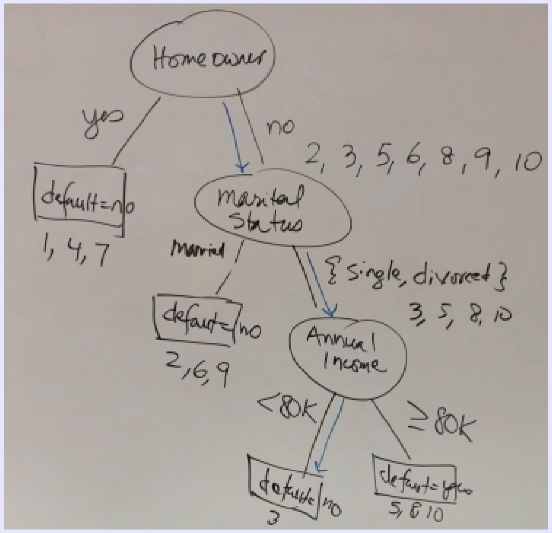
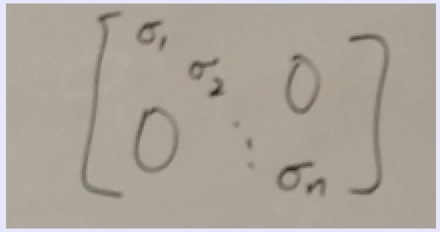
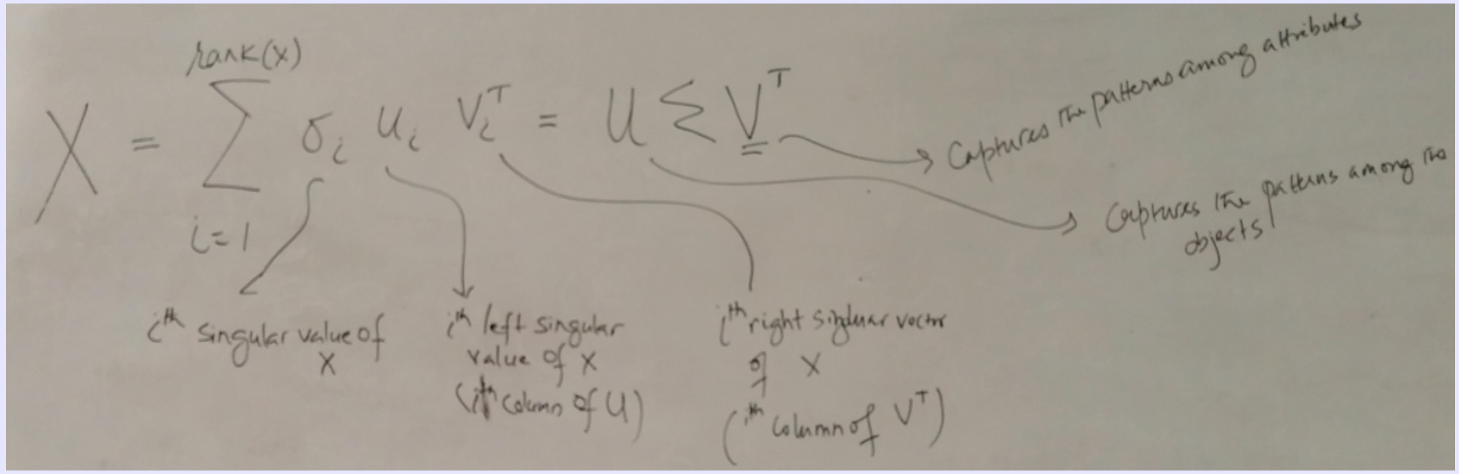
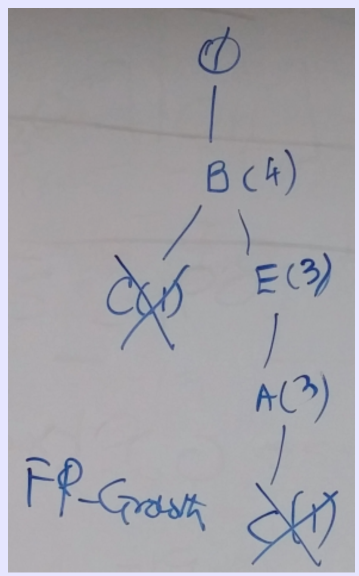
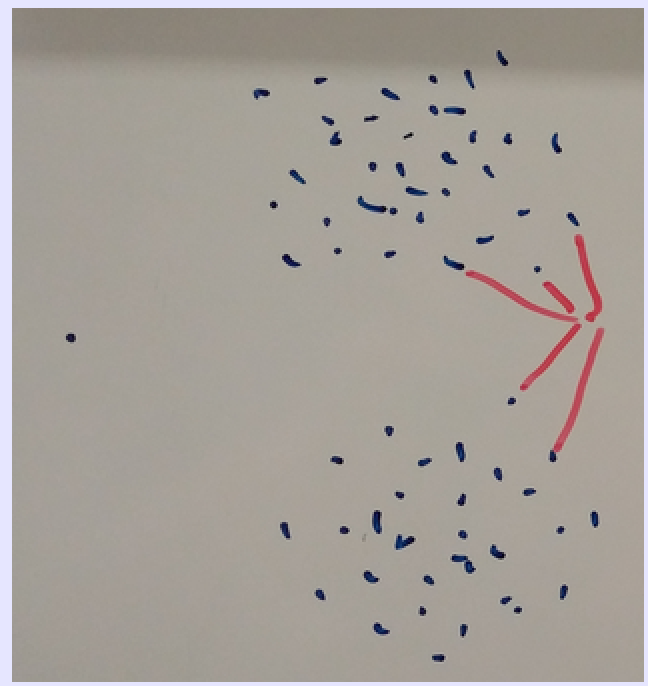










CS 422: Data Mining
Vijay K. Gurbani, Ph.D.,
Illinois Institute of Technology

Lecture: **Linear regression**

Linear regression: Example

- Let's check the effect of radio advertising on the sales through linear regression:

```
> model.radio <- lm(sales ~ radio, data=df)
> summary(model.radio)

Call:
lm(formula = sales ~ radio, data = df)

Residuals:
     Min       1Q   Median       3Q      Max
-15.7305  -2.1324   0.7707   2.7775   8.1810

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  9.31164    0.56290  16.542   <2e-16 ***
radio        0.20250    0.02041   9.921   <2e-16 ***

Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.275 on 198 degrees of freedom
Multiple R-squared:  0.332,     Adjusted R-squared:  0.3287
F-statistic: 98.42 on 1 and 198 DF,  p-value: < 2.2e-16
```

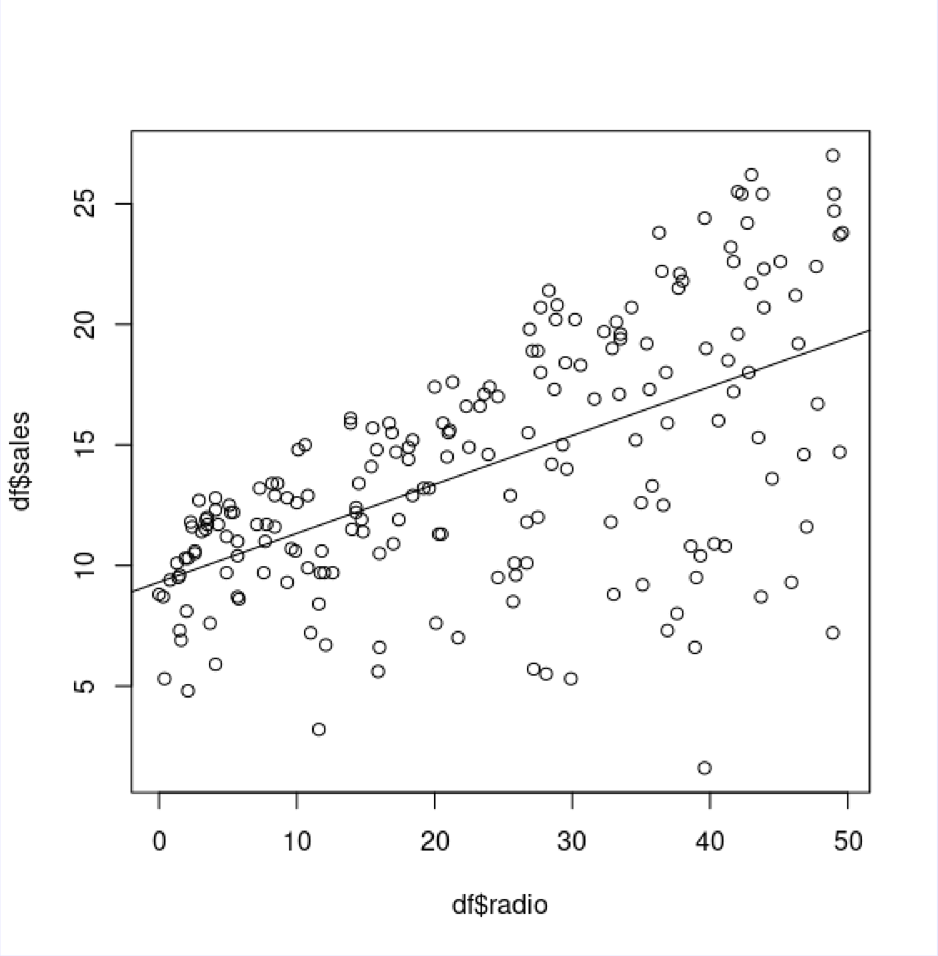
Interpretation:
A $1000 increase in spending on radio advertisement yields an average increase in sales of about 212 units.

- Regression equation: sales = $\beta_0 + \beta_1$*radio
  = 9.312 + 0.203*radio

Linear regression: Example

- Let's see how the regression line looks like for a single regressor (radio):

```
> plot(df$radio, df$sales)
> abline(model.radio)
```



Linear regression: Example

- Let's check the effect of all advertising media on the sales:

```
> model <- lm(sales ~ ., data=df)
> summary(model)

Call:
lm(formula = sales ~ ., data = df)

Residuals:
    Min      1Q  Median      3Q     Max 
-8.8277 -0.8908  0.2418  1.1893  2.8292 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept)  2.938889   0.311908   9.422   <2e-16 ***
TV           0.045765   0.001395  32.809   <2e-16 ***
radio        0.188530   0.008611  21.893   <2e-16 ***
newspaper   -0.001037   0.005871  -0.177     0.86    
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.686 on 196 degrees of freedom
Multiple R-squared:  0.8972,	Adjusted R-squared:  0.8956 
F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

- Regression equation:

$$\text{sales} = \beta_0 + \beta_1 * \text{TV} + \beta_2 * \text{radio} + \beta_3 * \text{newspaper}$$

$$= 2.939 + 0.046 * \text{TV} + 0.189 * \text{radio} - 0.001 * \text{newspaper}$$

Linear regression: Hypothesis testing

- p-values, statistical significance (α), and hypothesis tests.
  - H_0 (Null hypothesis) vs. H_1 (or H_a, alternative hypothesis)
  - α: P(rejecting H_0 | H_0 is true)
  - p-value: P(getting a result as extreme as you have | H_0 is true)
- Hypothesis test:
  - p-value ≤ α: result does not support H_0, **result statistically significant.**
  - p-value > α: result supports H_0.

Linear regression: Hypothesis testing

- p-values, statistical significance (α), and hypothesis tests.
  - H_0 (Null hypothesis) vs. H_1 (or H_a, alternative hypothesis)
  - α: P(rejecting H_0 | H_0 is true)
  - p-value: P(getting a result as extreme as you have | H_0 is true)
- Hypothesis test:
  - p-value ≤ α: result does not support H_0, **result statistically significant.**
  - p-value > α: result supports H_0.

The final conclusion once the test has been carried out is always given in terms of the null hypothesis. We either "reject H_0 in favor of H_1" or "do not reject H_0"; we never conclude "reject H_1", or even "accept H_1".

Linear regression: Analysis

- To understand the *fit* of a regression model, we need to ask some important questions.

1) Is at least one of the predictors useful in predicting the response?
2) Do all the predictors help explain the response, or is only a subset of predictors useful?
3) How well does the model fit the data?
4) Given a set of predictor values (the β's), what response value should we predict and how accurate is our prediction?

Linear regression: Analysis

```
Residuals:
    Min      1Q  Median      3Q     Max 
-8.8277 -0.8908  0.2418  1.1893  2.8292 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept)  2.938889   0.311908   9.422   <2e-16 ***
TV           0.045765   0.001395  32.809   <2e-16 ***
radio        0.188530   0.008611  21.893   <2e-16 ***
newspaper   -0.001037   0.005871  -0.177     0.86    
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.686 on 196 degrees of freedom
Multiple R-squared:  0.8972,	Adjusted R-squared:  0.8956 
F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

1. Is at least one of the predictors useful in predicting the response?

$H_0 = \beta_1 = \beta_2 = \ldots \beta_p = 0$

H_1 = At least one β is non zero

To answer the above question regarding H_0 and H_1, we define a F-statistic:

$$F = \frac{(TSS - RSS)/p}{RSS/(n-p-1)}$$

If there is no relationship between response and predictors, F-statistic is close to 1 (H_0 is true).

If H_1 is true, F-statistic > 1.

RSS: $\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$

TSS: $\sum_{i=1}^{n}(y_i - \bar{y})^2$

The *p-value* associated with the F-statistic is very small, giving us strong evidence that one of the predictors is associated with increased sales.

Linear regression: Analysis

```
Residuals:
    Min      1Q  Median      3Q     Max
-8.8277 -0.8908  0.2418  1.1893  2.8292

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.938889   0.311908   9.422   <2e-16 ***
TV           0.045765   0.001395  32.809   <2e-16 ***
radio        0.188530   0.008611  21.893   <2e-16 ***
newspaper   -0.001037   0.005871  -0.177     0.86
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.686 on 196 degrees of freedom
Multiple R-squared:  0.8972,    Adjusted R-squared:  0.8956
F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

2. Do all the predictors help explain the response, or is only a subset of predictors useful?

As before, we observe the p-values of the predictors in context of the following hypothesis test:

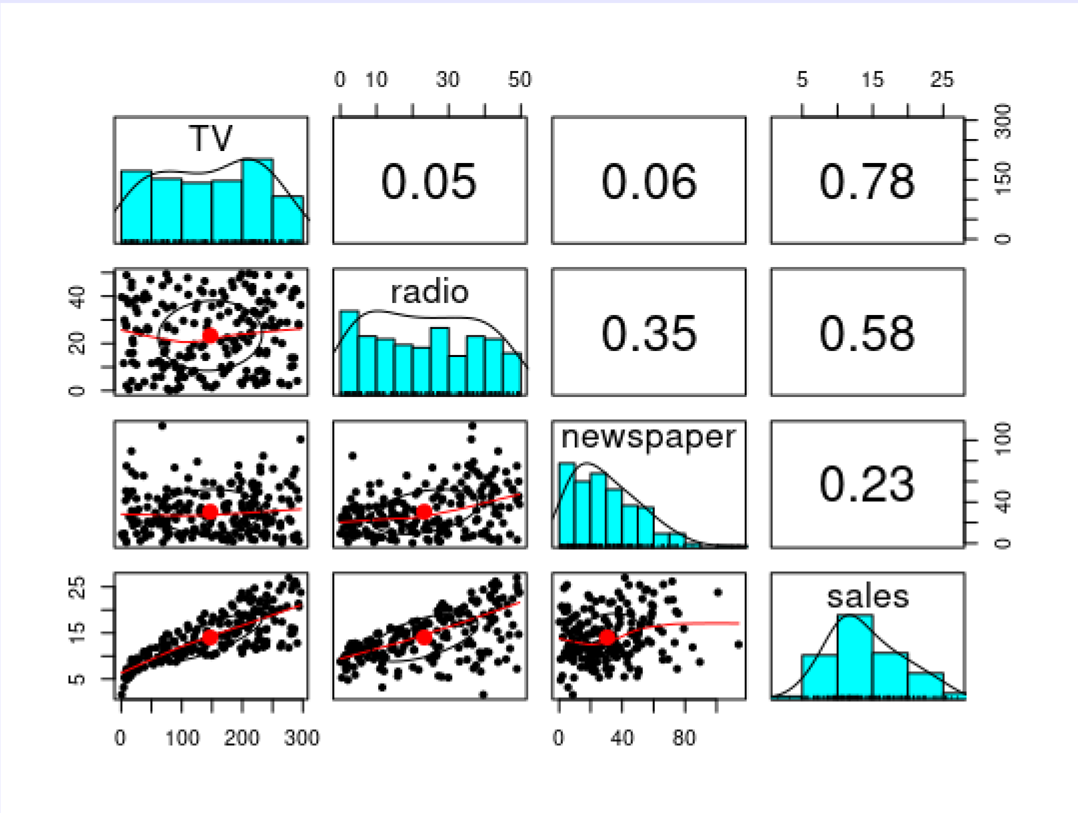
$$H_0 : \beta_i = 0 \text{ (There is no relationship between predictor and response)}$$
$$H_1 : \beta_i \neq 0 \text{ (There is some relationship between predictor and response)}$$

The p-values for TV and radio appear to imply that there is a relationship between these predictor and sales (p-values low).

The p-value for newspaper appears to imply that there may not be a relationship between newspaper and sales; i.e., H_0 cannot be rejected.

Linear regression: Analysis

Let’s drill down into the news-paper.




```
Residuals:
    Min      1Q  Median      3Q     Max
-8.8277 -0.8908  0.2418  1.1893  2.8292

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.938889   0.311908   9.422   <2e-16 ***
TV           0.045765   0.001395  32.809   <2e-16 ***
radio        0.188530   0.008611  21.893   <2e-16 ***
newspaper   -0.001037   0.005871  -0.177     0.86
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.686 on 196 degrees of freedom
Multiple R-squared:  0.8972,    Adjusted R-squared:  0.8956
F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

Linear regression: Analysis

```
Residuals:
    Min      1Q  Median      3Q     Max
-8.8277 -0.8908  0.2418  1.1893  2.8292

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.938889   0.311908   9.422   <2e-16 ***
TV           0.045765   0.001395  32.809   <2e-16 ***
radio        0.188530   0.008611  21.893   <2e-16 ***
newspaper   -0.001037   0.005871  -0.177     0.86
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.686 on 196 degrees of freedom
Multiple R-squared:  0.8972,    Adjusted R-squared:  0.8956
F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

Generally, what we are trying to do is *feature selection*, which is a hard problem.

Given *p* predictors, we will have 2^p models! (> 1 million models for p=30!)

Strategies:

- Forward selection: Start with null model (intercept) and add predictors.
- Backward selection: Start with all predictors and remove variables with largest p-values.

Linear regression: Analysis

```
Residuals:
    Min      1Q  Median      3Q     Max
-8.8277 -0.8908  0.2418  1.1893  2.8292

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.938889   0.311908   9.422   <2e-16 ***
TV           0.045765   0.001395  32.809   <2e-16 ***
radio        0.188530   0.008611  21.893   <2e-16 ***
newspaper   -0.001037   0.005871  -0.177     0.86
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.686 on 196 degrees of freedom
Multiple R-squared:  0.8972,    Adjusted R-squared:  0.8956
F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

3. How well does the model fit the data?

The following statistics describe this: Residual Standard Error (RSE), R^2, and RMSE.

$$R^2 = Cor(Y, \hat{Y})^2$$

R^2 for model that uses all three predictors = 0.8972
R^2 for model that uses TV and radio only = 0.8971

$$RSE = \sqrt{\frac{1}{n-p-1} RSS}$$

$$\mathrm{RMSE} = \sqrt{\frac{\sum (y_i - \hat{y}_i)^2}{N}}$$

Linear regression: Analysis

3. How well does the model fit the data?

```
Residuals:
    Min      1Q  Median      3Q     Max
-8.8277 -0.8908  0.2418  1.1893  2.8292

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.938889   0.311908   9.422   <2e-16 ***
TV           0.045765   0.001395  32.809   <2e-16 ***
radio        0.188530   0.008611  21.893   <2e-16 ***
newspaper   -0.001037   0.005871  -0.177     0.86
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.686 on 196 degrees of freedom
Multiple R-squared:  0.8972,	Adjusted R-squared:  0.8956
F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

The following statistics describe this: Residual Standard Error (RSE), R^2 and RMSE.

R^2 for model that uses all three predictors = 0.8972

R^2 for model that uses TV and radio only = 0.8971

$$RSE = \sqrt{\frac{1}{n-p-1} RSS}$$

$$R^2 = Cor(Y, \hat{Y})^2$$

$$\mathrm{RMSE} = \sqrt{\frac{\sum (y_i - \hat{y}_i)^2}{N}}$$

```
> anova(model)
Analysis of Variance Table

Response: sales
           Df Sum Sq Mean Sq  F value Pr(>F)
TV          1 3314.6  3314.6 1166.7308 <2e-16 ***
radio       1 1545.6  1545.6  544.0501 <2e-16 ***
newspaper   1    0.1     0.1    0.0312 0.8599
Residuals 196  556.8     2.8
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Linear regression: Analysis

```
Residuals:
    Min      1Q  Median      3Q     Max
-8.8277 -0.8908  0.2418  1.1893  2.8292

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  2.938889   0.311908   9.422   <2e-16 ***
TV           0.045765   0.001395  32.809   <2e-16 ***
radio        0.188530   0.008611  21.893   <2e-16 ***
newspaper   -0.001037   0.005871  -0.177     0.86
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.686 on 196 degrees of freedom
Multiple R-squared:  0.8972,    Adjusted R-squared:  0.8956
F-statistic: 570.3 on 3 and 196 DF,  p-value: < 2.2e-16
```

3. How well does the model fit the data?

The following statistics describe this: Residual Standard Error (RSE), R^2 and RMSE.

$$RSE = \sqrt{\frac{1}{n-p-1}RSS}$$

$$R^2 = Cor(Y, \hat{Y})^2$$

$$\mathrm{RMSE} = \sqrt{\frac{\sum(y_i - \hat{y}_i)^2}{N}}$$

RSE	Predictors
3.26	TV
1.681	TV + radio
1.686	TV + radio + newspaper

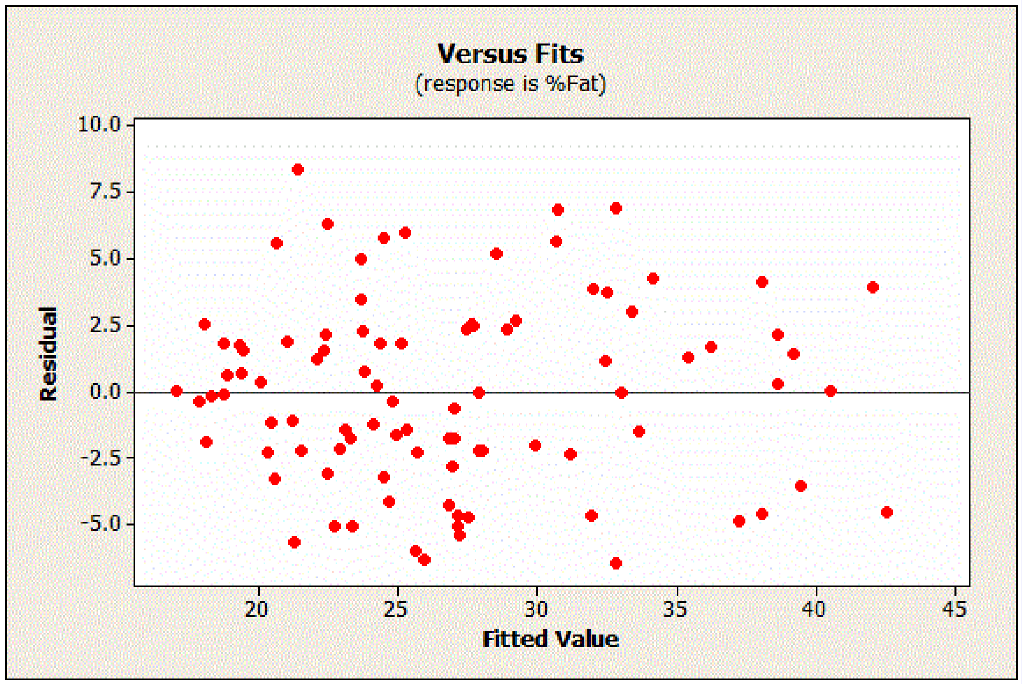
Linear regression: Analysis

Residual analysis is one more tool to see how well does the model fit the data.

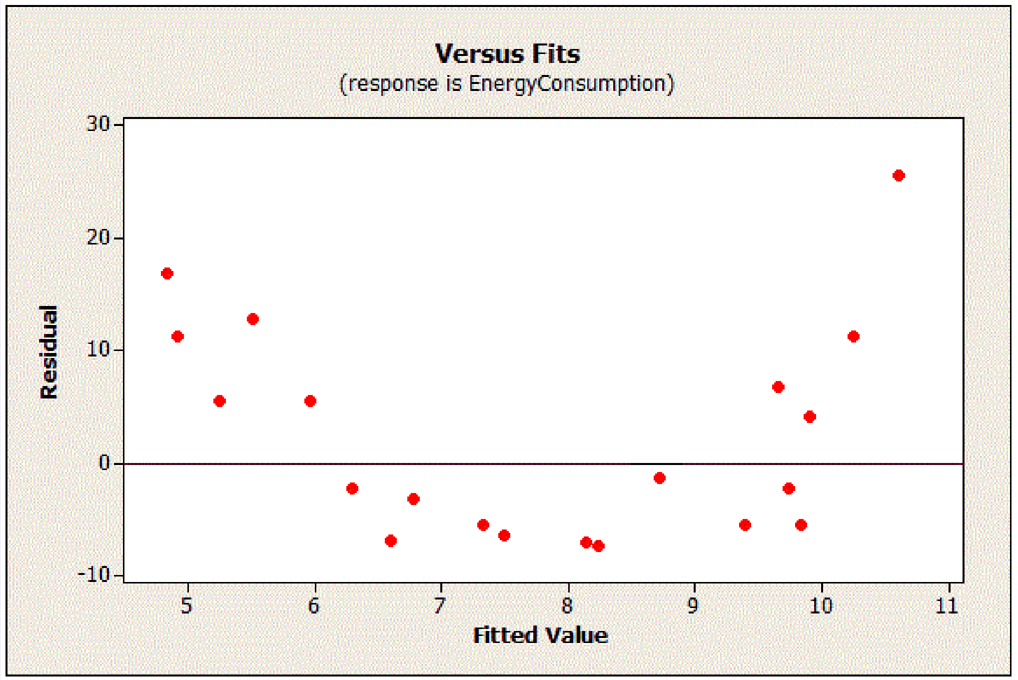
- Residuals should be homosceadastic (variance in the residual must not vary too much between low and high values of the fitted values).
- Centered on 0 through the range of fitted values ($\mu = 0.0$).
- Residuals should be normally distributed.
- Residuals must be uncorrelated to each other.

Linear regression: Analysis

Graphic source: http://blog.minitab.com/blog/adventures-in-statistics-2/why-you-need-to-check-your-residual-plots-for-regression-analysis



A good residual plot



A problematic residual plot

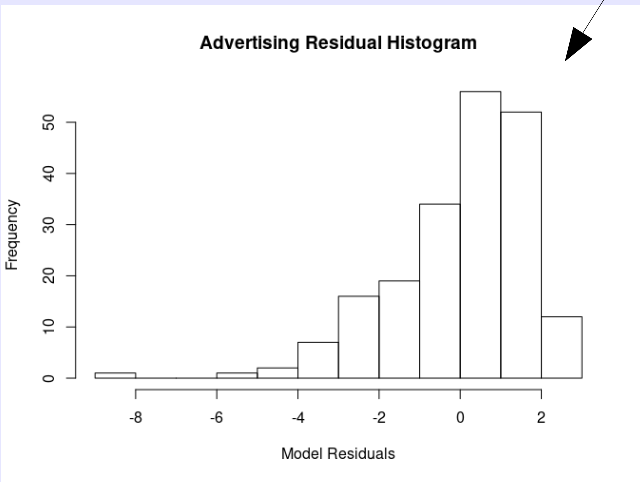
We can predict the residuals for fitted values. Residuals for 6.5-9.9 are negative, while those for 5, 6, and > 10 are positive.

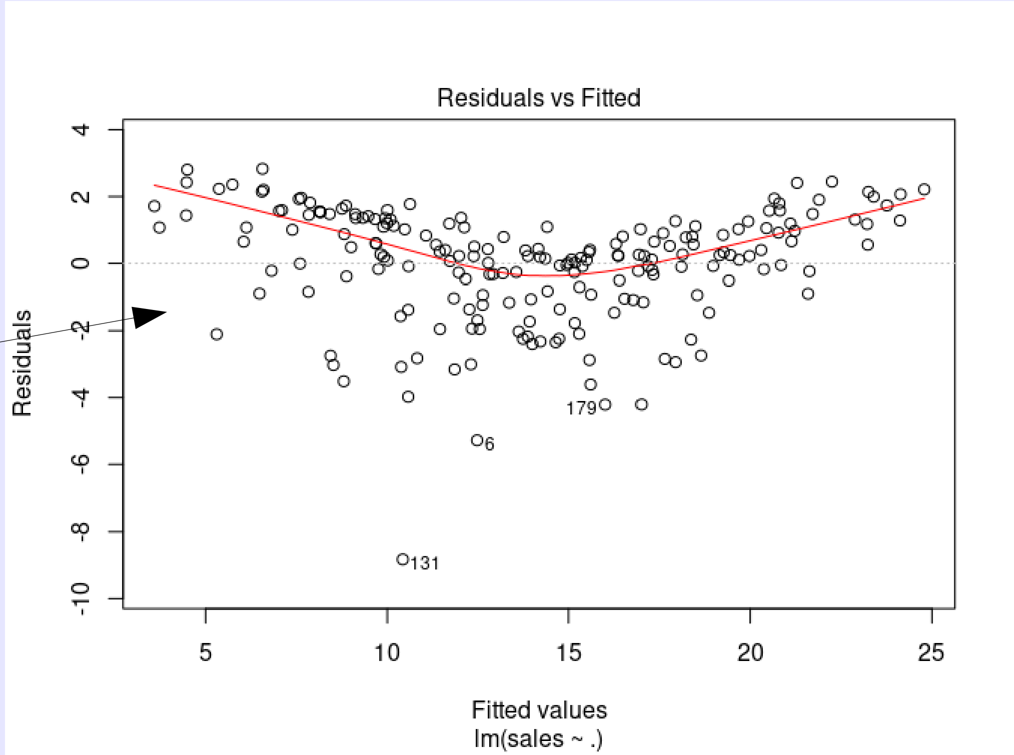
Linear regression: Analysis

What do the residual plot from the Advertising data look like?

```
# Plotting residuals ...
plot(model, 1) # The easy way using plot()
# Or the manual way
plot(model$fitted.values, model$residuals,
     xlab = "Fitted values\nlm(sales ~.)",
     ylab="Residuals",
     main="Residuals vs. Fitted");
abline(0, 0)
```

- Looks reasonably good as residuals appear homosceadastic and clustered around 0. (Though they don't appear to be normally distributed.)





- Note a slight convex shape (concave upward), which may indicate some non-linearity in the data.

Linear regression: Analysis

4. Given a set of predictor coefficiets (the β's), what response value should we predict and how accurate is our prediction?

- The least squares plane of a fitted model is an *approximation* to the true population regression plane: $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + ... + \hat{\beta}_n X_n \approx Y = f(X) = \beta_0 + \beta_1 X_1 + ... + \beta_n X_n$
- The approximation introduces some error.
- To account for the error, we compute confidence intervals in order to determine how close $\hat{Y}$ will be to $f(X)$.

Linear regression: Analysis

4. Given a set of predictor values (the β's), what response value should we predict and how accurate is our prediction?

To understand this, we need to understand:

- Significance level (α): Probability of making the wrong decision, given H_0 is true.
- Confidence interval: The range of results that would be expected to contain the population parameter of interest.
- Confidence level: Probability that if an experiment was repeated multiple times, results will be the same. Confidence level = 1 – α.

Linear regression: Analysis

Confidence Interval

Gallup poll before 2020 elections: Two in three Americans (66%) saying prior to the election "...that they are "very" or "somewhat confident" that votes will be cast and counted accurately across the country." The margin of sampling error was ±6 percentage points at the 95% confidence level.

(Here ±6 percentage points is the Margin of Error.)

Linear regression: Analysis

Confidence Level

Probability that if an experiment was repeated multiple times, results will be the same. Confidence level = 1 – α.

95% Confidence level means if the poll was to be repeated 20 times, Gallup would expect similar results 19 times (19/20 = 0.95).

What is the relationship between Confidence Level and Confidence Interval?

Linear regression: Analysis

Console ~/IIT/CS422/lectures/regression/

```
> # Let's see how we predict using the fitted model:
> # Find out what is our maximum sales:
> indx <- which.max(df$sales)
> # This will give you an index that we use to get that observation
> df[indx,]
       TV radio newspaper sales
176 276.9  48.9      41.8    27
```

Linear regression: Analysis

```
Console ~/IIT/CS422/lectures/regression/
> # Let's see how we predict using the fitted model:
> # Find out what is our maximum sales:
> indx <- which.max(df$sales)
> # This will give you an index that we use to get that observation
> df[indx,]
       TV radio newspaper sales
176 276.9  48.9      41.8    27
> # What happens if we double our TV budget to $500K, add ~20% to our radio
> # budget but keep the newspaper budget the same?
> predict.lm(model, data.frame(TV=500, radio=58.68, newspaper=41.8),
+            interval="confidence")
       fit      lwr      upr
1 36.84079 35.71472 37.96685
```

Asks R to calculate the CI at 95% level (default).

Linear regression: Analysis

Console ~/IIT/CS422/lectures/regression/

```
> # Let's see how we predict using the fitted model:
> # Find out what is our maximum sales:
> indx <- which.max(df$sales)
> # This will give you an index that we use to get that observation
> df[indx,]
       TV radio newspaper sales
176 276.9  48.9      41.8    27
> # What happens if we double our TV budget to $500K, add ~20% to our radio
> # budget but keep the newspaper budget the same?
> predict.lm(model, data.frame(TV=500, radio=58.68, newspaper=41.8),
+            interval="confidence")
       fit      lwr      upr
1 36.84079 35.71472 37.96685
> # Hmmm...we know from our analysis that newspaper is not a statistically
> # significant predictor.  (Sad, it should be!)  So, what if we zero out
> # the newspaper budget?
> predict.lm(model, data.frame(TV=500, radio=58.68, newspaper=0),
+            interval="confidence", level=0.99)
       fit     lwr      upr
1 36.88415 35.2061 38.56221
```

Note: You can specify the confidence level as a parameter

Linear regression: Analysis

Console ~/IIT/CS422/lectures/regression/

```
> # Let's see how we predict using the fitted model:
> # Find out what is our maximum sales:
> indx <- which.max(df$sales)
> # This will give you an index that we use to get that observation
> df[indx,]
       TV radio newspaper sales
176 276.9  48.9      41.8    27
> # What happens if we double our TV budget to $500K, add ~20% to our radio
> # budget but keep the newspaper budget the same?
> predict.lm(model, data.frame(TV=500, radio=58.68, newspaper=41.8),
+            interval="confidence")
       fit      lwr      upr
1 36.84079 35.71472 37.96685
> # Hmmm...we know from our analysis that newspaper is not a statistically
> # significant predictor.  (Sad, it should be!)  So, what if we zero out
> # the newspaper budget?
> predict.lm(model, data.frame(TV=500, radio=58.68, newspaper=0),
+            interval="confidence", level=0.99)
       fit     lwr      upr
1 36.88415 35.2061 38.56221
```

To get prediction without an interval, you can simply say:
 > res <- predict(model, newdata=x)
where x is a dataframe containing observations to be predicted.

Linear regression: Analysis

- Note:
  - Confidence interval: Tells you about the likely location of the true population parameter. For example, given that $100,000 is spent on TV advertising and $20,000 is spent on radio advertising in each city, the 95 % confidence interval is [10985, 11528].
  - Prediction interval (in R, to get this, specify interval="prediction" parameter to predict.lm()): This interval must account for both the uncertainty in knowing the value of the true population parameter, plus the variance of the data. Given that $100,000 is spent on TV advertising and $20,000 is spent on radio advertising in that city, the 95 % prediction interval is [7930, 14580].
  - Note that both intervals are centered at 11,256, but that the prediction interval is substantially wider than the confidence interval.

Linear regression: Analysis

- Factors that influence regression:
  - Multi-collinearity
  - In presence of multi-collinearity:
    - Cannot trust the learned coefficients (weights), thus making it tedious to assess the relative importance of predictors in explaining the variation caused to the response.
    - Standard errors of the affected coefficients are likely to be high.
      - Which also means that p-values for those coefficients become small, making it difficult to reject the null hypothesis.
- Resources:
  - Olsrr: Tools for building OLS regression models (see https://olsrr.rsquaredacademy.com/index.html).