Introduction
to
Deep Operator Networks

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Universal Approximation Theorem for operators

Let,

- σ - a continuous non-polynomial function
- X - Banach space
- $K_1 \in X,\ K_2 \in \mathbb{R}^d$ - two compact sets
- $V \in C(K_1)$ - a compact set in functional space of K_1
- G - non-linear continuous operator that maps $V \rightarrow C(K_2)$

Then for any $\epsilon > 0$, there are positive integers n, p, m, constants $c_i^k, \varepsilon_{ij}^k, \theta_i^k, \zeta_k \in \mathbb{R},\ w_k \in \mathbb{R}^d,\ x_j \in K_1,\ i = 1, .., n,\ k = 1, ..., p,\ j = 1, ..., m$ such that

$$\left| G(u)(y) - \sum_{k=1}^{p} \underbrace{\sum_{i=1}^{n} c_i^k \sigma \left(\sum_{j=1}^{m} \varepsilon_{ij}^k u(x_j) + \theta_i^k \right)}_{\text{branch}} \underbrace{\sigma\left(w_k . y + \zeta_k\right)}_{\text{trunk}} \right| < \epsilon \tag{1}$$

holds for $u \in V$ and $y \in K_2$.

Univeral Approx. Thm for operators

In other form,[1]

$$\left| G(u)(y) - \sum_{k=1}^{p} \underbrace{br_k\left(u(x_1), u(x_2), ..., u(x_m)\right)}_{\text{branch}} \underbrace{tr_k\left(y\right)}_{\text{trunk}} \right| < \epsilon \quad (2)$$

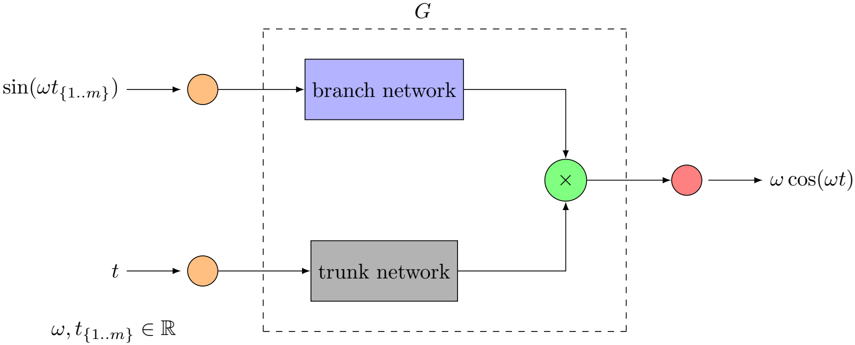
where, $x_1, ..., x_m$ are called sensor points.

The functions $br()$, $tr()$, can be approximated by neural networks[2]

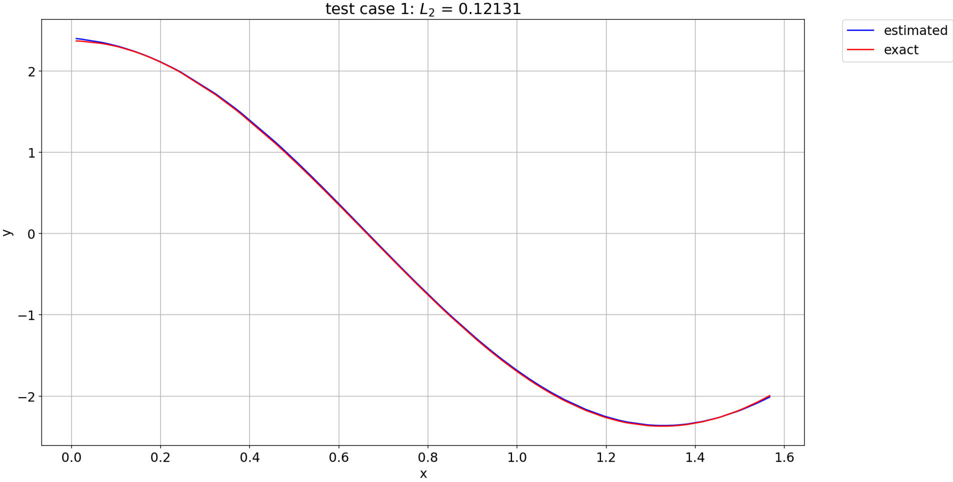
[1] Goswami et al., "Physics-informed deep neural operator networks".
[2] Hornik, Stinchcombe, and White, "Multilayer feedforward networks are universal approximators".

Data-driven deep-o-net example

The operator $G : sin(\omega t) \rightarrow \omega cos(\omega t)$



Data-driven deep-o-net example - results



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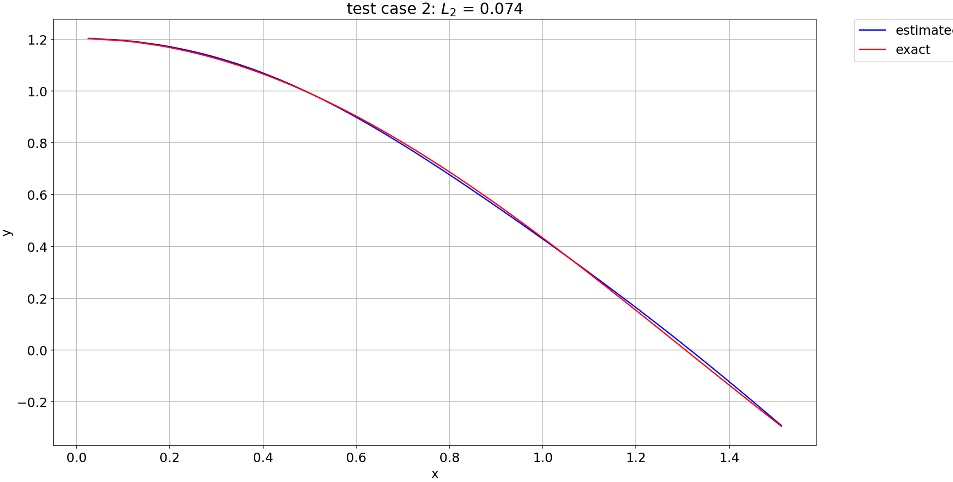

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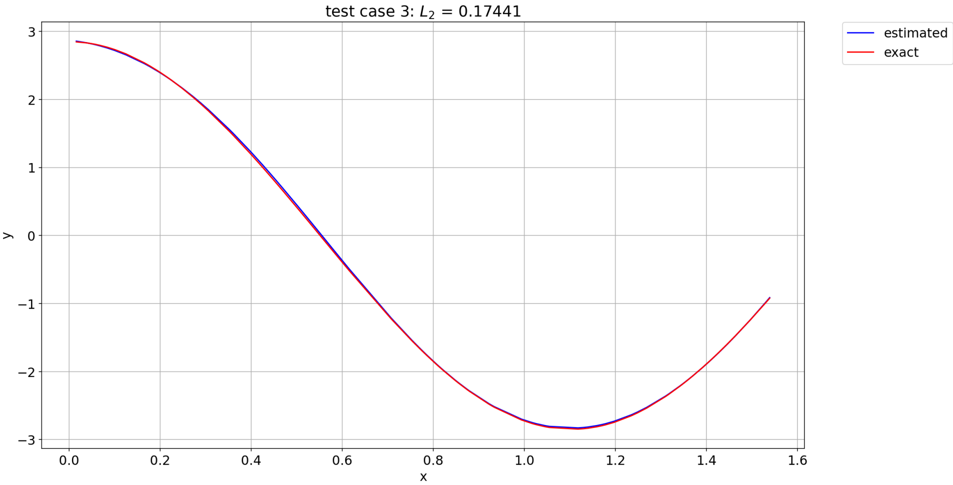

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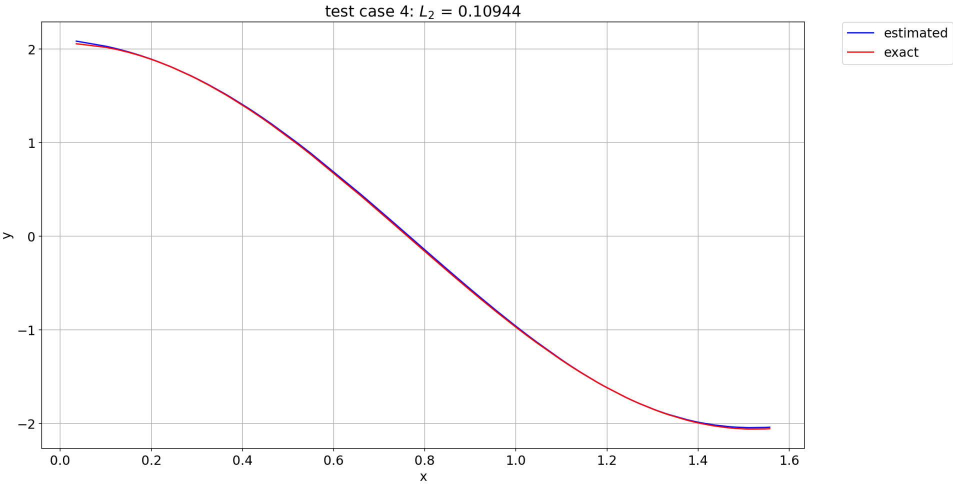

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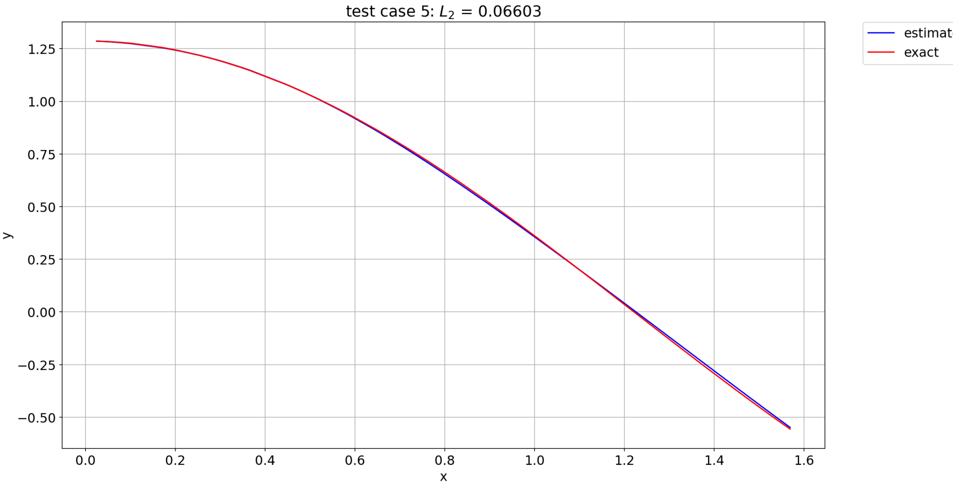

Data-driven deep-o-net Experiment

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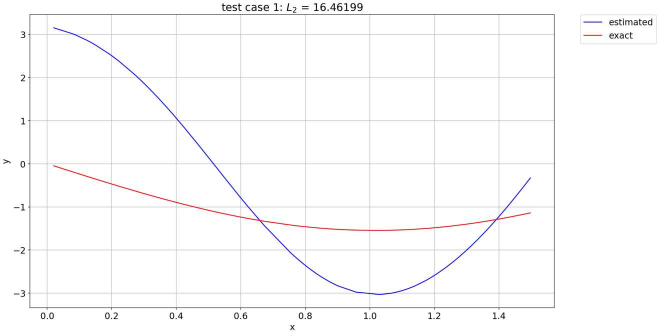

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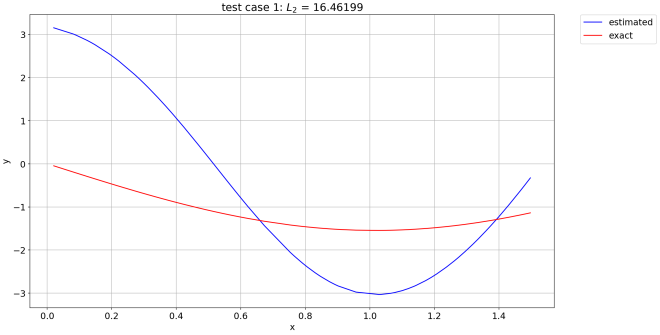

Network approximates Operator with encoded input function

Physics Informed Deep-o-net

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Tried with operator, $G : \cos(\omega t) \rightarrow x$

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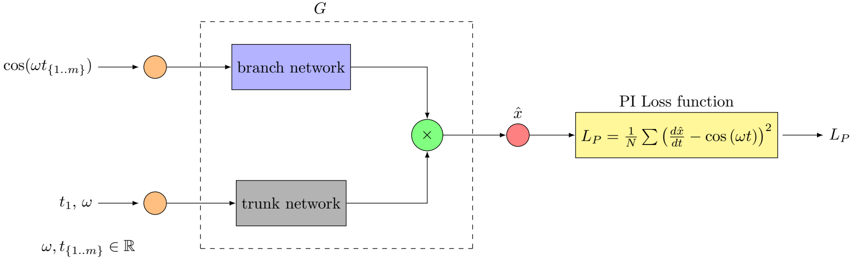

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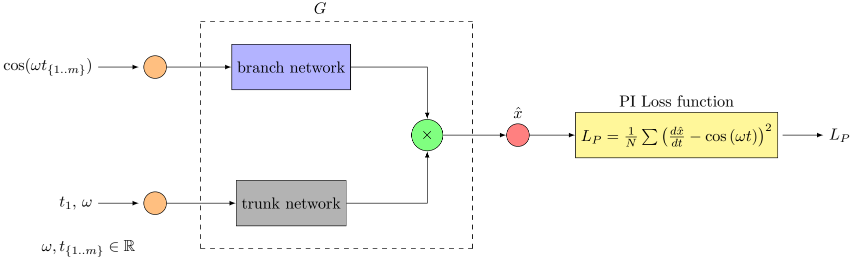

This includes automatic differentiation, $\dot{x}$

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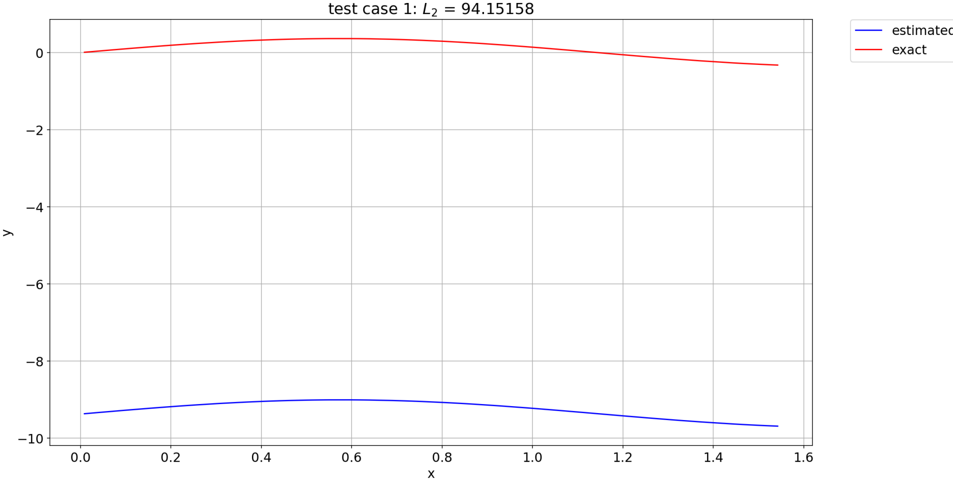

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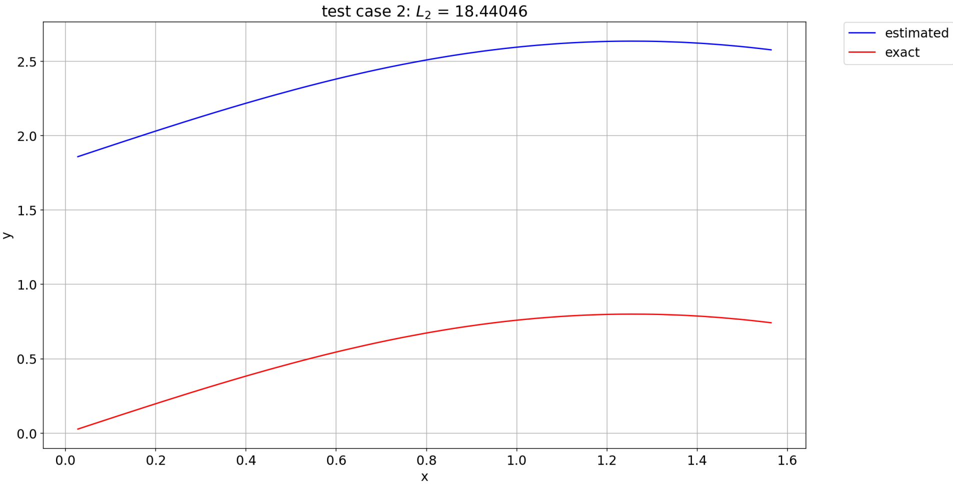

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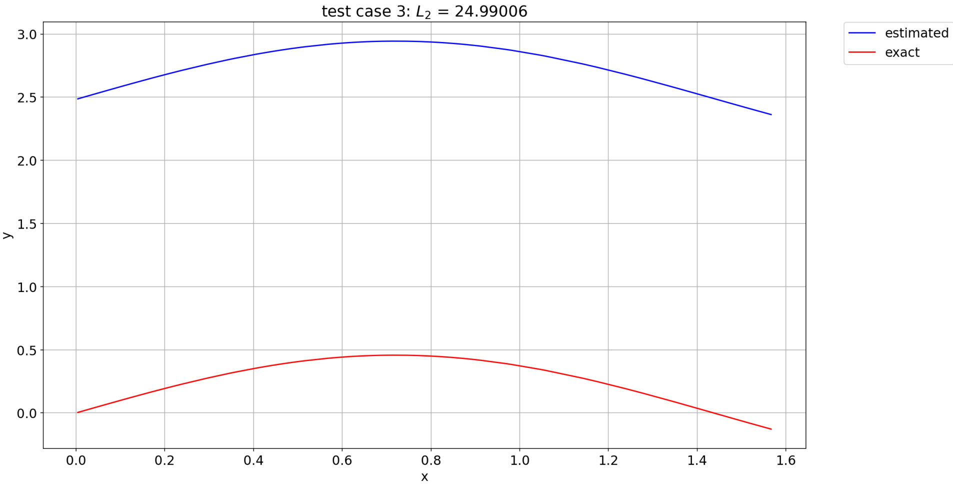

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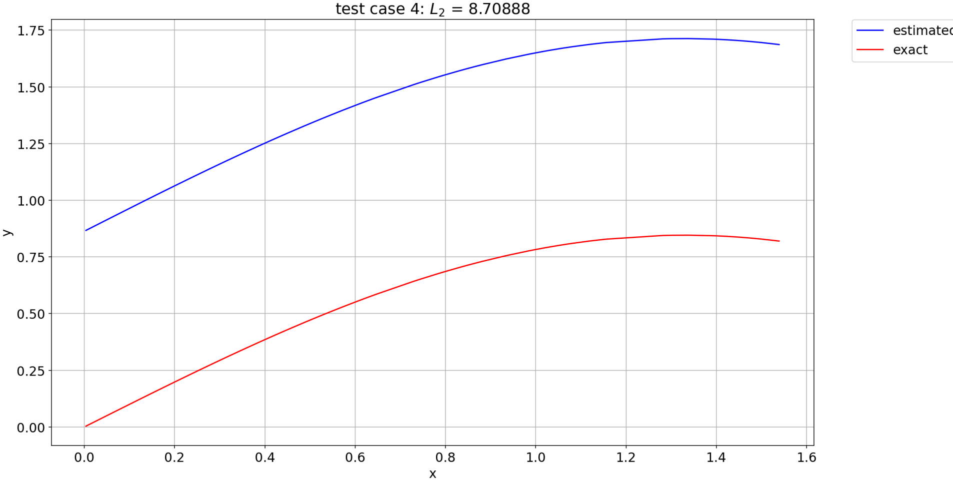

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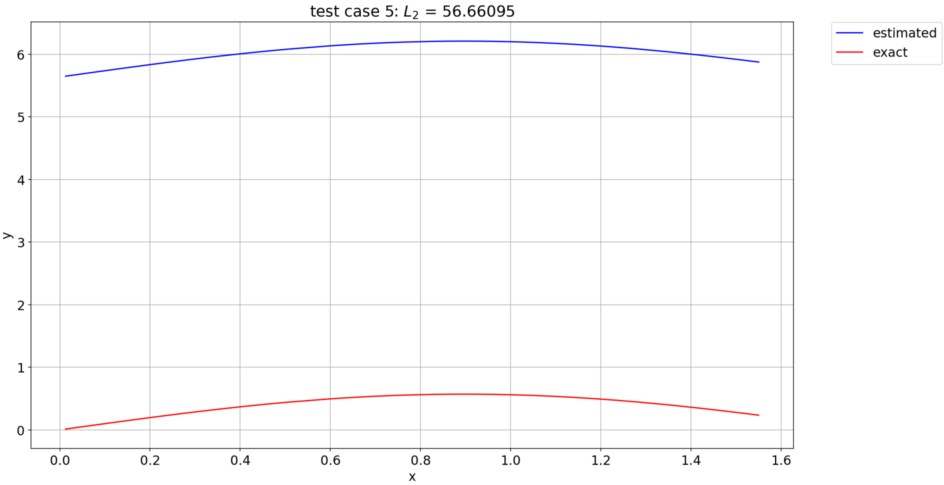

Physics Informed Deep-o-net - Results not mached

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Approximating, $\hat{G} : \cos(\omega t) \to x$

With constraint, $\frac{dx}{dt} = \cos(\omega t)$

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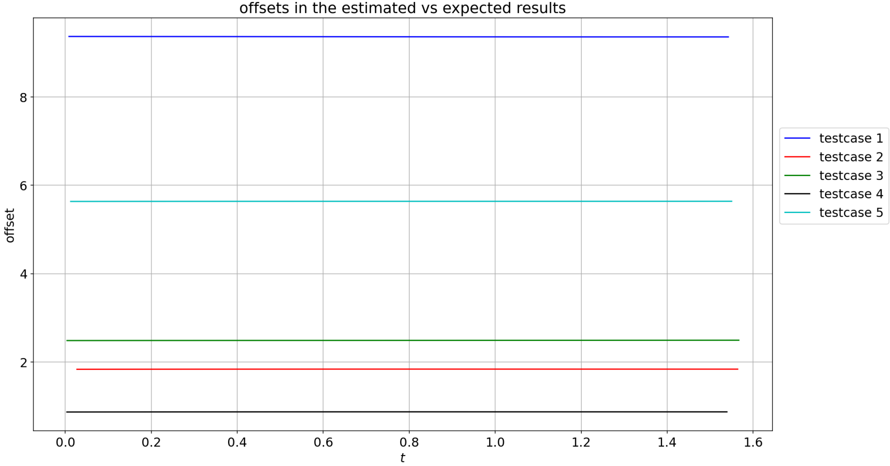
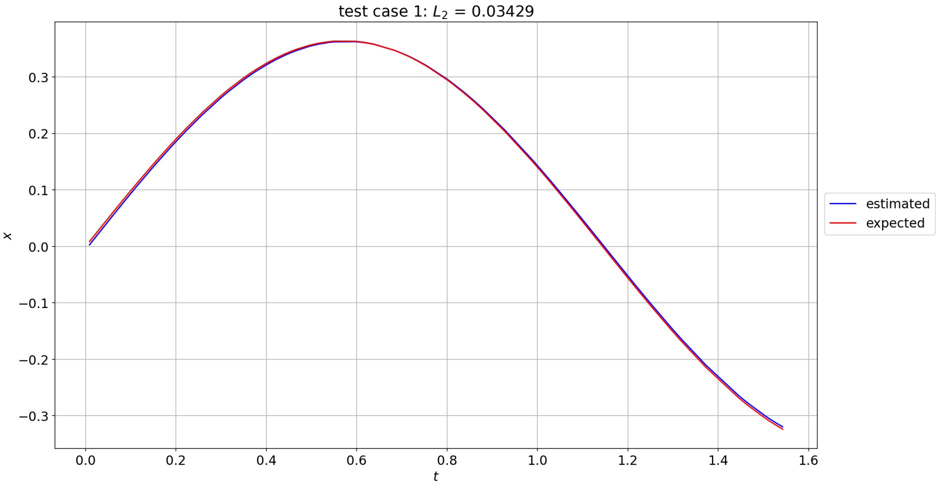
$x = \int cos(\omega t) \quad + \quad \boldsymbol{C}$

Physics Informed Deep-o-net - integral constant



Physics Informed Deep-o-net - corrected results



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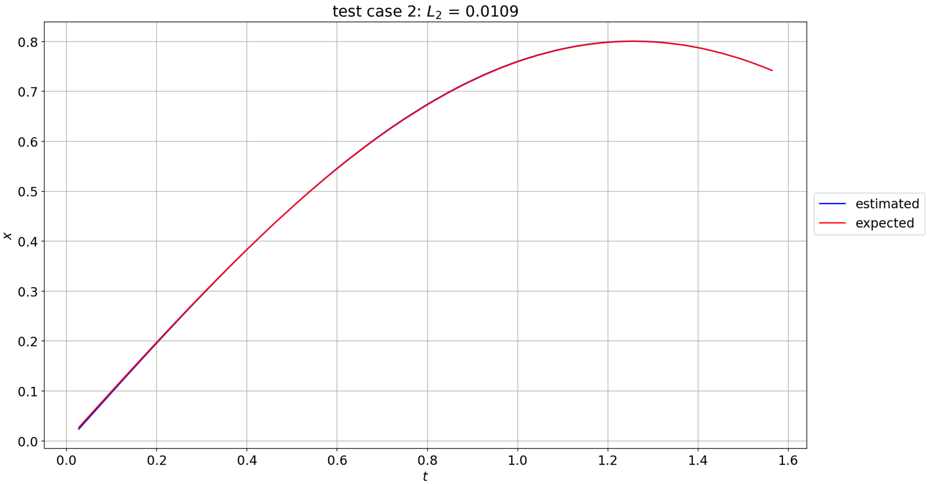

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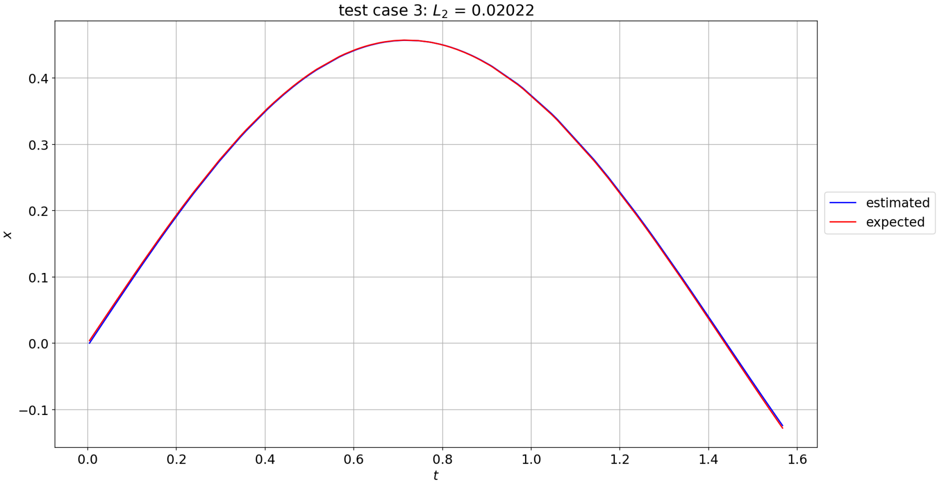

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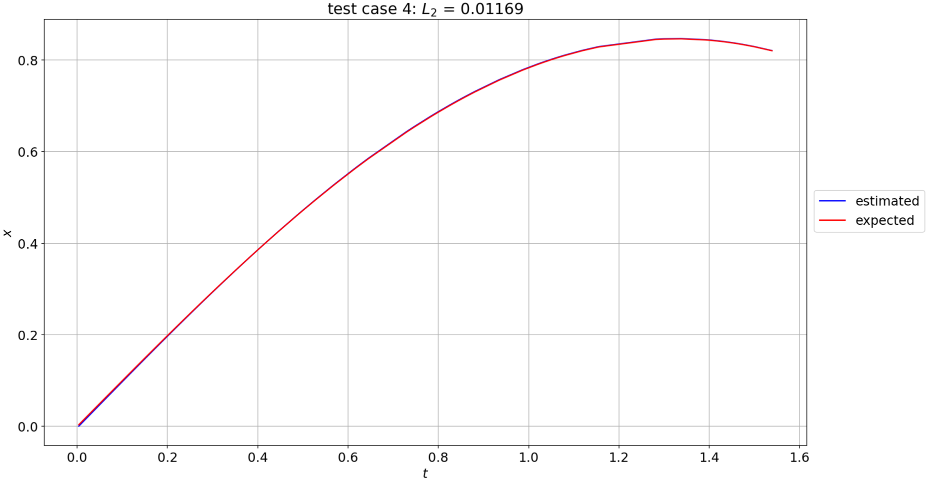

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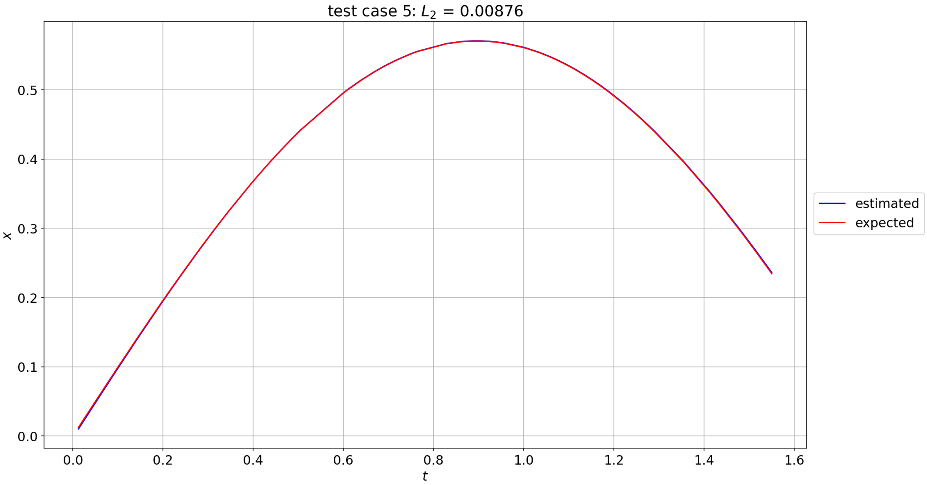

Source code

Available on github

https://github.com