Lab 4 Report

Robotics Integration Group Project I

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Abstract

See Resources on github.com/RamessesN/Robotics_MIT.

1 Introduction

2 Procedure

2.1 Individual Work

2.1.1 Single-segment trajectory optimization

Consider the following minimum velocity (r = 1) single-segment trajectory optimization problem:

$$\min_{P(t)} \int_0^1 \left(P^{(1)}(t)\right)^2 dt \tag{1}$$

s.t.

$$P(0) = 0, \tag{2}$$

$$P(1) = 1, \tag{3}$$

with $P(t) \in \mathbb{R}[t]$, i.e., $P(t)$ is a polynomial function in t with real coefficients:

$$P(t) = p_N t^N + p_{N-1} t^{N-1} + ... + p_1 t + p_0 \tag{4}$$

Note that because of constraint (2) $P(0) = p_0 = 0$, and we can parametrize $P(t)$ without a scalar part p_0.

1. **Suppose we restrict $P(t) = p_1 t$ to be a polynomial of degree 1, what is the optimal solution of problem (1)? What is the value of the cost function at the optimal solution?**

   $\because P(t) = p_1 t$

   Let $t = 1 \therefore P(1) = p_1 \cdot 1 = p_1$

$\because P(1) = 1 \therefore p_1 = 1$

$\therefore$ optimal solution: $P(t) = t$.

$\because P(t) = t \therefore P^{(1)}t = \frac{d}{dt}t = 1$

$\therefore$ Cost $= \int_0^1 (1)^2 dt = 1$.

2. **Suppose now we allow $P(t)$ to have degree 2, i.e., $P(t) = p_2t^2 + p_1t$.**

- **Write $\int_0^1 \left(P^{(1)}(t)\right)^2 dt$, the cost function of problem (1), as p^TQp, where $p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ and $Q \in S^2$ is a symmetric 2×2 matrix.**

  $\because P(t) = p_2t^2 + p_1t \ \therefore P^{(1)}(t) = 2p_2t + p_1$

  $\therefore$ Cost $= \int_0^1 (2p_2t + p_1)^2 dt = p_1^2 + 2p_1p_2 + \frac{4}{3}p_2^2$

  In order to write into a 2×2 matrix as p^TQp, we have

  $[p_1, p_2] \cdot \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = p_1^2 + 2p_1p_2 + \frac{4}{3}p_2^2$

  $\therefore Q_{11} = 1, Q_{22} = \frac{4}{3}$

  $\because Q_{12} = Q_{21} \ \therefore 2Q_{12} = 2$

  $\therefore Q_{12} = Q_{21} = 1 \Rightarrow Q = \begin{bmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{bmatrix}$.

- **Write $P(1) = 1$, constraint (3), as $Ap = b$, where $A \in \mathbb{R}^{1\times 2}$ and $b \in \mathbb{R}$.**

  $\because P(1) = 1$
  $\therefore P(1) = p_2(1)^2 + P_1(1) = p_1 + p_2 = 1$

  $\because Ap = b$ and $p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \ \therefore A = [1\ 1], b = 1$.

- **Solve the Quadratic Program (QP):**

$$\min_p p^TQp \ s.t. \ Ap = b \tag{5}$$

  You can solve it by hand, or you can solve it using numerical QP solvers (e.g., you can easily use the `quadprog` function in Matlab). What is the optimal solution you get for $P(t)$, and what is the value of the cost function at the optimal solution? Are you able to get a lower cost by allowing $P(t)$ to have degree 2?

  $\because$ we have $\min_p p^TQp \Leftrightarrow \min_{p_1,p_2}\left(p_1^2 + 2p_1p_2 + \frac{4}{3}p_2^2\right)$

  and $Ap = b \Leftrightarrow p_1 + p_2 = 1$

  Let $p_1 = 1 - p_2 \therefore$ Cost $= (1-p_2)^2 + 2(1-p_2)p_2 + \frac{4}{3}p_2^2 = 1 + \frac{1}{3}p_2^2$

In order to make `Cost` minimum $\Rightarrow \begin{cases} p_1=1 \\ p_2=0 \end{cases} \therefore \text{Cost}_{\text{minimal}} = 1.$

No, it remains the same value even though `P(t)` has degree 2.

3. **Now suppose we allow $P(t) = p_3t^3 + p_2t^2 + p_1t$:**

- **Let $p = [p_1 . p_2, p_3]^T$, write down $Q \in S^3, A \in \mathbb{R}^{1\times 3}, b \in \mathbb{R}$ for QP (5).**

$\because P(t) = p_3t^3 + p_2t^2 + p_1t$

$\therefore P_t^{(1)} = 3p_3t^2 + 2p_2t + p_1$

$\therefore \left[P_t^{(1)}\right]^2 = 9p_3^2t^4 + 4p_2^2t^2 + p_1^2 + \underbrace{12p_2p_3t^3}_{p_2p_3} + \underbrace{6p_1p_3t^1}_{p_1p_3} + \underbrace{4p_1p_2t}_{p_1p_2}$

$\therefore$ we have

$$\begin{cases} \text{item } p_3^2 : \int_0^1 9t^4dt = \frac{9}{5} \\ \text{item } p_2^2 : \int_0^1 4t^2dt = \frac{4}{3} \\ \text{item } p_1^2 : \int_0^1 1dt = 1 \\ \text{item } p_2p_3 : \int_0^1 12t^3dt = 3 \\ \text{item } p_1p_3 : \int_0^1 6t^2dt = 2 \\ \text{item } p_1p_2 : \int_0^1 4tdt = 2 \end{cases}$$

$$\therefore Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{9}{5} \end{bmatrix}$$

$\because p_3(1)^3 + p_2(1)^2 + p_1(1) = 1 \Rightarrow 1 \cdot p_1 + 1 \cdot p_2 + 1 \cdot p_3 = 1$
$\therefore A = [1 \ 1 \ 1], \ b = 1.$

- **Solve the QP, what optimal solution do you get? Do this example agree with the result we learned from Euler-Lagrange equation in class?**

From the above, we have the path that connects two points and minimizes the change in speed (energy) is always a straight line. That's regardless of the inclusion of higher-degree terms like t^2 or t^3, the optimization drives their coefficients to 0. The curve connecting the two points that minimizes the velocity cost is always a straight line. Consequently, the value of the cost function remains 1.

Yes. By `Euler-Lagrange` equation, we have $\frac{\partial L}{\partial P} - \frac{d}{dt}\frac{\partial L}{\partial P'} = 0$

Since $L = (P')^2 \ \therefore \frac{d}{dt}(2P') = 0 \Rightarrow P''(t) = 0$

The condition $P''(t) = 0$ implies that the optimal function must be linear. The QP result is indeed a linear function, which confirms that the theroretical result derived from calculus of variations.

4. **Now suppose we are interested in adding one more constraint to problem (1):**

$$\min_{P(t)} \int_0^1 \left(P^{(1)}(t)\right)^2 dt,\ s.t. P(0) = 0,\ P(1) = 1,\ P^{(1)}(1) = -2 \tag{6}$$

Using the QP method above, find the optimal solution and optimal cost of problem (6) in the case of:
- $P(t) = p_2t^2 + p_1t$**, and**
- $P(t) = p_3t^3 + p_2t^2 + p_1t$**.**

> **Case I. If** $P(t) = p_2t^2 + p_1t$**,**

$\begin{cases} \text{For } P(1)=1 : p_1+p_2=1 \\ P^{(1)}(1)=-2 : p_1+2p_2=-2 \end{cases}$

$\therefore$ we have $\begin{cases} p_1=4 \\ p_2=-3 \end{cases}$

$\therefore \begin{cases} P(t)=-3t^2+4t \\ \text{Cost } = p_1^2+2p_1p_2+\frac{4}{3}p_2^2=4 \end{cases}$

> **Case II. If** $P(t) = p_3t^3 + p_2t^2 + p_1t$**,**

$\begin{cases} \text{For } P(1)=1 : p_1+p_2+p_3=1 \\ P^{(1)}(1)=-2 : p_1+2p_2+3p_3=-2 \end{cases}$

$\therefore \begin{cases} p_1=4+p_3 \\ p_2=-3-2p_3 \end{cases}$

From (3), we have: $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{9}{5} \end{bmatrix}$

$$p = \begin{bmatrix} 4+p_3 \\ -3-2p_3 \\ p_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}}_{p_{\text{base}}} + p_3 \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}_{d}$$

For $\text{Cost}(p_3) = Ap_3^2 + Bp_3 + C$, we have $\begin{cases} A=d^TQd \\ B=2p_{\text{base}}^TQd \end{cases}$

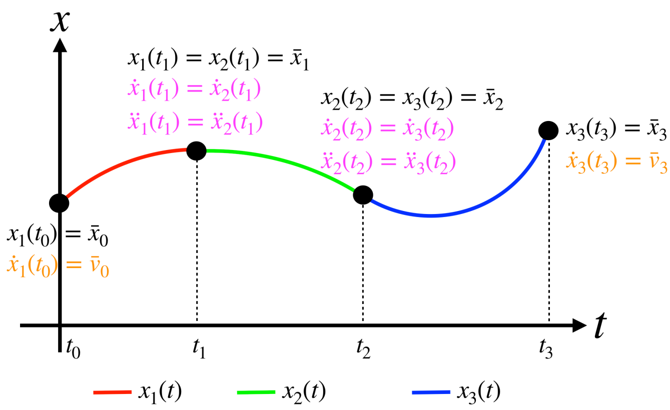
$$\therefore Qd = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{9}{5} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{6} \\ -\frac{1}{5} \end{bmatrix}$$

$\therefore A = \frac{2}{15},\ B = 1 \ \therefore \text{Cost} = \frac{2}{15}p_3^2 + p_3 + 4$

In order to make `Cost` minimum $\Rightarrow p_3 = -\frac{15}{4} \Rightarrow \text{Cost}_{\text{minimal}} = \frac{17}{8}$.

2.1.2 Multi-segment trajectory optimization

1. **Assume our goal is to compute the minimum snap trajectory ($r = 4$) over k segments. How many and which type of constraints (at the intermediate points and at the start and end of the trajectory) do we need in order to solve this problem? Specify the number of waypoint constraints, free derivative constraints and fixed derivative constraints.**



2. **Can you extend the previous question to the case in which the cost functional minimizes the r-th derivative and we have k segments?**

2.2 Team Work

2.2.1 Drone Racing

3 Reflection and Analysis

4 Conclusion

5 Source Code

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