Functional Analysis Applied to PDEs (2024-1)

Google Classroom: kmk6w62
Telegram: `https://t.me/+bOM71NAzRMI2MDQx`

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Homework 3

Instructions: Solve the following exercises, justifying your answers carefully. Upload your written answers in LaTeX using the Google Classroom platform no later than **Monday, September 18**.

Exercises:

1. (2 points) Let $p, q, r \in (1, \infty)$ be such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Let $\Omega \subseteq \mathbb{R}^N$ be open and $f \in L^q(\Omega)$. Let's prove that $(Lu)(x) := f(x)u(x)$ defines a bounded linear operator L from $L^p(\Omega)$ to $L^r(\Omega)$.

Let $u \in L^p(\Omega)$, observe that this is equivalen to $u^r \in L^{\frac{p}{r}}(\Omega)$. By the same reason $f^r \in L^{\frac{q}{r}}(\Omega)$. By hypothesis made about p, q, r we have that $\frac{r}{p} + \frac{r}{q} = 1$, therefore Holder's inequality implies that $f^r u^r \in L^1(\Omega)$ and

$$\begin{aligned}
\|Lu\|_r^r &= \int_\Omega |f(x)|^r \, |u(x)|^r \, dx \\
&\leq \left(\int_\Omega |f(x)|^{r\frac{q}{r}} \, dx \right)^{\frac{r}{q}} \left(\int_\Omega |u(x)|^{r\frac{p}{r}} \, dx \right)^{\frac{r}{p}} = \|f\|_q^r \, \|u\|_p^r .
\end{aligned}$$

Therfore $\|Lu\|_r \leq \|f\|_q \, \|u\|_p$. This prooves that L is well defined and that is a bounded linear operator.

2. Let E be a Banach space and let[1]

$$X := \{A \in \mathcal{L}(E) \, : \, A \text{ is bijective and } A^{-1} \in \mathcal{L}(E)\}.$$

[1]Later in the course, we will see that if $A \in \mathcal{L}(E)$ is bijective and E is a Banach space, then it is always the case that $A^{-1} \in \mathcal{L}(E)$. Completeness hypothesis is important! In general, a bounded operator $A : E_1 \to E_2$ between two normed spaces E_1 and E_2 can have an unbounded inverse if E_1 and/or E_2 are not Banach spaces, see for example: `https://math.stackexchange.com/questions/1580369/do-there-exist-bounded-operators-with-unbounded-inverses`

(a) (2 Points) Let $A \in X$ and $B \in \mathcal{L}(E)$ be such that $\|I - BA^{-1}\| < 1$. Let's show that $B \in X$ and give a formula for B^{-1}.

The Neumann series theorem asserts that in a Banach space E if $A \in \mathcal{L}(E)$ and $\|A\| < 1$ then $I - A$ is invertible and it's inverse is given by $(I - A)^{-1} = \lim_{n\to\infty} A_n$, where $A_n = \sum_{k=0}^{n} A^k$. The operator $I - BA^{-1}$ is in $\mathcal{L}(E)$ since by hypothesis $B, A^{-1} \in \mathcal{L}(E)$, $I \in \mathcal{L}(E)$ and $\mathcal{L}(E)$ is a vectorial space closed under composition. In addition $\|I - BA^{-1}\| < 1$. Applying Neumann's series theorem $I - I + BA^{-1} = BA^{-1}$ is invertible and $AB^{-1} = (BA^{-1})^{-1} = \lim_{n\to\infty} \sum_{k=0}^{n} (I - BA^{-1})^k$. Then B is invertible and it's inverse is given by

$$B^{-1} = \lim_{n\to\infty} \sum_{k=0}^{n} A^{-1}(I - BA^{-1})^k. \tag{1}$$

(b) (2 Points) Let's show that X is open in $\mathcal{L}(E)$ and that the mapping

$$\begin{aligned} X &\to X, \\ A &\mapsto A^{-1} \end{aligned}$$

from an operator to its inverse is continuous.

Let $A \in \mathcal{L}(E)$ and $0 < \varepsilon < \frac{1}{\|A^{-1}\|}$. Suppose that $B \in \mathcal{L}(E)$ is such that $\|A - B\| < \varepsilon$. Take $x \in E$, since A is invertible, there is a $y \in E$ such that $x = Ay$. Therfore

$$\begin{aligned} \left\|I - BA^{-1}(x)\right\|_E &= \left\|(I - BA^{-1})(Ay)\right\|_E \\ &= \|Ay - By\|_E \\ &\leq \|A - B\| \, \|y\|_E \\ &\leq \varepsilon \left\|A^{-1}x\right\|_E \\ &\leq \varepsilon \left\|A^{-1}\right\| \|x\|_E \leq \|x\|_E \, . \end{aligned} \tag{2}$$

Therfore $\|I - BA^{-1}\| < 1$, and by a) B is invertible. By 1

$$\begin{aligned} \left\|B^{-1}\right\| &= \left\| \lim_{n\to\infty} \sum_{k=0}^{n} A^{-1}(I - BA^{-1})^k \right\| \\ &\leq \left\|A^{-1}\right\| \lim_{n\to\infty} \sum_{k=0}^{n} \left\|I - BA^{-1}\right\|^k < \infty \end{aligned}$$

since the last is a geometric series with reason < 1. Therefore B^{-1} is bounded. This proves that X is open in $\mathcal{L}(E)$.

To prove that $A \mapsto A^{-1}$ is continuous let's prove that it's continuous at $A \in \mathcal{L}(E)$. Let $\delta > 0$. In equation 2 taking ε such that $0 < \varepsilon < \frac{1}{2\|A^{-1}\|}$ then if

$\|B-A\| < \varepsilon$ we have that $\|I - BA^{-1}\| < \frac{1}{2}$. Therefore there is $n_0 \in \mathbf{N}$ such that $\sum_{k=n_0}^{\infty} \|I - BA^{-1}\|^k < \frac{\delta}{2}$ (is a convergent geometric series). Since multiplying by a bounded linear operator, taking it's opposite, adding the identity and raising to the power k is a continuous function we have for $\|B-A\|$ small enough, let's say $\|B-A\| < \tilde{\varepsilon}$, that $\left\|I - \sum_{k=0}^{n_0}(I-BA^{-1})^k\right\| < \frac{\delta}{2}$, because $I - \sum_{k=0}^{n_0}(I - AA^{-1}) = 0$. Now if we take $\|B-A\|$ small enough, let's say $\|B-A\| < \min\{\varepsilon, \tilde{\varepsilon}\}$, then

$$\begin{aligned}
\left\|A^{-1} - B^{-1}\right\| &= \left\|A^{-1} - A^{-1}\sum_{k=0}^{\infty}(I-BA^{-1})^k\right\| \\
&= \left\|A^{-1}\left(I - \sum_{k=0}^{\infty}(I-BA^{-1})^k\right)\right\| \\
&\leq \left\|A^{-1}\right\| \left\|\left(I - \sum_{k=0}^{\infty}(I-BA^{-1})^k\right)\right\| \\
&= \left\|A^{-1}\right\| \left\|\left(I - \sum_{k=0}^{n_0}(I-BA^{-1})^k\right) + \sum_{k=n_0}^{\infty}(I-BA^{-1})^k\right\| \\
&\leq \left\|A^{-1}\right\| \left[\left\|I - \sum_{k=0}^{n_0}(I-BA^{-1})^k\right\| + \sum_{k=n_0}^{\infty}\left\|I - BA^{-1}\right\|^k\right] \\
&\leq \left\|A^{-1}\right\| \left(\frac{\delta}{2} + \frac{\delta}{2}\right) = \left\|A^{-1}\right\| \delta.
\end{aligned}$$

Since delta was arbitrary we conclude that $B^{-1} \to A^{-1}$ when $\|B-A\| \to 0$, therefore $A \mapsto A^{-1}$ is continous.

3. (2 points) Let E, F, G be normed spaces, and let $A\colon E \times F \to G$ be a bilinear map: For all $x_0 \in E$ and $y_0 \in F$, the maps $A[x_0, \cdot\,]\colon F \to G$ and $A[\,\cdot\,, y_0]\colon E \to G$ are linear. Let's prove that A is continuous if and only if there exists $C \geq 0$ such that

$$\|A[x,y]\|_G \leq C\|x\|_E\|y\|_F \quad \text{for all } x \in E,\ y \in F. \tag{3}$$

First suppose that 3 holds. Let $(x_k, y_k) \subseteq E \times F$ such that $(x_k, y_k) \to (x, y) \in E \times F$. Then

$$\begin{aligned}
\|A[x,y] - A[x_k,y_k]\| &\leq \|A[x,y] - A[x,y_k]\| + \|A[x,y_k] - A[x_k,y_k]\| \\
&= \|A[x - x_k, y]\| + \|A[x, y - y_k]\| \\
&\leq C\left[\|x - x_k\|\,\|y\| + \|x_n\|\,\|y - y_n\|\right] \to 0
\end{aligned}$$

when $k \to \infty$ because $\|x_n\|$ is bounded and $x_k \to x$ and $y_k \to y$.

Now suppose that A is continuous. Let's prove first that it exists $r > 0$ such that $A(B_E(r,0) \times B_F(r,0)) \subseteq B_G(1,0)$, where $B_E(r,0)$ is the closed ball in E with radius r centered in 0. By contradiction, suppose that no such r exists. Then for every $r > 0$ there are $(x,y) \in B_E(r,0) \times B_F(r,0)$ and $\|A[x,y]\| > 1$. Letting r go to 0 we have that $(x,y) \to (0,0)$ but $A[x,y] \geq 1$, a contradiction with the continuity of A at $(0,0)$. Now let $(x,y) \in E \times F$, then

$$\|A[x,y]\| = \frac{\|x\| \|y\|}{r^2} A \left\| A \left[\frac{rx}{\|x\|}, \frac{ry}{\|y\|} \right] \right\| \leq \frac{\|x\| \|y\|}{r^2}.$$

4. (2 points) Let E be a Hilbert space and let F and G be closed linear subspaces of E such that $F \perp G$. Let's prove that $F \cap G = \{0\}$ and that $F \oplus G$ is closed.

   Let $x \in F \cap G$. Then $\|x\|^2 = \langle x, x \rangle = 0$ and therefore $x = 0$. This proves $F \cap G = \{0\}$. To prove that $F \oplus G$ is closed, take $w_k := x_k + y_k \subseteq F \oplus G$ such that $x_k + y_k = w_k \to w \in E$ and $x_k \in F$ and $y_k \in G$. Using the ortogonality $F \perp G$ we have that

$$\begin{aligned}
\|w_m - w_n\|^2 &= \langle w_m - w_n, w_m - w_n \rangle \\
&= \langle x_m + y_m - x_n - y_n, x_m + y_m - x_n - y_n \rangle \\
&= \langle x_m - x_n, x_m - x_n \rangle + \langle y_m - y_n, y_m - y_n \rangle \\
&= \|x_m - x_n\|^2 + \|y_m - y_n\|^2
\end{aligned}$$

   for all $m, n \in \mathbb{N}$. Since (w_k) is a Cauchy sequence in E, so are (x_n) and (y_n) in F and G respectively. Since F and G are closed in a complete space, then $x_n \to x \in F$ and $y_n \to y \in G$. Therefore $w_n = x_n + y_n \to x + y$. We new that $w_n \to w$ so $w = x + y \in F \oplus G$. We conclude that $F \oplus G$ is a closed linear subspace of E.