Functional Analysis Applied to PDEs (2024-1)

Google Classroom: kmk6w62
Telegram: `https://t.me/+bOM71NAzRMI2MDQx`

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Homework 5

Instructions: Solve the following exercises, justifying your answers carefully. Upload your written answers in LaTeX using the Google Classroom platform no later than **Monday, October 23**.

Exercises:

1. (5 points) Let E, F be Banach spaces, let $A, K \in \mathcal{L}(E, F)$ such that K is compact, and let $C \geq 0$ such that $\|Ax\| \leq C\|Kx\|$ for all $x \in E$. Lets prove that A is compact.

   Let $(x_n)_{n\in\mathbb{N}_0}$ be a bounded sequece on E. It follows from compactnes of K that there is a subsequence $(x_{n_k})_{k\in\mathbb{N}_0}$ of $(x_n)_{n\in\mathbb{N}_0}$ such that $(K(x_{n_k}))_{k\in\mathbb{N}_0}$ is convergent, and is therefore a Cauchy sequence. To keep notation simple we suppose $(K(x_n))_{n\in\mathbb{N}_0}$ is convergent. Observe that

   $$\|A(x_n - x_m)\| \leq C\,\|K(x_n - x_m)\|\,.$$

   Therfore $(A(x_n))_{\mathbb{N}_0}$ is a Cauchy sequence and therfore converges since F is a Banach space. Therfore A is compact.

2. (5 points) Let E be a Hilbert space and let $K \in \mathcal{L}(E)$ be compact. Denoting $A := I - K$, define $E_n := \mathcal{N}(A^n)$ for $n \in \mathbb{N}$. Lets proove that

   $$E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$$

   and that there exists n_0 such that for every $m \in \mathbb{N}$ it holds that $E_{n_0} = E_{n_0+m}$.

   First suppose $x \in E_k$ for some $k \in \mathbb{N}$. Then $A^{n+1}x = A(A^n x) = A(0) = 0$. Therfore $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$.

   Suppose by contradiction that for every $n \in \mathbb{N}$ there is a $m \in \mathbb{N}$ such that $E_n \subsetneq E_{n+m}$. Therfore we can construct a subsequence

   $$E_{n_1} \subsetneq E_{n_2} \subsetneq E_{n_3} \subsetneq \ldots.$$

Observe that the sets $E_n = (A^n)^{-1}\{0\}$ are closed. We also have that $(I-K)\,E_n \subseteq E_{n-1}$. Infact if $y \in (I-K)\,E_n$ then there is $x \in E_n$ susch that $(I-K)(x) = y$. Therfore $(I-K)^{n-1}(y) = (I-K)^{n-1}(I-K)(x) = (I-K)^n(x) = 0$. Then $y \in E_{n-1}$. We have the hypotesis required in (lemma 7.1) on the notes that allows us to conclude that K is not compact, a contradiction.