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Inequality with Ordinal Data

By FRANK A. COWELL† and EMMANUEL FLACHAIRE‡

†*LSE* ‡*Aix-Marseille University*

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The standard theory of inequality measurement assumes that the equalisand is a cardinal quantity, with known cardinalization. However, one often needs to make inequality comparisons where either the cardinalization is unknown or the underlying data are categorical. We propose an alternative approach to inequality analysis that is rigorous, has a natural interpretation, and embeds both the ordinal data problem and the well-known cardinal data problem. We show how the approach can be applied to the inequality of happiness and of health status.

INTRODUCTION

It is quite common to find problems of inequality comparison where conventional tools just will not work. A principal cause of this difficulty lies in the nature of the thing that is being studied. By contrast to the inequality analysis of income or wealth distributions where the underlying variable is measurable and interpersonally comparable, 'ordinal data' may be measurable but with an unknown scale of measurement, or the underlying concept may not be measurable at all. The term 'ordinal data' covers such things as educational achievement, happiness and health, each of which has a sub-literature on inequality in its own right.[1] This paper does two things: it develops a rigorous treatment of the generic ordinal data problem, and it provides a comprehensive approach to inequality analysis that embeds both the ordinal data problem and the better-known cardinal data problem as particular cases.

Why is there a problem with ordinal data? Many key concepts of distributional analysis are not well defined with ordinal data, and as a result, some standard tools of inequality analysis cannot be applied.[2] The literature on inequality in happiness, health and so on contains a number of work-rounds that try to address this problem, but none is entirely satisfactory. Some work-rounds are potentially misleading: arbitrary cardinalization can lead to arbitrary rankings of distributions. Other work-rounds—involving first-order dominance and the use of quantiles to characterize inequality comparisons—are more promising. But difficulties can arise even with these methods.[3] However, we show that as long as the underlying data can be ordered, it is possible to construct a fully-fledged approach to inequality analysis that has a natural intuition and that can be easily implemented empirically.

I. ORDINAL DATA: THE ISSUES

In an inequality measurement problem there are three basic ingredients: (1) the definition of the thing that is unequally distributed; (2) the definition of the receiving unit, the basic component of the population among which there is an unequal distribution; (3) a method of aggregation. Section II deals with ingredient (3). We will assume that there is agreement about ingredient (2)—in that inequality is to be seen as an unequal distribution among people. Here we focus on ingredient (1).

Ordinal data: specification

Obviously there are many ways in which this ingredient of the problem could be specified. Some of these specifications differ radically in terms of their informational requirements. Conventional inequality applications require a cardinally measurable and interpersonally comparable quantity such as income, wealth or expenditure (Cowell 2011).[4] Inequality with ordinal data can be modelled in several ways, of which we will consider two leading cases.

Transformation of a cardinal variable Suppose that we are interested in the inequality of utility u, which depends on a cardinal variable x, so that $u = U(x)$, where U is a monotonic function. If U is a function common to all people, then it may be possible to infer its shape using, say, people's attitude to risk. If so, then the inequality measurement problem just becomes an extension of the standard cardinal variable model. Otherwise, if the form of U is unknown or x is a measurable quantity that has no agreed valuation, then increased dispersion of x leads to increased inequality, but we cannot say by how much. If U differs across individuals, then we cannot say even that.

Categorical variable In many cases the information about individuals is in the form of categorical data. We need only to assume that the categories can be ordered unambiguously to provide a simple approach to inequality comparisons.

As an example, consider the problem of inequality in access to specific public facilities. Assume that there is general agreement that, other things being equal, a person is better off with access to both gas and electricity supplies than with access to electricity only, a person is better off with access to electricity only than gas only, and a person is better off with access to gas only than access to neither utility. We know nothing about how much energy is consumed or about how much could be afforded; so trying to assign a dollar equivalent may be inappropriate or meaningless. In Table 1, suppose that n_k is the number of people in category $k \in \{\mathrm{B, E, G, N}\}$. If there are 100 people in the population, then intuition suggests that Case 1 in Table 1 represents more amenity inequality than does Case 2.

Inequality of educational attainment or health status can be seen as other practical examples of this type of problem. We will also see (in the subsection below entitled 'The role of status') that the categorical variable case can be taken as a useful paradigm for analysing other types of ordinal data.

Approaches to the ordinal data problem

In the literature there are two main ways in which the inequality measurement problem has been tackled when dealing with ordinal data.

TABLE 1
ACCESS TO AMENITIES: CATEGORICAL VARIABLE

	Case 1	Case 2
Both Gas and Electricity	25	0
Electricity only	25	50
Gas only	25	50
Neither	25	0

Status The first approach is to force a solution on the problem by imputing a notion of 'status' to a categorical data structure and then examining the inequality of status. The imputation is achieved sometimes through subjective evaluation by individuals (for example on a Likert scale) and sometimes by official institutions (for example the Quality-adjusted Life Year or the Human Development Index). The same procedure can be applied explicitly or implicitly to entities that do not have a natural ordering, such as vectors of attributes or endowments; one uses the utility function to force an ordering of the data.

However, this version of the 'status approach' is unsatisfactory: it typically runs into objections of arbitrariness of the cardinalization or of the method of aggregating components. In the next subsection (entitled 'The role of status'), we investigate an alternative version of the status approach that is more appropriately founded on the logic of categorical data structures.

Dominance The second approach, common in the health literature,[5] involves a reworking of traditional inequality ranking approaches focusing on first-order dominance criteria. The median has been suggested as an equality concept corresponding to the use of the mean in conventional inequality analysis, although it has been noted that comparing distributions with different medians raises special issues (Abul Naga and Yalcin 2010). Clearly, such an approach may run into difficulty if quantiles are not well defined, as may happen in the case of categorical variables.[6]

The role of status

In the income distribution literature it is common to find a person's location in the distribution used as a concept of status in society: this concept has then been used to develop measures of individual and social deprivation, and has sometimes been incorporated into the measurement of inequality.

Denote the utility of person i by u_i, and let the distribution function of u for a population of size n be F. Each person's status s_i is uniquely defined for a given distribution as $s_i = \psi(u_i, F(\cdot), n)$, where the function ψ is such that status is independent of the cardinalization of utility. One simple way of specifying status for an individual is the standard definition of position, the proportion of the population that is no better off than oneself; so if the distribution cumulative function for utility in the population is F, then we could use $s_i = F(u_i)$ as the status measure.[7] Such a status measure is familiar to anyone who has ever taken a GRE or TOEFL test, and is similar to that sometimes used to measure opportunity—see de Barros *et al.* (2008). A version of this approach underlies the McMaster Health Utility Index,[8] which uses the distribution of the ordinal variable across categories to provide a 'natural' indicator of status.

The concept of status is central to the approach to ordinal inequality developed here; let us see how it may be implemented given the data specification considered in the first subsection of this section.

Categorical data For this type of data, where we know only the labelling of the categories k ($k = 1, 2, \ldots, K$) and the number of people n_k in each category, we must assume that every person within a given category k has the same status. In principle, the status of a person i could depend on the set of numbers $\{n_k\}$ and on the label of the particular category $k(i)$ to which i belongs.

However, the specification of the categories may be arbitrary, even where the data are ordered. As an example, suppose that we have a dataset with categories labelled in a natural order as {extremely low, low, fairly low, fairly high, high, extremely high}, and suppose that person i belongs to the category 'fairly low'. Now suppose that the presentation of data is modified by the insertion of additional category 'very high' between 'high' and 'extremely high', but that category 'very high' is empty; it seems reasonable that this new category should leave the status of person i unaffected. Likewise, it would seem reasonable that inserting an empty category between 'extremely low' and 'low' should leave the status of person i unchanged. Finally, suppose that the presentation of the dataset is altered so that the distinction between categories 'high' and 'extremely high' is abolished. Again it seems reasonable that this change in the data categorization changes nothing for the status of person i. In each of these cases the labelling of the ordered categories changes, but the facts of the data do not; this suggests that the category label $k(i)$ by itself is not useful as an indicator of status.

In the light of this example, we introduce the following principle for ordered categorical data.

Mergers principle If two adjacent categories are merged, then this has no effect on the status of any person outside these two categories.

This principle has two implications. The first is that we may ignore any empty categories and merge them into the next category below or above. To see the second implication, suppose that there are three non-empty categories and that person i belongs to category 1. If categories 2 and 3 are merged, then the mergers principle implies that the status of person i is given by

$$s_i = f_1(n_1, n_2, n_3) = f_1(n_1, n_2 + n_3, 0).$$

If there are $K > 3$ categories and categories 2, …, K are merged, then, applying the principle iteratively, that status of person i in category 1 is given by

$$s_i = f_1(n_1, n_2, \ldots, n_K) = f_1\left(n_1, \sum_{\ell=2}^{K} n_\ell, 0, \ldots, 0\right).$$

Define the sums

$$\underline{N}_k := \sum_{\ell=1}^{k-1} n_\ell \quad \text{for } k > 1 \qquad \text{and} \qquad \overline{N}_k := \sum_{\ell=k+1}^{K} n_\ell \quad \text{for } k < K,$$

with $\underline{N}_1 = \overline{N}_K = 0$; notice that $n = \underline{N}_k + n_k + \overline{N}_k$, where n is the total population. Applying the same reasoning as before, if i belongs to $k(i)$, then the mergers principle implies that status takes the form $s_i = f(\underline{N}_{k(i)}, n_{k(i)}, \overline{N}_{k(i)})$.

Now consider cases that fall outside the simple mergers principle. The effect on the status of person i of a merger of his own class $k(i)$ with an adjacent class could take one of several forms, which will then determine the shape of the function f. If the adjacent classes are not empty, then clearly a merger with one or other of them must affect the status of person i.[9] But if the status of person i were to be unaffected by a merger of $k(i)$

with the class above, then we would have

$$s_i = f(\underline{N}_{k(i)}, n_{k(i)}, \overline{N}_{k(i)}) = f(\underline{N}_{k(i)}, n_{k(i)} + \overline{N}_{k(i)}, 0).$$

If, alternatively, the status of person i were to be unaffected by a merger with the class below, then would we have instead

$$s_i = f(\underline{N}_{k(i)}, n_{k(i)}, \overline{N}_{k(i)}) = f(0, \underline{N}_{k(i)} + n_{k(i)}, \overline{N}_{k(i)}).$$

This means that the function f can be expressed in either of the following two basic forms:

$$g_1(\underline{N}_{k(i)}, n), \tag{1}$$

$$g_2(\overline{N}_{k(i)}, n). \tag{2}$$

In version (1), my status is determined by all those below me, up to a transformation involving the total population n; in version (2), it is determined by all those above me, up to a transformation involving n. However, $\underline{N}_{k(i)}$ and $\overline{N}_{k(i)}$ are peer-exclusive concepts. We could also express status in terms of the peer-inclusive counterparts:[10]

$$g_3(\underline{N}_{k(i)} + n_{k(i)}, n), \tag{3}$$

$$g_4(\overline{N}_{k(i)} + n_{k(i)}, n). \tag{4}$$

Which of these basic status concepts is appropriate for inequality measurement using ordinal data? In Section III we provide particular examples of *downward-looking status* based on equations (1) and (3), and of *upward-looking status* based on equations (2) and (4).

Other ordinal data The treatment of other forms of ordinal data follows naturally from this. Suppose that each member i of a population of size n has utility u_i, where utility is interpersonally comparable but defined only up to a monotonic increasing transformation, and that $u_1 \leq u_2 \leq \ldots \leq u_n$. One may consider each distinct member of $\{u_1, u_2, \ldots, u_n\}$ as an individual 'category' and then proceed as for the argument using categorical data.

This method for categorical data—which can be extended to other forms of ordinal data—forms the basis for our analysis in Section II.

II. Inequality Measurement: Theory

Approach

Here we offer a new approach to inequality measurement that is designed to deal with the ordinal data problem, but which does more. It provides a coherent general approach to inequality measurement, within which the ordinal data method is nested. The approach involves two steps.

1. *Define status*. The precise definition of status will depend on the structure of information and may also depend on the purpose of the inequality analysis. In some cases, a person's status is self-defining from the data: for example, if we want to focus on wealth inequality, then we use a measure of net worth. In some cases, status is defined once one is given additional distribution-free information: for example, if there are observations on some variable x and it is known that utility is $\log(x)$. In some cases, status requires information dependent on distribution of the underlying data: for example, it could be determined as one of the specifications (1)–(4). For the moment we need only assume that an individual's status is given by $s \in S \subseteq \mathbb{R}$.
2. *Summarise the status distribution*. The distribution of status is given by a vector $\mathbf{s} \in S^n$, where n is population size. To make inequality comparisons, one needs to aggregate the information provided by different $\mathbf{s}$ vectors; this step requires both a concept of equality and a way of characterizing departures from equality.

Consider the requirements for step 2. We define $e \in S$ as an equality reference point that, depending on the type of inequality measurement problem, could be exogenously given or could depend on the status vector $\mathbf{s}$. The specification and meaning of e is discussed in detail in the second subsection of Section III; for the formal results in the next two subsections we need only work an exogenously given parameter. To capture inequality, we could define a specific distance function $d : S^2 \to \mathbb{R}$, where $d(s,e)$ means the distance that a person with status s is from e, the reference point, and then introduce a number of principles (axioms) to characterize an inequality ordering $\succcurlyeq$ and the associated distance concept. However, we do not need to do as much as this because we can make progress without specifying an explicit function $d(\cdot)$ *a priori*; in characterizing $\succcurlyeq$, the distance-from-equality concept emerges.

Inequality ordering: basic structure

Consider inequality as a weak ordering[11] $\succcurlyeq$ on S^{n+1}; denote by $\succ$ the strict relation associated with $\succcurlyeq$, and denote by $\sim$ the equivalence relation associated with $\succcurlyeq$. For any vector $\mathbf{s}$, denote by $\mathbf{s}(\varsigma,i)$ the vector formed by replacing the ith component of $\mathbf{s}$ by $\varsigma \in S$. We first characterize the general structure of the inequality relation using just four axioms.

Axiom 1 (Continuity). $\succcurlyeq$ is continuous on S^{n+1}

Axiom 2 (Monotonicity in distance). If $\mathbf{s}, \mathbf{s}' \in S^n$ differ only in their ith components, then: (a) if $s'_i \geq e$, then $s_i > s'_i \Longleftrightarrow (\mathbf{s}, e) \succ (\mathbf{s}', e)$; (b) if $s'_i \leq e$, then $s_i < s'_i \Longleftrightarrow (\mathbf{s}, e) \succ (\mathbf{s}', e)$.

Axiom 3 (Independence). If $\mathbf{s}(\varsigma, i), \mathbf{s}'(\varsigma, i) \in S^n$ satisfy $(\mathbf{s}(\varsigma, i), e) \sim (\mathbf{s}'(\varsigma, i), e)$ for some ς, then $(\mathbf{s}(\varsigma, i), e) \sim (\mathbf{s}'(\varsigma, i), e)$ for all $\varsigma \in S$.

Axiom 4 (Anonymity). For all $\mathbf{s} \in S^n$ and any permutation matrix $\mathbf{\Pi}$, $(\mathbf{\Pi s}, e) \sim (\mathbf{s}, e)$.

Theorem 1.[12] Given Axioms 1–4, $\succcurlyeq$ is representable by the continuous function $I : S^{n+1} \to \mathbb{R}$ given by

$$(5) \qquad I(\mathbf{s}; e) = \Phi\left(\sum_{i=1}^{n} d(s_i, e), e\right),$$

where Φ is increasing in its first argument and $d : S \to \mathbb{R}$ is a continuous function that is strictly increasing (decreasing) in its first argument if $s_i > e$ ($s_i < e$).

It is clear that to obtain this result we need only standard and easily interpretable assumptions. Monotonicity (Axiom 2) gives meaning to the inequality relation: if two distributions differ only in respect of the status of person i, then the distribution that registers greater individual distance from equality for i is the distribution that exhibits greater inequality. Independence (Axiom 3) means the following: suppose that distributions $\mathbf{s}$ and $\mathbf{s}'$ are equivalent in terms of inequality, and that there is some person i who has the same status in $\mathbf{s}$ and in $\mathbf{s}'$; then the same change in the status of person i in both distributions $\mathbf{s}$ and $\mathbf{s}'$ still leaves $\mathbf{s}$ and $\mathbf{s}'$ as equivalent in terms of inequality. This assumption, widely used in the social welfare and related uncertainty literature, induces a simple additive structure. Although this rules out aggregations of status that incorporate distributional ranks in the aggregation formula (an objection that has some force in other fields where an independence axiom is applied), distributional rank is of course already taken into account in the specification of individual status s. Anonymity (Axiom 4) means that if all information relevant to inequality is embodied in the status measure, then permuting the labels on the individuals must leave inequality unchanged.

The resulting Theorem 1 establishes inequality as total 'distance' from the reference point; the function d is continuous and has the property that for a given e, $d(s, e)$ is increasing in status s if s is above the reference point, and decreasing in s if s is below the reference point.[13]

Inequality ordering: scale

Theorem 1 gives only a general outline of the type of inequality measures that can be applied to ordinal data. To make progress, we next introduce more structure on d and hence on $\succcurlyeq$. The key property is that inequality orderings remain unchanged when status is rescaled; note that it does not imply that the *level* of inequality should remain unchanged under rescaling.

Axiom 5 (Scale invariance). For all $\lambda \in \mathbb{R}_+$, if $\mathbf{s}, \mathbf{s}', \lambda\mathbf{s}, \lambda\mathbf{s}' \in S^n$ and $e, e', \lambda e, \lambda e' \in S$, then: (a) $(\mathbf{s}, e) \sim (\mathbf{s}', e') \Rightarrow (\lambda\mathbf{s}, e) \sim (\lambda\mathbf{s}', e')$; (b) $(\mathbf{s}, e) \sim (\mathbf{s}', e') \Rightarrow (\lambda\mathbf{s}, \lambda e) \sim (\lambda\mathbf{s}', \lambda e')$.

Notice, first, that Axiom 5 does not require that the reference point be the same in the two distributions being compared, and second, it covers cases (a) where the rescaling of status is considered independently of the reference point, and (b) where the reference point in each distribution is rescaled along with the status distribution itself.

This property—a version of homotheticity—is widely used in the literature on social welfare and inequality. It has a special attraction in the present context. Parts (a) and (b) of Axiom 5 cover the cases where the reference point is independent of n and where it depends on n. So, take the simplest form of (1)–(4), where status is given in terms of absolute numbers; then scale invariance means that inequality comparisons are unaffected by the size of the population.[14] In particular, it allows one to normalize status in a common-sense fashion: one can simply divide the absolute-number status concept by by the total population so as to work with cumulative population *proportions* as indicators of individual status.

Corresponding to the two parts of scale invariance we have the following two results:

Lemma 1. Given Axiom 5(a), the function d in equation (5) takes the form

$$d(s,e) = A(e)s^{\alpha(e)}, \tag{6}$$

where A and α are real numbers that may depend on e.

Lemma 2. Given Axiom 5(b), the function d in equation (5) takes the form

$$d(s,e) = e^{\beta}\phi\left(\frac{s}{e}\right), \tag{7}$$

where β is a constant and ϕ is an arbitrary function.

Lemmas 1 and 2 immediately yield a complete ordering of distributions by inequality, and the essentials for a family of inequality measures. Taken with Theorem 1, we have the following result:[15]

Theorem 2. Given Axioms 1–5, $\succcurlyeq$ is representable as

$$I_\alpha(\mathbf{s};e) := \frac{1}{\alpha(\alpha-1)}\left[\frac{1}{n}\sum_{i=1}^{n} s_i^\alpha - e^\alpha\right], \tag{8}$$

where $\alpha \in \mathbb{R}$, or by some strictly increasing transformation of (8) involving e.

III. Inequality Measures

As we noted earlier, the conventional approach to inequality measurement works only within a narrowly defined information structure. In our alternative approach, Theorem 2 provides us with the structure for inequality analysis; to get from here to an operational inequality measure with ordinal data requires some further steps. We need to check that measures of the form (8) appear to 'work' like inequality indices when used with ordinal data, to consider how the reference point for inequality comparisons is to be determined, and to clarify the way in which different members of the class (8) behave. These three issues are addressed in the following three subsections. The next subsection briefly considers some remaining open issues concerning categorical data, and the final subsection of this section provides a brief comparison with conventional inequality indices based on data with cardinal significance.

$I_\alpha(\mathbf{s}; e)$: *basic properties*

Two things that are generally known about inequality measures are that (1) they take their minimum value—conventionally zero—when there is perfect equality, and (2) they (usually) satisfy the transfer principle, whereby a mean-preserving transfer from a poorer to a richer person always increases inequality. What can we say in the case of ordinal inequality?

(1) Clearly, the representation in (8) uses a normalization that ensures that

$$I_\alpha(\mathbf{1}e; e) = 0, \tag{9}$$

meaning that inequality takes the value 0 if $s_i = e$ for all i (everyone's status is at the reference point). So that basic property is easily settled.

(2) Mean-preserving transfers do not immediately seem to make sense here since there is no mean to preserve in the case of ordinal data But given a measure of individual status to represent a categorical variable, we could consider changes in the distribution across categories that keep the mean of status constant. So take the example in Table 2, based on Table 1: n_k is the number of people in category $k \in \{\mathrm{B, E, G, N}\}$, and in each of the four scenarios there are 100 people. Take the two versions of peer-inclusive status introduced in the subsection of Section I entitled 'The role of status', which, making use of the scale invariance property (Axiom 5), can be written as

$$s_i = \frac{1}{n}\sum_{\ell=1}^{k(i)} n_\ell \text{ (downward looking)}, \tag{10}$$

$$s'_i = \frac{1}{n}\sum_{\ell=k(i)}^{K} n_\ell \text{ (upward looking)}. \tag{11}$$

TABLE 2
DISTRIBUTIONAL COMPARISONS FOR A CATEGORICAL VARIABLE

	Case 0			Case 1			Case 2			Case 3		
	n_k	s_i	s'_i	n_k	s_i	s'_i	n_k	s_i	s'_i	n_k	s_i	s'_i
B	0			25	1	1/4	0			25	1	1/4
E	50	1	1/2	25	3/4	1/2	50	1	1/2	25	3/4	1/2
G	25	1/2	3/4	25	1/2	3/4	50	1/2	1	50	1/2	1
N	25	1/4	1	25	1/4	1	0			0		
$\mu(\mathbf{s})$		11/16			5/8			3/4			11/16	

The last row in Table 2 contains the value of mean status,

$$\mu(\mathbf{s}) = \mu(\mathbf{s}') = \frac{1}{n}\sum_{i=1}^{n} s_i = \frac{1}{n}\sum_{i=1}^{n} s_i'. \tag{12}$$

If the relevant scenario changes from Case 0 to Case 1, then 25 people are promoted from category E to category B, and by the principle of monotonicity, if e were a constant equal to the maximum value of status (1 in this case), then inequality would increase.[16] If instead the relevant scenario changes from Case 0 to Case 2, then 25 people are promoted from category N to category G, and by the same principle, inequality would decrease. Now consider a criterion that appears to be related to the transfer principle. If e is a constant or depends only on $\mu(\mathbf{s})$, then the following is true for any α: if there is a change in the underlying distribution such that the status of person i increases by $\delta > 0$ and the status of person j decreases by δ (where $s_i < s_j$ and $s_i + \delta < s_j - \delta$), then, from (8), inequality is reduced. However, this 'transfer property' is not particularly attractive, as we can see from Table 2. If the relevant scenario changes from Case 0 to Case 3, then this exactly fits the balanced change-of-status story: 25 people promoted from N in Case 0 to G in Case 3 experience an increase in status of 1/4; because 25 people are promoted from E to B, this reduces the status of those left in category E by 1/4. But the change from Case 0 to Case 3 is more appropriately seen as a combination of an inequality-increasing change (Case 0 to Case 1) and inequality-decreasing change (Case 1 to Case 3).[17] What is important in the ordinal data case is the pattern of individual moves towards or away from the reference point.

The reference point

The importance of specifying a reference point is evident from the example given in Table 2. The behaviour of I_α in (8) may be sensitive to the choice of e: for example, although the monotonicity axiom ensures that a change in the distribution that moves people closer to a *given* reference point, if the reference point is endogenous, $e = \eta(\mathbf{s})$, then a change in the distribution might possibly shift the reference point in a way that has a counterintuitive effect on inequality. What values could or should e take? We consider four possibilities that appear to be reasonable *a priori*: two where reference point e is exogenous, and two where it depends on $\mathbf{s}$, so that (8) becomes

$$I_\alpha(\mathbf{s};\eta(\mathbf{s})) = \frac{1}{\alpha(\alpha-1)}\left[\frac{1}{n}\sum_{i=1}^{n} s_i^\alpha - \eta(\mathbf{s})^\alpha\right]. \tag{13}$$

We take the status of individual i to be his or her position in the distribution given by (10), (11) or their peer-exclusive counterparts.

- *Maximum status*. Consider what happens in the situation of perfect equality where the distance from the reference point must be zero for everyone. If status is peer-inclusive as in (10) or (11), then there is only one situation in which we can have $s_i = e$ ($i = 1, \ldots, n$) in (13): this is where $e = 1$, the maximum possible value of s. However, although it is clear that $I_\alpha(\mathbf{1};\ 1) = 0$ for all α (the measure is always zero for an equal distribution), there is no guarantee that I_α will be non-negative for every value of α.

- *Minimum status*. In the corresponding peer-exclusive version of status one modifies (10) and (11) by excluding category $k(i)$ from the summation. Again there is only one situation where $s_i = e$ $(i = 1, \ldots, n)$: this time it is where $e = 0$, the minimum value of s. Immediately we see from (13) that I_α will be defined only for $\alpha \geq 0$.
- *Mean status*. Using $e = \eta(\mathbf{s}) = \mu(\mathbf{s})$ (as defined in (12)), it is easy to show that $I_\alpha(\mathbf{s};\mu(\mathbf{s})) \geq 0$ and that $I_\alpha(\mathbf{s};\mu(\mathbf{s})) = 0$ if and only if $s_i = \mu(\mathbf{s})$, for all i.
- *Median status*. Finally, consider the median, $e = \eta(\mathbf{s}) = \text{med}(\mathbf{s})$, defined as $e \in S$ such that $\#(s_i \leq e) \geq n/2$ and $\#(s_i \geq e) \geq n/2$, as a possible reference point. A fundamental difficulty in using this concept is that for categorical data, the median is not well defined.[18] For example, in Table 2 the median could be any value in an interval $M(\mathbf{s})$, where $M(\mathbf{s}) = [1/2, 1)$ in Cases 0 and 2, and $M(\mathbf{s}) = [1/2, 3/4)$ in Cases 1 and 3. Even if we resolve this problem by picking one specific value $e \in M(\mathbf{s})$ as the reference point—for example, the lower bound of the interval $M(\mathbf{s})$—it is not clear that this provides an appropriate reference point with categorical data. Furthermore, there is nothing in the formula (13) that prevents the index taking a negative value.

To see whether each of these suggestions for the reference point is reasonable in practice, let us compare them using the amenity inequality example. The first three rows of Table 3 show the values of a number of reference points e for peer-inclusive downward-looking status: the mean $\mu(\mathbf{s})$ and two interpretations of the median, $\text{med}_1(\mathbf{s})$ (the midpoint of the interval $M(\mathbf{s})$) and $\text{med}_2(\mathbf{s})$ (the lower bound of $M(\mathbf{s})$). The four columns correspond to the four cases in Table 2; notice that $\mu(\mathbf{s})$ and $\text{med}_1(\mathbf{s})$ differ from one distribution to another. Rows 4–7 of Table 3 give the values of the index $I_0(\mathbf{s}; e)$ (see equation (14) below) for each of these three endogenous specifications of e and for the case $e = 1$. (Row 8 of Table 3 is discussed in the subsection below entitled 'Inequality comparisons and the nature of status'.)

The specifications $I_0(\mathbf{s};\mu(\mathbf{s}))$ and $I_0(\mathbf{s};\text{med}_1(\mathbf{s}))$ appear to produce counterintuitive results: inequality *decreases* when one person is promoted from E to B (Case 0 to Case 1, or Case 2 to Case 3). We can understand why this happens when we see that the movement of this person changes both the $\mu(\mathbf{s})$ and $\text{med}_1(\mathbf{s})$ reference points. By contrast, if we use the $\text{med}_2(\mathbf{s})$ specification, then the reference point does not change and the inequality changes are in the direction that accords with

TABLE 3
INEQUALITY COMPARISONS FOR A CATEGORICAL VARIABLE

	Case 0	Case 1	Case 2	Case 3
$\mu(\mathbf{s})$	11/16	5/8	3/4	11/16
$\text{med}_1(\mathbf{s})$	3/4	5/8	3/4	5/8
$\text{med}_2(\mathbf{s})$	1/2	1/2	1/2	1/2
$I_0(\mathbf{s};\mu(\mathbf{s}))$	0.1451	0.1217	0.0588	0.0438
$I_0(\mathbf{s};\text{med}_1(\mathbf{s}))$	0.2321	0.1217	0.0588	−0.0515
$I_0(\mathbf{s};\text{med}_2(\mathbf{s}))$	−0.1732	−0.1013	−0.3465	−0.2746
$I_0(\mathbf{s};1)$	0.5198	0.5917	0.3465	0.4184
$I_0(\mathbf{s}';1)$	0.4184	0.5917	0.3465	0.5198

intuition. The major problem, of course, is that $I_0(\mathbf{s};\ \mathrm{med}_2(\mathbf{s}))$ is negative for *all* the cases in this example; this is clearly problematic since the requirement (9) suggests that this minimum value for $I_\alpha(\mathbf{s};\ e)$ should be zero. Only one specification of the reference point produces an inequality measure that behaves consistently with what one might expect: where $e = 1$. Here we find that $I_0(\mathbf{s};\ 1)$ increases when one individual moves away from the (fixed) reference point (Case 0 to Case 1, Case 3 to Case 1), and $I_0(\mathbf{s};\ 1)$ decreases when one individual moves towards the reference point (Case 0 to Case 2, Case 3 to Case 2). Moreover, measured inequality is always positive if $\mathbf{s} \neq \mathbf{1}$.

Either definition of the median produces unsatisfactory results: some or all of the inequality values are negative in our example. Use of mean status produces strange results when the mean changes significantly as people's positions change. This appears to leave only maximum status as a candidate reference point in the case of a peer-inclusive definition of status. Correspondingly, we would find that if status is defined in a peer-exclusive way, then the appropriate reference point is minimum status, $e = 0$. To get further insight on this, examine the way the class (13) behaves for different values of α.

The sensitivity parameter α

The parameter α in the generic formula (13) captures the sensitivity of measured inequality to different parts of the distribution, for any reference point e. In the case of high values of α, the index is particularly sensitive to high-status inequality. For low and negative values of the parameter, the opposite is true. This is well known for the cardinal data context (income, wealth) in which case the mean is the appropriate reference point and it is easy to show that $I_\alpha(\mathbf{s};\ \mu(\mathbf{s})) \geq 0$ for all $\mathbf{s}$ and α. However, in the context of ordinal data and where the reference point is not the mean, more needs to be said.

Special cases First, it is necessary to be clear about the behaviour of the index $I_\alpha(\mathbf{s};\ e)$ in two special cases of the sensitivity parameter.

Theorem 3. For the class of measures $I_\alpha(\mathbf{s};\ e)$ given in (8), the limiting cases $\alpha = 0$ and $\alpha = 1$ are respectively equal to

$$(14)\qquad I_0(\mathbf{s}; e) = -\frac{1}{n}\sum_{i=1}^{n}\log s_i + \log e,$$

$$(15)\qquad I_1(\mathbf{s}; e) = \begin{cases} \frac{1}{n}\sum_{i=1}^{n} s_i \log s_i - e\log e, & \text{if } e = \mu, \\ \pm\infty, & \text{if } e \neq \mu, \end{cases}$$

where

$$\mu = \frac{1}{n}\sum_{i=1}^{n} s_i.$$

But (15) is not relevant for the analysis of ordinal data, for which the mean is inappropriate as a reference point. In fact, the behaviour of inequality for $\alpha \geq 1$ is problematic, as we will now show.

Negative values For any $e \neq \mu(\mathbf{s})$, the index I_α can be negative for some values of α and is undefined for $\alpha = 1$. This behaviour is illustrated in Figure 1 for $e = 1$; this figure plots values of $I_\alpha(\mathbf{s}; 1)$ for different values of α, using the distribution labelled Case 0 in Table 2. The problem with the index in the neighbourhood of $\alpha = 1$ is clear: $I_\alpha(\mathbf{s}; 1) \to +\infty$ when $\alpha \uparrow 1$, and $I_\alpha(\mathbf{s}; 1) \to -\infty$ when $\alpha \downarrow 1$.

In the case where $e = 1$, using (14) we may rewrite (13) as

$$(16) \qquad I_\alpha(\mathbf{s}, 1) = \begin{cases} \frac{1}{\alpha(\alpha-1)}\left[\frac{1}{n}\sum_{i=1}^{n} s_i^\alpha - 1\right], & \text{if } \alpha \neq 0, 1, \\ -\frac{1}{n}\sum_{i=1}^{n} \log s_i, & \text{if } \alpha = 0. \end{cases}$$

Parameter restriction Here we focus principally on the ordinal data case with peer-inclusive status, for which it is appropriate to take the exogenous reference point $e = 1$. Clearly, it would be appropriate to restrict the class of inequality measures so as to avoid the problems of negative values. To do this, notice that $I_\alpha(\mathbf{s},1)$ can also be written as

$$(17) \qquad I_\alpha(\mathbf{s}, 1) = \frac{1}{\alpha - 1}\left[\frac{1}{n}\sum_{i=1}^{n} \frac{s_i^\alpha - 1}{\alpha}\right], \quad \text{if } \alpha \neq 0, 1.$$

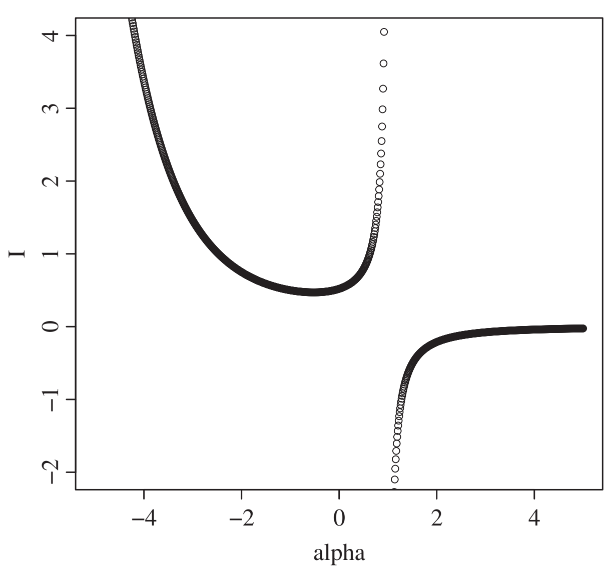

FIGURE 1. The behaviour of $I_\alpha(\mathbf{s};1)$ as α varies.

For any $\alpha \in \mathbb{R}$, it is clear that if $0 < s < 1$, then $[s^{\alpha}-1]/\alpha < 0$, and if $s = 1$, then $[s^{\alpha}-1]/\alpha = 0$. So it is evident from (17) that $I_{\alpha}(\mathbf{s};1)$ is well behaved only under the parameter restriction $\alpha < 1$, and analogously to Atkinson indices, our admissible class of inequality indices could be written in the ordinally equivalent form

$$(18) \qquad A_{\alpha}(\mathbf{s}) := \begin{cases} 1 - \left[\frac{1}{n}\sum_{i=1}^{n} s_i^{\alpha}\right]^{1/\alpha}, & \text{if } \alpha < 0 \text{ or } 0 < \alpha < 1, \\ 1 - \left[\prod_{i=1}^{n} s_i\right]^{1/n}, & \text{if } \alpha = 0. \end{cases}$$

Inequality comparisons and the nature of status

The framework developed in Section II takes as given the precise specification of status for ordinal data and the structure of categories. However, as we made clear in the subsection of Section I entitled 'The role of status', both 'downward-looking' and 'upward-looking' variants of status have a claim to validity in the analysis of ordinal data. It is useful to consider how inequality comparisons are affected by the choice of status variable.

For an illustration of the effect on measured inequality outcomes from using an upward-looking rather than downward-looking concept of status, examine the last two rows of Table 3, which report the values of I_0 for the peer-inclusive version of status where the reference status is $e = 1$ (the row $I_0(\mathbf{s};\, 1)$ reports the outcome for downward-looking status, and the row $I_0(\mathbf{s}';\, 1)$ reports the outcome for upward-looking status). It is not surprising that for both upward-looking and downward-looking status, inequality is high in Case 1 and low in Case 2. However, there is a clear contrast between upward-looking and downward-looking status when we consider Cases 0 and 3: for downward-looking status, $I_0(\mathbf{s};\, 1)$, inequality is higher when the distribution is skewed towards the higher categories (Case 0); for upward-looking status, $I_0(\mathbf{s}';\, 1)$, inequality is higher when the distribution is skewed towards the lower categories (Case 3).

The reason for this is suggested by equation (16): it is clear that the index decreases as the vector of status gets closer to the vector $\mathbf{1} := (1,\ldots,1)$. Label individuals in increasing order of underlying utility (increasing order of the categories to which they belong). Then for downward-looking status we have $0 < s_1 \le s_2 \le \ldots \le s_n \le 1$: the status vector becomes closer to $\mathbf{1}$ when people move to the lower categories. For upward-looking status we have $1 \ge s'_1 \ge s'_2 \ge \ldots \ge s'_n > 0$: the status vector becomes closer to $\mathbf{1}$ when people move to the higher categories. We can say more. Consider the associated vector of status across the *K categories* rather than for the n individuals (recall that every individual i in category k has the same status). Denote by $s_{[k]}$ the status of anyone in category k, and let

$$\mathbf{s}_{\uparrow} := (s_{[k]} : k = 1, \ldots, K),$$
$$\mathbf{s}_{\downarrow} := (s_{[k]} : k = K, \ldots, 1),$$

be the status vectors across categories when they are arranged in ascending order and in descending order, respectively. Again let $\mathbf{s}$ and $\mathbf{s}'$ denote the downward-looking and upward-looking status concepts derived from a particular distribution of individuals across categories. Then we have that inequality evaluated for $\mathbf{s}_{\uparrow}$ is the same as inequality evaluated for $\mathbf{s}'_{\downarrow}$. It is clear from Tables 2 and 3 that $I_0(\mathbf{s};1)$ in Case 0 has the same value (0.5198) as $I_0(\mathbf{s}';1)$ in Case 3. This property implies that ordinal inequality evaluated for downward-looking status is equal to inequality for upward-looking status if and only if the status vector by categories is symmetric so that $\mathbf{s}_{\uparrow} = \mathbf{s}_{\downarrow}$. This then allows us to make comparisons between the two versions (downward- and upward-looking status) to detect skewness, a feature that will be developed in the empirical application of Section V.

However, this comparison of the two types of status assumes a given structure of 'active' categories—that is, categories that are not empty. What happens if the number of active categories changes? For the case of downward-looking status, inequality must increase if a person migrates upwards from a category with multiple occupants to an empty category; for the upward-looking case, inequality must increase if a person migrates downwards from a category with multiple occupants to an empty category. We can see this if we consider what happens to inequality when there is a merger. Suppose that status is downward-looking, given by (10). Now let the non-empty categories k^* and $k^* + 1$ be merged. This is equivalent to increasing the size of category k^* and emptying category $k^* + 1$. For every i such that $k(i) = k^*$, we see that s_i increases by n_{k^*+1}/n; for every other i, status s_i remains unchanged. So the status of some people moves closer to e, and status remains unchanged for others. By Axiom 2, this means that inequality must fall. By contrast, if a person migrates from category k^* (where $n_{k^*} > 1$) to empty category $k^* + 1$, then this is the exact opposite of the merger just described, so inequality must rise. A similar argument can be constructed for upward-looking status.

Comparison with cardinal inequality measurement

The approach set out in Section II should not be seen as a niche theoretical development with only a very specialised scope for application. The approach is general and can also be applied to the conventional cardinal data problem. It requires only an interpretation of status that is appropriate to the information content of cardinal data.

In the cardinal data case, status can be identified with the thing that is to be equalized —call it 'income'—and the summation of status is well defined. Again taking the reference point as that corresponding to notional perfect equality, it is clear that in this case the reference point should be taken to be mean income (12); so inequality takes the form (8) with $\eta(\mathbf{s}) = \mu(\mathbf{s})$. The property of scale invariance implies that inequality can be measured in terms of income shares.[19]

As we noted in the first subsection of this section, the transfer principle has no meaning for $I_\alpha(\mathbf{s};\ e)$ in (8) when applied to ordinal data. So it is reassuring to note that if status is cardinal (income, wealth, earnings, and so on), then the transfer principle is satisfied by (8). This is because $I_\alpha(\mathbf{s};\ e)$ is increasing and convex in each s_i, so an infinitesimal transfer of δ from j to i must reduce $I_\alpha(\mathbf{s};\ e)$ if $s_i < s_j$.

Clearly, for cardinal data, the $I_\alpha(\mathbf{s};\ e)$ class in (8) subsumes the generalised entropy class of indices and therefore the two Theil indices as special cases—see equations (14) and (15). With a monotonic transformation, it also subsumes the class of Atkinson indices. So, considered together with the applications to ordinal data discussed in the previous four subsections, our approach, centred on the class $I_\alpha(\mathbf{s};\ e)$, covers a wide range of inequality measurement tools.

IV. Inequality Measurement: Implementation

For it to be used in practice, we need to establish the statistical properties of the inequality measure with ordinal data. In this section, we establish asymptotic distribution and finite sample performance of such inequality measures. In the light of the arguments in Section III, we focus on ordered categorical variables, for which the appropriate reference point is the maximum status $e = 1$, and take the family of inequality indices (16). We consider samples with independent observations on K ordered categories. Without loss of generality, any sample can be represented as

$$(19)\qquad x_i = \begin{cases} 1 & \text{with sample proportion } p_1, \\ 2 & \text{with sample proportion } p_2, \\ \vdots & \\ K & \text{with sample proportion } p_K, \end{cases}$$

where p_l is the number of observations in the lth category, divided by the sample size, so that $\sum_{l=1}^{K} p_l = 1$. For $K > 2$, the sample proportions $(p_1,\ p_2,\ \ldots,\ p_K)$ are known to follow a multinomial distribution with n observations and a vector of probabilities $(\pi_1,\ \pi_2,\ \ldots,\ \pi_K)$. The status of observation i is its position in the distribution, computed as the proportion of observations in the sample with a value less than or equal to x_i:[20]

$$(20)\qquad s_i = \frac{1}{n}\sum_{j=1}^{n} \iota(x_j \leq x_i) = \sum_{j=1}^{x_i} p_j,$$

where $\iota(z)$ equals 1 if z is true and 0 otherwise.

With a sample of categorical data (19) and status given by (20), we can rewrite the inequality measure (16) as

$$(21)\qquad I_\alpha = \begin{cases} \frac{1}{\alpha(\alpha-1)}\left[\sum_{i=1}^{K} p_i \left[\sum_{j=1}^{i} p_j\right]^{\alpha} - 1\right], & \text{if } \alpha \neq 0, 1, \\ -\sum_{i=1}^{K} p_i \log\left[\sum_{j=1}^{i} p_j\right], & \text{if } \alpha = 0. \end{cases}$$

This measure is expressed as a non-linear function of K parameter estimates $(p_1, p_2, \ldots, p_K)$ following a multinomial distribution.

Asymptotic distribution From the Central Limit Theorem, I_α asymptotically follows a Normal distribution with variance estimator

$$\widehat{\mathrm{Var}}(I_\alpha) = D\Sigma D^{\top},$$

where

$$\Sigma = \left[\iota(k = l) - p_k p_l\right]_{k,l=1,\ldots,K}, \quad D = \left[\frac{\partial I_\alpha}{\partial p_l}\right]_{l=1,\ldots,K}$$

and

$$\frac{\partial I_\alpha}{\partial p_k} = \begin{cases} \frac{1}{\alpha(\alpha-1)}\left(\left[\sum_{i=1}^{k} p_i\right]^{\alpha} + \alpha \sum_{i=k}^{K-1} p_i \left[\sum_{j=1}^{i} p_j\right]^{\alpha-1}\right), & \text{if } \alpha \neq 0, 1, \\ -\log\left[\sum_{j=1}^{k} p_j\right] - \sum_{i=k}^{K-1} p_i \left[\sum_{j=1}^{i} p_j\right]^{-1}, & \text{if } \alpha = 0. \end{cases}$$

We can use the variance estimators of I_α to compute test statistics and confidence intervals.

Finite sample The coverage error rate of a confidence interval is the probability that the random interval does not include the true value of the parameter. A method of constructing confidence intervals with good finite sample properties should provide a coverage error rate close to the nominal rate. For a confidence interval at 95%, the nominal coverage error rate is equal to 5%. We use Monte Carlo simulation to approximate the coverage error rate of asymptotic confidence intervals in several experimental designs.

In our experiments, samples are drawn from a multinomial distribution with probabilities $\pi = (\pi_1, \pi_2, \ldots, \pi_K)$. The status of observation i is its position in the distribution, computed as the proportion of observations in the sample with a value less than or equal to x_i. For fixed values of α, n, K and π, we draw 10,000 samples. For each sample, we compute $I_\alpha(\mathbf{s}, 1)$ and its confidence interval at 95%. The coverage error rate is computed as the proportion of times the true value of the inequality measure is not included in the confidence intervals.[21] Confidence intervals perform well in finite samples if the coverage error rate is close to the nominal value.

Table 4 shows coverage error rates of confidence intervals at 95% of I_α for different values of α ($\alpha = -1, 0, 0.5, 0.99, 1.01, 1.5, 2$) as the sample size increases ($n = 20, 50, 100, 200, 500, 1000$). We consider three ordered categories: samples are drawn from a multinomial distribution with probabilities $\pi = (0.3, 0.5, 0.2)$. If the asymptotic distribution is a good approximation to the exact distribution of the statistic, then the coverage error rate should be close to the nominal error rate, 0.05. The results show that asymptotic confidence intervals perform well in finite samples; they are still reliable for $\alpha = 0.99, 1.01$ when the index is undefined for $\alpha = 1$.

TABLE 4
COVERAGE ERROR RATE OF ASYMPTOTIC CONFIDENCE INTERVALS AT 95% OF I_α, 10,000 REPLICATIONS, $K = 3$ AND $X \sim$ MULTINOMIAL(0.3, 0.5, 0.2)

α	−1	0	0.5	0.99	1.01	1.5	2
$n = 20$	0.0606	0.0417	0.0598	0.0491	0.0491	0.0472	0.0431
$n = 50$	0.0553	0.0518	0.0704	0.0601	0.0603	0.0483	0.0394
$n = 100$	0.0499	0.0513	0.0684	0.0619	0.0619	0.0489	0.0416
$n = 200$	0.0544	0.0476	0.0617	0.0613	0.0607	0.0543	0.0451
$n = 500$	0.0523	0.0492	0.0521	0.0523	0.0526	0.0498	0.0466
$n = 1000$	0.0485	0.0540	0.0552	0.0549	0.0551	0.0546	0.0528

V. Application

What drives ordinal inequality indices in practice? In our application, we focus on the shape of the empirical distribution across categories and the resulting indicators of inequality. We do this for two important categorical indicators, health and happiness. We use the data from the fifth wave of the World Values Survey 1981–2008, conducted in 2005–8 over 56 countries.[22]

Our empirical examples focus on two questions concerning life satisfaction and health. The wording of the life satisfaction question is as follows:

All things considered, how satisfied are you with your life as a whole these days? Using this card on which 1 means you are ‘completely dissatisfied’ and 10 means you are ‘completely satisfied’ where would you put your satisfaction with your life as a whole? (code one number):

Completely dissatisfied—1 2 3 4 5 6 7 8 9 10—Completely satisfied

The wording of the health question is as follows:

All in all, how would you describe your state of health these days? Would you say it is (read out): 1 Very good, 2 Good, 3 Fair, 4 Poor.

We investigate the following questions:

- Do people in higher-income countries tend to report higher life and/or health satisfaction?
- What is the shape of the distribution across categories?
- How does this shape affect ordinal inequality?

Happiness–income paradox Easterlin (1974) shows that for a given country, people with higher incomes are likely to report higher life satisfaction, whereas for cross-country comparisons and for higher-income countries, the average level of life satisfaction does not vary much with income: this has come to be known as the Easterlin or happiness–income paradox. Several studies confirm the lack of impact of income on life satisfaction for higher-income countries,[23] while other studies find a significant positive impact.[24] To illustrate the issue, the left-hand panel of Figure 2 presents a cross-country comparison of the average of answers to the life satisfaction question in the World Values Survey and GDP per capita in 2005 from the Penn World Table 7. It is clear that the mean of life satisfaction is higher in countries with higher GDP per capita; it is also clear that the relationship between satisfaction and income is non-linear. However, it is not obvious how to choose between (1) a logarithmic relation and (2) a piecewise linear model with the slope of the second line not significantly different from zero beyond \$15,000 per head.[25] So the presence or absence of an income effect on life satisfaction among the higher-income countries is not clear, and the controversy remains unresolved.

Many empirical studies make comparisons of the averages of answers to questions, or use specific transformations to calculate a composite index of wellbeing.[26] In fact, they interpret the answers as cardinal with a linear scale: the values attached to each successive category are supposed to be equidistant. Such an interpretation is subject to disagreement in the literature on Likert-type scale questionnaires. Some consider that ordinal data provide information on ranks only and nothing else, so they should be treated as purely ordinal data with non-parametric statistics.[27] For example, a horse race result provides information on the rank order 1st, 2nd, 3rd, and so on, without any information on the arrival times and differences in time between horses. Others consider

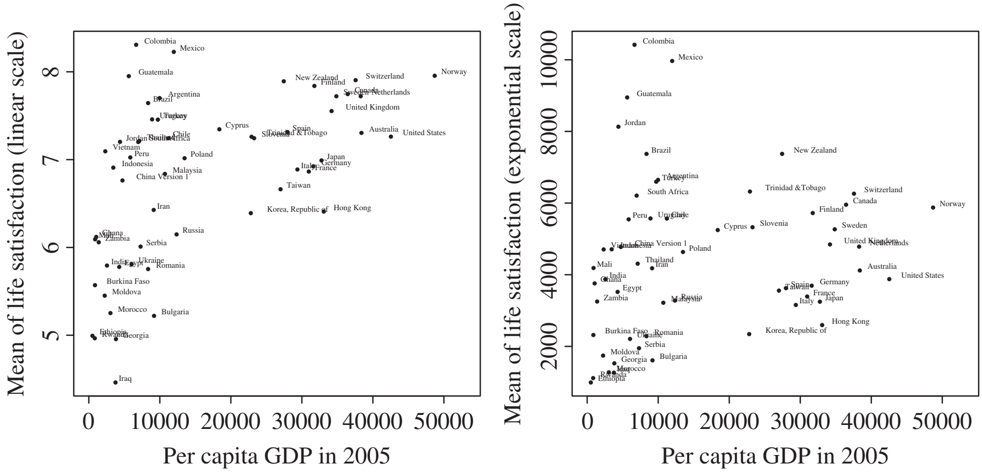

FIGURE 2. Mean of life satisfaction and GDP per capita: linear scale (left), exponential scale (right).

that ordinal data can be interpreted as cardinal, with a linear scale; in some contexts—where the scale presents a suitable symmetry of items around a clear middle category—the intervals between points could be considered approximately equal so they can be analysed parametrically.[28]

We can illustrate how sensitive the results are to this hypothesis. Let us consider an exponential scale rather than a linear scale.[29] The right-hand panel of Figure 2 presents a cross-country comparison of the average of the exponential transformation of the code numbers from the life satisfaction question and the GDP per capita. There is no clear relationship between the average of life satisfaction and national income.[30] Compared to the conclusions based on a linear scale (left-hand panel of Figure 2), we obtain very different results. Clearly, the results are sensitive to the cardinal interpretation of the data.[31]

Ordinal approach A better approach is to use the inequality index developed here; this is defined on individual ranks alone and is thus insensitive to any cardinal interpretation of the answers. If we require inequality to fall if answers are skewed towards more desirable categories, then for the life satisfaction question, where the categories are (Completely dissatisfied—1, 2, . . ., 10—Completely satisfied), an upward-looking version is required. The left-hand panel of Figure 3 presents a cross-country comparison of the inequality index of life satisfaction[32] with GDP per capita. A negative relationship seems to emerge, but it is far from clear-cut. Two countries, Colombia and Mexico, appear to be different from the others, with very small values of both inequality of life satisfaction and GDP per capita. The presence or absence of these two countries in the sample may lead to different conclusions.

Table 5 shows the number of respondents, by country, for each level of the life satisfaction question, and two inequality measures computed with $\alpha = 0$. The first inequality measure is based on downward-looking status, I_0^{down}, and the second is based on upward-looking status, I_0^{up}.[33] The first column is the country code.

To go further, we first look at the presence of empty categories. India is the only country with empty categories: nobody reports a life satisfaction of 4, 6, 8 or 9 over 1954

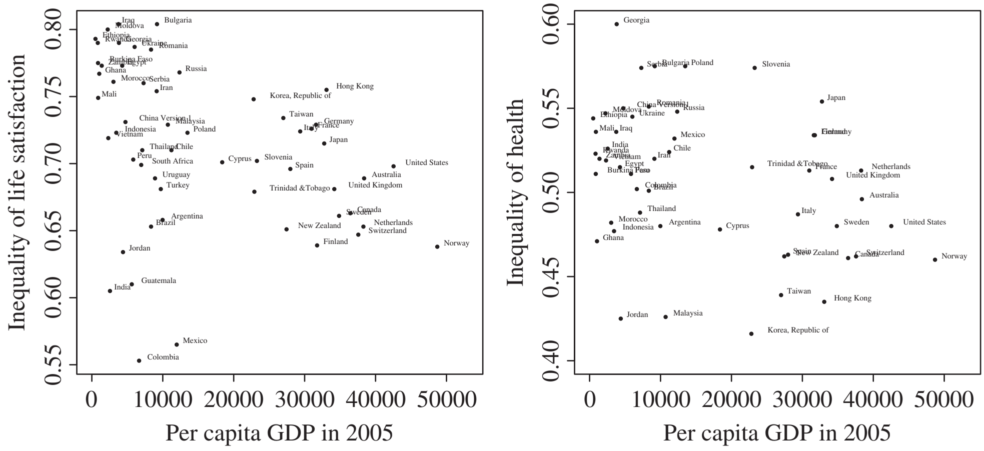

FIGURE 3. Inequality of life satisfaction (left) and of health satisfaction (right) vs. GDP per capita.

respondents. It is a very specific behaviour; every other country has non-empty categories. From the mergers principle, inequality falls with a smaller number of non-empty categories—see the fourth subsection of Section III.[34] A smaller value of the inequality index for India may then be partially due to the presence of four empty categories, compared to other countries.

We can elicit more information on the nature of the distribution of answers by comparing upward- and downward-looking versions of an inequality measure. Inequality falls if answers are skewed towards higher (smaller) categories with the upward (downward) version—see the fourth subsection of Section III. So the distribution is symmetric when upward and downward versions give similar values. When the upward-looking version provides significantly smaller (greater) values than the downward-looking version, the distribution is skewed to higher (smaller) categories. From Table 5, we have the following.

- $I_0^{\mathrm{down}} \approx I_0^{\mathrm{up}}$: The two versions of the inequality measure give the same value for Moldova (MD: $I_0^{\mathrm{up}} = I_0^{\mathrm{down}} = 0.8$), and the distribution of answers is symmetric around the middle category (labelled 5).
- $I_0^{\mathrm{down}} \ll I_0^{\mathrm{up}}$: Iraq (IQ) has a value of the downward-looking version smaller than that of the upward-looking version, $I_0^{\mathrm{down}} = 0.724$ and $I_0^{\mathrm{up}} = 0.804$. The distribution is clearly skewed to smaller categories, that is, people tend to report low life satisfaction levels.
- $I_0^{\mathrm{down}} \gg I_0^{\mathrm{up}}$: Colombia (CO) and Mexico (MX) provide values of the downward-looking version much greater than the upward-looking version, $I_0^{\mathrm{down}} = 0.764, 0.755$ and $I_0^{\mathrm{up}} = 0.553, 0.565$. Their distributions are strongly skewed towards higher categories.

Colombia and Mexico not only provide the biggest positive discrepancies, $\max(I_0^{\mathrm{down}} - I_0^{\mathrm{up}})$, but also provide the smallest values of the upward-looking inequality index, $\min(I_0^{\mathrm{up}})$. This suggests that their distributions are very skewed towards higher categories, and they are also very concentrated. Indeed, when we look at the detailed survey, we can see that more than 74% of individuals in Colombia, and more than 77% of individuals in Mexico, report a value of 8 or higher (out of 10) in the life satisfaction question.

TABLE 5
NUMBER OF PEOPLE PER CODE NUMBER IN THE LIFE SATISFACTION QUESTION AND THE INEQUALITY INDEX, WITH UPWARD- AND DOWNWARD-LOOKING STATUS, $\alpha = 0$

	– Life satisfaction+										Inequality	
	1	2	3	4	5	6	7	8	9	10	I_0^{down}	I_0^{up}
AR	14	9	19	15	75	53	197	279	131	203	0.774	0.658
AU	17	18	29	40	109	117	303	472	195	110	0.77	0.689
BF	55	71	103	186	399	233	171	126	48	107	0.781	0.775
BG	57	69	115	116	184	109	147	81	49	30	0.790	0.804
BR	29	15	21	34	162	130	185	352	201	366	0.791	0.653
CA	12	17	25	52	123	158	361	726	348	335	0.765	0.663
CH	8	7	12	16	64	78	170	445	241	191	0.746	0.647
CL	11	3	24	56	120	125	161	197	130	165	0.801	0.71
CN	72	72	86	85	222	278	244	427	198	275	0.814	0.731
CO	44	20	20	40	172	157	332	601	485	1147	0.764	0.553
CSS	30	53	84	97	177	170	262	210	54	38	0.800	0.760
CY	28	8	16	41	79	108	187	284	157	142	0.792	0.701
DE	24	27	86	112	262	208	370	530	290	141	0.799	0.729
EG	305	122	234	251	510	311	439	370	162	346	0.788	0.773
ES	4	2	16	25	84	166	324	335	170	69	0.746	0.696
ET	50	108	249	200	255	281	197	86	41	23	0.775	0.793
FI	7	18	19	20	40	34	137	347	290	102	0.744	0.639
FR	15	9	41	43	135	106	219	263	102	67	0.787	0.726
GB	9	3	11	20	73	100	208	346	157	111	0.757	0.681
GE	128	92	154	204	377	160	161	111	25	64	0.766	0.790
GH	97	84	117	157	114	181	221	273	120	164	0.811	0.767
GT	9	12	26	21	69	82	122	176	171	311	0.798	0.610
HK	23	22	49	77	191	272	236	222	85	67	0.789	0.755
ID	58	22	47	64	296	256	358	372	159	274	0.792	0.723
IN	71	1	308	**0**	821	**0**	437	**0**	**0**	316	0.624	0.605
IQ	465	221	232	318	616	310	207	149	63	89	0.724	0.804
IR	101	103	117	180	453	364	385	396	229	339	0.812	0.754
IT	12	10	16	47	93	182	280	216	84	66	0.771	0.724
JO	99	22	22	37	148	73	131	180	121	364	0.776	0.634
JP	10	12	41	38	92	171	219	301	146	50	0.784	0.715
KR	30	22	47	88	172	191	273	246	83	45	0.793	0.748
MA	27	31	115	195	373	206	130	58	22	40	0.746	0.761
MD	46	69	99	150	172	151	131	127	66	30	0.800	0.800
ML	113	31	79	95	328	154	184	148	96	202	0.780	0.749
MX	39	10	15	18	86	60	117	380	256	531	0.755	0.565
MY	17	9	28	41	145	230	287	265	92	86	0.774	0.729
NL	2	5	5	16	40	82	235	406	156	102	0.718	0.653
NO	7	3	7	12	47	51	142	387	245	122	0.725	0.638
NZ	7	8	16	20	50	67	149	224	182	204	0.788	0.651
PE	29	24	59	72	194	165	278	280	111	278	0.808	0.703
PL	14	15	34	34	152	117	149	238	112	124	0.797	0.723
RO	93	82	163	143	283	179	247	285	107	76	0.802	0.785
RU	117	74	104	151	362	217	326	342	146	177	0.802	0.768
RW	91	146	136	166	365	299	122	96	53	29	0.771	0.790
SE	5	5	16	22	47	61	188	367	177	114	0.746	0.661

TABLE 5
CONTINUED

	– Life satisfaction+										Inequality	
	1	2	3	4	5	6	7	8	9	10	I_0^{down}	I_0^{up}
SI	10	6	19	25	167	122	173	240	104	167	0.776	0.702
TH	12	11	36	53	137	217	307	429	172	158	0.783	0.710
TR	41	18	46	36	77	134	220	291	211	272	0.805	0.681
TT	35	7	14	29	120	124	176	195	80	219	0.789	0.679
TW	38	16	34	54	190	207	217	280	74	117	0.788	0.734
UA	45	51	97	81	147	147	170	149	66	43	0.806	0.787
US	9	11	31	54	91	117	275	358	224	71	0.779	0.698
UY	19	2	12	19	76	125	218	245	115	162	0.775	0.689
VN	6	10	41	61	163	287	278	292	138	206	0.787	0.719
ZA	96	75	97	141	257	298	419	588	435	571	0.818	0.699
ZM	107	52	89	114	204	189	249	229	99	131	0.802	0.773

Finally, comparing upward- and downward-looking status can help us to identify the shape of the underlying distribution. In our example, Colombia and Mexico are identified with a specific behaviour in self-reported life satisfaction: they are poor countries, but many people report being very happy. Figure 3 suggests that without these two countries, we would find that individuals in higher income countries tend to report higher life satisfaction.[35]

Health For the health question, the lowest category is assigned to the answer '1 Very good';[36] a downward-looking version ensures that inequality decreases when people tend to report better health states.[37]

The right-hand panel of Figure 3 presents a cross-country comparison of the inequality index of health[38] with GDP per capita. One country, Jordan, appears to be different from the others, with very small values of both inequality of health satisfaction and GDP per capita. The presence or absence of this country in the sample leads to different conclusions. With all countries, we find a correlation coefficient equal to -0.34, which is not significant at 1% ($p = 0.01381$). Excluding Jordan leads to a correlation coefficient equal to -0.39, which is statistically significantly different from zero ($p = 0.00467$).

Table 6 shows the number of respondents, by country, for each level of the health satisfaction question, and the inequality measure computed with $\alpha = 0$ and upward- and downward-looking status.[39] From comparisons between downward- and upward-looking versions of the inequality measure, we have the following.

- $I_0^{\text{down}} \approx I_0^{\text{up}}$ characterizes a country with a symmetric distribution by categories. It is the case for Japan (JP: $I_0^{\text{down}} = 0.554$ and $I_0^{\text{up}} = 0.561$), where the distribution is approximately symmetric ($\{148, 440, 405, 95\}$).
- $I_0^{\text{down}} \ll I_0^{\text{up}}$ characterizes a country with a distribution mostly skewed towards lower categories. The biggest negative discrepancies, $I_0^{\text{down}} - I_0^{\text{up}}$, are for Norway (NO: $I_0^{\text{down}} = 0.460$ and $I_0^{\text{up}} = 0.597$) and Jordan (JO: $I_0^{\text{down}} = 0.425$ and $I_0^{\text{up}} = 0.554$). The two discrepancies are *large*: people tend to report *strongly* higher health satisfaction levels (NO $\{426, 388, 145, 66\}$ and JO $\{524, 518, 130, 27\}$).

TABLE 6
NUMBER OF PEOPLE PER CODE NUMBER IN THE HEALTH SATISFACTION QUESTION AND THE INEQUALITY INDEX, WITH UPWARD- AND DOWNWARD-LOOKING STATUS, $\alpha = 0$

	+ Health –				Inequality	
	1	2	3	4	I_0^{down}	I_0^{up}
AR	326	463	185	27	0.480	0.567
AU	402	675	274	61	0.496	0.575
BF	427	670	350	72	0.511	0.585
BG	149	389	310	150	0.575	0.588
BR	408	700	355	36	0.501	0.556
CA	857	880	341	81	0.461	0.587
CH	400	635	173	33	0.462	0.546
CL	196	475	277	52	0.524	0.565
CN	619	617	486	290	0.550	0.622
CO	674	1517	760	72	0.502	0.540
CSS	226	431	406	147	0.574	0.592
CY	413	387	184	65	0.478	0.609
DE	458	908	538	148	0.534	0.589
EG	471	1578	766	236	0.515	0.564
ES	237	716	195	48	0.463	0.521
ET	386	582	390	139	0.544	0.608
FI	232	437	284	61	0.534	0.583
FR	300	418	225	58	0.513	0.598
GB	345	412	196	87	0.508	0.613
GE	189	439	574	296	0.600	0.568
GH	571	661	208	93	0.471	0.592
HK	68	723	407	48	0.435	0.487
ID	386	1144	408	63	0.477	0.526
IN	438	934	489	136	0.526	0.583
IQ	476	1285	592	317	0.536	0.586
IR	499	1294	680	156	0.520	0.568
IT	188	562	228	34	0.487	0.533
JO	524	518	130	27	0.425	0.554
JP	148	440	405	95	0.554	0.561
KR	175	809	208	8	0.416	0.437
MA	461	434	262	43	0.482	0.594
MD	107	443	354	117	0.547	0.565
ML	437	510	467	93	0.536	0.595
MX	344	654	503	54	0.532	0.554
MY	332	739	125	5	0.426	0.453
NL	247	497	262	42	0.513	0.566
NO	426	388	165	46	0.460	0.597
NZ	348	427	154	20	0.462	0.562
PE	112	606	723	59	0.511	0.473
PL	175	365	331	127	0.575	0.591
RO	175	776	569	254	0.551	0.570
RU	134	721	911	256	0.548	0.534
RW	37	469	740	255	0.523	0.512
SE	353	430	187	33	0.480	0.581
SI	220	369	302	144	0.574	0.605

TABLE 6
CONTINUED

	+ Health − 1	2	3	4	Inequality I_0^{down}	I_0^{up}
TH	331	834	295	66	0.488	0.548
TT	321	375	248	56	0.515	0.602
TW	280	772	117	58	0.439	0.493
UA	76	412	378	132	0.545	0.555
US	350	645	206	47	0.480	0.557
VN	155	736	473	131	0.519	0.554
ZM	415	618	282	128	0.520	0.606

- $I_0^{\text{down}} \gg I_0^{\text{up}}$ characterizes a country with a distribution mostly skewed towards higher categories. The biggest positive discrepancy is for Peru (PE: $I_0^{\text{down}} = 0.511$ and $I_0^{\text{up}} = 0.473$). The discrepancy is *slight*: people tend to report *slightly* lower health satisfaction levels ($\{112, 606, 723, 59\}$).

The smallest value of the inequality measure, $\min(I_0^{\text{down}}) = 0.416$, is given for the Republic of Korea (KR). Clearly, a single value of the inequality index does not tell us very much about the underlying distribution; we need to make comparisons. First, the discrepancy with the upward-looking version of the inequality index is quite small ($I_0^{\text{up}} = 0.437$), which suggests that the distribution is not highly skewed. Then, if not highly skewed, a small value of the inequality index suggests a highly concentrated distribution. It is clear from the detailed survey: more than two-thirds of people in the Republic of Korea give the same answer '2 Good' to the health satisfaction question (809 of 2000 respondents: $\{175, 809, 208, 8\}$).

Figure 4 presents a cross-country comparison between the (downward-looking) inequality of health and the (upward-looking) inequality of life satisfaction. A positive

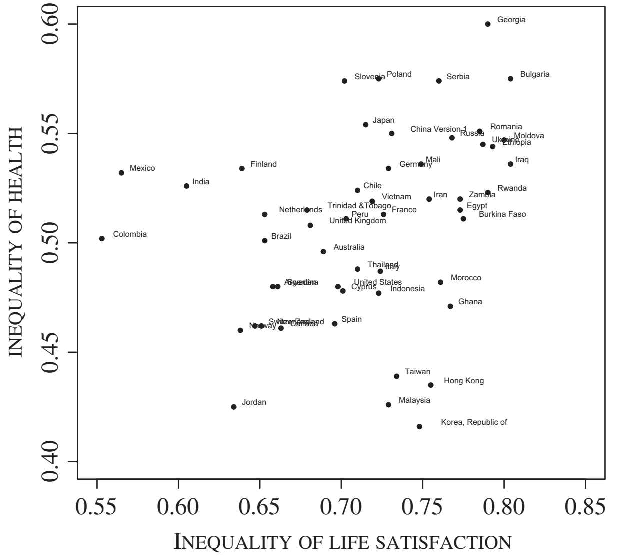

FIGURE 4. Inequality of life satisfaction vs. inequality of health satisfaction.

relationship seems to emerge, but it is far from clear-cut. With all countries, we do not find a significant correlation ($\rho = 0.35$ and $p = 0.0199$). However, removing Colombia, Mexico and India (with specific behaviour in the life satisfaction question—see above), we find a significant positive correlation coefficient ($\rho = 0.45$ and $p = 0.00123$). Again, the presence or absence of these countries in the sample may lead to different conclusions.

Finally, the application above illustrates that two features drive the ordinal inequality measure: (1) the dispersion and (2) the skewness of the underlying distribution. The discrepancy between downward- and upward-looking versions of the inequality index provides information on the skewness of the distribution. Moreover, the last discrepancy being equal between two countries, a smaller index value suggests a more concentrated distribution.

These features have been used in the present empirical study to identify the shape of a large diversity of distributions among many different countries, and to show that the relationships happiness–income, health–income and happiness–health are very sensitive to the presence of a very few countries, with specific behaviour, in the sample.

VI. Conclusion

There are three basic ingredients in our approach: the concept of status within a distribution, a reference point and a set of axioms. Status can be downward- or upward-looking depending on the context of the analysis. The axiomatization is general and provides an approach for a variety of data types, including the case of categorical data—it characterizes a family of indices that is conditional on a sensitivity parameter and a reference point. The specific class of inequality measures that emerges from the axiomatization is related to the generalised entropy and Atkinson classes; but, by contrast to conventional inequality analysis, the reference point for categorical data is not the mean of the distribution but either the maximum or minimum possible value of status.

We can provide a precise answer to the problem of measuring inequality of the distribution of ordinal data interpreted as categorical variables. The answer lies within a new analytical framework that is suitable for both the ordinal data case and the standard cardinal data case.

The approach is straightforward to implement empirically. As we have shown in Section V, how you treat categorical data in empirical studies really matters.

APPENDIX: PROOFS

Proof of Theorem 1 There are two cases to consider. (1) In the case where the data are categorical, S is the set of non-negative rational numbers $\mathbb{Q}_+$. (2) In the case where the data have cardinal significance, S can be taken as an interval in $\mathbb{R}$. In either case, $(S,+,\succ)$ forms a strictly ordered group (Krantz 1964; Luce and Tukey 1964; Wakker 1988), so from Theorem 5.3 of Fishburn (1970), Axioms 1–3 jointly imply that for a given e, $\succcurlyeq$ is representable by a continuous function $S^{n+1} \to \mathbb{R}$ given by

$$\text{(A1)} \qquad \sum_{i=1}^{n} d_i(s_i, e), \quad \text{for all } (\mathbf{s}, e) \in S^{n+1},$$

where for each i, $d_i : S \to \mathbb{R}$ is a continuous function. By Axiom 2 this is increasing in s_i if $s_i > e$, and vice versa. In view of Axiom 4, the functions d_i must all be identical. Clearly, the ordering $\succcurlyeq$ is also representable any monotonic transformation of the function in (1), possibly depending on e, so the result follows.

Proof of Lemma 1
From (5), $(\mathbf{s}, e) \sim (\mathbf{s}', e')$ implies that

$$\text{(A2)} \qquad \sum_{i=1}^{n} d(s_i, e) = \sum_{i=1}^{n} d(s'_i, e'),$$

so Axiom 5(a) implies that

$$\sum_{i=1}^{n} d(\lambda s_i, e) = \sum_{i=1}^{n} d(\lambda s'_i, e').$$

Therefore we have

$$\text{(A3)} \qquad \frac{\sum_{i=1}^{n} d(\lambda s_i, e)}{\sum_{i=1}^{n} d(s_i, e)} = \frac{\sum_{i=1}^{n} d(\lambda s'_i, e')}{\sum_{i=1}^{n} d(s'_i, e')}.$$

For any $\lambda > 0$ we have

$$\text{(A4)} \qquad \sum_{i=1}^{n} d(\lambda s_i, e) = \theta\left(\lambda, \sum_{i=1}^{n} d(s_i, e)\right),$$

where $\theta : \mathbb{R}^2 \to \mathbb{R}$ is increasing in its second argument. Consider the case where, for arbitrary distinct values j and k, we have $s_i = e$ for all $i \neq j, k$. This implies that for given values of e, λ, (A4) can be written as the functional equation

$$\text{(A5)} \qquad f(s_j) + f(s_k) = h\big(g(s_j) + g(s_k)\big),$$

where

$$\text{(A6)} \qquad f(s) := d(\lambda s, e) + \left[\frac{1}{2}n - 1\right] d(\lambda e, e),$$

$$\text{(A7)} \qquad g(s) := d(s, e) + \left[\frac{1}{2}n - 1\right] d(e, e)$$

and $h(x) := \theta(\lambda, x)$. Equation (A5) has the solution

$$\text{(A8)} \qquad f(s) = a_0 g(s) + a_1,$$

$$\text{(A9)} \qquad h(x) = a_0 x + 2a_1,$$

where a_0,a_1 are constants (Polyanin and Zaitsev 2004, Supplement S.5.5) that may depend on the arbitrary values of e and λ. Using (A6) and (A8) in the case $s = e$, we have

$$a_1 = \frac{1}{2}n\left[d(\lambda e, e) - a_0 d(e, e)\right].$$

Using this in (A8) and (A6), and writing $a_0 = \psi(\lambda, e)$ we have

$$d(\lambda s, e) = \psi(\lambda, e)d(s, e) + \frac{1}{2}n\left[d(\lambda e, e) - \psi(\lambda, e)d(e, e)\right],$$

so

$$d(\lambda e, e) = \psi(\lambda, e)d(e, e),$$

from which we may deduce

$$\text{(A10)} \qquad d(\lambda s, e) = \psi(\lambda, e)d(s, e).$$

Substituting from (A10) into (A3), we find that $\psi(\lambda, e) = \psi(\lambda, e')$. Therefore ψ is independent of e:

$$\text{(A11)} \qquad d(\lambda s, e) = \psi(\lambda)d(s, e).$$

For given e we may write (A11) as

$$\theta_1(u + v) = \theta_1(u) + \theta_2(v),$$

where $u := \log s$, $v := \log \lambda$, $\theta_1(u) := \log d(\exp(u),e)$, $\theta_2(v) := \log \psi(\exp(v))$. The solution of this equation is $\theta_1(u) = \alpha u + \beta$, $\theta_2(v) = \alpha v$, so equation (A11) implies that

$$d(s, e) = As^{\alpha}, \quad \psi(\lambda) = \lambda^{\alpha},$$

where α and A are constants that may depend on the exogenous value of e.

Outline proof of Lemma 2

The proof follows the proof of Lemma 1 from (A2) to (A10) closely, substituting (λs, λe) for (λs, e). The counterpart of equation (A10) is

$$d(\lambda s, \lambda e) = \psi(\lambda, e)d(s, e),$$

and once again we find that ψ is independent of e. So

$$d(\lambda s, \lambda e) = \psi(\lambda)d(s, e).$$

From Aczél and Dhombres (1989, p. 346), there must exist $\beta \in \mathbb{R}$ and a function $\phi : \mathbb{R}_+ \to \mathbb{R}$ such that (7) is satisfied.

Proof of Theorem 2

Consider the joint application of (6) and (7) in two cases corresponding to different values of e.

(1). If $e \neq 0$, then from (6) and (7) we have

$$d(s,e) = A(e)s^{\alpha(e)} = e^{\beta}\phi\left(\frac{s}{e}\right), \tag{A12}$$

and putting $s = e$, we have $A(e)e^{\alpha(e)} = ce^{\beta}$, where $c := \phi(1)$. So substituting back into (A12), we have $\phi(z) = cz^{\alpha(e)}$, where $z := s/e$. Since s is arbitrary, z is arbitrary and clearly $\alpha(e)$ must be a constant, independent of e, so that $\phi(z) = cz^{\alpha}$.

(2). If $e = 0$, then Lemma 1 implies that $d(s, e) = cs^{\alpha}$, where $c := A(0)$ and $\alpha := \alpha(0)$. Therefore in general we have

$$d(s,e) = cs^{\alpha}e^{\beta-\alpha}, \tag{A13}$$

with the additional restriction $\beta \geq \alpha$ if e can take the value 0.

Plugging (A13) into (5) gives the general functional form representing the inequality ordering. Given the general transformation Φ included in (5), it is possible to normalize so as to choose the scale and origin of I appropriately.

Proof of Theorem 3

L'Hôpital's rule states that the limit of an undefined ratio between two functions of the same variable is equal to the limit of the ratio of their first derivative. Let us define

$$f(\alpha) = \frac{1}{n}\sum_{i=1}^{n} s_i^{\alpha} - e^{\alpha} = \frac{1}{n}\sum_{i=1}^{n} \exp(\alpha \log s_i) - \exp(\alpha \log e) \quad \text{and} \quad g(\alpha) = \alpha(\alpha - 1),$$

from which we have

$$f'(\alpha) = \frac{1}{n}\sum_{i=1}^{n} s_i^{\alpha}\log s_i - e^{\alpha}\log e \quad \text{and} \quad g'(\alpha) = 2\alpha - 1.$$

For $\alpha = 0$, it follows from a simple application of l'Hôpital's rule that

$$I_0(\mathbf{s};e) = \lim_{\alpha\to 0}\frac{f(\alpha)}{g(\alpha)} = \frac{f'(0)}{g'(0)} = -\left(\frac{1}{n}\sum_{i=1}^{n}\log s_i - \log e\right).$$

L'Hôpital's rule applies only when the ratio $f(\alpha)/g(\alpha)$ is undefined. For the case $\alpha = 1$, this ratio is undefined only if

$$e = \frac{1}{n}\sum_{i=1}^{n} s_i = \mu;$$

otherwise, the ratio is defined and is equal to $\pm\infty$. Indeed, it is straightforward to see that $f(1) = 0$ if $e = \mu$; otherwise, $f(1) \neq 0$. It follows that for $\alpha = 1$, l'Hôpital's rule applies only when $e = \mu$:

$$I_1(\mathbf{s}; e = \mu) = \lim \alpha \to 1\frac{f(\alpha)}{g(\alpha)} = \frac{f'(1)}{g'(1)} = \frac{1}{n}\sum_{i=1}^{n} s_i\log s_i - e\log e.$$

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NOTES

1. On an approach to health inequality, see, for example, Van Doorslaer and Jones (2003). On life satisfaction and inequality of happiness, see Dutta and Foster (2012), Oswald and Wu (2011), Stevenson and Wolfers (2008b), and Yang (2008).
2. For example, the mean will depend on the particular cardinalization that is used, so one cannot use the standard principle of transfers.
3. See, for example, Abul Naga and Yalcin (2010), Allison and Foster (2004), and Gravel *et al*. (2014).
4. We return to this in the subsection entitled ‘Comparison with cardinal inequality measurement’ in Section III.
5. See Abul Naga and Yalcin (2008), Allison and Foster (2004), and Zheng (2011).
6. For examples of this, see note 18.
7. $F(u_i)$ and $1-F(u_i)$ correspond to the concepts of ‘Satisfaction’ and ‘Deprivation’ in Yitzhaki (1979).
8. See the discussion in Van Doorslaer and Jones (2003, p. 65).
9. Otherwise, for any i, $s_i = f(\underline{N}_{k(i)}, n_{k(i)}, \overline{N}_{k(i)}) = f(0, \underline{N}_{k(i)} + n_{k(i)} + \overline{N}_{k(i)}, 0) = f(0, n, 0)$, which is trivial.
10. We have $g_1(\underline{N}_{k(i)}, n) = g_1(n - [\overline{N}_{k(i)} + n_{k(i)}], n) =: g_4(\overline{N}_{k(i)} + n_{k(i)}, n)$ and $g_2(\overline{N}_{k(i)}, n) = g_2(n - [\underline{N}_{k(i)} + n_{k(i)}], n) =: g_3(\underline{N}_{k(i)} + n_{k(i)}, n)$.
11. In other words, the relation $\succcurlyeq$ is complete and transitive.
12. Proofs of theorems are given in the Appendix.
13. Note that d is not a conventional distance function since the symmetry property is not required.
14. The property is equivalent to the Dalton principle of population. See also Yaari (1988) for a similar assumption.
15. This result is superficially similar to results on inequality and other distributional comparisons such as Cowell (1985). However, the present treatment is more general. First, the axiomatization here does not impose *a priori* differentiability or additivity; second, the current treatment deals with any arbitrary representation of status—in particular the case of categorical data. The early literature dealt only with the case where the variable in question was something like income, with a well-defined mean.
16. The same is true if e is a constant equal to any of the values taken by $\mu(\mathbf{s})$, namely 5/8, 11/16 or 3/4.
17. Equivalently, we could see it as a combination of the inequality-decreasing change Case 0 to Case 2 and the inequality-increasing change Case 2 to Case 3.
18. Two examples illustrate the kind of problem that can arise. (1) With three ordered categories and the same proportion of individuals in each category, the median is ambiguous. The status vector is $\mathbf{s} = (1/3, 2/3, 1)$. The conventional definition of the median gives $\text{med}(\mathbf{s}) = m := 2/3$. But, nevertheless, we can see that 2/3 of the population has a status less or equal to m, and 2/3 of the population has a status greater than or equal to m. (2) With two ordered categories (where B is better than A) and a thousand persons, consider the following three distributions: (i) $n_A = 500$, $n_B = 500$; (ii) $n_A = 499$, $n_B = 501$; (iii) $n_A = 999$, $n_B = 1$. The corresponding status vectors are: (i) $\mathbf{s} = (0.5, 1)$, $\text{med}(\mathbf{s}) = 0.5$; (ii) $\mathbf{s} = (0.499, 1)$, $\text{med}(\mathbf{s}) = 1$; (iii) $\mathbf{s} = (0.999, 1)$, $\text{med}(\mathbf{s}) = 0.999$. Distributions (i) and (ii) have very different medians, but distributions (ii) and (iii) have almost the same median!
19. See, for example, Cowell and Kuga (1981a,b).
20. For upward-looking status, one just reverses the order of the categories.
21. The true values are

$$I_\alpha^{(0)} = \frac{1}{\alpha(\alpha-1)}\left[\sum_{i=1}^{K}\pi_i\left[\sum_{j=1}^{i}\pi_j\right]^{\alpha} - 1\right], \quad \alpha \neq 0$$

and

$$I_0^{(0)} = -\sum_{i=1}^{K} \log\left[\sum_{j=1}^{i} \pi_j\right].$$

22. The bibliographic citation of the datafile is: World Values Survey 1981–2008 Official Aggregate v.20090901, 2009, World Values Survey Association. Aggregate File Producer: ASEP/JDS, Madrid. It can be downloaded at www.worldvaluessurvey.org (accessed 18 February 2017).
23. See Easterlin (1995), Easterlin *et al*. (2010) and Clark and Senik (2011).
24. See Hagerty and Veenhoven (2003), Deaton (2008), Stevenson and Wolfers (2008a), and Inglehart *et al*. (2008).
25. Layard (2003, p. 23) notes that 'once a country has over $15 000 per head, its level of happiness appears to be independent of its income per head'. Deaton (2008) argues that different results are obtained from different datasets.
26. For instance, Deaton (2008) uses the average of life satisfaction, and Inglehart *et al*. (2008) use the following index of subjective wellbeing: SWB = life satisfaction$-2.5\times$happiness.
27. See Kuzon *et al*. (1996) and Jamieson (2004).
28. See Knapp (1990) and Norman (2010). The equidistance assumption is often used in happiness studies: see Ng (1997), Ferrer-i-Carbonell and Frijters (2004), and Kristoffersen (2011).
29. If $\log(x)$ is normally distributed and is strongly correlated to happiness, so that $\rho(\log x, \text{happiness}) \approx 1$, then it is natural to plot income x versus $\exp(\text{happiness})$. So in the case of a log-normal income distribution (which is empirically reasonable), it makes sense to represent the answers on an exponential scale rather than on a linear scale.
30. The correlation coefficient $\rho = 0.143$ is not significantly different from 0 ($p = 0.292$).
31. Replacing the mean by the median might seem to offer a solution, but the median is not well defined for ordinal data, in particular when there is a small number of categories (see the second subsection of Section III and note 18).
32. The inequality index is computed as defined in (21), with $\alpha = 0$ and upward-looking status.
33. Standard errors are not reported, because they are smaller than 0.0013.
34. One implication of the mergers principle is that we may ignore any empty categories. For India, the same value of the inequality index would then be obtained if we consider six non-empty categories with, respectively, 71, 1, 308, 821, 437, 316 respondents, rather than ten categories with, respectively, 71, 1, 308, 0, 821, 0, 437, 0, 0, 316 respondents.
35. With all countries, we find a correlation coefficient equal to -0.35, but a p-value very close to 1%, testing the null hypothesis of a correlation coefficient equal to zero ($p = 0.008145$). Excluding Colombia, Mexico and India leads to a correlation coefficient equal to -0.52, which is statistically significantly different from zero ($p = 0.00005$).
36. Out of all the countries, Guatemala, Turkey, Uruguay and South Africa report a few values equal to 5, associated with a 'Very poor' answer. From *Documentation of the Values Surveys*, for the other countries, 'Very poor' is not a possible answer in the questionnaire. We thus remove those four countries from the database.
37. In the health question—with options Very good, Good, Fair, Poor—the items are unlikely to be equidistant; they are not symmetric (only one item can receive a below-average rating), and a bias would be introduced in favour of better outcomes.
38. The inequality index is computed as defined in (21), with $\alpha = 0$ and downward-looking status.
39. Standard errors are not reported, they are are smaller than 0.0013.

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