COL703 - Assignment 2

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Contents

Question 1

To prove Strong Completeness, we will make use of the following four theorems. We omit their proofs since they have been covered in class.

Theorem 1. ***(Soundness Theorem for Propositional Logic)***
Let $\alpha \in \phi$. If $\vdash \alpha$ then $\vDash \alpha$.

Theorem 2. ***(Completeness Theorem for Propositional Logic)***
Let $\alpha \in \phi$. If $\vDash \alpha$ then $\vdash \alpha$.

Theorem 3. ***(Deduction Theorem)***
Let $X \subseteq \phi$ and $\alpha, \beta \in \phi$. Then, $X \cup \{\alpha\} \vdash \beta$ if and only if $X \vdash \alpha \supset \beta$.

Theorem 4. ***(Compactness)***
Let $X \subseteq \phi$ and $\alpha \in \phi$. Then, $X \vDash \alpha$ if and only if there exists a finite subset $Y \subseteq_{fin} X$ of X such that $Y \vDash \alpha$.

Theorem 5. ***(Strong Completeness)***
Let $X \subseteq \phi$ and $\alpha \in \phi$. Then, $X \vDash \alpha$ if and only if $X \vdash \alpha$.

Proof. **(if):** Assume $X \vdash \alpha$. Then we can exhibit a derivation $\alpha_1, \ldots, \alpha_n = \alpha$ of α from X. Let Y be the set of formulas of X which are actually used in the derivation, i.e., $Y = X \cap \{\alpha_1, \ldots, \alpha_n\}$. From the definition of Y we have $Y \subseteq_{\text{fin}} X$. Further, from the definition of $\vdash$ and definition of Y, we have $Y \vdash \alpha$ since $\alpha_1, \ldots, \alpha_n = \alpha$ is also a derivation of α from Y.

Suppose $Y = \{\beta_1, \ldots, \beta_m\}$. Then, from the Deduction Theorem, we have

$$\begin{aligned} & \{\beta_1, \ldots, \beta_m\} \quad \vdash \alpha \\ \implies & \{\beta_1, \ldots, \beta_{m-1}\} \vdash (\beta_m \supset \alpha) \\ \implies & \{\beta_1, \ldots, \beta_{m-2}\} \vdash (\beta_{m-1} \supset (\beta_m \supset \alpha)) \\ & \qquad \vdots \\ \implies & \qquad\qquad \vdash (\beta_1 \supset (\beta_2 \supset (\ldots (\beta_m \supset \alpha) \ldots))) \end{aligned}$$

From the Soundness Theorem for Propositional Logic, $\vdash (\beta_1 \supset (\beta_2 \supset (\ldots (\beta_m \supset \alpha) \ldots)))$ implies

$$\vDash (\beta_1 \supset (\beta_2 \supset (\ldots (\beta_m \supset \alpha) \ldots)))$$

We claim that this implies $\{\beta_1, \beta_2, \ldots, \beta_m\} \vDash \alpha$. The claim holds because if it didn't, then we would have a valuation v such that $v(\beta_1) = \cdots = v(\beta_m) = \top$ and $v(\alpha) = \bot$. But then we would have $v(\beta_1 \supset (\beta_2 \supset (\ldots (\beta_m \supset \alpha) \ldots))) = \bot$, which is a contradiction (since $\vDash (\beta_1 \supset (\beta_2 \supset (\ldots (\beta_m \supset \alpha) \ldots)))$). So, $Y = \{\beta_1, \ldots, \beta_m\}$ is a finite subset of X such that $Y \vDash \alpha$. From Compactness, this implies that $X \vDash \alpha$.

(Only If): Assume $X \vDash \alpha$. From Compactness, this implies that there is a finite subset $Y \subseteq_{\text{fin}} X$ such that $Y \vDash \alpha$. Suppose $Y = \{\beta_1, \ldots, \beta_m\}$. Then we have $\{\beta_1, \ldots, \beta_m\} \vDash \alpha$. This implies that $\vDash (\beta_1 \supset (\beta_2 \supset (\ldots (\beta_m \supset \alpha))))$ holds (if it didn't, then we would have a valuation v in which $v(\beta_1) = \cdots = v(\beta_m) = \top$ and $v(\alpha) = \bot$, which is not possible since $\{\beta_1, \ldots, \beta_m\} \vDash \alpha$). From the Completeness Theorem for Propositional Logic, this implies $\vdash (\beta_1 \supset (\beta_2 \supset (\ldots(\beta_m \supset \alpha)\ldots)))$. From the Deduction Theorem, we have

$$\begin{aligned}
& & \vdash (\beta_1 \supset (\beta_2 \supset (\ldots(\beta_m \supset \alpha)\ldots))) \\
&\Longrightarrow \{\beta_1\} & \vdash (\beta_2 \supset (\ldots(\beta_m \supset \alpha)\ldots)) \\
&\Longrightarrow \{\beta_1, \beta_2\} & \vdash (\beta_3 \supset \ldots(\beta_m \supset \alpha)\ldots) \\
& & \vdots \\
&\Longrightarrow \{\beta_1, \ldots, \beta_m\} \vdash \alpha & \qquad (1)
\end{aligned}$$

Since $\{\beta_1, \ldots, \beta_m\} \subseteq X$, from the definition of $\vdash$, equation 1 implies that $X \vdash \alpha$. $\square$

Question 2

Part (a)

Theorem 6. *Every FSS can be extended to a maximal FSS.*

Proof. Let X be an arbitrary FSS. Let $\{\alpha_0, \alpha_1, \dots\}$ be an enumeration of ϕ.

We define an infinite sequence of sets $X_0, X_1, X_2, \dots$ as follows.

- $X_0 = X$
- For $i \geq 0$, $X_{i+1} = \begin{cases} X_i \cup \{\alpha_i\} & \text{if } X_i \cup \{\alpha_i\} \text{ is an FSS} \\ X_i & \text{otherwise} \end{cases}$

Let $Y = \bigcup_{i \in \mathbb{N}} X_i$. By definition, we have that each X_i is an FSS and $X_0 \subseteq X_1 \subseteq \cdots \subseteq Y$. We claim that Y is a maximal FSS that extends X. Firstly, note that since $X = X_0 \subseteq Y$, it extends X.

Secondly, we will show that Y is an FSS. Assume, for the sake of contradiction, that it is not. Then there exists a finite subset $Z = \{\beta_1, \beta_2, \dots, \beta_n\}$ of Y which is not satisfiable. Let the formulas of Z in our enumeration of ϕ be $\{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_n}\}$. We then have that $Z \subseteq_{\text{fin}} X_{j+1}$, where $j = \max(i_1, i_2, \dots, i_n)$. Since Z is not satisfiable, this implies that X_{j+1} is not an FSS, which is a contradiction (as every X_i is an FSS by definition).

Thirdly, we will show that Y is a maximal FSS. Assume, for the sake of contradiction, that it is not. Then there exists $\beta \notin Y$ such that $Y \cup \{\beta\}$ is also an FSS. Let $\beta = \alpha_j$ in our enumeration. Since $\alpha_j \notin Y$, α_j wasn't added at step $j+1$ in our construction. This means that $X_j \cup \{\alpha_j\}$ is not an FSS. Therefore, there exists a finite subset $Z \subseteq_{\text{fin}} X_j \cup \{\alpha_j\}$ which is not satisfiable. Since $X_j \subseteq Y$, we also have $Z \subseteq_{\text{fin}} Y \cup \{\alpha_j\}$. So Z being not satisfiable is a contradiction since we had assumed that $Y \cup \{\alpha_j\} = Y \cup \{\beta\}$ is an FSS and every finite subset of an FSS is satisfiable. Hence, our assumption must have been wrong, and Y is a maximal FSS. □

Part (b)

Theorem 7. *Let X be a maximal FSS. Then, for every formula α, $\alpha \in X$ if and only if $\neg\alpha \notin X$.*

Proof. Let α be any formula. First, we will show that α and $\neg\alpha$ cannot both be in X, i.e., that we cannot have $\{\alpha, \neg\alpha\} \subseteq X$. To see why, note that the set $\{\alpha, \neg\alpha\}$ is not satisfiable since for any valuation v, $v(\alpha) = \top$ if and only if $v(\neg\alpha) = \bot$. Since X is an FSS, there exists no subset of X which is not satisfiable. Therefore, $\{\alpha, \neg\alpha\}$ cannot be a subset of X, i.e., α and $\neg\alpha$ cannot both belong to X.

Second, we will show that at least one of α and $\neg\alpha$ must be in X. This proof is by contradiction. Assume, if possible, that $\alpha \notin X$ and $\neg\alpha \notin X$. Since X is a *maximal* FSS, this implies that there exist finite subsets $B \subseteq_{\text{fin}} X$ and $C \subseteq_{\text{fin}} X$ such that $B \cup \{\alpha\}$ is not satisfiable and $C \cup \{\neg\alpha\}$ is not satisfiable. Let $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ and $C = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$. Let $\widehat{\beta}$ abbreviate the formula $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n$ and $\widehat{\gamma}$ abbreviate the formula $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_m$ Since $B \cup \{\alpha\}$ is not satisfiable, we have that $\neg(\widehat{\beta} \wedge \alpha)$ is valid. Since $C \cup \{\neg\alpha\}$ is not satisfiable, we have that $\neg(\widehat{\gamma} \wedge \neg\alpha)$ is valid. In other words, we have $\vDash \neg(\widehat{\beta} \wedge \alpha)$ and $\vDash \neg(\widehat{\gamma} \wedge \neg\alpha)$. Rewriting using DeMorgan's Laws, we have $\vDash \neg\widehat{\beta} \vee \neg\alpha$ and $\vDash \neg\widehat{\gamma} \vee \alpha$. Now, for any valuation v,

- If $v(\alpha) = \top$, $\vDash \neg\widehat{\beta} \vee \neg\alpha$ gives us $v(\neg\widehat{\beta}) = \top$.
- If $v(\alpha) = \bot$, $\vDash \neg\widehat{\gamma} \vee \alpha$ gives us $v(\neg\widehat{\gamma}) = \top$.

In either case, we will have $v(\neg\widehat{\beta} \vee \neg\widehat{\gamma}) = \top$. Since this holds for any valuation, we can conclude $\vDash \neg\widehat{\beta} \vee \neg\widehat{\gamma}$. Rewriting using DeMorgan's Laws, we get $\vDash \neg(\widehat{\beta} \wedge \widehat{\gamma})$. This implies that the set $B \cup C$ is not satisfiable. Since $B, C \subseteq_{\text{fin}} X$, we have that $B \cup C \subseteq_{\text{fin}} X$ is a finite subset of X which is not satisfiable, contradicting the fact that X is an FSS. Therefore, our assumption must be wrong, and at least one of $\alpha, \neg\alpha$ must be in X.

Since both cannot be in X together and at least one must be there, if $\alpha \in X$ then $\neg\alpha$ cannot be in X and if $\neg\alpha \notin X$ then α must be in X. This completes the proof. $\square$

Part (c)

Theorem 8. *Let X be a maximal FSS. For all formulas α, β, $(\alpha \vee \beta) \in X$ if and only if $(\alpha \in X$ or $\beta \in X)$.*

Proof. Let α, β be two arbitrary formulas. We will show that $\alpha \vee \beta \in X$ if and only if $\alpha \in X$ or $\beta \in X$.

(If): Assume, without loss of generality, that $\alpha \in X$. We will show that $\alpha \vee \beta \in X$. The proof is by contradiction. Assume, if possible, that $\alpha \vee \beta \notin X$. Since X is a maximal FSS, this implies that there exists a finite subset $Y \subseteq_{\text{fin}} X$ such that $Y \cup \{\alpha \vee \beta\}$ is not satisfiable. Let $Y = \{\alpha_1, \ldots, \alpha_n\}$ and let $\widehat{\alpha}$ abbreviate the formula $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n$. $Y \cup \{\alpha \vee \beta\}$ is not satisfiable means that $\neg(\widehat{\alpha} \wedge (\alpha \vee \beta))$ is valid, i.e., we have $\vDash \neg(\widehat{\alpha} \wedge (\alpha \vee \beta))$. We have

$$
\begin{aligned}
& \vDash \neg(\widehat{\alpha} \wedge (\alpha \vee \beta)) \\
\Longrightarrow & \vDash \neg((\widehat{\alpha} \wedge \alpha) \vee (\widehat{\alpha} \wedge \beta)) && \text{(Distributivity)} \\
\Longrightarrow & \vDash \neg(\widehat{\alpha} \wedge \alpha) \wedge \neg(\widehat{\alpha} \wedge \beta) && \text{(DeMorgan's Laws)} \\
\Longrightarrow & \vDash \neg(\widehat{\alpha} \wedge \alpha) && \text{(Defn. of } \wedge \text{)}
\end{aligned} \tag{2}
$$

Equation 2 implies that the set $Y \cup \{\alpha\}$ is not satisfiable. But $Y \subseteq_{\text{fin}} X$ and $\alpha \in X$, so $Y \cup \{\alpha\} \subseteq_{\text{fin}} X$ is a finite subset of X which is not satisfiable, contradicting the fact that X is an FSS. So our assumption must have been wrong, and we have $\alpha \vee \beta \in X$.

(Only If): Assume $(\alpha \vee \beta) \in X$. We will show that $\alpha \in X$ or $\beta \in X$. The proof is by contradiction. Assume, for the sake of contradiction, that $\alpha \notin X$ and $\beta \notin X$. Since X is a maximal FSS, this implies that there exist finite subsets $B, C \subseteq_{\text{fin}} X$ of X such that $B \cup \{\alpha\}$ is not satisfiable and $C \cup \{\beta\}$ is not satisfiable. Let $B = \{\beta_1, \beta_2, \ldots, \beta_n\}$ and $C = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$. Let $\widehat{\beta}$ abbreviate the formula $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n$ and $\widehat{\gamma}$ abbreviate the formula $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_m$ Since $B \cup \{\alpha\}$ is not satisfiable, we have that $\neg(\widehat{\beta} \wedge \alpha)$ is valid. Since $C \cup \{\beta\}$ is not satisfiable, we have that $\neg(\widehat{\gamma} \wedge \beta)$ is valid. In other words, we have $\vDash \neg(\widehat{\beta} \wedge \alpha)$ and $\vDash \neg(\widehat{\gamma} \wedge \beta)$. Rewriting using DeMorgan's Laws, we have $\vDash \neg\widehat{\beta} \vee \neg\alpha$ and $\vDash \neg\widehat{\gamma} \vee \neg\beta$. We claim that

$$\vDash \neg\widehat{\beta} \vee \neg\widehat{\gamma} \vee (\neg\alpha \wedge \neg\beta)$$

To see why, consider an arbitrary valuation v. We have two cases:

- If $v(\neg\widehat{\beta}) = \top$ or $v(\neg\widehat{\gamma}) = \top$, then $v \vDash \neg\widehat{\beta} \vee \neg\widehat{\gamma} \vee (\neg\alpha \wedge \neg\beta)$.
- If neither is true, then $\vDash \neg\widehat{\beta} \vee \neg\alpha$ and $\vDash \neg\widehat{\gamma} \vee \neg\beta$ tell us that we must have $v \vDash \neg\alpha$ and $v \vDash \neg\beta$. So, we will have $v \vDash (\neg\alpha \wedge \neg\beta)$ and thus $v \vDash \neg\widehat{\beta} \vee \neg\widehat{\gamma} \vee (\neg\alpha \wedge \neg\beta)$.

Therefore, we have $\vDash \neg\widehat{\beta} \vee \neg\widehat{\gamma} \vee (\neg\alpha \wedge \neg\beta)$. Rewriting this using DeMorgan's Laws, we get $\vDash \neg\widehat{\beta} \vee \neg\widehat{\gamma} \vee \neg(\alpha \vee \beta)$. Again rewriting using DeMorgan's Laws, we get $\vDash \neg(\widehat{\beta} \wedge \widehat{\gamma} \wedge (\alpha \vee \beta))$. This implies that the set $B \cup C \cup \{\alpha \vee \beta\}$ is not satisfiable. Since $B, C \subseteq_{\text{fin}} X$ and $\alpha \vee \beta \in X$, we have that $B \cup C \cup \{\alpha \vee \beta\} \subseteq_{\text{fin}} X$ is a finite subset of X which is not satisfiable, contradicting the fact that X is an FSS. Therefore, our assumption must be wrong, and we have $\alpha \in X$ or $\beta \in X$. $\square$

Part (d)

Let X be an arbitrary maximal FSS. Define the valuation v_X as setting every atomic proposition in X (and only the atomic propositions in X) to true. Formally, $v_X = \{p \in \mathcal{P} \mid p \in X\}$ or in other words, $v(p) = \top$ *iff* $p \in X$. We will now show that the valuation v_X has the property mentioned in Question 2(d). We have the following theorem:

Theorem 9. *For all formulas α, $v_X \vDash \alpha$ if and only if $\alpha \in X$.*

Proof. The proof is by structural induction on α.

Base Case: $\alpha = p$, a propositional atom. We have $v_X \vDash p$ iff $v_X(p) = \top$. By the definition of v_X, this happens iff $p \in X$.

Induction Step: We have the following two cases:

- $\alpha = \neg\beta$. We have

$$\begin{aligned}
&v_X \vDash \alpha \\
\text{iff } &v_X \nvDash \beta && \text{(by defn. of valuations)} \\
\text{iff } &\beta \notin X && \text{(by induction hypothesis)} \\
\text{iff } &\neg\beta \in X && \text{(by theorem 7)} \\
\text{iff } &\alpha \in X && (\alpha = \neg\beta)
\end{aligned}$$

- $\alpha = \beta \vee \gamma$. We have

$$\begin{aligned}
&v_X \vDash \alpha \\
\text{iff } &v_X \vDash \beta \text{ or } v_X \vDash \gamma && \text{(by defn. of valuations)} \\
\text{iff } &\beta \in X \text{ or } \gamma \in X && \text{(by induction hypothesis)} \\
\text{iff } &\beta \vee \gamma \in X && \text{(by theorem 8)} \\
\text{iff } &\alpha \in X && (\alpha = \beta \vee \gamma)
\end{aligned}$$

In both cases, $v_X \vDash \alpha$ if and only if $\alpha \in X$. This completes the induction step and our proof. □

We had started with an arbitrary maximal FSS X, and we have shown a valuation v_X which has the desired property. Therefore, every maximal FSS X generates a valuation v_X such that for every formula α, $v_X \vDash \alpha$ iff $\alpha \in X$.

Part (e)

We will now show that every FSS X is simultaneously satisfiable. Formally, we have the following theorem.

Theorem 10. *Let X be an FSS. There exists a valuation v such that $v \vDash X$.*

Proof. From theorem 6, X can be extended to a maximal FSS X'. Let $v_{X'}$ be the valuation generated by X', i.e., $v_{X'}(p) = \top$ iff $p \in X'$. Now, take an arbitrary formula $\alpha \in X$. Since X' extends X, we have $\alpha \in X'$. Then, from theorem 9, this implies that $v_{X'} \vDash \alpha$. Since α was arbitrary, we can conclude $v_{X'} \vDash \beta$ for all $\beta \in X$, i.e., $v_{X'} \vDash X$. So we have shown the existence of such a valuation. $\square$

Part (f)

Before proving the main theorem, we prove a lemma which will help us.

Lemma 1. *Let $Z \subseteq \phi$ and $\beta \in \phi$. $Z \vDash \beta$ if and only if $Z \cup \{\neg\beta\}$ is not satisfiable.*

Proof. **(If):** Suppose that $Z \cup \{\neg\beta\}$ is not satisfiable. Then, we will show that $Z \vDash \beta$. The proof is by contradiction. Assume, for the sake of contradiction, that $Z \nvDash \beta$. Then there exists a valuation v such that $v \vDash Z$ but $v \nvDash \beta$. Since $v \nvDash \beta$, $v \vDash \neg\beta$. Since $v \vDash Z$ and $v \vDash \neg\beta$, we have $v \vDash Z \cup \{\neg\beta\}$. This is a contradiction to the fact that $Z \cup \{\neg\beta\}$ is not satisfiable. Therefore, our assumption must be wrong, and we have $Z \vDash \beta$.

(Only If): Suppose that $Z \vDash \beta$. Then, we will show that $Z \cup \{\neg\beta\}$ is not satisfiable. The proof is by contradiction. Assume, for the sake of contradiction, that $Z \cup \{\neg\beta\}$ is satisfiable. Then there exists a valuation v such that $v \vDash Z \cup \{\neg\beta\}$. This implies that $v \vDash Z$ and $v \vDash \neg\beta$. Since $v \vDash \neg\beta$, $v \nvDash \beta$. So we have $v \vDash Z$ and $v \nvDash \beta$, which is a contradiction to the fact that $Z \vDash \beta$. Therefore, our assumption must be wrong, and $Z \cup \{\neg\beta\}$ is not satisfiable. □

Theorem 11. *Let $X \subseteq \phi$, $\alpha \in \phi$. $X \vDash \alpha$ iff there exists $Y \subseteq_{fin} X$ such that $Y \vDash \alpha$.*

Proof. **(If):** Assume there exists $Y \subseteq_{\text{fin}} X$ such that $Y \vDash \alpha$. To show that $X \vDash \alpha$, we have to show that for any valuation v, $v \vDash X$ implies $v \vDash \alpha$. Let v be an arbitrary valuation. If $v \vDash X$, then $v \vDash \beta$ for every $\beta \in X$. Since $Y \subseteq X$, $v \vDash \beta$ for every $\beta \in Y$, so we have $v \vDash Y$. Since $Y \vDash \alpha$, this implies that $v \vDash \alpha$. For any valuation v, we have shown that if $v \vDash X$ then $v \vDash \alpha$ (if there is no valuation v such that $v \vDash X$ then this holds vacuously). So we can conclude $X \vDash \alpha$.

(Only If): Assume $X \vDash \alpha$. We will show that there exists a finite subset $Z \subseteq_{\text{fin}} X$ of X such that $Z \vDash \alpha$. From lemma 1, $X \vDash \alpha$ implies that $X \cup \{\neg\alpha\}$ is not satisfiable. Now, since $X \cup \{\neg\alpha\}$ is not satisfiable, we claim that $X \cup \{\neg\alpha\}$ is not an FSS. This claim holds because if it were an FSS, then from theorem 10, we would have a satisfying valuation. Since it is not an FSS, by the definition of an FSS, there exists a finite subset $Y \subseteq_{\text{fin}} X \cup \{\neg\alpha\}$ such that Y is not satisfiable. Then, $(Y \setminus \{\neg\alpha\}) \cup \{\neg\alpha\}$ is not satisfiable either, where $(Y \setminus \{\neg\alpha\}) \subseteq_{\text{fin}} X$. From lemma 1, this implies that $(Y \setminus \{\neg\alpha\}) \vDash \alpha$. So there exists a finite subset Z of X such that $Z \vDash \alpha$ (namely, $(Y \setminus \{\neg\alpha\})$ is such a set). This completes the proof. □