How an historical *in re* account of mathematical structuralism sheds light on applicability

Rami Jreige

https://github.com/Rami-Jreige

For the development of mathematics cannot be isolated from the general history of the modes of understanding — however strenuous an effort may be made to shape mathematics into a self-contained, independent discipline.

Jacob Klein

Outline

- Structuralism
- Klein’s account
    - Ancient Greek mathematics
    - Diophantus of Alexandria
    - Francois Vieta’s translation
    - Descartes and Fermat
- 19th-20th Century
    - Dedekind’s axiomatisation
    - Frege-Hilbert Debate
- Conclusion

Structuralism

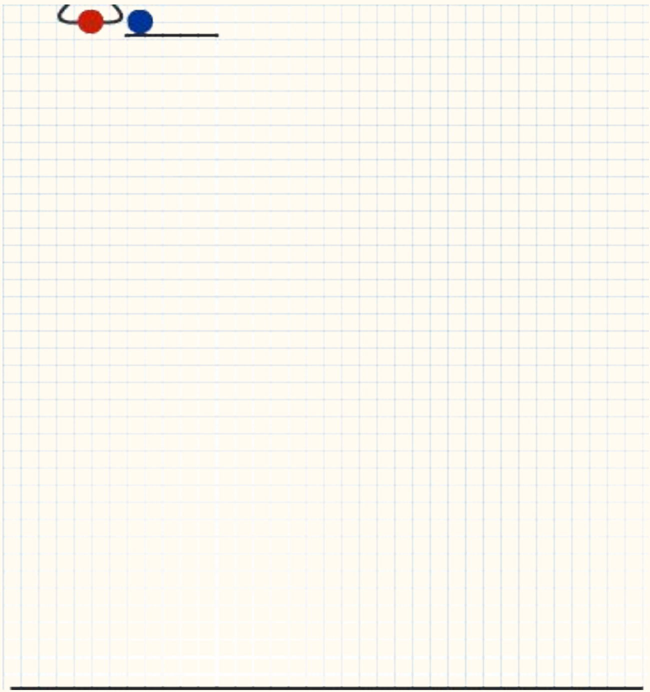
- Numbers are not objects
    - Positions in structure
- Government is a structure
    - Governmental position are to be filled by objects (people)
- Mathematical structuralism follows similarly with application
- *In re* structuralism:
    - Main Proponents: Charles Parsons, Michael Resnik (somewhat)
    - Proposes that mathematical structures are abstracted
    - They both rely on Quine and Putnam’s Indispensability Argument

Modification: The model

- In the concrete-abstract debate I propose to add another layer : the model (in the scientific sense)
- Brings it more in line with applicability
- Highlights a crucial intermediary step where the ‘positions’ are filled with abstract objects that can direct the model
- We now have
  - Concrete Form
  - Model
  - Structure
- This is not comprehensive, there are further ‘smaller’ steps along the way

Physics example

- Red Ball falling from initial height $y_0 = 20$m
- Final height $y = 0$m
- Initial vertical velocity 0 m/s
- Newtonian equation of motion: $y = 0.5at^2 + v_0t + y_0$
- a is gravity at -9.8 m/s^2
- Solving for this we get two solutions:
  - $\sim +4$secons and -4 second
- Negative result rejected



Logistic vs Arithmetic

- "This whole science, which thus concerns the behavior of numbers toward one another, i.e., their mutual relations, and which first enables us to relate numbers, i.e., to calculate with them, is called the "art of calculation" — "logistic." (Klein 1976, p.19)

- "Arithmetic" is … first and foremost the art of correct counting. As Plato says explicitly in the Theaetetus (198 A-B): "Through this art, I think, one is oneself master of the sciences of number and is able as a teacher to pass them on to another." (*Ibid.*)

The Monad

- “The “scientific” arithmetician and logistician deals with numbers of pure monads. … Plato stresses emphatically that there is “no mean difference” between these and the ordinary numbers. It would indeed seem strange if there should be numbers of — nothing.” (*Ibid*. p.50)
- “For even a “pure” number, i.e., a number of“pure” units, is no less “concrete” or “specified” than a number of apples.What distinguishes such a number is in both cases its twofold determinateness: it is, first of all, a number of objects determined in such and such a way, and it is, secondly, an indication that there are just so and so many of these objects.” (*Ibid*. p.48)

Thus an *unlimited* field of "pure" units presents itself to the view of the "scientific" arithmetician and logistician (cf. Aristotle, *Posterior Analytics* A 10, 76 b 4 f.). The particular multitudes which may be chosen from this field are precisely those "pure" numbers (of units) with which he deals. This is how the traditional "classical" definitions of *arithmos* are to be understood; Eudoxus (Iamblichus, *in Nicom.* 10, 17 f.): "A number is a *finite* multitude" [of units]" *(ἀριθμός ἐστιν πλῆθος ὡρισμένον)* — cf. Aristotle, *Metaphysics* Δ 13, 1020 a 13: "limited multitude" *(πλῆθος πεπερασμένον)*; Euclid (VII, Def. 2): "the multitude composed *of units*" *(τὸ ἐκ μονάδων συγκείμενον πλῆθος)* — cf. Aristotle, *Metaphysics* I 1, 1053 a 30: *πλῆθος μονάδων*; furthermore: "a set [composed] of units" *(μονάδων σύστημα* — Theon 18, 3; Nicomachus 13, 7 f.; Iamblichus 10, 9; Domninus 413, 5; *σύνθεσις μονάδων* — Aristotle, *Metaphysics* Z 13, 1039 a 12); "an aggregate in the realm of quantity composed of monads" *(ποσότητος χύμα ἐκ μονάδων συγκείμενον* — Nicomachus 13, 8). Numbers are, in short, many units: "for each number is many because it[consists of many] ones"[56] *(πολλὰ γὰρ ἕκαστος ὁ ἀριθμὸς ὅτι ἕνα . . .)*, i.e., because it represents nothing but several or many units (Aristotle, *Metaphysics* I 6, 1056 b 23 — cf. *Physics* Γ 7, 207 b 7).[57] "Multiplicity [manyness] is, as it were, the genus of number" *(τὸ δὲ πλῆθος οἷον γένος ἐστὶ τοῦ ἀριθμοῦ* — *ibid.*, 1057 a 2 f.; cf. Iamblichus 10, 18 f.). Because

of this, "number" and "one" are opposites (Aristotle, *Metaphysics* I 6, 1056 b 19 f.), although it is possible to speak of the one metaphorically as being "a certain, although a small, multitude" *(πλῆθός τι, εἴπερ καὶ ὀλίγον* — *ibid.*, 1056 b 13 f.), namely the multitude "one." Cf. the definition of Chrysippus in Iamblichus, *in Nicom.*, 11, 8 f. and Syrianus, *in Arist. metaph.*, Kroll, 140, 9 f.: "the unit is the multitude 'one'" *(μονάς ἐστι πλῆθος ἕν)*; also *Theaetetus* 185 C–D. On this possibility are founded the definitions of the series of numbers: "a progression of multitude beginning from the unit and a recession ceasing with the unit" *(προποδισμὸς πλήθους ἀπὸ μονάδος ἀρχόμενος καὶ ἀναποδισμὸς εἰς μονάδα καταλήγων* — Theon 18, 3 ff.; cf. Iamblichus, *in Nicom.* 10, 16 f.; this definition may go back to Moderatus, first century A.D., cf. Theon 18, note). Thus also Domninus (413, 5 ff.): "The whole realm of number is a progress from the unit to the infinite by means of the excess of one unit [of each successive number over the preceding]." *(ὁ δὲ σύμπας ἀριθμός ἐστι προκοπὴ ἀπὸ μονάδος κατὰ μονάδος ὑπεροχὴν ἄχρις ἀπείρου.)* The series of numbers may be understood as the result of a successive reproduction of the unit: "It is in consequence of the monad that all the successive numbers beginning with the dyad form aggregates and produce the well-ordered kinds of that which is multiple in accordance with the proper sequence [of the numbers]." *(πρὸς τὴν μονάδα πάντες οἱ ἐφεξῆς ἀριθμοὶ ἀπὸ δυάδος ἀρξάμενοι συγκρινόμενοι τὰ τοῦ πολλαπλασίου εὔτακτα εἴδη ἀπογεννῶσι τῇ οἰκείᾳ ἀκολουθίᾳ* — Nicomachus 46, 13 ff.) The truth is that the unit can be spoken of as a

Practical and Theoretical Logistic

- Practical Logistic is the calculations done by people everyday, e.g. Merchants calculating
- “Plato’s special demand for a theoretical logistic ...[that] there should also be a science addressed to the pure relations of numbers as such, which would correspond to the common art of calculation and provide its foundation.” (*Ibid*. p.28)

- “... the crucial question to be asked about theoretical logistic — whether that which distinguishes exact calculation, namely operation with fractional parts of the unit of calculation, can really be sufficiently grounded in the science of the possible relations of numbers, i.e., in the “pure” theory of relations, alone.” (*Ibid*. p.43)

Practical and Theoretical Arithmetic

- " Calculation," with all its presuppositions, must therefore be referred entirely to the realm of the practical arts and sciences, while the "pure" theory of relations loses its fixed place and comes to be assigned now to arithmetic as the theory of the kinds of numbers, now to harmonics as the theory of musical intervals based on ratios of numbers. (*Ibid*. 51 p.44)
- "Thus the first task of "scientific" arithmetic — as contrasted with that "practical" knowledge which is satisfied with "knowing" these numbers without understanding what this "knowing" implies — consists in finding such arrangements and orders of the assemblages of monads as will completely comprehend their variety under well-defined properties, so that their unlimited multiplicity may at last be brought within bounds (cf. Nicomachus I, 2)." (*Ibid*. p.54)
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Platonic vs Aristotelian Ontology

- Platonic
    - Monads as a separated object
    - Exists alongside the concrete
    - Separate realm of being
- Aristotelian
    - Monads are detached
    - They are abstracted
    - Associated with measure
    - Measure can always be switched
    - “For the number of the trees, i.e., “three,” has no proper, no independent, “nature”. Their being “so many,” just like their being, for instance, “green,” is dependent on their being trees.” (*Ibid.* p.101)

Euclid

- "...the Euclidean presentation is not symbolic. It always intends determinate numbers of units of measurement, and it does this without any detour through a "general notion" or a concept of a "general magnitude"... It does not identify the object represented with the means of its representation, and it does not replace the real determinateness of an object with a possibility of making it determinate..." (*Ibid.* p.123)
- "The "arithmetical" books of Euclid (VII, VIII, IX) directly mirror this ontological transformation...The "pure" units of which the numbers to be studied are composed are here understood precisely only as "units of measurement" such as can be represented most simply by straight lines which are directly measurable, quite independently of whether they form a "linear" (prime), "plane," or "solid" number." (*Ibid.* p.111)

Diophantus

- “First of all, it is immediately clear not only from “ Definition I” (2, 14 f.), which is reminiscent of Euclid VII, Def. 2 (cf. p. 51), and according to which “all numbers consist of a certain multitude of monads” but also from the way in which Diophantus designates the numbers which are presupposed as “known” (the determinates) in the calculation, that we are indeed dealing with numbers of pure monads.” (*Ibid.* p.130)
- “ The procedure [of finding the resolution] ‘ in the indeterminate [solution] ’ is such that, of however many monads someone might wish the unknown to be, when he sets [the indeterminate number] in the hypothetical expression, the problem is [at once] completed.” (*Ibid.* p.134)“

Greek Mathematics

- “In Greek science, concepts are formed in continual dependence on “natural,” prescientific experience, from which the scientific concept is “abstracted.” The meaning of this “abstraction,” through which the conceptual character of any concept is determined, is the pressing ontological problem of antiquity.” (*Ibid*. p.120)
- “To anticipate, [Vieta’s generalised procedure] can arise only on the basis of an insufficient distinction between the generality of the method and the generality of the object of investigation.” (*Ibid*. p.123)

Francois Vieta: Motivation

- Reviving Greek Mathematics
- Claims to have discovered the ‘previously buried genuine gold’ of ancient Greek mathematics.
- Racism

> “Those things which are new are wont in the beginning to be set forth rudely and formlessly and must then be polished and perfected in succeeding centuries. Behold, the art which I present is new, but in truth so old, so spoiled and defiled by the barbarians, that I considered it necessary, in order to introduce an entirely new form into it, to think out and publish a new vocabulary, having gotten rid of all its pseudo-technical terms, lest it should retain its filth and continue to stink in the old way....” (Klein 1976, p.153)

Francois Vieta: (mis)construing Greek logistic

- “Finally this “number” concept is able to obliterate the distinction, so fundamental for antiquity, between “continuous” geometric magnitudes and numbers divisible into “discrete” units (cf. pp. 10 and 53); this permits, for the first time, a “scientific” understanding of the approximation methods handled so masterfully by Vieta”.(Klein 1976, p.178)

- “With Viete’s algebra, an essentially new task of the mathematician comes to the fore: the investigation of the *constitutio aequationum*, the structure of equations.” (Mahoney 1980, p.144)

- “Viete, however, understood something else by ‘unknown’. True, following Diophantus he calls it a species, and he calls algebra *logistice speciosa*; but he also says that this ‘logistic of species’ shall be carried out using the ‘species or forms of things’”. (Mahoney 1980, p.143)

Francois Vieta: His unawareness

- “Zetetic (or theoretical search for the truth) was employed most subtly of all by Diophantus in those books which were written on the subject of arithmetic. But he exhibited it as though it were founded only on numbers and not also on species — although he himself used them — so that his subtlety and skill might be more admired, since things that appear very subtle and abstruse to one who calculates in numbers [i.e., practices *logistice numerosa*] appear very familiar and immediately obvious to one who calculates in species [i.e., practices *logistice speciosa*].” (Klein, 1976, p.170)

Rene Descartes and Pierre de Fermat

- “There is warrant for this reading Descartes’ later assertion that his teaching is concerned solely with the unfolding of relations among ‘measures’, so that the problems they present can be viewed as ones of Order.”(Schuster 1980, p.45)

- “The explanation for this tension may well lie in the fact that both [Fermat and Descartes] treated old problems by means of a new symbolic algebra, without themselves being clear on the extent to which the new means had changed not only the techniques of solution but also the very manner of posing problems. With the new algebra, the *ars analytica*, mathematicians thought at first that they had regained the mathematics of the Golden Age of antiquity.” (Mahoney 1980, p.141)

Side Consequences

- “The battle between the proponents of Peripatetic syllogistic and mathematical analysis about the primacy of their respective views concerning the framework of the world, which is thus initiated in the Seventeenth century, is still being waged today, now under the guise of the conflict between “formal logic” and “mathematical logic” or “logistic,” although its ontological presuppositions have been completely obscured.” (Klein 1976, p.169)

Dedekind

- “If in the consideration of a simply infinite system N set in order by a transformation ϕ we entirely **neglect the special character of the elements**, simply retaining their distinguishability and **taking into account only the relations to one another** in which they are placed by the order-setting transformation ϕ, then are these numbers called natural numbers or ordinal numbers or simply numbers . . . The relations or laws which are derived entirely from the conditions . . . are always the same in all ordered simply infinite systems, whatever names may happen to be given to the individual elements.” (Dedekind 1963, p. 75) (My emphasis)

- “...mathematics considers quantities only in their relation to one another... These different relations of quantities and the different means of representing quantities are the foundation of [arithmetic and geometry].” (Gauss 1999, p. 294)

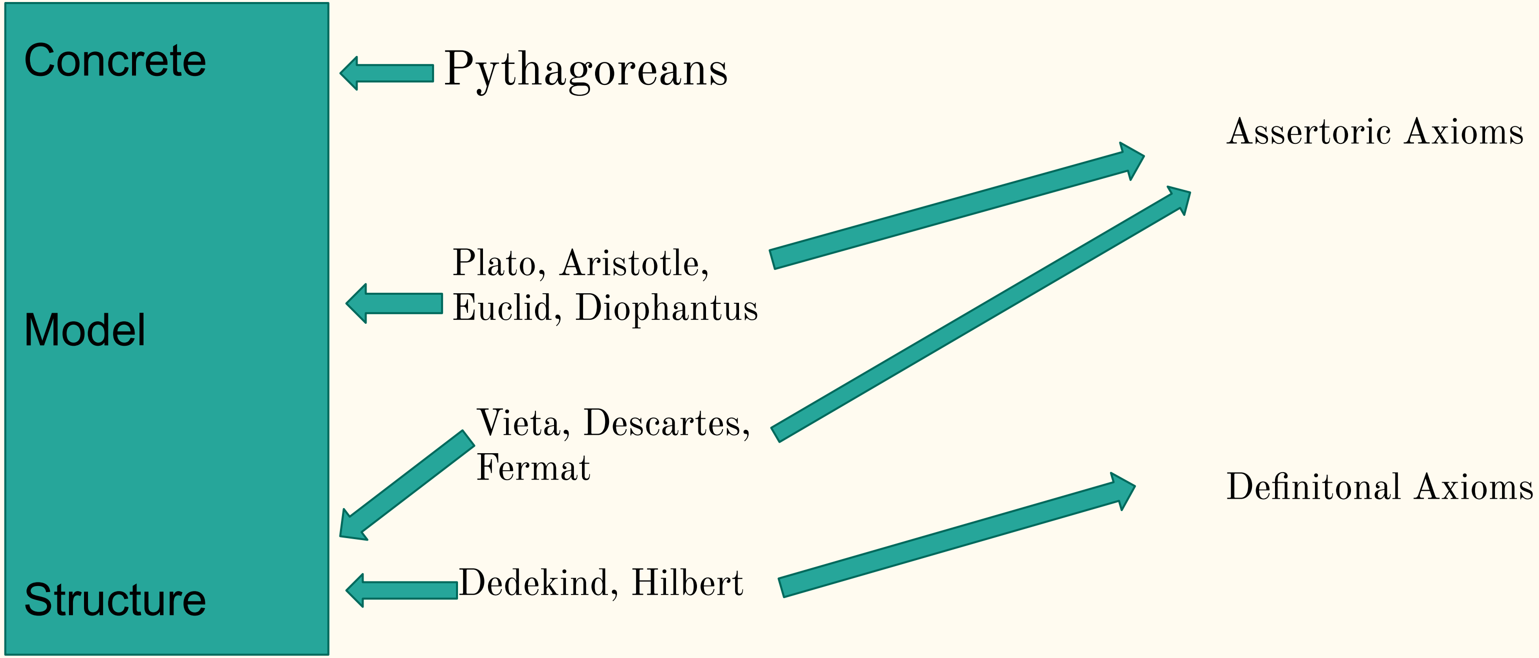
Frege to Hilbert

"...the meanings of the words "point," "line," "between" are not given, but are assumed to be known in advance. At least it seems so. But it is also left unclear what you call a point. . . Here the axioms are made to carry a burden that belongs to definitions. To me this seems to obliterate the dividing line between definitions and axioms in a dubious manner, and beside the old meaning of the word 'axiom', which comes out in the proposition that the axioms express fundamental facts of intuition, there emerges another meaning but one which I can no longer grasp."(Frege 1980, p. 35-36)

Hilbert's Response

"...it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points, I think of some system of things, e.g. the system: love, law, chimney-sweep,... and then assume all my axioms are relations between things, then my propositions, e.g. Pythagoras' theorem, are also valid for these things." (Hilbert in Frege 1980, p.40)

Conclusion


Concrete
Model
Structure
Pythagoreans
Plato, Aristotle,
Euclid, Diophantus
Vieta, Descartes,
Fermat
Dedekind, Hilbert
Assertoric Axioms
Definitonal Axioms

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