An Historical Account of Abstractionist Mathematical Structuralism

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A focus on objects

- Identity of objects
  - Problem of Identity of Indiscernibles
  - Incomplete Objects
  - No non-structural Properties
- Structural Properties
  - Definability
  - Invariance
- Implicit Definitions
- Ontology of mathematical objects in structuralism
  - Indispensability Argument

Traditional View of Relations

It is implicitly understood that relations are extensionally defined over objects:

- A differs than B
- C differs than D

If the relation “differs than” is extensionally defined then the above two relations are different in that their identity is based on the relata, the objects.

Intensional View of Relations

Russell (1903) *Principles of Mathematics*:

- "Relations do not have instances but are strictly the same in all propositions" (p.52)

Implies that relations have their own identity independent of their relata.

- A differs than B
- C differs than D

The relation "differs than" is the same in both propositions.

By looking at both, the relations and the objects, the history of mathematics provides strong evidence for an abstractionist view of mathematical structuralism.

- Relations must come prior to the structure

Abstracting Mathematical Structures

- Relational abstraction: Extensionally defined relations and intensionally defined objects that are abstracted from some physical state of affairs.
    - Solar system with gravitational relations between the objects

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    - $F = ma$

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- Relational Generalisation: Remove object ontology, define the relations intensionally and generalise them.
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- Categorical Axiomatisation: Intensionally defined relations are used to make (define) a structure and the objects are extensionally defined.

Relational Abstraction: Ancient Greek Mathematics

In this initial step, one needs to abstract from states of affairs to produce a system with abstract objects, whose relations are extensionally defined and results are constrained by the ontology of the objects.

Ancient Greek Mathematics: Monads and Magnitudes

- Mathematical Objects:
    - The monad: Indivisible units that constitutes the domain of arithmetic in the abstract
    - The magnitude: Continuous lines that are the domain of geometry

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- Mathematical Objects:
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- Object Definition
    - No account for their definition in mathematical practice (Mancosu 1996)
    - Can be accounted for in philosophical works (Molland 1991, Klein 1992)
    - Always showed ties to the concrete (Netz 1999, Dunham 1990)
    - Abstracted from the concrete (Klein 1992)

Ancient Greek Mathematics: Theory of Proportion

- About the relations multitudes (of monads) or magnitudes possess between each other
  - These are not mathematical objects
  - ½ is not a multitude or number, it is a relation between two numbers (one monad and two monads).
  - π is not a number or an object, it is a relation, a ratio between two magnitudes (circumference and diameter).

- Has no separate existence from the domain it is based on (Klein 1992)
  - It is beholden to the ontology of mathematical objects

Ancient Greek Mathematics: Theory of Proportion

Results obtained limited by ontology

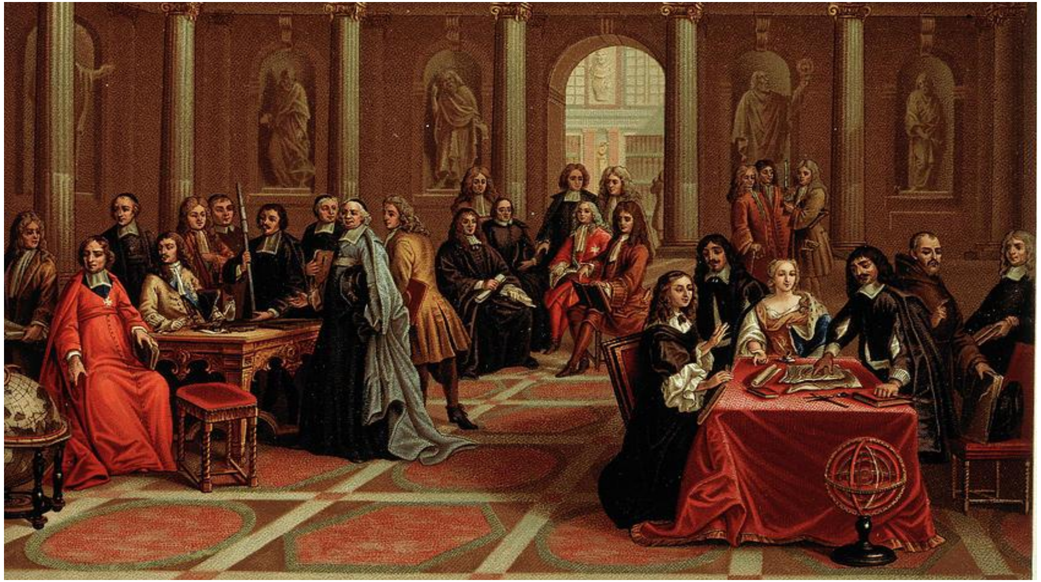
- Provided a plethora of results for geometry due incommensurable magnitudes
- Highly restricted in arithmetic
    - Irrational or negative numbers are not admissible due to ontology
    - Given its discrete nature results were limited to ensure acceptable answers (Malet 2006)
    - The only acceptable solutions to arithmetical problems were positive integers (Bos 2001)

Relational Abstraction

- Abstract objects and relations from physical states of affairs
- Objects are intensionally defined and the relations extensionally defined over the objects
- Relations are dependent, both ontologically and methodologically on the objects and systems they are in.
- Results are rejected if they are meaningless, ontologically speaking.
- Such as the case with parabolic trajectory and the rejection of negative distances

Relational generalisation: 17th century mathematics

- The removal of the ontological commitments of the objects.
- 17th Century mathematics was characterised by three main ideas (Mahoney 1980):
  - Symbolism
  - Reliance on relations (or order)
  - Lack of object ontology



17th century mathematics: Ontological confusion

- Translations of Euclid's elements and other Greek texts arrived to Europe in the 16th century. While the Greek methods became know, their ontological background was not.
- Attempts to make sense of it
    - Vieta attempted to rid Greek mathematics of its, as he called it, Arab filth, claiming at the end that he rediscovered the hidden Greek gold (Klein 1992, pp. 153-154)
    - "The ancient mathematicians used to employ only synthesis [as opposed to algebraic analysis] in their writings, not because they were simply ignorant of the other, but, as I see it, because they made so much of it that they reserved it as a secret for themselves alone" (Descartes in Mahoney 1980,p. 149).
    - Peter Ramus assumed that certain Euclidean definitions were defective, and he misinterpreted "every quantity and ratio as one-dimensional" (Sasaki 1985, p. 110)
        - Wallis-Barrow debate about the primacy of arithmetic/geometry/algebra (Sasaki 1985)

17th Century: Algebraisation of mathematics

"... the terms 'quantitas', 'quantity', 'quantita', 'quantité' on the one hand and 'magnitudo', 'magnitude', 'grandezza', 'grandeur' on the other hand were often used ambiguously ... because the same definition was adopted by different authors to convey distinct conceptions of mathematics. Moreover the emergence of that definition was strictly connected to the development of the concept mathesis universalis and to the preference for one or other of the words 'quantitas' and 'magnitudo' to translate the Euclidean term 'méghezos'"(Cantù 2010a, p. 226)

Ratios were plagued by a "systematic misreading of ratios as numbers (fractions) that the analytic techniques allowed . . . "(Mancosu 1996, p.36)

17th Century: Algebraisation of mathematics

Multiplication:

- For the Greeks:
    - Arithmetic: Standard definition
    - Geometry: multiplication of line = square
- For Vieta (extensionally defined relations):
    - for abstract magnitudes, multiplication in Vieta's work had no definite meaning (Bos 2001, p.148)
    - The method, though under the same concept now (multiplication), still relied on the objects' ontology in application
- For Descartes (ontology of geometrical object changed to accommodate the relation):
    - Multiplying the length of the lines via the introduction of a unit segment in geometry

17th Century: Algebraisation of mathematics

For Descartes and Vieta the theory of proportions is something more general than both arithmetic and geometry but common to both (Cantù 2010a)

Two stances in the literature (Mancosu 1996)

- Algebra was a tool for Descartes
- Algebra was the core of mathematics

17th Century: Infinitesimals were objects of concern

- Leibniz 1701 : “There is no need to take the infinite in a rigorous way, but only in the way in one says in optics that the rays of the sun come from an infinitely distant point and are therefore taken to be parallel” (in Mancosu 1996, p.171)

17th Century: Infinitesimals were objects of concern

Quand ils disputèrent en France avec l'Abbé Gallois , le Père Gouge & d'autres, je leur témoignai , que je ne croyois point qu’il y eut des grandeurs veritablement infinies ni véritablement infinitésimales, que ce n'étoient que des fictions , mais des fictions utiles pour abréger et pour parler universellement comme les racines imaginaires… (Leibniz 1717 p.500 in Opera omnia Tomus Tertius)

- Never used by Descartes on philosophical grounds (Mancosu 1996)

Relational Generalisation

- Marked by the loss of object ontology
- Paves the way for a reliance on methodology and generalisation of the method and relations irrespective of the ontology of the objects
- Though not explicit in the 17th century, relations necessarily must be defined intensionally to make sense of the results:
    - Imaginary numbers, irrational numbers, etc.
    - We arrive at these mathematical objects by using the methodology in a manner that defies the previously assumed ontology

- Justification, reliability?
- Rigour?

Categorical Axiomatisation: 19th and early 20th century

- Geometry was about (possible but at least well-defined) ‘abstract physical objects’ and was the sole representation of physical space.
  - Exclude the impossible and what is left, ~~however improbable~~, must be the truth (A. Conan Doyle).

- Arithmetic was problematic: science of quantities? Magnitudes? Concepts? Real? Fiction? … How can we justify the reliability of mathematics?
  - Berkeley’s Critiques (Folina 2012)

19th and early 20th century: Painting a picture

- The advent of non-Euclidean geometry threatened Euclidean geometry's epistemic certainty as the sole representation of our spatial experience (Torretti 2021)

- The end of the 19th century was marked by talk of a conceptual approach (Ferreirós, and Reck 2020)
    - In arithmetic, talk of quantities was replaced with talk of concepts

19th and early 20th century: Painting a picture

it is an intolerable offence against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely geometry [. . .] it is self-evident that the strictly scientific proof or the objective reason, of a truth which holds equally for all quantities, whether in space or not, cannot possibly lie in a truth which holds merely for quantities which are in space. On this view it may, on the contrary, be seen that such a geometrical proof is, in this as in most cases, really circular. (Bolzano 1817 in Ewald 2007, p. 228)

19th and Early 20th Century: Back to an empirical foundation?

Geometrical foundational programs to reclaim epistemic certainty

- Veronese: ‘mixed’ science’ whose objects are partially abstracted

- Bettazzi: “Every definition is an existential definition asserting the possibility of the attribution of certain properties”(Cantù 2010b, p.7) Those properties and concepts are derived from experience of the physical world.

19th and Early 20th Century: Back to an empirical foundation?

Also for arithmetic due to applicability concerns: Pasch's program

- To apply mathematics, the basic concepts must refer to something that is present in the world of experience and for which the content of the basic propositions is meaningful and valid. We acknowledge this connection with experience as soon as we consider analysis to be something else than . . . an internally consistent construction [einen Bau von innerer Folgerichtigkeit].(Pasch 1914,p. 138, quoted from Schlimm 2020, p. 94)

19th and early 20th century: The conceptual foundations

The attempt to ground mathematics in conceptual foundations, in categorical axioms emerged in that period

- Various foundational programs:
    - Grassman
    - Peano
    - Dedekind

These foundations were based solely on relations, not objects. The objects become defined via the relations in the axioms and gain meaning in an interpretation.

19th and early 20th century: The conceptual foundations

If in the consideration of a simply infinite system N set in order by a transformation ϕ **we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another** in which they are placed by the order-setting transformation ϕ, then are these numbers called natural numbers … (Dedekind 2007, p.33)

19th and early 20th century: The conceptual foundations

“Mathematicians do not study objects, but the relations between objects; to them it is a matter of indifference if these objects are replaced by others, **provided that the relations do not change**” (Poincaré 2011, p. 25, quoted from Ferreirós and Reck 2020).

19th and early 20th century: The conceptual foundations

Extended to Geometry:

- Geometry[’s]… facts can already be logically deduced from earlier ones….Nevertheless, with regards to its origins, **geometry is a natural science**.(Hilbert quoted in Corry 2004, p. 88) (My emphasis)

- If in speaking of my points, I think of some system of things, e.g. the system: love, law, chimney-sweep,... and then **assume all my axioms are relations between things**, then my propositions, e.g. Pythagoras’ theorem, are also valid for these things. (Hilbert quoted from (Frege 1980, p. 40)

19th and early 20th century: The conceptual foundations

- …is a new attempt to establish for geometry a complete, and as simple as possible, set of axioms and to **deduce from them the most important geometric theorems…** (Hilbert 1899 in Awodey and Reck 2002, p. 9) (My emphasis)
    - To recreate previous theorems in a new structure

Conclusion

To abstract a structure, history informs us of the (at least initial) need to:

Abstract states of affairs: extensionally defined relations over intensionally defined objects

- Discussions around the object ontology come prior to and inform the methodology
    - Dirac Delta function
- Methodology and reliability are assumed to hold on the basis of observations

Conclusion

Relational generalisation: remove objects ontology and relations are intensionally defined (somehow)

- Reliability is justified via methodology and success in science
    - Jones’ Knots Invariants
- Truth and existence of the objects are assumed from its scientific success
    - Or at least said objects are given the same status as scientific idealisations
    - E.g. Parallel rays of light
    - Reminiscent of Quine and Putnam’s IA

Conclusion

Categorical axiomatisation: Intensional relations are used to define a structure from which contains extensional objects.

- Truth and existence are relativised to the structure
    - Objects cannot exist outside the structure and ‘truth’ is not guaranteed extra-structurally.
- Reliability comes from the continuity of the relations
    - Change the relations and the objects change
    - ⇒ Results change

Conclusion

- In AG mathematics: Object definition came prior to mathematical practice.

- In structural mathematics: The definition of relations comes prior to the structure.

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Appendix: Modern Examples of Abstraction

- Dirac Delta function
  - Schwartz Theory of distribution
- Witten’s proof of Jones’ Knots Invariats from QFT
  - Subsequent debates
- Feynman Path integral
  - Still undecided, or not ‘properly’ mathematical
- Minhyong Kim’s approach to solutions of Diophantine equations
  - Trajectory of light on three holed-torus (gauge theory)
  - https://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201/

Appendix II: On Rigour - Jones' Knots Invariants

Witten (1989): Found a proof from QFT:

P.353: "To a physicist, a quantum field theory defined on a manifold M without any a priori choice of a metric on M is said to be generally covariant. Obviously, any quantity computed in a generally covariant quantum field theory will be a topological invariant. Conversely, a quantum field theory in which all observables are topological invariants can naturally be seen as a generally covariant quantum field theory."

P.365: "Were it not for the seeming nuisance that knots must be framed to define the Wilson lines as quantum observables, one would end up proving that the Jones knot invariants were trivial."

Appendix II: On Rigour - Jones' Knots Invariants

Jaffe and Quinn (1993):

P.8 "We also see that Witten, in giving a heuristic description of an extension of the Jones polynomial [Wi], was continuing in a long and problematic tradition even within topology."

P. 9 - " Also, it is a common rule of thumb now to regard any paper by Witten as theoretical. This short-changes Witten's work but illustrates the "better-safe-than-sorry" approach mainstream mathematics tends to take when questions arise."

Appendix II: On Rigour - Jones' Knots Invariants

Atiyah et. al. (1994):

Albert Schwartz: p.20

"All these papers are dealing with mathematical objects that have a rigorous definition. However a mathematician reading a textbook or a paper written by a physicist discovers often that the definitions are changed in the process of calculation. (This is true for example for the definition of scattering matrix in quantum field theory.)"

Appendix II: On Rigour - Diophantine Equations

https://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201/

Minhyong Kim: “Kim is working out a scheme in which “the problem of finding the trajectory of light and that of finding rational solutions to Diophantine equations are two facets of the same problem”

- Least-action principle for trajectories of light on a three holed-torus

“Kim long kept it to himself. “I was hiding it because for many years I was somewhat embarrassed by the physics connection,” he said. “Number theorists are a pretty tough-minded group of people, and influences from physics sometimes make them more skeptical of the mathematics.””

Appendix II: On Rigour - Proof lengths and Cut Eliminations

Jacques Dubucs (2024):

Some respectable principles are to be avoided. E.g. Pure mathematics (in the sense of Hilbert's Methodenreinheit) are certainly not tractable: a 3-line impure proof of a simple theorem of arithmetic requires, when purified, as many lines as the number of nano-seconds since the Big Bang (G. Boolos, Don't Eliminate Cut, in Logic, Logic, and Logic, Harvard UP, 1999)