MATH 569

Take home Exam

Problem 1:

Sol: To derive equation (5.55) from (5.54)

from (5.54), objective function ~~is~~ is in Matrix form.

$$J = (y - Ka)^T (y - Ka) + \lambda a^T K a$$

by taking derivate;

$$\frac{dJ}{da} = -2K^T (y - Ka) + 2\lambda K a = 0.$$

Solving for a,

$$a = (K^T K + \lambda I)^{-1} K^T y$$

by substituting this expression for a into equation for Prediction $\hat{a} = Ka$,

$$\hat{a} = K (K^T K + \lambda I)^{-1} K^T y$$

Using matrix identity

$$\hat{a} = K^+ y$$

finally, Substituting this ~~eqt~~ expression for $\hat{a}$ into original equation for $\hat{a} = (K + \lambda I)^{-1} y$

Prob 2
Sol:

Deriving MSE for least Square estimator.

least Square estimator of β =

$$\hat{\beta} = (X^T X)^{-1} X^T y.$$

least Square estimate of θ =

$$\hat{\theta} = \alpha^T (X^T X)^{-1} X^T y.$$

Mean Squared error

$$MSE(\hat{\theta}) = E\left[(\hat{\theta} - \theta)^2\right]$$

$$= E\left[(\alpha^T (X^T X)^{-1} X^T (y - X\beta) - \alpha^T \hat{\beta})^2\right]$$

Using linearity of expectation.

$$E[y - X\beta] = 0.$$

$$MSE(\hat{\theta}) = \alpha^T (X^T X)^{-1} X^T E\left[(y - X\beta)(y - X\beta)^T\right] X (X^T X)^{-1} \alpha$$

$$MSE(\hat{\theta}) = \alpha^T (X^T X)^{-1} X^T (\sigma^2 I + X Var(\beta) X^T) X (X^T X)^{-1} \alpha$$

where
σ^2 is Variance of error term
$Var(\beta)$ is Variance of Covariance matrix

Now, lets consider MSE $\{\hat{\theta}\} = c^T y$.

$$MSE(c^T y) = E[(\theta - \hat{\theta})^2]$$

$$= E[(\alpha^T \beta - c^T y)^2]$$

$$= E[(\alpha^T (\beta - \hat{\beta}) + \tau^T \hat{\beta} - c^T y)^2]$$

$$= E[(\alpha^T (\beta - \hat{\beta}))^2] + E[((\alpha^T \hat{\beta} - c^T y)^2] +$$
$$2E[\alpha^T (\beta - \hat{\beta})(\alpha^T \hat{\beta} - c^T y)]$$

$$= \alpha^T Var(\beta) \alpha + E[\alpha^T \hat{\beta} - c^T y)^2]$$

$$= \alpha^T (X^T X)^{-1} \alpha \tau^2 + E[(c^T y - \alpha^T \hat{\beta})^2]$$

$$= \alpha^T (X^T X)^{-1} \alpha \cancel{\sigma}^2 + c^T Var(y) c$$

$$= \alpha^T (X^T X)^{-1} \alpha \tau^2 + c^T \tau^2 I c$$

$$= (\alpha^T (X^T X)^{-1} \alpha + c^T c) \tau^2$$

Therefore

$$MSE(\tau^T \hat{\beta}) - MSE(c^T y) = \alpha^T (X^T X)^{-1} \alpha \tau^2 - (\alpha^T (X^T X)^{-1} \alpha$$
$$+ c^T c) \tau^2 c$$

$$= -c^T c \sigma^2$$

Since $c^T c$ is non-negative we have.

$$MSE(\alpha^T \hat{\beta}) - MSE(c^T y) \leq 0$$

which means $MSE(\alpha^T \hat{\beta})$ is no greater than MSE of any Unbaised estimator.