Prob 2

a) $\langle K(\cdot, x_i), f\rangle_{H_K} = f(x_i)$

To prove $\langle K(\cdot, x_i), f\rangle_{H_K} = f(x_i)$ we will use reproducing property of kernel $K(x,y)$ & inner product definition in the hilbert space H_K given by.

$$f(x) = \langle K(x_i), f\rangle H_K \quad -(1)$$

let's substitute $x = x_i$ in above eq.

$$f(x_i) = \langle K(x_i, \cdot), f\rangle H_K \quad -(2)$$

By using Symmetry Property $(K(x,y) = K(y,x))$

$$f(x_i) = \langle K(\cdot, x_i), f\rangle H_K \quad -(3)$$

Now,

$$\langle K(\cdot, x_i), f\rangle H_K =$$

$$= \sum_{i=1}^{\infty} \frac{c_i}{\gamma_i} \langle K(\cdot, x_i), \phi_i(\cdot)\rangle$$

$$= \sum_{i=1}^{\infty} \frac{c_i}{\gamma_i} [\gamma_i \phi_i(x_i)]$$

with $\gamma_i \geq 0$
$$\sum_{i=1}^{\infty} \gamma_i^2 < \infty$$

Hence, elements of H_k could have expression in terms of this egien fn.

$$\langle k(\cdot, x_i), f \rangle_{H_k} = \sum_{i=1}^{\infty} \frac{c_i}{r_i} \left[r_i \phi_i(x_i) \right]$$

$$= \sum_{i=1}^{\infty} c_i \phi_i(x_i)$$

$$\Rightarrow \boxed{\langle k(\cdot, x_i), f \rangle_{H_k} = f(x_i)}$$

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b)

$$\langle k(\cdot, x_i), k(\cdot, x_j) \rangle_{H_k} = k(x_i, x_j)$$

To show above eq. we need to used inner Product of function $k(\cdot, x_i)$ & $k(\cdot, x_j)$ in reproducing Kernal Hilbert Space, is equal to $k(x_i, x_j)$

let $h(x) = k(x, x_i)$ & $g(x) = k(x, x_j)$
By Reproducing Kernal Hilbert Space we have,

$$f(x) = \sum_i \alpha_i k(x, x_i) \quad \text{—(4)}$$

where $\alpha_i \rightarrow$ coefficient of expansion of $f(x)$
Using (4), we can rewrite $\langle f, h \rangle_{H_k}$ as

$$\langle f, h \rangle_{H_k} = \langle \sum_i \alpha_i k(\cdot, x_i) \cdot k(\cdot, x_i) \rangle$$

applying reproducing Property of RKHS.

we get

$$\langle f, h \rangle_{H_K} = \alpha_i \quad — (5)$$

Similarly $\langle f, g \rangle_{H_K} = \langle \sum_{j=} \alpha_j K(\cdot, x_j) K(\cdot, x_j) \rangle$

$$\langle f, g \rangle_{H_K} = \alpha_j . \quad — (6)$$

Now using fact that $K(x, y)$ satisfies the condition 5.45 from textbook

$$\sum_{ij} \alpha_i \alpha_j K(x_i, x_j) = \| f \|^2_{H_K}$$

~~The~~

$$\langle K(\cdot, x_i), K(\cdot, x_j) \rangle_{H_K} = \sum_{k=1}^{\infty} \frac{1}{\gamma_k} \langle K(\cdot, x_i), K(\cdot, x_j) \rangle$$

$$= \sum_{k=1}^{\infty} \frac{1}{\gamma_k} \left[\gamma_k (\gamma_k \phi_k (x_i) \gamma_k \phi_k (x_j)) \right.$$

with $\gamma_i \geq 0$, $\sum_{i=1}^{\infty} \gamma_i^2 < \infty$.

we can say that elements of H_K could have a expansion given in term of eigen fu.

$$\therefore \langle K(\cdot, x_i), K(\cdot, x_j) \rangle_{H_K} = \sum_{k=1}^{\infty} \gamma_k \phi_k (x_i) \phi_k (x_j)$$

$$= K(x_i, x_j)$$

$$\Rightarrow \boxed{\therefore \langle K(\cdot, x_i), K(\cdot, x_j) \rangle_{H_K} = K(x_i, x_j)}$$

c) if $g(x) = \sum_{i=1}^{N} \alpha_i K(x, x_i)$ then

$$J(g) = \sum_{i=1}^{N} \sum_{j=1}^{N} K(x_i, x_j) \alpha_i \alpha_j$$

Our goal is to minimize the empirical risk.

$$\min J(g) = \frac{1}{n} \sum_{i=1}^{N} L(y_i, g(x_i)) + \lambda \|g\|^2_{H_K}$$

Using regularized Parameter, we control trade off between Empirical risk & complexity of fn. $g(x)$

$\|g\|^2_{H_K}$ is Norm fn.

we can express fn. $g(x)$ in terms of kernel fn. $K(x,y)$ & the coefficient α_i as

$$g(x) = \sum_{i=1}^{N} \alpha_i K(x, x_i) \quad \text{—— (5)}$$

Plugging (5) in Regularized Empirical risk.

$$J(g) = \frac{1}{N} \sum_{i=1}^{N} L\left(y_i, \sum_{j=1}^{N} \alpha_j K(x_i, x_j)\right) + \lambda \left\| \sum_{i=1}^{N} \alpha_j K(\cdot, x_i) \right\|^2_{H_K}$$

Applying linearity of inner Product in RHS

$$\left\| \sum_{i=1}^{N} \alpha_i K(\cdot, x_i) \right\|^2_{H_K} = \left(\sum_{i=1}^{N} \alpha_i K(\cdot, x_i), \sum_{j=1}^{N} \alpha_j K(\cdot, x_j) \right)$$

Solving above where basis fn. $h_i(x) = K(x, x_i)$ as representer of evaluation at x_i in H_K, since for $f \in H_K$

It can easily seen that

$$\langle K(\cdot, x_i), f \rangle_{H_K} = f(x)$$

Similarly

$$\langle K(\cdot, x_i), K(\cdot, x_j) \rangle_{H_K} = K(x_i, x_j) \quad \text{—⑥}$$

$$\Rightarrow \boxed{\left(\sum_{i=1}^{N} \alpha_i K(\cdot, x_i) \sum_{j=1}^{N} \alpha_j K(\cdot, x_j)\right) = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j K(x_i, x_j)}$$

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d)

$$\sum_{i=1}^{N} L(y_i, \bar{g}(x_i)) + \lambda J(\bar{g}) \geq \sum_{i=1}^{N} L(y_i, g(x_i)) + \lambda J(g)$$

given $\bar{g}(x) = g(x) + p(x)$

because $p(x)$ orthogonal in H_K to each of $K(x_i, x_j)$

we have

$$\langle K(\cdot, x_i), P(x) \rangle_{H_K} = 0$$

$$\bar{g}(x_i) = \langle K(\cdot, x_i), \bar{g}(x_i) \rangle_{H_K}$$

$$= \langle K(\cdot, x_i), g(x_i) \rangle_{H_K} + \langle K(\cdot, x_i), P(x_i) \rangle_{H_K}$$

$$= g(x_i)$$

As it is given $p(x)$ orthogonal in H_K to each $K(x, x_i)$

$$J(\bar{g}) = \langle \bar{g}(x), \bar{g}(x) \rangle_{H_K}$$

$$= \langle g(x), g(x) \rangle_{H_K} + 2\langle g(x), P(x) \rangle_{H_K} + \langle P(x), P(x) \rangle_{H_K}$$

$$J(\bar{g}) = J(g) + J(p)$$

Now $\sum_{i=1}^{N} L(y, \bar{g}(x_i)) + \lambda J(\bar{g})$

$$= \sum_{i=1}^{N} L(y_i, g(x_i)) + \lambda(J(g) + J(p))$$

$$\geq \sum_{i=1}^{N} L(y, g(x_i)) + \lambda J(g)$$

$$\sum_{i=1}^{N} L(y_i, \bar{g}(x_i)) + \lambda J(\bar{g}) \geq \sum_{i=1}^{N} (y_i, g(x_i)) + \lambda J(g)$$

Now if $P(x) = 0$, $J(\bar{g}) = J(g)$

$$\Rightarrow \boxed{\sum_{i=1}^{N} L(y_i, \bar{g}(x_i)) + \lambda J(\bar{g}) = \sum_{i=1}^{N} L(y_i, g(x_i)) + \lambda J(g)}$$