中山大學
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Lecture 9
Shortest Path Algorithms

Algorithm Design

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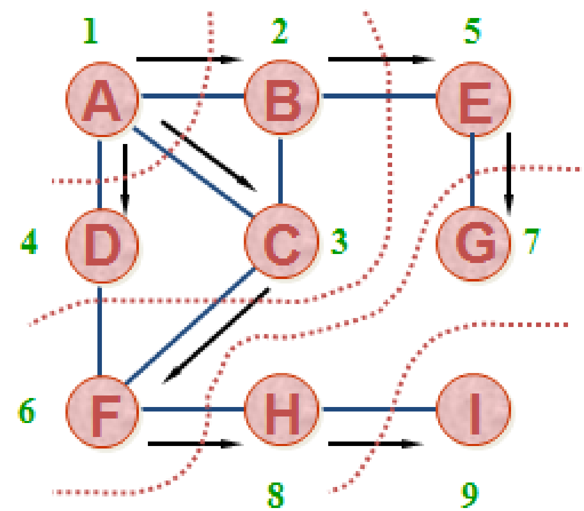
Shortest Path Algorithms

- Shortest Path Problem in Unweighted Graphs
  - Bread-First Search
- Single-Source Shortest Path Problem
  - Dijkstra algorithm
  - Bellman-Ford algorithm
- All-pairs Shortest Path Problem
  - Floyd-Warshall algorithm
- Application
  - A system of difference constraints

Breadth-first Traversal

- Breadth first traversal can be done by using a queue. The head is the vertex to be visited. When it is visited, all its adjacent vertices are put in the queue.
- Let v be the first vertex to be visited, and w_1, …, w_k be the adjacent vertices of v. We will first put v in the queue. Visit v and put w_1, …, w_k in the queue. Then visit w_1 and put the adjacent vertices of w_1 into the queue, …, until all the vertices have been visited.

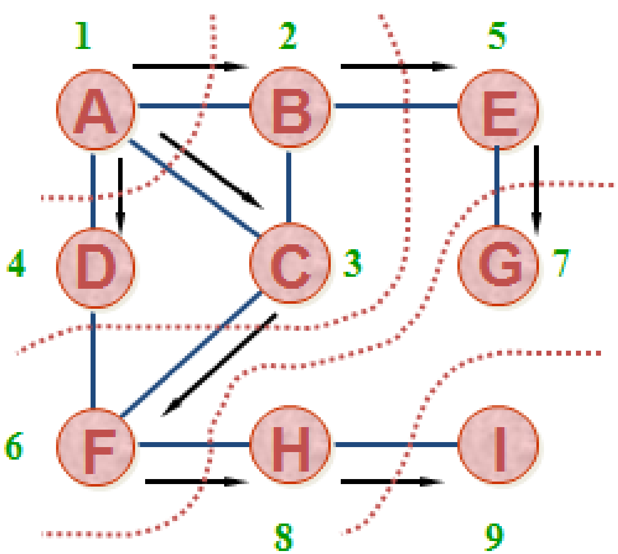


Shortest Paths in Unweighted Graph

- In BFS, vertices are discovered in the order of increasing distance from the root, so this tree has a very important property.
- The unique tree path from the root to any node $x \in V$ uses the smallest number of edges (or equivalently, intermediate nodes) possible on any root-to-x path in the graph.

Finding Paths

- The parent array set within BFS() is very useful for finding interesting paths through a graph.
- The vertex which discovered vertex i is defined as parent[i].
- The parent relation defines a tree of discovery with the the root of the tree.



vertex	A	B	C	D	E	F	G	H	I
Index	1	2	3	4	5	6	7	8	9
parent	−1	1	1	1	2	3	5	6	8

Breadth-first Search

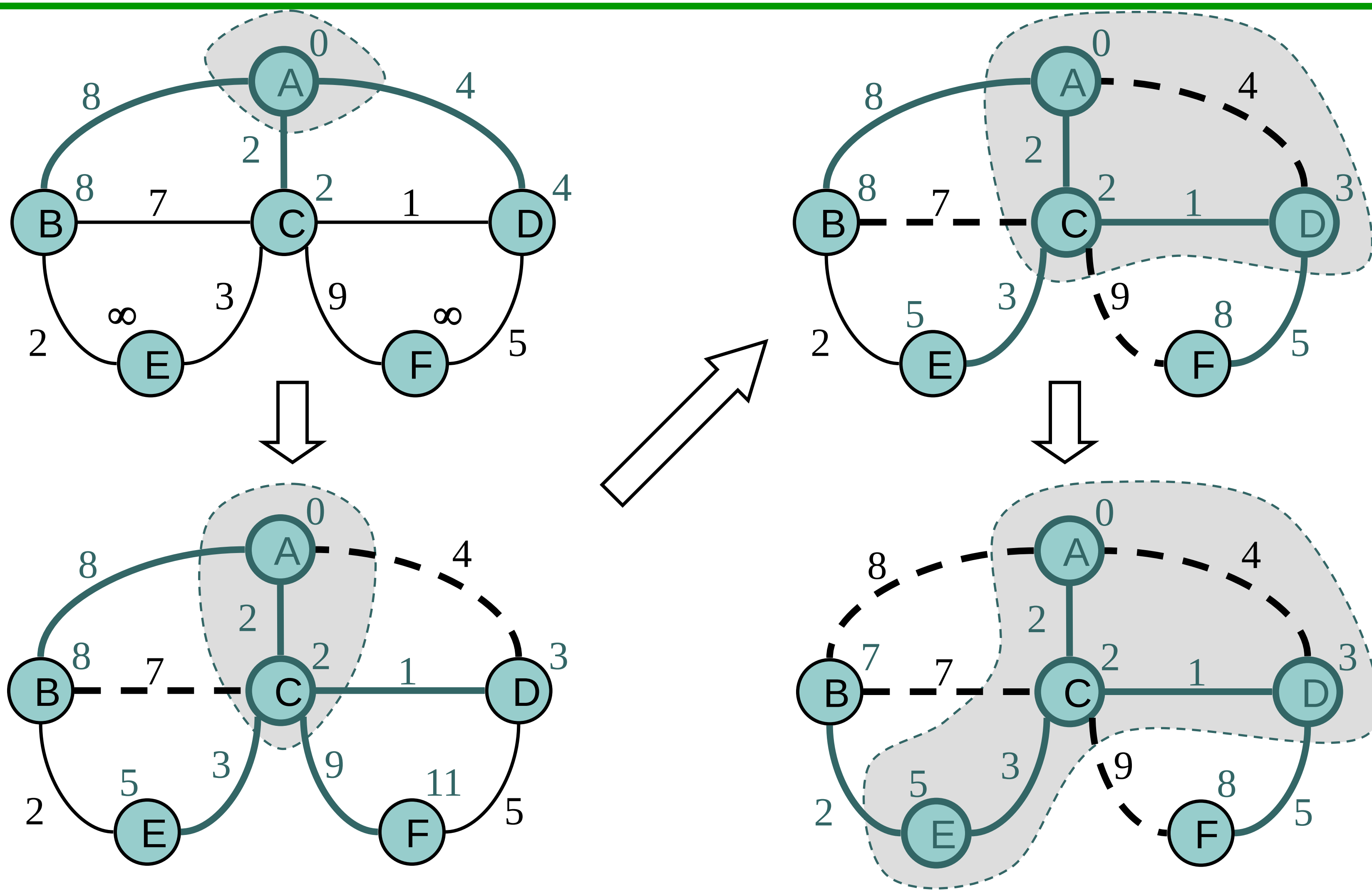
```
procedure BFS(G, v) is
    create a queue Q
    create a set V
    add v to V
    enqueue v onto Q
    while Q is not empty loop
        t ← Q.dequeue()
        if t is what we are looking for then
            return t
        end if
        for all edges e in G.adjacentEdges(t) loop
            u ← G.adjacentVertex(t, e)
            if u is not in V then
                add u to V
                enqueue u onto Q
            end if
        end loop
    end loop
    return none
end BFS
```

d[u]=d[t]+1;
parent[u]=t;

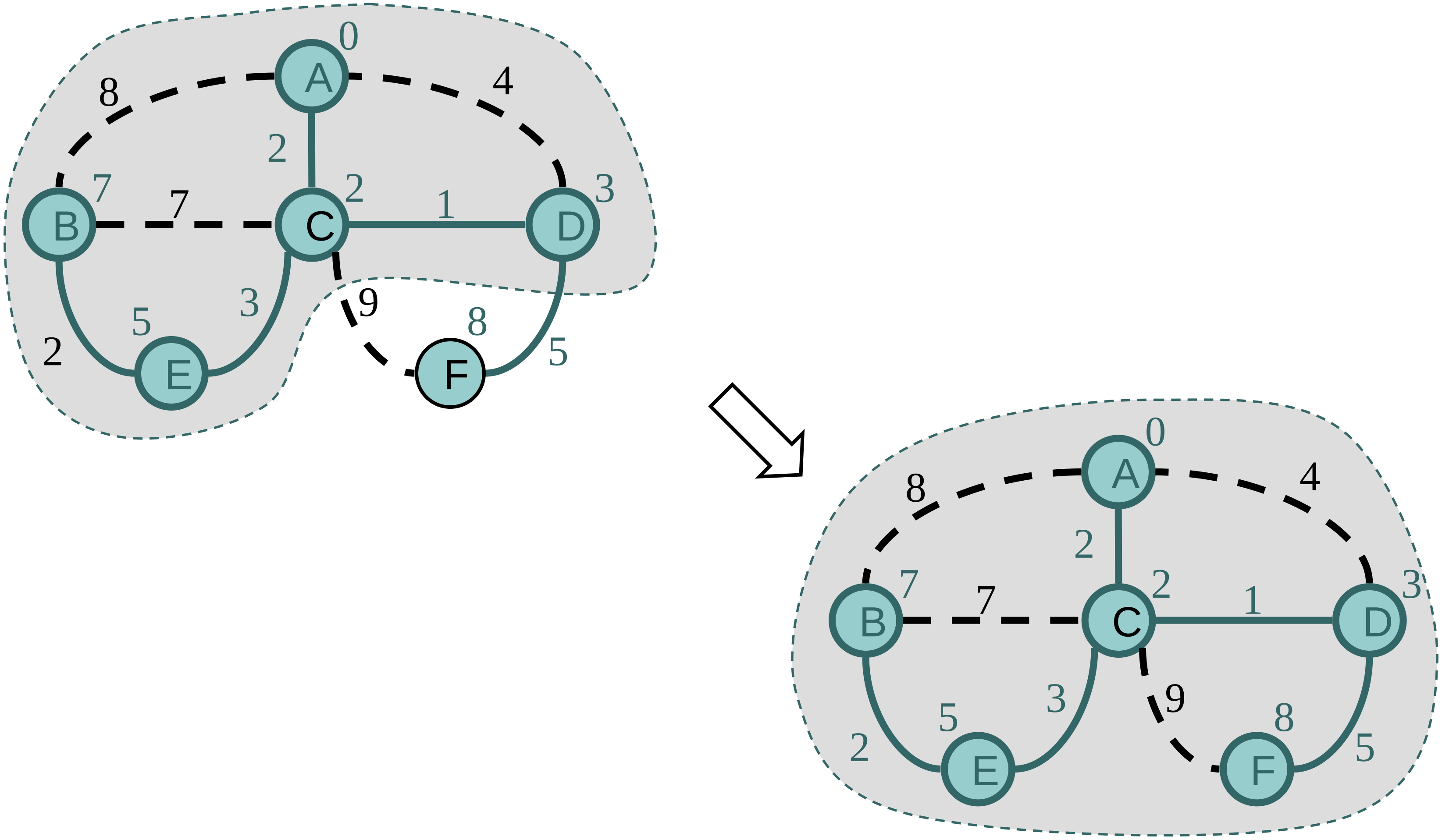
The Dijkstra's Algorithm

- Grow a "**cloud**" of vertices, beginning with s and eventually covering all the vertices.
- Store with each vertex v a label $d(v)$ representing the distance of v from s in the subgraph consisting of the cloud and its adjacent vertices.
- At each step
  - Add to the cloud the vertex u outside the cloud with the smallest distance label, $d(u)$.
  - Update the labels of the vertices adjacent to u.

Example of the Dijkstra's Algorithm



Example of the Dijkstra's Algorithm

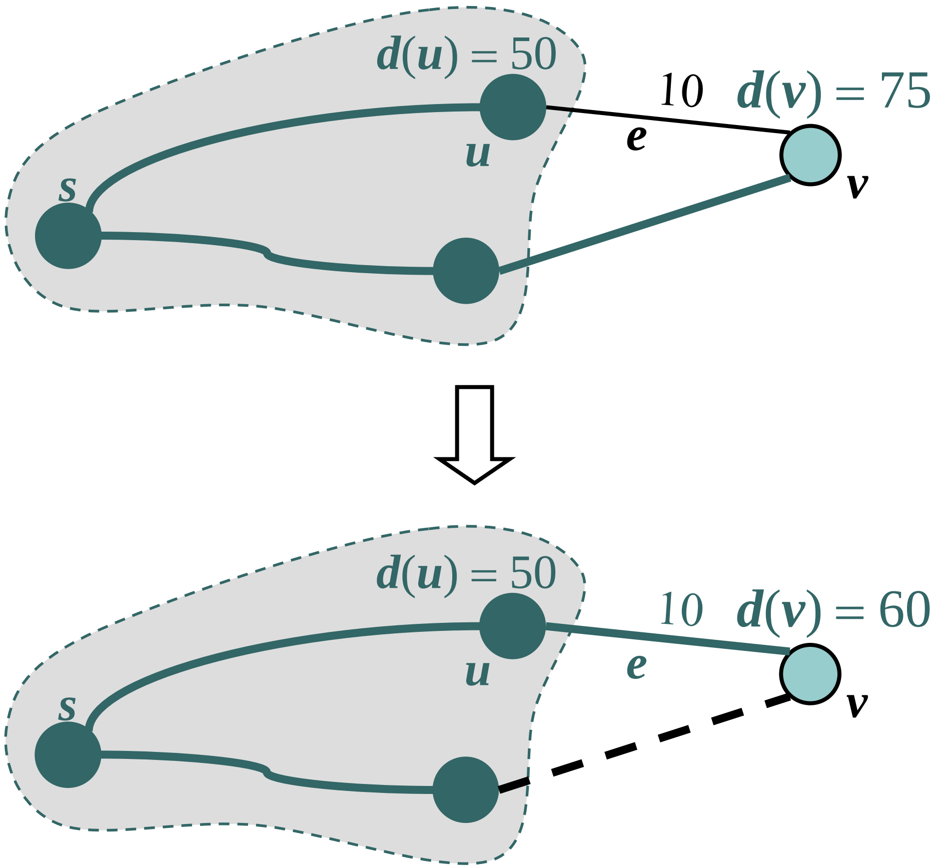


Relaxation

- Consider an edge $\boldsymbol{e} = (\boldsymbol{u},\boldsymbol{v})$ such that
  - $\boldsymbol{u}$ is the vertex most recently added to the cloud
  - $\boldsymbol{v}$ is not in the cloud
- The relaxation of edge $\boldsymbol{e}$ updates distance $\boldsymbol{d}(\boldsymbol{v})$ as follows:

Relax(u,v,w)

$\boldsymbol{d}(\boldsymbol{v}) \leftarrow \min\{\boldsymbol{d}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{u}) + \boldsymbol{w}(\boldsymbol{e})\}$



Implementation of Dijkstra's Algorithm

```
void Digraph<Weight, graph_size> :: set_distances(Vertex source,
                                                  Weight distance[ ]) const
/* Post: The array distance gives the minimal path weight from vertex source to each
         vertex of the Digraph. */
{ Vertex v, w; bool found[graph_size];  //  Vertices found in S
  for (v = 0; v < count; v++) {
    found[v] = false;
    distance[v] = adjacency[source][v];
  }
  found[source] = true;        //  Initialize with vertex source alone in the set S.
  distance[source] = 0;
  for (int i = 0; i < count; i++) {  //  Add one vertex v to S on each pass.
    Weight min = infinity;
    for (w = 0; w < count; w++) if (!found[w])
      if (distance[w] < min) {
        v = w;
        min = distance[w];
      }
    found[v] = true;
    for (w = 0; w < count; w++) if (!found[w])
      if (min + adjacency[v][w] < distance[w])
         distance[w] = min + adjacency[v][w];
  }
}
```

If there is no edge between (u,v), then adjacency[u][v]=+∞

Relaxation

Add "prev[w] = v;" here for recovering the path

Time Complexity

A Generic Single-Source Shortest Path Algorithm:

Initialize(G, s);
S := ∅;
Q := V[G];
while Q ≠ ∅ **do**
    u := Extract-Min(Q);
    S := S ∪ {u};
    for each v ∈ Adj[u] **do**
        Relax(u, v, w)
    od
od

Complexity:

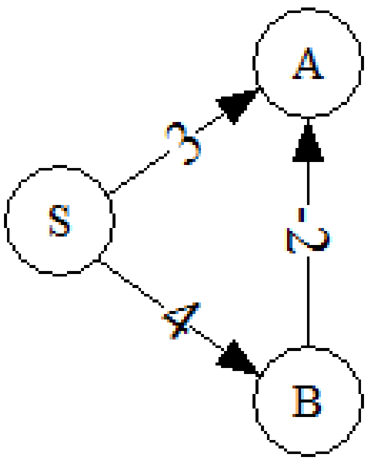
Naive implementation:
$O(V^2)$

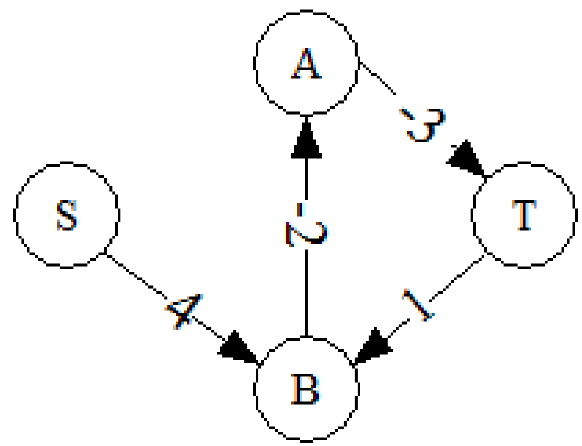
Using binary heaps:
$O((V+E) \lg V)$.

Using Fibonacci heaps:
$O(E + V \lg V)$.

Negative weight edges and cycles

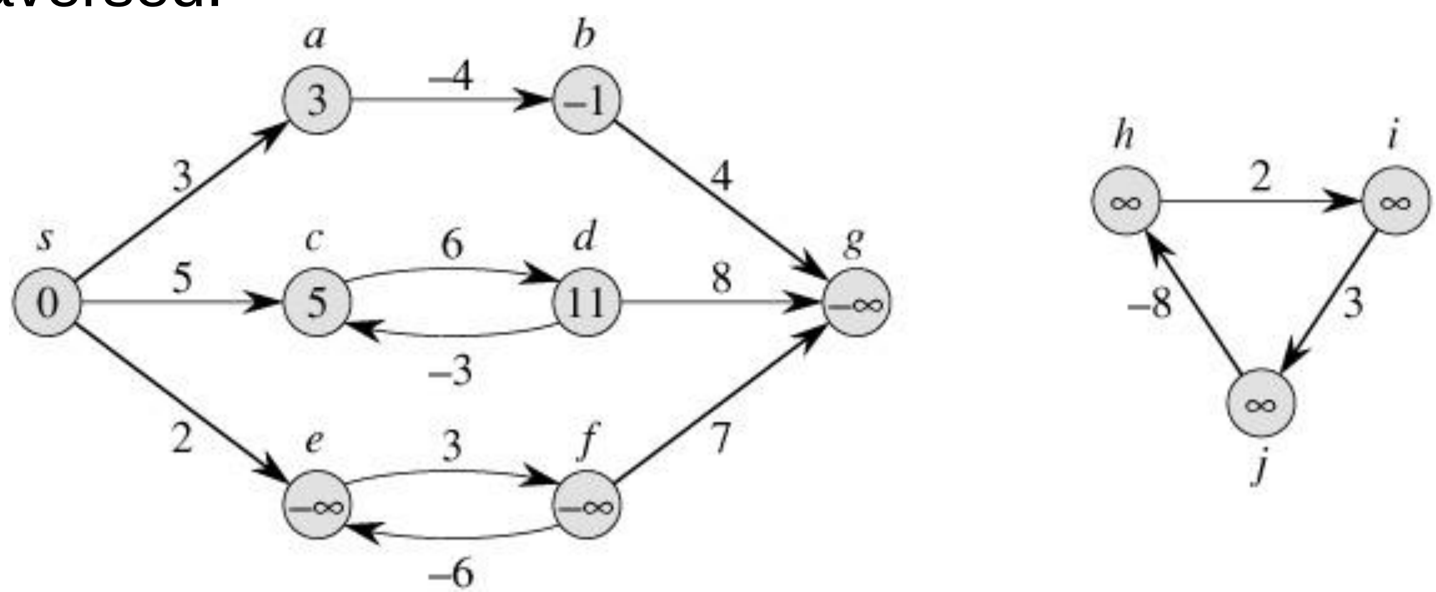
- If the graph contains negative weight edges, Dijkstra's algorithm cannot be directly used.
- Example:





Bellman–Ford algorithm

- The Bellman–Ford algorithm can be used on graphs with negative edge weights, as long as the graph contains no negative cycle reachable from the source vertex s. The presence of such cycles means there is no shortest path, since the total weight becomes lower each time the cycle is traversed.



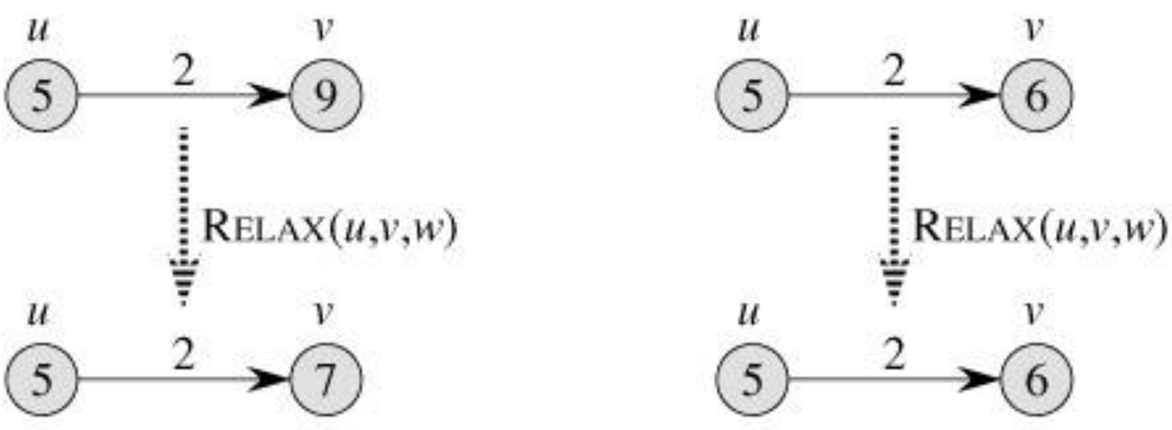
Revisit: relaxation

- d[v] is an upper bound on the weight of a shortest path from source s to v.

RELAX(u, v, w)

if $d[v] > d[u] + w(u, v)$

then $d[v] \leftarrow d[u] + w(u, v)$



(a) (b)

Important Lemma

- Path-relaxation property (Lemma 24.15):
- If $p = \langle v_0, v_1, \ldots, v_k \rangle$ is a shortest path from $s = v_0$ to v_k, and the edges of p are relaxed in the order (v_0, v_1), $(v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $d[v_k] = \delta(s, v_k)$.
- This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p.

Bellman–Ford algorithm

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i ← 1 to |V[G]| - 1
      do for each edge (u, v) ∈ E[G]
            do RELAX(u, v, w)
for each edge (u, v) ∈ E[G]
      do if d[v] > d[u] + w(u, v)
            then return FALSE
return TRUE
```

- Complexity: O(VE)

All-Pairs Shortest Path

- Notice that finding the shortest path between a pair of vertices (s, t) in worst case requires first finding the shortest path from s to all other vertices in the graph.
- Many applications, such as finding the center or diameter of a graph, require finding the shortest path between all pairs of vertices.
- We can run Dijkstra’s algorithm n times (once from each possible start vertex) to solve all-pairs shortest path problem in $O(n^3)$. Can we do better?

The Floyd-Warshall algorithm

- The Floyd-Warshall algorithm starts by numbering the vertices of the graph from 1 to n. Define $W[i,j]^k$ to be the length of the shortest path from i to j using only vertices numbered from 1,2,…,k as possible intermediate vertices.
- When k=0, we are allowed no intermediate vertices, so the only allowed paths are the original edges. Thus the initial all-pairs shortest-path matrix consists of the initial adjacency matrix.

The Floyd-Warshall algorithm

- We will perform n iterations, where the kth iteration allows only the first k vertices as possible intermediate steps on the path between each pair of vertices x and y.
- At each iteration, we allow a richer set of possible shortest path by adding a new vertex as a possible intermediary. Allowing the kth vertex as a stop helps only if there is a short path that goes through k, so
  $w[i,j]^{k} =$
  $\min(w[i,j]^{k-1} , w[i,k]^{k-1} + w[k,j]^{k-1})$

Implementation of the Floyd-Warshall algorithm

```
let dist be a |V| × |V| array of minimum distances initialized to ∞ (infinity)
for each vertex v
   dist[v][v] ← 0
for each edge (u, v)
   dist[u][v] ← w(u, v)  // the weight of the edge (u,v)
for k from 1 to |V|
   for i from 1 to |V|
      for j from 1 to |V|
         if dist[i][j] > dist[i][k] + dist[k][j]
             dist[i][j] ← dist[i][k] + dist[k][j]
         end if
```

- Time complexity: $O(V^3)$. The Floyd-Warshall algorithm is a good choice for computing paths between all pairs of vertices in **dense graphs**.

Difference constraints

- 问题：设需要安排5项任务T1~T5，有如下要求（为简单起见，我们忽略完成每项任务的所需的时间）：
  1. 先做T1再做T2；
  2. 做T1后至少1小时后才能做T5；
  3. 先做T5再做T2，且两者间隔时间不超过1小时；
  4. 先做T1再做T3，且两者间隔时间不超过5小时；
  5. 先做T1再做T4，且两者间隔时间不超过4小时.
- 如何安排任务？

Difference constraints

- A set of inequalities (difference constraints):
  - x1 - x2 <= 0,
  - x1 - x5 <= -1,
  - x2 - x5 <= 1,
  - x3 - x1 <= 5,
  - x4 - x1 <= 4.

A system of difference constraints

- In a *system of difference constraints*, each row of the matrix A contains one 1 and one -1, and all other entries of A are 0. Thus, the constraints given by $Ax \le b$ are a set of m *difference constraints* involving n unknowns, in which each constraint is a simple linear inequality of the form

  $x_j - x_i \le b_k$,

  where

  $1 \le i, j \le n$ and $1 \le k \le m$.

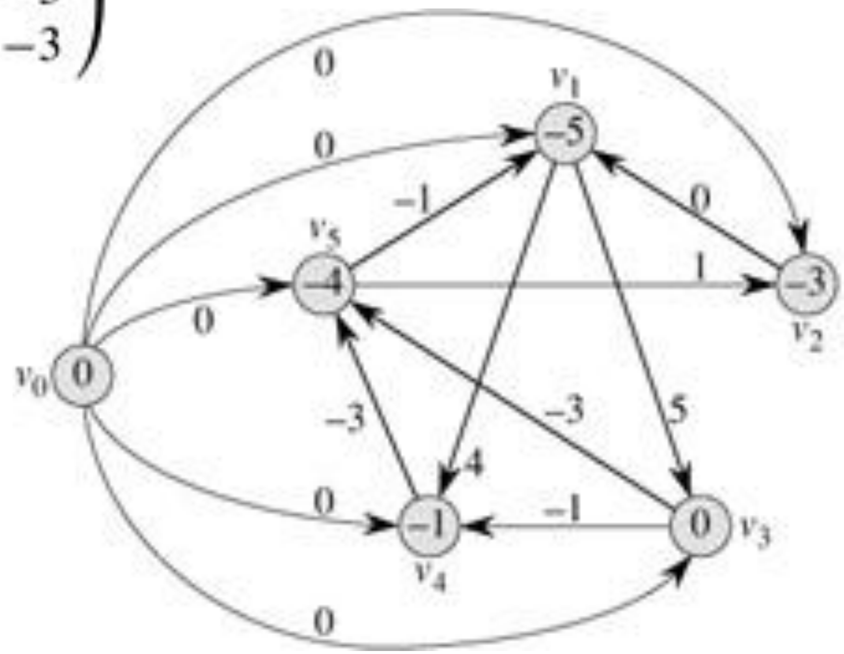
$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \le \begin{pmatrix} 0 \\ -1 \\ 1 \\ 5 \\ 4 \\ -1 \\ -3 \\ -3 \end{pmatrix} .$$

Constraint graphs

- We can interpret systems of difference constraints from a graph-theoretic point of view.
- The vertex set V consists of a vertex v_i for each unknown x_i , plus an additional vertex v_0.
- $E = \{(v_i, v_j) : x_j - x_i \leq b_k$ is a constraint$\} \cup \{(v_0, v_1), (v_0, v_2), (v_0, v_3), \ldots, (v_0, v_n)\}$.
- If $x_j - x_i \leq b_k$ is a difference constraint, then the weight of edge (v_i , v_j) is $w(v_i, v_j) = b_k$.
- The weight of each edge leaving v_0 is 0.

Constraint graphs

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \le \begin{pmatrix} 0 \\ -1 \\ 1 \\ 5 \\ 4 \\ -1 \\ -3 \\ -3 \end{pmatrix} .$$



Solving the system

- **Theorem**: Given a system $Ax \le b$ of difference constraints, let $G = (V, E)$ be the corresponding constraint graph. If G contains no negative-weight cycles, then

  $$x = (\delta(v_0,v_1), \delta(v_0,v_2), \delta(v_0,v_3), \ldots, \delta(v_0,v_n))$$

  is a feasible solution for the system. If G contains a negative-weight cycle, then there is no feasible solution for the system.
- Let $x=(x_1, x_2, \ldots, x_n)$ be a solution to a system $Ax \le b$ of difference constraints, and let d be any constant. Then

  $$x + d = (x_1 + d, x_2 + d, \ldots , x_n + d)$$

  is a solution to $Ax \le b$ as well.
- Solve the system of difference constraints using the Bellman-ford algorithm.

Thank you!