Technical University of Denmark

42104 Introduction to Financial Engineering

Final project

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Contents

1 Introduction

In this report we first and foremost investigate the Standard & Poor's depositary receipt exchange traded funds (SPDR ETFs) in correspondance to the S&P itself. Secondly, we will briefly look at a few select bonds and perform various calculations, such as convexity, maturity etc.

2 Portfolio Optimization

2.1 Diversification

In this section we select the portfolio we wish to analyze. We choose to look at the nine ETFs that comprise the S&P500 market. That is, we construct a portfolio consisting of the following:

- **XLE** - Energy
- **XLU** - Utilities
- **XLK** - Technology
- **XLB** - Materials
- **XLP** - Consumer Staples
- **XLY** - Consumer Discretionary
- **XLI** - Industrials
- **XLV** - Health Care
- **XLF** - Financials
- **(S&P500)** - The market as a whole

The S&P500 will not be a part of our portfolio but will be used for benchmarking.

2.1.1 Describing the data

For the above we have collected monthly data for the last 17 years, specifically from 01/01-1999 to 01/01-2016. This has been done by the use of the function `hist_stock_data` in `Matlab`.

```
stocks = hist_stock_data('01011999', '01012016', 'frequency', 'm' ...
                         'XLE','XLU','XLK','XLB','XLP','XLY','XLI','XLV','XLF','^GSPC');
```

We note that, the data acquired through `hist_stock_data` is stored in such a way that the earliest time point is stored in the last entry of the struct array. Take e.g. the very last entry in the vector with adjusted close prices for the XLE ETF. This entry will be the value corresponding to the very first time selected, in this case 01/01-1999. Because of this, all analyses and calculations have been performed on the flipped data.

After acquiring the desired data we calculate and store the log returns of the adjusted close prices. That is, returns R_t of the indices are calculated by

$$R_t = \log(P_t/P_{t-1})$$

where P_t is the net return in period t.

The final data consists of a 203×9 matrix with log returns for the nine ETFs together with a separate 203×1 vector with log returns for the S&P500. We chose the SPDR ETFs because they can be adequately benchmarked against the S&P500 given that the former (together as a whole) constitute the latter. This allows us to compare the performance of our optimal portfolio with that of the market more robustly.

2.2 Expected returns estimation

In this section we will estimate the expected returns and covariance matrices of our portfolio within a rolling window of 10 years. However, since we're only able to extract 17 years of historical data of the ETFs, (since most of the ETFs were created around 1999), we will be expanding the rolling window for the first 5 iterations. More precisely, we estimate expected returns and covariance matrices in the following time spans:

- **Window 1** - 6 year window from 1/1-1999 to 1/1-2005
- **Window 2** - 7 year window from 1/1-1999 to 1/1-2006
- **Window 3** - 8 year window from 1/1-1999 to 1/1-2007
- **Window 4** - 9 year window from 1/1-1999 to 1/1-2008
- **Window 5** - 10 year window from 1/1-1999 to 1/1-2009
- **Window 6** - 10 year window from 1/1-2000 to 1/1-2010
- **Window 7** - 10 year window from 1/1-2001 to 1/1-2011
- **Window 8** - 10 year window from 1/1-2002 to 1/1-2012
- **Window 9** - 10 year window from 1/1-2003 to 1/1-2013
- **Window 10** - 10 year window from 1/1-2004 to 1/1-2014
- **Window 11** - 10 year window from 1/1-2005 to 1/1-2015

We don't consider a window from 01/01-2006 to 01/01-2016, since, as we will see later, the last year (2016) will be used in backtesting our portfolio.

To estimate the expected yearly returns we simply take the mean of the log returns for each ETF and multiply by 12, since we collected monthly data. Similarly, we take the covariance of the log returns for each window and multiply by 12 to get the yearly covariance matrices. Below in table 1 we have listed the expected returns from the following windows. We leave it to the reader to check the covariance matrices for the 11 windows in the supplied Matlab code.

	$\bar{R}_{\text{XLE}}$	$\bar{R}_{\text{XLU}}$	$\bar{R}_{\text{XLK}}$	$\bar{R}_{\text{XLB}}$	$\bar{R}_{\text{XLP}}$	$\bar{R}_{\text{XLY}}$	$\bar{R}_{\text{XLI}}$	$\bar{R}_{\text{XLV}}$	$\bar{R}_{\text{XLF}}$
Window 1:	10.79%	2.86%	-10.19%	7.45%	-0.84%	4.17%	4.86%	1.84%	5.46%
Window 2:	15.55%	4.66%	-7.48%	8.07%	-0.47%	3.48%	5.00%	3.10%	5.97%
Window 3:	13.86%	6.11%	-5.27%	9.08%	1.51%	5.29%	6.18%	3.78%	7.40%
Window 4:	14.45%	6.50%	-4.83%	9.37%	1.81%	2.71%	6.04%	3.28%	4.19%
Window 5:	9.21%	3.18%	-8.83%	2.29%	-0.17%	-2.64%	-0.17%	0.64%	-7.31%
Window 6:	8.49%	4.19%	-8.01%	4.94%	3.13%	1.64%	1.72%	1.56%	-2.59%
Window 7:	9.83%	3.70%	-2.67%	8.46%	2.94%	3.72%	3.16%	2.17%	-3.61%
Window 8:	11.68%	6.18%	2.39%	7.63%	4.62%	4.51%	5.18%	4.45%	-3.90%
Window 9:	14.24%	10.38%	8.34%	9.63%	8.85%	9.34%	9.15%	6.50%	0.01%
Window 10:	12.40%	8.65%	6.29%	7.83%	8.70%	8.28%	8.13%	7.63%	-1.01%
Window 11:	8.59%	8.99%	8.29%	7.42%	9.74%	8.67%	7.94%	10.37%	-0.41%

Table 1: *Expected returns of 9 ETFs for each window.*

2.3 Efficient Frontier and Tobin Separation

In this section we find the efficient frontier with a 1% riskfree rate (Tobin separation) and without a riskfree rate in each of the 11 windows.

By taking linear combinations of two optimal portfolios (each derived from a different riskfree rate (we used $RF_1 = 1\%$, $RF_2 = 25\%$)), we can delineate the efficient frontier curve:

$$\text{New Optimal Portfolio} = (\lambda)\text{opt}P_1 + (1-\lambda)\text{opt}P_2$$

where λ can take any value from 0 to 1, and the number of iterations (for different λ values) depends on the resolution desired. The two initial optimal portfolios $(\text{opt}P_1, \text{opt}P_2)$ are derived using the supplied `highest_slope_portfolio` `MATLAB` function.

Figure 1 below shows the 11 efficient frontiers curves (without riskless asset) along with their corresponding 11 efficient frontier lines (when riskless asset is available); the highest slope portfolios are denoted with crosses.

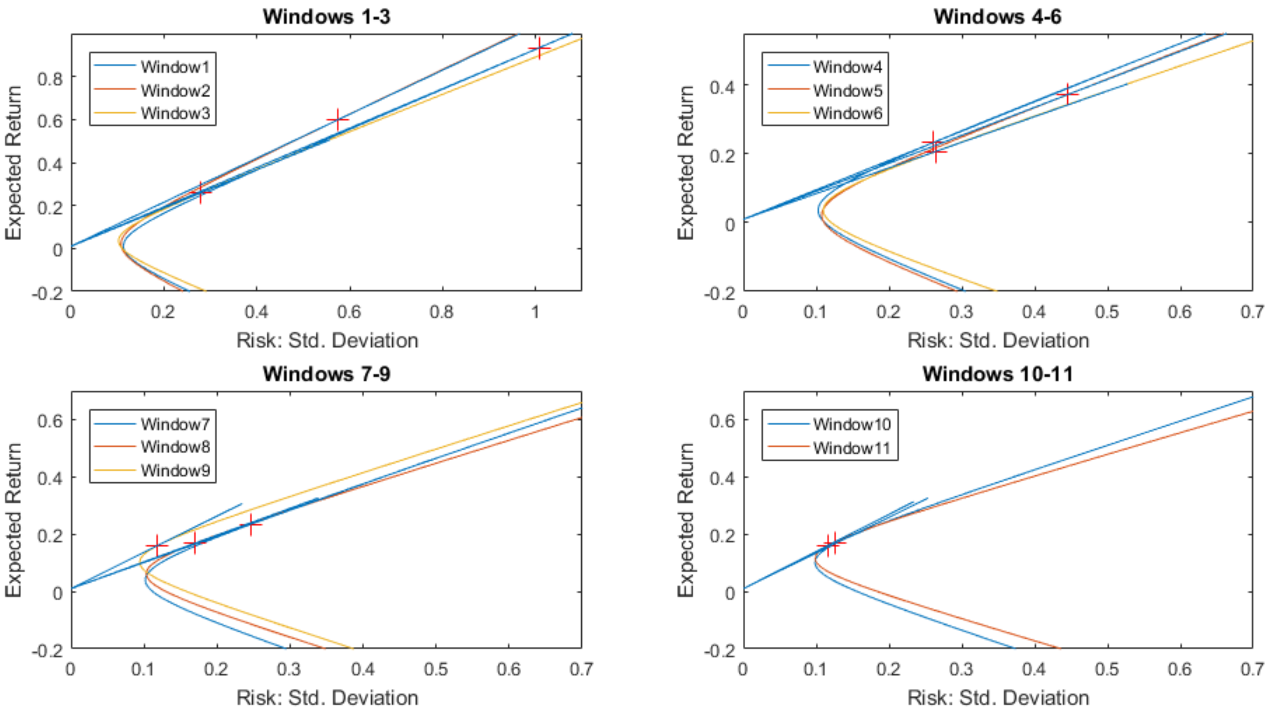


Figure 1: *11 Efficient frontiers, with and without riskfree rate.*

It can be assumed that assets with high potential returns also carry higher risk. Conversely, assets with low risk offer less potential return. To match an investor's risk tolerance, it may not necessarily entail picking assets with similar risk values. Rather, an optimal portfolio balances a variety of assets that on aggregate match the investor's risk preference. Through diversification, the resulting portfolio's risk is less than the sum of its weighted individual assets. Assets that perform poorly are counteracted by assets that perform favorably. Through this reduction of diversifiable risk, we obtain the benefit of having an optimal portfolio that offers greater return for a given level of risk, or conversely, less risk for a given level of return.

The efficient frontier curves in Figure 1 outline the optimal portfolios in the return-risk space (without riskfree assets). Any portfolios in the interior of the curve are considered suboptimal for they don't offer a better return-to-risk ratio than those in the frontier. In general, a portfolio can be moved closer to the efficient frontier by diversifying it.

Diversification Benefit Ultimately, the benefit of diversification can be summed up with the following insight: a portfolio's return is simply a weighted average of its securities. The same however, cannot be said of a portfolio's standard deviation due to cross-terms:

$$\sigma_p = [x_1^2\sigma_A^2 + x_2^2\sigma_B^2 + 2x_1x_2\sigma_{AB}]^{1/2}, \text{ where } \sigma_{AB} = \varrho_{AB}\sigma_A\sigma_B$$

The equation above shows the standard deviation of a portfolio with two assets. When the correlation coefficient (ϱ_{AB}) equals 1, the portfolio's standard deviation is simply a weighted average.

$$\sigma_p^2 = (x_1\sigma_A + x_2\sigma_B)^2 = [x_1^2\sigma_A^2 + x_2^2\sigma_B^2 + 2x_1x_2\sigma_A\sigma_B \cdot 1]$$

In any other case however ($\varrho_{AB} < 1$), the cross-terms become less than 1 therefore reducing the overall risk of the portfolio.

This reduction of risk also applies on a portfolio with more than 2 assets. In fact, as more assets are incorporated, the more a portfolio's risk is reduced (albeit in a diminishing returns fashion). The equation below represents the multi-asset case:

$$\sigma_p^2 = \sum_{n=1}^{N} x_n^2\sigma_n^2 + \sum_{n=1}^{N}\sum_{m}^{N} x_n x_k \sigma_{mn}$$

If we assign equal weights to all assets, then:

$$\sigma_p^2 = (\frac{1}{N})\sum_{n=1}^{N}(\frac{1}{N})\sigma_n^2 + (\frac{1}{N})\sum_{n=1}^{N}\sum_{m=1}^{N}(\frac{1}{N})\sigma_{mn}$$

and by introducing an $\frac{N-1}{N-1}$ in the second summation we get:

$$\sigma_p^2 = (\frac{1}{N})\sum_{n=1}^{N}(\frac{\sigma_n^2}{N}) + (\frac{N-1}{N})\sum_{n=1}^{N}\sum_{m=1}^{N}(\frac{\sigma_{mn}}{N(N-1)})$$

Notice that the terms after the summations are averages of the individual securities and the covariance terms, respectively. The terms outside the summation tell us that as $N \to \infty$, the variance stemming from the individual securities approaches 0, and thus is diversifiable. In contrast, the variance stemming from the covariance terms approaches the average of the covariances as $N \to \infty$, and is thus not diversifiable. More discussion of diversifiable and systemic risk in Section 2.6.

Theoretically, a portfolio's standard deviation could approach 0 if we had negatively correlated assets or a sufficiently large number of independent assets. In practice, however, this is highly unlikely. For instance, in our first window the range of correlation coefficients is from 0.17 to 0.82. Nonetheless, even within this range, the benefit of diversification is apparent. By combining different assets, diversification reduces the risk of a portfolio, and this is a huge benefit that can be leveraged in portfolio management.

Tobin Separation Without a riskfree asset, we are limited to investing within the constraints of the curved efficient frontier. With access to a riskless asset though (e.g. savings account/loan), we can distribute our capital between an optimal portfolio and a savings account. Linear combinations of these two assets form a line stretching from the y-intercept (since risk is zero) to the highest slope portfolio in the efficient frontier. We chose the highest slope portfolio (denoted as "+"s in Figure 1) because as rational investors we seek the largest Sharpe Ratio possible (more discussion on this in Black-Litterman model section).

Assuming we have access to unlimited loans at this riskless rate, we can then use this loaned capital to grow the size of our portfolio investment. In a sense, we could say we are shortening the riskless asset. This would be graphically illustrated as a ray stretching upwards from the highest slope portfolio in tandem with the borrowing line. Therefore, all combinations of riskless borrowing and lending with our ETF portfolio form a line tangent to the highest slope porfolio of the efficient frontier curve (see Figure 1). The benefit of such a world is that we can hold the best possible Sharpe Ratio we can find in the frontier curve, and scale it up (i.e. borrowing/shorting) or down (i.e. lending) according to our risk or expected return preference.

2.4 Asset Allocation

Now that we have found an optimal portfolio and have access to a riskless asset ($RF = 1\%$), we can calculate how much we will need to borrow or lend to meet our desired risk or return preference. We choose a constant required return ($\bar{R}_{req.}$) of 10%. Since the constant required return is derived as:

$$\bar{R}_{req.} = X_i \cdot \bar{R}_{p,i} + (1 - X_i) \cdot \mathrm{RF}, \qquad i = 1, \ldots, 11$$

We can calculate X_i, the fraction of our capital invested in our ETF portfolio (as opposed to invested in a savings account), for any given $\bar{R}_{req.}$ as follows:

$$X_i = \frac{\bar{R}_{req} - RF}{\bar{R}_{p,i} - RF}$$

where $RF = 1\%$ is the risk free rate, X_i is the percentage invested in our ETF portfolio in window i, and $\bar{R}_{p,i}$ is the expected return of our portfolio in window i. Notice that if $X_i > 1$, this means we will borrow money at the riskless rate (1%) and invest it in our ETF portfolio. Conversely, if $X_i < 1$, then we will invest that X_i fraction of our capital into our ETF porfolio, and the remaining capital $(1-X_i)$ will be placed in a savings account at the riskless rate.

Finally, to calculate the weights of each ETF that meet our desired constant required return of 10%, we scale as follows:

$$\textit{Asset Allocation} = X_i * \mathrm{opt}P_i$$

where $\mathrm{opt}P_i$ is the optimal portfolio vector in window i, and whose elements correspond to the weights for each of the nine ETFs. Notice that the proportion or ratio between the ETFs stays the same; they are simply scaled up or down the efficient frontier line by multiplying them with X_i. Figure 2 shows the calculated asset allocations for the 2005-2015 time period. For numerical values, the reader is referred to the Appendix.

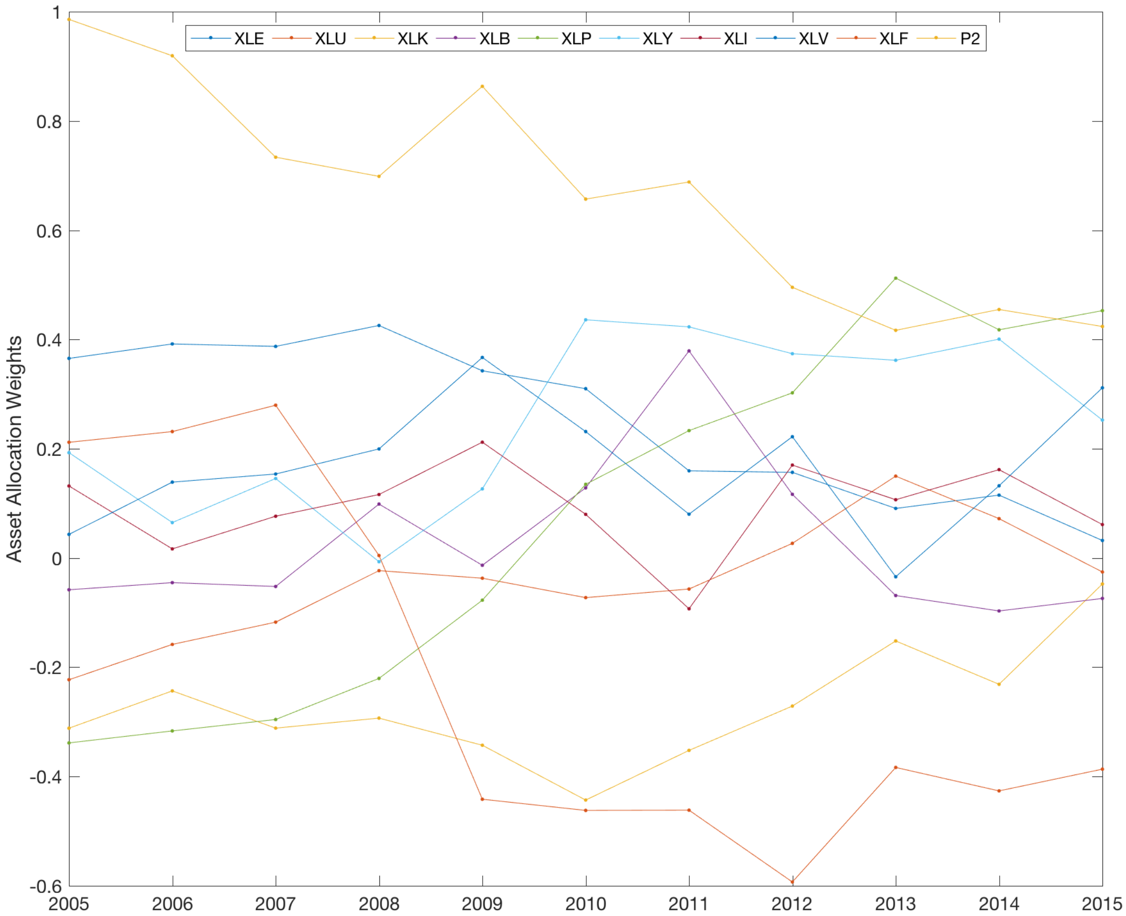


Figure 2: *The asset allocation per year for a constant required return of* 10% *is shown above. Here, P2 is the fraction of our capital invested in the riskless rate. Given our desired return of* 10%*, it is not surprising much of our capital is channeled to a riskless asset. If we had chosen* 25%*, P2 would be less and potentially even negative - meaning we would short it to leverage our portfolio. Note the drastic drop of XLF (Financials -and which also consists of real estate assets-) from 2007 to 2009 and onwards, interestingly coinciding with "the great recession."*

Next, we investigate the turnover needed each year to maintain a constant required return of 10%.

Let optP_i be the optimal portfolio vector in window i, and whose elements correspond to the weights for each ,jth, ETF. From one year to the next we find the turnover as:

$$\text{Turnover}_i = \begin{cases} ||X_i \cdot \text{opt}P_i||_1 = \sum_{j=1}^{9} |X_i \cdot \text{opt}P_{i,j}|, & \text{if } i = 1 \\ ||X_i \cdot \text{opt}P_i - X_{i-1} \cdot \text{opt}P_{i-1}||_1 = \sum_{j=1}^{9} |X_i \cdot \text{opt}P_{i,j} - X_{i-1} \cdot \text{opt}P_{i-1,j}|, & \text{if } i > 1 \end{cases}$$

where $||\cdot||_1$ is the taxi cab norm. Table 2 below shows the calculated turnovers.

Year:	'05	'06	'07	'08	'09	'10	'11	'12	'13	'14	'15
Turnover:	187.6%	55.3%	34.5%	89.0%	124.5%	112.0%	94.3%	108.5%	124.5%	60.7%	89.0%

Table 2: *Turnover for each year.*

The average turnover per year then becomes the average of the above, which is 98.18%. Note that the turnover rate is a direct measure of the net transactions (buying and selling or trading activity) of ETFs needed to maintain the constant required return. Because in this optimization task transaction costs are not taken into consideration (i.e. there's no penalty), the annual rebalancing of our portfolios can result in high turnover rates (more discussion on this in Black-Litterman section). All else being equal, a lower turnover rate is preferable as brokerage fees can reduce a portfolio's returns.

2.5 Backtest

We will now conduct a backtest for our optimized portfolio with annual re-balancing. The way we do this, is by utilizing the optimal asset allocation which we found in the previous section. That is, for each window we use that specific optimal asset allocation to calculate expected returns and standard deviations one year ahead of each respective window (i.e. out of sample). We calculate the expected returns and standard deviations as follows:

$$\bar{R}_{P,i} = \bar{R}^T_{ASSETS,i} \cdot \text{opt}P_i \cdot X_i + (1 - X_i) \cdot RF$$
$$\bar{\sigma}_{P,i} = \sqrt{\text{opt}P_i^T \cdot COV(Assets)_i \cdot \text{opt}P_i}$$

Where $\text{opt}P_i$ is as in section 2.4, $\bar{R}_{ASSETS,i}$ is the expected returns of the ETFs and $COV(Assets)_i$ is the covariance matrix of the log returns in window i multiplied by 12 to get the yearly covariance matrix. Notice that the standard deviation stemming from the RF term is not included as it's a riskless asset, and thus it's standard deviation is zero. The expected returns out of sample together with the standard deviations and expected returns of the S&P500 are shown in table 3.

Year:	'05	'06	'07	'08	'09	'10	'11	'12	'13	'14	'15
$\bar{R}_P$:	12.64%	-2.31%	-2.76%	-15.13%	-9.03%	12.10%	12.78%	6.49%	7.66%	9.66%	6.55%
$\bar{\sigma}_P$:	7	7.5	6.3	14	9.2	8.3	9.4	6.4	8.7	7	8.1
$\bar{R}_M$:	8.03%	11.65%	-4.24%	-51.23%	26.26%	18.04%	2.02%	13.23%	17.39%	11.26%	2.64%

Table 3: *Expected returns out of sample of optimal portfolio with annual rebalancing.*

In figure 3 we illustrate how we performed compared to the yearly expected returns of the S&P500 in the given years.

3.

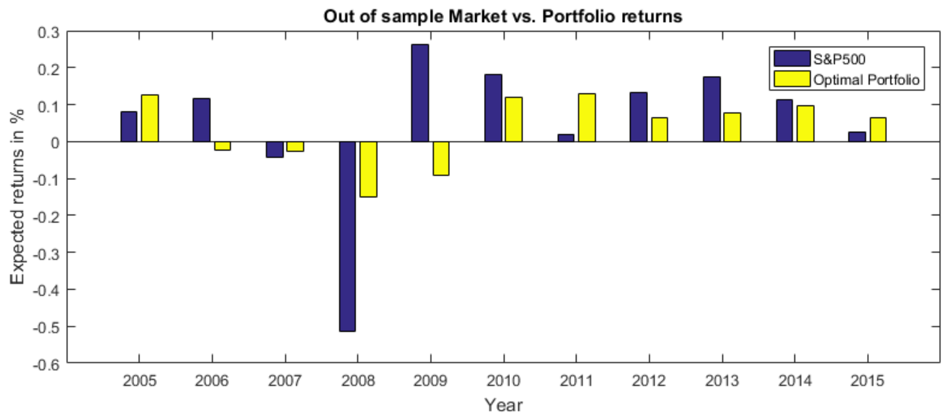


Figure 3: *Out of sample yearly mean returns of optimal portfolio vs. market returns. Notice the crash of the S&P500 in 2008. In comparison, the optimized portfolio faired well.*

2.6 Beta

We should check whether our portfolio is in line with the CAPM prediction. If the CAPM holds, one can describe the expected returns of our portfolio by the following relationship:

$$\bar{R}_P = RF + (\bar{R}_M - RF)\beta_P$$

However, it could be that our portfolio has over or under performed compared to the CAPM. If that is the case we should be able to describe that "excess" return by the α in the following equation:

$$\bar{R}_P = \alpha + RF + (\bar{R}_M - RF)\beta_P$$

This is also known as Jensen's alpha. To investigate this, note that we can rewrite the CAPM:

$$\bar{R}_P = RF + (\bar{R}_M - RF)\beta_P \Leftrightarrow \bar{R}_P - RF = (\bar{R}_M - RF)\beta_P$$

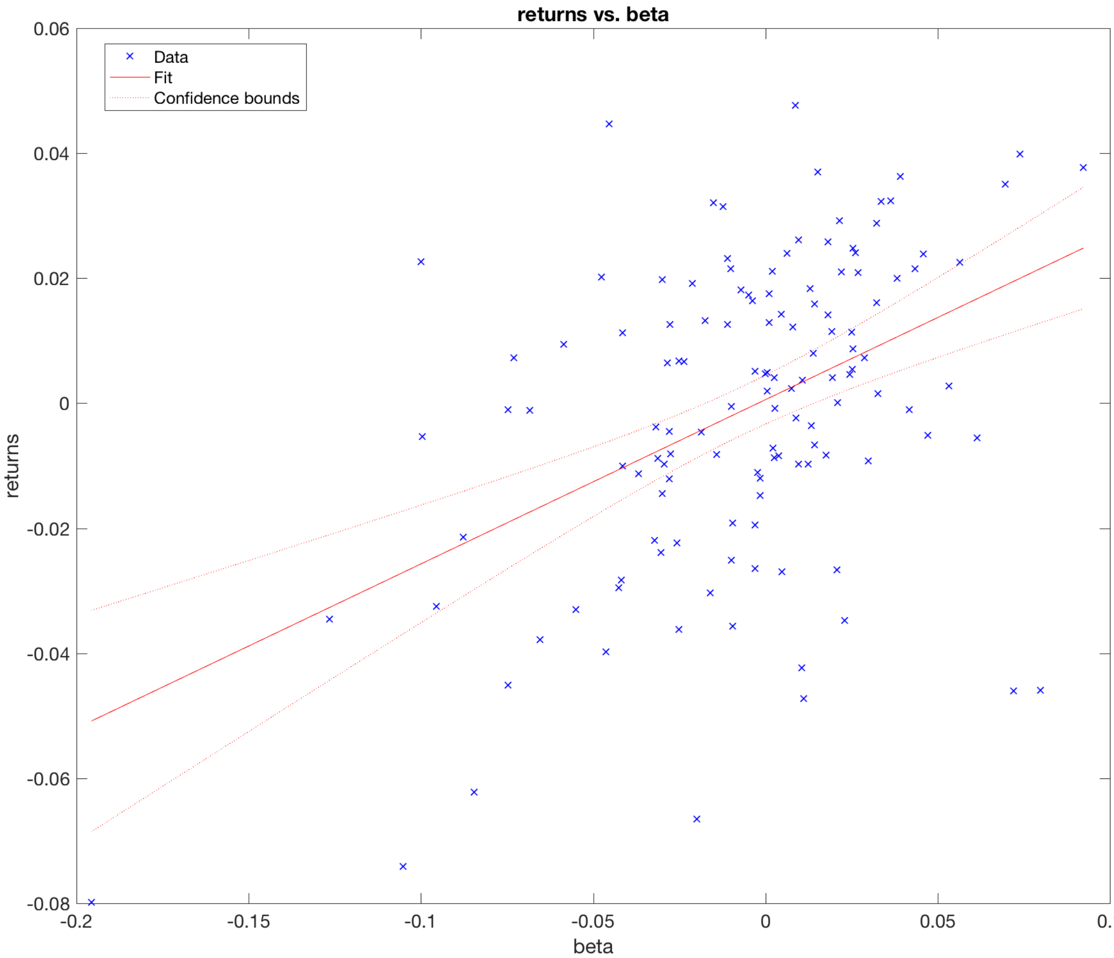
such that we can perform a linear regression with the response being $R_P - RF$ and the independent variable being $R_M - RF$. Thus a `fitlm` call in `Matlab` will produce our α (being the intercept) and our β_P. The output statistics are shown below:

```
Linear regression model:
    returns ~ 1 + beta

Estimated Coefficients:
                    Estimate        SE         tStat       pValue
                   __________    _________    _______    __________

    (Intercept)    0.00056881    0.0019748    0.28804       0.77378
    beta              0.26279     0.045998      5.713    7.2937e-08


Number of observations: 131, Error degrees of freedom: 129
Root Mean Squared Error: 0.0224
R-squared: 0.202, Adjusted R-Squared 0.196
F-statistic vs. constant model: 32.6, p-value = 7.29e-08
```



The question is now whether or not we have been able to produce an α which is significant. From the output statistics we see that $\alpha = 0.0005$ with a p-value of 77.4%. From this we can't reject the null hypothesis that our α is equal to zero (i.e. that there's no difference between the S&P500 and the optimized portfolio), hence we deem the α insignificant. This is perhaps not surprising when the excess return is only 0.05%. In contrast to the α, the β is highly significant with a p-value of essentially zero.

The β gives us a measure of how sensitive the portfolio is to "the market;" in this case we use the S&P500 as a representation of the market. A β of 0.263 tells us that for every unit of movement by the S&P500, our portfolio *should* (based on historical data) move 0.263 units in the same direction. Since the sensitivity (β) is less than 1, the historical data tells us that our portfolio should be less volatile than the S&P500. Resuming the discussion on risk from Section 2.3 on the benefits of diversification, it's noteworthy to highlight that β represents the systemic risk (i.e. non-diversifiable risk). In other words, while we can reduce the risk stemming from the individual securities through diversification, we cannot diversify systemic risk. Therefore, for well-diversified portfolios, the investor/manager is rewarded solely for the level of risk she/he takes as measured by β.

2.7 Black-Litterman

In this section we investigate how our asset allocation and the backtest done in section 2.5 change when we use the implied expected returns found by using the Black Litterman approach. Under the efficient frontier method, it is possible that the optimal asset allocation can include extreme -and particularly unintuitive- weights that although posses the highest Sharpe ratio, can nonetheless be vulnerable/risky bets given the extreme weights. To accommodate for this, the Black-Litterman approach sets the "market portfolio" as a neutral starting point that reflects the returns implied by the CAPM. If -given the investor's domain knowledge- the investor disagrees with those expected returns, then she/he can introduce her/his views (can be relative or absolute) of the market and produce optimal portfolios based on those views. As such, unintuitive or extreme weights are avoided, and the portfolio mirrors more closely the investor's views.

In order to calculate the implied expected returns we need the market weights of the 9 ETFs. The market weights can be found on e.g. `www.sectorspdr.com`. The weights together with our postulated views on the SPDR ETFs are given in table 4 below. The views we give in this report are not based on any particular market analysis on the SPDR ETFs. Rather, they are chosen arbitrarily to demonstrate the rationale behind the Black Litterman approach.

	XLE	XLU	XLK	XLB	XLP	XLY	XLI	XLV	XLF
Weights:	7.33%	3.11%	23.11%	2.85%	9.4%	12.16%	10.43%	14.25%	17.35%
Views:	-	$\searrow$	$\searrow$	-	$\searrow$	-	$\nearrow$	$\nearrow$	$\nearrow$

Table 4: *Weights of the 9 ETFs.*

Note that we use a fixed 3% increase or decrease in our views above. Lastly, note that the real-estate ETF, XLRE, is a part of the XLF (Financials) ETF and so we add the weight from XLRE to XLF which results in a weight of 17.35%.

In the coming we let $\gamma = 1.9$ be our risk aversion coefficient and let $\tau = 0.3$ be our precision factor. These values are taken as the default values presented in the exercise sessions.

First off, we find the expected returns based on the market weights as

$$\Pi = \omega \Omega \gamma$$

where Π is the expected returns based on the market weights, Ω is the covariance matrix of our yearly expected returns, ω is our market weights of the 9 ETFs and lastly γ is the risk aversion coefficient as stated above. We can now find the expected returns as

$$E(R) = \Pi + \tau \Omega P^T (P \tau \Omega P^T)^{-1} (V - P\Pi)$$

where P is our views on how the ETFs will change and V is the 3% change we expect.

With the implied expected returns we can then recalculate our weights ω as

$$\omega = (\gamma\Omega)^{-1}E(R)$$

and use these weights as our new "optimal portfolio". The above procedure is with so called certain views, as in, we don't let our view of a 3% increase/decrease vary with some error. With uncertain views the calculations change a bit in that we add a bit of uncertainty to our view. With uncertain views we can find the expected returns as

$$E(R) = [(\tau\Omega)^{-1} + P^T\Sigma^{-1}P]^{-1}[(\tau\Omega)^{-1}E(R) + P^T\Sigma^{-1}V]$$

where Σ is the uncertainty in our views, which in our case is set to 0.04. One can easily extend this to a matrix of various uncertainties if desired.

We then redo sections 2.4 through 2.6 with these new found weights. Note that, in this case our portfolio sums to one and hence we do not include a risk free rate in our portfolio.

Table 5 below shows the expected returns and standard deviations out of sample for the Black Litterman approach with certain and uncertain views.

Year:	'05	'06	'07	'08	'09	'10	'11	'12	'13	'14	'15
$\bar{R}_P$:	9.82%	12.61%	-8.49%	-69.88%	32.86%	19.33%	4.95%	14.83%	15.96%	15.55%	4.3%
$\bar{\sigma}_P$:	7.78	6.34	11.71	30.32	22.75	18.05	15.34	10.18	9.12	7.58	12.53
$\bar{R}_P(U)$:	9.85%	12.75%	-5.16%	-58.33%	32.39%	19.36%	4.45%	15.46%	18.30%	14.42%	4.41%
$\bar{\sigma}_P(U)$:	7.83	5.92	11.45	26.18	22.21	18.31	15.91	10.53	9.25	7.77	13.16

Table 5: *Expected returns of the portfolio derived from the Black Litterman model.*

Figures 4 shows the expected returns out of sample for the S&P500, the optimal portfolio derived through the efficient frontier (with riskless asset) and the Black Litterman approach with certain and uncertain views. We can see that with the Black Litterman approach we follow more closely the tendency of the S&P500. This makes sense as the BL method is essentially the S&P500 with the investors views incorporated in them. Therefore, unless the investors views are radically different from the implied returns, the BL portfolio should have exposure very similar to that of the market. The year of 2008 serves as a good illustration: both the S&P500 and the BL portfolios experienced similar losses while the mean-variance portfolio did not. The β of the portfolios tell us a similar story. While the β of the mean-variance portfolio is 0.263 (less sensitive to S&P500), the β of the BL portfolio is 1.089 (more sensitive to portfolio). Results from BL regression are not shown.

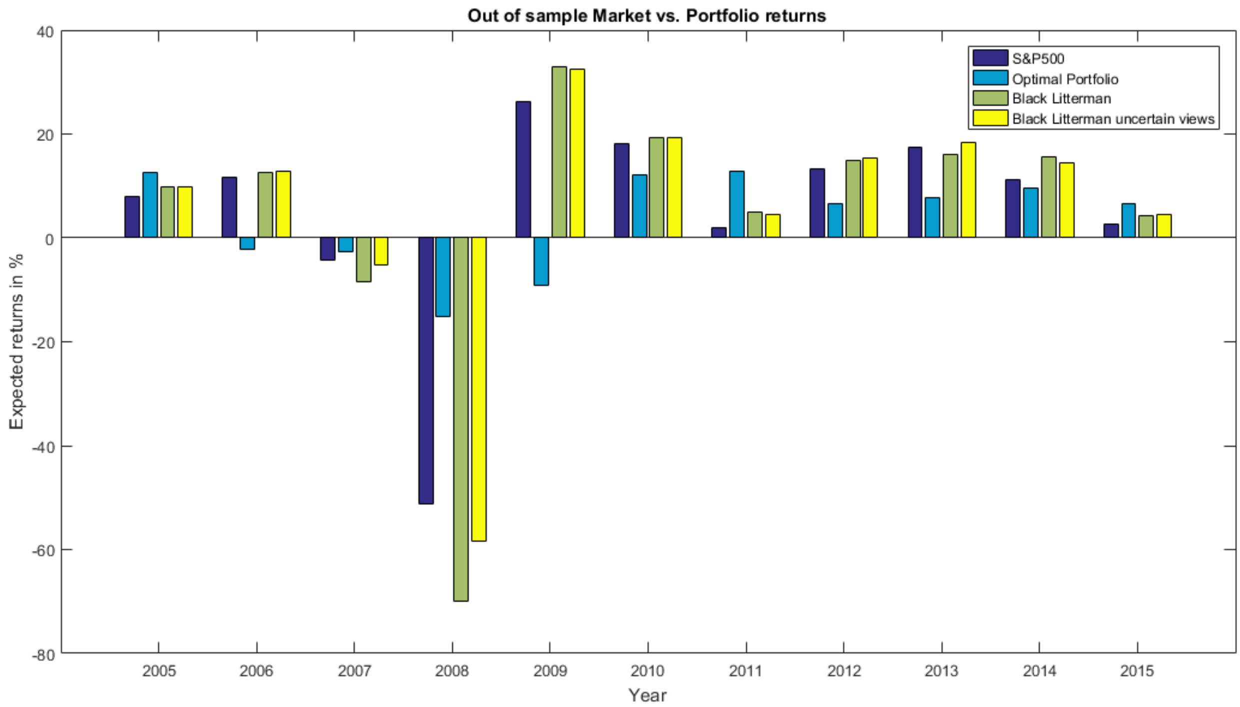


Figure 4: *Expected returns out of sample for the S&P500, optimal portfolio and Black Litterman approach.*

Figure 5 below illustrates how the turnover of our portfolio changes when considering the Black Litterman approach. Note the substantial reduction in turnover for the Black Litterman approach. Since the BL starts with the "market weights" and is constrained by the investor's views, it is less prone to radically different asset allocations from period to period compared to the mean-variance approach. By reducing turnover, costs stemming from purchasing/selling ETFs are reduced as well.

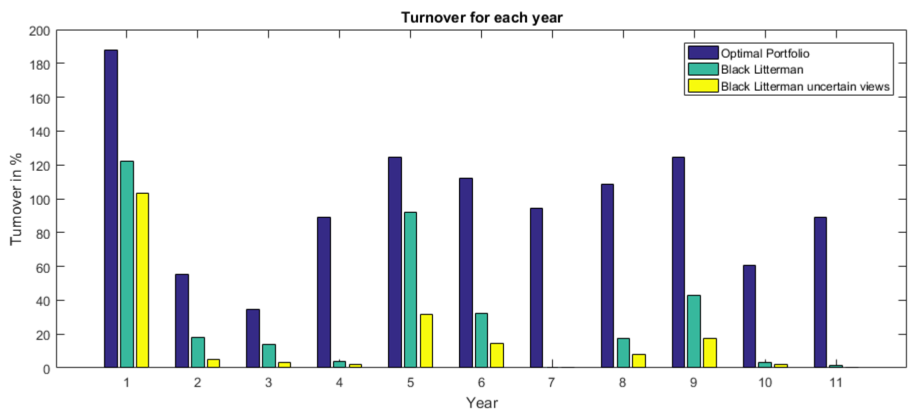


Figure 5: *Turnovers for the optimal portfolio and the Black Litterman approach with certain and uncertain views.*

2.8 Timing

In this section we investigate whether or not we've been able to time the market. That is, we test for curvature in our data (returns). This is done by calculating the Treynor-Mazuy measure. As before, let R_P be the returns of our portfolio and let R_M be the the returns of the market. Then by adding a quadratic term to the linear regression from section 2.6 we get the following model:

$$(R_P - RF) = \alpha + \beta(R_M - RF) + \gamma(R_M - RF)^2 + \varepsilon$$

If we have been able to time the market, the quadratic term should be significant and preferably with a positive estimate. To give an interpretation of the curvature, assume that we indeed have been able to time the market. In this case it would mean that we as portfolio managers have been able to anticipate say, an increase in the market and hence we increase our sensitivity to the market and thereby perform better. Likewise, we have been able to anticipate a decline in the market and so we decrease our sensitivity to the market and thereby decrease our portfolios decline. This would yield a quadratic pattern in our portfolio returns compared to the market returns.

The output statistics of the regression is shown below. We see that the squared term has a p-value of 7.2% which means that we can't reject the hypothesis that the squared term is equal to zero. Hence, we deem the squared term insignificant at at 5% level. What we can conclude from this is that we indeed haven't been able to time the market. However, if one considered the 10% level instead of the typical 5% we would indeed have a significant quadratic term indicating some success with timing the market.

```
Linear regression model:
    returns ~ 1 + beta + gamma(^2term)

Estimated Coefficients:
                       Estimate       SE         tStat       pValue
                       _________    _________    _______    __________

    (Intercept)        0.0021325    0.0021387    0.99706       0.32061
    beta                 0.20783     0.054737     3.7968    0.00022528
    gamma(^2 term)       -1.0217      0.56304    -1.8147      0.071918


Number of observations: 131, Error degrees of freedom: 128
Root Mean Squared Error: 0.0222
R-squared: 0.222, Adjusted R-Squared 0.21
F-statistic vs. constant model: 18.3, p-value = 1.06e-07
```

3 Bonds

In this section we take a look at five different bonds which we have selected through `https://screener.finance.yahoo.com/bonds.html`.

3.1 Bonds selection

We have chosen the following bonds as shown below (links to each bond you can find in the MATLAB file):

- **Bond 1** — CHICAGO ILL GO BDS.
- **Bond 2** — COCA COLA CO.
- **Bond 3** — ALPHABET HOLDING CO INC.
- **Bond 4** — DISNEY WALT CO MTNS BE.
- **Bond 5** — SHELL INTERNATIONAL FIN BV.

We have tried to diversify our bonds (with different maturity dates, payment frequency and coupons).

The raw data from the five bonds (acquired on the 30th of November 2016) can be seen in table 6.

	Price	Coupon (%)	Payment Frequency	Maturity Date	First Coupon Date
Bond 1	107.90	5.000	Semi-Annual	1-Jan-2027	1-Jan-2008
Bond 2	101.10	1.650	Semi-Annual	1-Nov-2018	1-May-2014
Bond 3	94.50	7.750	Semi-Annual	1-Nov-2017	1-Nov-2013
Bond 4	100.54	0.537	Quarterly	30-May-2019	30-Aug-2014
Bond 5	102.06	2.000	Semi-Annual	15-Nov-2018	15-May-2014

Table 6: *Raw data for the five bonds.*

Because we chose bonds which already exist and some payments happened recently we should transform the following formula for yield to maturity:

$$P = \sum_{n=1}^{T} \frac{C_t}{1+y}$$

to this:

$$P = \frac{C \times \mathcal{T}}{(1+y)^{\mathcal{T}}} + \sum_{t=1}^{T} \frac{C_t}{(1+y)^{t+\mathcal{T}}},$$

where C_t — payment in period t, T — number of future payments, P — *clean* price, $\mathcal{T}$ — time to next payment and y — yield to maturity. We first term to take into account dirty price and shifted time by $\mathcal{T}$ because of recent payment.

The same transformation has been applied to the duration formula $\mathcal{D}$ and convexity $\mathcal{C}$ (we evaluate dollar duration and convexity):

$$\mathcal{D} = \frac{C \times \mathcal{T} \times \mathcal{T}}{(1+y)^{\mathcal{T}}} + \sum_{t=1}^{T} \frac{C_t \times (t+\mathcal{T})}{(1+y)^{t+\mathcal{T}}}$$

$$\mathcal{C} = \frac{C \times \mathcal{T} \times [\mathcal{T} \times (1+\mathcal{T})]}{(1+y)^{\mathcal{T}}} + \sum_{t=1}^{T} \frac{C_t \times (t+\mathcal{T}) \times (t+\mathcal{T}+1)}{(1+y)^{t+\mathcal{T}}}$$

Formulas for duration and convexity were obtained from Taylor expansion of price as a function of yield. So, from meaning of a derivative follows that duration as first derivative and convexity as second derivative describe sensitivity of a bond price in responce to interest rate change. (Convexity was introduced for better approximation of price). Yield to maturity shows rate of return of holding a bond.

Table 7 below shows the evaluated characteristics.

	Yield To Maturity (%)	Duration	Convexity
Bond 1	4.036	2537.5	25067
Bond 2	1	347.8	765.7
Bond 3	16.34	147.8	178.3
Bond 4	0.32	1007.4	5529.5
Bond 5	0.637	310.3	625.0

Table 7: *Evaluated characteristics.*

So, we can conclude that *Bond 3* is the most profitable investment. But the most sensitive bonds to interest rate change are *Bond 1* and *Bond 4*.

3.2 Portfolio duration and convexity

We now look at the duration and convexity of the bond portfolio consisting of the five bonds from section 3.1. Taking into account that first and second derivatives are linear operators we can conclude that duration $\mathcal{D}$ and convexity $\mathcal{C}$ of a bond portfolio are equal sums of durations $\mathcal{D}_i$ and convexities $\mathcal{C}_i$ respectively.

So,

$$\mathcal{D} = \sum_{i=1}^{5} x_i \mathcal{D}_i$$

$$\mathcal{C} = \sum_{i=1}^{5} x_i \mathcal{C}_i$$

where x_i is the number of bonds i in the portfolio. In our case all $x_i = 106480/P_i,\ i \in \{1,2,3,4,5\}$ because we invest equal amount of money (106 480 EUR) in each bond i, P_i — price of the bond i.

We get the following values:

$$\mathcal{D} = 4427600$$
$$\mathcal{C} = 32253000$$

3.3 Market value

In this final section we investigate the potential decline in the market value of our portfolio, if the yield increases by 150 basis points. To that end, let $P(y)$ be the function of market value of a portfolio with argument y — yield. Assuming that function $P(y)$ is smooth enough we can find Taylor expansion around arbitrary point y_0 (Δy — change in yield).

$$P(y_0 + \Delta y) = P(y_0) + \frac{1}{1!}\frac{\mathrm{d}P(y_0)}{\mathrm{d}y}\Delta y + \frac{1}{2!}\frac{\mathrm{d}^2 P(y_0)}{\mathrm{d}y^2}(\Delta y)^2 + o((\Delta y)^2)$$

where (P_0 is a portfolio price):

$$\frac{\mathrm{d}P(y_0)}{\mathrm{d}y} = \mathcal{D} - \text{Duration} \qquad \frac{1}{2}\frac{\mathrm{d}^2 P(y_0)}{\mathrm{d}y^2} = \mathcal{C} - \text{Convexity}$$

Without error term $o((\Delta y)^2)$:

$$P(y_0 + \Delta y) \approx P(y_0) + \frac{1}{1!}\frac{\mathrm{d}P(y_0)}{\mathrm{d}y}\Delta y + \frac{1}{2!}\frac{\mathrm{d}^2 P(y_0)}{\mathrm{d}y^2}(\Delta y)^2$$

Change of the market value ΔP:

$$\Delta P = P(y_0 + \Delta y) - P(y_0) = \mathcal{D}\Delta y + \mathcal{C}(\Delta y)^2$$

Using obtained values in previous points we get that:

$$\Delta P = 73671 \text{ EUR}$$

So, this value of ΔP seems quite adequate for 106 480 EUR investment in each of 5 bonds.

4 Appendix

A Title

	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
XLE	0.37	0.39	0.39	0.43	0.34	0.31	0.16	0.16	0.09	0.12	0.03
XLU	-0.22	-0.16	-0.12	-0.02	-0.04	-0.07	-0.06	0.03	0.15	0.07	-0.03
XLK	-0.31	-0.24	-0.31	-0.29	-0.34	-0.44	-0.35	-0.27	-0.15	-0.23	-0.05
XLB	-0.06	-0.05	-0.05	0.10	-0.01	0.13	0.38	0.12	-0.07	-0.10	-0.07
XLP	-0.34	-0.32	-0.30	-0.22	-0.08	0.13	0.23	0.30	0.51	0.42	0.45
XLY	0.19	0.06	0.15	-0.01	0.13	0.44	0.42	0.37	0.36	0.40	0.25
XLI	0.13	0.02	0.08	0.12	0.21	0.08	-0.09	0.17	0.11	0.16	0.06
XLV	0.04	0.14	0.15	0.20	0.37	0.23	0.08	0.22	-0.03	0.13	0.31
XLF	0.21	0.23	0.28	0.00	-0.44	-0.46	-0.46	-0.59	-0.38	-0.43	-0.39
P2	0.99	0.92	0.73	0.70	0.86	0.66	0.69	0.50	0.42	0.45	0.42

Table 8: Optimal Asset Allocations with Tobin Separation ($RF = 1\%$) to meet a constant required return of 10%. Notice that $P2$ is the fraction of our capital placed in the riskless asset. The rest of the variables, the 9 ETFs, comprise alltogether the , $P1$, fraction that is invested in them (i.e. the ETFs).