FOURTH SEMESTER EXAMINATION - 2011

DISCRETE MATHEMATICS

1.(a) State De Morgan's Law and proves it by Truth table.

Ans. The Demorgan's law is $(A+B)' = A' \cdot B'$

$(A+B)' = A' + B'$

The truth table is

A	B	A'	B'	A + B	(A + B)'	A' − B'
0	0	1	1	0	1	1
1	0	0	1	1	0	0
0	1	1	0	1	0	0
1	1	0	0	1	0	0

Similarly

A	B	A − B	(A · B)'	A'	B'	A' + B'
0	0	0	1	1	1	1
1	0	0	1	0	1	1
0	1	0	1	1	0	1
1	1	1	0	0	0	0

(b) Express the statement "The firewall is in a diagnostic state only if the proxy server is in a diagnostic state" using predicates, quantifiers, and logical connectives.

Ans. Let p : The fire wall is in a diagnostic state only.

q : Proxy server is in a diagnostic state.

Ans. $q \Rightarrow p$

(c) Give a proof by contradiction of the 'If 3n+2 is odd, then n is odd'.

Ans. By contradiction let n is even i.e. n =2k we have to show 3n + 2 is not odd.

Since n = 2k

So $3n + 2 = 6k + 2 = 2(3k + 1)$ is even

Which contradicts to the fact that 3n+2 is odd.

So we have the proof if 3n+2 is odd then n is odd.

(d) Find R^2, where R = {(1, 2), (2, 2),(2, 4), (4, 1), (3, 2)} is a relation.

Ans. R = {(1, 2), (2, 2), (2, 4), (4, 1), (3, 2)}

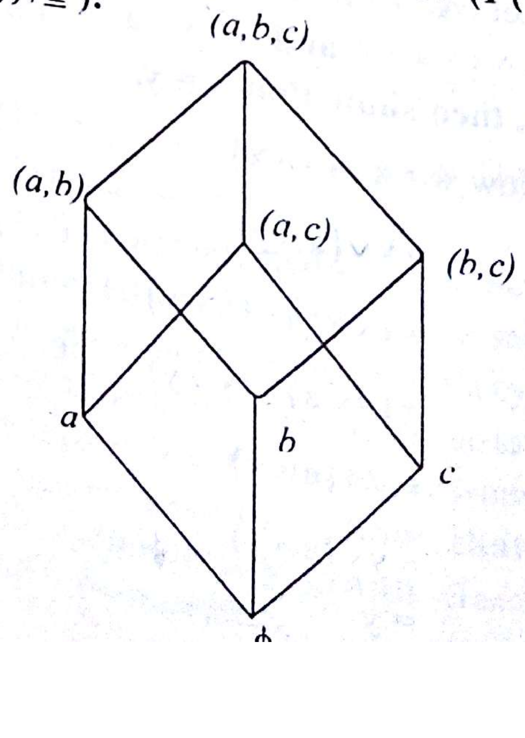
$R^2 = R \times R$ = {(1, 2), (1,4), (2,2), (2,4), (2,1), (4, 2), (3, 2), (3, 4)}

(e) Find the value of extended Binomial coefficient of $\binom{-1}{3}$.

Ans. $\binom{-1}{3} = \frac{-1.....2.....(-1-3+1)}{3!} = \frac{-1\times2\times3}{3!}$

$= \frac{-6}{6} = -1$

(f) Draw Hasse diagram of (P({a, b, c}), ⊆).



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(g) **Let the recurrence relation $a_n - 5a_{n-1} = 2a_{n-2}$, where $a_0 = 1$ and $a_1 = 2$, then find a_5.**

Ans. The given recurrence relation is

$a_n - 5a_{n-1} = 2a_{n-2}$

If $n = 5, a_5 - 5a_4 = 2a_3 a_4 - 5a_3 = 2a_2$

$a_3 - 5a_2 = 2a_1$ $\quad a_2 = 12, a_3 = 64$

$a_2 - 5a_1 = 2a_0$ $\quad a_4 = 344 a_5 = 1848$

(h) **What is the generating function for the sequence 1,1,1,1,1?**

Ans. Generating function for the sequence 1,1,1,1,1 is

$$a(x) = 1 + x + x^2 + x^3 + x^4 = \sum_{n=0}^{4} a_n x^n$$

where $a_n = 1$ since $a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$

(i) **What is a complete binary tree?**

Ans. A binary tree is said to be complete if every internal nodes has exactly two children except the leaf node.

(j) **In any bounded distributive lattice if an element has a complement, then that complement is unique.**

Ans. Let an element 'a' has two complement let it b and c say

So $a \vee b = 1$ $\quad a \wedge b = 0$

$a \vee c = 1$ $\quad a \wedge c = 0$

Now $b = b \wedge 1 = b \wedge (a \vee c)$

$= (b \wedge a) \vee (b \wedge c)$

$= 0 \vee (b \wedge c) = (a \wedge c) \vee (b \wedge c)$

$= (a \wedge b) \wedge c = 1 \wedge c = c$

so $b = c$

Hence the complement is unique.

2.(a) **Show that $(p \to q) \wedge (q \to r)$ and $q \to (q \wedge r)$ are logically equivalent.**

Ans. To show by the help of truth table.

The truth value of $(p \to q) \wedge (q \to r)$ and $q \to (q \wedge r)$ must be same.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$q \wedge r$	$q \to (q \wedge r)$
T	T	T	T	T	T	T	T
T	F	T	F	T	F	F	F

Similarly proceed accordingly for different cases of p.q.r.

(b) **Show that $3 + 3(5) + 3(5)^2 + ... + 3(5^n) = \dfrac{3(5^{n+1} - 1)}{4}$ by method of induction, where n is a positive integer.**

Ans. Let $p_n : 3 + 3(5) + 3(5)^2 + ... + 3(5^n)$

$$= \frac{3(5^{n+1} - 1)}{4}$$

For $n = 0$, LHS of $p_0 = 3$

RHS of $p_0 = \dfrac{3(5-1)}{4} = 3$

Let p_k is true

So $3 + 3.5 + 3.5^2 + ... + 3.5^k = \dfrac{3(5^{k+1} - 1)}{4}$

To verify p_{k+1} is true

To show $3 + 3.5 + 3.5^2 + ... + 3.5 + 3.5^{k+1}$

$$= \frac{3(5^{k+2} - 1)}{4}$$

LHS and p_{k+1}

$3 + 3.5 + 3.5^2 + ... + 3.5^k + 3.5^{k+1}$

$= \dfrac{3(5^{k+1} - 1)}{4} + 3.5^{k+1}$

$= \dfrac{15.5^k - 3}{4} + 15.5^k = \dfrac{75.5^k - 3}{4}$

$$= \frac{25 \cdot 3 \cdot 5^k - 3}{4} = \frac{3 \cdot 5^{k+2} - 3}{4} = \frac{3(5^{k+2} - 1)}{4}$$

$= \text{RHS of } P_{k+1}$

Since p_{k+1} is true, so p_n is true for all n.

Hence the proof follows.

3.(a) Solve the recurrence relation.

$a_n - 4a_{n-2} = n3^n$ **with** $a_0 = 1, a_1 = 0$

Ans. Let $a_n = r^n$ be the solution

Hence $r^n - 4r^{n-2} = 0$

$r^2 - 4 = 0$ since $r^n \neq 0$

So $r = \pm 2$

So $a_n^{(c)} = A_0 2^n + A_1(-2)^n$

Since $f(n) = n3^n$ so $a_n^{(p)} = (A_2 + A_3 n)3^n$

Now $(A_2 + A_3 n)3^n - 4A_2 \cdot 3^{n-2}$

$-4nA_3 3^{n-2} + 8A_3\, 3^{n-2} = n3^n$.

So $A_3 = 1, 8A_3 = -4A_2 = 0$

So $4A_2 = 8A_2 = 2$

$a_n = a_n^c + a_n^p = A_0 2^n + A_1(-2)^n + (2+n)3^n$

Since $a_0 = 1$ $1 = A_0 + A_1 + 2A_0 + A_1 = 1$

$a_1 = 0$ $0 = 2A_0 - 2A_1 + 9A_0 - A_1 = -\frac{9}{2}$

Solving we obtain A_0 and A_1.

(b) How many positive integers not exceeding 1000 are divisible by 7 or 11 or 13 by principle of inclusion-exclusion?

Ans. Let A be the number of integer divisible by 7

B be the number of integer divisible by 11

C be the number of integer divisible by 13

So $|A| = \left|\frac{1000}{7}\right| = 142$

$|B| = \left|\frac{1000}{11}\right| = 90$, $|C| = \left|\frac{1000}{13}\right| = 76$

$|A \cap B| = \left|\frac{1000}{7 \times 11}\right| = 12$

$|A \cap C| = \left|\frac{1000}{7 \times 13}\right| = 10$

$|B \cap C| = \left|\frac{1000}{11 \times 13}\right| = 6$

$|A \cap B \cap C| = \left|\frac{1000}{7 \times 11 \times 13}\right| = 0$

The number of integers divisible

So the positive integers not exceeding 1000 are divisible by 7 or 11 or 13 is

$|A \cup B \cup C| = 142 + 90 + 76 - 12 - 10$

$-6 + 0 = 280$

So there are 280 positive integer n = f exceeding 100 are divisible b 7 or 11 or 13.

4.(a) Use generating function to solve the recurrence relation

$a_n - 2a_{n-1} = 3^n, a_0 = 1$

Ans. Let $a(x) = \Sigma a_n x^n$ where $a(x)$ is the generating function for the sequence $|a_n|$. Multiply each term in the given recurrence relation by x^n and summing from 1 to ∞.

$$\sum_{n=1}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} 3^n x^n$$

$$\Rightarrow (a(x) - a_0) - 2xa(x) = \left(\frac{1}{1-3x} - 1\right)$$

Since $xa(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$

So $a(x) - 2x\,a(x) = a_0 + \frac{3x}{1-3x}$

$$a(x)(1-2x)=1+\frac{3x}{1-3x}=\frac{1}{1-3x}$$

$$a(x)=\frac{1}{(1-2x)(1-3x)}=\frac{-2}{1-2x}+\frac{3}{1-3x}$$

$$\sum_{n=0}^{\infty}a_n x^n=-2\sum_{n=0}^{\infty}2^n x^n+3\sum_{n=0}^{\infty}3^n x^n$$

$$\Rightarrow a_n=-2\cdot 2^n+3^n$$

$$\Rightarrow a_n=3^n-2\cdot 2^n=3^n-2^{n+1}$$

(b) If G is simple connected graph with sum of degree of vertices $n \geq 3$, and total number of edges is greater than equal to (C(n–1,2)+2), then G is Hamiltonian.

Ans. If is given G is a simple connected graph with sum of degree of vertices $n \geq 3$ and total number of edges

$m \geq (c(n-1,2)+2)$

Now $C(n-1,2)+2=\frac{(n-1)!}{2!(n-3)!}+2$

$$=\frac{(n-1)(n-2)}{2}+2=\frac{(n-1)(n-2)+4}{2}$$

$$=\frac{n^2-3n+6}{2}$$

So $m \geq \frac{n^2-3n+6}{2}$

Let U and V be two vertices of G which are not adjacent.

Let H be the subgraph of G obtained by removing U and V and the edges incident and U and V.

The number of edges in H are m-deg u-deg V

The maximum number of edges H can have is ${}^{n-2}C_2=n^2-5n+6$

So $m-\deg u-\deg V \leq 1/2(n^2-5n+6)$

$$\Rightarrow \deg U+\deg V \geq m-\tfrac{1}{2}(n^2-5n+6)$$

$$\Rightarrow \deg U+\deg V \geq \tfrac{1}{2}(n^2-3n+6)$$

$$-\tfrac{1}{2}(n^2-5n+6)=n$$

So G has a hamittonian circuit since for any two vertices U and V of G which are not adjacent. deg U + deg V $\geq$ n

5.(a) State and Prove Handshaking Theorem and draw a simple graph with degree of vertices are {3,4,2,3,6,4,2}.

Ans. Handshaking Theorem :

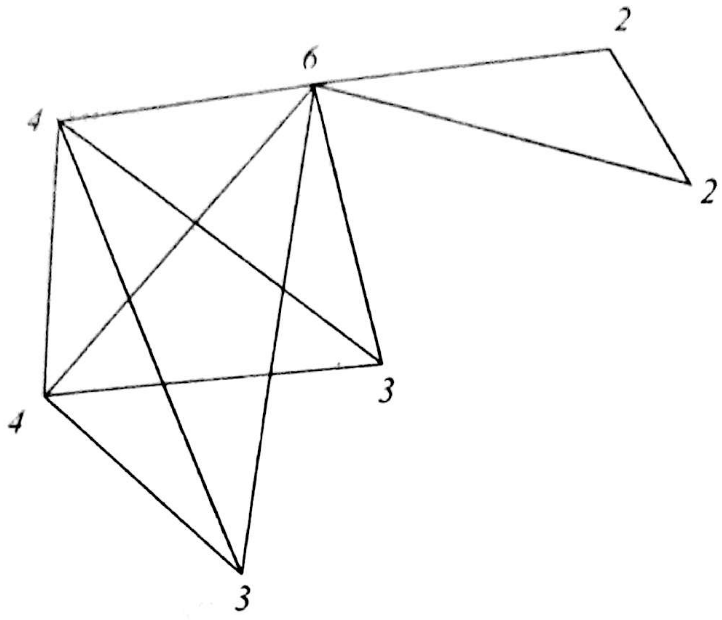
Let G be an undirected graph with |E| edges and |V| = n vertices then

$$\sum_{i=1}^{n}\deg(V_i)=2|E|$$

Proof : Let G be graph |E| edges and n vertices V_1, V_2 –V_n. When we sum over the degrees of all the vertices, we count each edge (V_i, V_j) twice : Once when we count it as (V_i, V_j) in the degree of V_i and again we count it as (V_j, V_i) in the degree of V_j. Then the conclusion follows

$$\sum_{i=1}^{n}\deg(V_i)=2|E|$$

Draw a simple graph with degrees of vertices are {3,4,2,3,6,4,2}



(b) Write an algorithm to find out shortest of path using Warshall's algorithm.

Ans. Warshall's algorithm to find the path matrix from A(G)

Given the adjancy matrix A(G) of a simple digraph, the following steps produce.

Step -1 $P^{(0)} = A$

Step -2 K = 1

Step -3 i = 1

Step -4 $p_{ij}^{(K)} = P_{ij}^{(K-1)} \vee \left(P_{ik}^{(K-1)} \wedge P_{kj}^{(K-1)}\right)$

Step -5 $i = i+1$ if $i \le n$ go to stop 4

Step -6 K = K + 1 if $K \le n$ go to step 3 otherwise stop.

Warshall's Algorithm can be nodified further to obtain a matrix which given the lengths of shortest paths between the vertices.

For this purpose let A is the adjancy matrix of the graph. Replace all those elements of A which are zero by ∞.

Following algorithm shows.

Start with the adjancy matrix A(G) replace the zero elements in the adjancy matrix A by ∞ or by some very large number. Denote this matrix by M

Step -1 $C^{(0)} = M$

Step -2 K = 1

Step -3 i = 1

Step -4 $C_{ij}^{(K)} = \text{Min}\left\{C_{ij}^{K-1}, C_{ik}^{K-1} + C_{kj}^{K-1}\right\}$

Step -5 $i = i+1$ if $i \le n$ go to step 4

Step -6 K = K + 1 if $K \le n$ go to step 3 otherwise stop.

6.(a) Prove that a simple graph G with n vertices is connected if it has more than $\frac{(n-1)(n-2)}{2}$ 2 edges.

Ans. Consider a simple graph of n vertices choose n–1 vertices $V_1, V_2, \ldots V_{n-1}$ of G. So the maximum number of edges only can drawn between these vertices is

$^{n-1}C_2 = \frac{(n-1)(n-2)}{2}$ so if we have more than $\frac{(n-1)(n-2)}{2}$ edges atleast one edge should be drawn between the nth vertex V_n to some vertex V_i, $1 \le i \le n-1$ of G. Hence G must be connected.

(b) If (G, *) is a group, then $(a^{-1})^{-1} = a$, and $(ab^{-i}) = b^{-1}a^{-1}$ for all a, b $\in$ G

Ans. We know $\left(a^{-1}\right)^{-1} * a^{-1} = c$ and $a * a^{-1} = c$

So $\left(a^{-1}\right)^{-1} * a^{-1} = a * a^{-1}$

$\Rightarrow \left(a^{-1}\right)^{-1} = a$

Let $a, b \in G$ so we have $a * b \in G$

So $(a*b)*(a*b)^{-1} = e$ ....(1)

Since $a^{-1}, b^{-1} \in G$ so

$(b^{-1}*a^{-1})(a*b) = b^{-1}*(a^{-1}*a)*b$

$= b^{-1}*e*b = e$

So $(b^{-1}*a^{-1})(a*b) = e$ ...(2)

From (1) and (2) we have

$(a*b)^{-1}*(a*b) = (b^{-1}*a^{-1})(a*b)$

$\Rightarrow (a*b)^{-1} = b^{-1}*a^{-1}$ i.e., $(ab)^{-1} = b^{-1}a^{-1}$

7.(a) Write the Boolean expression in both disjunctive and conjunctive normal forms over the two valued Boolean lattice where Boolean expression

$E(x,y,z) = (\overline{x \wedge y}) \vee (z \wedge \bar{y}) \vee (x \wedge z)$

Ans. Now it is given.

$E(x,y,z) = (xy)' + zy' + xz$

DNF : (sum of the product)

$(xy)'(z+z') + zy'(x+x') + xz(y+y')$

$= (xy)'z + (xy)'z' + xy'z + x'y'z + xyz + xy'z$

CNF : Conjuctive normal form

(Product of the sum)

$(xy)' + zy' + xz = x' + y' + zy' + xz$

$= (x'+z)(x'+y') + (y'+x)(y'+z)$

$= ((x'+z)(x'+y') + y'+x)$

$= ((x'+z)(x'+y') + y'+z)$

Using the rule

$a + b \cdot c = (a+b) \cdot (a+c)$

(b) If G is a group, and N is a normal subgroup of G then $G_n = \{xN \mid x \in G \text{ and } xN \cap yN = \phi\}$ is a group under the binary operation coset product.

Ans. Since $G_n = \{xN \mid x \in G \text{ and } xN \cap yN = \phi\}$

Binary operation is the coset product.

Let $xN, yN \in G_n$

So $xN \cdot yN = xyN$ since product of two right coset is again a right coset and $xy \in G$.

So closure axiom is verified.

Let $xN \cdot yN \in G_n$

So $xN \cdot yN = xyN = Nxy = Nyx$

Since N is normal subgroup so every left coset is right coset.

$= Ny \cdot Nx$

So commulative axiom is verified since closure are and commutative axioms are well satisfied so associative axiom will be also verified.

Since $x \in G$ and G is a group, letting x = e we have $eN \in G_n$ as the identity element, because $xN.eN = (xe)N = xN$

Identity axiom is verified.

As $x \in G$ so $x^{-1} \in G$.

$x^{-1}N \in G_n$ and

$(xN)(x^{-1}N) = (xx^{-1})N = eN$

the identity element.

So inverse axiom is verified.

Hence G_n is a group under the binary operation coset product.

8.(a) A semigroup (G, *) is a group if and only if for a given $a \in G$ there exist a unique $b \in G$ s.t. a*b*a = a.

Ans. Necessary part : Let (G, *) is a semigroup so closure and associative axioms are verified under the operation*. To claim that there exist a unique $b \in G$ s.t. a*b*a = a.

Let $a*b*a \neq a$ and $a*b*a = c$

$\Rightarrow c \neq a \Rightarrow c*a \neq a*c$

$\Rightarrow (a*b*a)*a \neq a*(a*b*a)$

$a*((b*a)*a) \neq (a*(a*b))*a$

So $(a*b)*a \neq a*(a*b)$

Associative property is not true which is a contradiction so $a*b*c = a$.

Sufficient part :

Let for unique $b \in G$ we have

$a*b*a = a$

To claim that G is a semigroup.

Let $a, b \in G$ So $a*b*a = a$

It says that a*b*G otherwise the restriction of binary operation is not satisfied.

So closure axiom is verified.

Since closure axiom is verified so associative axiom is verified.

So G is semigroup.

(b) If $(B, \vee, \wedge, -)$ is a Boolean lattice, then $\overline{x \wedge y} = \bar{x} \vee \bar{y}$ and $\overline{x \vee y} = x \vee y$ for any x, $y \in B$.

Ans. If $(B, \vee, \wedge, -)$ is a Boolean lattice then for any $x, y \in B$ we have

$\overline{x \vee y} = \bar{x} \wedge \bar{y}$ and $\overline{x \wedge y} = \bar{x} \vee \bar{y}$

Proof : We have

$$(x \vee y) \vee (\bar{x} \wedge \bar{y}) = ((x \vee y) \vee \bar{x}) \wedge ((x \vee y) \vee \bar{y})$$

$$= ((x \vee \bar{x}) \vee y) \wedge (x \vee (y \vee \bar{y}))$$

$$= 1 \wedge 1 = 1$$

and $(x \vee y) \wedge (\bar{x} \wedge \bar{y}) = (x \wedge (\bar{x} \wedge \bar{y})) \vee (y \wedge (\bar{x} \wedge \bar{y}))$

$$= ((x \wedge \bar{x}) \wedge \bar{y}) \vee ((y \wedge \bar{y}) \wedge \bar{x})$$

$$= 0 \vee 0$$

$$= 0$$

So is the complement of $x \vee y$ that is $\overline{x \vee y} = \bar{x} \wedge \bar{y}$

Again $\overline{x \wedge y} = \bar{x} \vee \bar{y}$ follows from the principle of duality.

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