Sheet (5)
Sequences and Mathematical Induction

Question 1:

Write the first four terms of the sequences defined by the following formulas:

1. $a_k = \dfrac{k}{10-k}$, for all integers $k \geq 1$.
2. $b_j = 1 + 2^j$, for all integers $j \geq 0$.
3. $c_i = \dfrac{(-1)^i}{3^i}$, for all integers $i \geq 0$.
4. $d_m = 1 - \left(\dfrac{1}{10}\right)^m$, for all integers $m \geq 1$.

Question 2:

Find explicit formulas for the following sequences with the initial terms given:

1. 0, 1, -2, 3, -4, 5.
2. $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}$.
3. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$.

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2. $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}$.
3. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$.
4. $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}$.

Question 3:

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$, $a_4 = 0$, $a_5 = -1$ and $a_6 = -2$. Compute each of the summations and products below.

1. $\sum_{i=0}^{0} a_i$.
2. $\sum_{j=1}^{3} a_j$.
3. $\prod_{k=0}^{6} a_k$.
4. $\prod_{k=2}^{2} a_k$.

Question 4:

Write each of the following summation or product notations.

1. $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2$.
2. $(1^3 - 1)(2^3 - 1)(3^3 - 1)(4^3 - 1)$.
3. $n(n-1)(n-2)\ldots\ldots\ldots\ldots1$.
4. $n + \frac{n-1}{2!} + \frac{n-2}{3!} + \frac{n-3}{4!} + \ldots\ldots\ldots\ldots + \frac{1}{n!}$

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Question 5:
Transform each of the following by making the change of variable $j = i - 1$.

1. $\sum_{i=1}^{n+1} \frac{(i - 1)^2}{i}$.
2. $\sum_{i=3}^{n+1} \frac{i}{i + n - 1}$.

Question 6:

a) Prove that $n! + 2$ is divisible by 2, for all integers $n \geq 2$.
b) Prove that $n! + k$ is divisible by k, for all integers $n \geq 2$ and $k = 1, 2, 3, \ldots, n$.

Question 7:
Using theorem 4.2.2 to solve the following:

$$2 + 4 + 6 + \ldots + 2n = n^2 + n.$$

For all integers $n \geq 1$.

Question 8:
Without using theorem 4.2.2, use mathematical induction to prove that:

$$1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1).$$

For all integers $n \geq 1$.

Question 9:
Without using theorem 4.2.3, use mathematical induction to prove that: $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$.

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Question 8:

Without using theorem 4.2.2, use mathematical induction to prove that:

$$1 + 5 + 9 + \ldots\ldots\ldots\ldots\ldots + (4n - 3) = n(2n - 1).$$

For all integers $n \geq 1$.

Question 9:

Without using theorem 4.2.3, use mathematical induction to prove that: $1 + 2 + 2^2 + \ldots\ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 1$.

For all integers $n \geq 0$.

Question 10:

Prove each of the following statements by mathematical induction:

1. $1^2 + 2^2 + \ldots\ldots\ldots\ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$, for all integers $n \geq 1$.
2. $1^3 + 2^3 + \ldots\ldots\ldots\ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$, for all integers $n \geq 1$.
3. $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \ldots\ldots\ldots\ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$, for all integers $n \geq 1$.
4. $\sum_{i=1}^{n-1} i(i+1) = \dfrac{n(n-1)(n+1)}{3}$, for all integers $n \geq 2$.

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Sheet (4)

Elementary Number Theory and Methods of Proof

Question 1:

Assume that r and s are particular integers. Justify your answers to each of the following:

a) Is 6rs even?

b) Is $6r + 8s2 + 7$ odd?

c) If r and s are both positive, is $r2 + 2rs + s2$ composite?

Question 2:

Prove the following statements:

a) There is an integer $n > 5$ such that $2n - 1$ is prime.

b) There are integers m and n such that $m > 1$ and $n > 1$ and is an integer $1/m + 1/n$

c) There are positive integers the sum of whose reciprocals is an integer.

d) There are real numbers a and b such that

Question 3:

Prove the statements that are true and give counterexamples to disprove those that are false.

a) For all integers m and n, if $2m + n$ is odd then m and n are both odd.

b) The product of any two odd integers is odd.

c) The sum of any even and any odd integer is odd.

d) The difference of any even integer minus any odd integer is odd.

a) The product of any even integer and any integer is even.

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Question 3:

Prove the statements that are true and give counterexamples to disprove those that are false.

a) For all integers m and n, if 2m + n is odd then m and n are both odd.

b) The product of any two odd integers is odd.

c) The sum of any even and any odd integer is odd.

d) The difference of any even integer minus any odd integer is odd.

a) The product of any even integer and any integer is even.

b) The difference of any two even integers is even.

Question 4:

If today is Tuesday, what day of the week will it be 1,000 days from today?

Question 5:

If m, n, and d are integers and m mod d = n mod d, does it necessarily follow that m = n? That m − n is divisible by d? Prove your answers.

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Sheet (3)

The Logic of Quantified Statements

Question 1:

Let R (m, n) be the predicate "If m is a factor of n2 then m is a factor of n," with domain for both m and n being the set Z of integers.

(a) Explain why R(m, n) is false if m = 25 and n = 10.
(b) Give values different from those in part (a) for which R(m, n) is false.
(c) Explain why R(m, n) is true if m = 5 and n = 10.
(d) Give values different from those in part (c) for which R(m, n) is true.

Question 2:

Find the truth set of each predicate.

(a) predicate: 6/d is an integer, domain: Z
(b) predicate: 6/d is an integer, domain: Z+
(c) predicate: $1 \leq x2 \leq 4$, domain: R
(d) predicate: $1 \leq x2 \leq 4$, domain: Z

Question 3:

Let R be the domain of the predicate variables a, b, c, and d. Which of the following are true and which are false? Give counterexamples for the statements that are false.

(a) $a > 0$ and $b > 0 \Rightarrow ab > 0$
(b) $a < 0$ and $b < 0 \Rightarrow ab < 0$

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(a) $a > 0$ and $b > 0 \Rightarrow ab > 0$
(b) $a < 0$ and $b < 0 \Rightarrow ab < 0$
(c) $ab = 0 \Rightarrow a = 0$ or $b = 0$
(d) $a < b$ and $c < d \Rightarrow ac < bd$

Question 4:
Find Counterexamples to show that the following statements are false :

(a) $\forall$ positive integers m and n, $m \,.\, n \geq m + n$.
(b) $\forall$ real numbers x and y, $x + y = x + y$.
(c) $\forall x \in R, x > 1/x$.
(d) $\forall a \in Z, (a - 1) / a$ is not an integer.

Question 5:
Write negations for each of the following statements:

(a) $\forall$ fish x, x has gills.
(b) $\forall$ computers c, c has a CPU.
(c) a movie m such that m is over 6 hours long.
(d) a band b such that b has won at least 10 Grammy awards.

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diagrams.

(a) All dogs are carnivorous.
Felix is not a dog.
∴ Felix is not carnivorous.

(b) All people are mice.
All mice are mortal.
∴ All people are mortal.

(c) All honest people pay their taxes.
Darth is not honest.
∴ Darth does not pay his taxes.

(d) All discrete mathematics students can tell a valid argument from an invalid one.
All thoughtful people can tell a valid argument from an invalid one.
∴ All discrete mathematics students are thoughtful

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4

If Today is Tuesday, what day of the week Will it be 1,000 days from today?

6 → days

⇒ Monday.

Saturday
Sunday
Monday
Tuesday
Wensday
thursday
Fraiday

5 Report.

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(d) the difference of any even integer minus any odd → is odd.

$m = 2k$ even. $\quad m - n = 2k - (2k+1)$

$n = 2k+1$ odd. $\quad = \cancel{2k} - \cancel{2k} - 1 = -1$

True.

(e) The product of any even integer and any integer is even.

$m = 2k$ Case 1: $m \cdot n = 2k \cdot 2k = 4k^2 = 2\underbrace{(2k^2)}_{k}$

$n = 2k$ even.

$m = 2k$ Case 2: $m \cdot n = 2k \cdot (2k+1) = 4k^2 + 2k$

$n = 2k+1$ $= 2\underbrace{(2k^2 + k)}_{k}$

even.

True.

(f) the difference of any two even integers is even.

$m = 2k$ $\quad m - n = 2k - 2S = 2\underbrace{(k - S)}_{k} = 2k$

$n = 2S$

even.

True.

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ⓒ Report

ⓓ $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

assume that $a=0$ and $b=9$.

$\sqrt{0+9} = \sqrt{0} + \sqrt{9} = \pm 3$

$\boxed{3}$

ⓐ $2m+n \rightarrow$ odd ∴ m, n is also odd .

false if ~~m~~ $2m+n = 5$ odd

even $\leftarrow m=2$ $n=1 \rightarrow$ odd .

ⓑ The product of any two odd $\rightarrow$ is odd

True ∴ $m = 2k+1$ ∴ $m \cdot n = (2k+1)(2k+1)$

$n = 2k+1$

$= 4k^2 + 4k + 1 = 2\underbrace{(2k^2+2k)}_{k}+1$

$2k+1 \rightarrow$ odd .

ⓒ the Sum of any even and any odd is $\rightarrow$ odd .

$m = 2k+1$ odd . $2k+1+2k = 4k+1$

$n = 2k$ even

true . $\underbrace{2(2k)+1}_{\text{odd}}$.

$\boxed{2}$

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sheet 4

$\boxed{1}$

(a) $6rs = 2(\underbrace{3rs}_{k}) = 2k$ even.

yes.

(b) $6r + 8s^2 + 7 \quad 2(\underbrace{3r + 4s^2 + 3}_{k}) + 1$

$= 2k + 1$ odd

yes.

(c) $r^2 + 2rs + s^2 = (r+s)(r+s)$

Report

$\boxed{2}$ (a) $n > 5$

$2n - 1 \longrightarrow$ is prime.

assume $n = 6$ $\therefore$ $2n - 1 = 12 - 1 = 11 \rightarrow$ Prime.

(b) $m > 1$ & $n > 1$

$\frac{1}{m} + \frac{1}{n} \rightarrow$ integer

assume $n = 2$ and $n = 2$ $\therefore \frac{1}{2} + \frac{1}{2} = \underline{\underline{1}}$

$\boxed{1}$

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Subject................. Date.................

(c)

All honest People Pay their taxes.
Darth isn't honest.
∴ Darth does not Pay his taxes.

this is invalid

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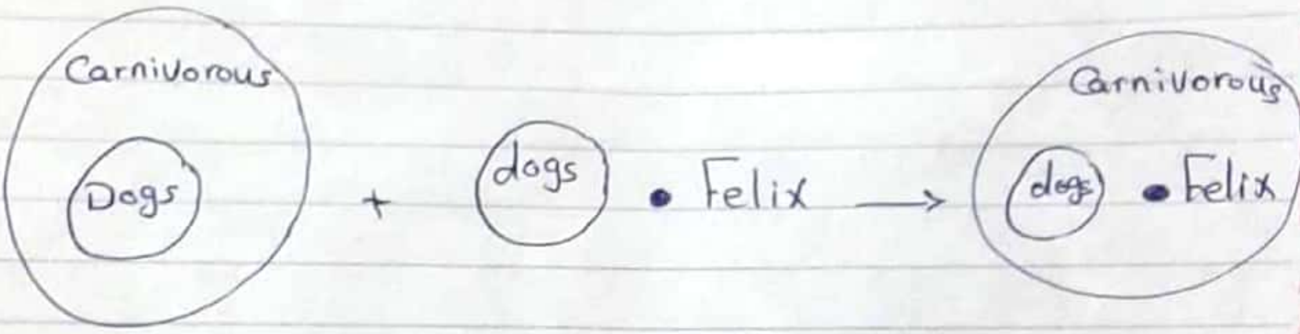
(d) $\exists$ a band b such that b has won at least 10 grammy awards.

* $\forall$ bands b, b has won fewer than 10 grammy awards.

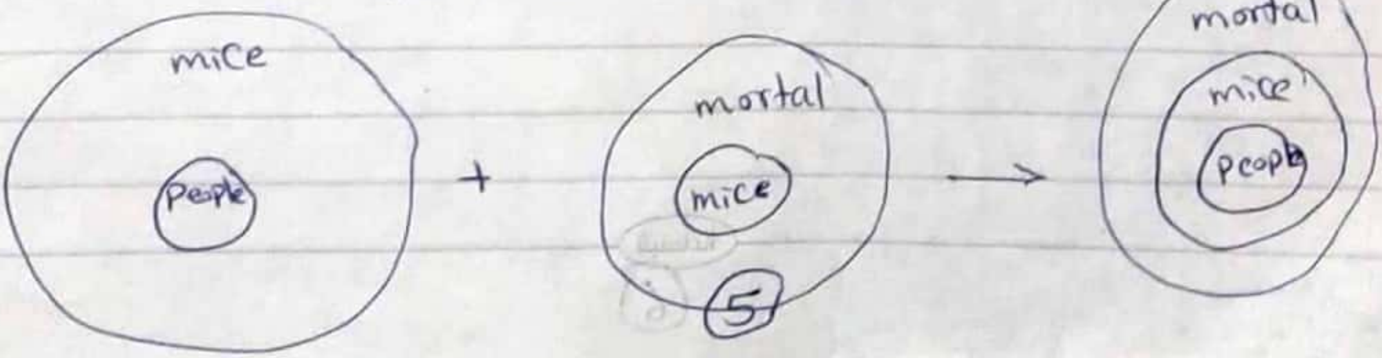
Question 5:

(a) All dogs are Carnivorous.
Felix is not a dog.
$\therefore$ Felix is not Carnivorous.



invalid $\Longrightarrow$ the premises are true and the Conclusion is false.

(b) All People are mice   valid.
All mice are mortal
$\therefore$ All People are mortal



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ⓒ $\forall x \in \mathbb{R}$, $x > 1/x$

∴ let $x = -2$ ∴ $-2 \not> \frac{-1}{2}$

ⓓ $\forall a \in \mathbb{Z}$ $(a-1)/a$ isn't an integer.

Counterexample:

let $a = 1$ ∴ $\frac{a-1}{a} = \frac{1-1}{1} = \frac{0}{1} = 0$

Question 5:

ⓐ $\forall$ fish x , x has gills.

negation:

$\exists$ a fish x such that x dosen't have gills.

∴ $\exists$ fish x , x has no gills.

ⓑ $\forall$ Computer c , c has a cpu.

$\exists$ Computer c, c has no cpu.

ⓒ $\exists$ a movie m such that m is over 6 hours long.

$\forall$ movies m, m is less than or equal to 6 hours long.

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Page :
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(d) $a<b$ and $c<d \Rightarrow ac<bd$

The statement is false.

Counterexample.

let $a=-2$, $b=2$, $c=-3$ and $d=3$

then $a*c < b*d = (-2)(-3) < (2)(3) = 6 \not< 6$

Questions 4

(a) $\forall$ Positive integers m and n, $m.n \geq m+n$

Counterexample.

let $m=2$ and $n=1$ $\therefore$ $2*1 \geq 2+1$

$2 \not\geq 3$

(b) $\forall$ Real numbers x and y, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$

Counterexample.

let $x=16$ and $y=9$.

$\therefore \sqrt{x+y} = \sqrt{16+9} = \sqrt{25} = \pm 5$.

and $\sqrt{x} = \sqrt{16} = \pm 4$

and $\sqrt{y} = \sqrt{9} = \pm 3$ $\therefore$ $\sqrt{16+9} \neq \sqrt{16} + \sqrt{9}$.

KHUFU

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ⓒ Predicate : $1 \le x^2 \le 4$, domain : R

Domain : R $x \in R \mid 1 \le x^2 \le 4$

$= [-2,-1] \cup [1,2]$

ⓓ predicate : $1 \le x^2 \le 4$, domain : Z

Domain : Z $x \in Z \mid 1 \le x^2 \le 4$

$= \{-2,-1,1,2\}$

Question 3 :

ⓐ $a > 0$ and $b > 0 \Rightarrow ab > 0$

The statement is true . Since a and b are Positive .

ⓑ $a < 0$ and $b < 0 \Rightarrow ab < 0$

the Statement is false

$\boxed{ex}$:

let $a = -5$ (<0) and $b = -6$ (<0)
Then $ab = (-5)(-6) = 30 > 0$
30 isn't less than 0 .

ⓒ $ab = 0 \Rightarrow a = 0$ or $b = 0$

The Statement is true

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Question 1: sheet 3

(a)

for $m = 25$ and $n = 10$

* m is a factor of n^2 (100) but it isn't a factor of 10

(b) $m = 4$, $n = 6$

(c) for $m = 5$ and $n = 10$ m is a factor of n^2 (100) and it is also a factor of 10.

(d) $m = 2$, $n = 4$.

Question 2:

(a) predicate $\frac{6}{d}$ is an integer, domain $\mathbb{Z}$

Domain : $\mathbb{Z} \{x \in \mathbb{Z} \mid 6x \in \mathbb{Z}\}$

$= \{-6, -3, -2, -1, 1, 2, 3, 6\}$

(b) predicate $\frac{6}{d}$ is an integer, domain $\mathbb{Z}^+$.

Domain : $N \{x \in N \mid 6x \in \mathbb{Z}\}$

$= \{1, 2, 3, 6\}$.

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