AE601 Mathematical Methods in Aerospace Engineering

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1. Find the general solution of the given second-order differential equation

$$12y'' - 5y' - 2y = 0$$

Solution: The characteristic equation is

$$12\lambda^2 - 5\lambda - 2 = 0$$

Solving the quadratic equation, we get $\lambda_1 = 0.6667$ and $\lambda_2 = -0.25$.

Since the roots are real and distinct, the general solution is

$$y(x) = Ae^{0.6667x} + Be^{-0.25x}$$

2. Solve the initial value problem

$$y'' + y' - 6y = 0,$$
$$y(0) = 10,\ y'(0) = 0$$

and plot the solution.

Solution:

The characteristic equation is

$$\lambda^2 + \lambda - 6 = 0$$

Solving the quadratic equation, we get $\lambda_1 = 2$ and $\lambda_2 = -3$. Since the roots are real and distinct, the general solution is

$$y = Ae^{2x} + Be^{-3x}$$

To find the constants A and B, we substitute the initial conditions

$$A = 6\ and\ B = 4$$

The general solution is

$$y(x) = 6e^{2x} + 4e^{-3x}$$

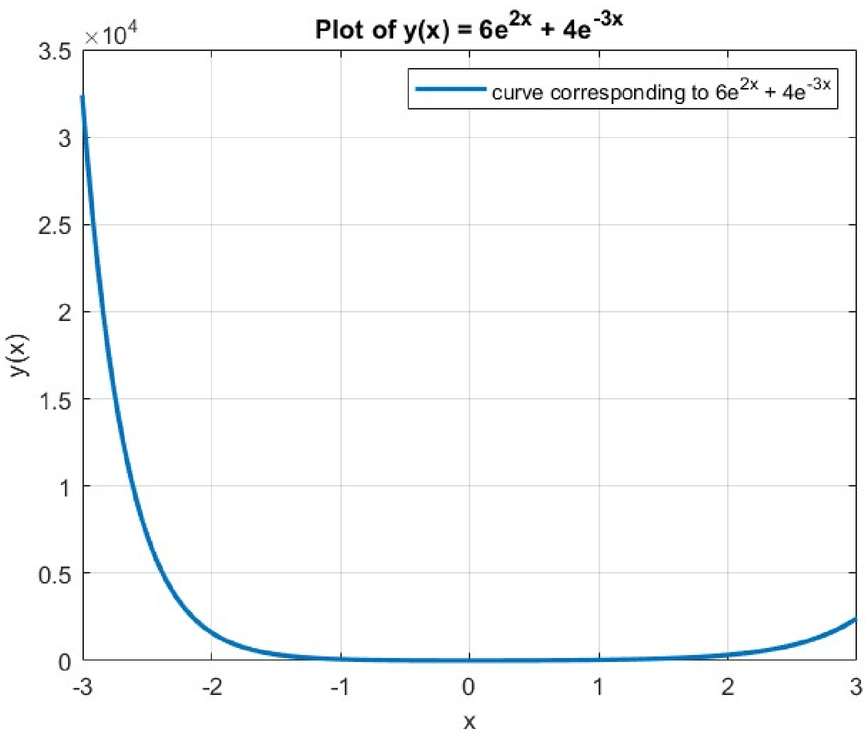

Figure 1: Plot of $y(x) = 6e^{2x} + 4e^{-3x}$

3. Solve the initial value problem

$$x^2y'' + xy' - 4y = 0$$
$$y(1) = 2,\ y'(1) = 0$$

and plot the solution.

Solution:

Comparing our equation with the Euler-Cauchy equation

$$x^2y'' + xy' - 4y = 0 \; and \; x^2y'' + axy' - by = 0$$
$$a = 1,\ b = -4$$

The auxiliary equation is

$$m^2 + (a - 1)m + b = 0$$
$$m^2 + (0)m - 4 = 0$$

solving the quadratic equation, we get

$$m_1 = 2 \; and \; m_2 = -2$$

The general solution is

$$y = Ax^2 + Bx^{-2}$$

substituting the initial conditions,

$$A = 1 \; and \; B = 1$$

Substituting A and B, we get the general solution

$$y(x) = x^2 + x^{-2}$$

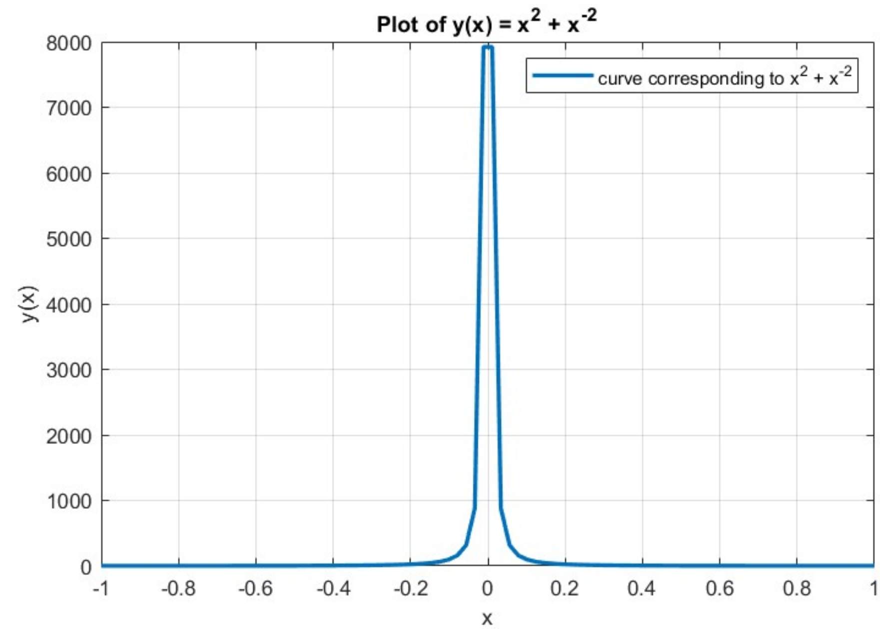

Figure 2: Plot of $y(x) = x^2 + x^{-2}$

4. Two roots of a cubic auxiliary equation with real coefficients are $\lambda_1 = -0.5$ and $\lambda_2 = 3 + i$. What is the corresponding homogeneous linear differential equation? Is your answer unique?

   Solution: We know that the complex roots come in conjugate pairs, so $\lambda_3 = 3 - i$.
   To get the auxiliary equation,

   $$(\lambda + 0.5)(\lambda - (3 + i))(\lambda + (3 - i)) = 0$$

   Multiplying the factors to get the cubic auxiliary equation,

   $$\lambda^3 - 5.5\lambda^2 + 7\lambda + 5 = 0$$

   The corresponding homogeneous linear differential equation is,

   $$y''' - 5.5y'' + 7y' + 5y = 0$$

5. Find a particular solution of the differential equation:

   $$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2)$$

   and plot it.

   Solution: To find the particular solution of the differential equation, since the RHS is in the form of quadratic equation multiplied with an exponential term, we assume,

   $$y_p = (Ax^2 + Bx + C)e^{3x}$$

   differentiating once,

   $$y_p{}' = e^{3x}(2Ax + B) + 3e^{3x}(Ax^2 + Bx + C)$$

   differentiating twice

   $$y_p{}'' = 9Ae^{3x} + (12A + 9B)xe^{3x} + (2A + 6B + 9C)e^{3x}$$

substituting the values of y_p'', y_p' and y, we get

$$e^{3x}[2Ax^2 + (6A + 2B)x + 2A + 3B + 2C] = (x^2 + 2x - 1)e^{3x}$$

Comparing the coefficients of x^2, x and constant terms,

$$A = 1/2,\ B = -1/2,\ C = -1/4$$

Substituting the values in the y_p, we get

$$y_p(x) = e^{3x}(\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4})$$

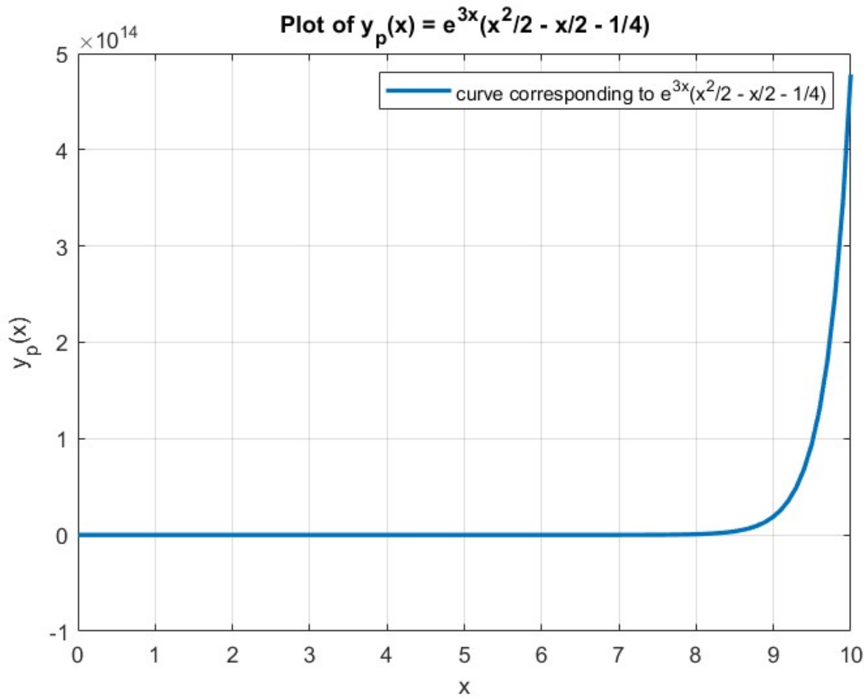

Figure 3: Plot of $y_p(x) = e^{3x}(x^2/2 - x/2 - 1/4)$

6. Solve the following differential equation by undetermined coefficients

$$y'' - 2y' + 5y = e^x \cos(2x)$$

Solution: The characteristic equation is

$$\lambda^2 - 2\lambda + 5 = 0$$

Solving the quadratic equation, we get $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Since the roots are complex, the general solution of the homogeneous ODE is

$$y_c = e^x(C_1 \cos(2\,x) + C_2 \sin(2\,x))$$

Since the RHS is a trigonometric function multiplied with an exponential function, we choose the particular solution as

$$y_p = e^x(A \cos(2\,x) + B \sin(2\,x))$$

Since we have e^x in our solution for the homogeneous equation, we multiply y_p with an x

$$y_p = xe^x(A\,\cos(2\,x) + B\,\sin(2\,x))$$

Differentiating once,

$$y_p' = ((2\,B + A)\;x + A)\;e^x\;\cos(2\,x) + ((B - 2\,A)\;x + B)\;e^x\;\sin(2\,x)$$

Differentiating twice,

$$y_p'' = ((4\,B - 3\,A)\;x + 4\,B + 2\,A)\;e^x\;\cos(2\,x) + ((-3\,B - 4\,A)\;x + 2\,B - 4\,A)\;e^x\;\sin(2\,x)$$

Substituting y_p'', y_p' and y in the differential equation

$$4\,B\,e^x\;\cos(2\,x) - 4\,A\,e^x\;\sin(2\,x) = e^x\;\cos(2\,x)$$

Comparing the coefficients to find A and B,

$$A = 0,\; and\, B = \frac{1}{4}$$

Substituting the values of A and B to the particular solution,

$$y_p = \frac{x\,e^x\;\sin(2\,x)}{4}$$

The general solution is $y = y_c + y_p$,

$$y(x) = C_1\,e^x\;\cos(2\,x) + C_2\,e^x\;\sin(2\,x) + \frac{x\,e^x\;\sin(2\,x)}{4}$$

7. Find the frequency of oscillation of a pendulum of length L, neglecting air resistance and the weight of the rod, and assuming the amplitude of oscillation θ to be so small that $\sin\theta$ practically equals θ.

Solution: The equation of motion for a simple pendulum can be given as:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

where g is the acceleration due to gravity and L is the length of the pendulum.

Assuming a small amplitude of oscillation, we can approximate $\sin\theta \approx \theta$. Therefore, the equation becomes:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

This is a second-order linear homogeneous differential equation with constant coefficients. Its characteristic equation is:

$$\lambda^2 + \frac{g}{L} = 0$$

Solving for λ, we find two complex roots:

$$\lambda = \pm i\sqrt{\frac{g}{L}}$$

The general solution for the differential equation:

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{L}}t\right) + B\sin\left(\sqrt{\frac{g}{L}}t\right)$$

The frequency of oscillation f, is related to the angular frequency ω by the formula:

$$\omega = 2\pi f$$

For the pendulum, ω is:

$$\omega = \sqrt{\frac{g}{L}}$$

The frequency of oscillation is:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

8. Show that the ratio of two consecutive maximum amplitudes of a damped free oscillation governed by the equation

$$my'' + cy' + ky = 0$$

is constant. Show also that the natural logarithm of this ratio, called the logarithmic decrement, is given by $\Delta = \frac{2\pi\alpha}{\omega^*}$ where

$$\alpha = \frac{c}{2m} \; and \; \omega^* = \frac{\sqrt{4mk - c^2}}{2m}$$

Find Δ for the solutions of

$$y'' + 2y' + 5y = 0$$

Solution: If the system is under-damped it will swing back and forth with decreasing size of the swing until it comes to a stop. Its amplitude will decrease exponentially.
The phase angle form of the amplitude is given by

$$x(t) = C\, e^{-\alpha\, t} \cos(\omega^*\, t - \phi)$$

let t_0 and t_1 be the time between two maximum amplitudes,

$$x(t_0) = C\, e^{-\alpha\, t_0} \cos(\omega^*\, t_0 - \phi)$$

$$x(t_1) = C\, e^{-\alpha\, t_1} \cos(\omega^*\, t_1 - \phi)$$

we know that the time period(T) is,

$$T = \frac{2\pi}{\omega^*}$$

So,

$$t_1 = t_0 + \frac{2\,\pi}{\omega^*}$$

substituting in $x(t_1)$,we get

$$x(t_1) = C\, e^{-\alpha\, t_1} \cos(\omega^*\, t_0 - \phi + 2\pi)$$

Dividing $x(t_0)$ by $x(t_1)$, we get

$$\frac{x(t_0)}{x(t_1)} = e^{\frac{2\pi\alpha}{\omega^*}}$$

So, the ratio of two consecutive maximum amplitudes is constant.
Taking natural logarithm of this ratio,

$$\Delta = \ln\frac{x(t_0)}{x(t_1)} = \frac{2\pi\alpha}{\omega^*}$$

To find the Δ of the given ODE $y'' + 2y' + 5y = 0$,

$$m = 1,\, c = 2,\, k = 5$$

Substituting and solving for ω^*, we get,

$$\omega^* = 2$$

Substituting the values in Δ, we get,

$$\Delta = \pi$$

9. How does the frequency of the harmonic oscillation change if we: (i) Double the mass? (ii) Take a spring of twice the modulus? First find qualitative answers by physics, then look at formulas.

Solution:

Qualitative answer by physics:

1. When the mass is doubled, the inertia of the system increases. This means that it will be more resistant to acceleration and deceleration. As a result, the system's natural frequency will decrease. In other words, it will oscillate more slowly.

2. When the spring modulus is doubled, it indicates that the spring becomes stiffer. A stiffer spring will exert a stronger force for a given displacement from the equilibrium position. This increased force will lead to a higher acceleration of the mass, causing it to oscillate more quickly. Therefore, doubling the spring constant will result in an increase in the natural frequency of the system.

From formulas,When
(i) Doubling the mass?
Let F_1 be the original frequency and F_2 be the frequency after doubling the mass,

$$F = \frac{\omega_0}{2\pi}$$

$$F_1 = \frac{\sqrt{\frac{k}{m_1}}}{2\pi}$$

$$m_2 = 2\,m_1$$

$$F_2 = \frac{\sqrt{\frac{k}{m_2}}}{2\pi}$$

Substituting m_2, we get,

$$F_1 = \sqrt{2}\,F_2$$

After doubling the mass, frequency is lowered by the factor of $\sqrt{2}$.

(ii) Take a spring of twice the modulus?

$$k_2 = 2\,k_1$$

$$F_1 = \frac{F_2}{\sqrt{2}}$$

After doubling the modulus, frequency is higher by the factor of $\sqrt{2}$.

10. The natural length of a spring is 1 m. An object is attached to it, and the length of the spring increases to 102 cm when the object is in equilibrium. Then the object is initially displaced downward 1 cm and given an upward velocity of 14 cm/s. Find the displacement for $t > 0$. Also, find the natural frequency, period, amplitude, and phase angle of the resulting motion.

Solution:
Given data: $l_1 = 100\,cm$ and $l_2 = 102\,cm$, $\Delta l = 2\,cm, y'(0) = 14$, $y(0) = -1$
The differential equation can be written as,

$$y'' + 490y = 0$$

The auxiliary equation is

$$\lambda^2 + 490 = 0$$

Solving for λ,

$$\lambda_1 = 7\sqrt{10}i,\ \lambda_2 = -7\sqrt{10}i$$

The general solution,

$$y(t) = C_1\, \cos 7\sqrt{10}\,t + C_2\, \sin 7\sqrt{10}\,t$$

Differentiating and applying the initial conditions,

$$y'(t) = 7\,\sqrt{10}\,(-C_1\, \sin 7\sqrt{10}\,t + C_2\, \cos 7\sqrt{10}\,t)$$

$$C_1 = -1 \text{ and } C_2 = \frac{2}{\sqrt{10}}$$

The frequency is $7\sqrt{10}\,rad/s$, and the time period is $T = \frac{2\,\pi}{(7\,\sqrt{10})}\; s$.
The amplitude is

$$R = \sqrt{{C_1}^2 + {C_2}^2}$$

$$R = \sqrt{-1^2 + \frac{2}{\sqrt{10}}^2}$$

$$R = \sqrt{\frac{7}{5}}\; cm.$$

The phase angle,

$$\cos\phi = -\sqrt{\frac{5}{7}} \text{ and } \sin\phi = \sqrt{\frac{2}{7}}$$

$$\phi = \cos^{-1}(-\sqrt{\frac{5}{7}})$$

$$\phi \approx 2.58\,radians.$$

11. The mass ($m = 0.1\,\text{kg}$) attached to a spring ($k = 50\,\text{N/m}$) is initially stretched to $1\,\text{m}$ length from the equilibrium position and released without any initial velocity. The motion is damped ($c = 2\,\text{kg/s}$) and is being driven by an external periodic ($T = \frac{\pi}{10}\,\text{s}$) force beginning at $t = 0$. Determine the displacement of mass as a function of time. Plot the displacement versus time graph.

Solution:
Given data: $m = 0.1\text{kg}$, $c = 2\,\text{kg/s}$, $k = 50\,\text{N/m}$, $T = \frac{\pi}{10}\,\text{s}$, $x(0) = 1\; x'(0) = 0$, $F_0 = 50\,\text{N}$.
From T, we can get $\omega = 20$ The differential equation can be written as,

$$0.1x'' + 2x' + 50x = 50\, \cos 20\,t$$

Dividing the equation by 0.1,

$$x'' + 20x' + 500x = 500\, \cos 20\,t$$

The characteristic equation is

$$\lambda^2 + 20\lambda + 500 = 0$$

Solving the quadratic equation, we get $\lambda_1 = -10 + 20i$ and $\lambda_2 = -10 - 20i$. Since the roots are complex, the general solution of the homogeneous ODE is

$$x_c = e^{-10\,t}(C_1\, \cos(20\,t) + C_2\, \sin(20\,t))$$

Since the RHS is a trigonometric function, we choose the particular solution as

$$x_p = A\, \cos(20\,t) + B\, \sin(20\,t)$$

differentiating once,

$$x_p' = -20\,A\, \sin(20\,t) + 20\,B\, \cos(20\,t)$$

differentiating twice

$$x_p'' = -400\,A\,\cos(20\,t) - 400\,B\,\sin(20\,t)$$

substituting the values of x_p'', x_p' and x, we get

$$(100\,A + 400\,B)\,\cos 20\,t + (-400\,A + 100\,B) = 20\,\cos 20\,t$$

Comparing the coefficients of $\cos 20\,t$ and $\sin 20\,t$,

$$A = \frac{5}{17} \;\; and \;\; B = \frac{20}{17}$$

Substituting the values in the x_p, we get

$$x_p = (\frac{5}{17})\,\cos 20\,t + (\frac{20}{17})\,\sin 20\,t$$

The general equation is,

$$x(t) = e^{-10\,t}\,(C_1\,\cos 20\,t + C_2\,\sin 20\,t) + (\frac{5}{17})\,\cos 20\,t + (\frac{20}{17})\,\sin 20\,t$$

Differentiating the general solution,

$$x'(t) = e^{-10\,t}\,(-20\,C_1\,\sin 20\,t + 20\,C_2\,\cos 20\,t) - 10\,e^{-10\,t}\,(C_1\,\cos 20\,t + C_2\,\sin 20\,t) - \frac{100}{17}\,\sin 20\,t + \frac{400}{17}\,\cos 20\,t$$

Substituting the initial conditions and solving for C_1 and C_2,

$$C_1 = \frac{12}{17} \;\; and \;\; C_2 = \frac{-14}{17}$$

Substituting the values in the general solution,

$$x(t) = e^{-10\,t}\,((\frac{12}{17})\,\cos 20\,t - (\frac{14}{17})\,\sin 20\,t) + (\frac{5}{17})\,\cos 20\,t + (\frac{20}{17})\,\sin 20\,t$$

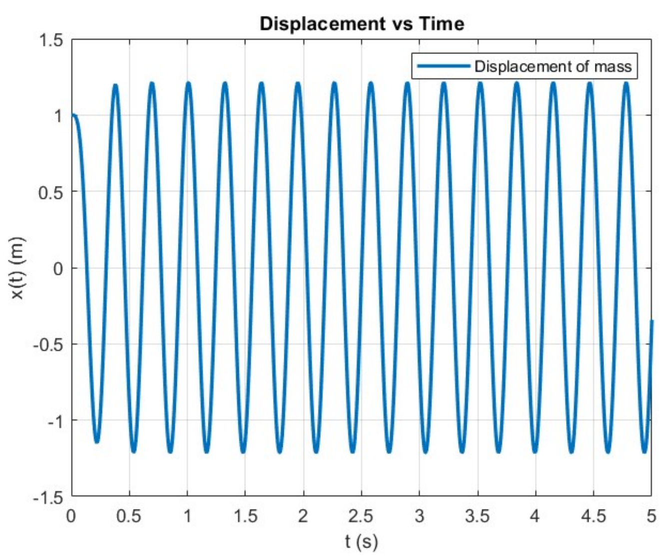

Figure 4: Plot of Displacement vs Time

12. Solve the initial value problem

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \cos(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

for $\omega_0 = 50$, $\omega = 20,\ 40,\ 48,\ 52,\ 50$, where ω_0 and ω are in rad/s. Plot the graph x vs t for each of these cases.

Solution:
From the differential equation, $m = 1 \quad k = {\omega_0}^2, \ \ x(0) = 0, \ \ x'(0) = 0, \ \ \omega_0 = 50, \ \ F_0 = 1$
The characteristic equation is

$$\lambda^2 + {\omega_0}^2 = 0$$

Solving the quadratic equation, we get $\lambda_1 = \omega_0\, i$ and $\lambda_2 = -\omega_0\, i$. Since the roots are complex, the general solution of the homogeneous ODE is

$$x_c = C_1\, \cos(\omega_0\, t) + C_2\, \sin(\omega_0, t)$$

Since the RHS is a trigonometric function, we choose the particular solution as

$$x_p = (A\, \cos(\omega\, t) + B\, \sin(\omega\, t))$$

differentiating once,

$$x_p' = -\omega\, A\, \sin(\omega\, t) + \omega\, B\, \cos(\omega\, t)$$

differentiating twice,

$$x_p'' = -\omega^2\, A\, \cos\,(\omega\, t) - \omega\, B\, \sin\,(\omega\, t)$$

Substituting and comparing coefficients,

$$(-\omega^2\, A\, + \omega^2\, AB)\, \cos\,(\omega\, t) + (-\omega^2\, B\, + \omega^2\, B)\, \sin\,(\omega\, t) = \cos\,(\omega\, t)$$

$$A = \frac{1}{{\omega_0}^2 - \omega^2}$$

Substituting in x_p,

$$x_p = (\frac{\cos \omega\, t}{{\omega_0}^2 - \omega^2})$$

General solution $x(t) = x_c + x_p$,

$$x(t) = C_1\, \cos(\omega_0\, t) + C_2\, \sin(\omega_0, t) + (\frac{\cos(\omega\, t)}{{\omega_0}^2 - \omega^2})$$

Differentiating the general solution,

$$x'(t) = -C_1\, \omega_0\, \sin(\omega_0\, t) + C_2\, \omega_0\, \cos(\omega_0, t) - (\frac{\omega\, \sin(\omega\, t)}{{\omega_0}^2 - \omega^2})$$

Applying the initial conditions $x(0) = 0, \ \ x'(0) = 0$ and solving for C_1 and C_2

$$C_1 = -\,(\frac{1}{({\omega_0}^2 - \omega^2})$$

General solution,

$$x(t) = \frac{\cos\,(\omega\, t) - \cos\,(\omega_0\, t)}{{\omega_0}^2 - \omega^2}$$

Substituting $\omega_0 = 50$,

$$x(t) = \frac{\cos\,(\omega\, t) - \cos\,(50\, t)}{2500 - \omega^2}$$

When $\omega = 20$,

$$x(t) = \frac{1}{2100}\,(\cos 20\,t - \cos 50\,t)$$

When $\omega = 40$,

$$x(t) = \frac{1}{900}\,(\cos 40\,t - \cos 50\,t)$$

When $\omega = 48$,

$$x(t) = \frac{1}{196}\,(\cos 48\,t - \cos 50\,t)$$

When $\omega = 52$,

$$x(t) = \frac{-1}{204}\,(\cos 52\,t - \cos 50\,t)$$

When $\omega = 50$, i.e $\omega = \omega_0$ The solution becomes,

$$x(t) = (\frac{1}{2\,\omega_0})\,t\,\sin(\omega_0\,t)$$

Because of the factor t, the amplitude of the vibration becomes larger and larger. Practically the system would fail.

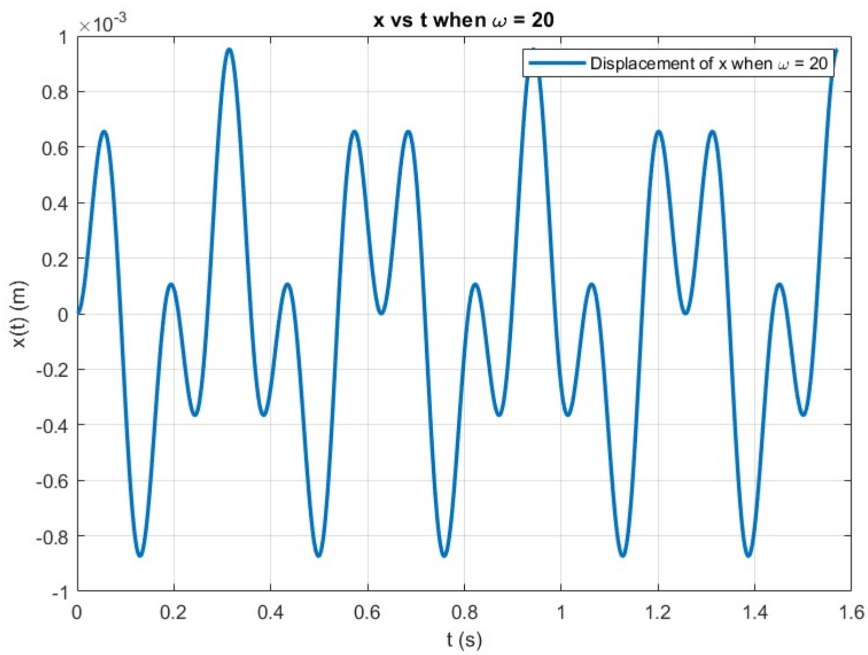

Figure 5: Displacement of x when $\omega = 20$

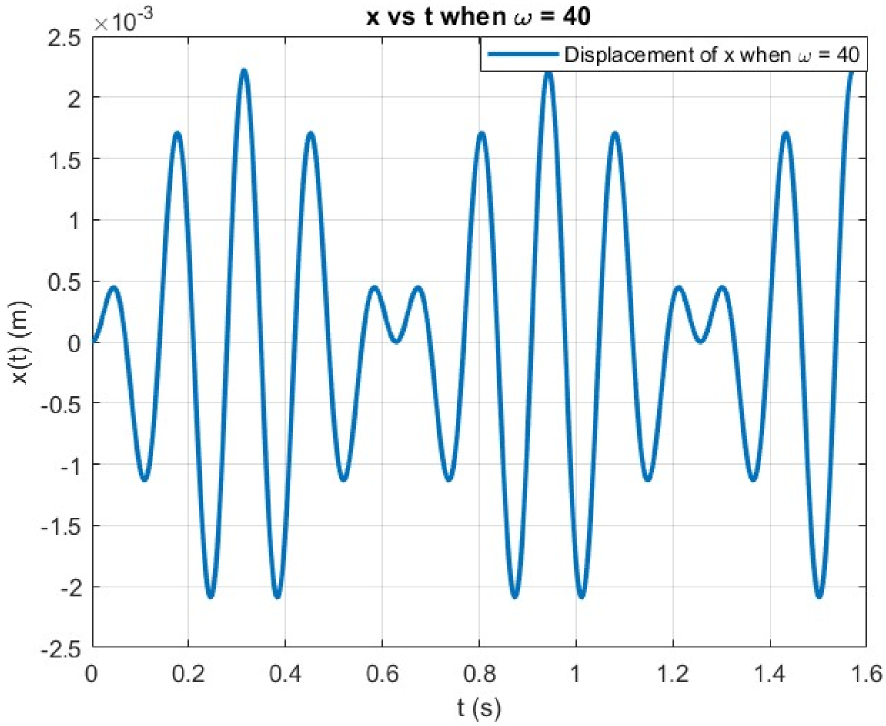

Figure 6: Displacement of x when $\omega = 40$

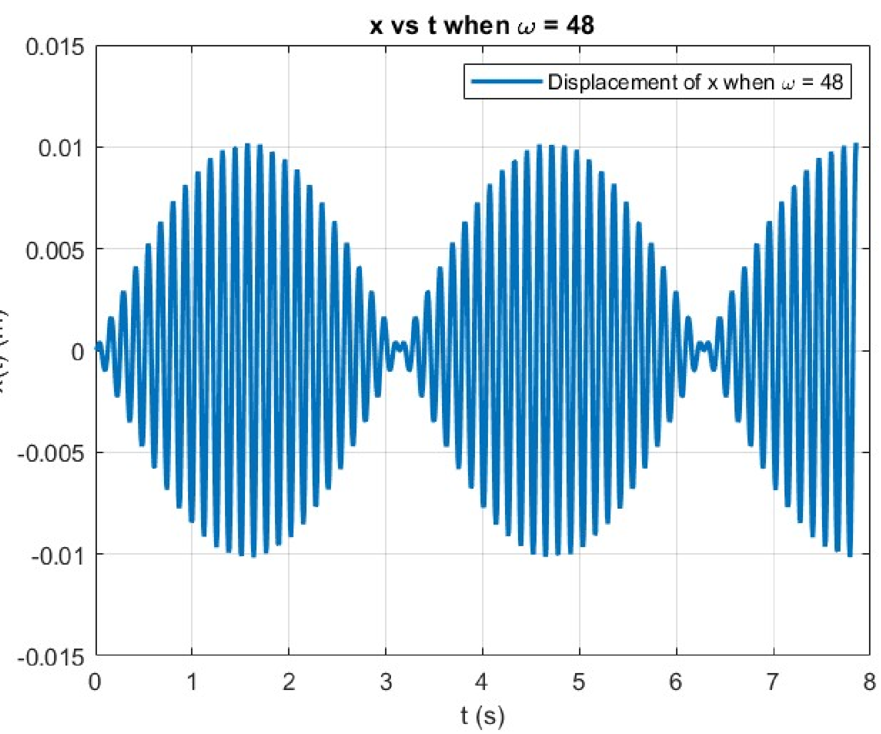

Figure 7: Displacement of x when $\omega = 48$

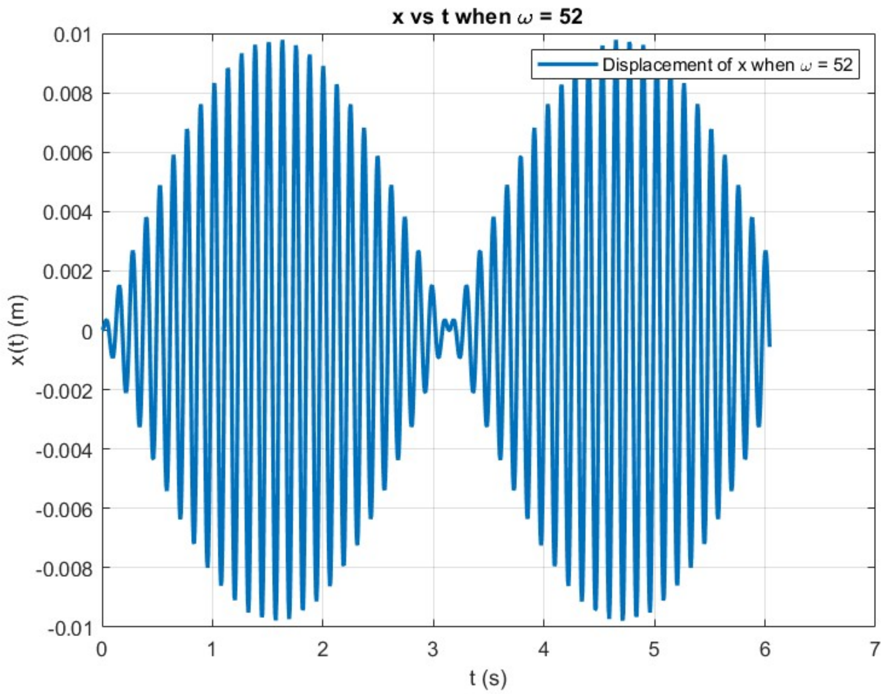

Figure 8: Displacement of x when $\omega = 52$

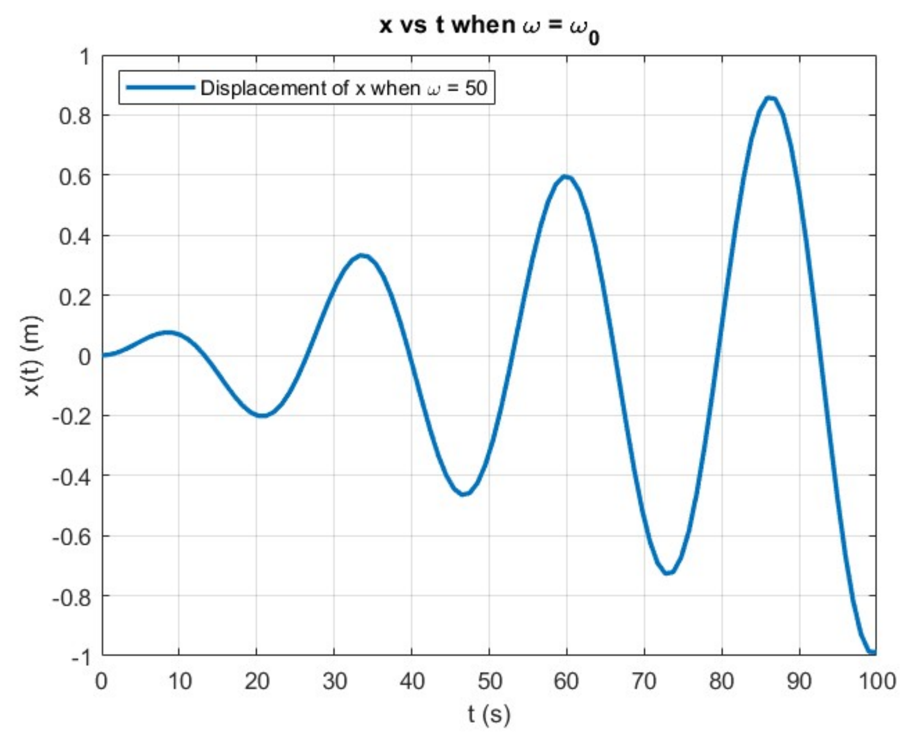

Figure 9: Displacement of x when $\omega = 50$