Classical two body problem:

The problem is easier to handle if we replace $\underbrace{\vec{r_1} \,\&\, \vec{r_2}}_{\text{position vectors of turtles}}$ by $\vec{r} = \vec{r_1} - \vec{r_2}$ (separation vector) & $\vec{R}$ → centre of mass vector

For an isolated system,

$\ddot{\vec{R}} = 0 \Rightarrow \vec{R} = \vec{R_0} + \vec{V}t$

Choose initial conditions such that $\vec{R_0} = 0$ & $\vec{V} = 0 \Rightarrow \vec{R} = 0$

$m_1\ddot{\vec{r_1}} = f(r)\hat{r} \qquad m_2\ddot{\vec{r_2}} = -f(r)\hat{r}$] Governing eq's

$\ddot{\vec{r_1}} - \ddot{\vec{r_2}} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\hat{r}$

$\Rightarrow \underset{\text{reduced mass}}{\mu}\ddot{\vec{r}} = f(r)\hat{r}$ } $\mu(\ddot{r} - r\dot{\theta}^2) = f(r)$, $\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$

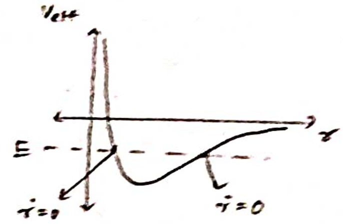
$\underset{\text{constant}}{L} = \mu r^2\dot{\theta}$ (conservation of Angular momentum) $\Rightarrow \dot{\theta} = \frac{L}{\mu r^2}$ → From this we get $\theta(t)$

Conservation of Energy $\Rightarrow E = \frac{1}{2}\mu v^2 + V(r) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$

$\Rightarrow E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{1}{2}\frac{L^2}{\mu r^2} + V(r)}_{V_{eff}(r)}$ → From this we get $r(t)$

Eliminating t we get $r(\theta)$

$$\frac{d\theta}{dr} = \frac{L}{\mu r^2}\frac{1}{\sqrt{\frac{2}{\mu}(E - V_{eff})}}$$



$\dot{r}=0$ turning pts

Here,

$V(r) = -\frac{Gm_1m_2}{r} \Rightarrow V_{eff}(r) = \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$

Consider, $m_1 = m_2 = m \Rightarrow \mu = \frac{m}{2}$ Let $m = 1u$

Let $L^2 = 9u \Rightarrow L = 3u$ & $G = 9u$

Consider E such that, $(V_{eff})_{min} < E < 0$

$\therefore E = \frac{\dot{r}^2}{4} + \underbrace{\frac{9}{r^2} - \frac{9}{r}}_{V_{eff}(r)}$

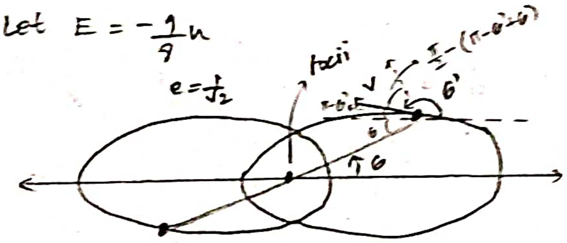
$(V_{eff})_{min} \Rightarrow \frac{-2}{r^3} + \frac{1}{r^2} = 0 \Rightarrow r = 2$

@ $r=2$, $(V_{eff})_{min} = -\frac{9}{4}u$

$A = \frac{L^2}{m} = 9u$

$C = Gm^2 = 9u$

Let $E = -\frac{9}{4}u$



At turning pts,

$E = \frac{A}{r^2} - \frac{C}{r}$

$-\frac{9}{4} = 9\left(\frac{1}{r^2} - \frac{1}{r}\right)$

upon solving,

$r_{max} = 4 + 2\sqrt{2}$

$r_{min} = 4 - 2\sqrt{2}$

$r(\theta) \Rightarrow r = \frac{(L^2/\mu C)}{1 - \underbrace{\sqrt{1 + \left(\frac{2EL^2}{\mu C^2}\right)}}_{\text{eccentricity}}\sin(\theta - \theta_0)}$ $\qquad \theta_0 = -\frac{\pi}{2}$

$r = \frac{2}{1 - \frac{1}{\sqrt{2}}\cos\theta}$

NOTE: r → separation b/w turtles.

$1 - \frac{1}{\sqrt{2}}\cos\theta = \frac{2}{r} \Rightarrow \frac{1}{\sqrt{2}}\cos\theta = 1 - \frac{2}{r} \Rightarrow \theta = \cos^{-1}\left(\sqrt{2}\left(1 - \frac{2}{r}\right)\right)$

Conservation of Angular momentum:

$\underset{\text{Angular momentum of each turtle}}{\frac{L}{2}} = m\left(\frac{r}{2}\right)v\cos\left(\frac{\pi}{2} - (\pi - \theta' + \theta)\right) = m\left(\frac{r}{2}\right)v\sin(\pi - (\theta' - \theta)) = m\left(\frac{r}{2}\right)v\sin(\theta' - \theta)$

$\Rightarrow 3 = vr\sin(\theta' - \theta)$

θ' → we can get from /turtle/rose

$\Rightarrow \underset{\text{Linear speed}}{v} = \frac{3}{r\sin(\theta' - \theta)}$

$$\dot{\theta} = \frac{L}{\mu r^2} \Rightarrow \dot{\theta} = \frac{3 \times 2}{r^2} = \frac{6}{r^2}$$

Angular speed.