Vectors

Vectors

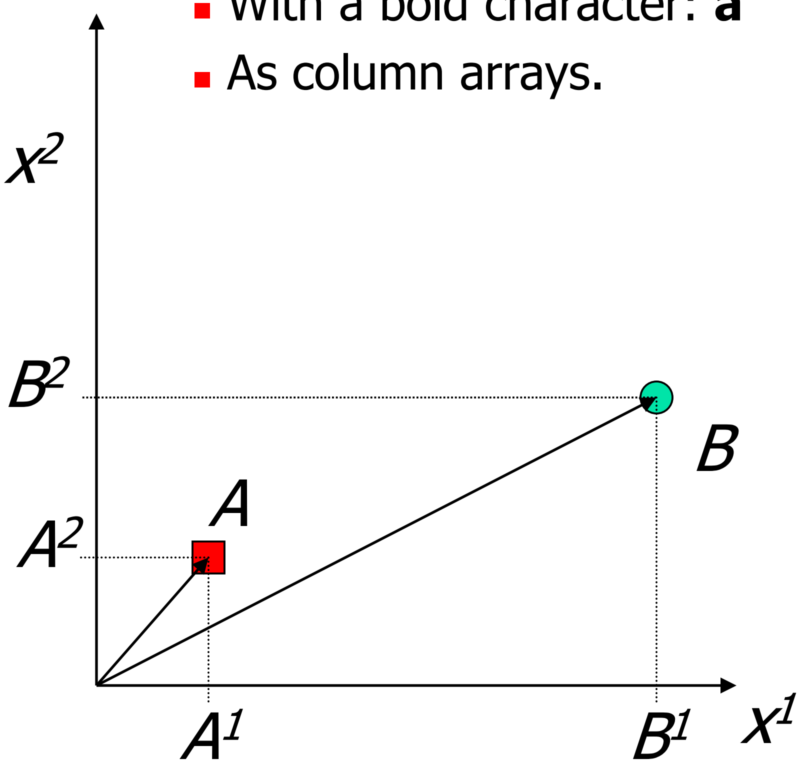
A vector space (also called a linear space) is a collection of objects called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context.

Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by complex numbers, rational numbers, or generally any field.

The operations of vector addition and scalar multiplication must satisfy certain requirements, called axioms

Vectors: notations

- A vector in a n-dimensional space in described by a n-uple of real numbers
- Vector symbols can be written:
  - With an arrow up to the vector names: $\vec{a}$
  - With a bold character: **a**
  - As column arrays.



$$A = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} \qquad A^T = \begin{pmatrix} A^1 & A^2 \end{pmatrix}$$

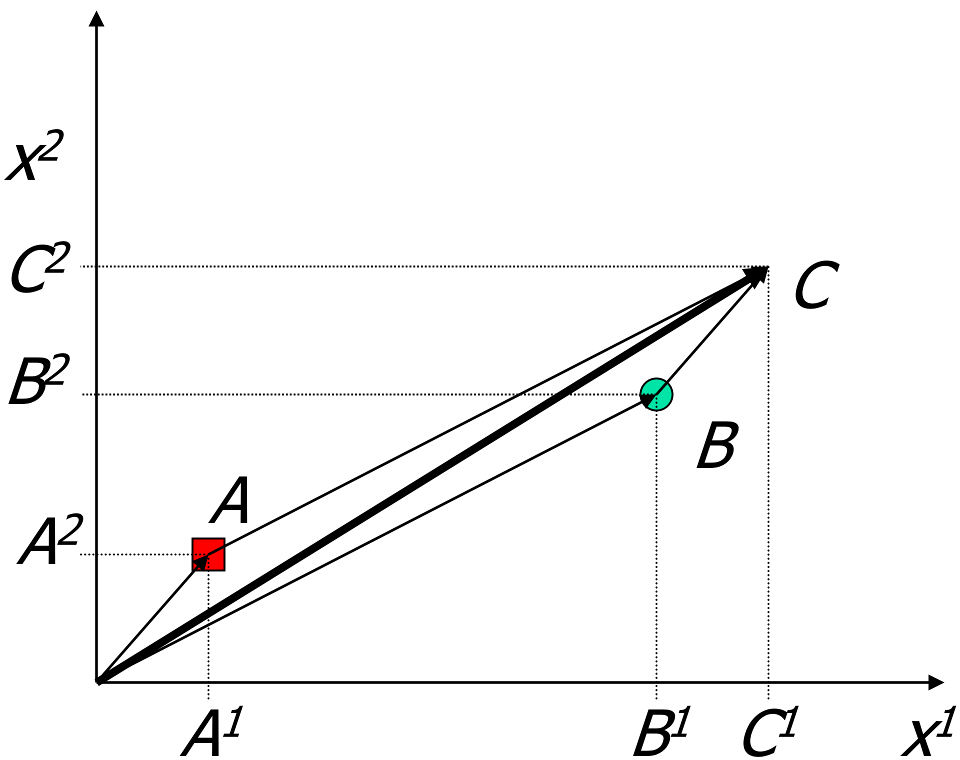
$$B = \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} \qquad B^T = \begin{pmatrix} B^1 & B^2 \end{pmatrix}$$

Vectors: sum

- The components of the sum vector are the sums of the components

$$C = A + B$$

$$\begin{pmatrix} C^1 \\ C^2 \end{pmatrix} = \begin{pmatrix} A^1 + B^1 \\ A^2 + B^2 \end{pmatrix}$$

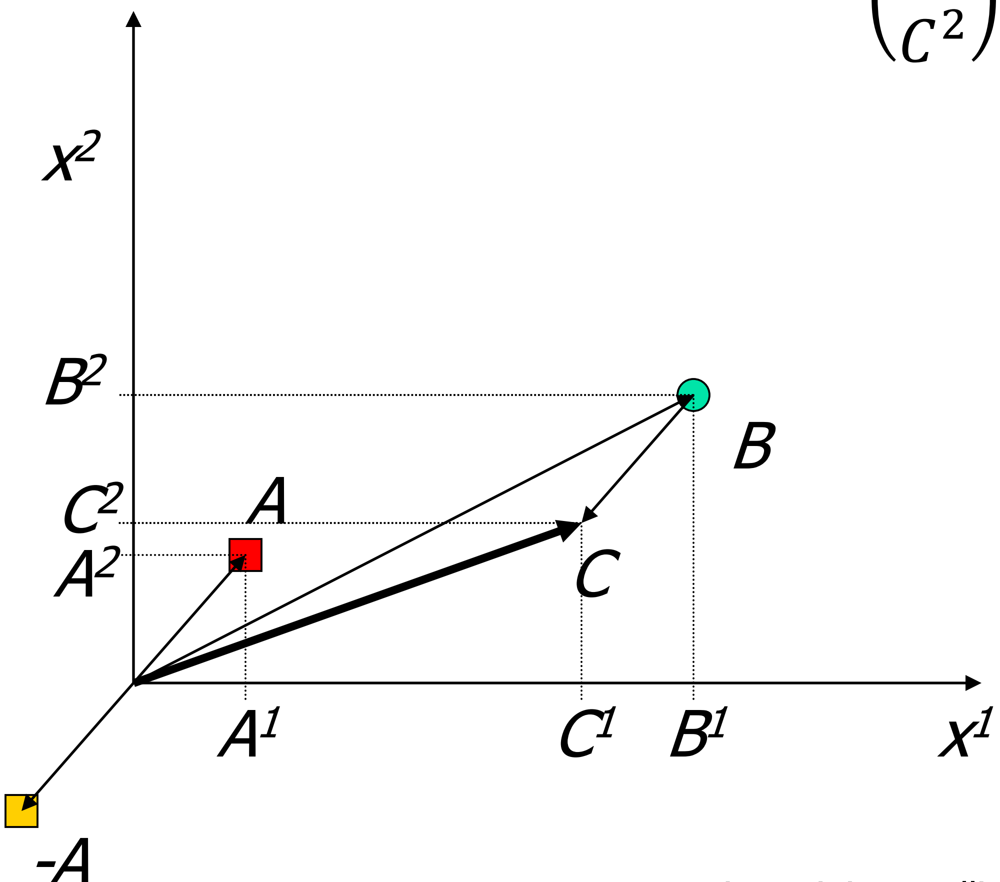


Vectors: difference

- The components of the sum vector are the difference of the components

$$C = B - A$$

$$\begin{pmatrix} C^1 \\ C^2 \end{pmatrix} = \begin{pmatrix} B^1 - A^1 \\ B^2 - A^2 \end{pmatrix}$$

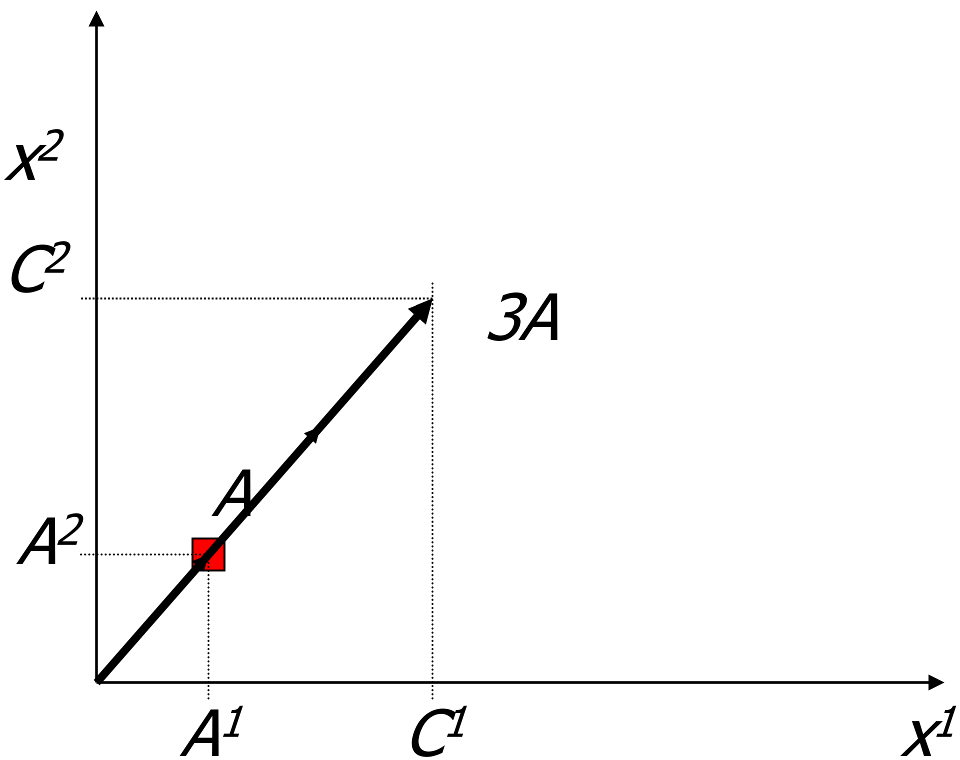


Vectors: product by a scalar

- The components of the rescaled vector are the rescaled components

$$C = a \cdot A$$

$$\begin{pmatrix} C^1 \\ C^2 \end{pmatrix} = \begin{pmatrix} a \cdot A^1 \\ a \cdot A^2 \end{pmatrix}$$



Axioms

Axiom	**Meaning**
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of addition	For every $\mathbf{v} \in \mathrm{V}$, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

Exercises

$$A = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \quad B = \begin{pmatrix} -1 \\ +3 \end{pmatrix} \quad C = \begin{pmatrix} -1 \\ +3 \\ +4 \end{pmatrix} \quad D = \begin{pmatrix} -1 \\ +3 \\ +4 \\ 0 \end{pmatrix} \quad E = \begin{pmatrix} 1 \\ -3 \\ -4 \\ 0 \end{pmatrix}$$

For which pair(s) of vectors it is possible to compute the vector:

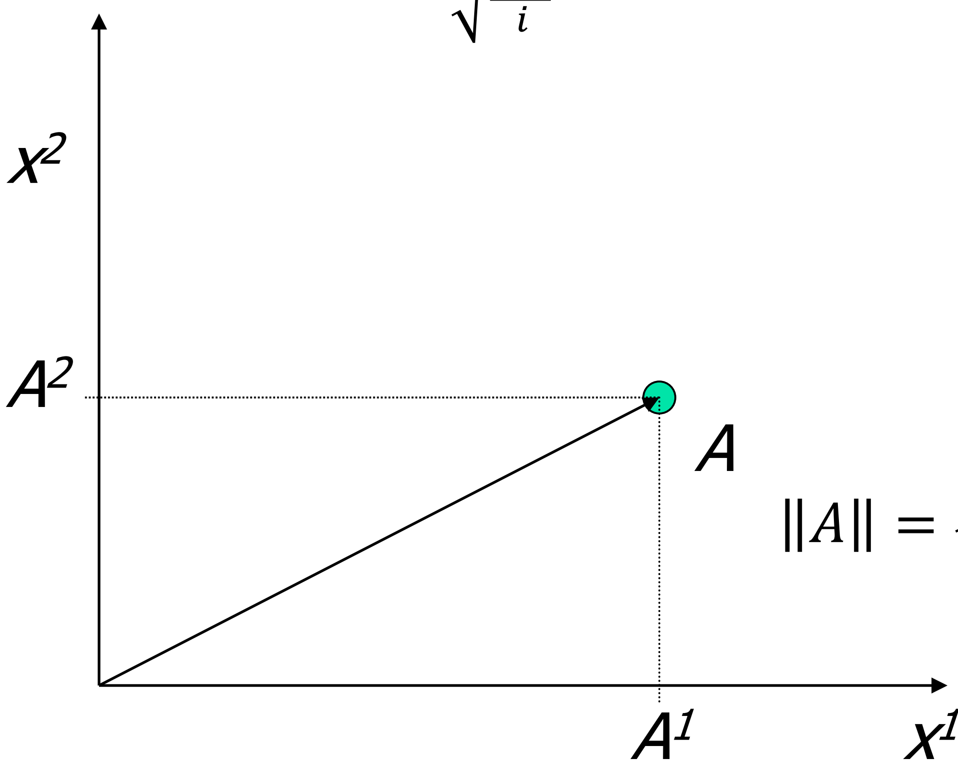
$$W^1 = 3 \cdot V - 2U$$
$$W^2 = V + U$$
$$W^3 = 4\left(\frac{1}{2}V + \frac{1}{4}U\right)$$

Compute W^1 , W^2 and W^3 for all that pairs.

Introducing a metric: Norm

- The most simple definition for a norm is the euclidean module of the components

$$\|A\| = \sqrt{\sum_i (A^i)^2}$$



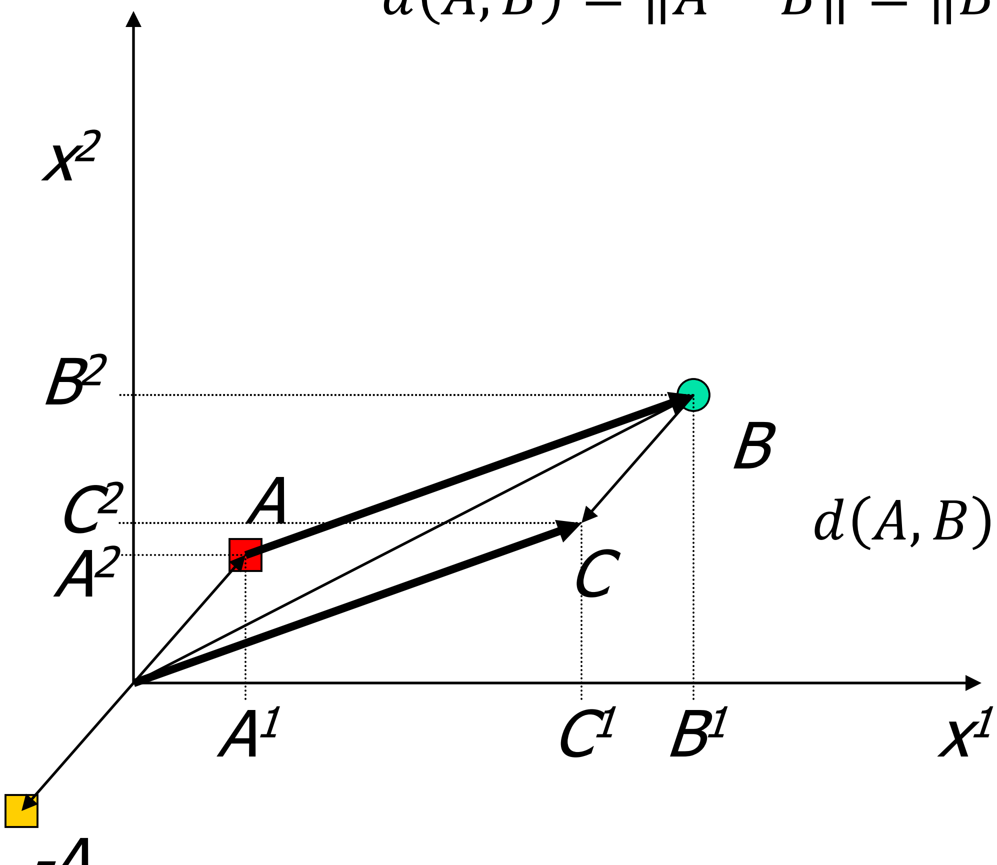
1. $||X + Y|| \leq ||X|| + ||Y||$
2. $||\lambda X|| = \lambda||X||$
3. $||X|| > 0 \quad \text{se } X \neq \mathbf{0}$

$$\|A\| = \sqrt{(A^1)^2 + (A^2)^2}$$

Distance between two points

- The distance between two points is the norm of the difference vector

$$d(A,B) = \|A - B\| = \|B - A\|$$



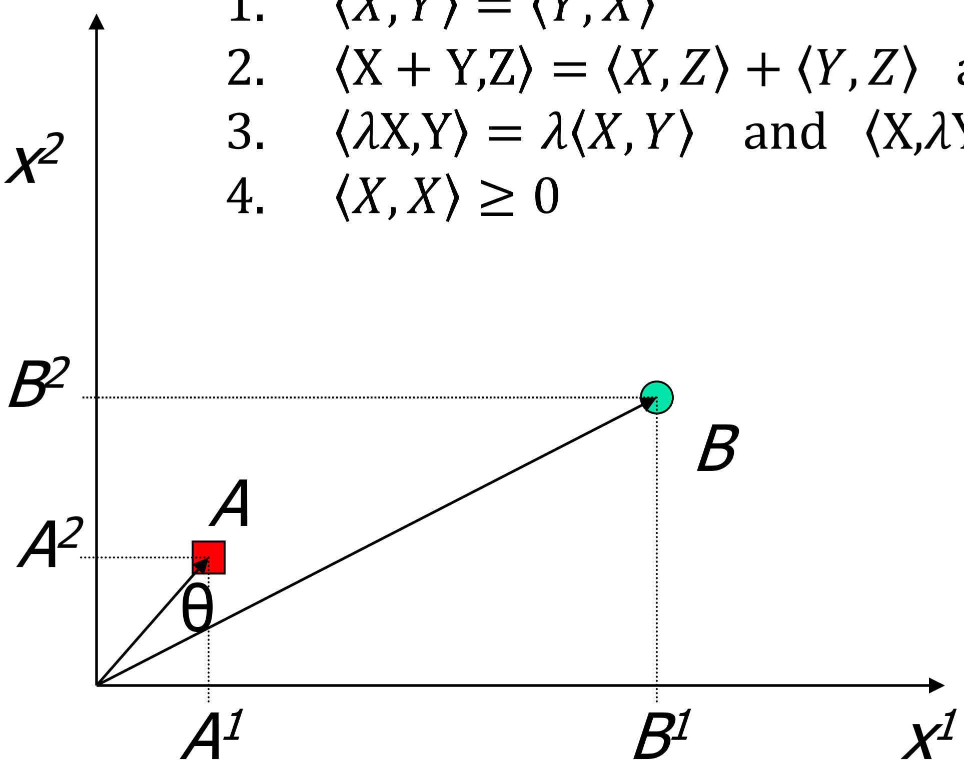
$$d(A,B) = \sqrt{(B^1 - A^1)^2 + (B^2 - A^2)^2}$$

Scalar [or "dot" or "inner"] product

$$c: \Re^n \times \Re^n \to \Re$$

$$c = A \cdot B = \langle AB \rangle = \sum_i A^i B^i$$

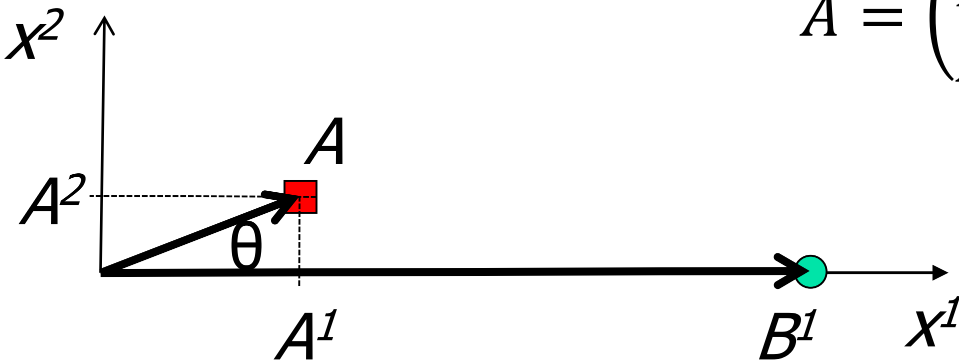
1. $\langle X, Y \rangle = \langle Y, X \rangle$
2. $\langle \mathrm{X} + \mathrm{Y}, \mathrm{Z} \rangle = \langle X, Z \rangle + \langle Y, Z \rangle$ and $\langle \mathrm{X}, \mathrm{Y} + \mathrm{Z} \rangle = \langle X, Y \rangle + \langle X, Z \rangle$
3. $\langle \lambda \mathrm{X}, \mathrm{Y} \rangle = \lambda \langle X, Y \rangle$ and $\langle \mathrm{X}, \lambda \mathrm{Y} \rangle = \lambda \langle X, Y \rangle$
4. $\langle X, X \rangle \geq 0$



$$c = \|A\| \cdot \|B\| \cdot \cos\vartheta$$

Scalar product: geometrical interpretation

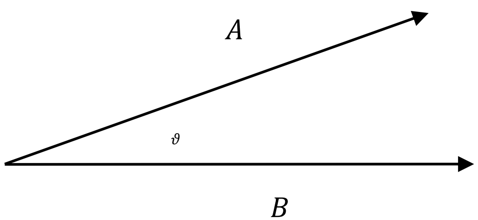
- Consider a reference frame where B is collinear to the x axis (you can ALWAYS find it)



$$A = \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} \qquad B = \begin{pmatrix} B^1 \\ 0 \end{pmatrix}$$

$$c = \langle AB \rangle = \sum_i A^i\, B^i = A^1 B^1 = \|B\| \cdot \|A\| \cos\theta$$

Scalar product and ortogonality



$\vartheta < 90 \quad \langle A, B \rangle > 0$

$\vartheta > 90 \quad \langle A, B \rangle < 0$

$\vartheta = 90 \quad \langle A, B \rangle = 0$

No cancellation rule

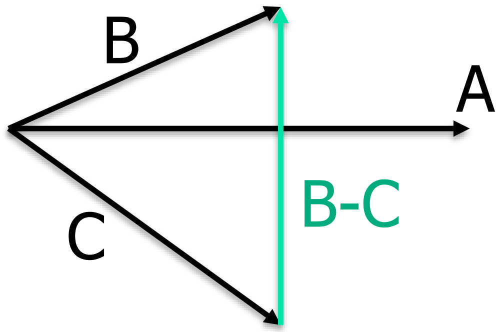
If a, b and c are real number and considering the usual product

$$a \cdot b = a \cdot c \Leftrightarrow b = c$$

If A, B and C are real number and considering the scalar product

$$A \cdot B = A \cdot C \not\Rightarrow B = C$$

The equation only says that the vector (B-C) is ortogonal to A



Norm and scalar product

- The square norm is equal to the scalar product of a vector with itself

$$\|A\| = \sqrt{\sum_i (A^i)^2} = \sqrt{\langle A, A \rangle}$$

Exercises

$$A = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \qquad B = \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix}$$

Compute the norm of both the vectors

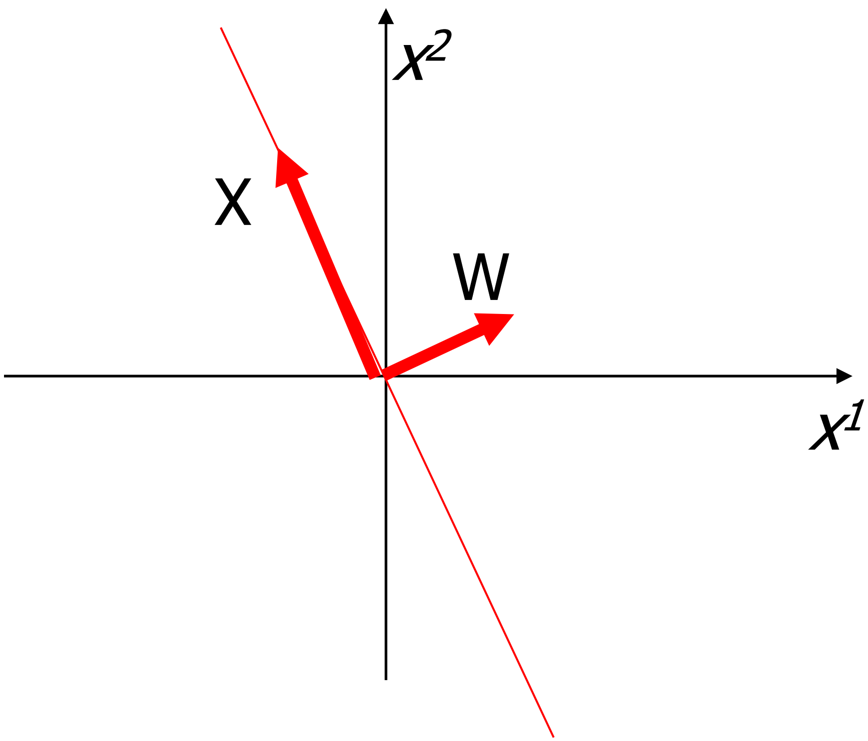
Compute the scalar product between them

Compute the angle between the two vectors

Definition of a hyperplane passing through the origin

In $\mathbf{R^2}$, a hyperplane is a line

A line passing through the origin can be defined with as the set of points defined by the vectors that are perpendicular to a given vector W

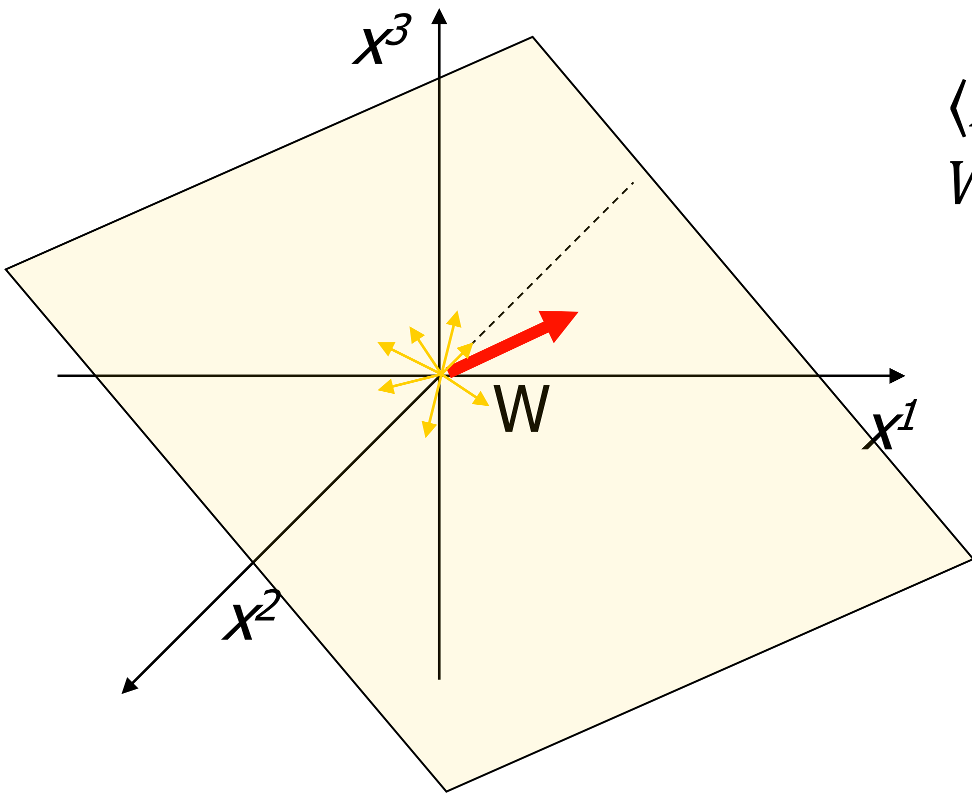


$$\langle XW \rangle = 0$$
$$W^1X^1 + W^2X^2 = 0$$

Definition of a hyperplane passing through the origin

In **R³** , a hyperplane is a plane

A plane passing through the origin can be defined with as the set of the vectors that are perpendicular to a given vector W



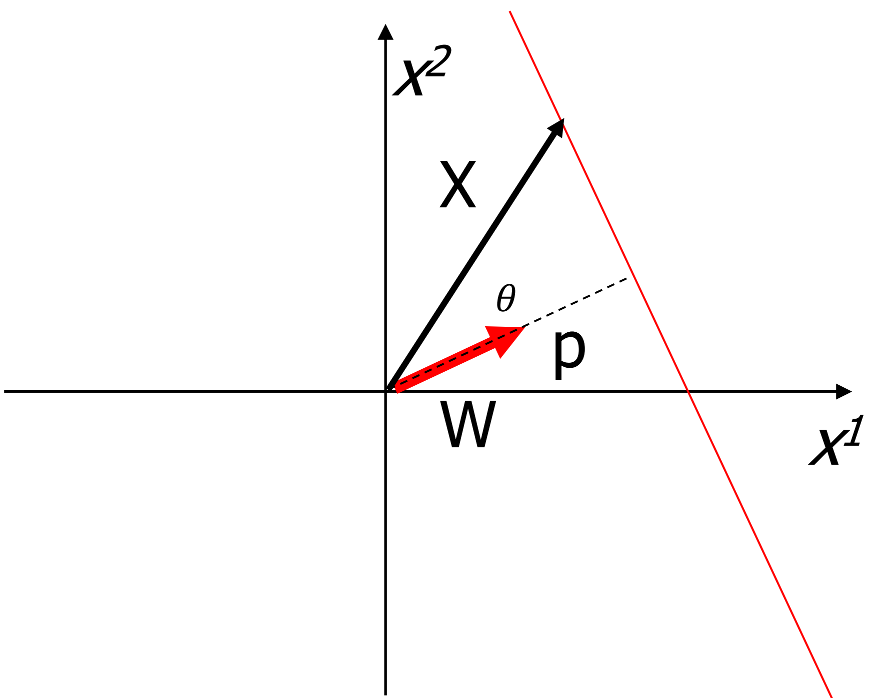
$$\langle XW \rangle = 0$$
$$W^1X^1 + W^2X^2 + W^3X^3 = 0$$

Definition of a hyperplane passing out the origin

In **R^2** , a hyperplane is a line

Consider a vector W: it defines an infinity of straight lines perpendicular to it.

A particular line is fixed when the projection of points of the line on vector W is fixed to a value p:



$$\|X\| \cos(\theta) = p$$

$$\frac{\langle XW \rangle}{\|W\|} = \frac{W^T X}{\|W\|} = p$$

$$p > 0$$

Definition of a hyperplane passing out the origin

In $\mathbf{R^2}$, a hyperplane is a line

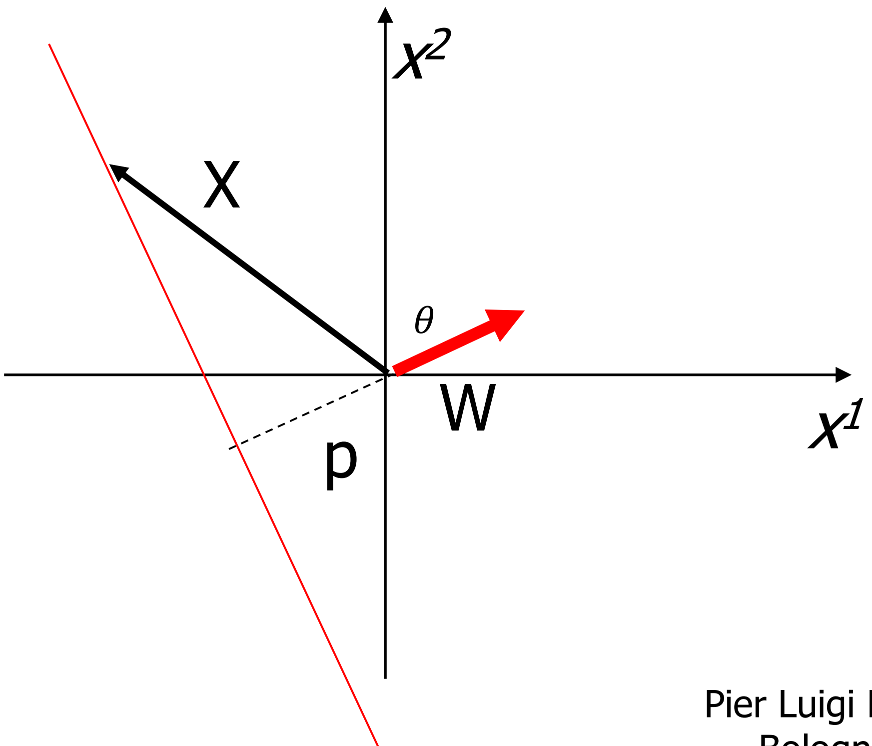
Consider a vector W: it defines an infinity of straight lines perpendicular to it.

A particular line is fixed when the projection of points of the line on vector W is fixed to a value p:

$$\|X\|\cos(\theta) = p$$

$$\frac{\langle XW \rangle}{\|W\|} = \frac{W^T X}{\|W\|} = p$$

$$p < 0$$



Definition of a hyperplane passing out the origin

In $\mathbf{R}^2$, a hyperplane is a line

$$\frac{\langle XW \rangle}{\|W\|} = p$$

Calling: $$p = \frac{-b}{\|W\|}$$

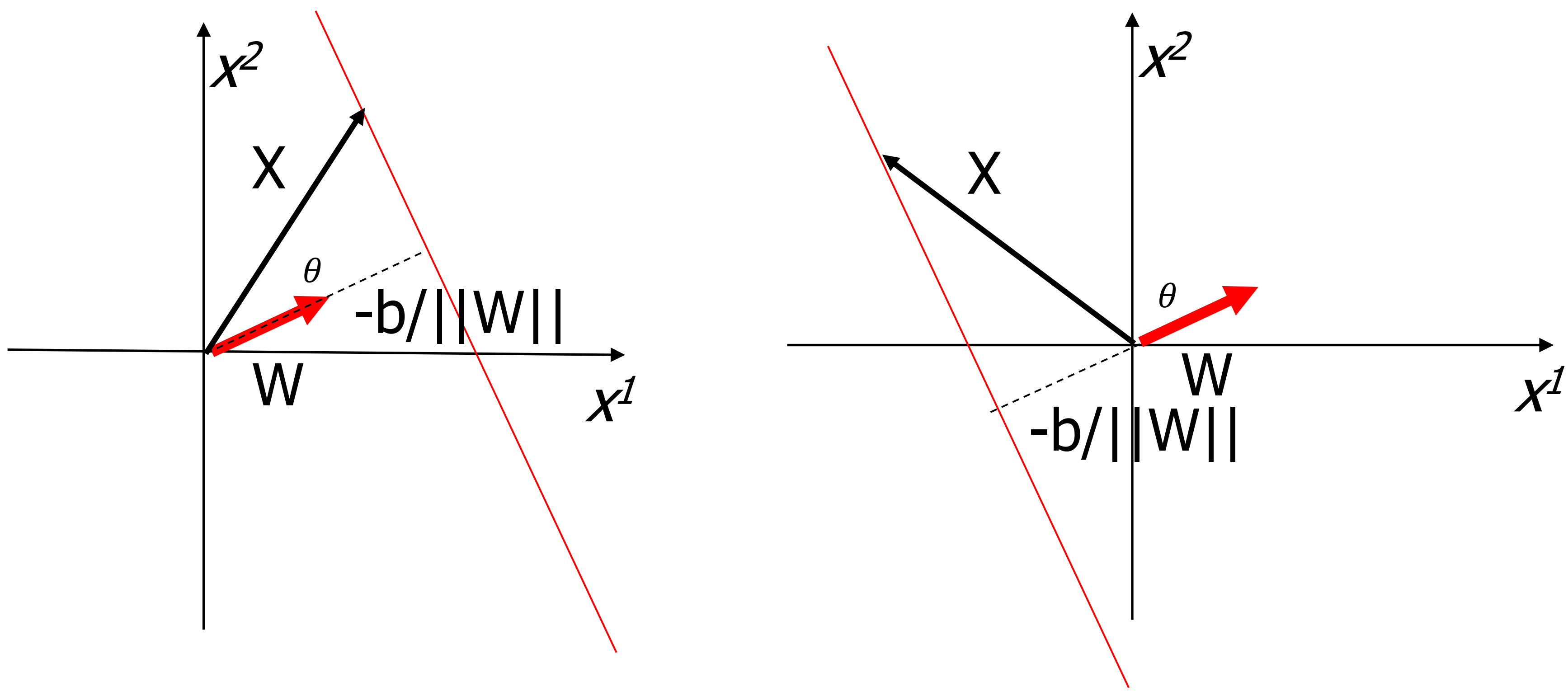
$$\frac{\langle XW \rangle}{\|W\|} = \frac{-b}{\|W\|}$$

$$W^1X^1 + W^2X^2 + b = 0$$

Definition of a hyperplane passing out the origin

In $\mathbf{R}^2$, a hyperplane is a line

$$W^1X^1 + W^2X^2 + b = 0$$



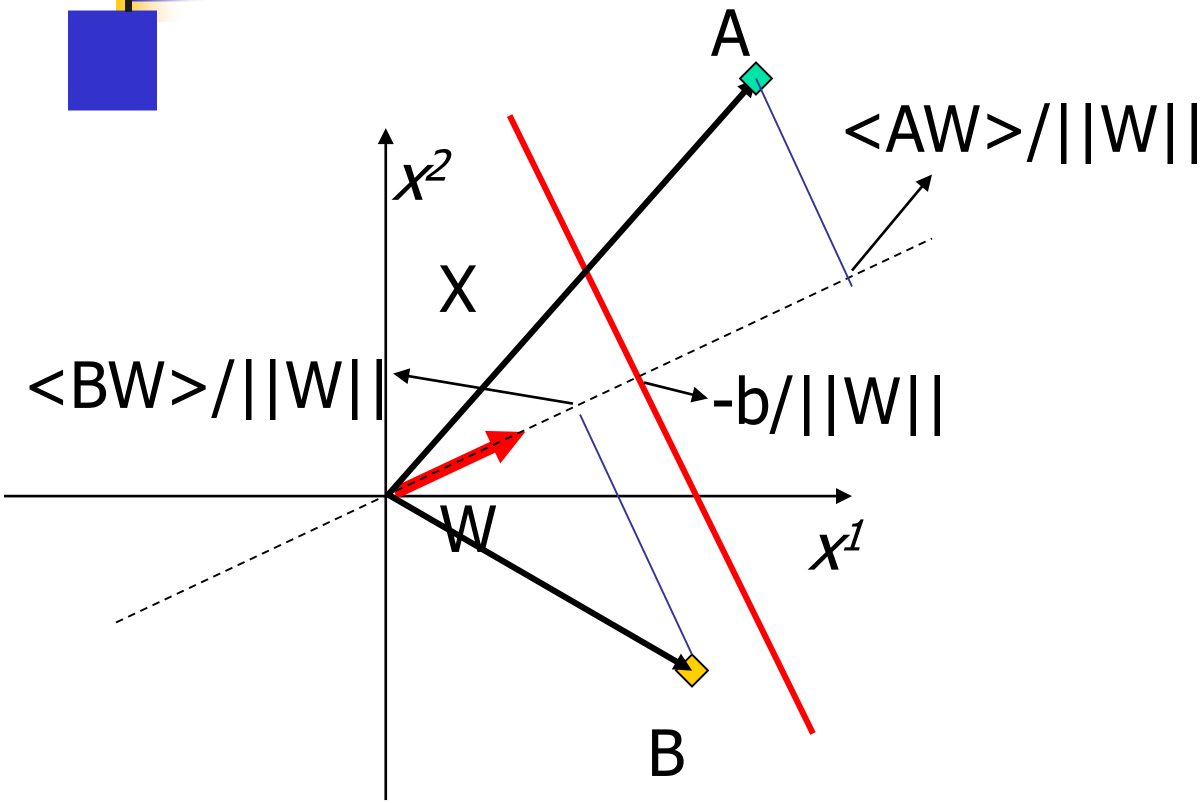
b<0 b>0

Generale definition of a hyperplane

In $\mathbf{R}^n$, an hyperplane is defined by

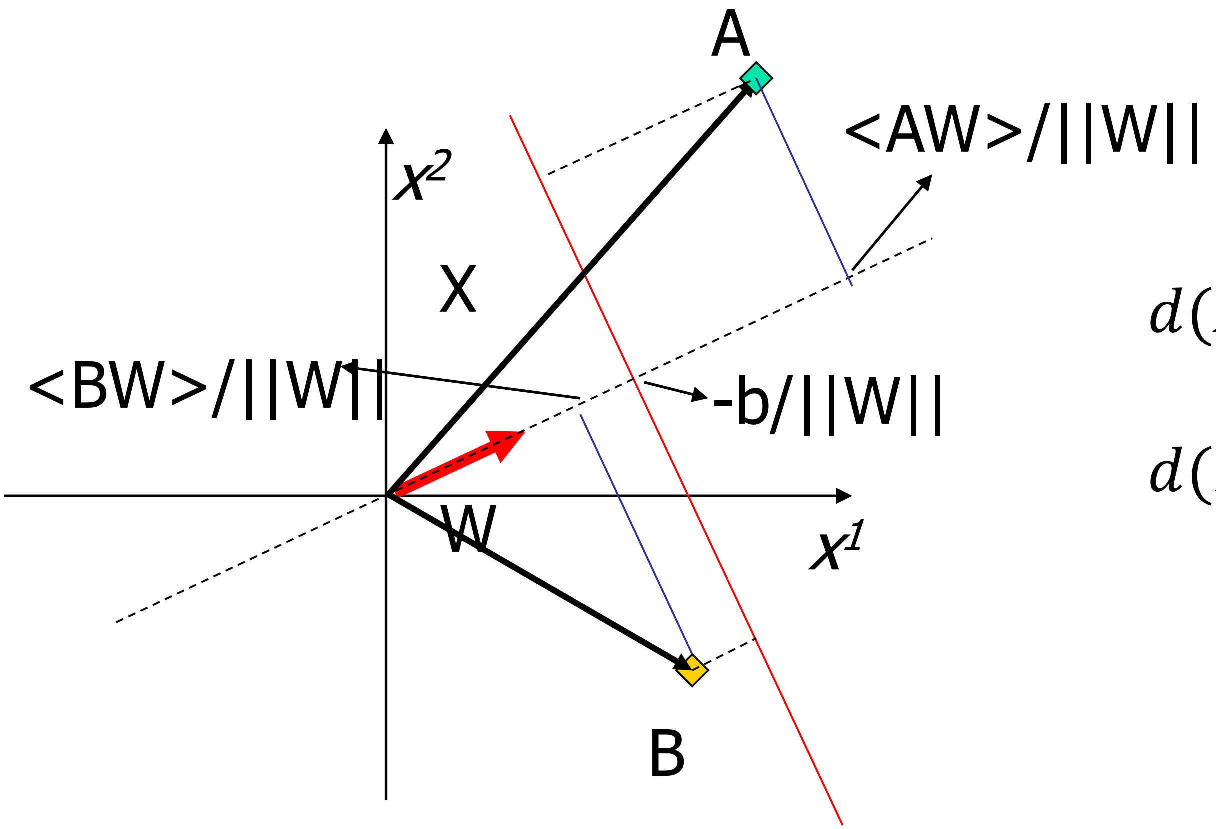
$$\langle XW \rangle + b = W^T X + b = 0$$

An hyperplane divides the space



$$\langle AW \rangle > -b \Rightarrow \langle AW \rangle + b > 0$$
$$\langle BW \rangle < -b \Rightarrow \langle BW \rangle + b < 0$$

Distance between a hyperplane and a point



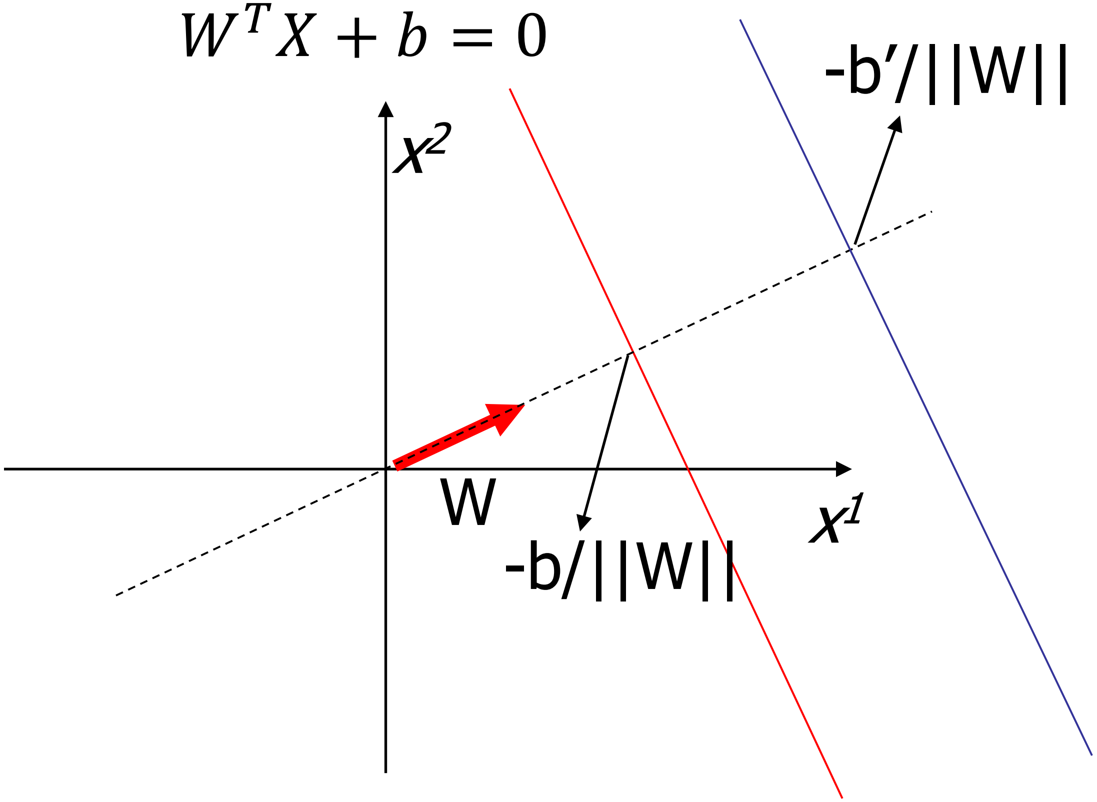
$$d(A,r) = \frac{|\langle AW \rangle + b|}{\|W\|}$$

$$d(B,r) = \frac{|\langle BW \rangle + b|}{\|W\|}$$

Distance between two parallel hyperplanes

$$W^T X + b' = 0$$

$$W^T X + b = 0$$



$$d(r, r') = \frac{|b - b'|}{\|W\|}$$

Classification problems

- Many problems are related to the search of a (hyper)plane that separate points in a space