Subject Gamma Date / /

Gama: generalization of factor fn.

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx \qquad \boxed{n>0}$$

Properties:

① $\Gamma(n+1) = n! \longrightarrow \mathbb{Z}^+$ $\boxed{4}$

recurrence relation ② $\Gamma(n+1) = n\,\Gamma(n) \longrightarrow \mathbb{R}^+$ $\boxed{3/2}$

③ $\Gamma(n)\,\Gamma(1-n) = \dfrac{\pi}{\sin(n\pi)}$ $\qquad \Gamma(\frac{1}{8})\,\Gamma(\frac{7}{8}) = \dfrac{\pi}{\sin(\frac{\pi}{8})}$

عددين مجموعهم "واحد"

④ $\Gamma(1) = 1$

⑤ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

⑥ $\Gamma(n) = \infty$ for: $n \leq 0$ → must be int

recognization

① $\int_0^\infty$

② $\boxed{e^{-}}$ التركيز على expo.

③ no Factor → x^0

tips

① to use (Γ)

- set the "$\int$" to the standard Form:

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx$$

by adjusting the power of the expo. terms

② if the "$\int$" in the Form of log, ln:

adjust it to the standard form by: $\ln(\frac{1}{x}) = -t$, $\ln(x) = -t$

OSCAR

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$$\int_0^1 \frac{1}{\sqrt{-\ln x}}\,dx$$

let $-\ln x = t$ $\quad \therefore \ln x = -t$

$\because e^{\ln(x)} = e^{-t}$ $\quad \therefore x = e^{-t}$

$\therefore dx = -e^{-t}\,dt$

$x = 0 \longrightarrow t = \infty$

$x = 1 \longrightarrow t = 0$

$$\int_\infty^0 \frac{1}{\sqrt{t}} * -e^{-t}\,dt$$

$$-\int_\infty^0 t^{-\frac{1}{2}}\, e^{-t}\,dt$$

$$= \int_0^\infty e^{-t}\, t^{-\frac{1}{2}}$$

$n-1 = -\frac{1}{2} \longrightarrow n = 1 - \frac{1}{2} = \frac{1}{2}$

$= \Gamma(\frac{1}{2}) = \sqrt{\pi}$

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$$\int_0^\infty (1+2x^2)^2 e^{-x}\, dx \qquad \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$$

let $t = 1+2x^2 \longrightarrow 2x^2 = t-1$

$x^2 = \frac{t-1}{2} \longrightarrow x = \left[\frac{t-1}{2}\right]^{\frac{1}{2}}$

$dx = \frac{1}{2}\left[\frac{t-1}{2}\right]^{-\frac{1}{2}} * dt$

$$= \int_0^\infty t^2 \, e^{-\left[\frac{1}{2}(t-1)\right]^{\frac{1}{2}}} * \frac{1}{2}\left[\frac{1}{2}(t-1)\right]^{-\frac{1}{2}} dt$$

$$= \left[\frac{1}{2}\right]^{\frac{1}{2}} \int_0^\infty t^2 \, e^{-\left[\frac{1}{2}(t-1)\right]^{\frac{1}{2}}} * [$$

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② $\int_0^\infty x^3 e^{-2x^5} dx$

let $2x^5 = t$ $\quad x^5 = \frac{t}{2} \rightarrow x = \left[\frac{1}{2}t\right]^{\frac{1}{5}}$

$dx = \frac{1}{5} t^{-\frac{4}{5}} * \left[\frac{1}{2}\right]^{\frac{1}{5}}$

$\int_0^\infty \left[\left(\frac{1}{2}t\right)^{\frac{1}{5}}\right]^3 e^{-t} * \frac{1}{5} t^{-\frac{4}{5}} * \left[\frac{1}{2}\right]^{\frac{1}{5}} dt$

$\left[\frac{1}{2}\right]^{\frac{1}{5}} * \left(\frac{1}{2}\right)^{3/5} * \frac{1}{5}\int_0^\infty t^{3/5} e^{-t} t^{-4/5} dt$

$\left[\frac{1}{2}\right]^{4/5} * \frac{1}{5}\int_0^\infty t^{-1/5} e^{-t} dt$

$n-1 = -\frac{1}{5} \rightarrow n = 1 - \frac{1}{5} = \frac{5}{5} - \frac{1}{5} = \frac{4}{5}$

$= \boxed{\left(\frac{1}{2}\right)^{4/5} * \frac{1}{5} * \Gamma\left(\frac{4}{5}\right)}$ #

③ $\int_0^\infty x^{5/2} e^{-2x} dx$

let $2x = t$ $\quad x = \frac{1}{2}t \quad dx = \frac{1}{2}dt$

$\int_0^\infty \left[\frac{1}{2}t\right]^{5/2} e^{-t} \frac{1}{2} dt$

$= \left[\frac{1}{2}\right]^{\frac{7}{2}} \int_0^\infty t^{5/2} e^{-t} dt$

$n-1 = \frac{5}{2} \quad n = \frac{5}{2} + \frac{2}{2} = \frac{7}{2}$

$= \left[\frac{1}{2}\right]^{\frac{7}{2}} \Gamma\left(\frac{7}{2}\right) = \left[\frac{1}{2}\right]^{\frac{7}{2}} \frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \Gamma\left(\frac{1}{2}\right)$

$\boxed{\frac{15}{64}\sqrt{\frac{\pi}{2}}}$

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④ $\int_0^\infty x^3 e^{-x^3} dx$

$x^3 = t \qquad x = (t)^{\frac{1}{3}} \qquad dx = \frac{1}{3} t^{-2/3} dt$

$\int_0^\infty t\, e^{-t} \times \frac{1}{3} t^{-2/3}\, dt$

$\frac{1}{3}\int_0^\infty e^{-t}\, t^{\frac{1}{3}}\, dt$

$n-1 = \frac{1}{3} \qquad n = \frac{1}{3} + \frac{3}{3} = \frac{4}{3}$

$\boxed{\frac{1}{3}\Gamma\left(\frac{4}{3}\right)}$ #

⑤ $\int_0^1 x^2 \ln\left(\frac{1}{x}\right)^3 dx$

let $\ln\left(\frac{1}{x}\right)^3 = t$

$e^{\ln\left(\frac{1}{x}\right)^3} = e^t \rightarrow \left(\frac{1}{x}\right)^3 = e^t \rightarrow \frac{1}{x} = [e^t]^{\frac{1}{3}}$

$x = [e^t]^{-\frac{1}{3}} = e^{-\frac{t}{3}}$

$dx = -\frac{1}{3} e^{-t/3}\, dt$

$\int_\infty^0 [e^{-t/3}]^2\, t \times -\frac{1}{3} e^{-t/3}\, dt$

$\frac{1}{3}\int_0^\infty e^{-\frac{2}{3}t - \frac{1}{3}t} \times t\, dt$

$\frac{1}{3}\int_0^\infty e^{-t}\, t\, dt$

$\boxed{\frac{1}{3}}$ #

Sol^n 2

$3\int_0^1 x^2 \ln\left(\frac{1}{x}\right) dx$

$\ln\left(\frac{1}{x}\right) = t$

$x = e^{-t}$

$dx = -e^{-t} dt$

$3\int_0^\infty e^{-2t} \cdot t\, dt$

let $2t = y$,

$\therefore t = \frac{y}{2}$

$dt = \frac{1}{2} dy$

$\frac{3}{4}\int e^{-y}\, y\, dy$

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$$\int_0^1 \ln\left[\left(\frac{1}{x}\right)^3\right] dx$$

let $\ln\left[\left(\frac{1}{x}\right)^3\right] = t$

$e^{\ln\left[\left(\frac{1}{x}\right)^3\right]} = e^t$

$\left(\frac{1}{x}\right)^3 = e^t \rightarrow \frac{1}{x} = \left[e^t\right]^{\frac{1}{3}}$

$\boxed{x = e^{\frac{-t}{3}}} \rightarrow \cancel{dx = -\frac{1}{3} e^{-t/3}}$

$dx = -\frac{1}{3} e^{-\frac{t}{3}-1} dt$

$0 \rightarrow \infty$

$1 \rightarrow 0$

$\int_\infty^0 t * -\frac{1}{3} e^{-\frac{t}{3}-1} dt$

$= \frac{1}{3}\int_0^\infty t * e^{-\frac{t}{3}-1} dt$

let $\frac{t}{3}+1 = y$ $\quad \therefore t = (y-1)3$

$dt = 3\,dy$

$= \frac{1}{3}\int_0^\infty (3y-3) *$

let $\ln\frac{1}{x} = t$

$\frac{1}{x} = e^t$

$\therefore x = e^{-t}$

$dx = -e^{-t} dt$

$0 \longrightarrow \infty$

$1 \longrightarrow 0$

$= 3\int_0^1 \ln\left(\frac{1}{x}\right) dx$

$= 3 * -\int_\infty^0 t\, e^{-t} dt$

$= 3\int_0^\infty e^{-t}\, t\, dt$

$= 3 * \Gamma(2) = \boxed{3}$

4) $\int_0^\infty e^{-ax^n} x^m \, dx$

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}} dx$$

$t^{\frac{1}{n}} = \frac{1}{t^{-n}}$

ضرب a عشان a=1

let $t = a x^n$

$\therefore x = \frac{t}{a} \rightarrow x = \left(\frac{t}{a}\right)^{\frac{1}{n}}$

$dx = \left[\frac{1}{n} t^{\frac{1}{n}-1} dt\right] * \left(\frac{1}{a}\right)^{\frac{1}{n}}$

$$= \int_0^\infty e^{-t} \left[\left(\frac{t}{a}\right)^{\frac{1}{n}}\right]^m * \frac{1}{(a)^{\frac{1}{n}}} * \frac{1}{n} t^{\frac{1}{n}-1} dt$$

$$= \frac{1}{a^{\frac{1}{n}} \cdot n} \cdot \frac{1}{a^{\frac{m}{n}}} \int_0^\infty e^{-t} t^{\frac{m}{n}} \cdot t^{\frac{1}{n}-1} dt$$

$$= \frac{1}{a^{\frac{m+1}{n}} * n} \int_0^\infty e^{-t} t^{\frac{m+1}{n}-1} dt$$

$\frac{m}{n} * \left[\frac{1}{n} - 1\right]$

$\frac{m+1}{n} - \frac{m}{n}$

$$= \frac{1}{n * a^{\frac{m+1}{n}}} \Gamma\left(\frac{m+1}{n}\right)$$

---

5) $\int_0^\infty e^{-x^3} \sqrt{x} \, dx$

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} * x^{n-1}} dx$$

let $t = x^3 \rightarrow x = t^{\frac{1}{3}}$

$dx = \frac{1}{3} t^{\frac{1}{3}-1} dt$

$= \frac{1}{3} t^{\frac{1}{3}-\frac{3}{3}} dt = \frac{1}{3} t^{\frac{-2}{3}} dt$

$$= \int_0^\infty e^{-t} \sqrt{t^{\frac{1}{3}}} * \frac{1}{3} t^{-\frac{2}{3}} dt$$

$$= \frac{1}{3}\int_0^\infty e^{-t} * (t^{\frac{1}{3}})^{\frac{1}{2}} * t^{-\frac{2}{3}} dt$$

$$= \frac{1}{3}\int_0^\infty e^{-t} \; t^{-\frac{1}{2}} dt$$

$$\frac{1}{3} * \frac{1}{2} - \frac{2}{3}$$

$$\frac{1}{6} - \frac{2}{3}$$

$$\frac{1-4}{6} = \frac{-3}{6} = \frac{-1}{2}$$

$$= \cancel{\Gamma(n)} \; n-1 = -\frac{1}{2}$$

$$n = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \boxed{\frac{-\sqrt{\pi}}{3}}$$

---

6) $\int_0^\infty \frac{x^9}{a^x} dx$ $\quad \boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx}$

$$= \int_0^\infty x^9 a^{-x} dx = \int_0^\infty x^9 * e^{\ln(a^{-x})} dx$$

$$= \int_0^\infty x^9 e^{-x\ln(a)} dx$$

let $x \ln(a) = t$

$$\therefore x = \frac{t}{\ln(a)} \qquad dx = \frac{1}{\ln(a)} dt$$

$$\int_0^\infty \left(\frac{t}{\ln a}\right)^9 e^{-t} * \frac{1}{\ln a} dt$$

$$= \left(\frac{1}{\ln a}\right)^{a+1} \int_0^\infty e^{-t} \cdot t^{a} \, dt$$

$\exists \; n-1 = a$

$\cancel{\Gamma(n)} = \quad \therefore n = a+1$

$$\boxed{= \left[\frac{1}{\ln a}\right]^{a+1} \Gamma(a+1)}$$

7) $\int_0^\infty 2^{-3x^2} \, dx$ $\quad \boxed{\Gamma(n) = \int_0^\infty e^{-x} \, x^{n-1}}$

$$= \int_0^\infty e^{\ln(2^{-3x^2})} \, dx = \int_0^\infty e^{-3x^2 \ln(2)} \, dx$$

let $t = 3x^2 \ln(2)$

$$\therefore x^2 = \frac{t}{3\ln(2)} \Rightarrow x = \left[\frac{t}{3\ln(2)}\right]^{\frac{1}{2}}$$

$$dx = \left[\frac{1}{3\ln(2)}\right]^{\frac{1}{2}} * \frac{1}{2} t^{-\frac{1}{2}} \, dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} \left[\frac{1}{3\ln(2)}\right]^{\frac{1}{2}} * t^{-\frac{1}{2}} \, dt$$

$$= \frac{1}{2} * \frac{1}{\sqrt{3\ln(2)}} \int_0^\infty e^{-t} \, t^{-\frac{1}{2}} \, dt$$

$n-1 = -\frac{1}{2} \quad (n = \frac{1}{2})$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

11) show : $\int_0^1 \left(\ln \frac{1}{y}\right)^{n-1} dy = \Gamma(n)$

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx}$$

let $x = \ln \frac{1}{y}$

$y = e^x = e^{\ln \frac{1}{y}}$

$e^x = \frac{1}{y} \Rightarrow \boxed{y = \frac{1}{e^x} = e^{-x}}$

$\boxed{dy = -e^{-x}} dx$

$$\boxed{\begin{array}{l} \text{when } y = 0 \rightarrow x = \infty \\ \quad\;\; y = 1 \rightarrow x = 0 \end{array}}$$

$$= \int_\infty^0 - x^{n-1} e^{-x} dx$$

$$= -\int_\infty^0 x^{n-1} e^{-x} dx = \int_0^\infty e^{-x} x^{n-1} dx$$

$$= \Gamma(n)$$

2) $\int_0^1 x^m \left(\ln \frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$

5

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx}$$

نفتح في عدل هشتقات

- هذا الطالب ... محسوب
- الاخرى ... في مضاده

let $\ln \frac{1}{x} = t$

$e^t = e^{\ln\left(\frac{1}{x}\right)}$

$e^t = \frac{1}{x} \Rightarrow \boxed{x = \frac{1}{e^t} = e^{-t}}$

$\boxed{dx = -e^{-t}\, dt}$

when: $x = 0 \longrightarrow t = \infty$

$x = 1 \longrightarrow t = 0$

$$\int_\infty^0 \left[e^{-t}\right]^m t^n * -e^{-t}\, dt$$

check

$$\int_0^\infty e^{-t(m+1)}\, t^n\, dt$$

let $t(m+1) = y \Rightarrow \boxed{t = \frac{y}{m+1}}$

$\boxed{dt = \frac{dy}{m+1}}$

$$\int_0^\infty e^{-y} \left[\frac{y}{m+1}\right]^n \frac{dy}{m+1}$$

check

$$\frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-y} y^n \, dy$$

$n-1 = n$

$\therefore n = n+1$

$$\Gamma(n+1) * \frac{1}{(m+1)^{n+1}}$$

1) $\int_0^1 \frac{1}{\sqrt{-\ln x}} \, dx$

$$= \int_0^1 \frac{1}{(-\ln x)^{\frac{1}{2}}} \, dx \quad \boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx}$$

let $-\ln x = t$

$e^{\ln x} = e^{-t}$

$\boxed{x = e^{-t}}$ $\boxed{dx = -e^{-t} dt}$

when: $x = 0 \longrightarrow t = \infty$

$x = 1 \longrightarrow t = 0$

$$\int_\infty^0 \frac{1}{(t)^{\frac{1}{2}}} * \left[e^{-t} \right] dt$$

$$\int_0^\infty t^{-\frac{1}{2}} e^{-t} \, dt$$

$n-1 = -\frac{1}{2} \longrightarrow n = 1 - \frac{1}{2} = \frac{1}{2} = \boxed{\sqrt{\pi}}$

2) $\int_0^1 \ln\left(\frac{1}{x}\right)^3 dx$

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx}$$

let $\ln\left(\frac{1}{x}\right) = t$

$e^{\ln\left(\frac{1}{x}\right)} = e^t$

$\frac{1}{x} = e^t$

$x = \frac{1}{e^t} \rightarrow \boxed{x = e^{-t}} \Rightarrow \boxed{dx = -e^{-t} dt}$

only the -ve

$x = 0 \rightarrow t = \infty$

$x = 1 \rightarrow t = 0$

$-3\int_\infty^0 t^3 \times e^{-t} dt$

$$\boxed{\ln(a^n) = n\ln(a)}$$

$3\int_0^\infty t^3 e^{-t} dt$

$n - 1 = 1 \rightarrow \boxed{n = 1 + 1 = 2}$

$\Gamma(2) = 1! = \cancel{1 \times \frac{1}{3} \times 2} \cancel{\boxed{24}}$ 1

$3 \times 1 = \boxed{3}$ #

3 $\int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^3 dx$

let $\ln\frac{1}{x} = t$

$\therefore e^{\ln\frac{1}{x}} = e^{t}$

$\frac{1}{x} = e^{-t}$

$\therefore \boxed{x = e^{-t}}$ $\boxed{dx = -1 \times e^{-t}\, dt}$

$x = 0 \longrightarrow t = \infty$

$x = 1 \longrightarrow t = 0$

$-\int_{\infty}^{0} t^3 e^{-t}\, dt$

$\int_0^{\infty} t^3 e^{-t}\, dt$

$n - 1 = 3 \rightarrow n = 3 + 1 = 4$

$\Gamma(4) = 3! = 3 \times 2 = \boxed{6}$ #

$\int_0^1 x^2 \ln\left(\frac{1}{x}\right)^3 dx$

let $\left(\ln\frac{1}{x}\right)^3 = t$

$\left[e^{\ln\left(\frac{1}{x}\right)}\right]^3 = e^{t}$

$\frac{1}{x} = e^{\frac{t}{3}}$

$\boxed{x = e^{-\frac{t}{3}}}$

$dx = -\frac{1}{3} e^{-t/3}\, dt$

~~let $\left[\ln\left(\frac{1}{x}\right)\right]^3 = t$~~

~~$\ln\left(\frac{1}{x}\right) = t^{\frac{1}{3}}$~~ ~~$x = e^{-t}$~~

~~$x = e^{-\frac{1}{3}}$~~

$0 \longrightarrow \infty$

$1 \longrightarrow 0$

$\int_0^{\infty} e^{-t}\, t\, dt$

$n - 1 = 1$

$\boxed{n = 2}$

$\frac{1}{3}\Gamma(2) = \boxed{\frac{1}{3}}$ #

$\int_{\infty}^{0} -\int^{0} e^{-\frac{2}{3}t}\, t * \frac{1}{3} e^{-\frac{t}{3}}\, dt$

Subject Beta

Beta #1:

$$\beta(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1} \, dx \quad (p,q) > 0$$

Beta #2:

$$\beta(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(\theta) \cdot \cos^{2q-1}(\theta) \, d\theta$$

Properties:

① $\beta(p,q) = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = \beta(q,p)$

② $\beta(p+1,q) = \dfrac{p}{p+q}\,\beta(p,q)$

$\beta(p,q+1) = \dfrac{q}{p+q}\,\beta(p,q)$

recognition

① $\int_0^1$ ② x, $x-1$ ③ sin, cos

Trignomatry

⊛ $\sin 2\theta = 2 \sin\theta \cos\theta$

⊛ $\cos 2\theta = 1 - 2\sin^2\theta$

$= 2\cos^2\theta - 1$

$= \cos^2\theta - \sin^2\theta$

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Important Examples

⑦ $\int_0^{\pi/2} \frac{1}{\sqrt{\operatorname{cosec}(2\theta)}}\, d\theta$

$\boxed{\operatorname{cosec} = \frac{1}{\sin}}$ $\boxed{\beta(p,q) = \frac{1}{2}\int_0^{\pi/2} \sin^{2p-1}\cos^{2q-1}\, d\theta}$

$= \int_0^{\pi/2} \frac{1}{\frac{1}{[\sin(2\theta)]^{\frac{1}{2}}}} = \int_0^{\pi/2} [\sin(2\theta)]^{\frac{1}{2}} = \int_0^{\pi/2} 2\sin(\theta)^{\frac{1}{2}}\cos(\theta)^{\frac{1}{2}}$

$p = q = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$

$\frac{1}{2}\beta\left(\frac{3}{4},\frac{3}{4}\right) = \frac{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4})2}{\Gamma(\frac{3}{2})}$ //

8) $\int_0^{\infty} \frac{1}{1+t^4}\, dt$

let $t^2 = \tan(\theta)$

$1 + \tan^2\theta = \sec^2(\theta)$

$t = \tan^{\frac{1}{2}}(\theta)$

$dt = \frac{1}{2}\tan^{-\frac{1}{2}}(\theta)\sec^2(\theta)\, d\theta$

$\frac{1}{2}\int_0^{\infty} \frac{1}{\sec^2(\theta)} \tan^{-\frac{1}{2}}(\theta)\sec^2(\theta)\, d\theta$

$\frac{1}{2}\int_0^{\pi/2} \frac{\sin^{-\frac{1}{2}}(\theta)}{\cos^{-\frac{1}{2}}}\, d\theta$

$\frac{1}{2}\int_0^{\pi/2} \cos^{\frac{1}{2}}(\theta)\sin^{-\frac{1}{2}}(\theta)\, d\theta$

$2p-1 = -\frac{1}{2} \rightarrow p = \frac{1}{4}$

$2q-1 = -\frac{1}{2} \rightarrow q = \frac{1}{4}$

$\frac{1}{2}\beta\left(\frac{1}{4},\frac{3}{4}\right) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{\Gamma(1)}$

$= \frac{\pi}{\sin(\frac{1}{4})}$ //

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① $\int_0^{\pi/2} \cos(2\theta) \cos^4(\theta)\, d\theta$

$\cos(2\theta) = \boxed{\cos^2(\theta) - \sin^2(\theta)}$

$= 1 - 2\sin^2(\theta)$

$= 2\cos^2\theta - 1$

$= \int_0^{\pi/2} (\cos^2(\theta) - \sin^2\theta) \cos^4(\theta)\, d\theta$

$= \int_0^{\pi/2} (\cos^2(\theta)\cos^4(\theta) - \sin^2(\theta)\cos^4(\theta))\, d\theta$

$= \int_0^{\pi/2} \cos^6\theta - \int_0^{\pi/2} \sin^2(\theta) \cdot \cos^4(\theta)\, d\theta$

$2p - 1 = 0 \rightarrow p = \frac{1}{2}$ | $2p - 1 = 2 \rightarrow p = \frac{3}{2}$

$2q - 1 = 6 \rightarrow q = \frac{7}{2}$ | $2q - 1 = 4 \rightarrow q = \frac{5}{2}$

$\frac{\Gamma(\frac{1}{2})\Gamma(\frac{7}{2})}{\Gamma(4)} - \frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(4)}$

$\frac{\sqrt{\pi} \cdot \frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \sqrt{\pi}}{3!} - \frac{\frac{1}{2}\sqrt{\pi} * \frac{3}{2} * \frac{1}{2}\sqrt{\pi}}{3!}$

$= \boxed{\frac{\pi}{8}}$

$$\beta(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1} \, dx$$

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(2) $\displaystyle\int_0^2 \frac{x^2}{\sqrt{2-x}} \, dx$

$$= \int_0^2 x^2 \, (2-x)^{-1/2} \, dx$$

$$= \int_0^2 x^2 \left[2\left(1-\frac{x}{2}\right)\right]^{-\frac{1}{2}} dx$$

$$= (2)^{-\frac{1}{2}} \int_0^2 x^2 \left[1-\frac{x}{2}\right]^{-\frac{1}{2}} dx$$

let $\frac{x}{2} = t$ $\quad \therefore x = 2t \quad \therefore dx = 2\,dt$

$x \to 0 \quad t \to 0$

$x \to 2 \quad t \to 1$

$$(2)^{-\frac{1}{2}} \int_0^1 (2t)^2 \, [1-t]^{-\frac{1}{2}} \times 2 \, dt$$

$$= 2^{\frac{5}{2}} \int_0^1 t^2 \, (1-t)^{-\frac{1}{2}} \, dt$$

$$= 2^{\frac{5}{2}} * \frac{\Gamma(3)\,\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})}$$

$x = 0 \longrightarrow y = 0$

$x = 3 \longrightarrow y = 1$

$$= 3\int_0^1 3\, y^{\frac{1}{3}} (1-y)^{\frac{1}{3}} y^{-\frac{2}{3}}\, dy$$

$$= 9\int_0^1 y^{-\frac{1}{3}} (1-y)^{\frac{1}{3}}\, dy$$

$p - 1 = -\frac{1}{3} \rightarrow \boxed{p = \frac{2}{3}}$

$q - 1 = \frac{1}{3} \rightarrow \boxed{q = \frac{4}{3}}$

$\frac{1}{3} - \frac{2}{3}$

$\boxed{-\frac{1}{3}}$

$$9\,\beta\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)}$$

$$= \frac{\Gamma\left(\frac{2}{3}\right)\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{1}$$

$\frac{2}{3} - \frac{3}{3}$

$$9\,\beta\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{9}{3} * \underbrace{\Gamma\left(\frac{2}{3}\right) * \Gamma\left(\frac{1}{3}\right)}_{\text{sum}=1} = \boxed{\frac{9}{3}\,\frac{\pi}{\sin\frac{1}{3}\pi}}$$

(3) $$\int_0^a y^4 \sqrt{a^2 - y^2}\, dy = \frac{\pi a^6}{32}$$

$$\boxed{\beta(p,q) = x^{p-1}(1-x)^{q-1}\, dx}$$

$\sqrt{a^2 - y^2}$

$\sqrt{a^2\left(1 - \frac{y^2}{a^2}\right)}$

$a\left[1 - \frac{y^2}{a^2}\right]^{\frac{1}{2}}$

$$= a\int_0^a y^4 \left[1 - \left(\frac{y}{a}\right)^2\right]^{\frac{1}{2}} dy$$

let $\left(\frac{y}{a}\right)^2 = x \rightarrow y^2 = a^2 x$

$\boxed{y = a\, x^{\frac{1}{2}}}$

$\boxed{dy = \frac{a}{2}\, x^{-\frac{1}{2}}\, dx}$

$\boxed{\begin{array}{ll} y = 0 & x = 0 \\ y = a & x = 1 \end{array}}$

$$= a \int_0^1 a^4 x^2 (1-x)^{\frac{1}{2}} \frac{a}{2} x^{-\frac{1}{2}} dx$$

$$= \frac{a^6}{2} \int_0^1 x^{\frac{3}{2}} (1-x)^{\frac{1}{2}} dx$$

$$\boxed{\beta(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx}$$

$$p-1 = \frac{3}{2} \longrightarrow p = \frac{3}{2} + \frac{2}{2} = \frac{5}{2}$$

$$q-1 = \frac{1}{2} \longrightarrow q = \frac{1}{2} + \frac{2}{2} = \frac{3}{2}$$

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$= \frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)}$$

$$= \frac{\frac{3}{2} * \frac{1}{2} * \cancel{\Gamma(\frac{1}{2})} * \frac{1}{2} \cancel{\Gamma(\frac{1}{2})}}{3 * 2 * 1} \rightarrow \pi$$

$$= \frac{\frac{3}{2} * \frac{1}{4} * \pi}{6} * \frac{a^6}{2}$$

$$= \frac{a^6 \pi}{16}$$

$$\frac{\frac{1}{8} * \frac{3}{2}}{6} = \frac{\frac{1}{8} + \frac{12}{8}}{6}$$

4 $\int_0^{\pi/2} \sin^6(\theta) \cos^7(\theta)\, d\theta$

$$\boxed{\beta(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(\theta) \cos^{2q-1}(\theta)\, d\theta}$$

$2p-1 = 6 \longrightarrow 2p = 7 \longrightarrow p = \frac{7}{2}$

$2q-1 = 7 \longrightarrow 2q = 8 \longrightarrow q = 4$

$$\frac{\beta(\frac{7}{2},4)}{2} = \frac{\Gamma(\frac{7}{2}) * \Gamma(4)}{\Gamma(\frac{15}{2})} \div 2$$

$$= \frac{\cancel{\frac{5}{2}} * \cancel{\frac{3}{2}} * \cancel{\frac{1}{2}} * \cancel{\sqrt{\pi}} * 3 * 2 * 1}{\frac{13}{2} * \frac{11}{2} * \frac{9}{2} * \frac{7}{2} * \cancel{\frac{5}{2}} * \cancel{\frac{3}{2}} * \cancel{\frac{1}{2}} * \cancel{\sqrt{\pi}}}$$

$$= \boxed{\frac{16}{3003}} \quad \#$$

5 $\int_0^{\pi/2} \cos^8(\theta)\, d\theta$

$$\boxed{\beta(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(\theta) \cos^{2q-1}(\theta)\, d\theta}$$

$2p-1 = 0 \longrightarrow \boxed{p = \frac{1}{2}}$

$2q-1 = 8 \longrightarrow \boxed{q = \frac{9}{2}}$

$$\frac{1}{2}\beta(p,q) = \frac{1}{2} * \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{9}{2})}{\Gamma(5)}$$

$$= \frac{1}{2} * \frac{\sqrt{\pi} * \frac{7}{2} * \frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \sqrt{\pi}}{4 * 3 * 2}$$

$$= \boxed{\frac{35}{256}\pi} \quad \#$$

(9)

(6) $\int_0^{\pi/2} \sqrt{\tan(\theta)}\, d\theta$

[A1]

$$\boxed{\beta(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\, d\theta}$$

$$= \int_0^{\pi/2} \left[\frac{\sin(\theta)}{\cos(\theta)}\right]^{\frac{1}{2}} d\theta$$

$$\boxed{= \int_0^{\pi/2} \sin^{\frac{1}{2}}(\theta) \cdot \cos^{-\frac{1}{2}}(\theta)\, d\theta}$$

$2p-1 = \frac{1}{2}$ $\quad 2p = \frac{3}{2} \rightarrow p = \frac{3}{2} \times \frac{1}{2} \boxed{\frac{3}{4}}$

$2q-1 = -\frac{1}{2}$ $\quad 2q = \frac{1}{2} \rightarrow \boxed{q = \frac{1}{4}}$

$$\frac{1}{2}\beta(\tfrac{3}{4}, \tfrac{1}{4}) = \frac{1}{2} \times \frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{4})}{\Gamma(1)}$$

$$= \frac{1}{2} \times \frac{1}{2}\sqrt{\pi} \times \Gamma(\tfrac{1}{4}) \qquad \frac{1}{2}\,\frac{\pi}{\sin\frac{1}{4}\pi}$$

$$= \frac{1}{4}\sqrt{\pi} \times \Gamma(\tfrac{1}{4})$$

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(7) $\int_0^{\pi/2} \frac{1}{\sqrt{\operatorname{cosec}(2\theta)}}\, d\theta$ $\quad \boxed{\beta(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\, d\theta}$

$$\int_0^{\pi/2} \left[\frac{1}{\operatorname{cosec}(2\theta)}\right]^{1/2} d\theta = \int_0^{\pi/2} \sin^{\frac{1}{2}}(2\theta)\, d\theta$$

$$= \int_0^{\pi/2} [2\sin\theta\cos\theta]^{\frac{1}{2}}\, d\theta = \sqrt{2}\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta \cos^{\frac{1}{2}}\theta\, d\theta$$

$2p-1 = \frac{1}{2}$ $\quad \boxed{p = \frac{3}{4}}$

$2q-1 = \frac{1}{2}$ $\quad \boxed{q = \frac{3}{4}}$

$$\rightarrow \boxed{\frac{\sqrt{2}}{2}\beta(\tfrac{3}{4}, \tfrac{3}{4})} \quad \frac{\Gamma(\frac{3}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{3}{2})}$$

(B)

(8) $\int_0^\infty \frac{1}{1+t^4}\, dt$

$\boxed{\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\, d\theta}$

let $t^2 = \tan\theta$

$t = \tan^{\theta}\frac{1}{2}$

$dt = \frac{1}{2}\tan^{-\frac{1}{2}}(\theta) * \sec^2(\theta)\, d\theta$

$t=0 \longrightarrow \theta = 0$

$t=\infty \longrightarrow \theta = \pi/2$

$= \int_0^{\pi/2} \frac{1}{1+\tan^2(\theta)} * \frac{1}{2}\tan(\theta)^{-\frac{1}{2}} * \sec^2(\theta)\, d\theta$

$\boxed{\sec^2 - \tan^2 = 1}$

$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\cancel{\sec^2(\theta)}} \tan(\theta)^{-\frac{1}{2}} * \cancel{\sec^2(\theta)\, d\theta}\, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \tan^{-\frac{1}{2}}(\theta)\, d\theta$

$\boxed{\frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}}$

$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin^{-\frac{1}{2}}(\theta)}{\cos^{-\frac{1}{2}}(\theta)}\, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \cancel{\sin^{\frac{1}{2}}(\theta)} \frac{\cos^{\frac{1}{2}}(\theta)}{\sin^{\frac{1}{2}}(\theta)}\, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \cos^{\frac{1}{2}}(\theta) * \sin^{-\frac{1}{2}}(\theta)\, d\theta$

$2p-1 = -\frac{1}{2} \longrightarrow p = \frac{1}{4}$

$2q-1 = \frac{1}{2} \longrightarrow q = \frac{3}{4}$

$\frac{1}{2}\times\frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{1} = \frac{1}{8}\sqrt{\pi} * \Gamma\left(\frac{1}{4}\right)$

$$\int_0^4 u^{3/2}(4-u)^{5/2}\,du \qquad \boxed{\Gamma(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}}\,dx$$

$$= \int_0^4 u^{3/2}\left[4\left(1-\frac{u}{4}\right)\right]^{5/2} du$$

$$= 4^{5/2}\int_0^4 u^{3/2}\left[1-\frac{u}{4}\right]^{5/2} du$$

let $\frac{u}{4} = t$

$u = 4t$

$du = 4\,dt$

$u = 0 \longrightarrow t = 0$

$u = 4 \longrightarrow t = 1$

$$= 4^{5/2}\int_0^1 (4t)^{3/2}(1-t)^{5/2}\,4\,dt$$

$$= 4^5\int_0^1 t^{3/2}(1-t)^{5/2}\,dt$$

$p-1 = 3/2 \rightarrow p = \frac{5}{2}$

$q-1 = \frac{5}{2} \rightarrow q = \frac{7}{2}$

$$= 4^5 * \frac{\frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \sqrt{\pi} * \frac{3}{2} * \frac{1}{2} * \sqrt{\pi}}{\Gamma(6)}$$

$$= \boxed{12\pi}$$

$$\int_0^1 (1-x^3)^{-1/2}\, dx \qquad \Gamma(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1}\, dx$$

let $x^3 = t$

$\therefore x = t^{1/3}$

$dx = \frac{1}{3} t^{-\frac{2}{3}}\, dt$

$$\frac{1}{3}\int_0^1 (1-t)^{-1/2} * t^{-\frac{2}{3}}\, dt$$

$$= \frac{1}{3} * \frac{\Gamma(\frac{1}{3}) * \sqrt{\pi}}{\Gamma(\frac{5}{6})} \quad \#$$

$p-1 = \frac{-2}{3}$

$p = \frac{1}{3}$

$q-1 = \frac{-1}{2}$

$q = \frac{1}{2}$

$$\int_0^1 \sqrt{\frac{(1-x)}{x}}\, dx$$

$$= \int_0^1 [x^{-1} (1-x)]^{\frac{1}{2}}\, dx$$

$$= \int_0^1 x^{-\frac{1}{2}} (1-x)^{\frac{1}{2}}$$

$p-1 = -\frac{1}{2} \longrightarrow p = \frac{1}{2}$

$q-1 = \frac{1}{2} \longrightarrow q = \frac{3}{2}$

$$\Gamma(p,q) = \frac{\sqrt{\pi} * \frac{1}{2} * \sqrt{\pi}}{\Gamma(2)} = \boxed{\frac{\pi}{2}} \quad \#$$

$$\int_0^2 (4-x^2)^{3/2}\,dx$$

$$= \int_0^2 \left[4\left(1-\frac{x^2}{4}\right)\right]^{3/2} dx$$

$$= \quad \text{let } \frac{x^2}{4} = t$$

$$x^2 = 4t$$

$$x = 4^{\frac{1}{2}}\, t^{\frac{1}{2}}$$

$$\boxed{dx = \frac{2}{2}\, t^{-\frac{1}{2}}\, dt}$$

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$$= 4\int_0^1 (1-t)^{3/2} * t^{-\frac{1}{2}}\, dt$$

$$p-1 = -\frac{1}{2} \longrightarrow p = \frac{1}{2}$$

$$q-1 = \frac{3}{2} \longrightarrow q = \frac{5}{2}$$

$$= 4\,\frac{\sqrt{\pi} * \frac{3}{2} * \frac{1}{2} * \sqrt{\pi}}{\Gamma(3)}$$

$$= \cancel{2}\,\frac{\frac{3}{2} * \frac{1}{2} * \pi}{\cancel{2}} = \boxed{3\pi} \#$$

$$\int_0^\infty \sqrt{y}\; e^{-y^2}\, dy \qquad \Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx$$

$$\text{let } y^2 = x \longrightarrow y = x^{\frac{1}{2}} \longrightarrow dy = \frac{1}{2} x^{-\frac{1}{2}} dx$$

$$= \frac{1}{2}\int_0^\infty \left[x^{\frac{1}{2}}\right]^{\frac{1}{2}} e^{-x} * x^{-\frac{1}{2}}\, dx$$

$$= \frac{1}{2}\int_0^\infty x^{\frac{1}{4}} * x^{-\frac{1}{2}} * e^{-x}\, dx \qquad \Big| \; n-1 = -\frac{1}{4}$$

$$n = -\frac{1}{4} + \frac{4}{4} = \boxed{\frac{3}{4}}$$

$$\frac{1}{2} * \Gamma\left(\frac{3}{4}\right) * \frac{3}{2}$$