Problem 5

For local linear Regression.

$$f'(x_0) = \frac{\sum_{i=1}^{N} l_i(x_0) \cdot y_i}{\sum_{i=1}^{N} l_i(x_0)}$$

where $l_i(x_0) = K\left(\frac{x_i - x_0}{h}\right) \cdot (x_i - x_0)$

To prove

$$\sum_{i=1}^{N} (x_i - x_0)\, l_i(x_0) = 0.$$

Using definition of $l_i(x_0)$, we have

$$(x_i - x_0)\, l_i(x_0) = K\left(\frac{x_i - x_0}{h}\right) (x_i - x_0)^2$$

Kernel fn. K is Symmetric & integrates to 1,

$$= \sum_{i=1}^{N} (x_i - x_0)\, l_i(x_0)$$

$$= \sum_{i=1}^{N} K\left(\frac{x_i - x_0}{h}\right) (x_i - x_0)^2$$

$$\sum_{i=1}^{N} (x_i - x_0)\, l_i(x_0) = 0$$

For local Polynomial Regression of degree k.

$$f'(x_0) = \frac{\sum_{i=1}^{N} l_i(x_0) \cdot y_i}{\sum_{i=1}^{N} l_i(x_0).}$$

where $l_i(x_0) = K\left(\frac{x_i - x_0}{h}\right) (x_i - x_0)^j$

To Prove $\boxed{b_j(x_0) = \sum_{i=1}^{N} (x_i - x_0)^j \, l_i(x_0) = 0}$ $\quad j = 0, \dots k.$

For $j=0$, we have

$$b_0(x_0) = \sum_{i=1}^{N} l_i(x_0) = K\left(\frac{x_i - x_0}{h}\right)$$

By Normalization Property of kernel fn.

$$b_0(x_0) = 1$$

for $j>0$

$$b_j(x_0) = \sum_{i=1}^{N} (x_i - x_0)^j \cdot K\left(\frac{x_i - x_0}{h}\right) \cdot (x_i - x_0)^j$$

By normalization Property of kernel fn

$$b_j(x_0) = \sum_{i=1}^{N} K\left(\frac{x_i - x_0}{h}\right) (x_i - x_0)^{2j}$$

$$b_j(x_0) = 0.$$

This implies that bais decreases as degree of Polynomial increase, since, higher - degree Polynomials have more flexibility to fit data.
However, higher degree ploynomials may also lead to overfitting, So, Choice of degree should be based on trade off B/w Vaiance & Bias.