Stability of stationary solutions

Chapter 10 Thierry Cazenave, Alain Haraux, Yvan Martel - An intoduction to semilinear evolution equations.([2])

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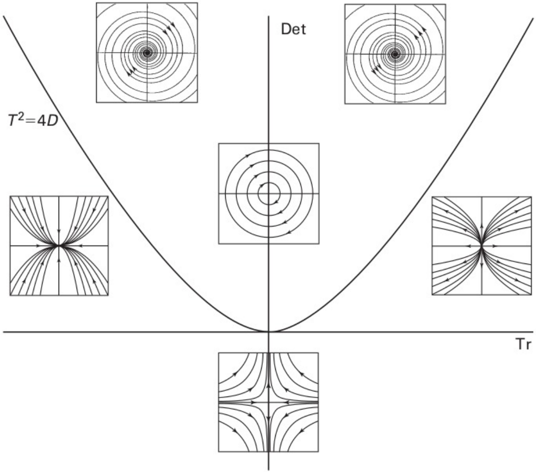
Introduction

The main goal of this chapter is to extend Liapunov linearization method of ODEs to establish the (local or global) asymptotic stability of equilibria for somme PDEs.



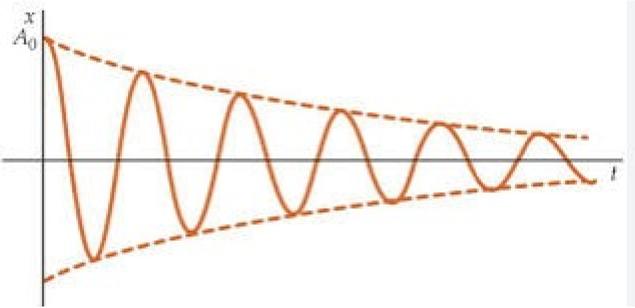


Figure: Ocilador harmónico, ejemplo de estabilidad.

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Let (X, d) be a complete metric space.

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Definition(9.1.1)

A dinamical system on X is a family $\{S_t\}_{t\geq 0}$ of mappings on X such that

1. $S_t \in C(X, X)$, $\forall t \geq 0$.
2. $S_0 = I$,
3. $S_{t+s} = S_t \circ S_s$, $\forall s, t \geq 0$,
4. the function $t \mapsto S_t x$ is in $C\,[0, \infty)$ for all $x \in X$.

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Definition(10.1.1)

A trajectory $v(t) = S_t a$ of the dinamical system $\{S_t\}_{t\geq 0}$. Is called positively stable in the sense of Liapunov if

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{such that}$$
$$x \in X \quad \text{and} \quad d(x, a) \leq \delta \Rightarrow \forall t \geq 0, \;\; d(S_t x, v(t)) \leq \varepsilon.$$

Definition(10.1.1)

A trajectory $v(t) = S_t a$ of the dinamical system $\{S_t\}_{t \geq 0}$. Is called asymptotically stable in the sense of Liapunov if it is stable in the sense of Liapunov and

$$\exists \delta_1 \geq 0, \quad \text{such that} \quad x \in X, d(x,a) \leq \delta_1 \Rightarrow \lim_{t \to \infty} d(S_t x, v(t)) = 0.$$

Theorem(10.1.3) (Liapunov)

Let X be a finite dimentional normes space and $f \in C^1(X, X)$ a vector field on X. Let $a \in X$ be such that $f(a) = 0$ and assume that all eigenvalues s_j, $1 \leq j \leq k$ of $Df(a)$ have negative real parts. Then a is asymptotically Liapunov stable equilibrium solution of the equation

$$u'(t) = f(u(t)), \quad t \geq 0, \tag{1}$$

in the following sense: for each $\delta < \nu = \min_{1 \leq j \leq k} \{-Re(s_j)\}$, there exists $\rho = \rho(\delta) > 0$ and $M(\delta) \geq 1$ such that. if $\|x - a\| \leq \rho(\delta)$, the solution u of (1) such that $u(0) = x$ is global with

$$\forall t \geq 0, \|u(t) - a\| \leq M(\delta) \|x - a\| e^{-\delta t}.$$

Proposition (10.1.4)

Let X be a finite dimentional normed space and $f \in C^1(X, X)$ a vector field on X. Let $a \in X$ be such that $f(a) = 0$ and assume that all eigenvalues s_j, $1 \leq j \leq k$ of $Df(a)$ have positive real parts. Then a is a completely unstable equilibrium solution of the equation

$$u'(t) = f(u(t)), \quad t \geq 0, \tag{2}$$

in the following sense: there is a neighbourhood ω of a such that, for each $b \in X$, $b \neq a$, the unique solution of (1) with initial condition b leaves ω for $t \geq T$ large enough.

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General dissipative operators

Definition 2.1.1

A linear unbounded operator in a normed space X is a pair (D, A), where D is a linear subspace of X and $A : D \to X$ is a linear mapping. Such that

$$\sup_{\{x \in D \mid \|x\|=1\}} \|Au\| = \infty. \tag{3}$$

Definition2.2.1

An operator A in X is dissipative if

$$\|u - \lambda Au\| \geq \|u\|, \tag{4}$$

for all $u \in D(A)$ and all $\lambda > 0$.

Definition 2.2.2

An operator A in X is m-dissipative if

1. A is dissipative;
2. for all $\lambda > 0$ and all $f \in X$, there exists $u \in D(A)$ such that $u - \lambda A u = f$.

Definition 2.2.4

Observe that for a m-dissipative operator the equation $u - \lambda A u = f$ has a unique solution. Therefore the following notation has sense. $J_\lambda f := (I - \lambda A)^{-1} f$. Observe that $\|J_\lambda\| \leq 1$ since $\|u\| \leq \|f\|$. We also define $A_\lambda := A J_\lambda$ and we have that $\|A_\lambda\|_{\mathcal{L}(X)} \leq \frac{2}{\lambda}$.

Proposition 2.2.12

If A is m-dissipative and if $\overline{D(A)} = X$, then $A_\lambda u \to Au$ for all $u \in D(A)$.

Proposition 2.4.2

Suppose X is now a Hilbert space then an operator A with dense domain is disipative if and only if $\langle Au, u \rangle \leq 0$ for all $u \in D(A)$. In adition A is m-dissipative if and only if A^* (Hilbert adjoint) is dissipative and the graph of A is closed.

The heat equation

Let Ω be any open subset of $\mathbb{R}^n$, and let $Y = L^2(\Omega)$. Consider the operator B in Y given by

$$\begin{cases} D(B) = \{u \in H_0^1(\Omega); \Delta u \in L^2(\Omega)\} \\ Bu = \Delta u, \quad \forall u \in D(B). \end{cases} \tag{5}$$

Proposition 2.6.1

B is m-dissipative with dense domain. More presisely, B is self adjoint and $B \leq 0$.

...to be continued

The semigroup generated by an *m*-dissipaative operator

Let X be a Banach space and A an m-dissipative operator in X. For $\lambda > 0$ consider the operators J_λ and A_λ defined as before. $T_\lambda(t) = e^{tA_\lambda}$, for $t \geq 0$.

Theorem 3.1.1

For all $x \in X$, the sequence $u_\lambda(t) = T_\lambda(t)x$ converges uniformly on bounded intervales of $[0, T]$ to a function $u \in C([0,\infty), X)$, as $\lambda \to 0^+$. We set $T(t)x =: u(x)$, for all $x \in X$ and $t \geq 0$. Then

1. $T(t) \in \mathcal{L}(X)$ and $\|T(t)\| \leq 1$, $\forall t \geq 0$;
2. $T(0) = I$;
3. $T(s+t) = T(t)T(s), \forall s, t \geq 0$.

In addition, for all $x \in D(A)$, $u(t) = T(t)x$ is the unique solution of the problem

$$\begin{cases} u \in C([0,\infty), D(A)) \cap C^1((0,\infty), X); \\ u'(t) = Au(t), \quad \forall t \geq 0; \\ u(0) = x. \end{cases}$$

Such family $\{T(t)\}_{t\geq 0}$ is nown as a contraction semigroup in X and A is it's generator.

Theorem 3.4.4 (The Hille-Yosida-Phillips Theorem)

A linear operator A is the generator of a contraction semigroup in X if and only if A is m-dissipative with densse domain.

Proposition 3.4.5

Let A be an m-dissipative operator with densse domain. Assume that A is the generator of a contraction semigroup $(S(t))_{t\geq 0}$. Then $(S(t))_{t\geq 0}$ is the semigroup corresponding to A given by (Theorem 3.1.1.)

Proposition 3.5.2

Let $\phi \in L^2(\Omega)$and let $u(t) = S(t)\phi$ for $t \geq 0$. Then u is the unique solution of the problem

$$\begin{cases} u \in C\left([0,\infty), L^2(\Omega)\right) \cap u \in C^1\left((0,\infty), L^2(\Omega)\right), \Delta u \in C\left((0,\infty, L^2(\Omega))\right); \\ u'(t) = \Delta u(t), \forall t \geq 0 \quad ; \\ u(0) = \phi. \end{cases}$$

Proposition 3.5.5

Let $\lambda = \inf\left\{ \frac{\|\nabla u\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} \mid u \in H_0^1(\Omega), u \neq 0 \right\}$. Then

$$\|S(t)\|_{\mathcal{L}(L^2)} \leq e^{-\lambda t}.$$

Semilinear heat eqution

Given $\varphi \in X = C_0(\Omega)$ we look for $T > 0$ and u solving the problem

$$\begin{cases} u \in C([0,T], X) \cap C((0,T], H_0^1(\Omega)) \cap C^1((0,T], L^2(\Omega)); \\ \Delta u \in C((0,T], L^2(\Omega)); \\ u_t - \Delta u = F(u), \quad \forall t \in (0,T]; \\ u(0) = \varphi. \end{cases} \tag{6}$$

Theorem 5.2.1

For all $\varphi \in X$, there exists a unique function u, defined on a maximal interval $[0, T(\varphi))$, wich is a solution of (6).

idea of the prove

Solving (6) is equivalent to solving

$$u(t) = \mathcal{T}(t)\varphi + \int_0^t \mathcal{T}(t-s)F(u(s))ds, \quad \forall t \in [0, T],$$

wich is solved by a Banach fixed point argument.

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Let X be a real Banach space , let $\mathcal{T}(t) = e^{ct}S(t)$ with $c \in \mathbb{R}$, and let $(S(t))_{t\geq 0}$ be a contraction semigroup on X, and $F : X \to X$ locally Lipschitz continous on bounded subsets. For any $x \in X$, we consider the unique maximal solution $u \in C\left([0, \tau(x)), X\right)$ of the equation

$$u(t) = \mathcal{T}(t)x + \int_0^t \mathcal{T}(t-s)F(u(s))ds, \quad \forall t \in [0, \tau(x)). \tag{7}$$

By a stationary solution of (7), we mean a constant vector $a \in X$ such that

$$a = \mathcal{T}(t)a + \int_0^t \mathcal{T}(t-s)F(a)ds, \quad \forall t \geq 0.$$

Theorem(10.2.2)

Assume that, for some constants $\delta > 0, M \geq 1$, we have

$$\forall t \geq 0, \ \ \|\mathcal{T}(t)\| \leq Me^{-\delta t}.$$

Let $a \in X$ a stationary solution of (7) such that

$$\exists R_0 > 0, \exists \nu > 0 : \quad \|F(u) - F(a)\| \leq \nu \|u - a\| \text{ for } \|u - a\| \leq R_0,$$

with $\nu < \frac{\delta}{M}$.

...

Then for all $x \in X$ such that

$$\|x - a\| \leq R_1 = \frac{R_0}{M},$$

The solution u of (7) is global and satifies

$$\forall t \geq 0, \quad \|u(t) - a\| \leq M \|x - a\| e^{-\gamma t}, \tag{8}$$

with $\gamma = \delta - \nu M > 0$.

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Stability for the heat equation

We first consider the semilinear heat equation

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where Ω is a bounded domain in $\mathbb{R}^N$. And f is a function of class $C^1 : \mathbb{R} \to \mathbb{R}$ such that

$$f(0) = 0 \text{ and } f'(0) > -\lambda_1, \tag{10}$$

where $\lambda_1 = \lambda_1(\Omega)$ is the smallest eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

Proposition(10.3.1)

Under above hypotheses, the stationary solution $u \equiv 0$ of (9) is exponentially stable in $X = C_0(\Omega)$ in the following sense: for each $\gamma \in (0, \lambda_1 + f'(0))$, there exists $R = R(\gamma)$ such that for all $x \in X$ with $\|x\| \leq R$, the solution u of (9) such that $u(0) = x$ exists and is global, and satisfies

$$\forall t \geq 0, \quad \|u(t)\| \leq M \|x\| e^{-\gamma t},$$

with M independent of γ and x.

Another situation, this time we assume some conditions which are in a sense opposite to (10) is when we have that

$$f \text{ is strictly convex on } [0,\infty), f(0) = 0 \text{ and } , f_d'(0) < -\lambda_1(\Omega).$$

Here the solution 0 is unstable and we have the following

Theorem(10.3.2)

1. There exists one and only one positive solution φ of the problem:

$$\phi \in X \cap H_0^1(\Omega), \quad -\Delta\phi + f(\varphi) = 0. \tag{11}$$

2. For each $u_0 \in X$, $u_0 \geq 0$ and not identically 0, the solution u of the semilienar heat equation such that $u(0) = u_0$ tends to φ as $t \to \infty$. Morover, we have

$$\forall t \geq 0, \|u(t,\cdot) - \varphi(\cdot)\|_{L^\infty(\Omega)} \leq C(u_0)e^{-\gamma t},$$

where $\gamma > 0$ is independent of u_0.

Remarks 10.3.5

The main result of Theorem (10.3.2) can be viewed as a property of *global exponential stability* of the positive stationary solution in the metric space $Z \setminus \{0\} = \{u \in C_0(\Omega) \mid u \geq 0, u \neq 0\}$. The following remarks are in order

1. The constant $C(u_0)$ does not remain bounded with $\|u_0\|_{L^\infty}$.
2. Theorem 10.3.2 is applicable, as a tipical case to the nonlinearity

$$f(u) = c\,|u|^\alpha\, u - \lambda u$$

for some positive constants c, α and λ.

Stability for the wave equation

We consider the semilinear wave equation

$$u_{tt} - \Delta u - f(u) + \lambda u_t = 0 \text{ in } \mathbb{R}^+ \times \Omega \quad u = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \tag{12}$$

where Ω is a bounded domain in $\mathbb{R}^N$, f is a $C^1 : \mathbb{R} \to \mathbb{R}$ function such that

$$f(0) = 0 \text{ and } f'(0) > \lambda_1,$$

and satisfies the growth condition

$$|f(u)| \leq C(1 + |u|^r), \text{ on } \mathbb{R},$$

$r \geq 0$ arbitrary if $N = 1, 2$ and $0 \leq r \leq \frac{N}{N-2}$ if $N \geq 3$.

Proposition 10.3.6

Under above hypotheses, the stationary solution $(u, v) \equiv 0$ of (12) is exponentially stable in $X = H_0^1(\Omega) \times L^2(\Omega)$ in the following sense: there exist $\delta < 0$ and $R := R(\delta)$ such that, for all $x \in X$ with $\|x\| \leq R$, the solution u of (12) such that $u(0) = x$ is global and satisfies

$$\forall t \geq 0, \quad \|x\| \leq M(\delta)\, \|x\|\, e^{-\delta t}.$$

Stability and positivity

Consider the one dimentional heat equation

$$\begin{cases} u_t - u_{xx} + f(u) = 0 \text{ in } \mathbb{R}^+ \times (0, L), \\ u(t,0) = u(t,L) = 0, \text{ on } \mathbb{R}^+, \end{cases} \tag{13}$$

where f is a $C^1 : \mathbb{R} \to \mathbb{R}$ function. Any solution u of this problem wich is global and uniformly bounded on $\mathbb{R}^+ \times (0, T)$ converges to a solution ϕ of the elliptic problem

$$\phi \in H_0^1(0, L), \quad -\varphi_{xx} + f(\phi) = 0.$$

Proposition 10.4.1

If φ is a stable solution of (13), then φ has constant sign on $(0, L)$.

Limits of the theory

In higher dimentions relationship between stability and the absence of zeroes seems much more intricate. For instance, in the case of Neumann boundary conditions, a result of Casten and Holland ([1]) asseerts that if Ω is star shaped, any non constant solution is unstable. Counter examples of stable solutions changing sign in Ω for both Dirichlet and Neumann boundary conditions in nonconvex domains. On the other hand, even for Ω convex there is no general instability result for solutions chainging sign in the case of Dirichlet boundary conditions.

Proposition 10.4.3

Let φ be a solution of (11) in a rectangle $\Omega = (0, a) \times (0, b)$ and assume that there is a sub-rectangle $R = (a, \frac{a}{p}) \times (0, \frac{b}{q})$ with $p, q \geq 1$ and $\max(p, q) > 1$ such that φ has constant sign in R and $\varphi = 0$ on ∂R. Then φ is unstable.

Proposition 10.4.4

If Ω is a ball in $\mathbb{R}^N$,$N \geq 2$, and φ is a Liapunov-stable solution of (9), we have either $\phi = 0$, or ϕ is spherically symmetric with constant sign in Ω.

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Thanks!