Correction of DE (July 2023)

$\boxed{\text{Convex optimization}}$

Exercise 1: see the chapter 1 and 2 of the lesson.

Exercise 2:

$$f(x,y) = x^2 + y^2 + xy - 2x - 2y$$

1) f is of class C^2 on $\mathbb{R}^2$

$$\nabla f = \begin{pmatrix} 2x + y - 2 \\ 2y + x - 2 \end{pmatrix} \text{ and } \boxed{\underbrace{\nabla^2 f}_{A''} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}$$: constant and sym matrix.

Is $\nabla^2 f$ definite positive on $\mathbb{R}^2$? Compute the associated eingenvalues

$$|A - \lambda Id| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = (2-\lambda-1)(2-\lambda+1) = 0$$

$\Rightarrow \boxed{\lambda_1 = 1 > 0}$ and $\boxed{\lambda_2 = 3 > 0}$

$\Rightarrow \nabla^2 f$ is definite positive on $\mathbb{R}^2$ (as it's a constant matrix with strictly positive eingenvalues).

$\Rightarrow f$ is strictly convex on $\mathbb{R}^2$.

Rem: one can also use Sylvester's criteria: as:

$m_1 = 2 > 0$ and $m_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$

$\Rightarrow A$ is def positive - - - - -

2) • To determine a stationary point, $\boxed{\text{solve } \nabla f = 0}$

$$\Leftrightarrow \begin{cases} 2x + y - 2 = 0 \\ 2y + x - 2 = 0 \end{cases} \Rightarrow \begin{cases} y = 2 - 2x \\ 2(2-2x) + x - 2 = 0 \end{cases} \Rightarrow \begin{cases} \bar{y} = \frac{2}{3} \\ \bar{x} = \frac{2}{3} \end{cases}$$

$\Rightarrow \boxed{(\bar{x}, \bar{y}) = \left(\frac{2}{3}, \frac{2}{3}\right)}$ is the unique stationary pt of f in $\mathbb{R}^2$.

• As f is strictly convex on $\mathbb{R}^2$, this stationary pt $(x^*, y^*) = \left(\frac{2}{3}, \frac{2}{3}\right)$ is the unique global minimum of f on $\mathbb{R}^2$.

Exercise 3:

$$\begin{cases} \min f(x,y,z) = (x-4)^2 + y^2 + z^2 \\ x + y + z \leq -2 \end{cases}$$

f is Non linear, one constraint $\boxed{g(x,y,z) = x + y + z + 2 \leq 0}$

$$\nabla f = \begin{pmatrix} 2(x-4) \\ 2y \\ 2z \end{pmatrix} \quad , \quad \underset{A}{\nabla^2 f} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ : diagonal (so sym) and sonstant}$$

$\Rightarrow$ The eingenvalues of A are $\boxed{\lambda_1 = \lambda_2 = \lambda_3 = 2 > 0}$

$\Rightarrow$ A is def positive constant matrix $\Rightarrow$ $\boxed{f \text{ is strictly convex.}}$

Moreover The constraint $g(x,y,z)$ is linear (so convex).

Slater Condition: There exists a pt $\hat{X}$ such That $g(\hat{x}) < 0$.

With $\hat{X} = (\hat{x}, \hat{y}, \hat{z})$: many possibilities : exple:

$(-8,0,0)$ as $-8+0+0 < -2$

or $(-4,0,0)$ as $-4+0+0 < -2$

.....

Now, Using The Karush-Kuhn-Tucker Theorem (as The problem is convex and slater Condition is satisfied) : The optimal sol of our problem is The solution of The following system of Kuhn-Tucke:

$$\begin{cases} \nabla f(x,y,z) + \lambda \cdot \nabla g(x,y,z) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \lambda \cdot g(x,y,z) = 0 \\ g(x,y,z) \leq 0 \\ \lambda \geq 0 \end{cases}$$

So : Replace:
$$\begin{cases} \overbrace{\begin{pmatrix} 2(x-4) \\ 2y \\ 2z \end{pmatrix}}^{\nabla f} + \lambda \overbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}^{\nabla g} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \lambda \overbrace{(x+y+z+2)}^{g} = 0 \quad \leftarrow \lambda \cdot g(x,y,z) = 0 \\ x+y+z+2 \leq 0 \quad \leftarrow g(x,y,z) \leq 0 \\ \lambda \geq 0 \end{cases}$$

Now, consider The system to analyze :
$$\begin{cases} 2(x-4) + \lambda = 0 & (1) \\ 2y + \lambda = 0 & (2) \\ 2z + \lambda = 0 & (3) \\ \lambda(x+y+z+2) = 0 & (4) \\ x+y+z+2 \leq 0 & (5) \\ \lambda \geq 0 & (6) \end{cases}$$

Case 1: $\boxed{\lambda = 0}$ (so ⑥ is satisfied)

① $\Rightarrow \bar{x} = 4$

② $\Rightarrow \bar{y} = 0$

③ $\Rightarrow \bar{z} = 0$

④ is satisfied.

No

But ⑤ is not satisfied as: $0+0+0+2 \leq 0$ IMPOSSIBLE.

Case 2: $\boxed{\lambda \neq 0}$

① $\Rightarrow \bar{x} = \frac{-\lambda+8}{2}$

② $\Rightarrow \bar{y} = -\frac{\lambda}{2}$

③ $\Rightarrow \bar{z} = -\frac{\lambda}{2}$

④ $\Rightarrow x+y+z+2=0$: replace: $\left(\frac{-\lambda+8}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} + 2\right) = 0$

$\Rightarrow \frac{-3\lambda}{2} + 6 = 0 \Rightarrow \lambda = 4 > 0$ (⑥ is satisfied).

$\bar{x} = \frac{-4+8}{2} = 2$, $\bar{y} = -\frac{4}{2} = -2$, $\bar{z} = -2$

Replace in ⑤: $2 - 2 - 2 + 2 = 0 \leq 0$ is true so

This candidate $\boxed{(\bar{x}, \bar{y}, \bar{z}) = (2, -2, -2)}$ is accepted.

Conclusion: $(+2, -2, -2)$ is the optimal sol of this pb.

it's a global and unique minimum as this pS is convex.

Exercise 4: $P_1(x_1, y_1), \dots, P_n(x_n, y_n)$ given in $\mathbb{R}^2$

a quadratic function is given by $y = ax^2 + bx + c$, $x \in \mathbb{R}$

So we have to determine (a, b, c) sol of:

$$\begin{cases} \min f(a,b,c) = \underbrace{\left(y_1 - a x_1^2 - b x_1 - c\right)^2}_{\text{for } P_1(x_1, y_1)} + \dots + \underbrace{\left(y_n - a x_n^2 - b x_n - c\right)^2}_{\text{for } P_n(x_n, y_n)} \\ (a,b,c) \in \mathbb{R}^3 \end{cases}$$

You are not asked to solve this pb only to write it.