Functional Analysis Applied to PDEs (2024-1)

Google Classroom: kmk6w62
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Assignment 4

Instructions: Solve the following exercises, justifying your answers carefully. Upload your written answers in LaTeX using the Google Classroom platform no later than **Monday, October 9**.

Exercises:

1. (4 points) Demonstrate the existence of a (weak) solution to the problem

$$-u''(x) + xu'(x) = 1 \quad \text{for } x \in (0, \tfrac{1}{2}), \qquad u(0) = u(\tfrac{1}{2}) = 0. \tag{1}$$

Consider the previous Cauchy Problem. We will look for week solutions in $H_0^1(1, \frac{1}{2})$. The blilinear form asociated to the problem is

$$B[u, v] = \int_0^{\frac{1}{2}} u'v' + xu'vdx.$$

To proove it's coercive we write

$$B[u, u] = \int_0^{\frac{1}{2}} (u')^2 + xu'udx. \tag{2}$$

Asume for now that $u \in C_0^\infty(1, \frac{1}{2})$. Integration by parts gives

$$\int_0^{\frac{1}{2}} xu'udx = \frac{1}{2}\int_0^{\frac{1}{2}} x\frac{d}{dx}\left(u(x)^2\right)dx = -\frac{1}{2}\int_0^{\frac{1}{2}} u(x)^2dx.$$

Replacing this in 2 we have

$$B[u, u] = \int_0^{\frac{1}{2}} (u')^2 - \frac{1}{2}u^2dx.$$

Poincare in equality asserts that there is a constant $C > 0$, such that

$$\|u\|_{L^2(0,\frac{1}{2})} \leq C\|u'\|_{L^2(0,\frac{1}{2})}.$$

Therfore

$$B\left[u,u\right]\geq\int_0^{\frac{1}{2}}\left(u'\right)^2-\frac{C^2}{2}\left(u'\right)^2=\left(1-\frac{C^2}{2}\right)\left\|u'\right\|^2_{L^2(0,\frac{1}{2})}.$$

Observe that if $\left(1-\frac{C^2}{2}\right)>0$ or equivalently if $\sqrt{2}>C>0$ in Pointcaré's then B is coercive. Let's proove that this is ther case.

We are asumoing that $u\in C_c^\infty(0,\frac{1}{2})$, so by the fundamental theorem of calculus

$$u(x)=\int_0^x u'(t)dt.$$

Using this fact and applying Jensen's inequality we have that

$$\begin{aligned}
\left\|u\right\|^2_{L^2(1,\frac{1}{2})} &= \int_0^{\frac{1}{2}}\left|u(x)\right|^2dx\\
&= \int_0^{\frac{1}{2}}\left|\int_0^x u'(t)dt\right|^2dx\\
&\leq \int_0^{\frac{1}{2}}\left(\int_0^{\frac{1}{2}}\left|u'(t)\right|dt\right)^2dx\\
&\leq \int_0^{\frac{1}{2}}\frac{1}{2}\int_0^{\frac{1}{2}}\left|u'(t)\right|^2dtdx=\frac{1}{4}\left\|u'\right\|^2_{L^2(0,\frac{1}{2})}.
\end{aligned}$$

Therefore we obtain the pointcaré constant $C=\frac{1}{2}$ wich satifies $0<C<\sqrt{2}$ and therfore, B is coercive over all $u\in C_c^\infty(0,\frac{1}{2})$. We can extend this property by coninuity of$B$ to all $H_0^1(0,\frac{1}{2})$. This means that

$$B\left[u,u\right]\geq\tilde{C}\left\|u\right\|^2_{L^2(0,\frac{1}{2})}$$

for all $u\in H_0^1(0,\frac{1}{2})$ and somme constant $\tilde{C}>0$. Observe that $\tilde{C}=\frac{7}{8}$. Therefore by Lax Millgram's theorem there is a unique $u\in H_0^1(1,\frac{1}{2})$ such that

$$B\left[u,v\right]=\left(1,v\right)_{L^2(0,\frac{1}{2})},$$

or more explicitly

$$\int_0^{\frac{1}{2}}u'vdx=\int_0^{\frac{1}{2}}vdx$$

for all $v\in H_0^1(1,\frac{1}{2})$. That proves that (1) has a unique weak solution.

2. (3 points) Let $\Omega \subset \mathbb{R}^N$ be a smooth and bounded domain, and let $f \in L^2(\Omega)$. Consider the problem

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = \partial_\nu u = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where Δ^2 is the bilaplacian operator (apply the Laplacian twice) and ∂_ν is the normal derivative on $\partial\Omega$. Lets first provide a weak formulation of the problem (3).

Suppose $u \in C^\infty(\Omega)$ and $\phi \in C_0^\infty(\Omega)$. Since $supp(\phi) \Subset \Omega$, ϕ and $\partial_\nu \phi$ are vanishes on $\partial\Omega$. Integration bay parts gives

$$\int_\Omega \Delta^2 u \phi dx = \int_\Omega \Delta(\Delta u)\phi dx = \int_\Omega \Delta u \Delta \phi dx.$$

This suggests to define a week solution of 3 as a function $u \in H_0^2(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v dx = (f, v)_{L^2(\Omega)},$$

for all $v \in H_0^2(\Omega)$.

Let's proove the existence of a weak solution and show it can be characterized as the minimum of a functional in a suitable Hilbert space. Let's asume (for now) the Poincaré inequality in $H_0^2(\Omega)$, i.e., there exists $C > 0$ such that

$$\|u\|_{H^2(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)} \qquad \text{for all } u \in H_0^2(\Omega).$$

Observe that

$$\langle u, v\rangle = \int_\Omega \Delta u \Delta v dx,$$

defines a scalar product on $H_0^2(\Omega)$. It's clearly linear on each argument because of the linearity of the Laplacian and the integral. Simetry is clear to. To prove $\langle v, v\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0$, we can use the Pointacré inequality mentioned above wich forces u to be 0 if $\Delta u = 0$.

Let's proove that this scalar product induces an equivalent norm on $H_0^2(\Omega)$ to the usual one. First note that for every $u \in H_0^2(\Omega)$, using integration by parts and the fact that the order in wich partial derivatives are taken doesn't matter we have that

$$\begin{aligned}
\int_\Omega (\Delta u)^2\, dx &= \int_\Omega \left(\sum_i^N \partial_i^2 u \right)^2 dx \\
&= \int_\Omega \sum_{i,j}^N (\partial_i^2 u)(\partial_j^2 u) dx \\
&= \sum_{i,j}^N \int_\Omega (\partial_i^2 u)(\partial_j^2 u) dx \\
&= \sum_{i,j}^N \int_\Omega (\partial_j^2 \partial_i^2 u) u dx \\
&= \sum_{i,j}^N \int_\Omega (\partial_i \partial_j \partial_i \partial_j u) u dx \\
&= \sum_{i,j}^N \int_\Omega (\partial_i \partial_j u)^2 u dx \\
&= \int_\Omega \sum_{i,j}^N |\partial_i \partial_j u|^2\, dx.
\end{aligned}$$

Therfore,

$$\|u\|_{H^2(\Omega)}^2 = \int_\Omega |u|^2 + \sum_{i=1}^N |\partial_i u|^2 + \sum_{i,j}^N |\partial_i \partial_j u|^2\, dx \geq \int_\Omega (\Delta u)^2\, dx$$

Using this and Pointacré inequality we obtain the equivalence.

$$\|\Delta u\|_{L^2(\Omega)} \leq \|u\|_{H^2(\Omega)} \leq C \|\Delta u\|_{L^2(\Omega)}.$$

Observe that as in the case of $H_0^1(\Omega)$ we can now define the norm $\|u\|_{H_0^2(\Omega)} := \|\Delta u\|_{L^2(\Omega)}$ in $H_0^2(\Omega)$.

To conclude we observe that $\phi : H_0^2(\Omega) \to \mathbb{R}$ given by $\phi(v) = (f, v)_{L^2(\Omega)}$ is continuous (since the scalar product on a Hilbert space is continuous). The Riesz representation theorem applied to the Hilbert space $H_0^2(\Omega)$ endowed with the new scalar product given above gives the existence and uniquenes of an element $u \in H_0^2(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v dx = (f, v)_{L^2(\Omega)},$$

for all $v \in H_0^2(\Omega)$. This is there exists a unique weak solution $u \in H_0^1(\Omega)$ of the problem (3). For the more the Riesz representation theorem asserts that u can be characterised as

$$u = \min_{u \in H_0^2(\Omega)} \left\{ Ju := \frac{1}{2} \|\Delta u\|_{L^2(\Omega)}^2 - (f,u)_{L^2(\Omega)} \right\}.$$

3. (3 points) Let $F : L^2(0,1) \to \mathbb{R}$ be defined by

$$Fu := \int_0^{\frac{1}{2}} u(t)\, dt.$$

To proove F is bounded we use Holder's inequality in $L^2(0, \frac{1}{2})$ for u and the constant 1, to obtain that

$$|Fu| \leq \int_0^{\frac{1}{2}} |u(x)|\, dx \leq \frac{1}{\sqrt{2}} \left(\int_0^{\frac{1}{2}} |u(x)|^2 \right)^{\frac{1}{2}} dx.$$

Therefore F is continous and $|F| \leq \frac{1}{\sqrt{2}}$. Now take the function $\delta = \sqrt{2}\mathbb{1}_{[0,\frac{1}{2}]} \in L^2(0,1)$. Observe that

$$\|\delta\|_{L^2(0,1)} = \left(\int_0^{\frac{1}{2}} \sqrt{2}^2 dx \right)^{\frac{1}{2}} = 1. \tag{4}$$

For the more

$$|F(\delta)| = \int_0^{\frac{1}{2}} \sqrt{2} dx = \frac{1}{\sqrt{2}}.$$

Therfore $|F| \geq \frac{1}{\sqrt{2}}$ and $|F| = \frac{1}{\sqrt{2}}$.

Another way to compute the norm of F is via the Riesz representation theorem. We prooved F is continous on the Hilber space $L^2(0,1)$, in other words $F \in L^2(0,1)^*$. The Riesz representation theorem asserts that there is a unique function $\gamma \in L^2(0,1)$ such that

$$Fu = \int_0^1 \gamma u dx$$

for all $u \in L^2(0,1)$. Observe that the functión $\mathbb{1}_{[0,\frac{1}{2}]}$ has this exact property. So $\gamma = \mathbb{1}_{[0,\frac{1}{2}]}$. Riesz representation's lemma asserts that $\|\gamma\|_{L^2(0,1)} = |F|$, this is

$$\frac{1}{\sqrt{2}} = \left(\int_0^{\frac{1}{2}} dx \right)^{\frac{1}{2}} = \|\gamma\|_{L^2(0,1)} = |F|.$$