Relations in Structuralism

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Structuralism

Philosophical structuralism/Mathematical practice issues

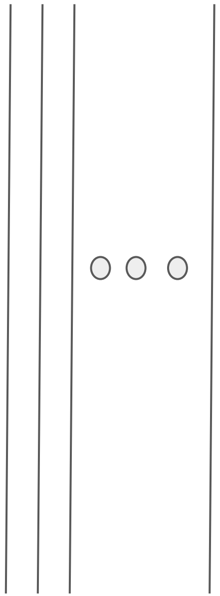
- "...does the view allow for a face-value interpretation of mathematical statements?" (Hellman and Shapiro, 2018)
- Problem of Identity of Indiscernibles
    - Shapiro's mathematical practice response
- MacBride(2005) and MacLarty(2008) Critique

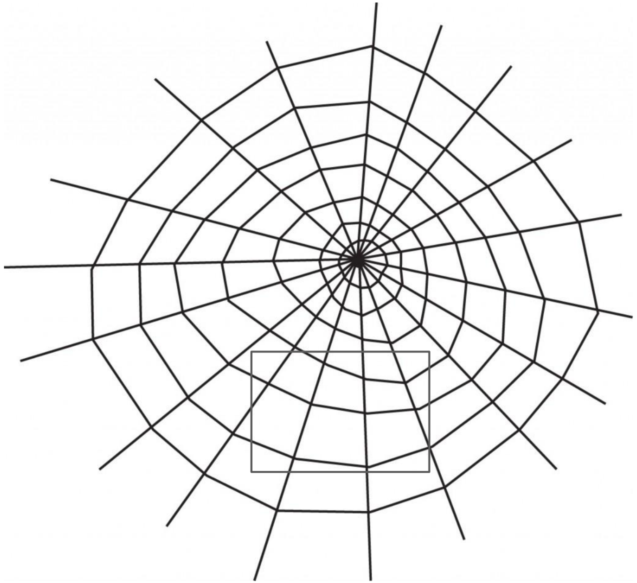
Mathematical Structuralism

- Set Theory Structuralism
- Category Theory Structuralism
    - Bourbaki Group

Motivation

Take 25 strings





The Metaphysics of Relations

Given relation R and objects a, b

Directionalism or Standard View (Russell 1903)

- aRb is a directed relation from a to b, different from the converse bR*a where the direction is the opposite
- a on top of b
- State s of ‘a on top of b’ and state t of ‘b below a’ are the same state
- Problem: One state of affairs gives rise to two formalisms

Positionalism (Fine 2000)

- Relation R has two empty positions α and β such that $\alpha R \beta$
- Objects a and b need to be slotted into α and β respectively
- ‘a is adjacent to b’
- Symmetrical: a can be slotted in both α and β simultaneously
- Problem: One state of affairs giving rise to two formalisms

The Metaphysics of Relations

Donnelly's Relative Postionalism (Donnelly 2016,2021)

- Positions are not construed as objects as Fine does
- They are relative unary properties
- Problem: "…relative positionalist must accept relative property instantiation as an undefined notion in her theory." (p.97, Donnelly 2016)

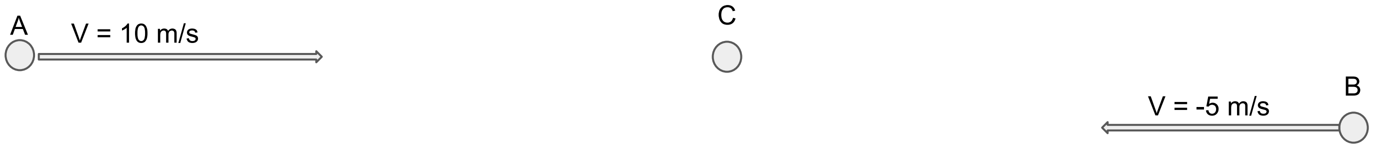
Antipositionalism (Fine 2000)

- R has no positions
- Symmetry solved
- Two relations relating objects 'in the same manner' are identical relations
- "Under these transitions, the concept of relation has become successively simpler and the concept of application successively more complex." (p.32, Fine 2000)

Frames of Reference: or how I learned to stop worrying and love Relativity

- One state of affairs necessarily gives rise to more than one formalism
- Not an issue as long as they're translatable to each other (covariant)

Example:



- $x = vt + x_0$
- This equation is not contingent on one frame of reference but holds true in all non-inertial frames
- Shows that one state of affairs can give rise to several formalisms

Mechanism of FoR

- Set a reference, e.g. the observer (arbitrary)
- Set a metric, e.g. metre (arbitrary)
- Need a translation scheme
- Covariant results from different FoR
- Generally Covariant Form Example: three objects **a**, **b** and **c** in a straight line with **b** between **a** and **c**
  - **b** is 5 cm from **c** (ref), **a** is 6 cm from **c** ⇒ **a** and **b** are 1 cm apart
  - **c** is 5 cm from **b** (ref), **a** is 1 cm from **b** ⇒ **a** and **c** are 6 cm apart

The structural case in mathematics

- Mathematical structuralism: generalisation of relations, postionalism
  - Entities are defined mathematically, either axiomatically (e.g. number 0 and successor function) or in the process (e.g. function, operator, etc…)
- The focus becomes relations
  - Dedekind, Hilbert, etc.
- Objects are dealt with as positions, defined axiomatically.
- Have structural properties of positions
  - Linnebo (2008), Korbmacher and Schiemer(2018)
- Proofs have to be traced from axioms
  - This last proof can rightly be called “formal” since we act on the formulas, not on the objects (p.11, Cartier 2000)

Hilbert-Bourbaki Formalism

“The implicit philosophical belief of the working mathematician is today the Hilbert-Bourbaki formalism. Ideally, one works within a closed system: the basic principles are clearly enunciated once for all, including (that is an addition of twentieth century science) the formal rules of logical reasoning clothed in mathematical form. The basic principles include precise definitions of all mathematical objects…” (p.3, Cartier 2000)

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The case of physics

- Objects are entities that ‘exist’ in the outside world
- Relations are between these objects
- Truth comes from experimentation
- Object abstraction is from those in the physical world
    - Anomalous Objects
    - “The entities that appear in these calculations, such as infinitesimals or representative variable, are logically anomalous objects.” (p.429, Urquhart 2008)
    - Dirac Delta function
    - These (so far) ill-defined [Feynman] integrals are powerful tools to evaluate infinite series and infinite products. (P.66, Cartier 2000)
- Mathematical proofs in physics
    - Lack of rigour (according to some mathematicians)
        - If it works, it works
    - Based on physical constraints
    - Tied to experimentation, which is messy
        - “...*science often requires methods that ‘eliminate both detail and, in some sense, precision.”* (p.13, Batterman 2002)

Edward Witten’s QFT solution to Jone’s knot invariants

- “… understanding these [mathematical] theories as quantum field theories involves constructing theories in which all of the **observables** are topological invariants… Something that can be computed from a manifold M as a topological space (perhaps with a smooth structure) without a choice of metric is called a "topological invariant" (or a "smooth invariant") by mathematicians. To a physicist, a quantum field theory defined on a manifold M without any a priori choice of a metric on M is said to be generally covariant. Obviously, any quantity computed in a generally covariant quantum field theory will be a topological invariant.” (p.352-3, Witten 1989) (my emphasis)

- “Were it not for the seeming nuisance that knots must be framed to define the Wilson lines as quantum observables, one would end up proving that the Jones knot invariants were trivial.” (p.365, Witten 1989)

The subsequent debate

- “We also see that Witten, in giving a heuristic description of an extension of the Jones polynomial [Wi], was continuing in along and problematic tradition even within topology.” (p.8, Jaffe and Quinn, 1993)

- Albert Schwarz: “All these [heuristic mathematical] papers are dealing with mathematical objects that have a rigorous definition. However a mathematician reading a textbook or a paper written by a physicist discovers often that the definitions are changed in the process of calculation.” (p.21, Atiyah et. al. 1994)

The subsequent debate

“The basic insight is that a mathematical structure is a scientific structure but one which has many different empirical realizations…The axioms needed for this formulation must then be such as to hold in all the examples. And how is it with the consequences of these axioms? They are not established by example. They are those established by proof….the theorem itself is not about any one use and so must be established by a formal proof from the definitions.” (p. 150, Mac Lane 1997)

The subsequent debate

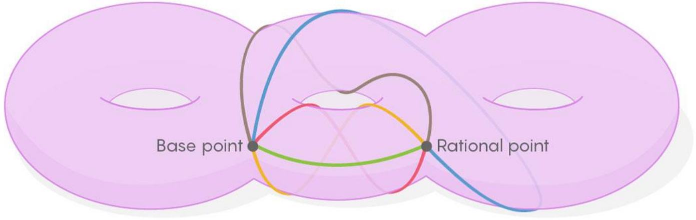
- Armand Borrel: “I also feel that what mathematics needs least are pundits who issue prescriptions or guidelines for presumably less enlightened mortals” (p.3, Atiyah et. al. 1994)

- Michael Atiyah: “Perhaps we now have high standards of proof to aim at but, in the early stages of new developments, we must be prepared to act in more buccaneering style.” (p.1, Atiyah et. al. 1994)

- “However, a buccaneer is a pirate, and a pirate is often engaged in stealing… One such was the notorious Captain William Kidd; before the bibliography we append a poetic summary of the style of his doing. We do not need such styles in mathematics.” (p.150, Mac Lance 1997)

Minhyong Kim’s Diophantine Geometry

- To Kim, rational solutions are like the trajectory of light on a torus
- Uses space of spaces from gauge theory to find rational solutions for Diophantine equations
- Sociological aspect
    - “I was hiding it because for many years I was somewhat embarrassed by the physics connection,” he said. “Number theorists are a pretty tough-minded group of people, and influences from physics sometimes make them more skeptical of the mathematics.” (Hartnett, 2017)



THREE-HOLED TORUS: Paths connect the base point with a rational point.

Conclusion

- By focusing on relations in mathematics, one begins to get a grasp of the mathematical-physical distinction with regards to practice
- Objects are an issue
  - For the mathematician they come from axioms
  - For physicists, they are abstraction from physical objects, states, degrees of freedom, etc.
- Relations with(out) objects as generalised procedures
  - For physics: from physical phenomena with no axiomatic trace
  - For mathematicians: needs to be traced back to the axioms (Hilbert-Bourbaki Formalism)

- Further questions:
  - Proof from examples: e.g. are proofs in physics generalisable to mathematics
    - "Do not bother students with the proof of the Pythagorean theorem; just let them measure sides and hypotenuse of a few right triangles and so have them see that Pythagoras is correct." (p.147, Mac Lane 1997)
  - Does a generalisation of a relation 'hold'?

“At a great distance from its empirical source, or after much ‘abstract’ inbreeding, a mathematical subject is in danger of degeneration… Whenever this stage is reached, the only remedy seems to me to be the rejuvenating return to the source: the reinjection of more or less empirical ideas.” (Von Neumann 1947)

Bibliography

- Atiyah, Michael et al. (1994), ‘Responses to “Theoretical mathematics: towards a cultural synthesis of mathematics and theoretical physics”, by A. Jaffe and F. Quinn’, Bulletin of the American Mathematical Society, 30, 178–207.
- Batterman, Robert W. (2002), The Devil in the Details: Asymptotic Reasoning in Explanation, Reduction and Emergence (Oxford: Oxford University Press).
- Cartier, Pierre (2000), ‘Mathemagics (A tribute to L. Euler and R. Feynman)’, in Michel Planat (ed.), Noise, Oscillators and Algebraic Randomness: From Noise in Communications Systems to Number Theory (Berlin: Springer-Verlag). Lectures of a school held in Chapell des Bois, France, April 5–10, 1999.
- Donnelly, Maureen (2016), ‘Positionalism Revisited’, in Anna Marmadoro and David Yates, The Metaphysics of Relations, Oxford University Press, 80-99
- Fine, Kit (2000), ‘Neutral Relations’, The Philosophical Review, 109(1), 1-33
- Hartnett, Kevin (2017), ‘Secret Link Uncovered Between Pure Math and Physics’, Quantum Magazine https://www.quantamagazine.org/secret-link-uncovered-between-pure-math-and-physics-20171201/
- Hellman, Geoffrey and Shapiro, Stewart (2018), ‘Mathematical Structuralism’, Cambridge University Press.
- Jaffe, Arthur and Quinn, Frank (1993), ‘“Theoretical mathematics”: Toward a cultural synthesis of mathematics and theoretical physics’, Bulletin of the American Mathematical Society, 29, 1–12.
- Korbmacher, Johannes and Schiemer, Georg (2018) ‘What Are Structural Properties, Philosophia Mathematica, 1-29.
- Linnebo, Øystein (2008) ‘Structuralism and the Notion of Dependence’, The Philosophical Quarterly, 58, 230, pp. 59-79.
- Macbride, Fraser (2005), ‘Structuralism reconsidered’, in S. Shapiro (ed.), The Oxford Handbook of Philosophy of Mathematics and Logic (Oxford: Oxford University Press) pp. 563–589.
- Mac Lane, S. (1997), ‘Despite physicists, proof is essential in mathematics’, Synthese, 111, 147–154.
- McLarty, Colin (2008), ‘What Structuralism Achieves’, in Paolo Mancosou, The Philosophy of Mathematical Practice, Oxford Scholarship Online, 354-369
- Russell, Bertrand (1903), ‘The Principles of Mathematics’, Oxford University Press.
- Urquhart, Alasdair (2008), ‘Mathematics and Physics: Strategies of Assimilation’, in Paolo Mancosou, The Philosophy of Mathematical Practice, Oxford Scholarship Online, 417-440
- Witten, Edward (1989), ‘Quantum Field Theory and the Jones Polynomial’, Communications in Mathematical Physics, 121, 351-399