Homework Assignment (Problem Set) 2:

Xinyu (Jerry) Zhao

Question 1:
1A. Write the general dual problem associated with the given LP.
(Do not transform or rewrite the primal problem before writing the general dual)

Maximize $4x_1 - 2x_2$
Subject To
$4x_1 + x_2 + x_3 = -20$
$2x_1 - x_2 \geq 6$
$x_1 - x_2 + 5x_3 \geq 5$
$-3x_1 + 2x_2 + x_3 \leq -4$
$x_1 \leq 0$, $x_2 \geq 0$, x_3 unrestricted

1B. Given the following information for a product-mix problem with three products and three resources.
Primal Decision Variables: x_1 = number of unit 1 produced; x_2 = # of unit 2 produced; x_3 = # of unit 3 produced

Primal Formulation:

Max Z (Rev.) =	$25x_1$	$+ 30x_2$	$+ 20x_3$		
Subject To	$8x_1$	$+ 6x_2$	$+ x_3$	≤ 50	(Res. 1 constraint)
	$4x_1$	$+ 2x_2$	$+ 3x_3$	≤ 20	(Res. 2 constraint)
	$2x_1$	$+ x_2$	$+ 2x_3$	≤ 25	(Res. 3 constraint)
		x_1, x_2, x_3		≥ 0	(Nonnegativity)

Dual Formulation:

Min W =	$50\pi_1$	$+ 20\pi_2$	$+25\pi_3$	
Subject To	$8\pi_1$	$+ 4\pi_2$	$+2\pi_3$	≥ 25
	$6\pi_1$	$+ 2\pi_2$	$+\pi_3$	≥ 30
	π_1	$+ 3\pi_2$	$+2\pi_3$	≥ 20
	π_1, π_2, π_3			≥ 0

Optimal Solution:

Optimal Z = Revenue = $268.75

$x_1 = 0$ (Number of unit 1)	Dual Var. Optimal Value = 22.5 (Surplus variable in 1st dual constraint)
$x_2 = 8.125$ (Number of unit 2)	Dual Var. Optimal Value = 0 (Surplus variable in 2nd dual constraint)
$x_3 = 1.25$ (Number of unit 3)	Dual Var. Optimal Value = 0 (Surplus variable in 3rd dual constraint)
Resource Constraints:	
Resource 1 = 0 leftover units	Dual Var. Optimal Value = $3.125 = \pi_1$
Resource 2 = 0 leftover units	Dual Var. Optimal Value = $5.625 = \pi_2$
Resource 3 = 14.375 leftover units	Dual Var. Optimal Value = $0 = \pi_3$

1Bi. What is the fair-market price for one unit of Resource 3?

1Bii. What is the meaning of the surplus variable value of 22.5 in the 1st dual constraint with respect to the primal problem?

Solution:
1A.

Dual Problem (D):
Minimize $-20y_1+6y_2+5y_3+4y_4$
(Coefficient is the right hand side of the primal problem constraints)

Subject to (coefficient is the primal problem coefficient of each variables):
$4y_1 + 2y_2 + y_3 - 3y_4 \geq 4$ (5)
$y_1 - y_2 - y_3 + 2y_4 \leq -2$ (6)
$y_1 + 5y_3 + y_4 = 0$ (7)

With the conditions:
$y_2 \geq 0$, $y_3 \geq 0$, $y_4 \leq 0$, y_1 unrestricted (sign of constrain)

1Bi.
A shadow price indicates how much the objective function value would increase if the available quantity of a resource were increased by one unit. The variable $\pi 3$ represents the **shadow price** for one unit of Resource. The optimal value of $\pi 3$ (the dual variable associated with Resource 3) is 0. Increasing the amount of Resource 3 does not improve the revenue from the product mix, likely because there is excess availability of Resource 3 (as noted in the resource constraint, with 14.375 units leftover). Thus, adding more of Resource 3 does not affect the optimal revenue.

1Bii.
In the dual formulation, the surplus variable $\pi 1$ corresponds to the **shadow price** of the first resource constraint (Resource 1). The surplus variable value of **22.5** means the following:

- For **Resource 1**, each additional unit of Resource 1 would **increase** the revenue by **22.5 units**.
- This reflects the **marginal value** of having one more unit of Resource 1 available to produce the products.
- Since the **resource constraint** for Resource 1 is fully used, this value of 22.5 shows that Resource 1 is fully utilized and that increasing its availability by one unit would directly increase the revenue by **$22.5**.

Question 2:
Carco manufactures cars and trucks. Each car contributes $300 to profit and each truck, $400; these profits do not consider machine rental. The resources required to manufacture a car and a truck are shown below. Each day Carco can rent up to 98 Type 1 machines at a cost of $50 per machine. The company now has 73 Type 2 machines and 260 tons of steel available. Marketing considerations dictate that at least 88 cars and at least 26 trucks be produced.

Part A: Formulate the problem as a Linear Program.
Part B: Solve the LP (provide exact values for all variables and the optimal objective function).
Hint: The optimal objective function value is $32540
[Note, I am providing this hint because having the optimal solution is necessary to do Part C.]

Part C: Answer the following questions from your output. *(Note, do not simply rerun the model – use the Linear Programming output and Sensitivity Analysis to explain your answers.)*
i) If cars contributed $310 to profit, what would be the new optimal solution to the problem?
ii) What is the most that Carco should be willing to pay to rent an additional Type 1 machine for 1 day?
iii) What is the most that Carco should be willing to pay for an extra ton of steel?
iv) If Carco were required to produce at least 86 cars, what would Carco's profit become?
v) Carco is considering producing jeeps. A jeep contributes $600 to profit and requires 1.2 days on machine 1, 2 days on machine 2, and 4 tons of steel. Should Carco produce any jeeps?

Table:

Vehicle Type	Days on Machine 1	Days on Machine 2	Tons of Steel
Car	0.8	0.6	2
Truck	1	0.7	3

Solution:

Part A:

- Let x_1 be the number of cars produced per day.
- Let x_2 be the number of trucks produced per day.

Maximize $Z=300x_1+400x_2-50*(0.8x_1+x_2)$

$0.8x_1+x_2 \leq 98$ (Machine constraint for Type 1)

$0.6x_1+0.7x_2 \leq 73$ (Machine constraint for Type 2)

$2x_1+3x_2 \leq 260$ (Steel availability constraint)

$x_1 \geq 88$ (At least 88 cars)

$x_2 \geq 26$ (At least 26 trucks)

$x_1 \geq 0,\ x_2 \geq 0$

Part B:

Solving this equation using python:

```python
import pulp

# Create a problem variable (Maximization problem)
prob = pulp.LpProblem("Car_Production_Optimization", pulp.LpMaximize)

# Decision variables
x1 = pulp.LpVariable('x1', lowBound=0, cat='Continuous')  # number of cars produced
x2 = pulp.LpVariable('x2', lowBound=0, cat='Continuous')  # number of trucks produced

# Objective function
prob += 300 * x1 + 400 * x2 -50*(0.8*x1+x2), "Total Profit"

# Constraints
prob += 0.8 * x1 + 1 * x2 <= 98, "Type 1 machine constraint"
prob += 0.6 * x1 + 0.7 * x2 <= 73, "Type 2 machine constraint"
prob += 2 * x1 + 3 * x2 <= 260, "Steel availability constraint"
prob += x1 >= 88, "At least 88 cars"
prob += x2 >= 26, "At least 26 trucks"

# Solve the problem
prob.solve()

# Print the results
print("Status:", pulp.LpStatus[prob.status])
print("Number of Cars to Produce (x1):", pulp.value(x1))
print("Number of Trucks to Produce (x2):", pulp.value(x2))
print("Total Profit:", pulp.value(prob.objective))
```

```
Status: Optimal
Number of Cars to Produce (x1): 88.0
Number of Trucks to Produce (x2): 27.6
Total Profit: 32540.0
```

Part C:

```python
# Sensitivity Analysis for Constraints
print("\nSensitivity Analysis Results:")

# Shadow prices (dual values) and allowable ranges for the constraints
for name, c in prob.constraints.items():
    print(f"\nConstraint: {name}")
    print(f"  Shadow Price: {c.pi}")

# Sensitivity analysis for the objective function coefficients (Profit per unit)
print("\nObjective Function Sensitivity Analysis:")
for var in prob.variables():
    print(f"\nVariable: {var.name}")
    print(f"  Current Value: {var.varValue}")
    print(f"  Reduced Cost: {var.dj}")
```

```
Sensitivity Analysis Results:

Constraint: Type_1_machine_constraint
  Shadow Price: 350.0

Constraint: Type_2_machine_constraint
  Shadow Price: -0.0

Constraint: Steel_availability_constraint
  Shadow Price: -0.0

Constraint: At_least_88_cars
  Shadow Price: -20.0

Constraint: At_least_26_trucks
  Shadow Price: -0.0

Objective Function Sensitivity Analysis:

Variable: x1
  Current Value: 88.0
  Reduced Cost: -1.4210855e-14

Variable: x2
  Current Value: 27.6
  Reduced Cost: 0.0
```

i): The **reduced cost** for x1 (number of cars) is very close to zero (-1.4210855e-14), which indicates that x1=88 is currently at its optimal value. Any increase in the profit from cars would still leave x1=88 as the optimal number of cars to produce.

ii): The shadow price for the Type 1 machine constraint is 350.0. This value indicates that for each additional Type 1 machine Carco rents, the total profit will increase by $350.00.

Thus, the most that Carco should be willing to pay for an additional Type 1 machine for 1 day is $350.00. This is the marginal value of one more Type 1 machine to the company.

iii): The **shadow price** for the steel availability constraint is **-0.0**, which means that the company currently has no marginal benefit from adding more steel. Since the shadow price is 0, increasing the amount of steel does not contribute to a higher profit.

iv): The **shadow price** for the "at least 88 cars" constraint is **-20.0.** This means the company would likely produce fewer cars (at the new minimum of 86 cars) and therefore have more resources available to produce trucks, potentially increasing total profit. The new profit would be 32580, $40 more.

v):

After 88 cars and 26 trucks requirement, there are 96.4 days of Type I machine used and 71 days of Type II machine used, and 254 tons of steel used. Therefore 1.6 days of Type I machine, 2 days of Type II, and 6 tons of steel left. Since the shadow price for car is -20, producing additional car is not a good choice, so we compare truck and jeep.

A Truck contributes $400 to profit and requires 1 days on machine 1, 0.7 days on machine 2, 3 tons of steel. The company can produce 1.6 truck (Type I machine is the constraint), it would make 1.6*400 – 1.6*50 = 560 profit

A jeep contributes $600 to profit and requires 1.2 days on machine 1, 2 days on machine 2, and 4 tons of steel. The company can only produce 1 jeep (Type II machine is the constraint), it would make 1*600 – 1*50 = 550 profit

Since jeep’s profit is lower, the company should produce jeep after satisfying the production requirement of car and truck.

Question 3:

A catering company must have the following number of clean napkins available at the beginning of each of the next four days: day 1: 15, day 2: 12, day 3: 18, and day 4: 6. After being used, a napkin can be cleaned by one of two methods: fast service or slow service. Fast service costs $0.10 per napkin, and a napkin cleaned via fast service is available for use the day after it is last used. Slow service costs $0.06 per napkin, and a napkin cleaned via slow service is available two days after they were last used. New napkins can be purchased for a cost of $0.20 per napkin.

Part A: Formulate the problem as a minimum cost transportation problem.
Part B: Solve the problem (provide exact values for all variables and the optimal objective function).

Solution:

Formulation of the Problem:

- x1: Number of napkins cleaned via fast service on Day 1 and available on Day 2.
- x2: Number of napkins cleaned via slow service on Day 1 and available on Day 3.
- x3: Number of napkins cleaned via fast service on Day 2 and available on Day 3.
- x4: Number of napkins cleaned via slow service on Day 2 and available on Day 4.
- x5: Number of napkins cleaned via fast service on Day 3 and available on Day 4.
- x6: Number of napkins purchased on Day 1.
- x7: Number of napkins purchased on Day 2.
- x8: Number of napkins purchased on Day 3.
- x9: Number of napkins purchased on Day 4.

Objective Function:

The objective function is to minimize the total cost, which includes the cost of cleaning napkins and purchasing new napkins:

Minimize Z= 0.10(x1+x3+x5) + 0.06(x2+x4) + 0.2(x6+x7+x8+x9)

Constraints:

The constraints represent the napkin usage and availability on each day:

1. Day 1: The total napkins needed is 15, so the available napkins on Day 1 must be bought:

   x6 >= 15

   Sent for cleaning cannot exceed total used:

   x1+x2 <= 15

2. Day 2: The napkins needed on Day 2 is 12. Napkins available on Day 2 come from those cleaned on Day 1 via fast service, leftover new napkins from Day 1, and new napkins that were purchased on Day 2:

   x1 + x7 + (x6-15) >= 12

Sent for cleaning cannot exceed total used:

x3+x4<= 12

3. Day 3: The napkins needed on Day 3 is 18. Napkins available on Day 3 come from those cleaned on Day 1 via slow service, on Day 2 via fast service, leftover of Day 2 clean Napkin, and newly purchased napkins on Day 3:

   x2+x3+x8+(x1+x7+(x6-15)-12) >= 18

   Sent for cleaning cannot exceed total used:

   x5 <= 18

4. Day 4: The napkins needed on Day 4 is 6. Napkins available on Day 4 come from those cleaned on Day 2 via slow service, on Day 3 via fast service, leftover new napkins from Day 3, and newly purchased napkins on Day 4:

   x4+x5+x9 + (x2+x3+x8+(x1 + x7 + (x6-15) – 12)-18) >= 6

Solve the problem:

```python
import pulp

# Create the linear programming problem
prob = pulp.LpProblem("Napkin_Production", pulp.LpMinimize)

# Decision variables
x1 = pulp.LpVariable('x1', lowBound=0, cat='Continuous')  # Fast service, Day 1 -> Day 2
x2 = pulp.LpVariable('x2', lowBound=0, cat='Continuous')  # Slow service, Day 1 -> Day 3
x3 = pulp.LpVariable('x3', lowBound=0, cat='Continuous')  # Fast service, Day 2 -> Day 3
x4 = pulp.LpVariable('x4', lowBound=0, cat='Continuous')  # Slow service, Day 2 -> Day 4
x5 = pulp.LpVariable('x5', lowBound=0, cat='Continuous')  # Fast service, Day 3 -> Day 4
x6 = pulp.LpVariable('x6', lowBound=0, cat='Continuous')  # New napkins purchased
x7 = pulp.LpVariable('x7', lowBound=0, cat='Continuous')  # New napkins purchased
x8 = pulp.LpVariable('x8', lowBound=0, cat='Continuous')  # New napkins purchased
x9 = pulp.LpVariable('x9', lowBound=0, cat='Continuous')  # New napkins purchased


# Objective function: minimize the total cost
prob += 0.10 * (x1 + x3 + x5) + 0.06 * (x2 + x4) + 0.20 * (x6+x7+x8+x9), "Total Cost"

# Constraints
prob += x6 >= 15, "Day 1 Napkins"
prob += x1+x2<=15, "Day 1 cleaning cannot exceed used"

prob += x1+x7+(x6-15) >= 12, "Day 2 Napkins"
prob += x3+x4 <= 12, "Day 2 cleaning cannot exceed used"

prob += x2+x3+x8+(x1+x7+(x6-15)-12) >= 18, "Day 3 Napkins"
prob += x5 <= 18,"Day 3 cleaning cannot exceed used"

prob += x4+x5+x9+(x2+x3+x8+(x1+x7+(x6-15)-12)-18) >= 6, "Day 4 Napkins"

# Solve the problem
prob.solve()

# Print the results
print("Status:", pulp.LpStatus[prob.status])
print("Fast service, Day 1 -> Day 2 (x1):", pulp.value(x1))
print("Slow service, Day 1 -> Day 3 (x2):", pulp.value(x2))
print("Fast service, Day 2 -> Day 3 (x3):", pulp.value(x3))
print("Slow service, Day 2 -> Day 4 (x4):", pulp.value(x4))
print("Fast service, Day 3 -> Day 4 (x5):", pulp.value(x5))
print("New napkins purchased on Day 1(x6):", pulp.value(x6))
print("New napkins purchased on Day 2(x7):", pulp.value(x7))
print("New napkins purchased on Day 3(x8):", pulp.value(x8))
print("New napkins purchased on Day 4(x9):", pulp.value(x9))
print("Total Cost:", pulp.value(prob.objective))
```

```
Status: Optimal
Fast service, Day 1 -> Day 2 (x1): 9.0
Slow service, Day 1 -> Day 3 (x2): 6.0
Fast service, Day 2 -> Day 3 (x3): 12.0
Slow service, Day 2 -> Day 4 (x4): 0.0
Fast service, Day 3 -> Day 4 (x5): 6.0
New napkins purchased on Day 1(x6): 18.0
New napkins purchased on Day 2(x7): 0.0
New napkins purchased on Day 3(x8): 0.0
New napkins purchased on Day 4(x9): 0.0
Total Cost: 6.66
```

Question 4:
A university has three professors who each teach four courses per year. Each year, four sections of marketing, finance, and production must be offered. At least one section of each class must be offered during each semester (fall and spring). Each professor's time preferences and preference for teaching various courses are given below.

The total satisfaction a professor earns teaching a class is the sum of the semester satisfaction and the course satisfaction. Thus, professor 1 derives a satisfaction of 3 + 6 = 9 from teaching marketing during the fall semester.

Part A: Formulate the problem as a minimum cost network flow problem that can be used to assign professors to courses so as to maximize the total satisfaction of the three professors. Draw the network and identify the nodes and arcs.

Part B: Solve the problem (provide exact values for all variables and the optimal objective function).

Table:

	Professor 1	Professor 2	Professor 3
Fall Preference	3	5	4
Spring Preference	4	3	4
Marketing	6	4	5
Finance	5	6	4
Production	4	5	6

Solution:

Nodes:
Source Node (S): Represents the beginning of the assignment process.
Professor Nodes (P1, P2, P3): Represent each of the three professors.

Course Nodes (M1, M2, F1, F2, P1, P2): Represent the different course sections offered during the fall and spring semesters. Here, the "1" or "2" represents the fall and spring semester for marketing (M), finance (F), and production (P).

Sink Node (T): Represents the endpoint where the flow ends (i.e., where all assignments are finalized).

Arcs:
From Source (S) to Professors (P1, P2, P3): Each professor can be assigned to any of the course sections, so there will be arcs from the source to the professors.

From Professors (P1, P2, P3) to Course Sections (M1, M2, F1, F2, P1, P2): Arcs represent the assignment of professors to course sections.

From Course Sections (M1, M2, F1, F2, P1, P2) to Sink (T): Represents the course sections being taught.

Flow Capacities and Costs:
Each professor has a preference for teaching each course in the fall and spring. The satisfaction for assigning a professor to a specific course during a specific semester is given in the table. These values will be used as the costs in the network flow. There are four sections of marketing, finance, and production, which means each course has a capacity of 1. Thus, each course section must be assigned exactly one professor.

Objective:
The objective is to maximize the total satisfaction of the professors. In the context of a minimum cost flow problem, we want to maximize the total flow along the network, where the flow represents the satisfaction of assigning a professor to a course.

Constraints:

- Each course section must be assigned exactly one professor.
- Each professor can be assigned to a maximum of four course sections.

```
import pulp

# Create the Linear Program (Maximization problem)
prob = pulp.LpProblem("Professor_Assignment", pulp.LpMaximize)

# Decision variables: professor-course assignments
# These represent if professor i is assigned to course j
professors = ['Professor1', 'Professor2', 'Professor3']
courses = ['Marketing Fall', 'Marketing Spring', 'Finance Fall', 'Finance Spring', 'Production Fall', 'Production Spring']  # M1

# Satisfaction table for each professor and course
satisfaction = {
    ('Professor1', 'Marketing Fall'): 3 + 6, ('Professor1', 'Marketing Spring'): 4 + 6,
    ('Professor1', 'Finance Fall'): 3 + 5, ('Professor1', 'Finance Spring'): 4 + 5,
    ('Professor1', 'Production Fall'): 3 + 4, ('Professor1', 'Production Spring'): 4 + 4,

    ('Professor2', 'Marketing Fall'): 5 + 4, ('Professor2', 'Marketing Spring'): 3 + 4,
    ('Professor2', 'Finance Fall'): 5 + 6, ('Professor2', 'Finance Spring'): 3 + 6,
    ('Professor2', 'Production Fall'): 5 + 5, ('Professor2', 'Production Spring'): 3 + 5,

    ('Professor3', 'Marketing Fall'): 4 + 5, ('Professor3', 'Marketing Spring'): 4 + 5,
    ('Professor3', 'Finance Fall'): 4 + 4, ('Professor3', 'Finance Spring'): 4 + 4,
    ('Professor3', 'Production Fall'): 4 + 6, ('Professor3', 'Production Spring'): 4 + 6,
}

# Decision variables for assignment of professors to courses (binary)
assignments = pulp.LpVariable.dicts("assign", (professors, courses), cat="Binary")

# Objective function: maximize the total satisfaction
prob += pulp.lpSum(satisfaction[(p, c)] * assignments[p][c] for p in professors for c in courses), "Total Satisfaction"

# Constraints: Each course must have one professor, and each professor can teach at most 4 courses
for c in courses:
    prob += pulp.lpSum(assignments[p][c] for p in professors) == 1, f"Course {c} must be assigned to exactly one professor"

for p in professors:
    prob += pulp.lpSum(assignments[p][c] for c in courses) <= 4, f"Professor {p} can teach at most 4 courses"

# Solve the problem
prob.solve()

# Output the results
print("Status:", pulp.LpStatus[prob.status])

# Output the assignments and the total satisfaction
for p in professors:
    for c in courses:
        if pulp.value(assignments[p][c]) == 1:
            print(f"{p} is assigned to {c} with satisfaction {satisfaction[(p, c)]}")

# Total satisfaction
print("Total Satisfaction:", pulp.value(prob.objective))
```

```
Status: Optimal
Professor1 is assigned to Marketing Spring with satisfaction 10
Professor2 is assigned to Finance Fall with satisfaction 11
Professor2 is assigned to Finance Spring with satisfaction 9
Professor3 is assigned to Marketing Fall with satisfaction 9
Professor3 is assigned to Production Fall with satisfaction 10
Professor3 is assigned to Production Spring with satisfaction 10
Total Satisfaction: 59.0
```