Topological Solitons in Chiral Nematic Liquid Crystals

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Chiral Nematic Liquid Crystals

Liquid crystals are phases of matter that lie between simple liquids and traditional crystals in their molecular order.

They are composed of rod-shaped molecules. The average locally preferred direction is known as the director $\mathbf{n}$.

Chiral nematics have a helical director field. The cost that splay, bend and twist distortions incur on a chiral nematic liquid crystal is given by the Frank free energy

$$F = \frac{1}{2}K\int_V dV\left\{|\nabla\mathbf{n}|^2 + q_0\,\mathbf{n}\cdot\nabla\times\mathbf{n}\right\}.$$

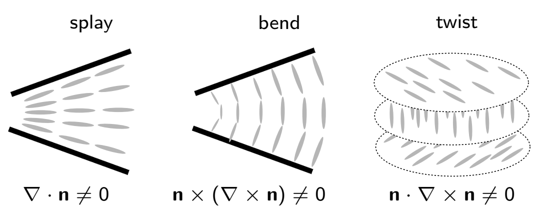

Figure: Liquid crystal fields undergo splay, bend and twist distortions.

Topological Solitons

Two vector fields $\mathbf{n_1}$ and $\mathbf{n_2}$ are homotopic if they can be continuously deformed to look the same.

If two liquid crystal director fields are not homotopic, an energy barrier forbids the transition from $\mathbf{n_1}$ to $\mathbf{n_2}$.

A Hopfion is a field configuration where the preimage of any orientation forms a closed loop. Any two such loops form a Hopf link. The Hopf charge Q counts how many times the preimages of any two orientations link.

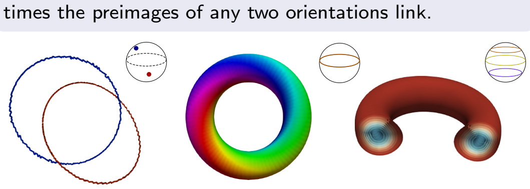

Figure: Preimages of two orientations, a circle of constant latitude, and several circles of constant latitude on S^2 under the standard Hopf map.

Rational Map Construction

In the literature, topological solitons are described using complex rational maps. The Hopf map is given by

$$\eta : \mathbb{C}^2 \to \mathbb{C}, \quad \eta(z_1, z_2) = \frac{z_1}{z_2}.$$

There is no direct method of deriving these maps.

Solid Angle Function

The solid angle ω of a curve K equals the magnetostatic potential of a K shaped wire carrying unit current.

We have used this to construct a vector field around a loop that is homotopic to the Hopf map.

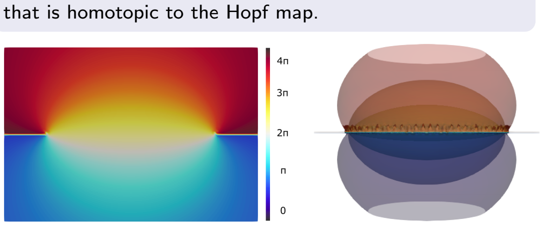

Figure: The solid angle function of a circle through a cross section of the circle and in the far field.

Hopfion Stability: Chiral Energy

The chirality pseudotensor

$$\chi_{ij} = \partial_i n^k \epsilon_{jkl} n^l$$

is designed so that $v^i\chi_{ij}u^j$ describes how $\mathbf{n}$ rotates about $\mathbf{u}$ when moving along $\mathbf{v}$. The trace of χ_{ij} equals the twist of $\mathbf{n}$.

Dirichlet energy measures variability

$$E^{(1)} = \sum_{i=1}^{3}\int |\partial_i n|^2.$$

Chiral energy measures chirality

$$E^{(2)} = \sum_{i=1}^{3}\int |n\times\partial_i n|^2.$$

We have proven that topological solitons are stabilised by increasing these energies. More precisely,

$$|Q| \leq c\cdot E^{(1)}E^{(2)}.$$

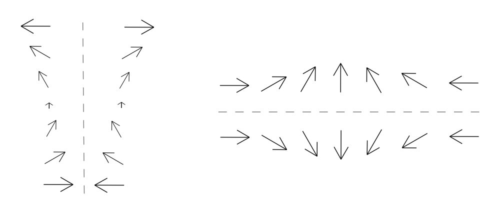

Figure: Helicoidal and planar chirality and their mirror images.

Hopfion Dynamics: Free Energy

Chiral nematic liquid crystals prefer a uniform sense of twist. The twist $\mathbf{n}\cdot\nabla\times\mathbf{n}$ of a Hopfion changes sign. There is an energetic drive to correct this, causing a contraction.

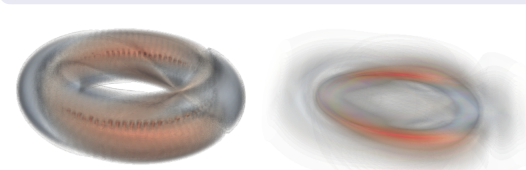

Figure: Twist isosurfaces of a Hopfion after 1 and 3000 relaxation time steps respectively. The incorrectly twisted region (red) has contracted.

Hopfion Dynamics: Chiral Energy

Hopfions have an energetic drive to maximise chiral energy. Zero twist surfaces minimise diagonal chiral energy density

$$\epsilon^{(1)} = \sum_{i,j}\chi_{ij}^2\delta_{ij},$$

so they must maximise off-diagonal chiral energy density

$$\epsilon^{(2)} = \sum_{i,j}\chi_{ij}^2(1-\delta_{ij}).$$

The far field of a Hopfion minimises chiral energy so that

$$\sum_{i,j,k}\frac{1}{2}u_k(\partial_i^2 u_k - \partial_i u_j) + \partial_i u_j(\partial_i u_j - \partial_i u_k) = 0.$$

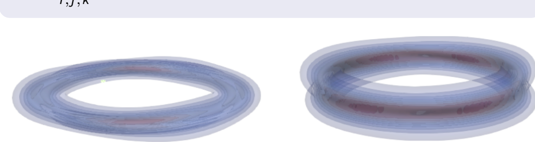

Figure: Isosurfaces of the diagonal and off-diagonal chiral energy densities for a relaxed Hopfion.

Uniform Topological Solitons

Uniform topological solitons are vector fields for which the preimage of any orientation yields the same knot and the preimages of any two orientations yields the same link.

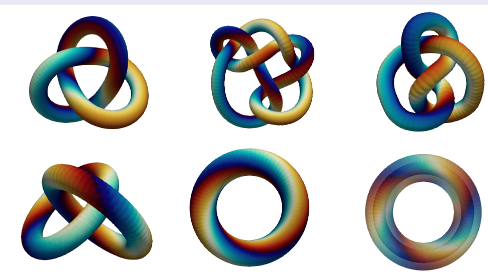

Figure: Equatorial preimages of uniform topological solitons constructed using the solid angle function.

Non-Uniform Topological Solitons

Non-uniform topological solitons exhibit a different linking of preimages for different orientations. These are constructed using the solid angle function by summing the vector fields of different knotted fields together.

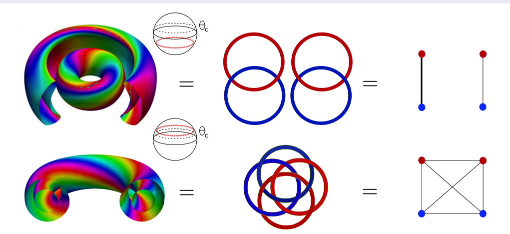

Figure: The linking of preimages in a non-uniform topological soliton changes as the orientation varies over a critical angle θ_c. This means their Hopf invariant can be undefined.

Conclusions

- The chiral energy of a system plays a central role in stabilising and deforming topological solitons irrespective of the field theory they are set in.
- The solid angle function can be used to construct arbitrary topological solitons and this supersedes the well established rational map construction.
- Non-uniform topological solitons which have not previously appeared in the theoretical literature can be constructed using the solid angle function.
- Further work includes a study of chiral energy, the modeling of singular topological solitons such as torons and twistions, and an investigation of non-uniform topological solitons with undefined Hopf charge.

References

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