Matrices

Matrices

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix}$$

Both A and B are examples of matrix. A matrix is a rectangular array of numbers enclosed by a pair of bracket.

Why matrix?

Matrices and linear systems of equations

Consider the following set of equations:

$$\begin{cases} x + y = 7, \\ 3x - y = 5. \end{cases}$$

It is easy to show that $x = 3$ and $y = 4$.

How about solving

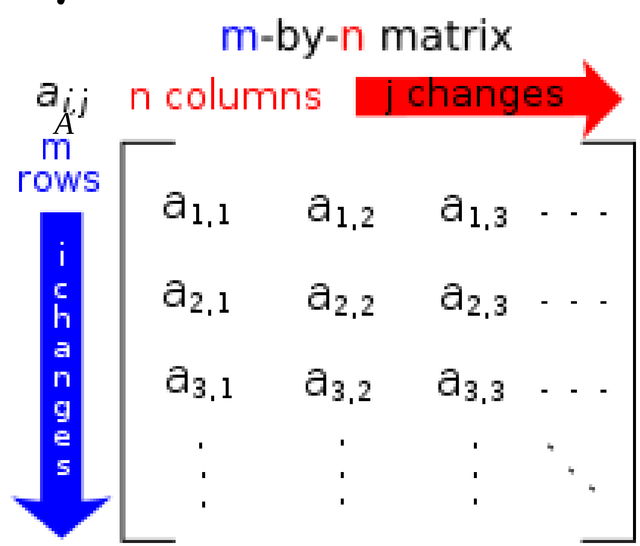
$$\begin{cases} x + y - 2z = 7, \\ 2x - y - 4z = 2, \\ -5x + 4y + 10z = 1, \\ 3x - y - 6z = 5. \end{cases}$$

Matrices can help…

Matrices: definitions

- A matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns that is interpreted and manipulated in certain prescribed ways.

$$A \in \Re^{m \times n}$$



Size of A = m x n

Matrices: definitions

In the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

- numbers a_{ij} are called *elements*. First subscript indicates the row; second subscript indicates the column. The matrix consists of mn elements
- It is called "the $m \times n$ matrix $A = [a_{ij}]$" or simply "the matrix A" if number of rows and columns are understood.

Square matrices

- When $m = n$, i.e., $$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$
- A is called a "square matrix of order n" or "n-square matrix"
- elements a_{11}, a_{22}, a_{33},..., a_{nn} called diagonal elements.
- $\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$ is called the *trace* of A.

Column and row matrices

- When $n = 1$, i.e., $$A = \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ a_{m1} \end{bmatrix}$$
- A is called a "column matrix"
- A n-valued vector is represented as a column matrix
- When $m = 1$, i.e., $$A = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \end{bmatrix}$$
- A is called a "row matrix"

Equal matrices

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal ($A = B$) iff
  - They have the same size

    AND
  - Each element of A is equal to the corresponding element of B, i.e., $a_{ij} = b_{ij}$ for $1 \le i \le m, 1 \le j \le n$.
- *iff* pronouns "if and only if"

  if $A = B$, it implies $a_{ij} = b_{ij}$ for $1 \le i \le m, 1 \le j \le n$;

  if $a_{ij} = b_{ij}$ for $1 \le i \le m, 1 \le j \le n$, it implies $A = B$.

Equal matrices

Example: $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Given that $A = B$, find a, b, c and d.

if $A = B$, then $a = 1$, $b = 0$, $c =$ -4 and $d = 2$.

Zero matrices

- If every element of a matrix is zero, it is called a zero matrix, i.e.,

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}$$

Operations: Sum of matrices

- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then $A + B$ is defined as a matrix $C = A + B$, where $C = [c_{ij}]$, $c_{ij} = a_{ij} + b_{ij}$ for $1 \le i \le m$, $1 \le j \le n$.

Example: if $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$

Evaluate $A + B$ and $A - B$.

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0+(-1) & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0-(-1) & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Operations: Sum of matrices

- Two matrices of the <u>same</u> size are said to be *conformable* for addition or subtraction.
- Two matrices of <u>different</u> size cannot be added or subtracted, e.g.,

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix}$$

are NOT conformable for addition or subtraction.

Operations: Scalar multiplication

- Let λ be any scalar and $A = [a_{ij}]$ is an $m \times n$ matrix. Then $\lambda A = [\lambda a_{ij}]$ for $1 \le i \le m, 1 \le j \le n$, i.e., each element in A is multiplied by λ.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$. Evaluate $3A$.

$$3A = \begin{bmatrix} 3\times 1 & 3\times 2 & 3\times 3 \\ 3\times 0 & 3\times 1 & 3\times 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 12 \end{bmatrix}$$

- In particular, $\lambda = -1$, i.e., $-A = [-a_{ij}]$. It's called the *negative* of A.
  - Note: $A - A = 0$ is a zero matrix

Operations: basic properties

Matrices A, B and C are conformable,

- $A + B = B + A$ (commutative law)
- $A + (B + C) = (A + B) + C$ (associative law)
- $\lambda(A + B) = \lambda A + \lambda B$, where λ is a scalar (distributive law)

Can you prove them?

Operations: basic properties

Example: Prove $\lambda(A + B) = \lambda A + \lambda B$.

Let $C = A + B$, so $c_{ij} = a_{ij} + b_{ij}$.

Consider $\lambda c_{ij} = \lambda\,(a_{ij} + b_{ij}) = \lambda a_{ij} + \lambda b_{ij}$, we have, $\lambda C = \lambda A + \lambda B$.

Since $\lambda C = \lambda(A + B)$, so $\lambda(A + B) = \lambda A + \lambda B$

Operations: matrix product

- If $A = [a_{ij}]$ is a $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then AB is defined as a $m \times n$ matrix $C = AB$, where $C = [c_{ij}]$ with

$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{ip} b_{pj}$ for $1 \le i \le m$, $1 \le j \le n$.

$$\mathfrak{R}^{m \times p} \times \mathfrak{R}^{p \times n} \rightarrow \mathfrak{R}^{m \times n}$$

Two matrices are conformable for the product if the number of columns of the first is equal to the number of rows of the second

Operations: matrix product

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$ and $C = AB$.

Evaluate c_{21}

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \qquad c_{21} = 0 \times (-1) + 1 \times 2 + 4 \times 5 = 22$$

Operations: matrix product

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$, Evaluate $C = AB$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \Rightarrow \begin{cases} c_{11} = 1\times(-1) + 2\times 2 + 3\times 5 = 18 \\ c_{12} = 1\times 2 + 2\times 3 + 3\times 0 = 8 \\ c_{21} = 0\times(-1) + 1\times 2 + 4\times 5 = 22 \\ c_{22} = 0\times 2 + 1\times 3 + 4\times 0 = 3 \end{cases}$$

$$C = AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 22 & 3 \end{bmatrix}$$

Operations: matrix product

Example: $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix}$, Evaluate $D=BA$.

$$D = AB = \begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix} = \begin{bmatrix}-1 & 0 & 5\\ 2 & 7 & 18\\ 5 & 10 & 15\end{bmatrix}$$

for each element do the row n of elemnts for that row of the first matrix * the col n col of the second matix

Note: AB ≠ BA

Row matrix by column matrix product

if it is row by column the product is just a scalar

- In particular, A is a $1 \times m$ matrix and B is a $m \times 1$ matrix, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}$$

then $C = AB$ is a scalar. $C = \sum_{k=1}^{m} a_{1k} b_{k1} = a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1m}b_{m1}$

Column matrix by row matrix product

- BUT BA is a $m \times m$ matrix!

$$BA = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} & b_{11}a_{12} & \dots & b_{11}a_{1m} \\ b_{21}a_{11} & b_{21}a_{12} & & b_{21}a_{1m} \\ \vdots & & \ddots & \\ b_{m1}a_{11} & b_{m1}a_{12} & & b_{m1}a_{1m} \end{bmatrix}$$

- Again, in general $AB \neq BA$!

Operations of matrices: properties

If matrices A, B and C are conformable,

- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $A(BC) = (AB)\,C$

A(f,g)*B(g,i)*C(i,j)

Pay attention to the order of the elements in the product

However

- $AB \neq BA$ in general
- $AB = 0$ NOT necessarily imply $A = 0$ or $B = 0$
- $AB = AC$ NOT necessarily imply $B = C$

Operations of matrices: properties

Example: Prove $A(B + C) = AB + AC$ where A, B and C are n-square matrices

Let $X = B + C$, so $x_{ij} = b_{ij} + c_{ij}$. Let $Y = AX$, then

$$y_{ij} = \sum_{k=1}^{n} a_{ik} x_{kj} = \sum_{k=1}^{n} a_{ik} (b_{kj} + c_{kj})$$

$$= \sum_{k=1}^{n} (a_{ik} b_{kj} + a_{ik} c_{kj}) = \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}$$

So $Y = AB + AC$; therefore, $A(B + C) = AB + AC$

$AB = 0$ NOT necessarily imply $A = 0$ or $B = 0$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

▪ $AB = AC$ NOT necessarily imply $B = C$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Triangular matrices

- A square matrix whose elements $a_{ij} = 0$, for $i > j$ is called upper triangular, i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

- A square matrix whose elements $a_{ij} = 0$, for $i < j$ is called lower triangular, i.e.,

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & & 0 \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

Diagonal matrices

- Both upper and lower triangular, i.e., $a_{ij} = 0$, for $i \neq j$, i.e.,

$$D = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

is called a diagonal matrix, simply

$$D = \mathrm{diag}[a_{11}, a_{22}, \ldots, a_{nn}]$$

Identity matrix

- In particular, $a_{11} = a_{22} = \ldots = a_{nn} = 1$, the matrix is called <u>identity matrix</u>.
- Properties: $AI = IA = A$

Examples of identity matrices: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

if you multiply anithing for an identiti matrix it dos not change

Commutative matrices

- $AB \neq BA$ in general. However, if two square matrices A and B such that $AB = BA$, then A and B are said to *commute*.

if one of the two is an identity

Can you suggest two matrices that must commute with a square matrix A?

Ans: A itself, the identity matrix, ..

- If A and B such that $AB = -BA$, then A and B are said to *anti-commute*.

Transpose of a matrix

- The matrix obtained by interchanging the rows and columns of a matrix A is called the transpose of A (write A^T).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The transpose of A is $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- For a matrix $A = [a_{ij}]$, its transpose $A^T = [b_{ij}]$, where $b_{ij} = a_{ji}$.
- $(A^T)^T = A$
- A^T and A are conformable for product in both directions

Symmetric matrix

- A square matrix A such that $A^T = A$ is called symmetric, i.e., $a_{ji} = a_{ij}$ for all i and j.
- $A + A^T$ must be symmetric. Why?

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.

A+AT=SYMMETRIC
A-AT=SKEW

- A square matrix A such that $A^T = -A$ is called skew-symmetric (or antisymmetric), i.e., $a_{ji} = -a_{ij}$ for all i and j.

ALL DIAGONAL ELEMENTS MUST BE 0

- $A - A^T$ must be skew-symmetric. Why?

Inverse matrix

- If two square, product conformable matrices A and B are such that $AB = BA = I$, then B is called the inverse of A (symbol: A^{-1}); and A is called the inverse of B (symbol: B^{-1}).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Show B is the the inverse of matrix A.

Ans: Note that $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Can you show the details?

Orthogonal matrix

RAPPRESENT A ROTATION IN THE SPACE

- A square matrix A is called orthogonal if $AA^T = A^TA = I$, i.e., $A^T = A^{-1}$

Example: prove that $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$ is orthogonal.

Since, $A^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$. Hence, $AA^T = A^TA = I$.

Can you show the details?

We'll see that orthogonal matrix represents a rotation in fact!

Properties of inverse and transpose matrices

ALL SQUARE MATRIX

- $(AB)^{-1} = B^{-1}A^{-1}$

A*A^-1=IDENTITY

- $(A^T)^T = A$ and $(\lambda A)^T = \lambda A^T$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Properties of inverse and transpose matrices

Example: Prove $(AB)^{-1} = B^{-1}A^{-1}$.

Since $(AB)\,(B^{-1}A^{-1}) = A(B\,B^{-1})A^{-1} = I$ and

$(B^{-1}A^{-1})\,(AB) = B^{-1}(A^{-1}\,A)B = I$.

Therefore, $B^{-1}A^{-1}$ is the inverse of matrix AB.

Determinants

Square matrix of order 2

Consider a 2×2 matrix: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- Determinant of A, denoted $|A|$, is a number and can be evaluated by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\Re^{2\times 2} \to \Re$$

Determinant square matrix of order 2

- easy to remember (for order 2 only)..

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$

\-   +

Example: Evaluate the determinant: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

Properties of determinants

- The following properties are true for determinants of square matrices of *any* order.

1. If every element of a row (column) is zero, e.g., $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 1\times 0 - 2\times 0 = 0$, then $|A| = 0$.

2. $|A^T| = |A|$

   determinant of a matrix = that of its transpose

3. $|AB| = |A||B|$

Properties of determinants

Example: Show that the determinant of any orthogonal matrix is either $+1$ or -1.

For any orthogonal matrix, $A A^T = I$.

Since $|AA^T| = |A||A^T| = 1$ and $|A^T| = |A|$, so $|A|^2 = 1$ or $|A| = \pm 1$.

Inverse of a square matrix of order 2

For any 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Its inverse can be written as $\boxed{A^{-1} = \frac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}$

Proof:

INVERT ELEMENTS OD DIAGONAL AND ELEMENTS TO THE SIDE OF IT

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & -a_{11}a_{12} + a_{12}a_{11} \\ a_{21}a_{22} - a_{22}a_{21} & -a_{21}a_{12} + a_{22}a_{11} \end{bmatrix} =$$

$$= \begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix}$$

NB: if A has null determinant, its inverse cannot be defined

Inverse of a square matrix of order 2

Example: Find the inverse of $A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$

The determinant of A is -2

Hence, the inverse of A is $A^{-1} = \begin{bmatrix} -1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

How to find an inverse for a 3x3 matrix?

Minors and cofactors

DETERMINANT OF THE MATRIX WITHOUT THE ROW\COLUMN OF THE CHOSEN ELEMENATS

Definition 1: Given a matrix *A*, a minor is the determinant of any square submatrix of *A*.

Definition 2: Given a matrix **A=[a$_{ij}$]** , the **cofactor** of the element **a$_{ij}$** is a scalar obtained by multiplying together the term **$(-1)^{i+j}$** and the minor obtained from **A** by removing the *i*th row and the *j*th column.

In other words, the cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

For example,

COFACTOR= (-1)^i+j*MINOR

if odd= -1
if even= 1

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \Rightarrow C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \Rightarrow C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

Determinant of a square matrix of any order

To find the determinant of a matrix *A* of arbitrary order,

a) Pick any one row or any one column of the matrix;

b) For each element in the row or column chosen, find its cofactor;

c) Multiply each element in the row or column chosen by its cofactor and sum the results. This sum is the determinant of the matrix.

In other words, the determinant of ***A*** is given by

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

*i*th row expansion

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

*j*th column expansion

Determinants of matrices of order 3

Consider an example: $A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$

Its determinant can be obtained by:

$$|A|=\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}=3\begin{vmatrix}4 & 5\\ 7 & 8\end{vmatrix}-6\begin{vmatrix}1 & 2\\ 7 & 8\end{vmatrix}+9\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}$$

$$=3(-3)-6(-6)+9(-3)=0$$

You are encouraged to find the determinant by using other rows or columns

$$|\mathbf{T}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

by expanding along the first row,

$$|\mathbf{T}| = 1\times(-)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2\times(-)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\times(-)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3+12-9=0$$

Or expand down the second column:

$$|\mathbf{T}| = 2\times(-)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5\times(-)^{2+2}\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8\times(-)^{3+2}\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 12-60+48=0$$

Example 2: (using a row or column with many zeroes)

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix} = 1\times(-)^{2+3}\begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix} = 16$$

Properties of determinants

Property 1: If one row of a matrix consists entirely of zeros, then the determinant is zero.

Property 2: If two rows of a matrix are interchanged, the determinant changes sign.

switch

Property 3: If two rows of a matrix are identical, the determinant is zero.

Property 4: If the matrix **B** is obtained from the matrix ***A*** by multiplying every element in one row of ***A*** by the scalar λ, then $|\mathbf{B}| = \lambda|\boldsymbol{A}|$.

if you multiply all the elements you get lambda^n*A
with n=dimension of the matrix

Property 5: For an $n \times n$ matrix ***A*** and any scalar λ, $\det(\lambda \boldsymbol{A}) = \lambda^n \det(\boldsymbol{A})$.

Properties of determinants

Property 6: If a matrix **B** is obtained from a matrix ***A*** by adding to one row of ***A***, a scalar times another row of ***A***, then $|\boldsymbol{A}|=|\mathbf{B}|$.

Property 7: $|\boldsymbol{A}| = |\boldsymbol{A}^T|$.

Property 8: The determinant of a triangular matrix, either upper or lower, is the product of the elements on the main diagonal.

Property 9: If ***A*** and **B** are of the same order, then

$$|\boldsymbol{A}\mathbf{B}|=|\boldsymbol{A}|\ |\mathbf{B}|.$$

Inversion

Theorem 1: A square matrix has an inverse if and only if its determinant is not zero.

Definition 1: The **cofactor matrix** associated with an n $\times$ n matrix **A** is an n $\times$ n matrix $\mathbf{A}^c$ obtained from **A** by replacing each element of **A** by its cofactor.

Definition 2: The **adjugate** of an n $\times$ n matrix **A** is the transpose of the cofactor matrix of **A**: $\mathbf{A}^a = (\mathbf{A}^c)^T$

Cofactor and adjugate matrices

- Find the adjugate of

$$A=\begin{bmatrix}-1 & 3 & 2\\ 0 & -2 & 1\\ 1 & 0 & -2\end{bmatrix}$$

Solution:

The cofactor matrix of *A*:

$$\begin{bmatrix}\begin{vmatrix}-2 & 1\\ 0 & -2\end{vmatrix} & -\begin{vmatrix}0 & 1\\ 1 & -2\end{vmatrix} & \begin{vmatrix}0 & -2\\ 1 & 0\end{vmatrix}\\ -\begin{vmatrix}3 & 2\\ 0 & -2\end{vmatrix} & \begin{vmatrix}-1 & 2\\ 1 & -2\end{vmatrix} & -\begin{vmatrix}-1 & 3\\ 1 & 0\end{vmatrix}\\ \begin{vmatrix}3 & 2\\ -2 & 1\end{vmatrix} & -\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix} & \begin{vmatrix}-1 & 3\\ 0 & -2\end{vmatrix}\end{bmatrix}=\begin{bmatrix}4 & 1 & 2\\ 6 & 0 & 3\\ 7 & 1 & 2\end{bmatrix}$$

$$A^a=\begin{bmatrix}4 & 6 & 7\\ 1 & 0 & 1\\ 2 & 3 & 2\end{bmatrix}$$

Inverse of a square matrix of order n

If $|A| \neq 0$, then A^{-1} may be obtained by dividing the adjugate of A by the determinant of A.

$$A^{-1} = \frac{1}{|A|} A^{adjugate} = \boxed{\frac{1}{|A|} \left(A^{cofactors}\right)^T}$$

For example, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$A^{-1} = \frac{1}{|A|} A^{a} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of a 3×3 matrix

Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

The cofactor for each element of matrix A:

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

Inverse of a 3×3 matrix

Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is then given by:

$$\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

Inverse of a 3×3 matrix

Inverse matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is given by:

$$A^{-1} = \frac{1}{\boxed{|A|}} \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}^T = \frac{1}{22} \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

NB the det

$$= \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$

Inversion using determinants

AGJUGATE= trasposition of the cofactor matrix

Use the adjugate of $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ to find A^{-1}

$$|A| = (-1)(-2)(-2) + (3)(1)(1) - (1)(-2)(2) = 3$$

$$\mathrm{A}^{a} = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|\mathrm{A}|} A^{a} = \frac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

Exercises

- Determine whether the following matrices are invertible and compute, if possible, the inverse matrices

$$\begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Matrices and systems of linear equations

- Systems of linear equations can be represented in matrix form

$$\begin{cases} x+y=7, \\ 3x-y=5. \end{cases} \qquad \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$\begin{cases} x+y-2z=7, \\ 2x-y-4z=2, \\ -5x+4y+10z=1, \\ 3x-y-6z=5. \end{cases} \qquad \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & -4 \\ -5 & 4 & 10 \\ 3 & -1 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 7 \\ 2 \\ 1 \\ 5 \end{bmatrix}$$

The matrix of coefficients has as many rows as the number of equations and as many columns as the number of variables

Linear systems with n equations of n variables

The matrix of coefficients is square of order n
So, in general, the equation is written as:

$$A\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

If A is invertible,
the solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = A^{-1}\begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}$$

V(solution)= a^-1*known factors vector

Linear systems with n equations of n variables

$$\begin{cases} x + y = 7, \\ 3x - y = 5. \end{cases} \qquad \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

The matrix has non-null determinant, so it is invertible

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}^{-1} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix}$$

The solution is then given by:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Exercises

Determine whether the following systems are solvable and, if yes, find the solutions (using matrices)

$$\begin{cases} x+2y=3 \\ x-y=0 \end{cases} \qquad \begin{cases} -2x+2y=3 \\ x-y=0 \end{cases} \qquad \begin{cases} 4x+2y=3 \\ x+2y=0 \end{cases}$$

$$\begin{cases} x+2y+z=3 \\ x-y-z=0 \\ 3x-z=1 \end{cases} \qquad \begin{cases} x+2y+z=3 \\ x-y=0 \\ 3x+z=1 \end{cases} \qquad \begin{cases} x+2y+z=2 \\ x-y+z=2 \\ 3x+z=1 \end{cases}$$