# Functional Analysis Applied to PDEs (2024-1)

Google Classroom: kmk6w62
Telegram: `https://t.me/+bOM71NAzRMI2MDQx`

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# Homework 7

Instructions: Solve the following exercises justifying your answers carefully. Upload your written answers in LaTeX format using the Google Classroom platform by **Monday, November 27th** at the latest.

Exercises:
Let $\Omega \subset \mathbb{R}^N$ be a bounded domain.

1. Using energy methods (3 points) and maximum principles (3 points), prove that the Neumann problem

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

only has constant solutions.

(Using maximum principles) Suppose that (1) has a nonconstant solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. The strong maximum principle implies that there is $z_0 \in \partial\Omega$ such that $u(z_0) > u(x)$ for all $x \in \Omega$. Take a small ball centred in $x \in \Omega$ such that $B_\varepsilon(x) \subseteq \Omega$ and $z_0 \in \partial B_\epsilon(x)$, this is possible since we suppose that $\partial\Omega$ is flat enough since the outwards normal vector exists. Now $u$ is a solution and therfore a subsolution in the ball $B_\epsilon(x)$. Hopf lemma implies that $\partial_\nu u(z_0) > 0$ contradicting Neumann's boarder conditions.

2. (5 points) Let $\lambda_1$ be the first eigenvalue of the Laplacian with Dirichlet conditions in $\Omega$. Prove that

$$\lambda_1 = \sup_{u \in U} \inf_{x \in \Omega} \frac{-\Delta u(x)}{u(x)},$$

where

$$U := \{u \in C^\infty(\overline{\Omega}) \,:\, u > 0 \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\}.$$

(Hint: By regularity theory, you can use that the first eigenfunction of the Laplacian $\varphi_1$ belongs to $U$).

Using the hint, lets take $\varphi_1 \in U$ the firs eigenfunction of $-\Delta$. Since

$$\lambda_1 = \frac{-\Delta\varphi_1(x)}{\varphi_1(x)}, \quad \forall x \in \Omega,$$

we have that

$$\lambda_1 \leq \sup_{u\in U} \inf_{x\in\Omega} \frac{-\Delta u(x)}{u(x)}.$$

To prove the reverse inequality, let $u \in U$. Let's prove that ther is an $x^* \in \Omega$ such that

$$\lambda_1 \geq \frac{-\Delta u(x^*)}{u(x^*)}.$$

Suppose that this is not the case. This means that

$$\frac{-\Delta\varphi_1(x)}{\varphi_1(x)} = \lambda_1 < \frac{-\Delta u(x)}{u(x)},$$

for all $x \in \Omega$. In other words

$$0 < \frac{-\Delta u(x)\varphi_1(x) - (-\Delta\varphi_1(x)u(x))}{\varphi_1(x)u(x)} \quad \forall x \in \Omega.$$

Since $u, \varphi_1 \in U$ are positive in $\Omega$ the previous condition is equivalent to

$$0 < -\Delta u(x)\varphi_1(x) - (-\Delta\varphi_1(x)u(x)) \quad \forall x \in \Omega.$$

Integrating over $\Omega$ and using integration by parts (using the fact that $u$ and $\varphi_1$ vanishes on $\partial\Omega$) we obtain

$$\begin{aligned} &\int_\Omega -\Delta u(x)\varphi_1(x) - (-\Delta\varphi_1(x)u(x))\,dx \\ &= \int_\Omega \nabla u(x)\cdot\nabla\phi_1(x) - \nabla\varphi_1(x)\cdot\nabla u(x)dx = 0. \end{aligned}$$

A contradiction. Then the existence of such $x^*$ is proved. Since $u$ was an arbitrary function in $U$ then

$$\lambda_1 \geq \sup_{u\in u} \inf_{x\in\Omega} \frac{-\Delta u(x)}{u(x)},$$

wich completes the prove.

3. (5 points) Let $\lambda_k$ be the $k$-th eigenvalue of the Laplacian with Dirichlet boundary conditions. Prove that

$$\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{u \in S^\perp} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx},$$

where $\Sigma_{k-1}$ denotes the collection of all subspaces of $H_0^1(\Omega)$ with dimension $k-1$.