0$)

$$P(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

If we want to center the series (and it's graph) at some point other than zero, we get the Taylor Series:

Taylor Series:

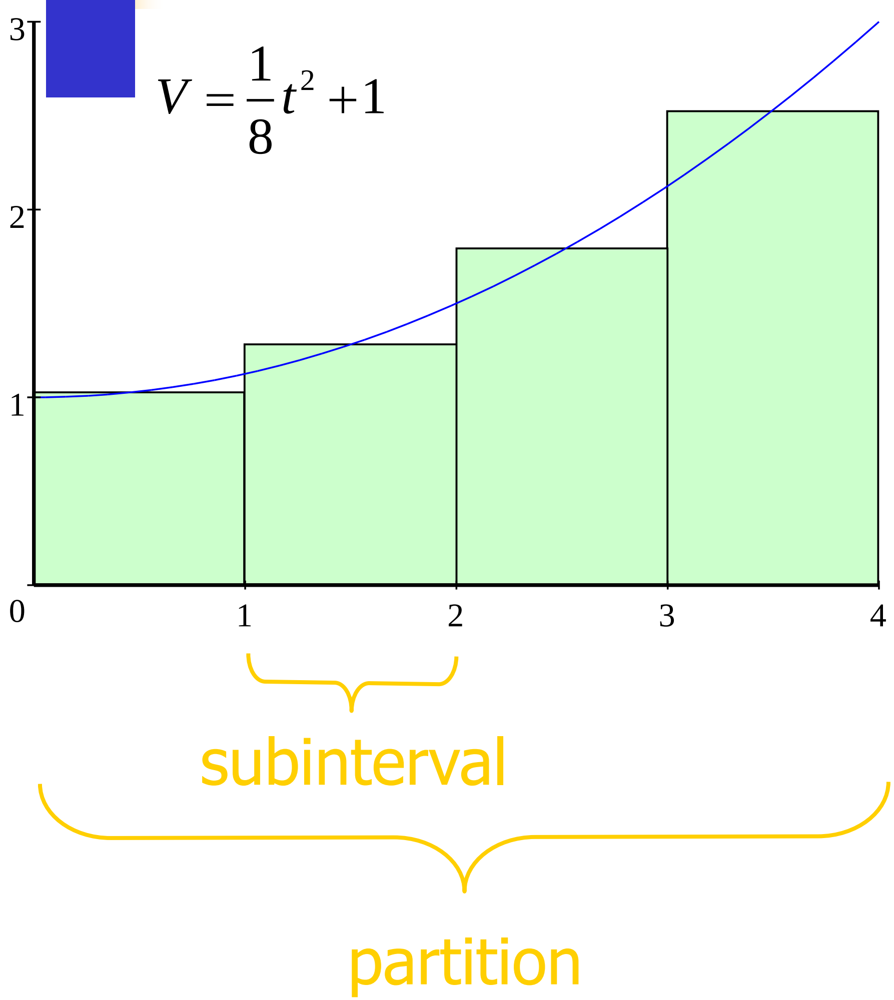
(generated by f at $x=a$)

a=the value we want our function aproximated at

$$P(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

Definite Integrals

Riemann sum



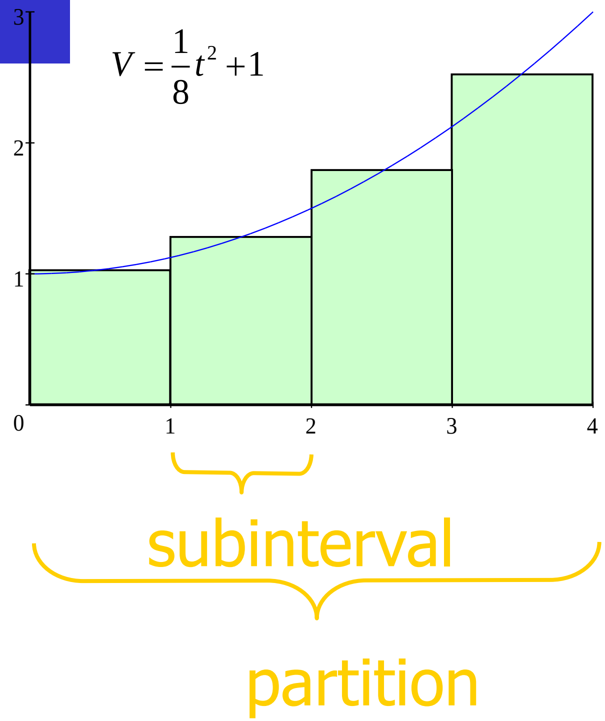
When we find the area under a curve by adding rectangles, the answer is called a **Riemann sum**.

The width of a rectangle is called a **subinterval**.

The entire interval is called the **partition**.

Subintervals do not all have to be the same size.

Riemann sum



If the partition is denoted by P, then the length of the longest subinterval is called the **norm** of P and is denoted by $\|P\|$.

As $\|P\|$ gets smaller, the approximation for the area gets better.

$$\text{Area} = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f\left(c_k\right)\Delta x_k$$ if P is a partition of the interval $[a, b]$

Definite integrals

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f\left(c_k\right)\Delta x_k$$ is called the **definite integral** of f over $\left[a,b\right]$.

If we use subintervals of equal length, then the length of a subinterval is: $\Delta x = \dfrac{b-a}{n}$

The definite integral is then given by:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(c_k\right)\Delta x = \int_a^b f\left(x\right)dx$$

Definite integrals

upper limit of integration

Integration Symbol

$$\int_a^b \underbrace{f(x)}_{\text{integrand}}\,dx$$

lower limit of integration

variable of integration
(dummy variable)

It is called a dummy variable because the answer does not depend on the variable chosen.

Area $= \lim_{\|P\| \to 0} \sum_{k=1}^{n} f\left(c_k\right) \Delta x_k$

$$= \int_a^b f\left(x\right) dx$$ Definite integral

$$= F(b) - F(a)$$

Where F is a function : $\dfrac{dF}{dx} = f\left(x\right)$

F is called indefinite integral

→

The Fundamental Theorem of Calculus

If f is continuous on $[a,b]$, then the function

$$F(x)=\int_a^x f(t)\,dt$$

has a derivative at every point in $[a,b]$, and

$$\frac{dF}{dx}=\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$$

If a function is coantinue it has a integrals and if you derivate it gives back the original function

$\rightarrow$

Proof:

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

$$= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}$$

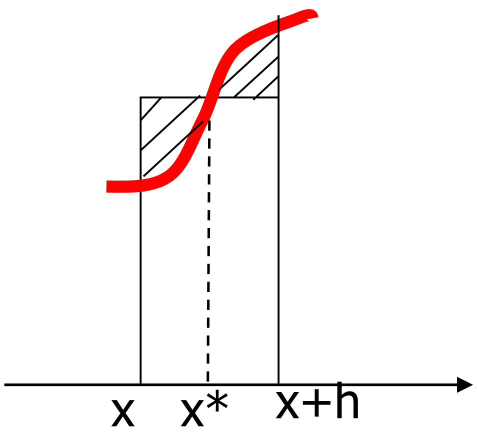
$$= \lim_{h\to 0}\frac{f(x^*)\Delta x}{h}$$

$$= \lim_{h\to 0}\frac{f(x^*)h}{h}$$

$$= \lim_{h\to 0} f(x^*)$$

$$= f(x),$$

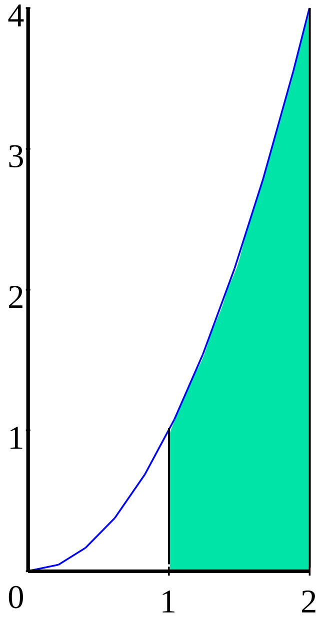
Where x^* is a point between x and $x+h$



Example

$$y = x^2$$

Find the area under the curve from x=1 to x=2.



$$A = \int_1^2 x^2 dx$$

A=f(2)-f(1)

$$F = \frac{x^3}{3} + c$$

The indefinite integral is defined up to a constant c

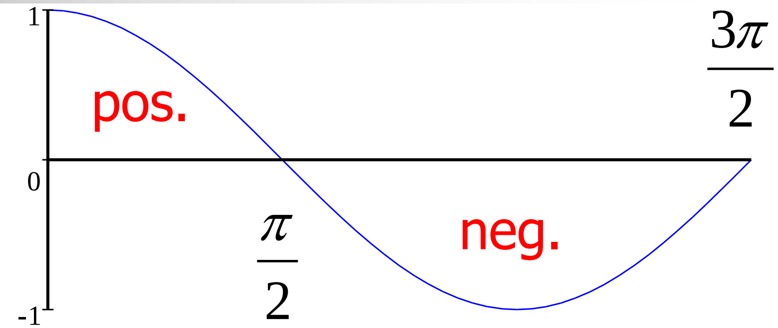
Proof: $$\frac{dF}{dx} = 3\frac{x^2}{3} + 0 = x^2 = f(x)$$

$$A = \int_1^2 x^2 dx = F(2) - F(1) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

if the funcion is both positive and negative you have to divide the two integral or the areas will cancel out

Example

Find the area between the x-axis and the curve $y = \cos x$ from $x = 0$ to $x = \dfrac{3\pi}{2}$.



$$\int_0^{\frac{\pi}{2}} \cos x \, dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, dx = \sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{3\pi/2} =$$

$$= \left(\sin\frac{\pi}{2} - \sin 0 \right) - \left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2} \right) = 3$$

This because

$F = \sin x + c$ Proof: $\dfrac{dF}{dx} = \cos x = f(x)$

Rules for integrals

1. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Reversing the limits changes the sign.

2. $$\int_a^a f(x)\,dx = 0$$

If the upper and lower limits are equal, then the integral is zero.

3. $$\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$$

Constant multiples can be moved outside.

4. $$\int_a^b \left[f(x) + g(x) \right] dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Integrals can be added and subtracted.

5. $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

<u>Intervals</u> can be added (or subtracted.)

Integration by Substitution

The chain rule allows us to differentiate a wide variety of functions, but we are able to find antiderivatives for only a limited range of functions. We can sometimes use substitution to rewrite functions in a form that we can integrate.

You need to be able to remove X from the funtion but also from the trasformation of Dx in Du or it wont work

Example 1:

$$\int \left(\underline{x+2}\right)^5 dx$$

$$\text{Let } u = x+2$$

$$du = dx$$

$$\int u^5 du$$

$$\frac{1}{6}u^6 + C$$

$$\frac{\left(x+2\right)^6}{6} + C$$

<u>Don't</u> forget to substitute the value for u back into the problem!

Example 2

$$\int \sqrt{1+x^2} \cdot 2x \, dx$$

Let $u = 1 + x^2$

The derivative of $1+x^2$ is 2x

$\cdot$ $du = 2x\, dx$

$$\int u^{\frac{1}{2}} \, du$$

$$\frac{2}{3} u^{\frac{3}{2}} + C$$

$$\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + C$$

Example 3:

$$\int \sqrt{4x-1}\, dx$$

Let $u = 4x-1$

$$du = 4\, dx \qquad \frac{1}{4}du = dx$$

in the trasformation of dx in du we managed to remove X so it is ok to substitute

$$\int u^{\frac{1}{2}} \cdot \frac{1}{4} du$$

$$\frac{2}{3}u^{\frac{3}{2}} \cdot \frac{1}{4} + C$$

$$\frac{1}{6}u^{\frac{3}{2}} + C$$

$$\frac{1}{6}(4x-1)^{\frac{3}{2}} + C$$

→

Example 4

$$\int \cos(7x+5)\ dx$$

Let $u = 7x+5$

$$du = 7\ dx \qquad \frac{1}{7}du = dx$$

usefull to remove coplex variable from funtion

$$\int \cos u \cdot \frac{1}{7} du$$

$$\frac{1}{7}\sin u + C$$

$$\frac{1}{7}\sin(7x+5) + C$$

$\rightarrow$

Example 5

$$\int x^2 \sin\left(x^3\right)\, dx$$

Let $u = x^3$

$du = 3x^2\, dx$ $\qquad \frac{1}{3} du = \underbrace{x^2\, dx}$

We solve for $x^2\, dx$ because we can find it in the integrand.

$$\frac{1}{3}\int \sin u\, du$$

$$-\frac{1}{3}\cos u + C$$

$$-\frac{1}{3}\cos x^3 + C$$

Example 6

$$\int \sin^4 x \cdot \cos x \, dx$$

$$\int (\sin x)^4 \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$\int u^4 \, du$$

$$\frac{1}{5}u^5 + C$$

$$\frac{1}{5}\sin^5 x + C$$

Integration By Parts

Start with the product rule:

$$\frac{d}{\cancel{dx}}(uv) = u\frac{dv}{\cancel{dx}} + v\frac{du}{\cancel{dx}}$$

$$d(uv) = u\ dv + v\ du$$

$$d(uv) - v\ du = u\ dv$$

$$u\ dv = d(uv) - v\ du$$

$$\int u\ dv = \int \left(d(uv) - v\ du\right)$$

$$\int u\ dv = \int \left(d(uv)\right) - \int v\ du$$

$$\boxed{\int u\ dv = uv - \int v\ du}$$

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)dx$$

This is the Integration by Parts formula.

→

Integration By Parts

$$\int u(x)\frac{dv(x)}{dx}dx = u(x)v(x) - \int \frac{du(x)}{dx}v(x)dx$$

$$\boxed{\int u\ dv = uv - \int v\ du}$$

Start with the product rule:

$$\frac{d}{dx}[u(x)v(x)] = \frac{du(x)}{dx}v(x) + u(x)\frac{dv(x)}{dx}$$

$$u(x)\frac{dv(x)}{dx} = \frac{d}{dx}[u(x)v(x)] - \frac{du(x)}{dx}v(x)$$

$$\int u(x)\frac{dv(x)}{dx}dx = \int \frac{d}{dx}[u(x)v(x)]dx - \int \frac{du(x)}{dx}v(x)dx$$

$$\int u(x)\frac{dv(x)}{dx}dx = u(x)v(x) - \int \frac{du(x)}{dx}v(x)dx$$

u can be always differentiated

v is easy to integrate.

The Integration by Parts formula is a "product rule" for integration.

Example 1

$$\int x \cdot \cos x \, dx$$

Easy to integrate

$$u = x \qquad \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = 1 \qquad v = \sin x$$

$$\int u(x)\frac{dv(x)}{dx}dx = u(x)v(x) - \int \frac{du(x)}{dx}v(x)dx$$

$$\int x \cos x \cdot dx = x \sin x - \int 1 \cdot \sin x \cdot dx =$$

$$= x \sin x + \cos x + C$$

Example 2

$$\int \boxed{1} \ln x \, dx \quad \Rightarrow \quad u = \ln x \qquad \frac{dv}{dx} = 1$$

put o 1 infront of it to have somrthing to derivate

$$\frac{du}{dx} = \frac{1}{x} \qquad v = x$$

$$\int u(x)\frac{dv(x)}{dx}dx = u(x)v(x) - \int \frac{du(x)}{dx}v(x)dx$$

$$\int \ln x \cdot 1 \cdot dx = \ln x \cdot x - \int \frac{1}{x} \cdot x \cdot dx =$$

$$= x \ln x - x + C$$

Example 3 double part integration

$$u = x^2 \qquad \frac{dv}{dx} = e^x$$

$$\int x^2 e^x dx =$$

Easy to integrate

$$\frac{du}{dx} = 2x \quad v = e^x$$

$$= x^2 e^x - \int 2x \cdot e^x dx =$$

Easy to integrate

$$u^* = x \qquad \frac{dv^*}{dx} = e^x$$

$$\frac{du^*}{dx} = 1 \quad v^* = e^x$$

$$= x^2 e^x - 2\left[x \cdot e^x - \int 1 \cdot e^x dx \right] =$$

$$= x^2 e^x - 2x \cdot e^x + 2e^x + C$$