Major Project: AM3813A: Nonlinear ODE's

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1 Introduction

Find the approximate solutions of the following differential equation

$\ddot{x} + x - \varepsilon x\dot{x} - \frac{1}{2}\varepsilon^2 x^2 \dot{x} = 0$ Up to the 3th order

2 Solve using Lindstedt-Poincarè Method

Let $\omega(\epsilon) = w_0 + \epsilon w_1 + ...$

and let the timescale $\tau = \omega(\epsilon)t$

Assume solution is in the form $x(\tau, \epsilon) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + ...$

Additionally, we have $\dot{x} = \frac{\partial x}{\partial t} = \omega(\epsilon)\frac{\partial x}{\partial \tau}$

and so, $\ddot{x} = \frac{\partial^2 x}{\partial t^2} = \omega^2(\epsilon)\frac{\partial^2 x}{\partial \tau^2}$

Mind you for the duration of this section, we will continue to use $\ddot{x}$ notation with the implication that derivative is now with respect to τ simply to lessen the confusion (and decrease writing)

with this our original DE becomes: $\omega^2(\epsilon)\ddot{x} + x - \epsilon x\omega(\epsilon)\dot{x} - \frac{\epsilon^2}{2}x^2\omega(\epsilon)\dot{x} = 0$

note we will allow $w_0 = 1$ to get the harmonic oscillator

Expanding our equation

$(((1+\epsilon\omega_1+\epsilon^2\omega_2+\epsilon^3\omega_3)(1+\epsilon\omega_1+\epsilon^2\omega_2+\epsilon^3\omega_3))(\ddot{x}_0+\epsilon\ddot{x}_1+\epsilon^2\ddot{x}+\epsilon^3\ddot{x}_3))+(x_0+\epsilon x_1+\epsilon^2 x_2+\epsilon^3 x_3)-((\epsilon(x_0+\epsilon x_1+\epsilon^2 x_2+\epsilon^3 x_3)(w_0+\epsilon w_1+\epsilon^2 w_2+\epsilon^3 w_3)(\dot{x}_0+\epsilon\dot{x}_1+\epsilon^2\dot{x}_2+\epsilon^3\dot{x}_3))-(\frac{\epsilon^2}{2}((x_0+\epsilon x_1+\epsilon^2 x_2+\epsilon^3 x_3)(x_0+\epsilon x_1+\epsilon^2 x_2+\epsilon^3 x_3))(\dot{x}_0+\epsilon\dot{x}_1+\epsilon^2\dot{x}_2+\epsilon^3\dot{x}_3)(w_0+\epsilon w_1+\epsilon^2 w_2+\epsilon^3 w_3))$

Collecting Epsilon powered terms

ϵ^0: $\quad \ddot{x}+x_0=0$

ϵ^1: $\quad 2\omega_1\ddot{x}_0+\ddot{x}_1+x_1-x_0\dot{x}_0=0$

ϵ^2: $\quad 2\omega_2\ddot{x}_0+\omega_1^2\ddot{x}_0+2\omega_1\ddot{x}_1+\ddot{x}_2+x_2-x_0\dot{x}_1-w_1x_0\dot{x}_0-x_1\dot{x}_0$

$-\frac{1}{2}x_0^2\dot{x}_0=0$

ϵ^3: $\quad 2w_1w_2\ddot{x}_0+2w_3\ddot{x}_0+2w_2\ddot{x}_1+2w_1\ddot{x}_2+\ddot{x}_3+x_3-x_0\dot{x}_2-w_1x_0\dot{x}_1$

$-x_1\dot{x}_1-w_2x_0\dot{x}_0-w_1x_1\dot{x}_0-x_2\dot{x}_0-\frac{1}{2}w_1x_0^2\dot{x}_0-\frac{1}{2}x_0^2\dot{x}_1-\frac{1}{2}w_1x_0x_1\dot{x}_0$

$-\frac{1}{2}x_0x_1\dot{x}_1-\frac{1}{2}x_0x_1\dot{x}_0$

2.1 Solving: ϵ^0

ϵ^0: $\ddot{x} + x_0 = 0$

Supposing our conditions for the original DE is

$x(0) = A$ and $\frac{\partial x}{\partial t}(0) = B$

Then the solution to the ϵ^0:

is $x_0(\tau) = A\cos(\tau) + B\sin(\tau)$

2.2 Solving: ϵ^1

ϵ^1: $2\omega_1\ddot{x}_0 + \ddot{x}_1 + x_1 - x_0\dot{x}_0 = 0$

which becomes: $\ddot{x}_1 + x_1 = x_0\ddot{x}_0 - 2\omega_1\ddot{x}_0$

to compute the RHS first, we'll need:

$x_0(\tau) = A\cos(\tau) + B\sin(\tau)$

$\dot{x}_0(\tau) = -A\sin(\tau) + B\cos(\tau)$

$\ddot{x}_0(\tau) = -A\cos(\tau) - B\sin(\tau)$

then we have:

$(A\cos(\tau) + B\sin(\tau))(B\cos(\tau) - A\sin(\tau)) + 2\omega_1(A\cos(\tau) + B\sin(\tau)$

Applying trigonometric properties we obtain:

$[B^2 - A^2]\frac{\sin(2\tau)}{2} + AB\cos(2\tau) + 2\omega_1[A\cos(\tau) + B\sin(\tau)]$

As we have the freedom of choosing our w_n terms to remove secular terms, we are faced with the trivial solution in this case. i.e, to let $w_1 = 0$

Now the ϵ^1 equation becomes:

$\ddot{x} + x = [B^2 - A^2]\frac{\sin(2\tau)}{2} + AB\cos(2\tau)$

solve this using the *undetermined coefficient method*

using the initial conditions with

$x_1(0) = 0$: $\quad\quad \dot{x}_1(0) = 0$

yields:

$x_1(\tau) = \frac{AB}{3}\cos(\tau) + \frac{B^2-A^2}{12}\sin(\tau) - \frac{AB}{3}\cos(2\tau) + \frac{A^2-B^2}{6}\sin(2\tau)$

2.3 Solving: ϵ^2

ϵ^2: $\quad\quad 2\omega_2\ddot{x}_0 + \omega_1^2\ddot{x}_0 + 2\omega_1\ddot{x}_1 + \ddot{x}_2 + x_2 - x_0\dot{x}_1 - w_1x_0\dot{x}_0 - x_1\dot{x}_0$

$\quad\quad -\frac{1}{2}x_0^2\dot{x}_0 = 0$

Rewriting the equation, with $w_1 = 0$:

$\ddot{x}_2 + x_2 = x_0\dot{x}_1 + \dot{x}_0x_1 - 2\omega_2\ddot{x}_0 + \frac{1}{2}\dot{x}_0x_0^2$

To compute this, we'll need the following

$x_0(\tau) = A\cos(\tau) + B\sin(\tau)$

$\dot{x}_0(\tau) = -A\sin(\tau) + B\cos(\tau)$

$\ddot{x}_0(\tau) = -A\cos(\tau) - B\sin(\tau)$

and

$x_1(\tau) = \frac{AB}{3}\cos(\tau) + \frac{B^2-A^2}{12}\sin(\tau) - \frac{AB}{3}\cos(2\tau) + \frac{A^2-B^2}{6}\sin(2\tau)$

$\dot{x}_1(\tau) = -\frac{AB}{3}\sin(\tau) + \frac{B^2-A^2}{12}\cos(\tau) - \frac{AB}{3}\cos(2\tau) + \frac{A^2-B^2}{6}\sin(2\tau)$

The RHS becomes:

$-\frac{A^2B}{3}\sin(\tau)\cos(\tau)+\frac{AB^2-A^3}{12}\cos^2(\tau)-\frac{A^2B}{3}\cos(2\tau)\cos(\tau)+\frac{A^3-AB^2}{6}\sin(2\tau)\cos(\tau)-\frac{AB^2}{3}\sin^2(\tau)+\frac{B^3-A^2B}{12}\cos(\tau)sin(\tau)-\frac{AB^2}{3}\cos(2\tau)\sin(\tau)+\frac{A^2B-B^3}{6}\sin(2\tau)\sin(tau)-\frac{A^3B}{3}\sin(\tau)cos(\tau)+\frac{A^3-AB^2}{12}\sin^2(\tau)+\frac{A^3B}{3}\cos(2\tau)\sin(\tau)+\frac{AB^2-A^3}{6}\sin(2\tau)\sin(\tau)+\frac{AB^2-A^3}{6}\sin(2\tau)\sin(\tau)+\frac{AB^2}{3}\cos^2(\tau)+\frac{B^3-A^2B}{12}\sin(\tau)\cos(\tau)-\frac{AB^2}{3}\cos(2\tau)\cos(\tau)+\frac{A^2B-B^3}{6}\sin(2\tau)\cos(\tau) - 2\omega_2(-A\cos(\tau) - B\sin(\tau)) - \frac{1}{2}(A^3\cos^3(\tau)\sin(\tau) - AB^2\sin^2(\tau)-A^2B\sin(2\tau)\sin(\tau)+A^2B\cos^3(\tau)+B^3\sin^2(\tau)\cos(\tau)+AB^2\sin(2\tau)\cos(tau))$

Collecting like terms and using trig properties

$\frac{1}{24}[(2A+3B)(A^2+B^2)\cos(\tau) + (3A+2B)(A^2+B^2)\sin(\tau)+$

$2A(5B^2-A^2)\cos(2\tau) - 2B(5A^2-B^2)\sin(2\tau) + (3B(3A^2-B^2)+$

$6A(A^2-3B^2))\cos(3\tau) + ((3A(3B^2-A^2) - 6B(B^2-3A^2))\sin(3\tau)]$

$-2\omega_2(-A\cos(\tau) - B\sin(\tau))$

The terms causing secular terms are:

$\frac{(2A+3B)(A^2+B^2)\cos(\tau)+(3A+2B)(A^2+B^2)\sin(\tau)}{24} - 2\omega_2(-A\cos(\tau) - B\sin(\tau))$

$$2A\omega_2 = -\frac{(2A+3B)(A^2+B^2)}{2A}$$

$$2B\omega_2 = -\frac{(3A+2B)(A^2+B^2)}{24}$$

Dividing the first Equation by the second achieves:

$$\frac{A}{B} = \frac{2A+3B}{3A+2B}$$

$$3A^2 + 2AB = 2AB + 3B^2$$

$$A^2 = B^2$$

Now I believe this is similar to the condition that appears on page 150 of the notes on the section of Perturbation.

Using $A^2 = B^2$, we achieve the value for ω_2 to remove secular terms

$$\omega_2 = -\frac{A(2A+3B}{24} = -\frac{B(3A+2B}{24}$$

Thus, we need to solve:

$$\ddot{x}_2 + x_2 = 2A(5B^2 - A^2)\cos(2\tau) - 2B(5A^2 - B^2)\sin(2\tau)$$

$$+(3B(3A^2 - B^2) + 6A(A^2 - 3B^2))\cos(3\tau)$$

$$+((3A(3B^2 - A^2) - 6B(B^2 - 3A^2))\sin(3\tau)$$

Again, we use undet. coeff. Method with the complementary form guess:

$$C\cos(\tau)+D\sin(tau)+E\cos(2\tau)+F\sin(2\tau)+G\cos(3\tau)+H\sin(3\tau)$$