AE601 Mathematical Methods in Aerospace Engineering

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1. Solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L,\, t > 0$$

in a finite domain for the following initial and boundary conditions:
(a)

$$\begin{aligned} u(x,0) &= U, \quad 0 < x < L \\ u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

where U is a constant. Find the location of the maximum temperature for $t > 0$.

Solution:
We have two homogeneous boundary conditions, so using method of separation of variables to solve the problem:

$$u(x,t) = X(x) \cdot T(t)$$

Substituting this in the one dimensional heat equation we get,

$$X(x) = C_1 cos(kx) + C_2 sin(kx)$$

$$T(x) = C_3 e^{(-\alpha \cdot k^2 \cdot t)}$$

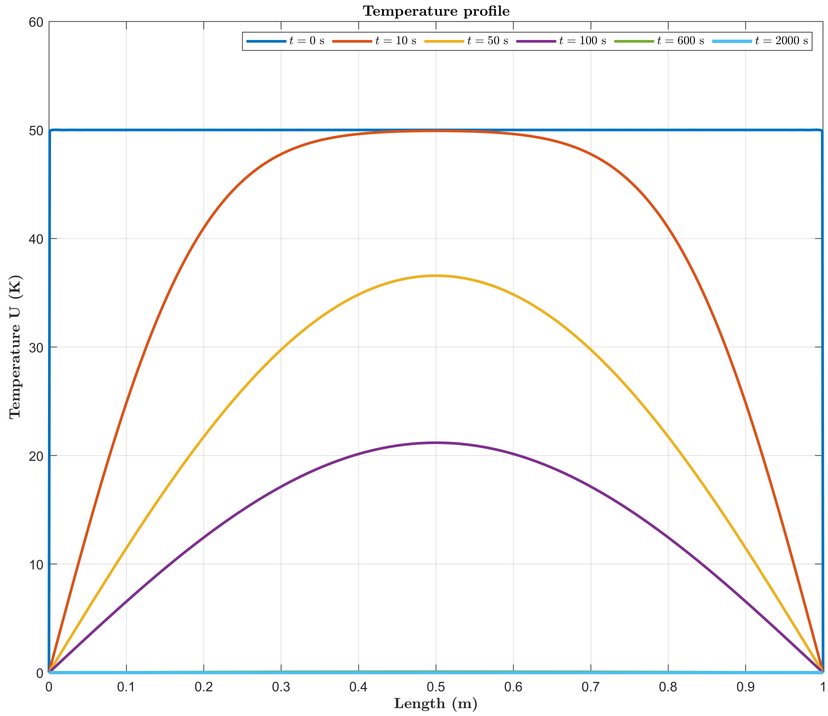
Solving for C_1, C_2, C_3 and k by applying the Boundary conditions,

$$k = \frac{n\pi}{L}$$

$$\boxed{B_n = \frac{2U}{\pi}(1-(-1)^n)}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2U}{\pi}(1-(-1)^n) \cdot e^{(-\alpha \cdot (\frac{n\pi}{2})^2 \cdot t)} \cdot sin(\frac{n\pi x}{L})}$$

The values are assumed for plotting,
Thermal diffusivity, $\alpha = 1.115 \times 10^{-3}\,\mathrm{m^2/s}$
Initial temperature, $U = 50\,\mathrm{K}$
Length of the domain, $L = 1\,\mathrm{m}$
The plot is shown below:



(b)

$$u(x,0) = \begin{cases} x, & 0 < x \leq \frac{L}{2} \\ L - x, & \frac{L}{2} < x < L \end{cases}$$
$$u(0,t) = 0$$
$$u(L,t) = 0$$

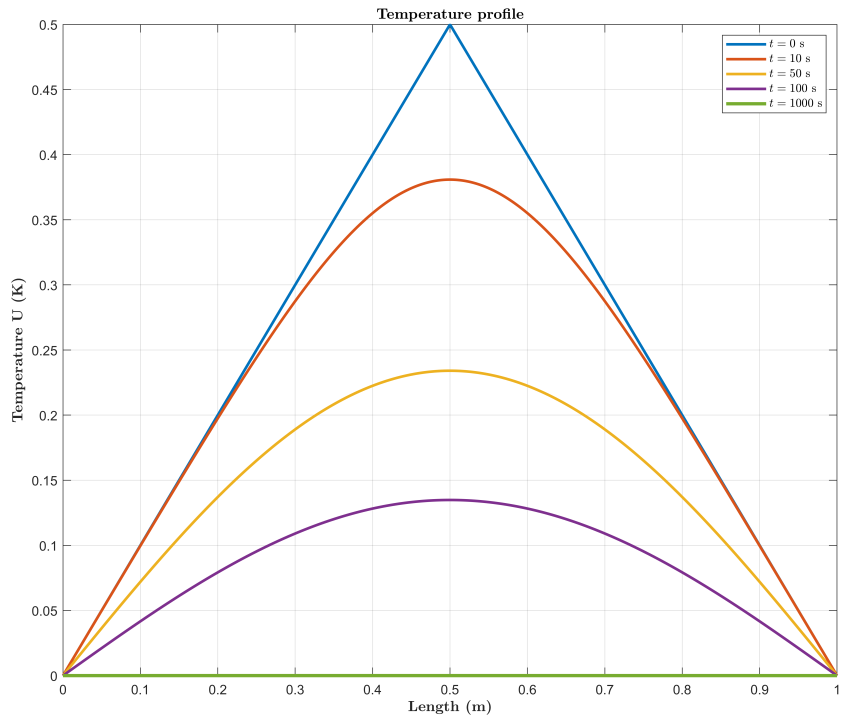
Solution:
Following the same procedure as a, B_n can be calculated,

$$B_n = \frac{4L}{(n\pi)^2} \cdot \sin\left(\frac{n\pi}{2}\right)$$

The total solution will be,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{4L}{(n\pi)^2} \cdot \sin\left(\frac{n\pi}{2}\right) \cdot e^{-\alpha \cdot \left(\frac{n\pi}{2}\right)^2 \cdot t} \cdot \sin\left(\frac{n\pi x}{L}\right)}$$

The values are assumed for plotting,
Thermal diffusivity, $\alpha = 1.115 \times 10^{-3}\,\mathrm{m}^2/\mathrm{s}$
Length of the domain, $L = 1\,\mathrm{m}$
The plot is shown below:



(c)

$$\begin{aligned} u(x,0) &= (u'_L - u'_o)\cdot\left(\frac{x}{L}\right) + u'_0, \quad 0 < x < L \\ u(0,t) &= u_0 \\ u(L,t) &= u_L \end{aligned}$$

Solution:
The solution is splitted into two parts, Steady state part and a time dependent part

$$u(x,t) = u_s(x) + \theta(x,t)$$

$$\theta(x,t) = u(x,t) - u_s(x)$$

Steady state governing equation:

$$\frac{d^2u_s}{dx^2} = 0$$

Boundary conditions,

$$u_s(0) = u_o \quad u_s(L) = u_L$$

$$u_s(x) = \frac{u_L - u_o}{L}x + u_o$$

$$\theta(x,0) = g(x) = f(x) - u_s(x)$$

$$g(x) = \frac{u'_L - u'_o}{L}x + u'_o$$

$$\boxed{\theta(x,t) = \sum_{n=1}^{\infty} B_n \cdot e^{-\alpha\cdot\left(\frac{n\pi}{2}\right)^2\cdot t}\cdot\sin\left(\frac{n\pi x}{L}\right)}$$

$$\boxed{B_n = \frac{2}{L}\int_0^L g(x)\cdot\sin\left(\frac{n\pi}{L}x\right)\,dx}$$

solving for B_n, it is split into B_1, B_2, B_3, and B_4 for convenience,

$$B_1 = \frac{(u'_o - u'_L)}{n\pi}\cdot L\cdot\cos(n\pi)$$

$$B_2 = \frac{L\cdot u'_o}{n\pi}\cdot(1-\cos(n\pi))$$

$$B_3 = \frac{(u_o - u_L)}{n\pi}\cdot L\cdot\cos(n\pi)$$

$$B_4 = \frac{L\cdot u_o}{n\pi}\cdot(1-\cos(n\pi))$$

$$B_n = \frac{2}{L}\cdot(B_1 + B_2 - B_3 - B_4)$$

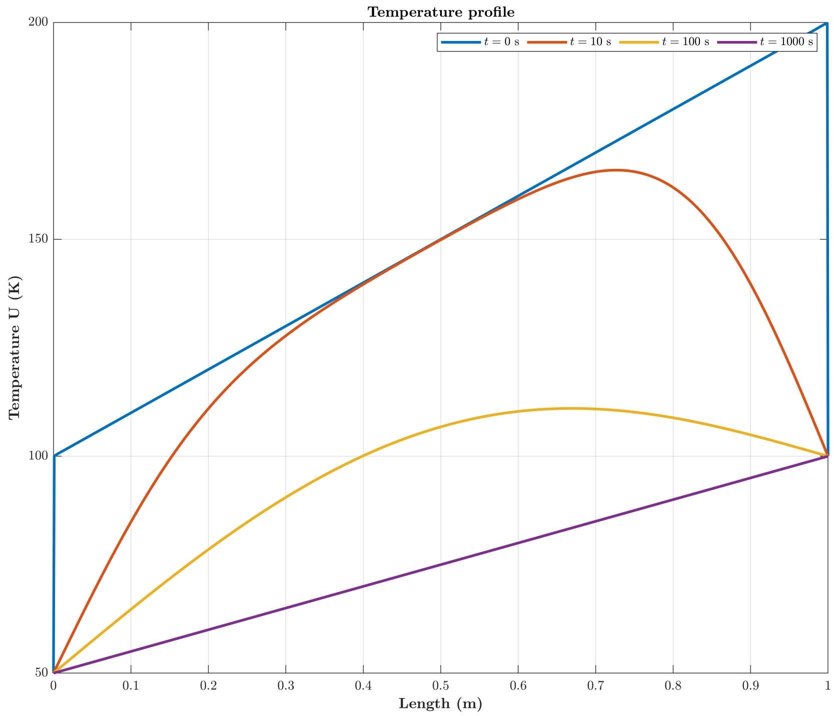
$$\boxed{B_n = \frac{2}{L}\cdot(\frac{(u'_o - u'_L)}{n\pi}\cdot L\cdot\cos(n\pi) + \frac{L\cdot u'_o}{n\pi}\cdot(1-\cos(n\pi)) - \frac{(u_o - u_L)}{n\pi}\cdot L\cdot\cos(n\pi) - \frac{L\cdot u_o}{n\pi}\cdot(1-\cos(n\pi)))}$$

The total solution,

$$u(x,t) = \frac{u_L - u_o}{L}x + u_o + \sum_{n=1}^{\infty} B_n\cdot e^{-\alpha\cdot\left(\frac{n\pi}{2}\right)^2\cdot t}\cdot\sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{\begin{aligned} u(x,t) = & \frac{u_L - u_o}{L}x + u_o \\ & + \sum_{n=1}^{\infty}\frac{2}{L}\cdot\left(\frac{(u'_o - u'_L)}{n\pi}\cdot L\cdot\cos(n\pi)\right. \\ & + \frac{L\cdot u'_o}{n\pi}\cdot(1-\cos(n\pi)) - \frac{(u_o - u_L)}{n\pi}\cdot L\cdot\cos(n\pi) \\ & \left. - \frac{L\cdot u_o}{n\pi}\cdot(1-\cos(n\pi))\right)\cdot e^{-\alpha\cdot\left(\frac{n\pi}{2}\right)^2\cdot t}\cdot\sin\left(\frac{n\pi x}{L}\right) \end{aligned}}$$

The values are assumed for plotting,
$u_o = 50\,\mathrm{K}$
$u_L = 100\,\mathrm{K}$
$u'_o = 100\,\mathrm{K}$
$u'_L = 200\,\mathrm{K}$
Thermal diffusivity, $\alpha = 1.115\times10^{-3}\,\mathrm{m^2/s}$
Length of the domain, $L = 1\,\mathrm{m}$
The plot is shown below:



(d)

$$u(x,0) = \begin{cases} u_l, & 0 < x \leq \frac{L}{2} \\ u_r, & \frac{L}{2} < x < L \end{cases}$$
$$u(0,t) = u_l$$
$$u(L,t) = u_r$$

Solution:
Following a similar procedure, we can find the steady state solution,

$$u_s(x) = \frac{u_r - u_L}{L}x + u_L$$

$$g(x) = \frac{u_r - u_L}{L}x \quad 0 < x \leq \frac{L}{2}$$

$$g(x) = u_r - \frac{u_r - u_L}{L}x - u_L \quad \frac{L}{2} < x \leq L$$

$$B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}\right)\, dx$$

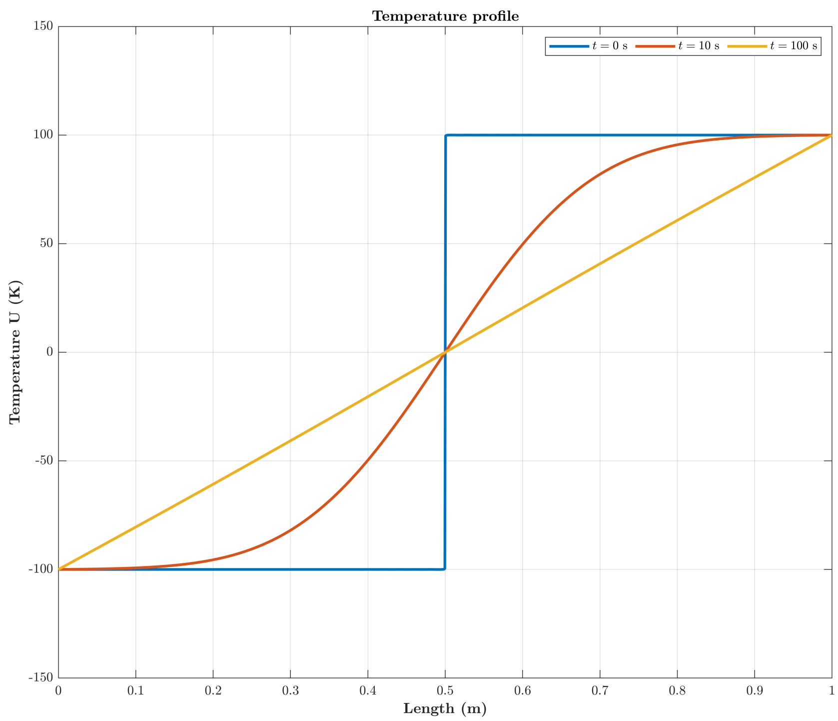
$$\boxed{B_n = \frac{2(u_r - u_L)}{n\pi}\cdot\cos\left(\frac{n\pi}{2}\right)}$$

$$\boxed{\theta(x,t) = \sum_{n=1}^{\infty} B_n \cdot e^{-\alpha\cdot\left(\frac{n\pi}{2}\right)^2\cdot t}\cdot\sin\left(\frac{n\pi x}{L}\right)}$$

The total solution,

$$\boxed{u(x,t) = \frac{u_r - u_L}{L}x + u_L + \sum_{n=1}^{\infty} \frac{2(u_r - u_L)}{n\pi} \cdot \cos\left(\frac{n\pi}{2}\right) \cdot e^{-\alpha \cdot \left(\frac{n\pi}{2}\right)^2 \cdot t} \cdot \sin\left(\frac{n\pi x}{L}\right)}$$

The values are assumed for plotting,
Thermal diffusivity, $\alpha = 1.115 \times 10^{-3}\,\mathrm{m^2/s}$
Length of the domain, $L = 1\,\mathrm{m}$
$u_r = 100\,\mathrm{K}$
$u_L = -100\,\mathrm{K}$
The plot is shown below:



(e)

$$u(x,0) = U, \quad 0 < x < L$$
$$u(0,t) = u_0$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

Solution:
Applying a transformation to make the boundary conditions homogeneous,

$$\theta(x,t) = u(x,t) - u_o$$

Governing equation

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}$$

Initial condition

$$\theta(x,0) = U - u_o$$

Boundary conditions

$$\theta(0,t) = 0$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0$$

Applying the method of separation of variables,

$$\theta(x,t) = X(x) \cdot T(t)$$

Substituting this into the governing equation and solving,

$$\boxed{\theta_n = \sum_{n=1}^{\infty} B_n \cdot e^{-\alpha \cdot \left(\frac{n\pi}{2}\right)^2 \cdot t} \cdot \sin(\lambda_n x)}$$

where λ_n is

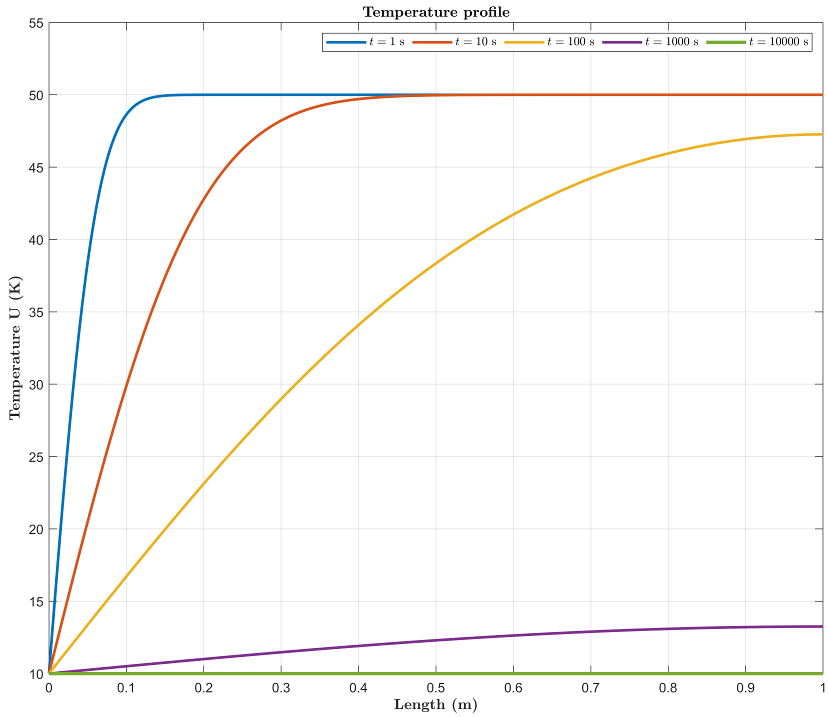
$$\boxed{\lambda_n = \frac{(2n-1)\pi}{2L}}$$

$$\boxed{B_n = \frac{2(U - u_o)}{L\lambda_n}(1 - \cos(\lambda_n L))}$$

The final solution,

$$\boxed{u(x,t) = u_o + \sum_{n=1}^{\infty} \frac{2(U - u_o)}{L\lambda_n}(1 - \cos(\lambda_n L)) \cdot e^{-\alpha \cdot \left(\frac{n\pi}{2}\right)^2 \cdot t} \cdot \sin(\lambda_n x)}$$

The values are assumed for plotting,
Thermal diffusivity, $\alpha = 1.115 \times 10^{-3}\,\mathrm{m^2/s}$
Length of the domain, $L = 1\,\mathrm{m}$
Initial temperature, $U = 50\,\mathrm{K}$
Temperature at $x = 0$. $u_o = 10\,\mathrm{K}$
The plot is shown below:



(f)

$$u(x,0) = \frac{U}{2}\left(1 + \cos\frac{\pi x}{L}\right), \quad 0 < x < L$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

Solution:

Following a similar procedure except for the boundary conditions from the previous problem e, we can get the final solution as,

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cdot e^{-\alpha\cdot\left(\frac{n\pi}{L}\right)^2\cdot t} \cdot \cos\left(\frac{n\pi x}{L}\right)$$

Only a_1 and a_0 exist,

$$a_1 = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{\pi x}{L}\right)\,dx$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

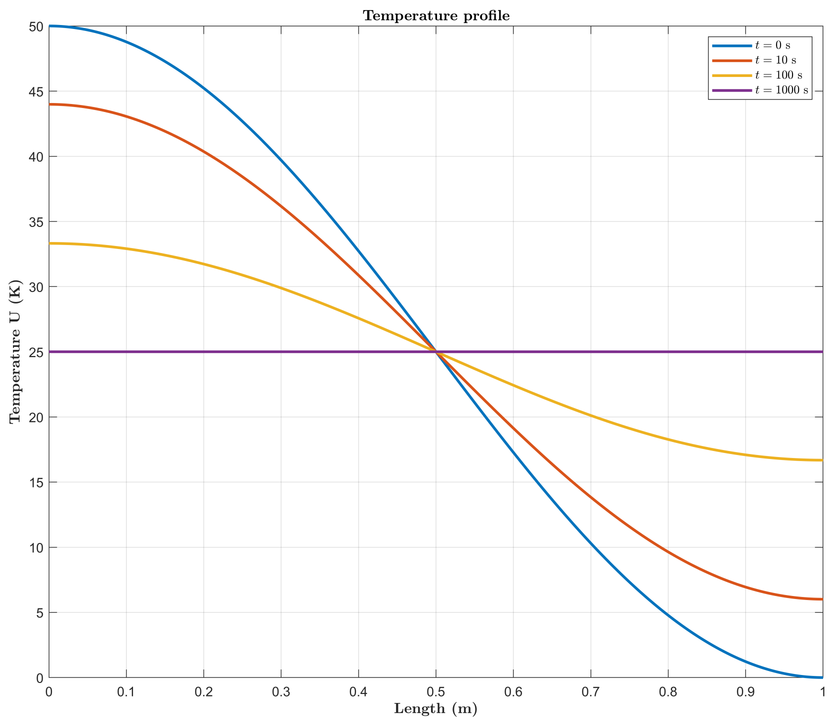
where $f(x)$ is

$$f(x) = \frac{U}{2}\left(1 + \cos\left(\frac{\pi x}{L}\right)\right)$$

Solving for a_1 and a_0 and substituting in the final solution,

$$\boxed{u(x,t) = \frac{U}{2}\left(1 + e^{-\alpha\cdot\left(\frac{\pi}{L}\right)^2\cdot t}\cdot\cos\left(\frac{\pi x}{L}\right)\right)}$$

The values are assumed for plotting,
Thermal diffusivity, $\alpha = 1.115 \times 10^{-3}\,\mathrm{m}^2/\mathrm{s}$
Length of the domain, $L = 1\,\mathrm{m}$
$U = 50\,\mathrm{K}$
The plot is shown below:



2. Solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \frac{\dot{q}_g}{c}, \quad 0 < x < L,\, t > 0$$

in a finite domain for the following initial and boundary conditions:

$$u(x,0) = f(x), \quad 0 < x < L$$

$$u(0,t) = u_0$$

$$u(L,t) = u_L$$

Solution:
The problem can be split into two parts,

$$U(x,t) = \theta(x,t) + u_s(x)$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{d^2 u_s}{dx^2} + \frac{\dot{q}_g}{\alpha c} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

$$\frac{d^2u_s}{dx^2}+\frac{\dot{q}_g}{\alpha c}=0$$

Boundary conditions

$$u_s(0)=u_o$$

$$u_s(L)=u_L$$

$$\theta(x,0)=f(x)-u_s(x)$$

$$\theta(0,t)=f(0)-u_o=0$$

$$\theta(0,t)=f(L)-u_L=0$$

Solving for u_s,

$$u_s(x)=u_o+(u_L-u_o)\frac{x}{L}+\frac{\dot{q}_g x}{2\alpha L}(L-x)$$

For the time-dependent solution $\theta(x,t)$, this can be solved using the method of separation of variables,

$$\theta(x,t)=X(x)\cdot T(t)$$

$$X(x)=A_n\sin(\lambda_n x)+B_n\cos(\lambda_n x)$$

$$T(t)=C_n e^{-\alpha\lambda_n^2 t}$$

$$\theta(x,t)=\sum_{n=1}^{\infty}a_n e^{-\alpha\lambda_n^2 t}\sin(\lambda_n x)$$

Substituting the boundary conditions and solving, we get,

$$\lambda_n=\frac{n\pi}{L}$$

$$\boxed{a_n=\frac{2}{L}\int_0^L (f(x)-u_s(x))\sin(\lambda_n x)\,dx}$$

The total solution is,

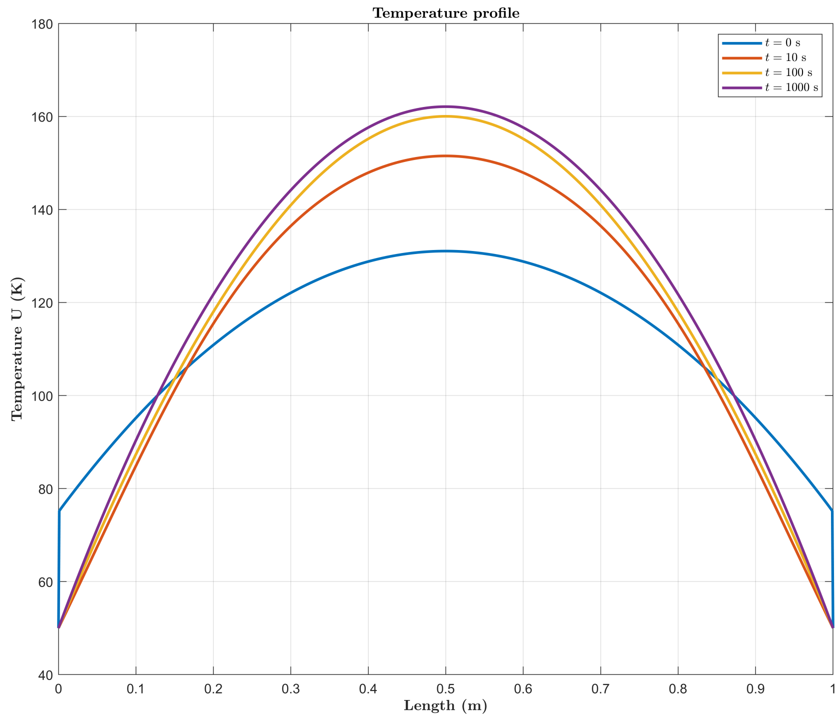
$$\boxed{u(x,t)=\sum_{n=1}^{\infty}a_n e^{-\alpha\lambda_n^2 t}\sin(\lambda_n x)+u_o+(u_L-u_o)\frac{x}{L}+\frac{\dot{q}_g x}{2\alpha L}(L-x)}$$

Let $f(x)=U$, calculating a_n

$$\boxed{\begin{aligned}a_n =&\frac{U-u_0}{\lambda_n}(1-\cos(\lambda_n))\\ &-\left(\frac{\sin(\lambda_n L)}{\lambda_n^2}-\frac{L\cos(\lambda_n L)}{\lambda_n}\right)\left(\frac{u_L-u_0}{L}+\frac{\dot{q}_g L}{2\alpha L}\right)\\ &-\left(\frac{\dot{q}_g}{2\alpha L}\right)\left(\frac{L^2\cos(\lambda_n L)}{\lambda_n}-\frac{2L\sin(\lambda_n L)}{\lambda_n^2}-\frac{2\cos(\lambda_n L)}{\lambda_n^3}+\frac{2}{\lambda_n^3}\right)\end{aligned}}$$

Substituting a_n in the $u(x,t)$, we can get a solution for plotting,
The values are assumed for plotting,
Thermal diffusivity, $\alpha=1.115\times10^{-3}\,\mathrm{m^2/s}$
Length of the domain, $L=1\,\mathrm{m}$
$U=100\,\mathrm{K}$
$u_o=50\,\mathrm{K}$
$u_L=50\,\mathrm{K}$
Heat generation, $\dot{q}_g=1\,\mathrm{W/m^3}$
$c=1$

The plot is shown below:



3. Solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\, t > 0$$

in an infinite domain for the following initial conditions:
(a)

$$u(x,0) = \begin{cases} \cos x, & \text{if } x \leq |\pi| \\ 0, & \text{if } x > |\pi| \end{cases}$$

Solution:
From

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x - k\sqrt{4\alpha t}) e^{-k^2}\, dk$$

For our problem,

$$u(x,t) = \frac{1}{\sqrt{4\alpha t \pi}} \int_{-\pi}^{\pi} \cos(x - k\sqrt{4\alpha t}) e^{-k^2}\, dk$$

Using $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ and simplifying,

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \cos(x)\cos(k\sqrt{4\alpha t}) e^{-k^2}\, dk$$

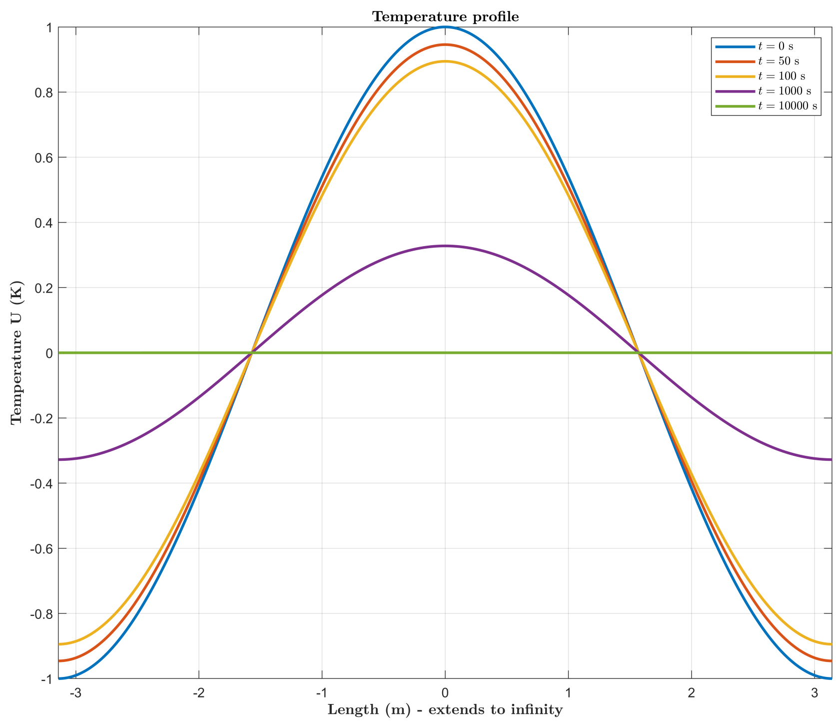
Integrating the right-hand side,

$$\boxed{u(x,t) = \frac{\cos(x)e^{-\alpha t}}{2}\left[\operatorname{erf}(\pi - i\sqrt{\alpha t}) - \operatorname{erf}(-\pi - i\sqrt{\alpha t})\right]}$$

For plotting, considering only real values, the equation becomes,

$$u(x,t) = \frac{\cos(x)e^{-\alpha t}}{2}$$

The values assumed for plotting,
Thermal diffusivity, $\alpha = 1.115 \times 10^{-3}\,\mathrm{m^2/s}$
The plot is shown below:



(b)

$$u(x,0) = \begin{cases} u_l, & \text{if } x \le 0 \\ u_r, & \text{if } x > 0 \end{cases}$$

Solution:
From

$$u(x,t) = \frac{1}{\sqrt{4\alpha t\pi}} \int_{-\infty}^{\infty} f(s) e^{\frac{-(s-x)^2}{4\alpha t}}\, ds$$

For our problem,

$$u(x,t) = \frac{1}{\sqrt{4\alpha t\pi}} \int_{-\infty}^{0} u_l e^{\frac{-(s-x)^2}{4\alpha t}}\, ds + \frac{1}{\sqrt{4\alpha t\pi}} \int_{0}^{\infty} u_r e^{\frac{-(s-x)^2}{4\alpha t}}\, ds$$

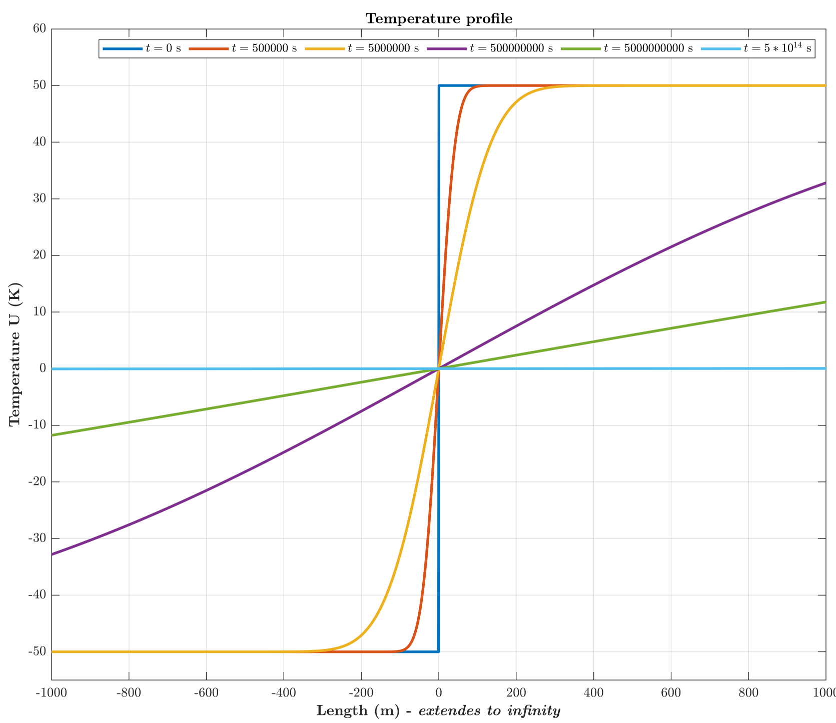
Transforming from s to k,

$$u(x,t) = \frac{u_l}{\sqrt{\pi}} \int_{-\infty}^{\frac{-x}{\sqrt{4\alpha t}}} e^{k^2}\,dk + \frac{u_r}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4\alpha t}}}^{\infty} e^{k^2}\,dk$$

After manipulating to get the error function,

$$\boxed{u(x,t) = \frac{u_l}{2}\left(1 - \mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)\right) + \frac{u_r}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)\right)}$$

The values are assumed for plotting,
Thermal diffusivity, $\alpha = 1.115 \times 10^{-3}\,\mathrm{m}^2/\mathrm{s}$
$u_l = -50\,\mathrm{K}$
$u_r = 50\,\mathrm{K}$
The plot is shown below:



4. Solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty,\, t > 0$$

in a semi-infinite domain for the following initial and boundary conditions

$$u(x,0) = U_i, \quad 0 < x < \infty$$
$$u(0,t) = U_0$$

Solution:
Using the method of similarity variables,

Given initial condition,

$$u(x,0) = U_i$$

Given boundary condition,

$$u(0,t) = U_o$$

Boundary condition at $x = \infty$,

$$u(\infty,t) = f(x)|_{x\to\infty}$$

Introducing the similarity variable η,

$$\eta = \frac{x}{\sqrt{\alpha t}}$$

Non-dimensionalizing,

$$f(\eta) = T = \frac{u(x,t) - u_o}{u_i - u_o}$$

Boundary conditions,

$$T(0,t) = 0$$
$$T(\infty,t) = 1$$

Initial condition,

$$T(x,0) = 1$$

Solving for $T(x,t)$ we get,

$$T(x,t) = \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

Transforming back to $u(x,t)$,

$$\boxed{u(x,t) = u_o + (u_i - u_o)\cdot \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)}$$

The values are assumed for plotting,
Thermal diffusivity, $\alpha = 1.115 \times 10^{-3}\,\mathrm{m^2/s}$
Initial temperature, $U_i = 20\,\mathrm{K}$
Temperature at $t > 0$, $U_0 = 50\,\mathrm{K}$

The plot is shown below: