Prob 4

Nadaraya Watson Kernel regression estimator is given by.

$$f(x) = \sum\left(K\left(\frac{(x-x_i)}{\lambda}\right)\right)y_i \Big/ \sum\left(K\left(\frac{(x-x_i)}{\lambda}\right)\right)$$

Gaussian kernal is given by

$$\cancel{K(u) = (1/\sqrt{2\pi})^{\frac{1}{2}} \exp}$$

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \quad \text{where}$$

$$u = \frac{(x-x_i)}{\lambda}.$$

By differentiating Nadaraya Watson kernel using Gaussian kernal.

$$f'(x) = d\left(\frac{\sum\left(K\left(\frac{x-x_i}{\lambda}\right)\right)\cdot y_i}{\sum\left(K\left(\frac{x-x_i}{\lambda}\right)\right)}\right) dx.$$

Gaussian kernal differenation with x.

$$\frac{dK(u)}{dx} = d\left(1/\sqrt{2\pi}\right)e^{-\frac{u^2}{2}}\, dx$$

$$= \left(-\frac{u}{\lambda}\right)\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{u^2}{2}}$$

From Nadaraya - Watson elimator lets differentiate Numerator & denominator with respect to x.

Numerator =

$$\sum \left(d\left(k\left(\frac{x-x_i}{\lambda}\right)\right) y_i \, dx \right.$$

$$= \sum \left(\frac{-u}{\lambda}\right)\left(\frac{1}{\sqrt{2\pi}}\right) e^{-u^2/2} \cdot y_i$$

Denominator =

$$\sum \left(d\left(k\left(\frac{x-x_i}{\lambda}\right)\right) dx \right.$$

$$= \sum \left(\frac{-u}{\lambda}\right)\left(\frac{1}{\sqrt{2\pi}}\right) e^{-u^2/2},$$

Now we apply quotient rule to find $f'(x)$

$$f'(x) = \left(\sum \left(\frac{-u}{\lambda}\right)\left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{u^2}{2}} \cdot y_i \right) \sum \left(k\left(\frac{x-x_i}{\lambda}\right) y_i \right) \cdot$$

$$\sum \left(\frac{-u}{\lambda}\right)\left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{u^2}{2}} \Big) \Big/ \sum k\left(\frac{x-x_i}{\lambda}\right)\Big)^2 .$$

Since above expression involves continous fn. & sums of continous fn. we can conclude that Nadaraya-Watson kernel smooth with Gaussian kernel is differentiable.

As for Epanechnikov kernel is given by

$$K(u) = \frac{3}{4}(1-u^2) \text{ for } |u| \leq 1 \text{ \& } 0 \text{ otherwise}$$

Regarding Epanechnikov kernel with an adaptive nearest - neighbor bandwidth $\lambda(x_0)$

Since kernal fn itself is not differentiable at $u = \pm 1$, the Same Conclusion applies.

Nadaraya-Watson estimator using Epanechnikov kernel with an adaptive bandwidth is not guaranteed to be differentiable everywhere.