MATH 569

Homework 4

Problem 1:

Truncated Power Series representation for Cubic ~~series~~ Splines with K interior knots can be written as

$$f(x) = \sum_{j=0}^{3} \beta_j x^j + \sum_{k=1}^{K} \theta_k \cdot (x - \xi_k)^3 \quad -①$$

The natural boundary condition for natural cubic splines require

$$f''(x_0) = 0 \ \& \ f''(x_{K+1}) = 0$$

Using eq ① & derivating $f(x)$ twice

$$f''(x) = 6 \cdot \sum_{k=1}^{K} \theta_k (x - \xi_k).$$

$$\therefore \theta_1 = \theta_K = 0$$

$$\Rightarrow \sum_{k=1}^{K} \theta_k = 0 \quad -②.$$

Using eq ①

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \sum_{k=1}^{K} \theta_k (x - \xi_k)^3$$

Taking 2nd derivative of above

$$f''(x) = 2 \cdot \beta_2 + 6 \sum_{k=1}^{K} \theta_k \cdot (x - \xi_k)$$

$\therefore \beta_2 = 0$ — (3)

Taking 3^{rd} derivate of $f(x)$ from eq (1)

$$f'''(x) = 6\beta_3 + 6\sum_{k=1}^{K} \theta_k$$

$$\therefore \beta_3 = 0 \quad \text{— (4)}$$

Using eq (1), taking derivate of $f(x)$ gives

$$f'(x) = \beta_1 + 3\beta_2 x^2 + 3\beta_3 x^2 + 3\sum_{k=1}^{K} \theta_k (x-\xi_k)^2$$

Multiplying both sides with $(x-\xi_k)$

$$(x-\xi_k)\cdot f'(x) = \beta_1(x-\xi_k) + 3\beta_2 x^2(x-\xi_k) + 3\beta_3 x^2(x-\xi_k) + 3\theta_k (x-\xi_k)^3$$

Integrating both sides over $[\xi_k, \xi_{k+1}]$ interval

we get,

$$\int_{\xi_k}^{\xi_{k+1}} (x-\xi_k) f'(x)dx =$$

$$= \beta_1/2 (\xi_{k+1})^2 - \frac{\beta_1}{2}\xi_k^2 + \frac{\beta_2}{2}(\xi_{k+1})^4 - \frac{\beta_2}{2}$$

$$\xi_k^4 + \frac{\beta_3}{2}(\xi_{k+1})^4 - \frac{\beta_3}{2}\xi_k^4 + \frac{\theta_k}{4}(\xi_{k+1} - \xi_k)^4$$

Using natural boundary $f'(x_0) = f'(x_{K+1}) = 0$.

we get

$\beta_1 = 0$ &

$$\beta_2 (x_{K+1})^2 + \beta_3 (x_{K+1})^3 + \frac{\theta_K}{4}(x_{K+1} - \xi_K)^4 = 0$$

$$\therefore \beta_3 (x_{K+1})^3 + \frac{\theta_K}{4}(x_{K+1} - \xi_K)^4 = 0.$$

This implies $\theta_K = 0$, or $\beta_3 = 0$

if $\theta_K = 0$ then eq (2) we have,

$$\sum_{K=1}^{K} \theta_K = \theta_1 + \theta_2 + \cdots \theta_{K-1} = 0.$$

This implies that at least one θ_K is zero, which contradicts assumption that all interior knots are non-degenerate.

Therefore, we must have

$$\beta_3 = 0 \quad -(5)$$

Using (3), (4), (5) equations in eq (1).

$$f(x) = \beta_0 + \beta_1 x + \sum_{K=1}^{K} \theta_K (x - \xi_K)^3$$

where $\beta_2 = \beta_3 = 0$ & atleast one θ_K is zero we can now derive basis fn's of natural cubic Splines with K Interior Knots, Using Constraints we derived earlier.

$$d_k(x) = \left((x-\xi_k)^3 - (x-\xi_K)^3\right) \Big/ (\xi_K - \xi_k)$$

Using above fun. we ~~get~~ can truncated Power Series representation as

$$f(x) = \beta_0 + \beta_1 x + \sum_{k=1}^{K} \theta_k \cdot d_k(x) (\xi_K - \xi_k)^3$$

Basis fn for Natural cubic Splines with K interior Knots are.

$$N_1(x) = 1$$

$$N_2(x) = x$$

$$N_{k+2}(x) = d_k(x) - d_k(K+1)$$

It can be shown that basis functions $N_1(x)$, $N_2(x)$ & $N_{k+2}(x)$ are linearly independent and span the space of all natural cubic splines with K interior knots.

Therefore, Any natural cubic spline with K interior Knots can be written as a linear combination of these basis fn.