# Functional Analysis Applied to PDEs (2024-1)

Google Classroom: kmk6w62
Telegram: `https://t.me/+b0M71NAzRMI2MDQx`

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## Homework 2

Instructions: Solve the following exercises, providing careful justifications for your answers. Submit your written responses in LaTeX using the Google Classroom platform no later than **Monday, September 4th**.

Exercises:

1. (2 points) In the proof of Theorem B.3 (global approximation by smooth functions) in Step 2, it is stated that the ball $U_\varepsilon(x_\varepsilon)$ is contained in $U \cap U_r(x^0)$. Prove that this is the case.

   Let $U$ an open bounded subset of $\mathbb{R}^N$ such that $\partial U$ is $C^1$. Let $x_0 \in \partial U$, $r > 0$ and $\gamma \in C^1(\mathbb{R}^{(}N-1)$ such that

$$U \cap U_r(x_o) = \left\{ x = (x_1, \dots, x_N) \in \mathbb{R}^N | x_N > \gamma(x_1, \dots, x_N - 1) \right\}.$$

   Let $V := U \cap U_{\frac{r}{2}}(x_0)$. Since $\gamma \in C^1(\mathbb{R}^{(}N-1)$ we have that

$$\|\gamma(x) - \gamma(y)\| \leq \sup_{z \in [x,y]} \|D\gamma_z\| \|x - y\| \leq C \|x - y\|$$

   for all $x, y \in U$, where $C := \sup_{z \in \overline{U}} \|D\gamma_z\|$wich is finite since $D\gamma$ is contnous and $\overline{U}$ is compact. Let $\lambda \in \mathbb{R}$ be such that $C + 1 < \lambda$. Then for every $\epsilon > 0$ and $h \in U_\epsilon(0)$

$$\begin{aligned} (1 + C)\epsilon &< \lambda\epsilon, \quad \text{therefore} \\ \|h\| (1 + C) &< \lambda\epsilon, \quad \text{then} \\ C \|h\| &< \lambda\epsilon - \|h\| < \lambda\epsilon + h_N. \end{aligned} \tag{1}$$

   Now we take $\epsilon > 0$ small enough that $(1 + \lambda)\epsilon < \frac{r}{2}$. For every $x \in V$, the translation $x^\epsilon$ is defined by $x^\epsilon = x + \lambda\epsilon e_N$.

   Let $y \in U_\epsilon(x^\epsilon)$. Then

$$
\begin{aligned}
\|y - x_0\| &\leq \|y - x^\epsilon\| + \|x^\epsilon - x_0\| \\
&\leq \|y - x^\epsilon\| + \|x - x_0\| + \lambda\epsilon \\
&< \epsilon + \frac{r}{2} + \lambda\epsilon \\
&= (1 + \lambda)\epsilon + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r.
\end{aligned}
$$

Therefore $U_\epsilon(x^\epsilon) \subseteq U_r(x_0)$. Since $y \in U_\epsilon(x^\epsilon)$ we can write $y = x^\epsilon + h$ for $\|h\| < \epsilon$. Since $x \in V$ $\gamma(x_1, \ldots, x_{N-1} < X_N$. Using 1 we have that

$$
\begin{aligned}
\gamma(y_1, \ldots, y_N - 1) &= \gamma(x_1^\epsilon + h_1, \ldots, x_{N-1}^\epsilon + h_{N-1}) \\
&= \gamma(x_1 + h_1, \ldots, x_{N-1} + h_{N-1}) \\
&< \gamma(x_1, \ldots, x_{N-1}) + C\,\|h\| \\
&< X_N + C\,\|h\| < \lambda + h_N + x_N = y_N.
\end{aligned}
$$

Therefore $y \in U \cap U_r(x_0)$ and $U_\epsilon(x^\epsilon) \subseteq U \cap U_r(x_0)$.

2. (2 points) In the proof of Theorem B.3, it is used that

$$
\lim_{\varepsilon \to 0} \|D^\alpha v^\varepsilon - D^\alpha u_\varepsilon\|_{L^p(V)} = 0.
$$

Prove it!

We suppose the same as in the past exercise. That means $U$ is an open bounded subset of $\mathbb{R}^N$ such that $\partial U$ is $C^1$. Let $x_0 \in \partial U$, $r > 0$ and $\gamma \in C^1(\mathbb{R}^(N-1)$ such that

$$
U \cap U_r(x_o) = \left\{x = (x_1, \ldots, x_N) \in \mathbb{R}^N | x_N > \gamma(x_1, \ldots, x_N - 1)\right\}.
$$

Let $V := U \cap U_{\frac{r}{2}}(x_0)$.

We define $u_\epsilon(x) = u(x^\epsilon)$, and observe that this definition make sense for every $x \in V + U_\epsilon(0)$, since for every $x \in V + U_\epsilon(0)$, $x^\epsilon \in U_\epsilon(\tilde{x}^\epsilon) \subseteq U \cap U_r(x_0)$ for somme $\tilde{x} \in V$ so $u_\epsilon(x)$ is well defined. To prove that, observe that if $x \in V_\epsilon$, then

$$
x = \tilde{x} + h,
$$

for somme $\|h\| < \epsilon$ . Hence

$$
x^\epsilon = x + \lambda\epsilon e_n + \tilde{x} + h = \tilde{x}^\epsilon + h \in U_\epsilon(\tilde{x}^\epsilon).
$$

Now define $\nu^\epsilon(x) = \eta_{\frac{\varepsilon}{2}} * u_\epsilon(x)$. Since $u_\epsilon$ is defined in $V + U_\epsilon(0)$, $\nu^\epsilon$ must be defined in $V + U_{\frac{\varepsilon}{2}}(0)$. Observe that $\nu \in C^\infty(V + U_{\frac{\varepsilon}{2}}(0)) \subseteq C^\infty(\overline{V})$. Let's prove that

$$\lim_{\varepsilon \to 0} \|D^\alpha v^\varepsilon - D^\alpha u_\varepsilon\|_{L^p(V)} = 0.$$

Observe that by construction $u^\varepsilon \in L^p(V + U_\varepsilon(0))$ and $u_\varepsilon \to u$ when $\varepsilon \to 0$ in $L^p(V)$ because $u_\varepsilon$ is a translation of $u$.

Take $\varepsilon_0$ small enough such that $\nu^\varepsilon$ is well defined for $\varepsilon < \varepsilon_0$. Let's prove that for $\varepsilon$ small enough and $g \in C(V + U_{\varepsilon_0}(0))$

$$\left\|\eta_{\frac{\varepsilon}{2}} * (u_\varepsilon - g)\right\|_{L^p(V)} \leq \|u_\varepsilon - g\|_{L^p(V+U_\varepsilon(0))} \tag{2}$$

for $\varepsilon$ small enough. Applying Hölder's inequality and using that $\eta_\epsilon$ integrats to 1 we obtain

$$\begin{aligned}
\left|\eta_{\frac{\varepsilon}{2}} * (u_\varepsilon - g)\right| &\leq \int_{B_{\frac{\varepsilon}{2}}(0)} \eta_{\frac{\varepsilon}{2}}(x-y) \left|u_\varepsilon(y) - g(y)\right| dy \\
&= \int_{B_{\frac{\varepsilon}{2}}(0)} \eta_{\frac{\varepsilon}{2}}^{1-\frac{1}{p}}(x-y) \eta_{\frac{\varepsilon}{2}}^{\frac{1}{p}}(x-y) \left|u_\varepsilon(y) - g(y)\right| dy \\
&= \left(\int_{B_{\frac{\varepsilon}{2}}(0)} \eta_{\frac{\varepsilon}{2}}(x-y) dy\right)^{\frac{p-1}{p}} + \left(\int_{B_{\frac{\varepsilon}{2}}(0)} \eta_{\frac{\varepsilon}{2}}(x-y) \left|u_\varepsilon(y) - g(y)\right|^p dy\right)^{\frac{1}{p}} \\
&= \left(\int_{B_{\frac{\varepsilon}{2}}(0)} \eta_{\frac{\varepsilon}{2}}(x-y) \left|u_\varepsilon(y) - g(y)\right|^p dy\right)^{\frac{1}{p}},
\end{aligned}$$

and therefore,

$$\begin{aligned}
\int_V \left|\eta_{\frac{\varepsilon}{2}} * (u_\varepsilon - g)\right|^p dx &\leq \int_V \int_{B_{\frac{\varepsilon}{2}}(0)} \eta_{\frac{\varepsilon}{2}}(x-y) \left|u_\varepsilon(y) - g(y)\right|^p dy dx \\
&= \int_V \int_{V+U_\varepsilon(0)} \eta_{\frac{\varepsilon}{2}}(x-y) \left|u_\varepsilon(y) - g(y)\right|^p dy dx \\
&\leq \int_{V+U_\varepsilon(0)} \left|u_\varepsilon(y) - g(y)\right|^p \int_V \eta_{\frac{\varepsilon}{2}}(x-y) dx dy \\
&= \|u_\epsilon - g\|^p_{L^p(V+U_\varepsilon(0))}
\end{aligned}$$

which proves equation 2.

Let $\delta > 0$ and $g \in C(V + U_{\varepsilon_0}(0))$ such that $\|g - u\|_{L^p(V)} < \delta$. Then we have that

$$
\begin{aligned}
\|\nu^\varepsilon - u_\varepsilon\|_{L^p(V)} &\leq \left\|\eta_{\frac{\varepsilon}{2}} * u_\varepsilon - \eta_{\frac{\varepsilon}{2}} * g\right\|_{L^p(V)} + \left\|\eta_{\frac{\varepsilon}{2}} * g - g\right\|_{L^p(V)} + \|g - u\|_{L^p(V)} + \|u - u_\varepsilon\|_{L^p(V)} \\
&\leq \|u_\varepsilon - g\|_{L^p((V)+U_{\varepsilon(0)})} + \left\|\eta_{\frac{\varepsilon}{2}} * g - g\right\|_{L^p(V)} + \|u - u_\varepsilon\|_{L^p(V)} + \delta.
\end{aligned}
$$

The second and 3rd term tend to 0 when $\epsilon \to 0$.

$$
\begin{aligned}
\|u_\varepsilon - g\|_{L^p((V)+U_{\varepsilon(0)})} &= \|u_\varepsilon - g\|_{L^p((V)} + \|u_\varepsilon - g\|_{L^p((V)+U_{\varepsilon(0)})\setminus(V)} \\
&\leq \|g - u\|_{L^p(V)} + \|u - u_\varepsilon\|_{L^p(V)} \\
&\quad + \|u_\varepsilon - g\|_{L^p((V)+U_{\varepsilon(0)}\setminus V)} \\
&\leq \|u - u_\varepsilon\|_{L^p(V)} \\
&\quad + \|u_\varepsilon - g\|_{L^p((V)+U_{\varepsilon(0)}\setminus V)} + \delta
\end{aligned}
$$

It remains to prove that $\|u_\varepsilon - g\|_{L^p((V)+U_{\varepsilon(0)}\setminus V)} \to 0$ when $\varepsilon \to 0$. It's done as follows

$$
\begin{aligned}
\|u_\epsilon - g\|_{L^p((V)+U_{\varepsilon(0)}\setminus V)} &= \int_{V+U_{\varepsilon(0)}\setminus V)} |u_\varepsilon(x) - g(x)|^p \, dx \\
&\leq \int_{V+U_{\varepsilon(0)}\setminus V} |u_\varepsilon(x)|^p \, dx + \int_{V+U_{\varepsilon(0)}\setminus V} |g(x)|^p \, dx \\
&= \int_{V_\epsilon} |u(x)|^p \, dx + \int_{V+U_{\varepsilon(0)}\setminus V} |g(x)|^p \, dx
\end{aligned}
$$

where $V_\epsilon = \left\{x \in U | x^\varepsilon \in V + U_{\varepsilon(0)} \setminus V\right\}$. Since $g \in C(V + U_{\varepsilon_0}(0))$ and $\left|V + U_{\varepsilon(0)} \setminus V\right| \to 0$ when $\varepsilon \to 0$, the second integral tends to 0 when $\varepsilon \to 0$. The same is true for the first integral, observing that $u \in L^p(U)$ and $|V_\varepsilon| \to 0$ when $\varepsilon \to 0$. Therefore

$$
\lim_{\epsilon \to 0} \|\nu^\varepsilon - u_\varepsilon\|_{L^p(V)} \leq 2\delta.
$$

since delta was arbitrary, the limit is 0. To prove that

$$
\lim_{\varepsilon \to 0} \|D^\alpha v^\varepsilon - D^\alpha u_\varepsilon\|_{L^p(V)} = 0.
$$

we just have to observe that $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon$, where $(D^\alpha u)_\varepsilon(x) := D^\alpha u(x^\varepsilon)$ and that $D^\alpha \nu^\varepsilon = \eta_{\frac{\varepsilon}{2}} * D^\alpha u_\varepsilon = \eta_{\frac{\varepsilon}{2}} * (D^\alpha u)_\varepsilon$. Then we have to prove that

$$
\lim_{\varepsilon \to 0} \left\|\eta_{\frac{\varepsilon}{2}} * (D^\alpha u)_\varepsilon - (D^\alpha u)_\varepsilon\right\|_{L^p(V)} = 0
$$

but then, the prove is exactly the same using $D^\alpha u$ instead of $u$.

3. (2 points) In the proof of Theorem B.4 (existence of the extension operator), Step 6, a partition of unity is used. Prove the existence of these functions.

We establish the existence of those functions in the next

**Theorem 1.** *Let $\Omega \subset\subset \mathbb{R}^{\mathbb{N}}$ and $\{W_i\}_{i=1}^n$ an open cover of $\overline{\Omega}$, each $W_i$ relatively compact. Then there is a partition of the unity subordinated to $\{W_i\}_{i=1}^n$, that is a family $\{\zeta_i\}_{i=1}^n$ such that $\zeta_i \in C_c^\infty(W_i)$, $0 \leq \zeta_i \leq 1$, and $\sum_{i=1}^n \zeta_i(x) = 1$ for all $x \in \Omega$.*

*Proof.* Let $\{W_i\}_{i=1}^n$ be an open cover of the compact set $\overline{\Omega}$, of realtively compact sets. We claim that there is an open cover $\{W_i^\varepsilon\}_{i=1}^n$ of $\overline{\Omega}$ such that $\overline{W_i^\varepsilon} \subset W_i$ for all $i = 1, \ldots, n$. Define $W_i^\varepsilon := \{x \in W_i | d(x, \partial W_i) > \varepsilon\}$. We want to prove that

$$\overline{U} \subseteq \bigcup_{i=1}^n W_i^\varepsilon,$$

for $\varepsilon$ small enough. Suppose instead that

$$\overline{U} \setminus \bigcup_{i=1}^n W_i^\varepsilon = \emptyset,$$

for all $\epsilon > 0$. Let $V_k = \overline{U} \setminus \bigcup_{i=1}^n W_i^{\frac{1}{k}}$. Observe that $(V_k)_{k \in \mathbb{N}_0}$ is an decreasing sequence of closed sets in a compact space therefore it's limit (intersection) is nonempty. But

$$\bigcap_{k=1}^\infty V_k = \bigcap_{k=1}^\infty \left( \overline{U} \setminus \bigcup_{i=1}^n W_i^{\frac{1}{k}} \right) = \overline{U} \setminus \bigcup_{i=1}^n W_i = \emptyset$$

a contradiction.

Applying lemma 2.8 seen in class we have that there are $\phi_i \in C_c^\infty(W_i)$, such that $\phi_i \equiv 1$ in $\overline{W_i^\varepsilon}$, for $i = 1, \ldots, n$. Let $r > 0$ such that $\bigcup_{i=0}^n \overline{W_i^\varepsilon} \subseteq U_r(0)$. Let $W_0 := B_r(0) \setminus \bigcup_{i=0}^n \overline{W_i^\varepsilon}$. Observe that $\overline{\Omega} \cap W_0 = \emptyset$, therefore there is a open set $U$ such that $W_0 \subseteq U$ and $U \cap \overline{\Omega} = \emptyset$. Again, using lemma 2.8, there is $\phi_0 \in C_c^\infty(U)$ such that $\phi_0 \equiv 1$ in $W_0$. Define $\zeta(x) = \sum_{j=0}^n \phi_i(x)$. We observe that $\zeta(x) \neq 0$ for $x \in \bigcup_{i=1}^n W_i$ since if $x \in W_i$ then $x \in \overline{W_j^\varepsilon}$ for somme $j = 1, \ldots, n$ in wich case $\phi_j(x) = 1$ or $x \in W_0 = B_r(0) \setminus \bigcup_{i=0}^n \overline{W_i^\varepsilon}$, wich implies that $\phi_0(x) = 1$. We define

$$\zeta_i(x) := \frac{\phi_i(x)}{\zeta(x)} = \frac{\phi_i(x)}{\sum_{j=0}^n \phi_j(x)}.$$

for $i = 1, \ldots, n$. Then $\zeta_i(x) \in C_c^\infty(W_i)$ is well defined and

$$\sum_{i=1}^n \zeta_i(x) = \frac{\sum_{i=1}^n \phi_i(x)}{\sum_{j=0}^n \phi_j(x)} = \frac{\sum_{i=1}^n \phi_i(x)}{\sum_{j=1}^n \phi_j(x)} = 1$$

for every $x \in \overline{\Omega}$, since $supp\phi_0 \subseteq W_0 \subseteq \overline{\Omega}^c$ and so $\phi_0(x) = 0$.

□

4. (2 points) In the proof of Theorem B.4, Step 7, it is stated that the extension $Eu$ is independent of the approximating sequence. Prove that this is the case.

The following result shows that in general, such extensions can always be made, and the proof demonstrates that the extension is independent of the chosen approximating sequence.

**Theorem 2.** *Let $E$ be a normed vector space, $F$ a Banach space, and $D \subseteq E$ a dense subset of $E$. Suppose there exists $T : D \to F$, linear and bounded, i.e., there exists $C > 0$ such that $\|Tx\|_E \leq C \|x\|_F$ for all $x \in D$. Then, $T$ can be extended (abusing notation) to a linear and bounded operator $T : E \to F$.*

*Note: The result is valid for a uniformly continuous mapping from a dense subset of a metric space to a complete metric space.*

*Proof.* Let $u \in E$, and $\{u_n\} \subseteq E$ such that $u_n \to u$. By hypothesis, we have

$$\|Tu_n - Tu_m\|_F \leq C \|u_n - u_m\|_E \,, \tag{3}$$

for all $n, m \in \mathbb{N}_0$. From here on, we will omit the subscripts $E$ and $F$ in $\|\cdot\|$. Since $u_n$ converges, it is a Cauchy sequence, and by (3), $Tu_n$ is a Cauchy sequence in $F$, which is a Banach space. Therefore, $Tu_n$ converges. We define $Tu := \lim_{n\to\infty} Tu_n$. Let's show that this definition does not depend on the approximating sequence.

Suppose there is another approximating sequence $v_n \to u$. Then,

$$\begin{aligned} \|Tu - Tv_n\| &\leq \|Tu - Tu_n\| + \|Tu_n - Tv_n\| \\ &\leq C\left(\|u - u_n\| + \|u_n - v_n\|\right) \\ &\leq C\left(\|u - u_n\| + \|u_n - u\| + \|u - v_n\|\right) \to 0. \end{aligned}$$

This means that $\lim_{n\to\infty} Tv_n = Tu$. Thus, the definition of $Tu$ is independent of the approximating sequence. The continuity of $T$ is obtained by taking limits in $\|Tu_n\| \leq C \|u_n\|$. □

5. (2 points) Prove the case $p = \infty$ of Theorem B.4.

6. (3 points) In the proof of Theorem B.7 (characterization of the homogeneous Sobolev space through the trace), in Step 2, it is stated that

   *Using partitions of unity and smoothing the boundary $\partial U$, we can assume that*

$$\begin{cases} u \in W^{1,p}\left(\mathbb{R}^N_+\right), u \textit{ has compact support in } \overline{\mathbb{R}}^N_+, \\ \qquad Tu = 0 \textit{ on } \partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}. \end{cases} \tag{4}$$

   Explain why considering (4) is sufficient. In other words, argue why, if $u$ satisfies (4) and it is proven that $u \in W^{1,p}_0(\mathbb{R}^N)$, then the statement of Theorem B.6 can be demonstrated.

If you have any questions about the homework, feel free to ask in the **Telegram chat**.