Assignment - Parameter Estimation

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Q1. Let $(X_1, X_2, \ldots)$ be a random sample of size n taken from a Normal Population with parameters : mean $= \theta_1$ and variance $= \theta_2$. Find the Maximum Likelihood Estimates of these two parameters.

Sol For normal population,

$$pmf = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\cdot\left(\frac{x-\theta_1}{\sqrt{\theta_2}}\right)^2}$$

Likelihood function : $L(\theta_1, \theta_2)$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\left(\frac{x_i-\theta_1}{\sqrt{\theta_2}}\right)^2}$$

$$\ln L(\theta_1, \theta_2) = \sum_{i=1}^{n}\left[\ln\frac{1}{\sqrt{2\pi}} + \ln\frac{1}{\sqrt{\theta_2}} - \frac{1}{2}\frac{(x_i-\theta_1)^2}{\theta_2}\right]$$

$$\ln L(\theta_1, \theta_2) = \sum_{i=1}^{n}\left[-\ln\sqrt{2\pi} - \ln\sqrt{\theta_2} - \frac{(x_i-\theta_1)^2}{2\theta_2}\right]$$

$$\ln L(\theta_1,\theta_2) = -n \ln \sqrt{2\pi} - n \ln \sqrt{\theta_2} - \sum_{i=1}^{n} \frac{(x_i-\theta_1)^2}{2\theta_2}$$

Taking partial derivative wrt θ_1.

$$\frac{\partial}{\partial \theta_1} \ln L(\theta_1,\theta_2) = \sum_{i=1}^{n} \frac{2(x_i-\theta_1)}{2\theta_2}$$

$$\frac{\partial}{\partial \theta_1} \ln L(\theta_1,\theta_2) = \sum_{i=1}^{n} \frac{(x_i-\theta_1)}{\theta_2}$$

For getting max value, $\frac{\partial}{\partial \theta_1} \ln L(\theta_1,\theta_2) = 0$

$$\Sigma x_i - \Sigma \theta_1 = 0$$

$$\Sigma x_i = n\theta_1$$

$$\theta_1 = \frac{\Sigma x_i}{n}$$

$$\boxed{\hat{\theta}_1 = \bar{x}} \quad \text{(mean)}$$

Taking double derivative

$$\frac{\partial^2}{\partial \theta_1^2} \ln L(\theta_1,\theta_2) = \frac{\Sigma(-1)}{\theta_2} = \frac{-n}{\theta_2} < 0$$

since it is < 0 so obtained value is maximum.

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Now, taking partial derivative wrt $\sqrt{\theta_2}$

Let $\sqrt{\theta_2} = \sigma$

$$\frac{\partial}{\partial\sigma} \ln L(\theta_1, \sigma^2) = \frac{-n}{\sigma} + \frac{\sum_{i=1}^{n} (x_i - \theta_1)^2}{\sigma^3}$$

To get max value, $\frac{\partial}{\partial\sigma} \ln L(\theta_1, \sigma^2) = 0$

$$\frac{n}{\sigma} = \frac{\Sigma (x_i - \theta_1)^2}{\sigma^3}$$

$$\sigma^2 = \frac{\Sigma (x - \theta_1)^2}{n}$$

$$\theta_2 = \frac{\Sigma (x_i - \theta_1)^2}{n}$$

Putting $\theta_1 = \bar{x}$,

$$\boxed{\theta_2 = \frac{\Sigma (x - \bar{x})^2}{n}}$$ (variance)

Taking double derivative,

$$\frac{\partial^2}{\partial\theta_2} \ln L(\theta_1, \theta_2) = \frac{n}{\theta_2} - \frac{3}{\theta_2^{\,2}} \Sigma (x_i - \theta_1)^2$$

$$= \frac{n}{\theta_2} - \frac{3}{\theta_2^{\,2}} n\theta_2$$

$$= \frac{-2n}{\theta_2} < 0$$

since double derivative is < 0. So obtained value is maximum.

Overall;

$$\boxed{\hat{\theta}_1 = \bar{x}}$$

$$\boxed{\hat{\theta}_2 = \sum_{i=1}^{n} \frac{(x_i - \theta_1)^2}{n}}$$

Q2 Let $X_1, X_2, \ldots, X_n$ be random sample from $B(m, \theta)$ distribution where $\theta \in \Theta = (0,1)$ is unknown and m is known positive integer. Compute value of θ using M.L.E.

Sol 2 For binomial distribution,

$$pmf = {}^{m}C_{x}\, \theta^{x} (1-\theta)^{m-x}$$

Likelihood function,

$$L(\theta) = \prod_{i=1}^{n} \left[{}^{m}C_{x_i} \cdot \theta^{x_i} (1-\theta)^{m-x_i} \right]$$

$$L(\theta) = \prod_{i=1}^{n} \left[{}^{m}C_{x_i} \right] \cdot \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{\sum_{i=1}^{n} (m-x_i)}$$

$$L(\theta) = \left(\prod_{i=1}^{n} {}^{m}C_{x_i} \right) \cdot \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{nm - \sum_{i=1}^{n} x_i}$$

$$\ln L(\theta) = \sum_{i=1}^{n} \ln {}^{m}C_{x_i} + \sum_{i=1}^{n} x_i \ln(\theta) + \left(nm - \sum_{i=1}^{n} x_i\right) \ln(1-\theta)$$

Taking derivative w.r.t θ

$$\frac{\partial}{\partial \theta} \ln L(\theta) = \frac{\sum x_i}{\theta} + \frac{nm - \sum x_i}{1-\theta} (-1)$$

$$\frac{\partial}{\partial \theta} \ln L(\theta) = \frac{\sum x_i - \theta \sum x_i - mn\theta + \theta \sum x_i}{\theta(1-\theta)}$$

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$$\frac{\partial}{\partial\theta}\ln L(\theta) = \frac{\Sigma x_i - nm\theta}{\theta(1-\theta)}$$

To get max value, $\frac{\partial}{\partial\theta}\ln L(\theta) = 0$

$$\theta = \frac{\Sigma x_i}{nm} = \frac{\bar{x}}{m}$$

$$\boxed{\hat{\theta} = \frac{\bar{x}}{m}}$$

Taking double derivative,

$$\frac{\partial^2}{\partial\theta^2}\ln L(\theta) = -\frac{\Sigma x_i}{\theta^2} - \frac{(nm - \Sigma x_i)}{(1-\theta)^2}$$

$$\frac{\partial^2}{\partial\theta^2}\ln L(\theta) = -\frac{\Sigma x_i - \theta^2\Sigma x_i + 2\theta\Sigma x_i - \theta^2 nm + \Sigma x_i\theta^2}{\theta^2(1-\theta)^2}$$

$$= -\left[\frac{\Sigma x_i + \theta^2 nm - 2\theta\,\Sigma x_i}{\theta^2(1-\theta)^2}\right]$$

$$= -nm\left[\frac{\theta + \theta^2 - 2\theta^2}{\theta^2(1-\theta)^2}\right]$$

$$= \frac{-nm\,(1-\theta)\,\theta}{\theta(1-\theta)^2}$$

$$\frac{\partial^2\ln L(\theta)}{\partial\theta^2} = \frac{-nm}{\theta(1-\theta)} < 0 \quad [\theta \in (0,1)]$$

Since double derivative is <0 so, obtained value is maximum.

$$\therefore \boxed{\hat{\theta} = \frac{\bar{x}}{m}}$$