Counting

Basic counting

- SUM RULE: *if a task can be done in n1 ways and a second task can be done in n2 ways and they cannot be done at the same time, then there are n1+n2 ways to do either tasks.*

Example: A student can choose a project from one of 3 lists, containing 23,15 and 19 projects. How many possible projects are there to choose from?

Solution: 23+15+19

Basic counting

- PRODUCT RULE: *if a procedure can be broken in two tasks and there are n1 ways to do the first part and n2 ways to do the second after the first has been completed, then there are n1*n2 ways to do the procedure.*

Example: How many DNA sequences of length 100 are possible?

Solution: 4^{100}

Basic counting

- THE PIGEONHOLE PRINCIPLE: *if k+1 or more objects are placed into k boxes, then there is at least one box containing 2 or more of the objects.*

Example: In a room there are 100 people. What is the minimal number of people were born in the same month?

Solution: [100/12]

Counting methods: Example 1

Example 1: You draw one card from a deck of cards. What's the probability that you draw an ace?

$$P(\text{draw an ace}) = \frac{\#\text{ of aces in the deck}}{\#\text{ of cards in the deck}} = \frac{4}{52} = .0769$$

Counting methods: Example 2

Example 2. What's the probability that you draw 2 aces when you draw two cards from the deck?

$$P(\text{draw ace on first draw}) = \frac{\#\text{ of aces in the deck}}{\#\text{ of cards in the deck}} = \frac{4}{52}$$

$$P(\text{draw an ace on second draw too}) = \frac{\#\text{ of aces in the deck}}{\#\text{ of cards in the deck}} = \frac{3}{51}$$

$$\therefore P(\text{draw ace AND ace}) = \frac{4}{52} \text{ x } \frac{3}{51}$$

Counting methods: Example 2

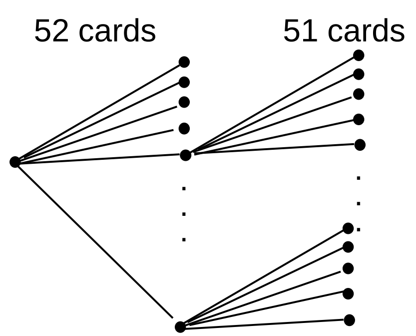
<u>Two counting method ways to calculate this:</u>

1. Consider order:

$$P(\text{draw 2 aces}) = \frac{\#\text{ of ways you can draw ace, ace}}{\#\text{ of different 2 - card sequences you could draw}}$$

Numerator: A♣A♦, A♣A♥, A♣A♠, A♦A♥, A♦A♠, A♦A♣, A♥A♦, A♥A♣, A♥A♠, A♠A♣, A♠A♦, or A♠A♥ = 12

Denominator = 52x51 = 2652 -- why?



$$\therefore P(\text{draw 2 aces}) = \frac{12}{52x51}$$

Counting methods: Example 2

2. Ignore order:

$$P(\text{draw 2 aces}) = \frac{\#\text{ of pairs of aces}}{\#\text{ of different two-card hands you could draw}}$$

Numerator: A♣A♦, A♣A♥, A♣A♠, A♦A♥, A♦A♠, A♥A♠ = 6

$$\text{Denominator} = \frac{52x51}{2} = 1326$$

Divide out order!

$$\therefore P(\text{draw 2 aces}) = \frac{6}{\frac{52x51}{2}}$$

Summary of Counting Methods

Counting methods for computing probabilities

- **Permutations— order matters!**
  - **With replacement**
    - always same dataset
  - **Without replacement**
    - each test influence the next
- **Combinations— Order doesn't matter**
  - **Without replacement**
  - **With repetitions**

Summary of Counting Methods

Counting methods for computing probabilities

- **Permutations—order matters!**
  - **With replacement**
  - **Without replacement**

Permutations—Order matters!

A permutation is an ordered arrangement of objects.

With replacement=once an event occurs, it can occur again (after you roll a 6, you can roll a 6 again on the same die).

Without replacement=an event cannot repeat (after you draw an ace of spades out of a deck, there is 0 probability of getting it again).

Summary of Counting Methods

Counting methods for computing probabilities

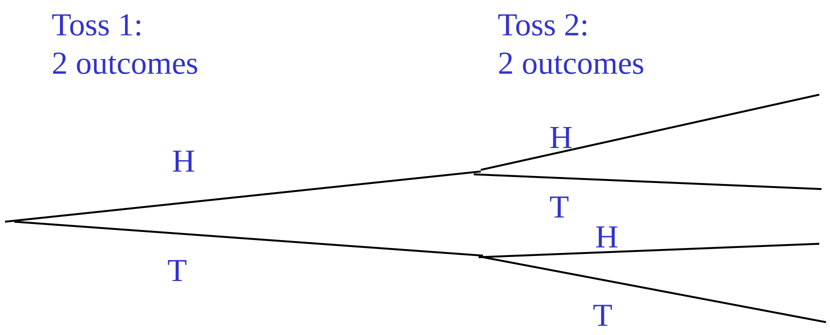
- **Permutations— order matters!**
  - **With replacement**

Permutations—with replacement

With Replacement – Think coin tosses, dice, and DNA.

"memoryless" – After you get heads, you have an equally likely chance of getting a heads on the next toss (unlike in cards example, where you can't draw the same card twice from a single deck).

What's the probability of getting two heads in a row ("HH") when tossing a coin?

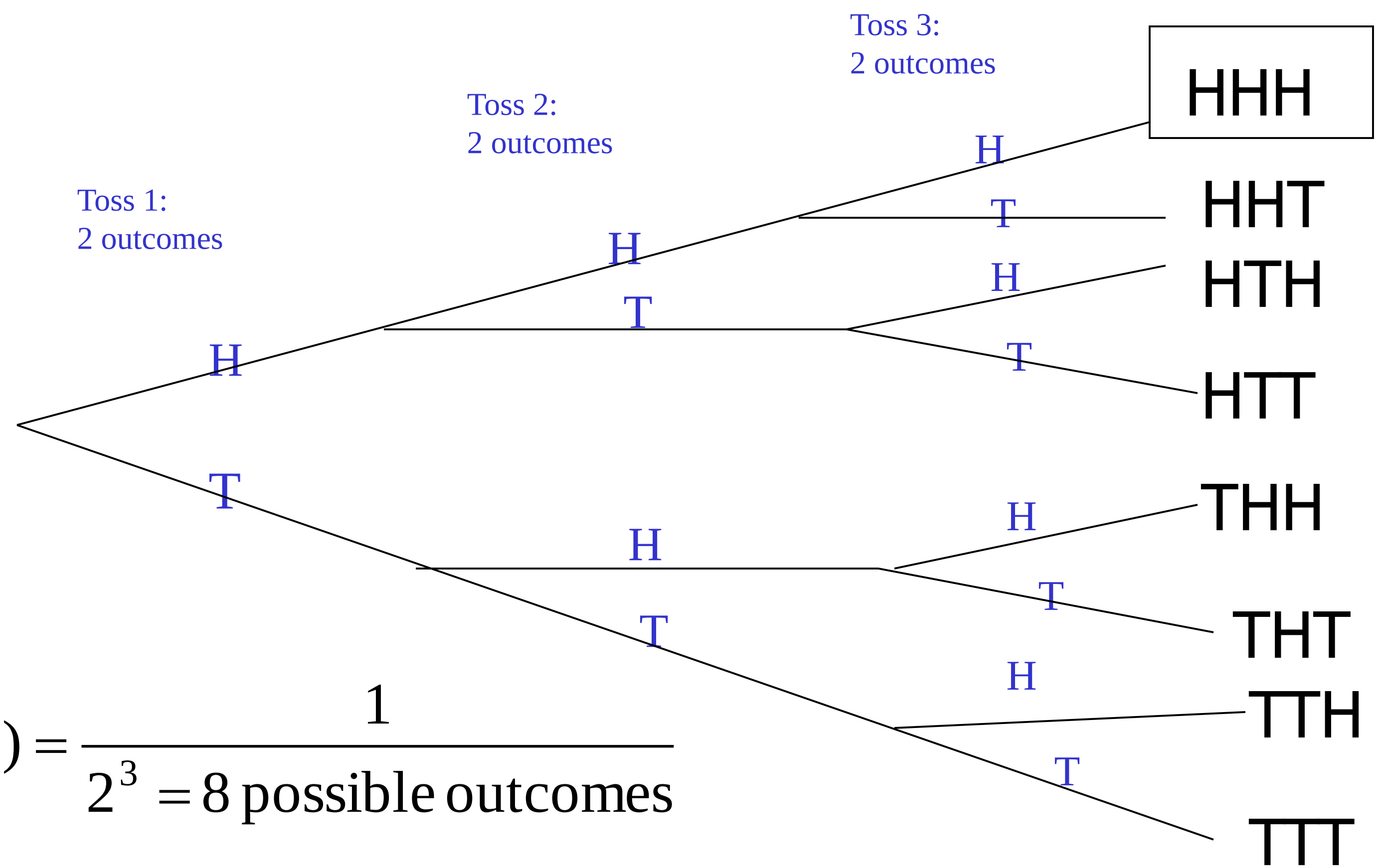


2^2 total possible outcomes: {HH, HT, TH, TT}

$$P(HH) = \frac{1 \text{ way to get HH}}{2^2 \text{ possible outcomes}}$$

Permutations—with replacement

What's the probability of 3 heads in a row?



$$P(HHH) = \frac{1}{2^3 = 8 \text{ possible outcomes}}$$

Permutations—with replacement

When you roll a pair of dice (or 1 die twice), what's the probability of rolling 2 sixes?

$$P(6,6)\frac{1\text{ way to roll } 6,6}{6^2} = \frac{1}{36}$$

What's the probability of rolling a 5 and a 6?

$$P(5\,\&\,6) = \frac{2\text{ ways}: 5,6\text{ or }6,5}{6^2} = \frac{2}{36}$$

Summary: order matters, with replacement

Formally, "permutations" and "with replacement"→ use powers→

outcomes^tryes

$$(\#\,\text{possible outcomes per event})^{\text{the \# of events}} = n^r$$

Summary of Counting Methods

Counting methods for computing probabilities

Permutations—order matters!

Without replacement

Permutations—without replacement

Without replacement—Think cards (w/o reshuffling) and seating arrangements.

Example: You are moderating a debate of gubernatorial candidates. How many different ways can you seat the panelists in a row? Call them Arianna, Buster, Camejo, Donald, and Eve.

Permutation—without replacement

→ "Trial and error" method:

Systematically write out all combinations:

A B C D E

A B C E D

A B D C E

A B D E C

A B E C D

A B E D C

.

.

.

Quickly becomes a pain!

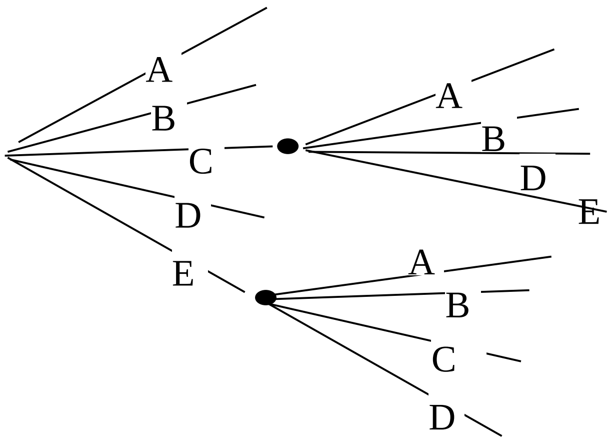
Easier to figure out patterns using a the probability tree!

Permutation—without replaceme

<u>Seat One:</u>
5 possible

<u>Seat Two:</u>
only 4 possible

Etc….



…….

# of permutations = 5 x 4 x 3 x 2 x 1 = 5!

There are 5! ways to order 5 people in 5 chairs
(since a person cannot repeat)

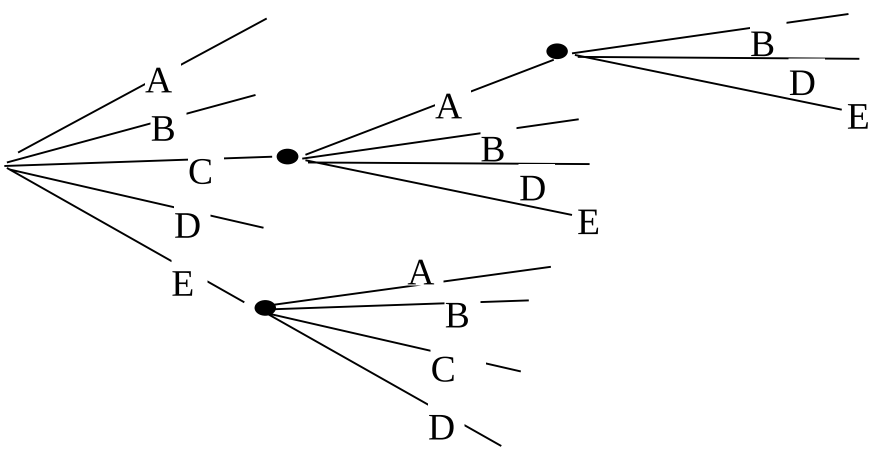
Permutation—without replacement

What if you had to arrange 5 people in only 3 chairs (meaning 2 are out)?

Seat One:
5 possible

Seat Two:
Only 4 possible

Seat Three:
only 3 possible



$$\boxed{5x4x3 =}$$

$$\boxed{\frac{5x4x3x2x1}{2x1} = \frac{5!}{2!} =}$$

$$\boxed{\frac{5!}{(5-3)!}}$$

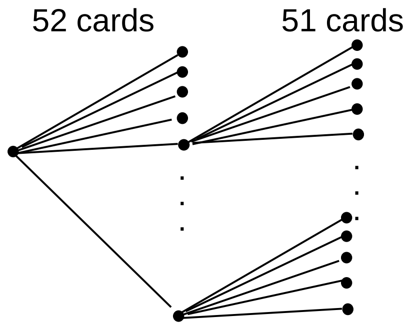
Permutation—without replacement

Note this also works for 5 people and 5 chairs:

$$\boxed{\frac{5!}{(5-5)!} = \frac{5!}{0!} = 5!}$$

Permutation—without replacement

How many two-card hands can I draw from a deck when order matters (e.g., ace of spades followed by ten of clubs is different than ten of clubs followed by ace of spades)



$$\frac{52!}{(52-2)!} = 52x51$$

Summary: order matters, without replacement

Formally, "permutations" and "without replacement"→ use factorials→

$$\frac{(n\text{ people or cards})!}{(n\text{ people or cards} - r\text{ chairs or draws})!} = \frac{n!}{(n-r)!}$$

$$\text{or } \; n(n-1)(n-2)\ldots(n-r+1)$$

Practice problems:

1. A wine taster claims that she can distinguish four vintages or a particular Cabernet. What is the probability that she can do this by merely guessing (she is confronted with 4 unlabeled glasses)? 1\4!
2. In some states, license plates have six characters: three letters followed by three numbers. How many distinct such plates are possible? 26^3+10^3 NB:without replacment

Answer 1

1. A wine taster claims that she can distinguish four vintages or a particular Cabernet. What is the probability that she can do this by merely guessing (she is confronted with 4 unlabeled glasses)? (hint: without replacement)

P(success) = 1 (there's only way to get it right!) / total # of guesses she could make

Total # of guesses one could make randomly:

glass one:	glass two:	glass three:	glass four:
4 choices	3 vintages left	2 left	no "degrees of freedom" left

= 4 x 3 x 2 x 1 = 4!

$\therefore$ P(success) = 1 / 4! = 1/24 = .04167

Answer 2

2. In some states, license plates have six characters: three letters followed by three numbers. How many distinct such plates are possible? (hint: with replacement)

26^3 different ways to choose the letters and 10^3 different ways to choose the digits

$\therefore$ total number = $26^3 \times 10^3 = 17{,}576 \times 1000 = 17{,}576{,}000$

Summary of Counting Methods

Counting methods for computing probabilities

Combinations—Order doesn't matter

Without replacement

2. Combinations—Order doesn't matter

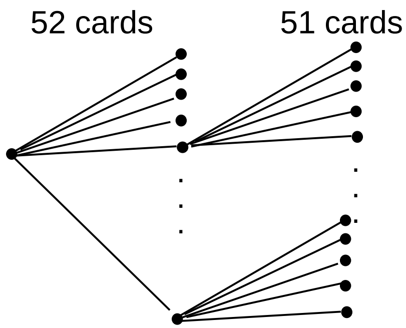
Introduction to combination function, or "choosing"

Written as: ${}_nC_r$ or $\binom{n}{r}$

Spoken: "*n* choose *r*"

Combinations

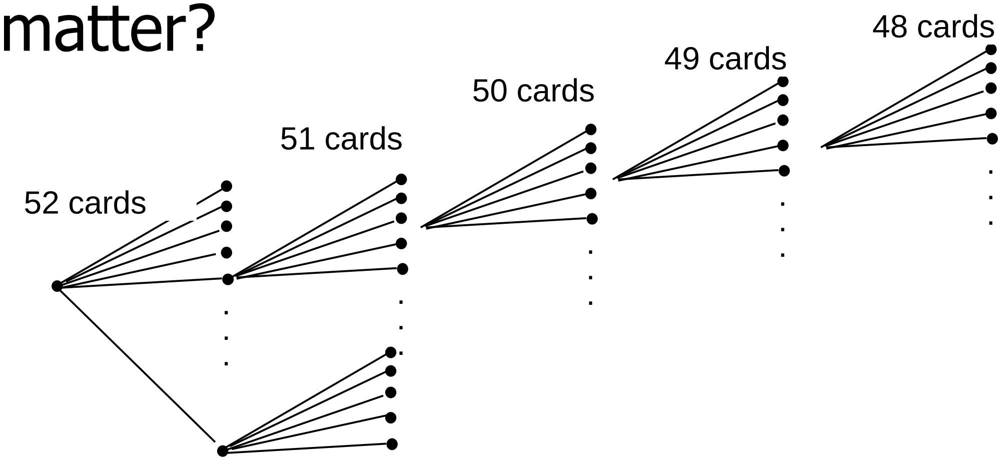
How many two-card hands can I draw from a deck when order does <u>not</u> matter (e.g., ace of spades followed by ten of clubs is <u>the same</u> as ten of clubs followed by ace of spades)



$$\frac{52x51}{2} = \frac{52!}{(52-2)!2}$$

Combinations

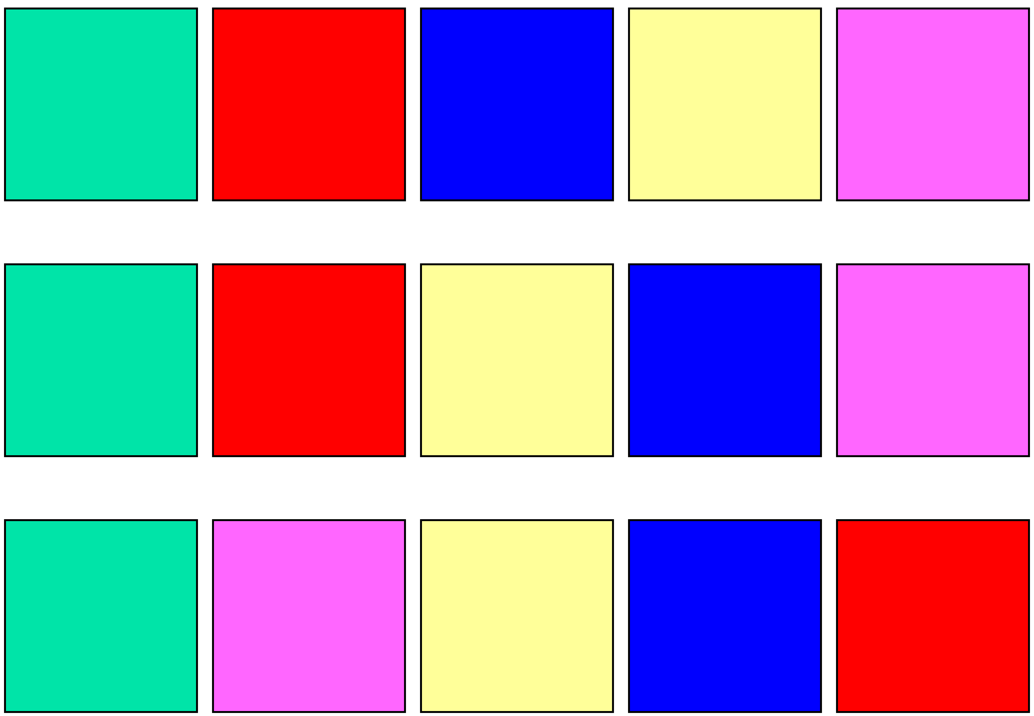
How many five-card hands can I draw from a deck when order does not matter?



$$\frac{52x51x50x49x48}{?}$$

Combinations

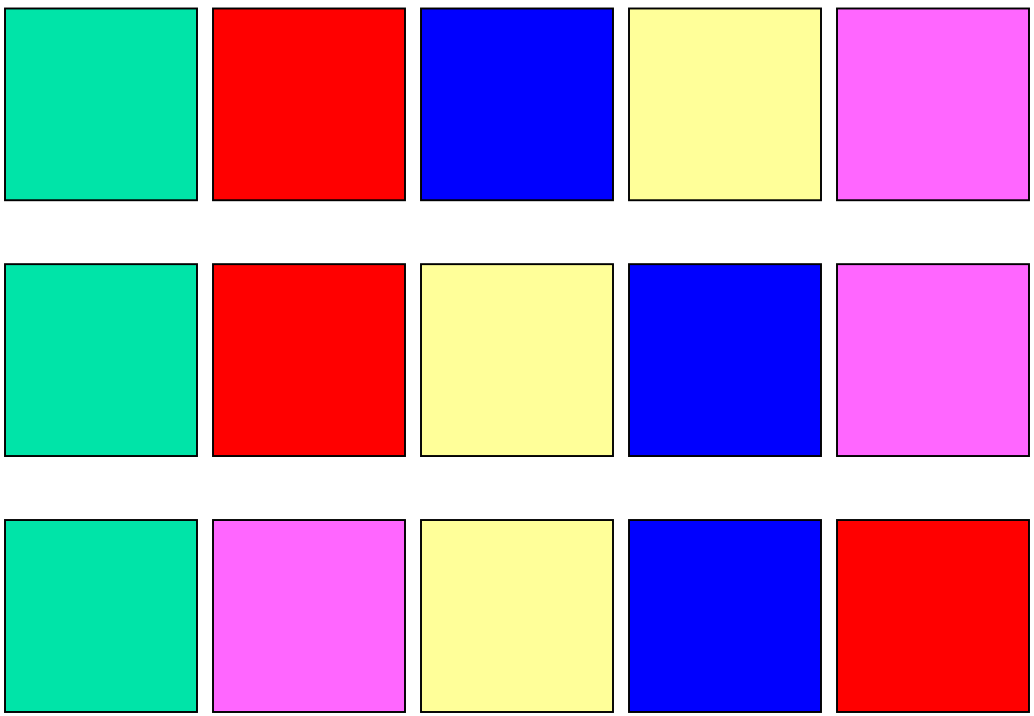
1.
2.
3.



….

How many repeats total??

Combinations



1.

2.

3.

….

i.e., how many different ways can you arrange 5 cards…?

Combinations

That's a permutation without replacement.

5! = 120

$$\text{total \# of 5-card hands} = \frac{52x51x50x49x48}{5!} = \frac{52!}{(52-5)!5!}$$

Combinations

- How many unique 2-card sets out of 52 cards? $$\frac{52x51}{2} = \frac{52!}{(52-2)!2!}$$
- 5-card sets? $$\frac{52x51x50x49x48}{5!} = \frac{52!}{(52-5)!5!}$$
- r-card sets? $$\frac{52!}{(52-r)!r!}$$
- r-card sets out of n-cards? $$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Combinations

Example 2: You are moderating a debate of 3 men and 2 women. How many different ways (in terms of gender) can you seat the candidates in a row?

Recall: Arianna, Buster, Camejo, Donald, and Eve. Obviously, if you only consider gender, there will be fewer arrangements.

consider it as sitting 2 people in five seats and the other three will stay accordingly

For example:

arrangement A B C D E (♀♂ ♂ ♂ ♀) =

arrangement E C B D A (♀♂ ♂ ♂ ♀)

This one arrangement: ♀♂ ♂ ♂ ♀ (women occupy ends, men center 3 seats) covers 12 distinct scenarios:

A B C D E

A B D C E

A C B D E

A C D B E

A D B C E

A D C B E

E B C D A

E B D C A

E C B D A

E C D B A

E D B C A

E D C B A

6 permutations of the 3 men (=3!) x 2 permutations of the women (=2!) = 12

12 permutations → 1 gender-based seating arrangement

Similarly: ♂ ♂ ♂ ♀ ♀ covers 3! x 2! permutations.

B C D E A
B D C E A
C B D E A
C D B E A
D B C E A
D C B E A
B C D A E
B D C A E
C B D A E
C D B A E
D B C A E
D C B A E

6 permutations of the 3 men (=3!) x 2 permutations of the women (=2!) = 12

∴ 5! possible arrangements of A, B, C, D, and E are reduced to 5!/12 or 5!/(3!2!)

Summary

This is also a "choosing" problem, since you are choosing 3 out of 5 seats to go to the men (the rest go to the women)

$_5C_3 = {}_5C_2 = \quad = 5!/(3!2!) = 10$

Summary: combinations

If r objects are taken from a set of n objects without replacement and disregarding order, how many different samples are possible?

Formally, "order doesn't matter" and "without replacement" → use choosing →

$$\binom{n}{r} = \frac{n!}{\boxed{(n-r)!r!}}$$

binomial coefficent

Examples—Combinations

A lottery works by picking 6 numbers from 1 to 49. How many combinations of 6 numbers could you choose?

$$\binom{49}{6} = \frac{49!}{43!6!} = 13{,}983{,}816$$

Which of course means that your probability of winning is 1/13,983,816

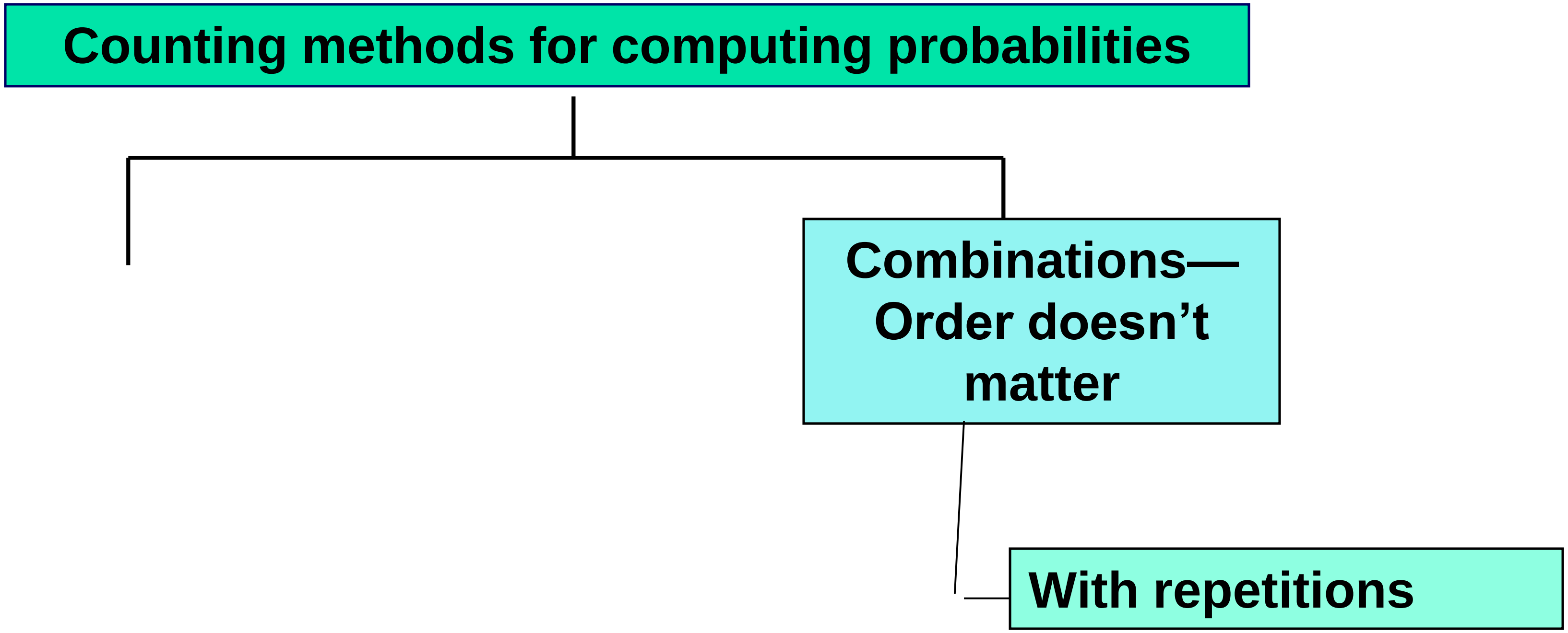
Examples

How many ways can you get 3 heads in 5 coin tosses?

$$\binom{5}{3} = \frac{5!}{3!2!} = 10$$

combination of three heads in five 'slot'

Counting Methods



Stars and bars

- Suppose you have three possible type of biscuits, say Chocolate, Nuts, Vanilla. You are picking 5 from a set that contains more than 5 biscuits of each type.
- How many ways to select the 5 biscuits do you have?

Stars and bars

- Chocolate, Nuts, Vanilla (three kinds)
- * * * * * = 5 biscuits
- * * *| *| * = 3 Chocolate, 1 Nuts, 1 Vanilla
  * * *| |* * = 3 Chocolate, 0 Nuts, 1 Vanilla
- …

Stars and bars

- * * *| *| *
- There are 5*+2|| = 7 objects and we can pick 7! possibilities. However, swapping two stars or two bars do not change the picture, then
- 7!/(5! 2!)= $\binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(k-1)!}$

N='slot'
K=different element

always the number of combinations divided by the factorial of the number of elemenats for wich we dont care of the order

Summary of Counting Methods

NB

Counting methods for computing probabilities

Permutations—order matters!

- **With replacement: n^r**
- **Without replacement:** ***n(n-1)(n-2)…(n-r+1)=***

$$\frac{n!}{(n-r)!}$$

Combinations—Order doesn't matter

- **Without replacement:**

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

- **With repetitions:**

$$\binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(k-1)!}$$