COL703 - Assignment 1

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Contents

Note: The Modus Ponens proof rule we will use in this assignment is

$$\frac{\phi \qquad \phi \to \psi}{\psi} \ (\to\text{e or MP})$$

Question 1

Part (b)

The sequent is valid. **To Show:** $\neg p, p \vee q \vdash q$.

1.	$\neg p$	premise
2.	$p \vee q$	premise
3.	p	assumption
4.	$\bot$	$\neg$e 3,1
5.	q	$\bot$e 4
6.	q	assumption
7.	q	copy 6
8.	q	$\vee$e 2,3-5,6-7

Part (d)

The sequent is valid. **To Show:** $p \wedge \neg p \vdash \neg(r \to q) \wedge (r \to q)$.

1.	$p \wedge \neg p$	premise
2.	p	$\wedge$e$_1$ 1
3.	$\neg p$	$\wedge$e$_2$ 1
4.	$\bot$	$\neg$e 2,3
5.	$\neg(r \to q) \wedge (r \to q)$	$\bot$e 4

Question 3

We design the following introduction and elimination rules for $\leftrightarrow$:

$$\frac{\phi \to \psi \qquad \psi \to \phi}{\phi \leftrightarrow \psi} \ (\leftrightarrow\mathrm{i})$$

$$\frac{\phi \leftrightarrow \psi \qquad \phi}{\psi} \ (\leftrightarrow\mathrm{e}_1) \qquad \frac{\phi \leftrightarrow \psi \qquad \psi}{\phi} \ (\leftrightarrow\mathrm{e}_2)$$

We also have the following two rules similar to Modus Tollens in Natural Deduction:

$$\frac{\phi \leftrightarrow \psi \qquad \neg\phi}{\neg\psi} \ (\leftrightarrow\mathrm{MT}_1) \qquad \frac{\phi \leftrightarrow \psi \qquad \neg\psi}{\neg\phi} \ (\leftrightarrow\mathrm{MT}_2)$$

We will show that each of these can be derived using the existing natural deduction rules if we replace $\phi \leftrightarrow \psi$ with $(\phi \to \psi) \wedge (\psi \to \phi)$.

$\leftrightarrow$i: We will show that $\phi \to \psi, \psi \to \phi \vdash (\phi \to \psi) \wedge (\psi \to \phi)$.

1.	$\phi \to \psi$	premise
2.	$\psi \to \phi$	premise
3.	$(\phi \to \psi) \wedge (\psi \to \phi)$	$\wedge$i 1,2

$\leftrightarrow\mathbf{e_1}$: We will show that $(\phi \to \psi) \wedge (\psi \to \phi), \phi \vdash \psi$.

1.	$(\phi \to \psi) \wedge (\psi \to \phi)$	premise
2.	ϕ	premise
3.	$\phi \to \psi$	$\wedge\mathrm{e}_1$ 1
4.	ψ	$\to$e 2,3

$\leftrightarrow\mathbf{e_2}$: We will show that $(\phi \to \psi) \wedge (\psi \to \phi), \psi \vdash \phi$.

1.	$(\phi \to \psi) \wedge (\psi \to \phi)$	premise
2.	ψ	premise
3.	$\psi \to \phi$	$\wedge\mathrm{e}_2$ 1
4.	ϕ	$\to$e 2,3

$\leftrightarrow\mathbf{MT_1}$: We will show that $(\phi \to \psi) \wedge (\psi \to \phi), \neg\phi \vdash \neg\psi$.

1.	$(\phi \to \psi) \land (\psi \to \phi)$	premise
2.	$\neg\phi$	premise
3.	ψ	assumption
4.	$\psi \to \phi$	$\land e_2$ 1
5.	ϕ	$\to e$ 3,4
6.	$\bot$	$\neg e$ 5,2
7.	$\neg\psi$	$\neg i$ 3-6

↔MT$_2$: We will show that $(\phi \to \psi) \land (\psi \to \phi), \neg\psi \vdash \neg\phi$.

1.	$(\phi \to \psi) \land (\psi \to \phi)$	premise
2.	$\neg\psi$	premise
3.	ϕ	assumption
4.	$\phi \to \psi$	$\land e_1$ 1
5.	ψ	$\to e$ 3,4
6.	$\bot$	$\neg e$ 5,2
7.	$\neg\phi$	$\neg i$ 3-6

Question 4

Part (c)

To Show: $(p \to r) \wedge (q \to r) \vdash p \wedge q \to r$.

1.	$(p \to r) \wedge (q \to r)$	premise
2.	$p \wedge q$	assumption
3.	p	$\wedge\mathrm{e}_1$ 2
4.	$p \to r$	$\wedge\mathrm{e}_1$ 1
5.	r	$\to$e 3,4
6.	$p \wedge q \to r$	$\to$i 2-5

Part (d)

To Show: $p \to q \wedge r \vdash (p \to q) \wedge (p \to r)$.

1.	$p \to q \wedge r$	premise
2.	p	assumption
3.	$q \wedge r$	$\to$e 2,1
4.	q	$\wedge\mathrm{e}_1$ 3
5.	$p \to q$	$\to$i 2-4
6.	p	assumption
7.	$q \wedge r$	$\to$e 6,1
8.	r	$\wedge\mathrm{e}_2$ 7
9.	$p \to r$	$\to$i 6-8
10.	$(p \to q) \wedge (p \to r)$	$\wedge$i 5,9

Question 7

No, $\{\leftrightarrow, \neg\}$ is not an adequate set of connectives. A counter-example is the formula $p \vee q$ (where p, q are propositional atoms), for which there is no equivalent formula formed only from the connectives $\{\leftrightarrow, \neg\}$. Formally, we prove this using the following theorem:

Theorem 1. *Let ϕ be any formula formed using the connectives $\{\leftrightarrow, \neg\}$ and containing at least 2 propositional atoms. Then, any sub-formula of ϕ evaluates to* `T` *in an even number of lines in the truth table of ϕ.*

Proof. Let ψ be a sub-formula of ϕ. The proof is by structural induction on ψ. Before beginning, note that since ϕ has at least 2 propositional atoms, the number of lines in its truth table is a multiple of 4.

Base Case: $\psi = p$ (A propositional atom). By definition of a truth table, p is `T` in half the lines in the truth table and `F` in the rest. Since the number of lines is a multiple of 4, ψ is `T` in an even number of lines.

Induction Step: We consider the following two cases:

- $\psi = \neg\psi_1$. Now, ψ_1 is also a sub-formula of ϕ, and by the Induction Hypothesis evaluates to `T` in an even number of lines (call it n_1). If the total number of lines in the truth table is n, this means it evaluates to `F` in $n - n_1$ lines. Since $\psi = \neg\psi_1$, by definition of $\neg$, it evaluates to `T` in exactly those lines in which ψ_1 evaluates to `F`, implying that it is `T` in $n - n_1$ lines. Now, $n \bmod 4 = 0$ and $n_1 \bmod 2 = 0$, therefore $(n - n_1) \bmod 2 = 0$, meaning ψ is `T` in an even number of lines.

- $\psi = \psi_1 \leftrightarrow \psi_2$. Now, ψ_1 and ψ_2 are also sub-formulae of ϕ, and by the Induction Hypothesis evaluate to `T` in an even number of lines in the truth table. Let n be the total number of lines in the truth table. Suppose ψ_1 is `T` in n_1 lines, of which ψ_2 is true in m_1 of them. Among the $n - n_1$ lines in which ψ_1 is `F`, suppose ψ_2 is `T` in m_2 of them. By definition of $\leftrightarrow$, ψ is `T` whenever ψ_1 and ψ_2 have the same truth value. So the number of lines in which ψ is `T` is $m_1 + (n - n_1 - m_2)$, call this k. Note that the number of lines in which ψ_2 is `T` is $m_1 + m_2$, which we know to be even (from I.H.). It follows that $m_1 \bmod 2 = m_2 \bmod 2$, which implies that $(m_1 - m_2) \bmod 2 = 0$. This implies that $k \bmod 2 = (n - n_1) \bmod 2$ since $k = m_1 - m_2 + n - n_1$. Finally, note that $n \bmod 4 = 0$ and $n_1 \bmod 2 = 0$, therefore $(n - n_1) \bmod 2 = 0$, meaning k is even, i.e., ψ is `T` in an even number of lines.

□

We can use the theorem above to show that $p \vee q$ cannot have an equivalent formula formed using the connectives $\{\leftrightarrow, \neg\}$. The proof is by contradiction. Suppose there is such an equivalent formula ϕ. Since equivalent formulae must have the same truth table, we know that ϕ evaluates to `T` for exactly the same valuations of p, q under which $p \vee q$ evaluates to `T`. From the truth table of $p \vee q$ (see the figure below), we know it evaluates to `T` in an odd number of lines. But, since every formula is a sub-formula of itself (and also because ϕ is formed from $\{\leftrightarrow, \neg\}$ and has at least two atoms p, q), we can apply the above theorem to ϕ to conclude that it evaluates to `T` in an even number of lines. This is a contradiction, and therefore, no such equivalent formula exists. This proves that $\{\leftrightarrow, \neg\}$ is not an adequate set of connectives for propositional logic.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Question 8

Part (b)

Choose the valuation v such that $v(p) = \mathtt{T}, v(q) = \mathtt{F}, v(r) = \mathtt{T}$. Then, $v(\neg r) = \mathtt{F}$, which implies that $v(\neg r \to (p \vee q)) = \mathtt{T}$. Further, since $v(r) = \mathtt{T}$ and $v(\neg q) = \mathtt{T}$, we have $v(r \wedge \neg q) = \mathtt{T}$. So both the formulae to the left of the $\vdash$ evaluate to $\mathtt{T}$. The formula to the right is $r \to q$. Since $v(r) = \mathtt{T}$ and $v(q) = \mathtt{F}$, we get $v(r \to q) = \mathtt{F}$. So the formula to the right of the $\vdash$ evaluates to $\mathtt{F}$. The existence of such a valuation implies that the sequent $\neg r \to (p \vee q), r \wedge \neg q \vdash r \to q$ is not valid.

Part (d)

Choose the valuation v such that $v(p) = \mathtt{T}, v(q) = \mathtt{F}, v(r) = \mathtt{T}$. Then, since $v(q) = \mathtt{F}$, we get $v(q \to r) = \mathtt{T}$. Since $v(p) = \mathtt{T}$ and $v(q \to r) = \mathtt{T}$, we get $v(p \to (q \to r)) = \mathtt{T}$. So the formula to the left of the $\vdash$ evaluates to $\mathtt{T}$. The formula to the right is $p \to (r \to q)$. Since $v(r) = \mathtt{T}$ and $v(q) = \mathtt{F}$, we get $v(r \to q) = \mathtt{F}$. Since $v(p) = \mathtt{T}$ and $v(r \to q) = \mathtt{F}$, we can conclude $v(p \to (r \to q)) = \mathtt{F}$. So the formula to the right of the $\vdash$ evaluates to $\mathtt{F}$. The existence of such a valuation implies that the sequent $p \to (q \to r) \vdash p \to (r \to q)$ is not valid.