Linear transformations of vectors

Vectors as mx1 matrices

An m-valued vector is represented as a mx1 matrix

$$V = \begin{bmatrix} v_{11} \\ v_{21} \\ . \\ v_{m1} \end{bmatrix}$$

Column vector

$$V^T = \begin{bmatrix} v_{11} & v_{21} & .. & v_{m1} \end{bmatrix}$$

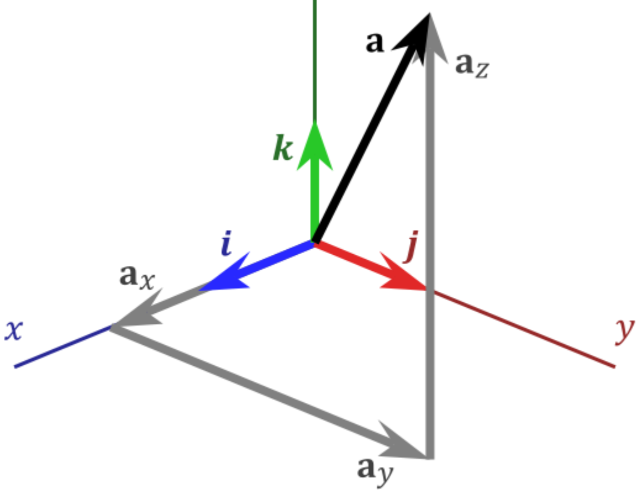
Row vector

only if you consider it a trasposition

$$\begin{bmatrix} 1 \\ 0 \\ . \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ . \\ 0 \end{bmatrix} \quad \ldots\ldots \quad \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix}$$

scalar product of vector=Vt*W

Basis vectors



Scalar product

Scalar product of two m-valued vectors is written as matrix product

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{m} A_i B_i = \sum_{i=1}^{m} a_{i1} b_{i1} = A^T \cdot B = B^T \cdot A$$

Remember that $A \cdot B^T, B \cdot A^T$ would be two (different) mxm matrices

Matrices as linear operators on vectors

- A nxm matrix transforms a m-valued vector in a n-valued vector

$$A\vec{v} = A^{(n\times m)}V^{(m\times 1)} = \begin{bmatrix} a_{11} & a_{12} & . & a_{1m} \\ a_{21} & a_{22} & . & a_{2m} \\ . & . & . & . \\ a_{n1} & a_{n2} & . & a_{nm} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \\ . \\ v_{m1} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{1i}v_{i1} \\ \sum_{i=1}^{m} a_{2i}v_{i1} \\ . \\ \sum_{i=1}^{m} a_{ni}v_{i1} \end{bmatrix}$$

$$A^{(n\times m)} : \Re^m \to \Re^n$$

It follows the linearity conditions

you can use a matrix to transform a vector

$$A(\vec{v} + \vec{u}) = A\vec{v} + A\vec{u}$$

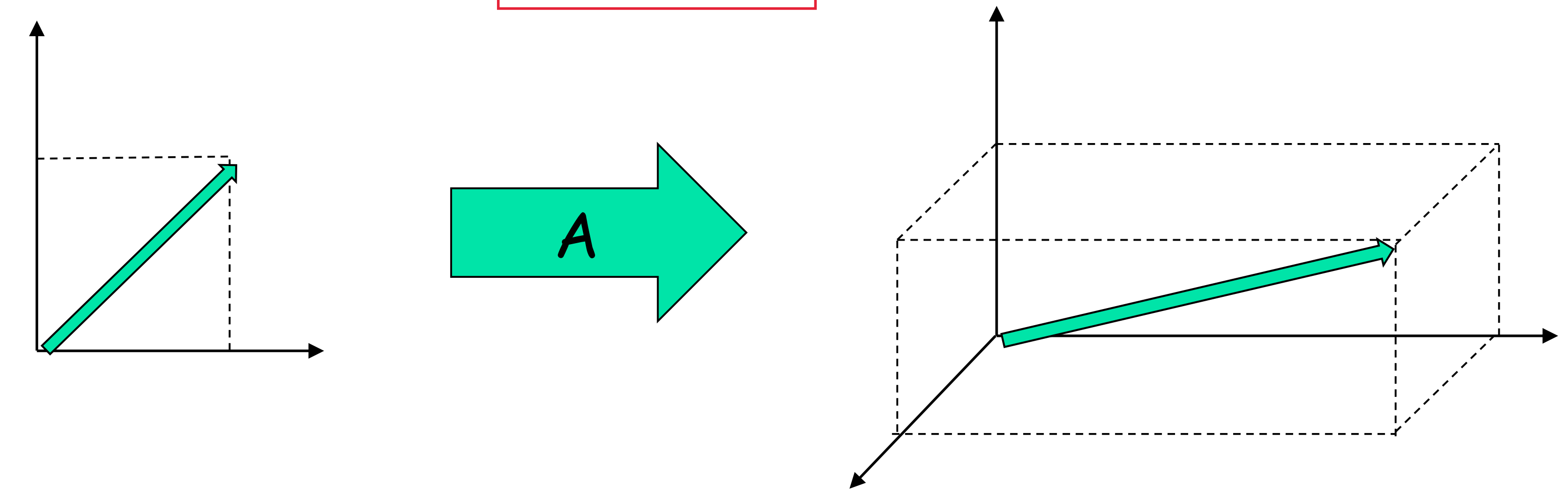
$$A(\lambda\vec{v}) = \lambda A\vec{v}$$

Matrices as linear operators on vectors

- A nxm matrix transforms a m-valued vector in a n-valued vector

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \\ 2 & -1 \end{bmatrix} \qquad A^{(3\times 2)} : \Re^2 \to \Re^3 \qquad \begin{bmatrix} 1 & 2 \\ 4 & -3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

transform 2D to 3D



Matrices as linear operators on vectors

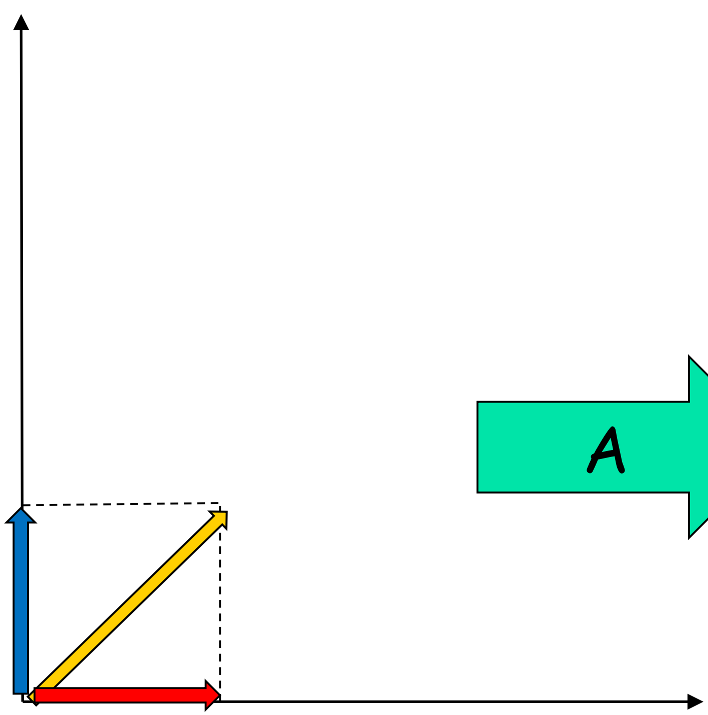
- A square matrix of order m transforms a m-valued vector in the same space

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

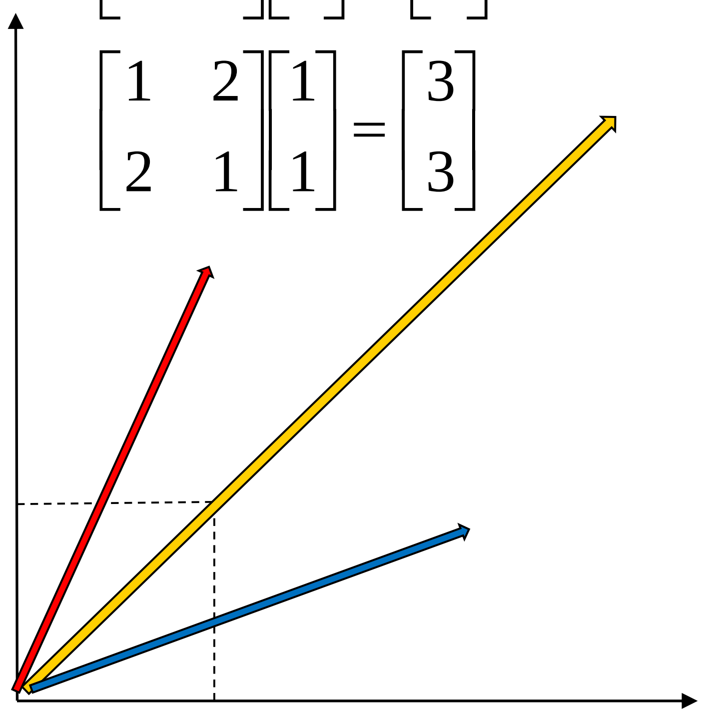
$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$



A

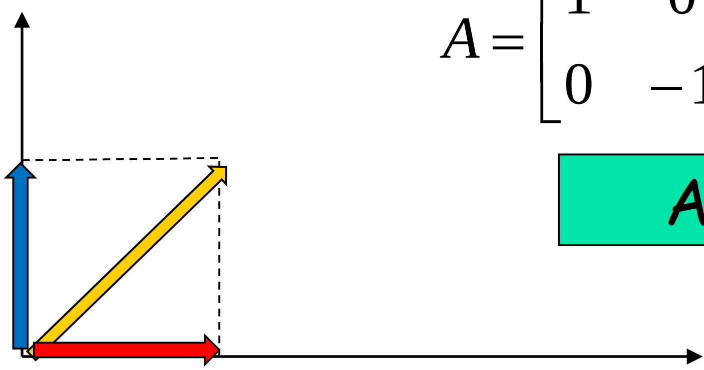


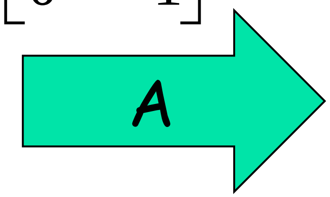
Inversions

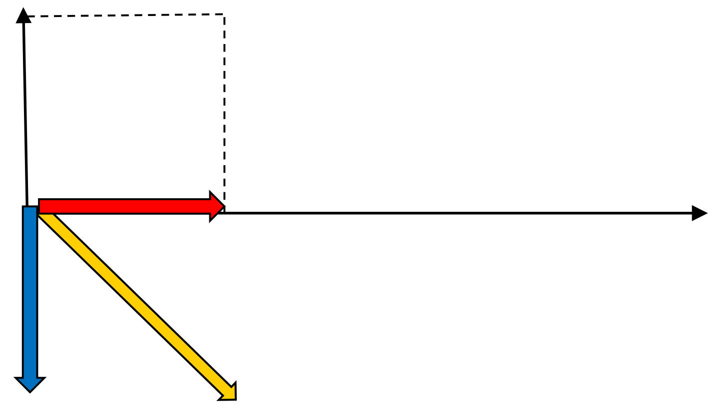
SPECIFIC MATRIX

- With respect to the x_1-axis (oppose the x_2 coordinate)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

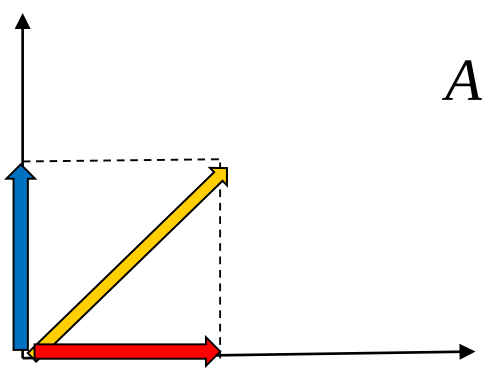


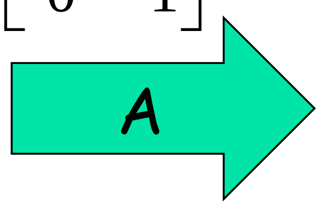


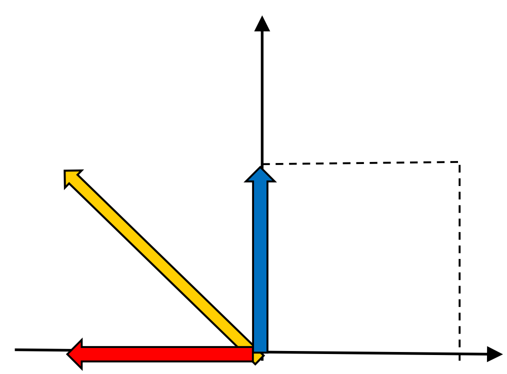


- With respect to the x_2-axis (oppose the x_1 coordinate)

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



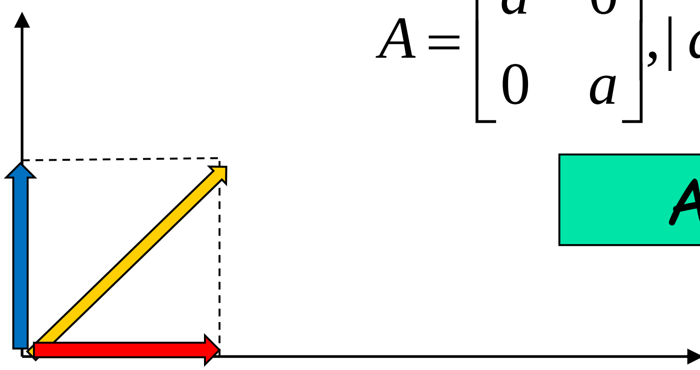


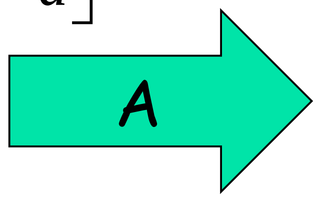


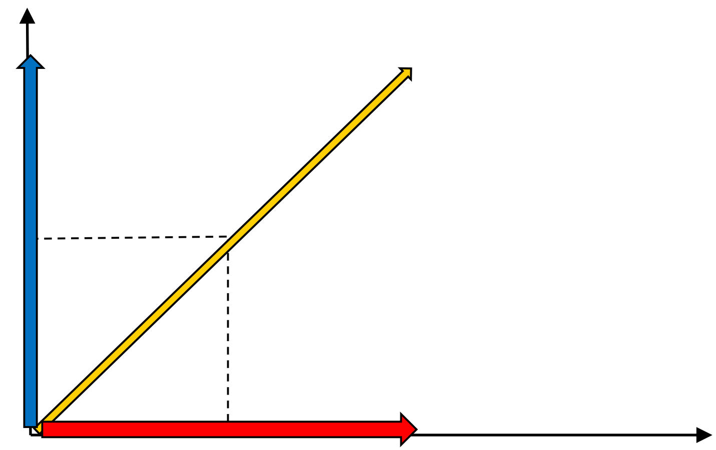
Rescalings

- Expansions

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, |a| > 1$$

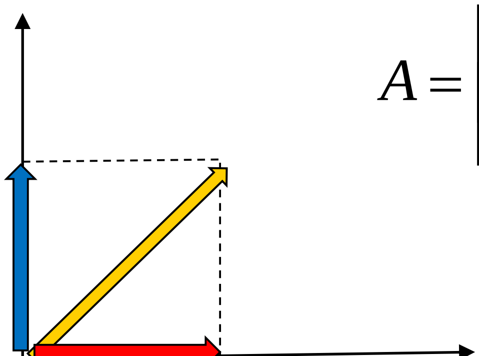


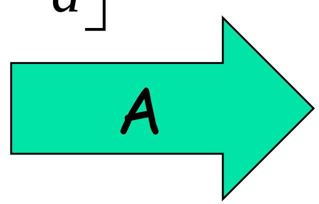


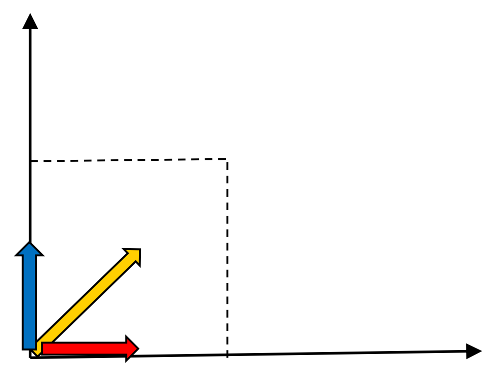


- Contractions

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, |a| < 1$$

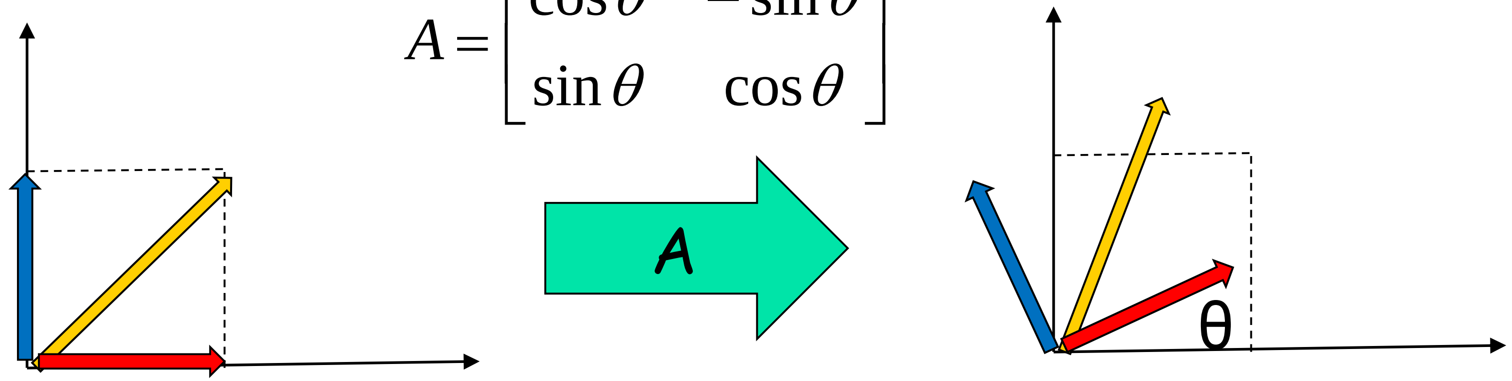






Rotations

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



To convince yourself, analyse the behavior on:

-) the x_1 axis (red arrow)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

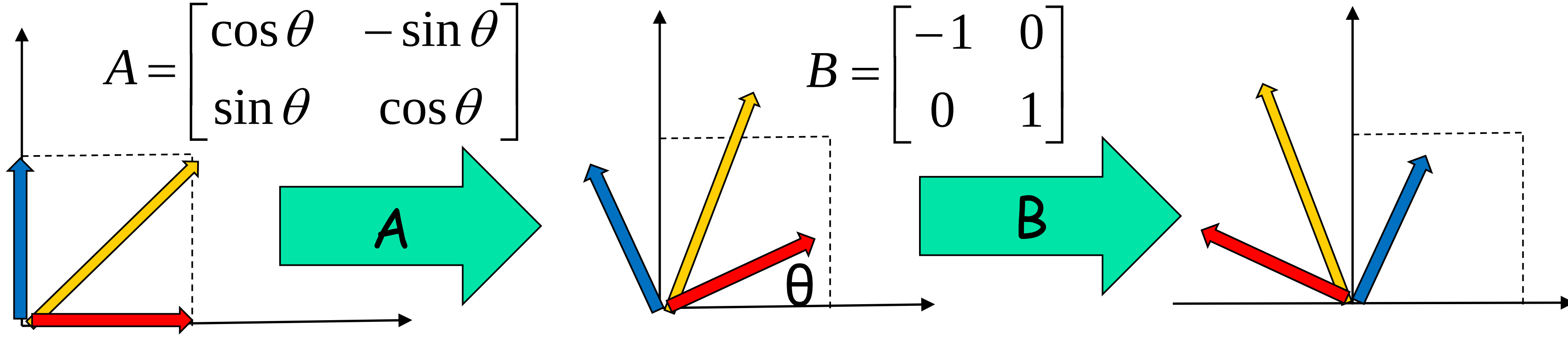
-)the x_2 axis (blue arrow)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Compositions of transformations

Perform the two following transformations, in order:
-) Rotation by an angle θ and then inversion with respect to the axis x_2

$$A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \qquad B=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$$



The overall effect on a vector X is

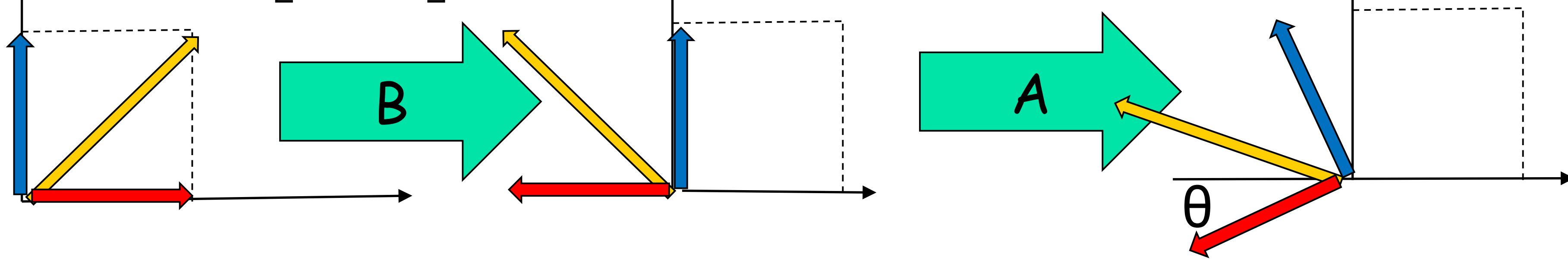
DO THE CALCULATION STARTING FROM RIGHT TO LEFT(NON COMMUTATIVR)

$$X'=B(AX)=(BA)X=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}X=\begin{bmatrix}-\cos\theta & +\sin\theta\\ +\sin\theta & +\cos\theta\end{bmatrix}X$$

Compositions of transformations

Perform the two following transformations, in order:
-) Inversion with respect to the axis x_2 and then rotation by an angle θ

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



The overall effect on a vector X is

$$X' = A(BX) = (AB)X = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} X$$

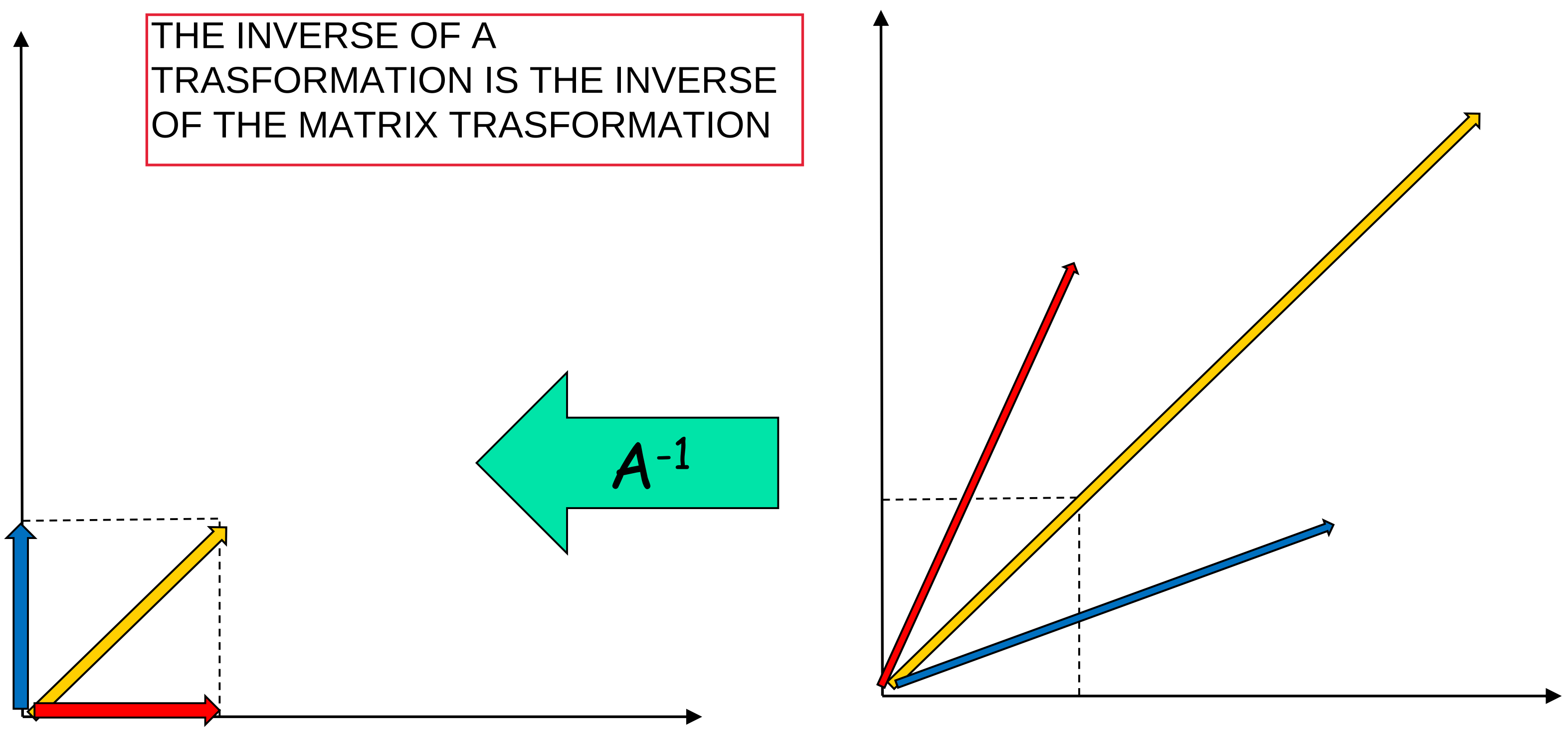
Notice that the order of application matters (and the matrices do not commute)

Inverse transformation

- If A is an invertible ($|A| \neq 0$) square matrix of order m, it is possible to define an inverse transformation and it is described by the matrix A^{-1}

$$A^{-1}(A\vec{v}) = (A^{-1}A)\vec{v} = I\vec{v} = \vec{v}, \forall \vec{v}$$

THE INVERSE OF A TRASFORMATION IS THE INVERSE OF THE MATRIX TRASFORMATION



Non Invertible transformations

$$A = \begin{bmatrix} +1 & -2 \\ -1 & +2 \end{bmatrix}$$

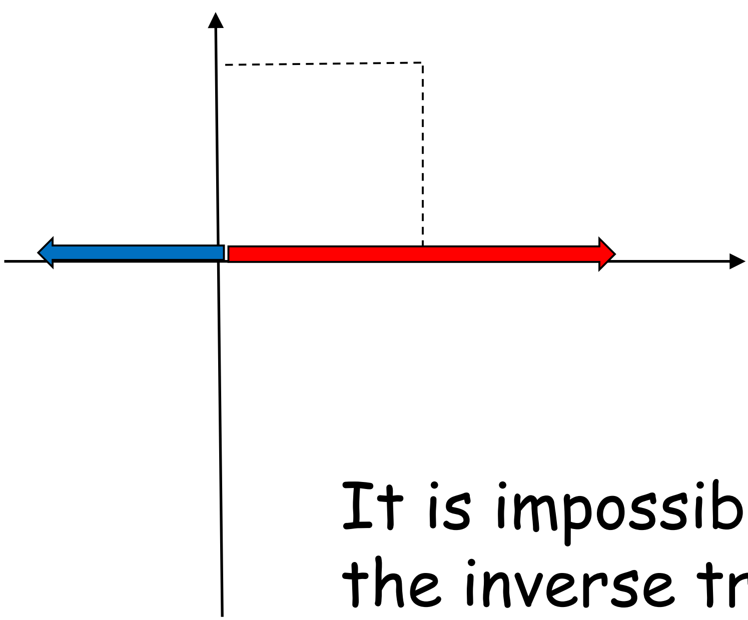
$|A|=0$

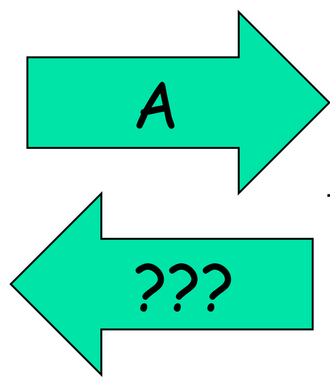
IF THE MATRIX IS NOT INVERTIBLE IT IS A NON INVERTIBLE TRASFORMATION

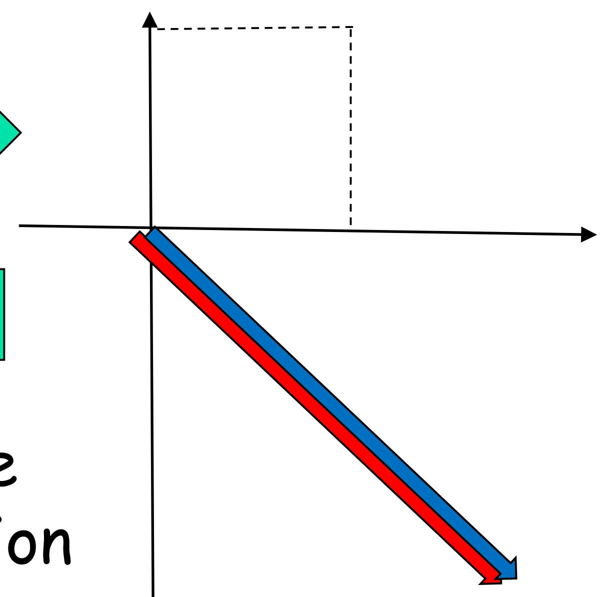
$$\begin{bmatrix} +1 & -2 \\ -1 & +2 \end{bmatrix} \begin{bmatrix} +2 \\ 0 \end{bmatrix} = \begin{bmatrix} +2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} +1 & -2 \\ -1 & +2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} +2 \\ -2 \end{bmatrix}$$

Two different vectors are transformed in the same vector







It is impossible to define the inverse transformation

Determinant as volume/area

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

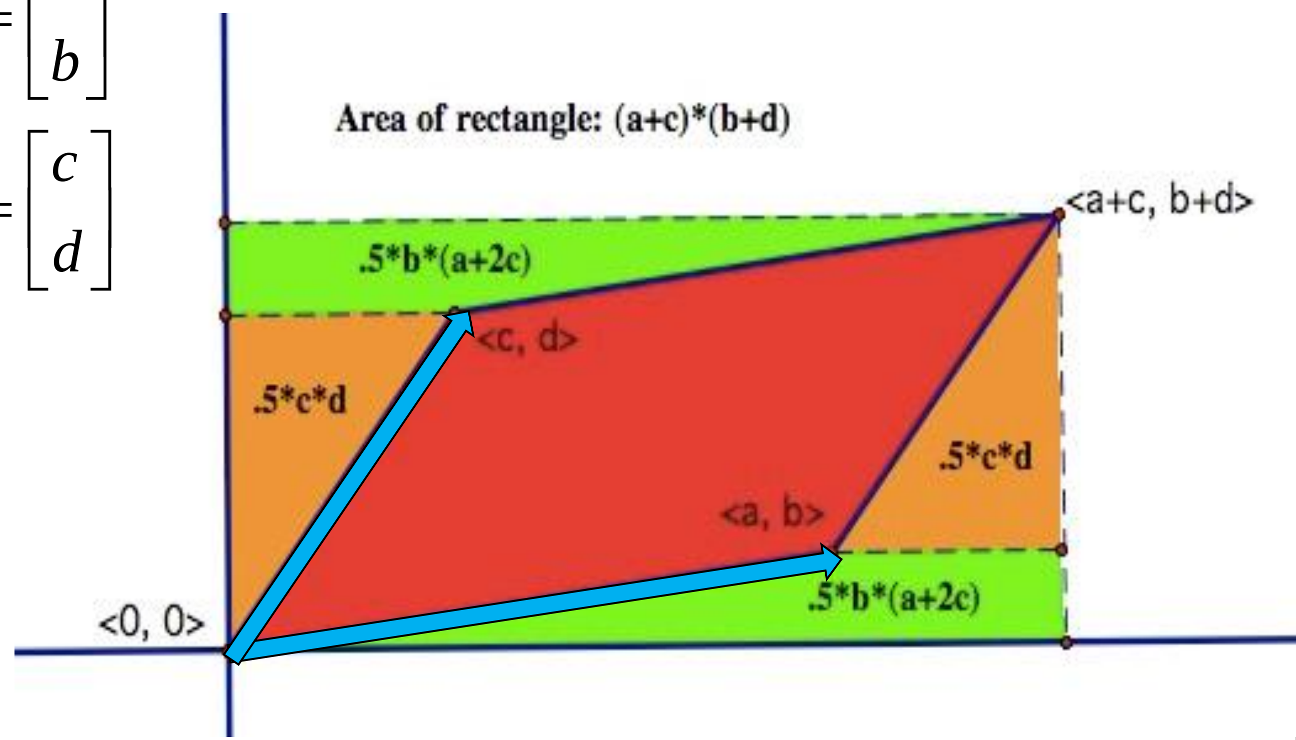
$$|A| = ad - bd$$

Det= area of the palallelogram made by the trasformation of BASIC vectors

Transformation of the axis basis

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$



Determinant as volume/area

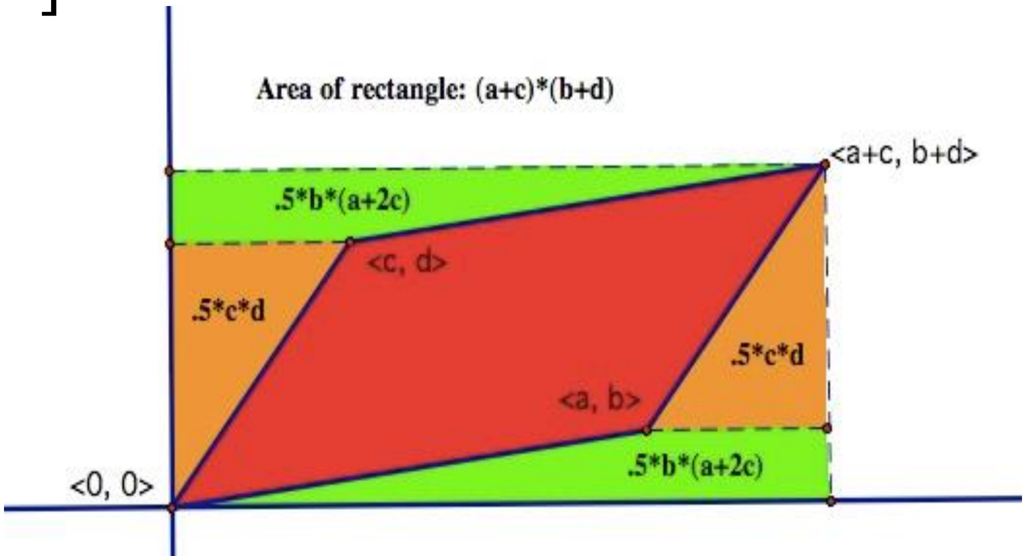
- Area of the parallelogram

$$Area = (a+c)(b+d) - 2\left[\frac{(a+c+c)b}{2}\right] - 2\left[\frac{cd}{2}\right] =$$

$$= ab + ad + bc + cd - 2\left[ab/2 + bc\right] - cd =$$

$$= +ad - bc =$$

$$= \det\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$



The area of the square defined by the basis vectors (=1) is rescaled after the transformation A to an area = |A|

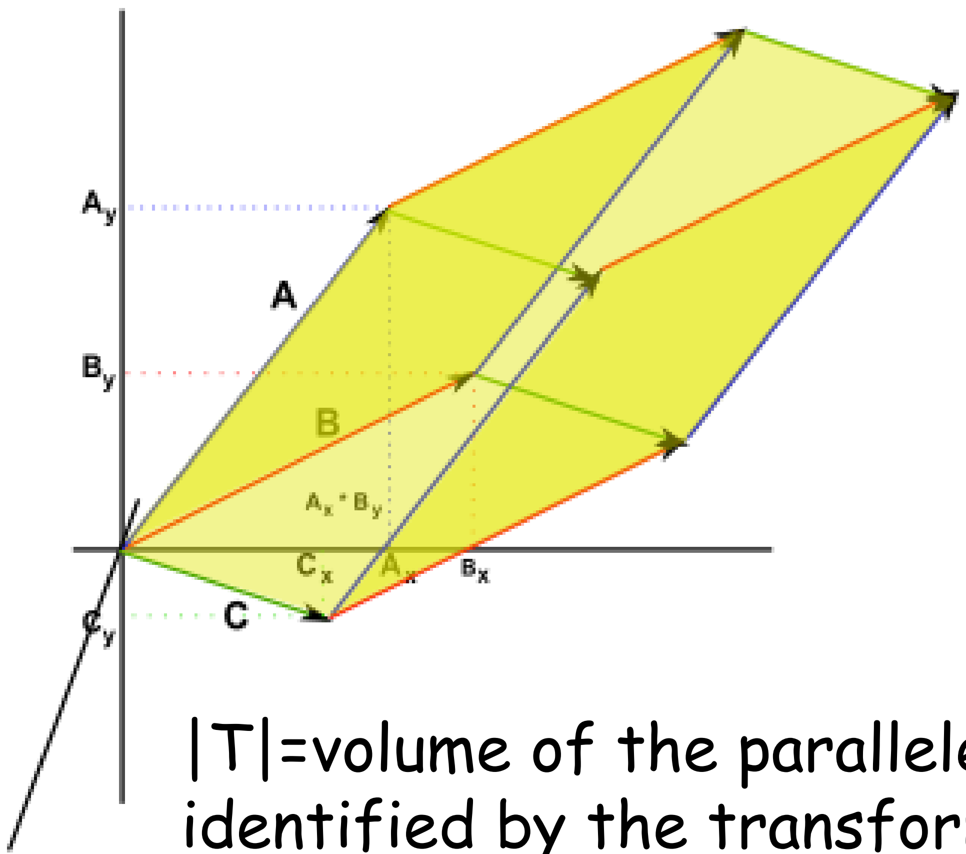
Determinant as volume/area

Given a linear transformation described by a 3x3 matrix T

$$\vec{A} = T\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\vec{B} = T\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\vec{C} = T\begin{bmatrix}0\\0\\1\end{bmatrix}$$



|T|=volume of the parallelepiped identified by the transformed of the basis vectors

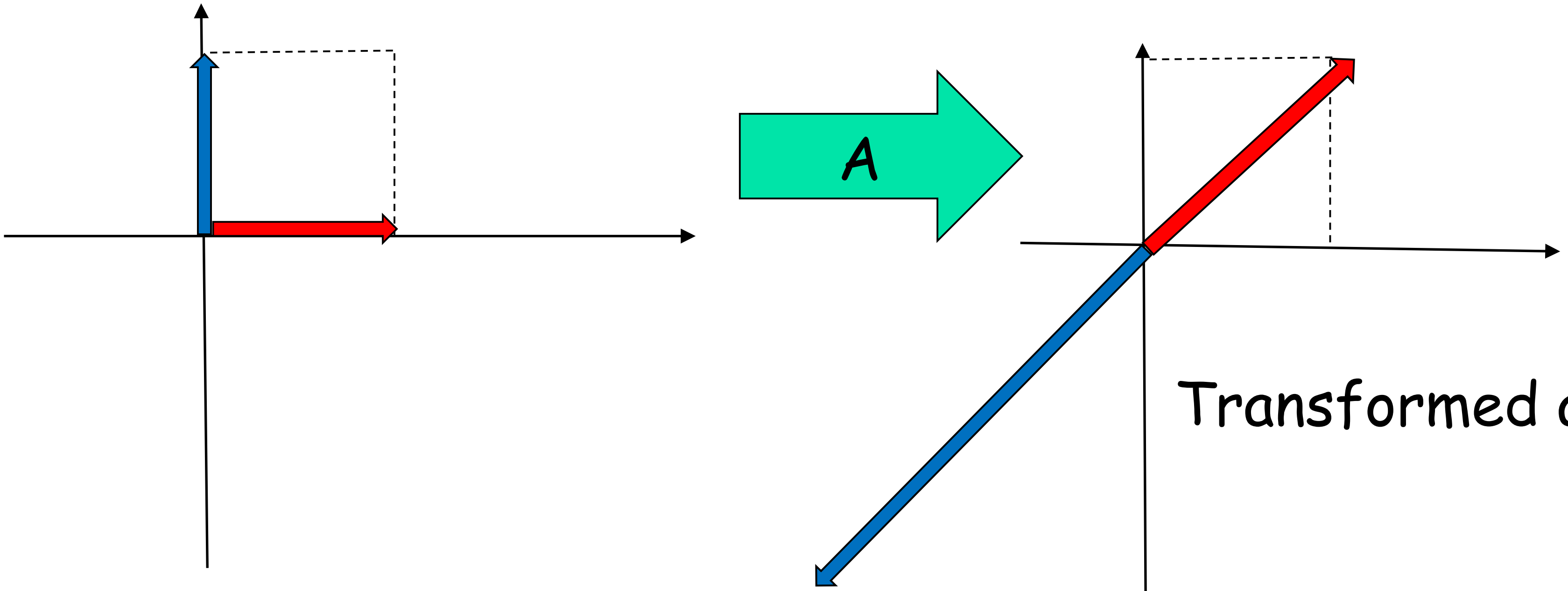
Non Invertible transformations

$$A = \begin{bmatrix} +1 & -2 \\ +1 & -2 \end{bmatrix}$$

Transformation of the basis vectors

$$\begin{bmatrix} +1 & -2 \\ +1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} +1 \\ +1 \end{bmatrix}$$

$$\begin{bmatrix} +1 & -2 \\ +1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$



Transformed area=0

Null spaces

- Given a mxn matrix, the null space is the set of n-valued vectors for which

$$A^{(m\times n)}\begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ .. \\ \\ 0 \end{bmatrix}$$

m elements

The n-dimensional 0-vector always satisfies the conditions.

Are there other non trivial solutions?

We will consider only the case of square matrices.

Null spaces of square matrices

- Given a square matrix of order n, the null space consists of vectors $\vec{b}$ that:

$$A\vec{b} = \vec{0}^{(n)}$$

If *A* is invertible

$$A^{-1}A\vec{b} = A^{-1}\vec{0}^{(n)} \Leftrightarrow I\vec{b} = A^{-1}\vec{0}^{(n)} \Leftrightarrow \vec{b} = \vec{0}^{(n)}$$

The condition to have non trivial null space is that *A* is not invertible (i.e. $|A|=0$)

Null spaces of square matrices

- Examples

Det different from 0

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow |A| = 0$$

Non trivial null space exist

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} b_1 + 2b_2 = 0 \\ 2b_1 + 4b_2 = 0 \end{cases}$$

if two equation of the system are lineary dependent you can have a null space because the trasformation is not invertible

The system is degenerate and all the vectors

$$\vec{b} = \lambda \begin{bmatrix} +2 \\ -1 \end{bmatrix}, \forall \lambda$$

all vector in null space trasform into 0 vectors

belong to the null space

Eigenvalues and Eigenvectors

- In general, a matrix acts on a vector by changing both its magnitude and its direction. However, a matrix may act on certain vectors by changing only their magnitude, and leaving their direction unchanged (or possibly reversing it). These vectors are the **eigenvectors** of the matrix.

A matrix acts on an eigenvector by multiplying its magnitude by a factor, which is positive if its direction is unchanged and negative if its direction is reversed. This factor is the **eigenvalue** associated with that eigenvector.

Eigenvector and Eigenvalues

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

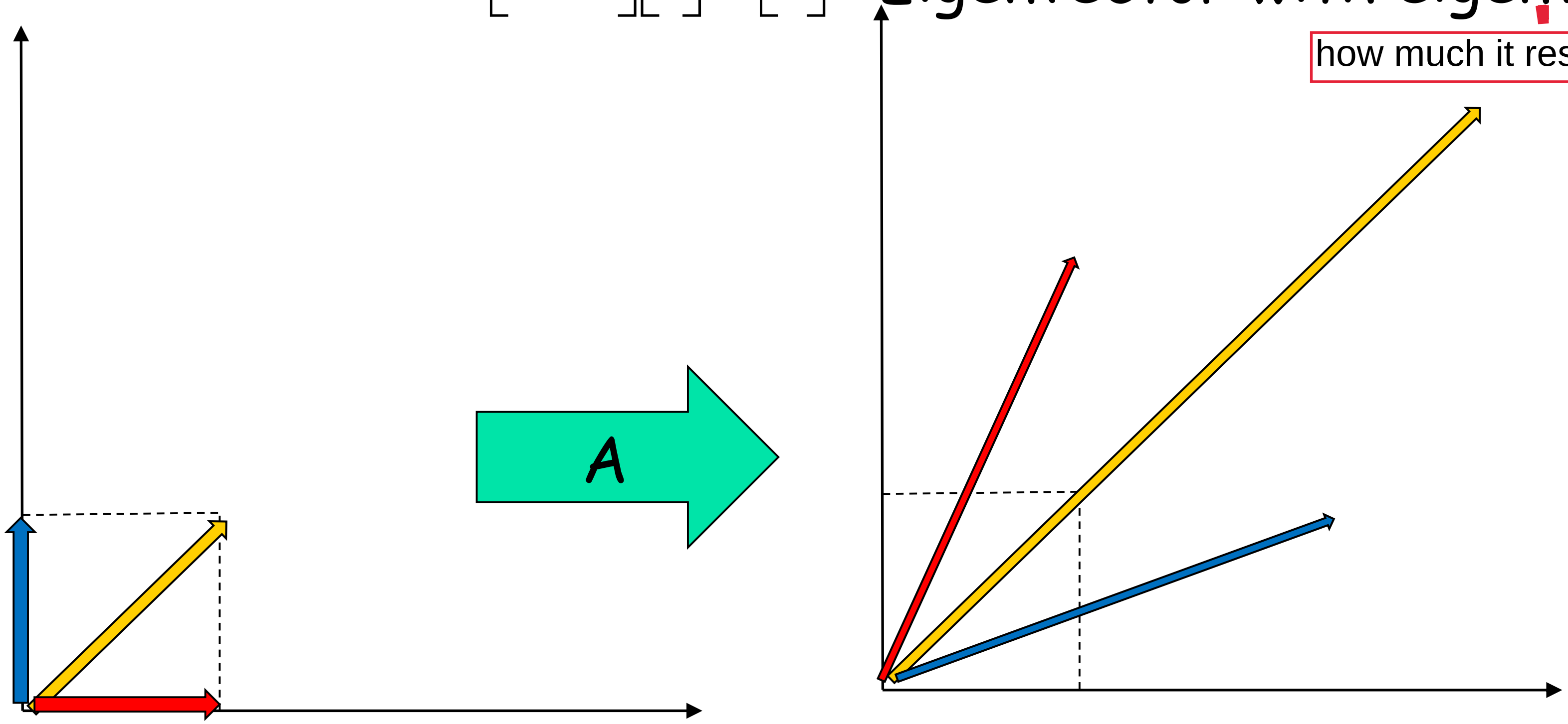
$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ Changes direction

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$ Changes direction

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$ Does not change direction

Eigenvector with eigenvalue = 3

how much it rescale the vector



Eigenvector and Eigenvalues

A nonzero vector **x** is an ***eigenvector*** of a square matrix ***A*** if there exists a scalar λ such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

Then λ is an ***eigenvalue*** of ***A***.

Note: The zero vector can not be an eigenvector even though $\mathbf{A}\mathbf{0} = \lambda\mathbf{0}$. But $\lambda = \mathbf{0}$ can be an eigenvalue.

Eigenvalues

Let **x** be an eigenvector of the matrix **A**. Then there must exist an eigenvalue λ such that

$$A\vec{x} = \lambda\vec{x}$$

or, equivalently,

$$A\vec{x} - \lambda\vec{x} = 0 \Leftrightarrow \left(A - \lambda I\right)\vec{x}$$

The eigenvectors are then the non-trivial elements of the null space of the matrix **A-λI**

The condition of existence of such vectors is that

det(A – λI) = 0

lamda=
eigenvalues

This is called the **characteristic equation** of **A**. Its roots determine the eigenvalues of **A**.

Eigenvalues: examples

Example 1: Find the eigenvalues of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = (\lambda - 2)(\lambda + 5) + 12$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

two eigenvalues: $-1, -2$

Example 2: Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$\text{A-lambda*I} = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

$\lambda = 2$ is an eigenvector of multiplicity 3.

Eigenvectors

are part of a null space

To each distinct eigenvalue of a matrix **A** there will correspond at least one eigenvector which can be found by solving the appropriate set of homogenous equations. If λ_i is an eigenvalue then the corresponding eigenvector $\mathbf{x}_i$ is the solution of

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = \mathbf{0}$$

IT IS GOOD TO USE THE EIGENVECTORS AS BASE FOR TRASFORMATION

IN A SQUARE MATRIX WE EXPECT A NUMBER OF EIGENVALUES EQUALS TO THE SIZE OF THE MATRIX IN THE COMPLEX NUMBERS

Example 1 (cont.)

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

$$\lambda = -1 \Rightarrow A - (-1)I = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -12 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - 4x_2 = 0 \Rightarrow x_1 = 4t,\ x_2 = t$$

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix},\ t \neq 0$$

$$\lambda = -2 : A - 2I = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{\mathbf{x}}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \end{bmatrix},\ s \neq 0$$

Example 2 (cont.)

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Recall that λ = 2 is an eigenvector of multiplicity 3.
Solve the homogeneous linear system represented by

$$(A-2I)\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A*v=lamda*v

Let $x_1 = s, x_3 = t$. The eigenvectors of λ = 2 are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

s and *t* not both zero.

you can normalize the eigenvector using pithagoras theorem

Properties of Eigenvalues and Eigenvectors

-The sum of the eigenvalues of a matrix equals the trace of the matrix.

- The product of the eigenvalues of a matrix (counting the multiplicity, equals the determinant of the matrix

- A matrix is singular (non invertible) if and only if it has a zero eigenvalue.

- The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.

and are always real and not complex

- If λ is an eigenvalue of A and A is invertible, then $1/\lambda$ is an eigenvalue of matrix A^{-1}.

Properties of Eigenvalues and Eigenvectors

- If λ is an eigenvalue of A then $k\lambda$ is an eigenvalue of kA where k is any arbitrary scalar.

- If λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for any positive integer k.

- If λ is an eigenvalue of A then λ is an eigenvalue of A^T.

if a matrix is symmetric the eigenvectors are ortogonal between them

and if the U matrix is normalize Ut=u^-1

Exercises

- Compute eigenvalues and eigenvectors of the following matrices

$$\begin{bmatrix} 1 & 0 \\ 5 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Eigenvalues exists always in complex field

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A - \lambda I = 0 \Rightarrow \det\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)^2 + 1 = 0$$

No solutions in the real field

Solution exixts in complex field

$$(1-\lambda)^2 = -1 \Rightarrow (1-\lambda) = \pm\sqrt{-1} = \pm i$$

$$\lambda = 1 \pm i$$

Complex numbers

- Complex numbers are written as

$z=a+ib$,

where

i is the imaginary unit (square root of -1)

a is the real number representing the real part of the number (a= Re(z))

b is the real number representing the imaginary part of the number (b = Im(z))

Complex numbers : operations

I. $(a+bi)+(\mathrm{c}+\mathrm{d}i)=(\mathrm{a}+\mathrm{c})+(\mathrm{b}+\mathrm{d})i$

Ex: $(2+3i)+(4+2i)=(2+4)+(3i+2i)=\boldsymbol{6+5i}$

II. $(a+bi)-(\mathrm{c}+\mathrm{d}i)=(\mathrm{a}-\mathrm{c})+(\mathrm{b}-\mathrm{d})i$

Ex: $(2+3i)-(4+2i)=(2-4)+(3i-2i)=\boldsymbol{-2+i}$

III. $(a+bi)*(\mathrm{c}+\mathrm{d}i)=(\mathrm{ac}-bd)+(\mathrm{ad}+bc)i$

Ex: $(2+3i)*(4+2i)=8+2i+12i+6i^2=$

$8+14i+6(-1)=8-6+14i=\boldsymbol{2+14i}$

Complex numbers : operations

$$\textbf{IV}.\quad \frac{a+bi}{c+di} = \frac{a+bi}{c+di} * \frac{c-di}{c-di} =$$

$$\frac{ac+bd}{c^2+d^2} + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

Ex:

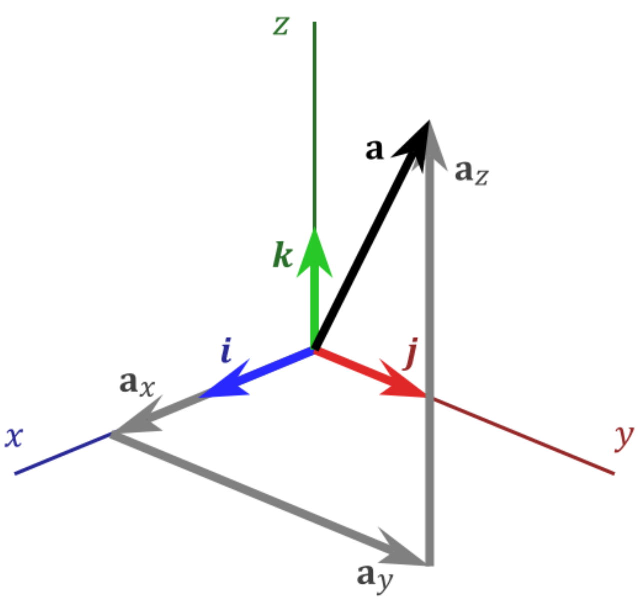
$$\frac{2+3i}{4+2i} = \frac{2+3i}{4+2i} * \frac{4-2i}{4-2i} = \frac{8-4i+12i-6i^2}{16-4i^2} =$$

$$\frac{8+8i+6}{16+4} = \frac{14+8i}{20} = \frac{\boldsymbol{7+4i}}{\boldsymbol{10}}$$

Basis of a vectorial space

- Vector are represented as n-uples with respect to a particular basis

you can use eigenvector as versors



$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

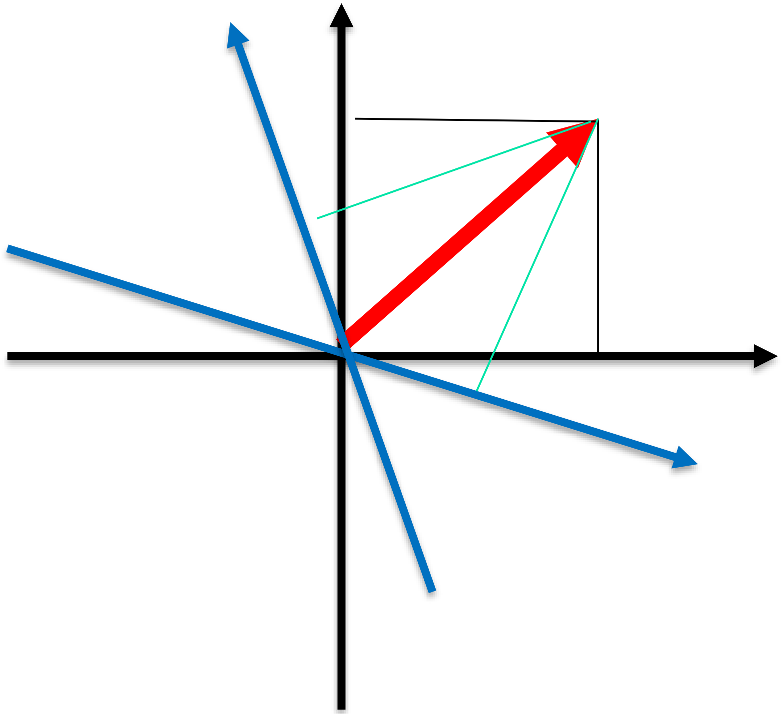
$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$

The same vector can be represented with respect to other basis

Change of basis

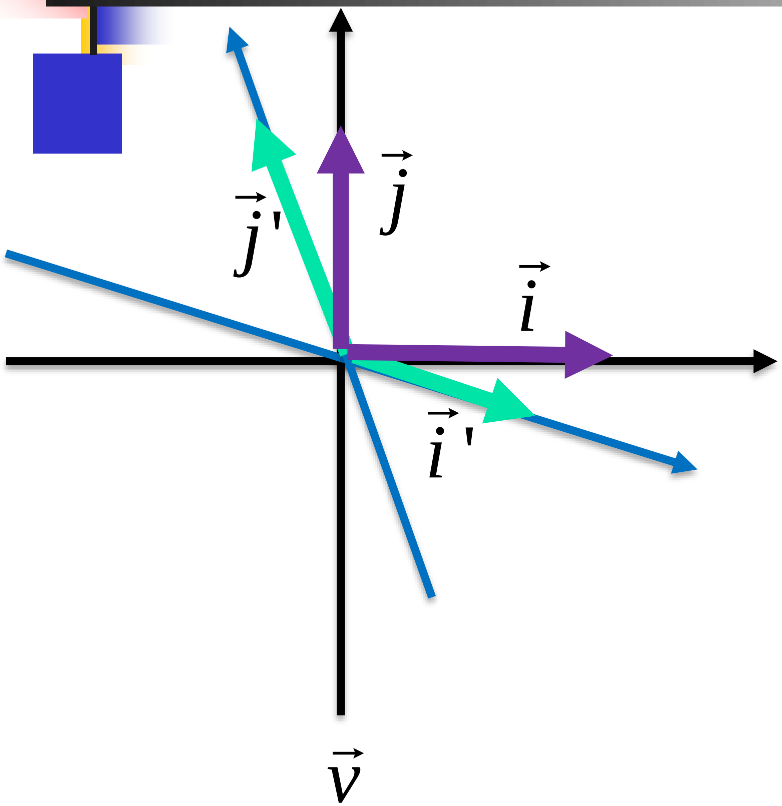
Red vector can be represented with respect to the black reference system or with respect to the blue reference system.

The vector does not change, its coordinates do



For operating the change of coordinates we need only to know the relationship between the basis vectors of the two reference systems

Change of basis



$$\vec{i}\,' = a \cdot \vec{i} + b \cdot \vec{j}$$

$$\vec{j}\,' = c \cdot \vec{i} + d \cdot \vec{j}$$

Representation of the {i',j'} basis with respect to the {i,j} basis

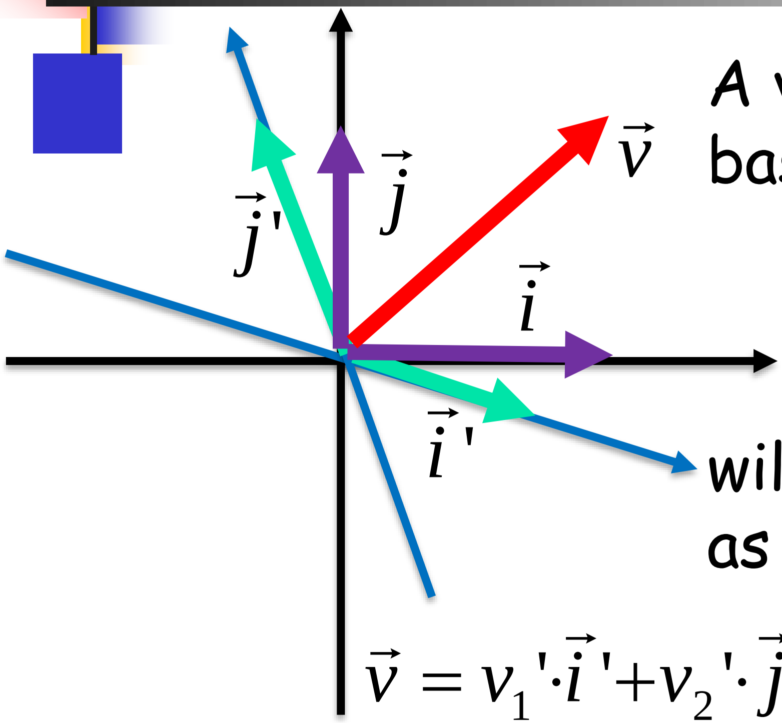
$$\vec{i}\,' = \begin{bmatrix} a \\ b \end{bmatrix}, \vec{j}\,' = \begin{bmatrix} c \\ d \end{bmatrix}$$

Define the matrix U as:

create a matrix with the rappresentation of the new basis vector in the normal space

$$U = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Change of basis



A vector $\vec{v}$ represented in the {i',j'} basis with with coordinates

$$\vec{v}\Big|_{\{\vec{i}\,',\vec{j}\,'\}} = \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}$$

will be represented in the basis {i,j} as

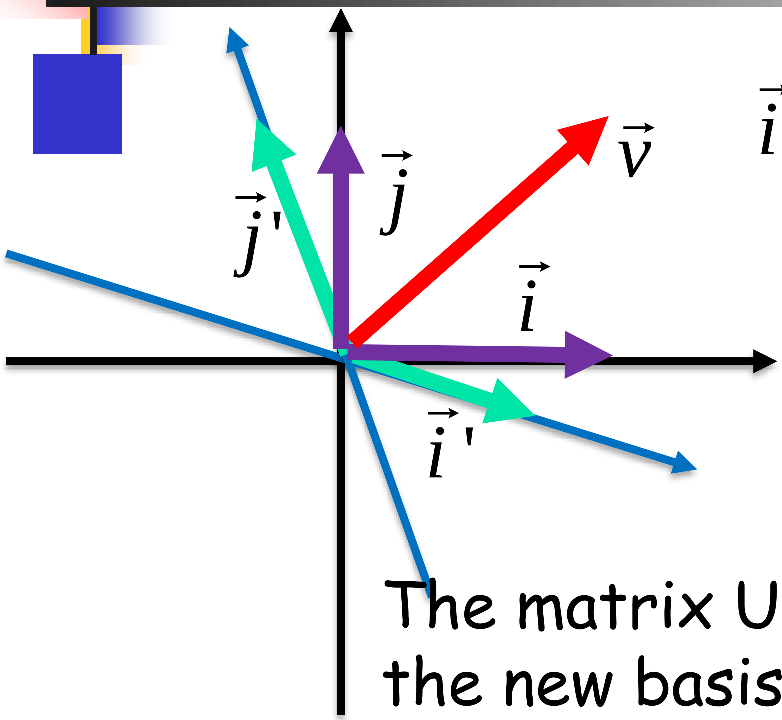
$$\vec{v} = v_1'\cdot\vec{i}\,' + v_2'\cdot\vec{j}\,' = v_1'\cdot\left(a\cdot\vec{i} + b\cdot\vec{j}\right) + v_2'\cdot\left(c\cdot\vec{i} + d\cdot\vec{j}\right) =$$

$$= \left(a\cdot v_1' + c\cdot v_2'\right)\cdot\vec{i} + \left(b\cdot v_1' + d\cdot v_2'\right)\cdot\vec{j}$$

nuove cord=U^-1*old cord

$$\vec{v}\Big|_{\{\vec{i},\vec{j}\}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a\cdot v_1' + c\cdot v_2' \\ b\cdot v_1' + d\cdot v_2' \end{bmatrix} = \boxed{\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix}} = U\vec{v}\Big|_{\{\vec{i}\,',\vec{j}\,'\}}$$

Change of basis



$$\vec{i}\,' = \begin{bmatrix} a \\ b \end{bmatrix}, \vec{j}\,' = \begin{bmatrix} c \\ d \end{bmatrix} \qquad U = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The matrix U transforms the coordinates from the {i',j'} to the coordinates in the basis {i,j}

The matrix U is invertible (|U| ≠0) otherwise the new basis would not span all the space:in case of non invertibility, i' and j' would be collinear (remind the geometrical intepretation of the determinant)

The matrix U^{-1} transforms the coordinates in the {i,j} to the coordinates in the basis {i',j'}

Diagonalization of a matrix

Consider a matrix A and its eigenvectors:
IF their number is equal to the order of A and they are linearly independent,
THEN they can serve as new basis.
The change of basis matrix is made of the eigenvectors written in column

$$U = \begin{bmatrix} \boxed{\begin{matrix} u_1^1 \\ u_2^1 \\ \cdot \\ u_n^1 \end{matrix}} & \boxed{\begin{matrix} u_1^2 \\ u_2^2 \\ \cdot \\ u_n^2 \end{matrix}} & \begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix} & \boxed{\begin{matrix} u_1^n \\ u_2^n \\ \cdot \\ u_n^n \end{matrix}} \end{bmatrix}$$

matrix made from the eigenvector of a matrix as a new system of reference

$\vec{u}^{\,1}$ =1st eigenvector

$\vec{u}^{\,n}$ =n-th eigenvector

$\vec{u}^{\,2}$ = 2nd eigenvector

Diagonalization of a matrix

By definition of eigenvector

$$A\vec{u}_i = \lambda_i \vec{u}_i$$

you can trasform a trasformation in a simply rescaling(diagonal matrix) if we use as system yhe eigenvalues

so

$$\boxed{AU} = \begin{bmatrix} \lambda_1 u_1^1 & \lambda_2 u_1^2 & . & \lambda_n u_n^1 \\ \lambda_1 u_2^1 & \lambda_2 u_2^2 & . & \lambda_n u_n^2 \\ . & . & . & . \\ \lambda_1 u_n^1 & \lambda_2 u_n^2 & . & \lambda_n u_n^n \end{bmatrix} = \begin{bmatrix} u_1^1 & u_1^2 & . & u_1^n \\ u_2^1 & u_2^2 & . & u_2^n \\ . & . & . & \\ u_n^1 & u_n^2 & . & u_n^n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix} = \boxed{U \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix}}$$

$$\boxed{U^{-1}AU = \begin{bmatrix} \lambda_1 & 0 & . & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_n \end{bmatrix}}$$

The transformation in the new coordinate system is diagonal

Example

N B

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

$$\lambda_1 = -1 \Rightarrow \quad \vec{\mathbf{x}}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \lambda_2 = -2 \Rightarrow \quad \vec{\mathbf{x}}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

normalize dividing by the sqr(first value^2 +second value^2)

$$U = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \qquad |U| = 1 \qquad U^{-1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

the diagonal matrix is the matrix of the eigenvalues

=LAMBDA

$$U^{-1}AU = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -8 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

is the same trasformation as A but using the two eigenvector aas the base of our space

Exercise

- Diagonalise, if possible, the following matrices

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

Orthonormal change of basis matrices

If a square matrix A of order n is symmetric ($A=A^T$),

1) Its eigenvalues are Real

2) Its eigenvectors are orthogonal (their scalar products are null).

3) If the eigenvectors are normalized (norm = 1), the following relation holds

$$U^{-1} = U^T$$

$$U^T A U = Diagonal$$

Example

all hortogonal matrixare rotation

$$U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Is an orthogonal matrix

To prove, prove that the transpose is the inverse

$$U^T U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} =$$

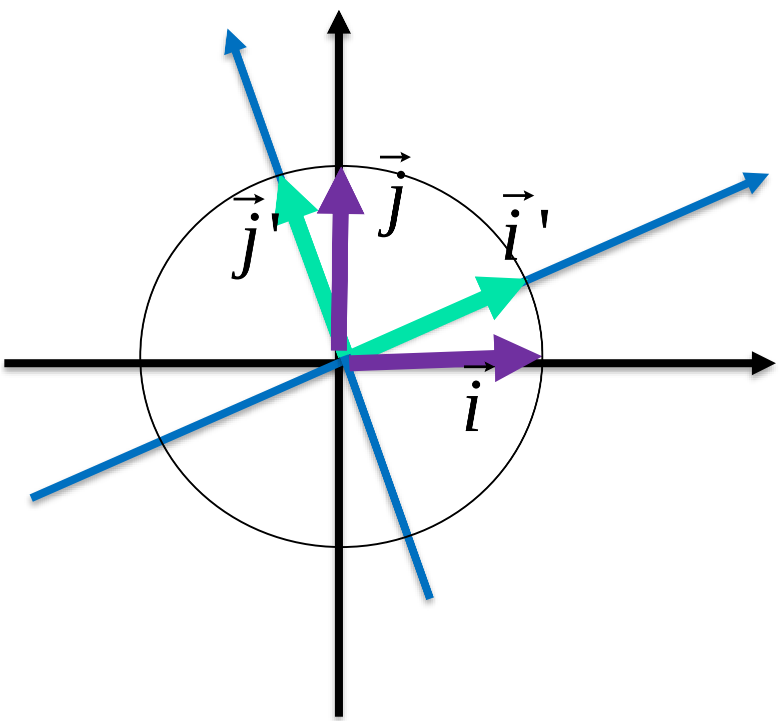
$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

What that matrix represent?

Remember that the column of U are the component of

$$U = \begin{bmatrix} \boxed{\begin{matrix}\cos\theta \\ \sin\theta\end{matrix}} & \boxed{\begin{matrix}-\sin\theta \\ \cos\theta\end{matrix}} \end{bmatrix}$$

$\vec{i}\,'$ $\vec{j}\,'$ in terms of the original basis



It is a counterclockwise rotation of coordinates by an angle θ

Linear forms

- We already know that in R^n a vector **w** can be used to define a linear form:

$$\langle wx \rangle + b = w^T x + b = 0$$

is a hyperplane, perpendicular to the vector **w** and with projection on W equal to –b/||**w**||.

Quadratic forms

- In R^n a (nxn) matrix A can be used to define a quadratic forms.

$$x^t A x + b = 0$$

In standard form (obtained by rescaling A)

$$x^t A x = 1$$

In explicit (for R^2)

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1$$

$$a_{11}x_1^{\ 2} + a_{22}x_2^{\ 2} + (a_{12} + a_{21})x_1x_2 = 1$$

Quadratic forms

- In R^2

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$a_{11}x_1^{\ 2} + a_{22}x_2^{\ 2} + (a_{12} + a_{21})x_1x_2 = 1$$

- Do you know particular cases?

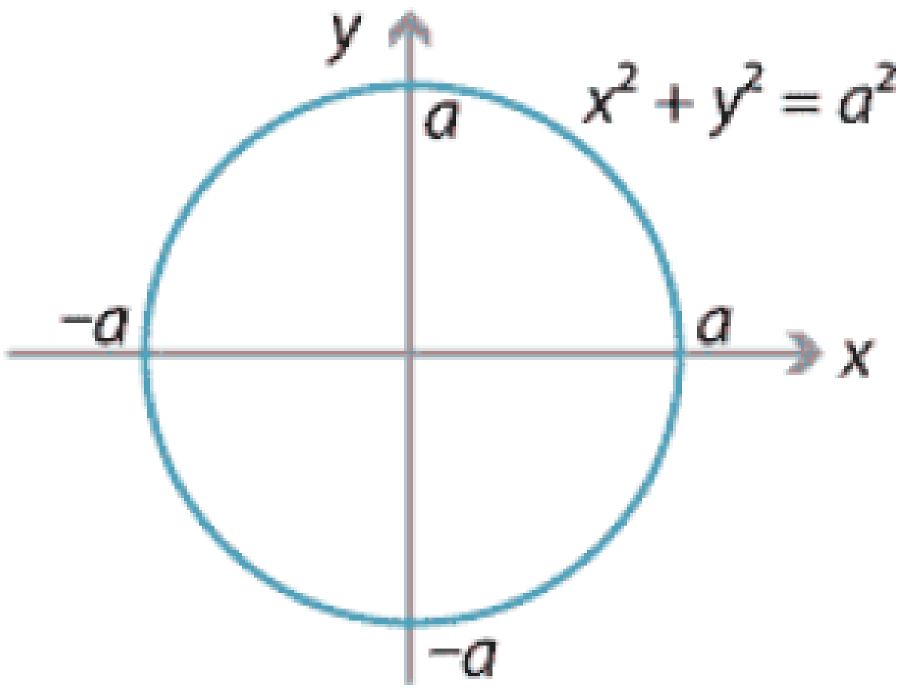
if the matrix id diagonal it rappresent a:circle,ellics,hyperbola

Quadratic forms

- In R^2

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad x_1^{\ 2} + x_2^{\ 2} = 1$$

Circle



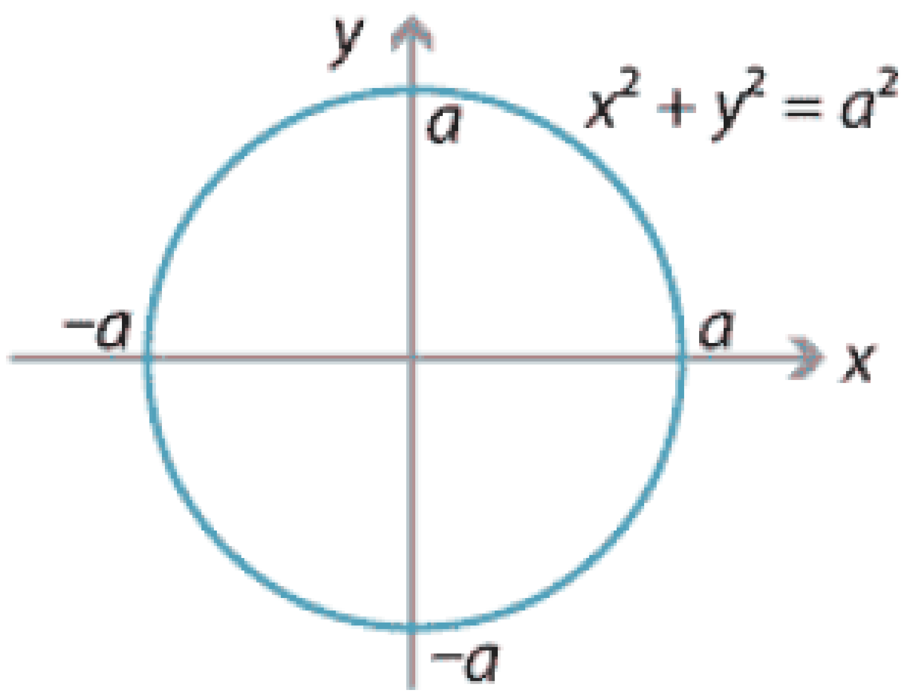
a=1

Quadratic forms

- In R^2

$$\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2} \end{pmatrix} \qquad x_1^{\ 2} + x_2^{\ 2} = a^2$$

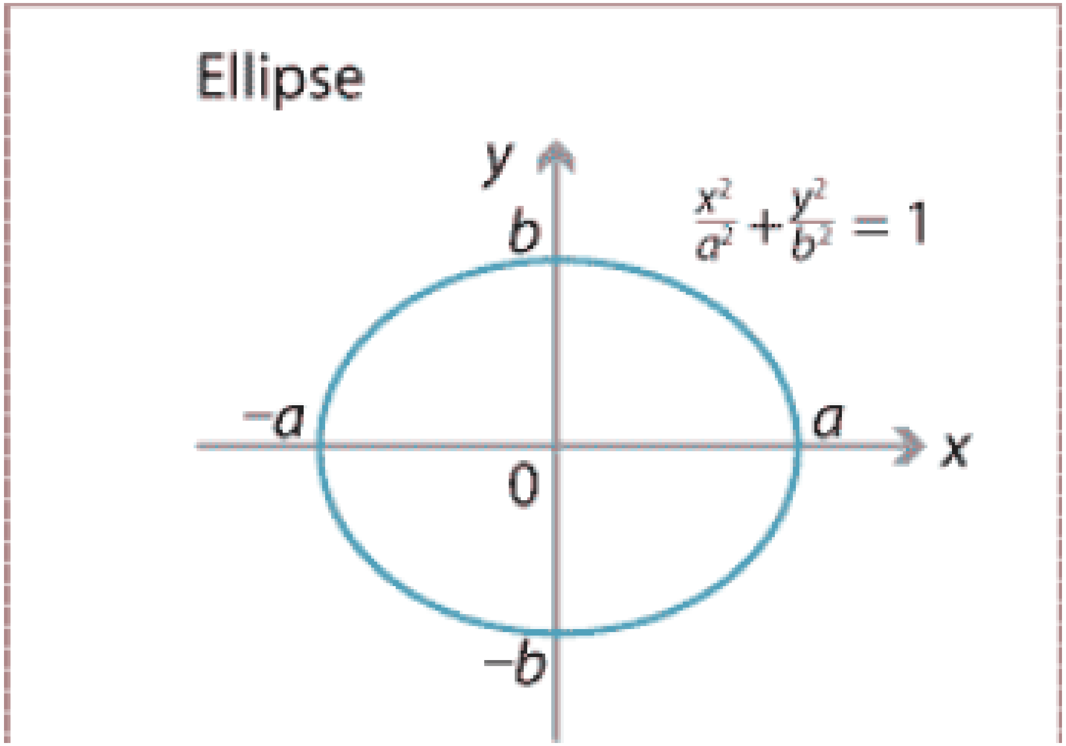
Circle



Quadratic forms

- In R^2

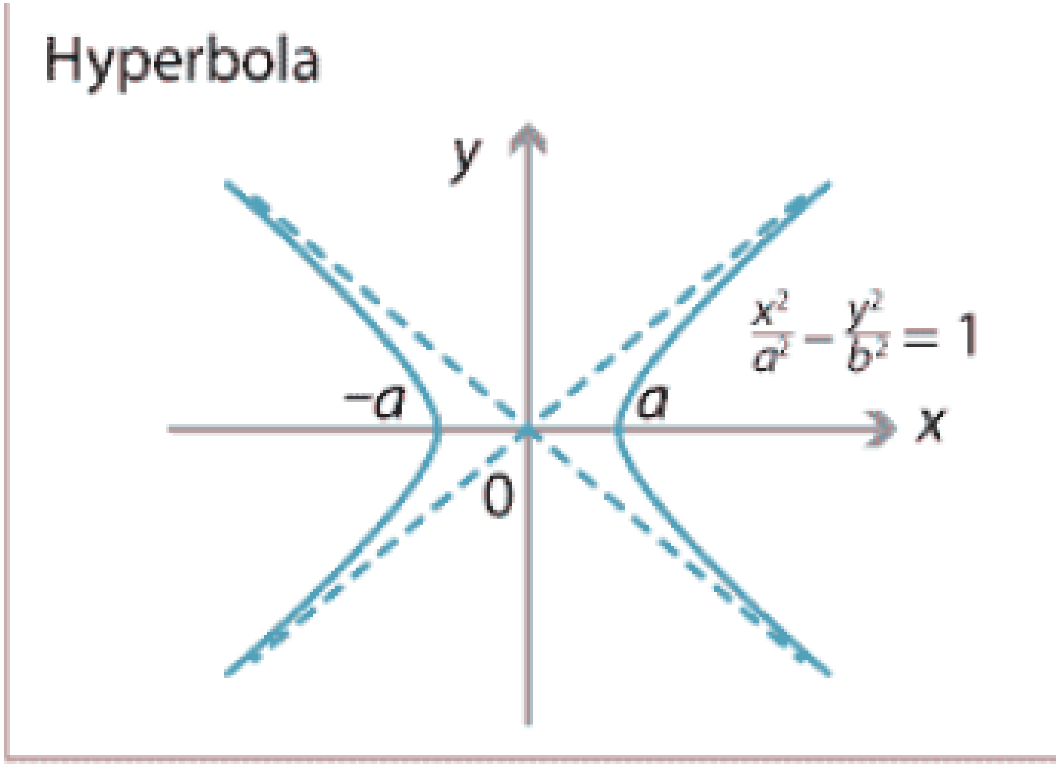
$$\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \qquad \frac{x_1^{\ 2}}{a^2} + \frac{x_2^{\ 2}}{b^2} = 1$$



Quadratic forms

- In R^2

$$\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{pmatrix} \qquad \frac{x_1^{\;2}}{a^2} - \frac{x_2^{\;2}}{b^2} = 1$$



Quadratic forms

- In R^n any pure quadratic form can be described with a <u>symmetric matrix</u>

=U (trasformation matrix)

- R^2 $$ax_1^{\ 2} + bx_2^{\ 2} + cx_1x_2 = 1 \qquad \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix}$$

'clean' a quadratic form in a way that will not be rotated:one you have U you can get the eigenvalue and trasform the original matrix into a diagonal one and then you can get the eigenvector that are the new basis

- R^3

$$ax_1^{\ 2} + bx_2^{\ 2} + cx_3^{\ 2} + dx_1x_2 + ex_1x_3 + fx_2x_3 = 1$$

$$\begin{pmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{pmatrix}$$

Quadratic forms

- In R^n any pure quadratic form can be described with a symmetric matrix → a diagonal representation is possible, upon an orthogonal transformation (IF A IS NOT SINGULAR)

$$U^T A U = \Lambda$$

- It exists an orthogonal matrix that can be used to put the system in better coordinates

$$x' = U^T x$$

$$x = U x'$$

Quadratic forms

- The quadratic form

$$x^t A x = 1$$

Becomes

$$(Ux')^t A (Ux') = 1$$

$$x'^t U^t A U x' = 1$$

$$x'^t \Lambda x' = 1$$

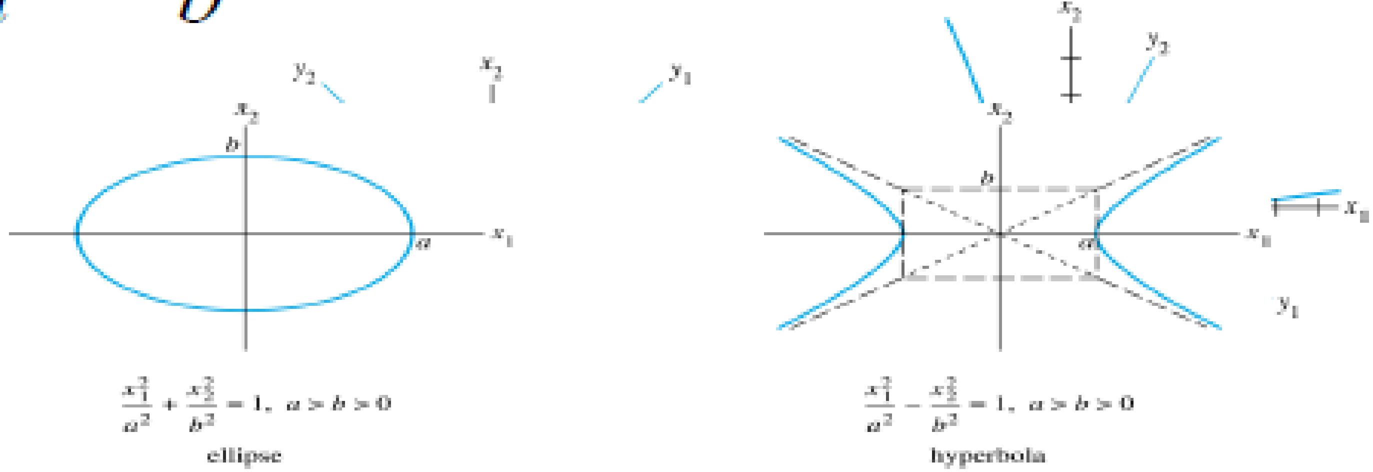
In the new system the quadratic form does not contain terms with mixed coordinates (Canonical form)

Canonical forms (R^2)

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

- Both the eigenvalues are positive

$$\frac{x_1^{\,2}}{a^2} + \frac{x_2^{\,2}}{b^2} = 1$$



An ellipse and a hyperbola in standard position.

An ellipse and a hyperbola *not* in standard position.

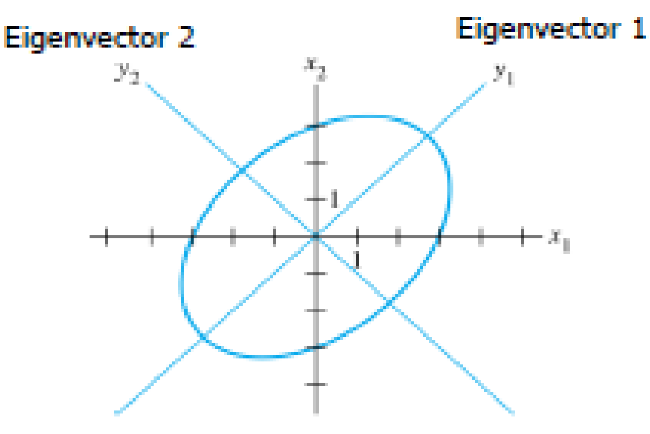
Canonical forms (R^2)

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

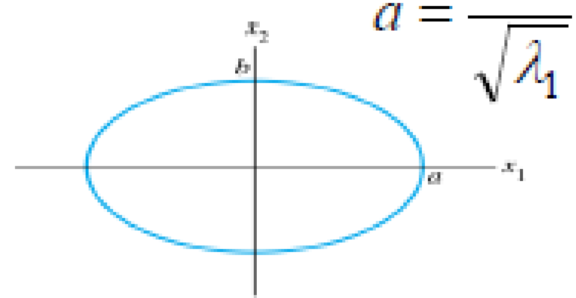
- Both the eigenvalues are positive

$$\frac{x_1^{\ 2}}{a^2} + \frac{x_2^{\ 2}}{b^2} = 1 \qquad \lambda_1 = \frac{1}{a^2}; \lambda_2 = \frac{1}{b^2}$$

$$a = \frac{1}{\sqrt{\lambda_1}}; b = \frac{1}{\sqrt{\lambda_2}}$$



(a) $5x_1^2 - 4x_1x_2 + 5x_2^2 = 48$



$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \ a > b > 0$

ellipse

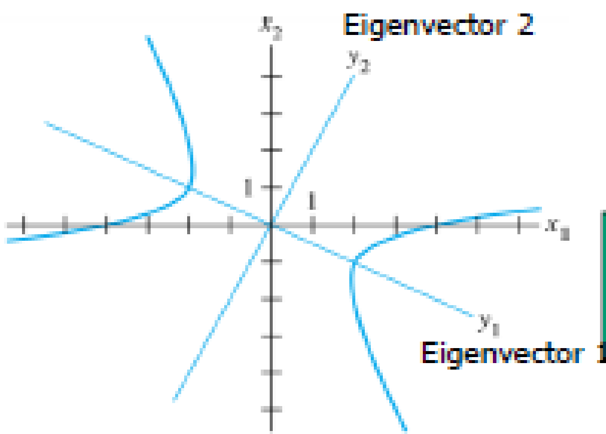
Canonical forms (R^2)

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

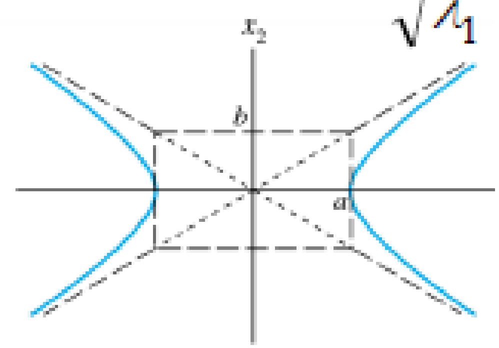
- Eigenvalues of opposite sign

$$\frac{x_1^{\;2}}{a^2} - \frac{x_2^{\;2}}{b^2} = 1 \qquad \lambda_1 = \frac{1}{a^2}; \lambda_2 = -\frac{1}{b^2}$$

$$a = \frac{1}{\sqrt{\lambda_1}}; b = \frac{1}{\sqrt{-\lambda_2}}$$



(b) $x_1^2 - 8x_1x_2 - 5x_2^2 = 16$



$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1,\ a > b > 0$

Canonical forms (R^2)

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

- Both the eigenvalues are negatives

$$-\frac{x_1^{\ 2}}{a^2} - \frac{x_2^{\ 2}}{b^2} = 1 \qquad \lambda_1 = -\frac{1}{a^2}; \lambda_2 = -\frac{1}{b^2}$$

No real solutions

Examples

$$x_1^{\ 2} - 5x_2^{\ 2} - 8x_1x_2 = 1$$

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

$$\lambda = 3 : \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} ; \lambda = -7 : \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

hyperbola

Examples

$$5x_1^{\ 2} + 5x_2^{\ 2} - 4x_1x_2 = 1$$

$$A = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$$

$$\lambda = 3 : \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}; \lambda = 7 : \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

Ellipse with semiaxes $\sqrt{3}$ and $\sqrt{7}$

3D

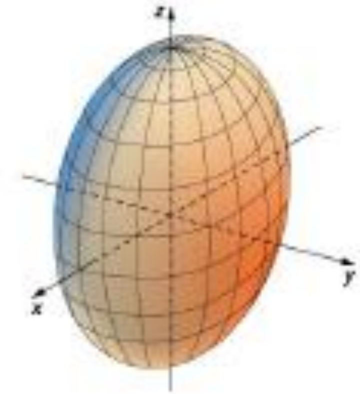
Characteristics of Common Quadric Surfaces

Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: an ellipse
In plane $x = r$: an ellipse

If $a = b = c$, then this surface is a sphere.

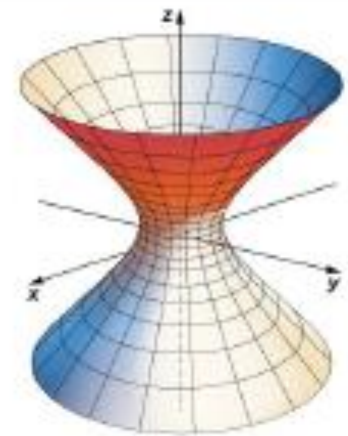


Hyperboloid of One Sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.



Hyperboloid of Two Sheets

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Traces
In plane $z = p$: an ellipse or the empty set (no trace)
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.