COL703 - Assignment 4

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Contents

Question 1

We have the following three first-order logic formulas representing points (a), (b) and (c) given in the question:

$$\begin{aligned}
F_1 :&\quad \forall x(B(x) \rightarrow \forall y(\neg S(y,y) \rightarrow S(x,y))) \\
F_2 :&\quad \neg\exists x\exists y(B(x) \wedge S(x,y) \wedge S(y,y)) \\
F_3 :&\quad \neg(\exists x B(x))
\end{aligned}$$

To show $F_1 \wedge F_2 \vDash F_3$, we will show that the formula $F_1 \wedge F_2 \wedge \neg F_3$ is unsatisfiable. First, we Skolemise each of the conjuncts and convert their matrices into CNF to get the following three formulas:

$$\begin{aligned}
G_1 :&\quad \forall x\forall y(\quad \neg B(x) \vee S(y,y) \vee S(x,y) \quad) \\
G_2 :&\quad \forall x\forall y(\quad \neg B(x) \vee \neg S(x,y) \vee \neg S(y,y) \quad) \\
G_3 :&\quad B(c)
\end{aligned}$$

where c is a fresh constant symbol.

The formula $F_1 \wedge F_2 \wedge \neg F_3$ is equisatisfiable with $G_1 \wedge G_2 \wedge G_3$. Thus it suffices to give a ground resolution refutation of $G_1 \wedge G_2 \wedge G_3$. Now we derive $\square$ from ground instances of clauses in the respective matrices of the formulas G_1, G_2, G_3.

1.	$\{\neg B(c), S(c,c)\}$	clause of G_1's matrix with $[c/x][c/y]$
2.	$\{\neg B(c), \neg S(c,c)\}$	clause of G_2's matrix with $[c/x][c/y]$
3.	$\{\neg B(c)\}$	1,2 resolution
4.	$\{B(c)\}$	clause of G_3's matrix
5.	$\square$	3,4 resolution

By the ground resolution theorem, $G_1 \wedge G_2 \wedge G_3$ is unsatisfiable, which implies that $F_1 \wedge \wedge F_2 \wedge \neg F_3$ is unsatisfiable. Therefore, we can conclude that $F_1 \wedge F_2 \vDash F_3$.

Question 2

Let σ be a signature and $\sim$ be a relation on σ-assignments which satisfies the following properties:

1. If $\mathcal{A} \sim \mathcal{B}$, then for every atomic formula F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.
2. If $\mathcal{A} \sim \mathcal{B}$, then for each variable x we have:
   A. For each $a \in U_{\mathcal{A}}$, there exists $b \in U_{\mathcal{B}}$ such that $A_{[x \mapsto a]} \sim B_{[x \mapsto b]}$.
   B. For each $b \in U_{\mathcal{B}}$, there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$.

We have the following theorem.

Theorem 1. *If $\mathcal{A} \sim \mathcal{B}$, then for every formula F we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.*

Proof. Suppose $\mathcal{A} \sim \mathcal{B}$. We will show by induction on the structure of F that for every formula F, $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.

- **Base Case:** F is an atomic formula. From property 1, we have $\mathcal{A} \models F$ iff $\mathcal{B} \models F$.
- **Induction Step:** We consider the following three cases:
  - $F = F_1 \wedge F_2$. We have:

$$
\begin{array}{ll}
\quad\mathcal{A} \models F & \\
\text{iff } \mathcal{A} \models F_1 \wedge F_2 & \\
\text{iff } \mathcal{A} \models F_1 \text{ and } \mathcal{A} \models F_2 & (\text{defn. of } \models) \\
\text{iff } \mathcal{B} \models F_1 \text{ and } \mathcal{B} \models F_2 & (\text{induction hypothesis}) \\
\text{iff } \mathcal{B} \models F_1 \wedge F_2 & (\text{defn. of } \models) \\
\text{iff } \mathcal{B} \models F &
\end{array}
$$

  - $F = \neg F_1$. We have:

$$
\begin{array}{ll}
\quad\mathcal{A} \models F & \\
\text{iff } \mathcal{A} \models \neg F_1 & \\
\text{iff } \mathcal{A} \not\models F_1 & (\text{defn. of } \models) \\
\text{iff } \mathcal{B} \not\models F_1 & (\text{induction hypothesis}) \\
\text{iff } \mathcal{B} \models \neg F_1 & (\text{defn. of } \models) \\
\text{iff } \mathcal{B} \models F &
\end{array}
$$

  - $F = \exists x F_1$. In this case, we show that $\mathcal{A} \models F$ iff $\mathcal{B} \models F$ by showing both the directions separately.
    * *(only if):* Suppose $\mathcal{A} \models F$. By definition of $\models$, there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \models F_1$. From Property 2A, we know that there exists $b \in U_{\mathcal{B}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$. Given this, we can apply the induction hypothesis on F_1 to conclude, from $\mathcal{A}_{[x \mapsto a]} \models F_1$, that $\mathcal{B}_{[x \mapsto b]} \models F_1$. From the definition of $\models$, this implies that $\mathcal{B} \models \exists x F_1$. Thus $\mathcal{B} \models F$.

* *(if):* Suppose $\mathcal{B} \vDash F$. By definition of $\vDash$, there exists $b \in U_{\mathcal{B}}$ such that $\mathcal{B}_{[x \mapsto b]} \vDash F_1$. From Property 2B, we know that there exists $a \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b]}$. Given this, we can apply the induction hypothesis on F_1 to conclude, from $\mathcal{B}_{[x \mapsto b]} \vDash F_1$, that $\mathcal{A}_{[x \mapsto a]} \vDash F_1$. From the definition of $\vDash$, this implies that $\mathcal{A} \vDash \exists x F_1$. Thus $\mathcal{A} \vDash F$.

Since we have shown $\mathcal{A} \vDash F$ iff $\mathcal{B} \vDash F$ in all the three cases and they are exhaustive, the induction step is complete.

From the principle of structural induction, we can conclude that for every formula F, $\mathcal{A} \vDash F$ iff $\mathcal{B} \vDash F$. $\square$

Question 3

We have the following four first-order logic formulas representing points (a), (b), (c) and (d) given in the question:

$$
\begin{aligned}
F_1 : &\quad \forall x(\ (\forall y(\ C(x,y) \rightarrow R(y)\ )) \rightarrow H(x)\ ) \\
F_2 : &\quad \forall x(\ G(x) \rightarrow R(x)\ ) \\
F_3 : &\quad \forall x(\ (\exists y(\ G(y) \wedge C(y,x)\ )) \rightarrow G(x)\ ) \\
F_4 : &\quad \forall x(\ G(x) \rightarrow H(x)\ )
\end{aligned}
$$

To show $F_1 \wedge F_2 \wedge F_3 \models F_4$, we will show that $F_1 \wedge F_2 \wedge F_3 \wedge \neg F_4$ is unsatisfiable. Skolemising each of the conjuncts and converting their matrices to CNF, we get the following formulas.

$$
\begin{aligned}
G_1 : &\quad \forall x(\ (C(x, f(x)) \vee H(x))\ \wedge\ (\neg R(f(x)) \vee H(x))\ ) \\
G_2 : &\quad \forall x(\ \neg G(x) \vee R(x)\ ) \\
G_3 : &\quad \forall x \forall y(\ \neg G(y) \vee \neg C(y,x) \vee G(x)\ ) \\
G_4 : &\quad G(c)\ \wedge\ \neg H(c)
\end{aligned}
$$

where f is a fresh function symbol and c is a fresh constant symbol.

The formula $G_1 \wedge G_2 \wedge G_3 \wedge G_4$ is equisatisfiable with $F_1 \wedge F_2 \wedge F_3 \wedge \neg F_4$. Thus it suffices to show a predicate logic resolution refutation of $G_1 \wedge G_2 \wedge G_3 \wedge G_4$. The proof is as follows. Note that we subscript the variables in line k with k to ensure we always resolve clauses with disjoint sets of variables.

1.	$\{\ G(c)\ \}$	clause of G_4
2.	$\{\ \neg G(y_2), \neg C(y_2, x_2), G(x_2)\ \}$	clause of G_3
3.	$\{\ \neg C(c, x_3), G(x_3)\ \}$	1,2 Res. $D_1 = \{G(c)\}, D_2 = \{\neg G(y_2)\}, \theta = [c/y_2].[x_3/x_2]$
4.	$\{\ \neg G(x_4), R(x_4)\ \}$	clause of G_2
5.	$\{\ \neg C(c, x_5), R(x_5)\ \}$	3,4 Res. $D_1 = \{G(x_3)\}, D_2 = \{\neg G(x_4)\}, \theta = [x_3/x_4].[x_5/x_3]$
6.	$\{\ C(x_6, f(x_6)), H(x_6)\ \}$	clause of G_1
7.	$\{\ R(f(c)), H(c)\ \}$	6,5 Res $D_1 = \{C(x_6, f(x_6))\} D_2 = \{\neg C(c, x_5)\} \theta = [c/x_6].[f(c)/x_5$
8.	$\{\ \neg R(f(x_8)), H(x_8)\ \}$	clause of G_1
9.	$\{\ H(c)\ \}$	7,8 Res. $D_1 = \{R(f(c))\}, D_2 = \{\neg R(f(x_8))\}, \theta = [c/x_8]$
10.	$\{\ \neg H(c)\ \}$	clause of G_1
11.	$\square$	9,10 Res. $D_1 = \{H(c)\}, D_2 = \{\neg H(c)\}, \theta = \text{Identity}$

Therefore, $G_1 \wedge G_2 \wedge G_3 \wedge G_4$ is unsatisfiable, which implies that $F_1 \wedge F_2 \wedge F_3 \wedge \neg F_4$ is unsatisfiable. Then we can conclude that $F_1 \wedge F_2 \wedge F_3 \models F_4$.

Question 4

Let $A(x_1, \ldots, x_n)$ be a formula with no quantifiers and no function symbols. We have the following theorem.

Theorem 2. *The formula $F = \forall x_1 \ldots \forall x_n A(x_1, \ldots, x_n)$ is satisfiable if and only if it is satisfiable in an interpretation with just one element in the universe.*

Proof. **(If):** If there is a model $\mathcal{A}$ such that $|U_{\mathcal{A}}| = 1$ and $\mathcal{A} \vDash F$, then F is satisfiable since there exists a satisfying model.

(Only If): Assume that F is satisfiable. Let $\mathcal{A}$ be a model that satisfies F. By definition of satisfaction, we have

$$\mathcal{A}_{[x_1 \mapsto a_1][x_2 \mapsto a_2]\ldots[x_n \mapsto a_n]} \vDash A(x_1, \ldots, x_n) \text{ for all } a_1, a_2, \ldots, a_n \in U_{\mathcal{A}}$$

Pick any $a \in U_{\mathcal{A}}$. In particular, we have

$$\mathcal{A}_{[x_1 \mapsto a][x_2 \mapsto a]\ldots[x_n \mapsto a]} \vDash A(x_1, \ldots, x_n) \tag{1}$$

We now define a new model $\mathcal{B}$ as follows (note that there are no constants or function symbols in the signature):

- $U_{\mathcal{B}} = \{a\}$.
- For each k-ary predicate symbol P in $A(x_1, \ldots, x_n)$, $(a, \ldots, a) \in P_{\mathcal{B}}$ iff $(a, \ldots, a) \in P_{\mathcal{A}}$.

Claim: $\mathcal{B}_{[x_1 \mapsto a][x_2 \mapsto a]\ldots[x_n \mapsto a]} \vDash A(x_1, \ldots, x_n)$.

Proof: Consider any atomic formula in $A(x_1, \ldots, x_n)$. Since there are no function symbols, it must be of the form $P(x_{i_1}, \ldots, x_{i_k})$ where $i_1, \ldots, i_k \in \{1, \ldots, n\}$. We have

$$\begin{array}{ll}
\quad\mathcal{B}_{[x_1 \mapsto a][x_2 \mapsto a]\ldots[x_n \mapsto a]} \vDash P(x_{i_1}, \ldots, x_{i_k}) & \\
\text{iff } (a, \ldots, a) \in P_{\mathcal{B}} & (\text{defn. of } \vDash) \\
\text{iff } (a, \ldots, a) \in P_{\mathcal{A}} & (\text{defn. of } \mathcal{B}) \\
\text{iff } \mathcal{A}_{[x_1 \mapsto a][x_2 \mapsto a]\ldots[x_n \mapsto a]} \vDash P(x_{i_1}, \ldots, x_{i_k}) & (\text{defn. of } \vDash)
\end{array}$$

So the models $\mathcal{B}_{[x_1 \mapsto a][x_2 \mapsto a]\ldots[x_n \mapsto a]}$ and $\mathcal{A}_{[x_1 \mapsto a][x_2 \mapsto a]\ldots[x_n \mapsto a]}$ assign the same truth value to each atomic formula occurring in $A(x_1, \ldots, x_n)$. Since $A(x_1, \ldots, x_n)$ is formed from atomic formulas using propositional connectives (no quantifiers), this implies that they both assign the same truth value to $A(x_1, \ldots, x_n)$. From equation 1, we know that $\mathcal{A}_{[x_1 \mapsto a][x_2 \mapsto a]\ldots[x_n \mapsto a]} \vDash A(x_1, \ldots, x_n)$. Thus we get that

$$\mathcal{B}_{[x_1 \mapsto a][x_2 \mapsto a]\ldots[x_n \mapsto a]} \vDash A(x_1, \ldots, x_n)$$

This completes the proof of the claim. Since a is the only element in $U_{\mathcal{B}}$, we further get

$$\mathcal{B}_{[x_1 \mapsto a_1][x_2 \mapsto a_2]\ldots[x_n \mapsto a_n]} \vDash A(x_1, \ldots, x_n) \text{ for all } a_1, a_2, \ldots, a_n \in U_{\mathcal{B}}$$

From the definition of satisfaction, we get

$$\mathcal{B} \models \forall x_1 \dots \forall x_n A(x_1, \dots, x_n)$$

Thus $\mathcal{B}$ is a model that satisfies F and has only one element in the universe. We started with a model $\mathcal{A}$ that satisfies F and constructed a model with only one element in the universe which satisfies F. Therefore, F is satisfiable **only if** it is satisfiable in a model with just one element in the universe. $\square$

Question 5

Part (a)

Let σ be a signature with only one binary relation symbol R. Let n be a positive integer. Define the formula F_n as follows:

$$\begin{aligned} F_n \stackrel{\text{def}}{=} (& \forall x(\neg R(x,x)) \\ & \wedge \forall x \forall y \forall z((R(x,y) \wedge R(y,z)) \rightarrow R(x,z)) \\ & \wedge \exists x_1 \exists x_2 \ldots \exists x_n(R(x_1,x_2) \wedge R(x_2,x_3) \wedge \cdots \wedge R(x_{n-1},x_n))) \end{aligned}$$

We have the following theorem.

Theorem 3. *Every model $\mathcal{A}$ which satisfies F_n has at least n elements.*

Proof. The proof is by contradiction. Assume, if possible, that there is a model $\mathcal{A}$ such that $\mathcal{A} \vDash F_n$ and $|U_{\mathcal{A}}| < n$. Since $\mathcal{A}$ satisfies the third conjunct of F_n, we know there exist $a_1, \ldots, a_n \in U_{\mathcal{A}}$ such that

$$\mathcal{A}_{[x_1 \mapsto a_1] \ldots [x_n \mapsto a_n]} \vDash (R(x_1,x_2) \wedge \cdots \wedge R(x_{n-1},x_n))$$

This is true if and only if we have $(a_1,a_2),(a_2,a_3),\ldots,(a_{n-1},a_n) \in R_{\mathcal{A}}$. Now, we claim that for any $i,j \in \{1,\ldots,n\}$ with $i \neq j$, we have $a_i \neq a_j$.

Proof: W.l.o.g, assume $i < j$. We know from above that $(a_i,a_{i+1}),(a_{i+1},a_{i+2}),\ldots,(a_{j-1},a_j) \in R_{\mathcal{A}}$. Since $\mathcal{A}$ satisfies the second conjunct of F_n, we have for all $a,b,c \in U_{\mathcal{A}}$, if $(a,b),(b,c) \in R_{\mathcal{A}}$, then $(a,c) \in R_{\mathcal{A}}$. We can apply this to the chain from a_i to a_j ($j-i-1$ times) to get $(a_i,a_j) \in R_{\mathcal{A}}$. Finally, since $\mathcal{A}$ satisfies the first conjunct of F_n, we have $(a,a) \notin R_{\mathcal{A}}$ for all $a \in U_{\mathcal{A}}$. Therefore, since $(a_i,a_j) \in R_{\mathcal{A}}$, we cannot have $a_i = a_j$. This completes the proof of the claim.

Given the claim, we have shown the existence of n elements (namely $a_1,\ldots,a_n$) in the universe $U_{\mathcal{A}}$, none of which are equal to any other among them. This implies that $|U_{\mathcal{A}}| \geq n$, contradicting our assumption that $|U_{\mathcal{A}}| < n$. Therefore, our assumption must be wrong, and every model satisfying F_n has at least n elements. $\square$

Part (b)

Theorem 4. *Let σ be a signature containing only unary predicate symbols $P_1, \ldots, P_k$. Let F be any satisfiable σ-formula. F has a model where the universe has at most 2^k elements.*

Proof. Let $\mathcal{A}$ be a σ-structure satisfying F (exists since F is satisfiable). For each $a \in U_{\mathcal{A}}$, we define a k-length binary string b_a as:

- For each $1 \leq i \leq k$, $b_a^i = 1$ if and only if $a \in P_{i\mathcal{A}}$.

We define a new σ-structure $\mathcal{B}$ as follows:

- $U_{\mathcal{B}} = \{b_a \mid a \in U_{\mathcal{A}}\}$, the set of k-length binary strings corresponding to all elements of $\mathcal{A}$.
- For each $1 \leq i \leq k$, $P_{i\mathcal{B}} = U_{\mathcal{B}} \bigcap (\{0,1\}^{i-1} \cdot \{1\} \cdot \{0,1\}^{k-i})$. In other words, $P_{i\mathcal{B}}$ contains exactly those strings of $U_{\mathcal{B}}$ which have 1 at the i-th position.
- For each variable x, if $x_{\mathcal{A}} = a$, then $x_{\mathcal{B}} = b_a$.

Claim: The relation $\sim$ between a σ-structure $\mathcal{A}$ and the corresponding σ-structure $\mathcal{B}$ as defined above satisfies the properties mentioned in question 2.

Proof:

- Any atomic formula is of the form $P_i(x)$ for some $i \in \{1, \ldots, k\}$ and variable x. By definition, we have $\mathcal{A} \vDash P_i(x)$ iff $x_{\mathcal{A}} \in P_{i\mathcal{A}}$ iff the string $b_{x_{\mathcal{A}}} = x_{\mathcal{B}}$ has a 1 at the i-th position iff $x_{\mathcal{B}} \in P_{i\mathcal{B}}$ iff $\mathcal{B} \vDash P_i(x)$.
- We have:
  - For any variable x and $a \in U_{\mathcal{A}}$, the σ-structure $\mathcal{B}'$ corresponding to $\mathcal{A}_{[x \mapsto a]}$ will be exactly the same as $\mathcal{B}$, except that it'll have $x_{\mathcal{B}'} = b_a$. Thus we wil have $\mathcal{B}' = B_{[x \mapsto b_a]}$, or in other words, $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b_a]}$.
  - Similarly, for each variable x and $b_a \in U_{\mathcal{B}}$, the σ-structure $\mathcal{A}'$ which $\mathcal{B}_{[x \mapsto b_a]}$ corresponds to is exactly the same as $\mathcal{A}$, except that $x_{\mathcal{A}'} = a$. Thus we wil have $\mathcal{A}' = A_{[x \mapsto a]}$, or in other words, $\mathcal{A}_{[x \mapsto a]} \sim \mathcal{B}_{[x \mapsto b_a]}$.

This completes the proof of the claim. Now, as we have proved in question 2, this implies that for any formula G, $\mathcal{A} \vDash G$ iff $\mathcal{B} \vDash G$. Since we started with the assumption that $\mathcal{A} \vDash F$, we get $\mathcal{B} \vDash F$. Finally, note that $U_{\mathcal{B}} \subseteq \{0,1\}^k$, i.e., all elements of $U_{\mathcal{B}}$ are k-length binary strings. Since there are only 2^k such strings, we must have $|U_{\mathcal{B}}| \leq 2^k$. Starting from an arbitrary model $\mathcal{A}$ of F, we have constructed a model $\mathcal{B}$ of F with no more than 2^k elements in its universe. Therefore, any satisfiable σ-formula F has a model with at most 2^k elements. $\square$