Functional Analysis Applied to PDEs (2024-1)

Google Classroom: kmk6w62
Telegram: `https://t.me/+bOM71NAzRMI2MDQx`

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Homework 6

Instructions: Solve the following exercises, justifying your answers carefully. Upload your answers written in LaTeX using the Google Classroom platform no later than **Monday, November 6**.

Exercises:

1. Let Ω be a bounded and smooth domain, $f \in L^2(\Omega)$, and L a uniformly elliptic operator in divergence form. Consider the problem

$$(\wp_f) = \begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In each case, argue your answers.

(a) (1 point) Explicitly exhibit an operator L such that $(\wp_f)$ has a solution for every $f \in L^2(\Omega)$.

Let $L = -\Delta$. Theorem 4.3 of the notes asserts that poisson problem has a unique weak solution $u \in H_0^1(\Omega)$ for every $f \in L^2(\Omega)$.

(b) (1 point) Explicitly exhibit an operator L and $f, g \in L^2(\Omega)$ such that $(\wp_f)$ has a solution but $(\wp_g)$ does not.

Consider $Lu = u'' + u$ and $\Omega = (0, \pi)$. The function $\sin : (0, \pi) \to [0, 1]$ is a nontrivial solution of the problem

$$(\wp_f) = \begin{cases} u'' + u = 0 & \text{in } (0, \pi), \\ u(0) = u(\pi) = 0. \end{cases} \tag{1}$$

The Theorem 7.9 (Fredholm's Alternative for equations) on the notes guarantees that the problem

$$(\wp_f) = \begin{cases} u'' + u = f & \text{in } (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases} \tag{2}$$

has a solution if and only if f is othogonal to every nontrivial solution of the adjoint equation wich in this case is the same given by (1). Theory of second order ordinary differential equations with constant coefficients gives that

$$u(x) = A\sin(x),$$

$A \in \mathbb{R}$. In other words, the problem 2 has a solution if and only if

$$\int_0^\pi f(x)\sin(x) = 0.$$

Therefore, if $f(x) = \cos(x)$, the problem has a solution, but if $f(x) = \sin(x)$, the problem does not have a solution.

(c) (1 point) Does there exist a uniformly elliptic operator in divergence form L such that $(\wp_f)$ has no solution for any $f \neq 0$?

No, for any unifomly eliptic operator in divergence form L, there is a $f \in L^2(\Omega) \setminus \{0\}$ for wich the problem $(\wp_f)$ has a solution.Indeed, Theorem 7.9 (Fredholm's Alternative for equations) asserts that $(\wp_f)$ has a unique solution for every $f \in L^2(\Omega)$, in wich case we are donne, or

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Has nontrivial solutions and u is a solution of $(\wp_f)$ if and only if f is orthogonal to every nontrivial solution of the adjoint problem

$$(\wp_f^*) = \begin{cases} L^*u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Therfore to assert that we can find a $f \in L^2(\Omega)\setminus\{0\}$ for wich $(\wp_f)$ has a solution we have to guarantee that $S^\perp := \left\{u \in H_0^1(\Omega) \mid u \text{ is a weak solution of } (\wp_f^*)\right\}^{\perp_{L^2(\Omega)}}$ is not $\{0\}$. Reasoning by contradiction suppose that $S^\perp = \{0\}$. Then $S = H_0^1(\Omega)$, meaning that every element in $H_0^1(\Omega)$ is a weak solution of $(\wp_f^*)$ wich is a contradiction for every nontrivial ellpitic operator.

2. Let λ_i be the i-th eigenvalue of the Laplacian in Ω with Dirichlet boundary conditions.

(a) (1 point) Show that for all $\lambda \in [0, \lambda_1)$, the problem

$$\begin{cases} -\Delta u = \lambda u + 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

has a weak solution $u_\lambda \in H_0^1(\Omega)$.

Theorem 4.8 a) of the notes gives that the spectrum of an elliptic operator is an increasing divergent sequence $\Sigma = \{\lambda_1, \lambda_2, \cdots\}$. Therfore if $\lambda \in [0, \lambda_1)$ then $\lambda \in \mathbb{R} \setminus \Sigma$. Theorem 4.8 b) of the notes gives that the problem 3 has a unique weak solution for any right hand side of the equation, in this case, for the constant 1.

(b) (2 points) Investigate the asymptotic behavior of u_λ as $\lambda \to \lambda_1$.
Using Theorem 4.10 of the notes we now that

$$\|u\|_{H_0^1(\Omega)} \leq C \|1\|_{L^2(\Omega)} = C |\Omega| .$$

Therfore u_λ is bounded and therfore by the Banach-Alaoglu theorem it contains a weakely convegent subsequence $\{u_{\lambda_k}\}_{k\in\mathbb{N}}$. The Rellich-Kondrashov embbeding theorem asserts that $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ compactly. Therfore $\{u_{\lambda_k}\}_{\mathbb{N}}$ is a convergent sequence in L^2. This proves in general that if $\lambda \to \lambda_i$ then $\{u_\lambda\}$ has a convergent subsequence $\{u_{\lambda_k}\}_{k\in\mathbb{N}}$ in L^2.

(c) (2 points) At least in a particular case, investigate the asymptotic behavior of u_λ as $\lambda \to \lambda_2$.

3. Let E be a Hilbert space and let $T \in \mathcal{L}(E)$ be symmetric.

(a) (2 points) Show that the following properties are equivalent.

i. $(Tu, u) \geq 0$ for every $u \in E$,

ii. $\sigma(T) \subset [0, \infty)$.

Suppose $(Tu, u) \geq 0$ for every $u \in E$. Since T is symmetric Theorem 6.46 of the notes asserts that $\sigma(T) \subseteq [m, M]$ where

$$m := \inf_{u\in S_1 E} (Tu, u) \quad \text{y} \quad M := \sup_{u\in S_1 E} (Tu, u) .$$

Therfore $m \geq 0$ and $\sigma(T) \subseteq [0, \infty)$. Reciprocally if we suppose that $\sigma(T) \subseteq [0, \infty)$ we have that $0 \leq m$. Let $u \in E$, and define $v = \frac{u}{\|u\|_E}$ then

$$(Tu, u) = \|u\|_E^2 (Tv, v) \geq m \geq 0.$$

(b) (2 points) Show that if $0 \in \rho(T)$ then

$$\sigma(T^{-1}) = \left\{ \frac{1}{\lambda} \ : \ \lambda \in \sigma(T) \right\} .$$

Suppose $0 \in \rho(T)$. Then T is bijective. Therfore $s \in \sigma(T^{-1})$ if and only if $s\mathbb{1} - T^{-1}$ is not bijective. If and only $sT - \mathbb{1}$ is not bijective, if and only if $\frac{1}{s}\mathbb{1} - T$ is not bijective, if and only if $\frac{1}{s} \in \sigma(T)$.

If you have any questions about the homework, do not hesitate to ask in the **Telegram chat**, where the assistants, the professor, and/or some of your classmates can respond.