中山大學 SUN YAT-SEN UNIVERSITY

Lecture 3
Divide-and-Conquer

Algorithm Design

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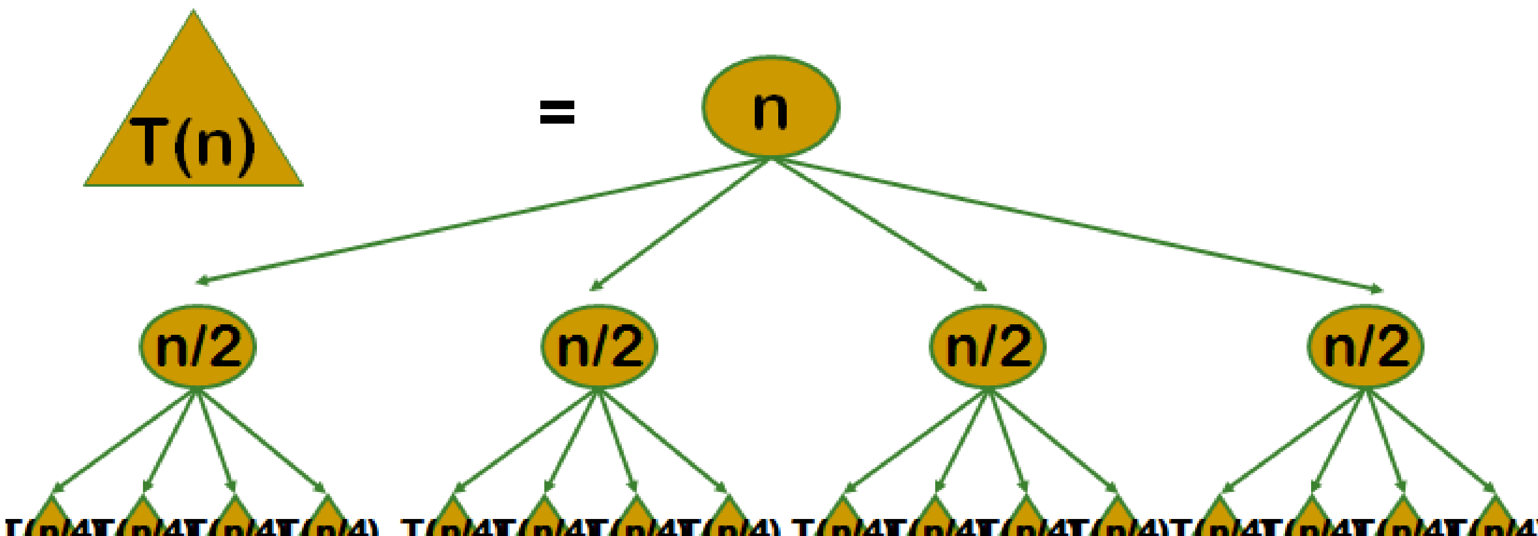
Definition

- A divide and conquer algorithm works by recursively breaking down a problem into two or more sub-problems of the same or related type, until these become simple enough to be solved directly. The solutions to the sub-problems are then combined to give a solution to the original problem.



Merge-sort

```
procedure MERGE-SORT(A, p, r)
    if p < r
        then q ← ⌊(p + r)/2⌋
            MERGE-SORT(A, p, q)
            MERGE-SORT(A, q + 1, r)
            MERGE(A, p, q, r)
```

```
procedure MERGE(A, p, q, r)
    n1 ← q − p + 1; n2 ← r − q
    allocate arrays L[1 ... n1 + 1] and R[1 ... n2 + 1]
    for i ← 1 to n1
        do L[i] ← A[p + i − 1]
    for j ← 1 to n2
        do R[j] ← A[q + j]
    L[n1 + 1] ← ∞; R[n2 + 1] ← ∞
    i ← 1; j ← 1
    for k ← p to r
        do if L[i] ≤ R[j]
            then A[k] ← L[i]
                i ← i + 1
            else A[k] ← R[j]
                j ← j + 1
```

Analysis of the Merge-sort algorithm

- Described by recursive equation
- Suppose $T(n)$ is the running time on a problem of size n.
- $T(n) = \begin{cases} \Theta(1) & \text{if } n \le n_c \\ aT(n/b)+D(n)+C(n) & \text{if } n > n_c \end{cases}$

where a: number of subproblems

n/b: size of each subproblem

$D(n)$: cost of divide operation

$C(n)$: cost of combination operation

Analysis of the Merge-sort algorithm

- **Divide**: $D(n) = \Theta(1)$
- **Conquer**: a=2,b=2, so $2T(n/2)$
- **Combine**: $C(n) = \Theta(n)$
- $$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2)+\Theta(n) & \text{if } n>1 \end{cases}$$
- $$T(n) = \begin{cases} c & \text{if } n=1 \\ 2T(n/2)+cn & \text{if } n>1 \end{cases}$$

Analysis of the Merge-sort algorithm

- The recursive equation can be solved by recursive tree.
- $T(n) = 2T(n/2) + cn,$
- lg n+1 levels, cn at each level, thus
- Total cost for merge sort is:

  $T(n) = cn\lg n + cn = \Theta(n\lg n)$.
- Question: best, worst, average?

Analysis of the Merge-sort algorithm

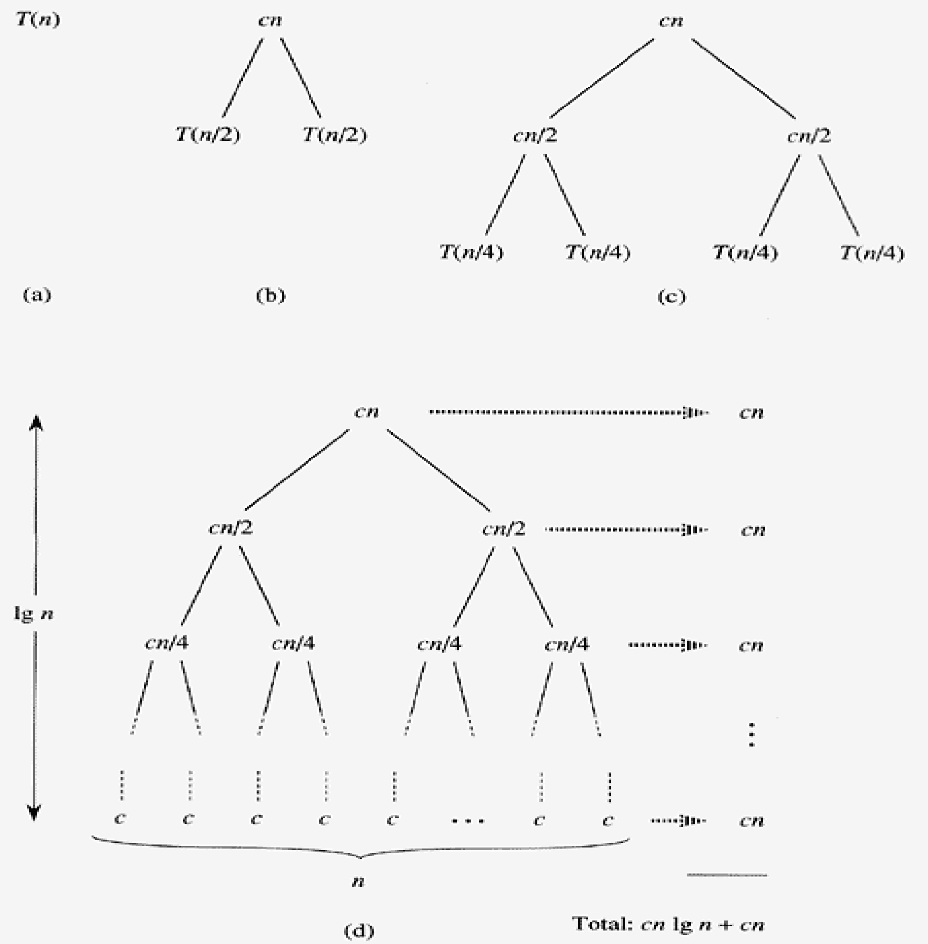


Figure 2.5 The construction of a recursion tree for the recurrence $T(n) = 2T(n/2) + cn$. Part **(a)** shows $T(n)$, which is progressively expanded in **(b)–(d)** to form the recursion tree. The fully expanded tree in part (d) has $\lg n + 1$ levels (i.e., it has height $\lg n$, as indicated), and each level contributes a total cost of cn. The total cost, therefore, is $cn \lg n + cn$, which is $\Theta(n \lg n)$.

Master theorem

Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n) \,,$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

Exercise

Use the master method to give tight asymptotic bounds for the following recurrences.

(a) $T(n)=2T(n/4)+1$

(b) $T(n)=2T(n/4)+\sqrt{n}$

(c) $T(n)=2T(n/4)+n$

(d) $T(n)=2T(n/4)+n^2$

Multiplication of Large Integers

Consider the problem of multiplying two (large) n-digit integers represented by arrays of their digits such as:

A = 12345678901357986429   B = 87654321284820912836

The grade-school algorithm:

$$
\begin{array}{r}
a_1\ a_2 \ldots\ a_n \\
\underline{b_1\ b_2 \ldots\ b_n} \\
(d_{10})\ d_{11} d_{12} \ldots d_{1n} \\
(d_{20})\ d_{21} d_{22} \ldots d_{2n} \quad\quad \\
\underline{\ldots \ldots \ldots \ldots \ldots \ldots \ldots} \quad\quad \\
(d_{n0})\ d_{n1} d_{n2} \ldots d_{nn} \quad\quad\quad\quad
\end{array}
$$

Efficiency: $\Theta(n^2)$ single-digit multiplications

First Divide-and-Conquer Algorithm

A small example: $A * B$ where $A = 2135$ and $B = 4014$

$A = (21 \cdot 10^2 + 35)$, $B = (40 \cdot 10^2 + 14)$

So, $A * B = (21 \cdot 10^2 + 35) * (40 \cdot 10^2 + 14)$

$$= 21 * 40 \cdot 10^4 + (21 * 14 + 35 * 40) \cdot 10^2 + 35 * 14$$

In general, if $A = A_1A_2$ and $B = B_1B_2$ (where A and B are *n*-digit, A_1, A_2, B_1, B_2 are $n/2$-digit numbers),

$A * B = A_1 * B_1 \cdot 10^n + (A_1 * B_2 + A_2 * B_1) \cdot 10^{n/2} + A_2 * B_2$

Recurrence for the number of one-digit multiplications $M(n)$:

$$M(n) = 4M(n/2), \quad M(1) = 1$$

Solution: $M(n) = n^2$

Second Divide-and-Conquer Algorithm

$A * B = A_1 * B_1 \cdot 10^n + (A_1 * B_2 + A_2 * B_1) \cdot 10^{n/2} + A_2 * B_2$

The idea is to decrease the number of multiplications from 4 to 3:

$(A_1 + A_2) * (B_1 + B_2) = A_1 * B_1 + (A_1 * B_2 + A_2 * B_1) + A_2 * B_2,$

i.e., $(A_1 * B_2 + A_2 * B_1) = (A_1 + A_2) * (B_1 + B_2) - A_1 * B_1 - A_2 * B_2,$ which requires only 3 multiplications at the expense of 3 extra add/sub.

Recurrence for the number of multiplications M(n):

$$M(n) = 3M(n/2), \quad M(1) = 1$$

Solution: $M(n) = 3^{\log_2 n} = n^{\log_2 3} \approx n^{1.585}$

Karatsuba Multiplication Algorithm

KARATSUBA-MULTIPLY(x, y, n)

IF ($n = 1$)

  RETURN $x \times y$.

ELSE

  $m \leftarrow \lceil n / 2 \rceil$.

  $a \leftarrow \lfloor x / 2^m \rfloor$; $b \leftarrow x \bmod 2^m$.

  $c \leftarrow \lfloor y / 2^m \rfloor$; $d \leftarrow y \bmod 2^m$.

  $e \leftarrow$ KARATSUBA-MULTIPLY(a, c, m).

  $f \leftarrow$ KARATSUBA-MULTIPLY(b, d, m).

  $g \leftarrow$ KARATSUBA-MULTIPLY($a - b$, $c - d$, m).

  RETURN $2^{2m} e + 2^m (e + f - g) + f$.

Example of Large-Integer Multiplication

2135 * 4014

= (21*10^2 + 35) * (40*10^2 + 14)

= (21*40)*10^4 + c1*10^2 + 35*14

where c1 = (21+35) *(40+14) - 21*40 - 35*14, and

21*40 = (2*10 + 1) * (4*10 + 0)

= (2*4)*10^2 + c2*10 + 1*0

where c2 = (2+1) *(4+0) - 2*4 - 1*0, etc.

- This process requires 9 digit multiplications as opposed to 16.

Matrix multiplication

- Given two *n-by-n* matrices *A* and *B*, compute *C* = *AB.*
- Grade-school. $\Theta(n^3)$ arithmetic operations.

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

Block matrix multiplication

C_{11} A_{11} A_{12} B_{11}

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

B_{11}

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

Matrix multiplication

- To multiply two *n-by-n matrices A and B:*
  - Divide: partition *A and B into 1/2n-by-1/2n* blocks.
  - Conquer: multiply 8 pairs of 1/2*n-by-1/2n* matrices, recursively.
  - Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= \left(A_{11} \times B_{11}\right) + \left(A_{12} \times B_{21}\right) \\ C_{12} &= \left(A_{11} \times B_{12}\right) + \left(A_{12} \times B_{22}\right) \\ C_{21} &= \left(A_{21} \times B_{11}\right) + \left(A_{22} \times B_{21}\right) \\ C_{22} &= \left(A_{21} \times B_{12}\right) + \left(A_{22} \times B_{22}\right) \end{aligned}$$

- Running time

$$T(n) = \underbrace{8T\left(n/2\right)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow T(n) = \Theta(n^3)$$

Strassen's method

- Key idea: multiply 2-by-2 blocks with only 7 multiplications. (plus 11 additions and 7 subtractions)

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_1 + P_5 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &\leftarrow A_{11} \times (B_{12} - B_{22}) \\ P_2 &\leftarrow (A_{11} + A_{12}) \times B_{22} \\ P_3 &\leftarrow (A_{21} + A_{22}) \times B_{11} \\ P_4 &\leftarrow A_{22} \times (B_{21} - B_{11}) \\ P_5 &\leftarrow (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &\leftarrow (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &\leftarrow (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

Pf. $$\begin{aligned} C_{12} &= P_1 + P_2 \\ &= A_{11} \times (B_{12} - B_{22}) + (A_{11} + A_{12}) \times B_{22} \\ &= A_{11} \times B_{12} + A_{12} \times B_{22}. \checkmark \end{aligned}$$

Strassen's algorithm

STRASSEN (n, A, B)

IF $(n = 1)$ RETURN $A \times B$.

assume n is a power of 2

Partition A and B into 2-by-2 block matrices.

keep track of indices of submatrices (don't copy matrix entries)

$P_1 \leftarrow$ STRASSEN $(n / 2, A_{11}, (B_{12} - B_{22}))$.

$P_2 \leftarrow$ STRASSEN $(n / 2, (A_{11} + A_{12}), B_{22})$.

$P_3 \leftarrow$ STRASSEN $(n / 2, (A_{21} + A_{22}), B_{11})$.

$P_4 \leftarrow$ STRASSEN $(n / 2, A_{22}, (B_{21} - B_{11}))$.

$P_5 \leftarrow$ STRASSEN $(n / 2, (A_{11} + A_{22}) \times (B_{11} + B_{22}))$.

$P_6 \leftarrow$ STRASSEN $(n / 2, (A_{12} - A_{22}) \times (B_{21} + B_{22}))$.

$P_7 \leftarrow$ STRASSEN $(n / 2, (A_{11} - A_{21}) \times (B_{11} + B_{12}))$.

$C_{11} = P_5 + P_4 - P_2 + P_6$.

$C_{12} = P_1 + P_2$.

$C_{21} = P_3 + P_4$.

$C_{22} = P_1 + P_5 - P_3 - P_7$.

RETURN C.

Analysis of Strassen's algorithm

- **Theorem**. Strassen's algorithm requires $O(n^{2.81})$ arithmetic operations to multiply two *n-by-n* matrices.

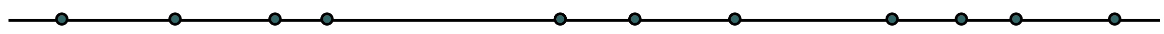
$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

- Q. What if n is not a power of 2 ?
- A. Could pad matrices with zeros.

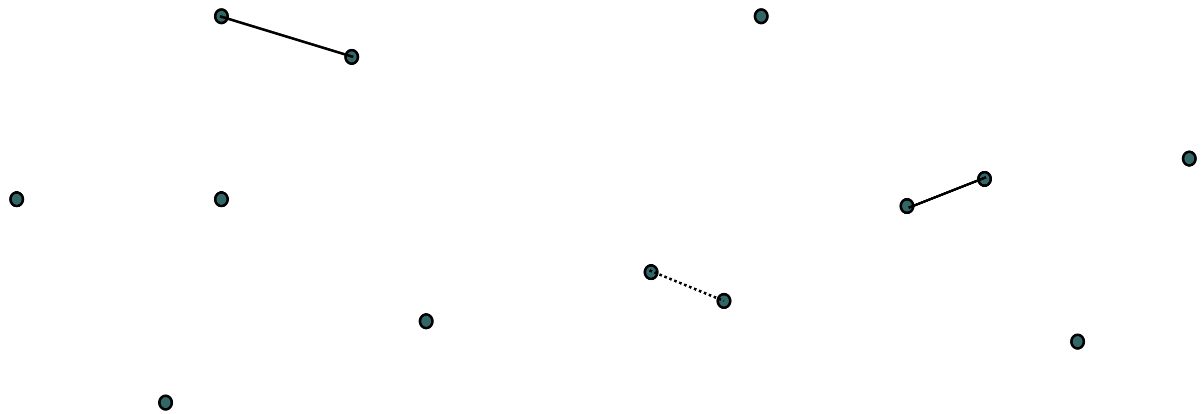
$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 10 & 11 & 12 & 0 \\ 13 & 14 & 15 & 0 \\ 16 & 17 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 84 & 90 & 96 & 0 \\ 201 & 216 & 231 & 0 \\ 318 & 342 & 366 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Closest Pair

- Given a set S = {p_1, p_2, ..., p_n} of *n* points in the plane find the two points of *S* whose distance is the smallest.
- 1-d



- 2-d



Closest Pair – Naïve Algorithm

Pseudo code

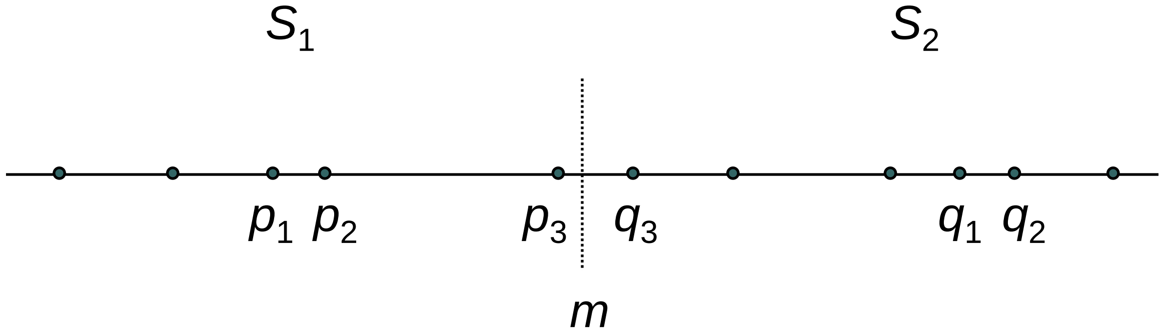
```
for each pt i∈S
    for each pt j∈S and i<>j
    {
      compute distance of i, j
      if distance of i, j < min_dist
          min_dist = distance i, j
    }
return min_dist
```

- Time Complexity– $O(n^2)$
- Can we do better?

1-D Closest Pair – Divide & Conquer

- We consider a divide-and-conquer algorithm for CLOSEST-PAIR in 1 dimension ($d = 1$).
- Partition S, a set of points on a line, into two sets S_1 and S_2 at some point m such that for every point $p \in S_1$ and $q \in S_2$, $p < q$.
- Solving CLOSEST-PAIR recursively on S_1 and S_2 separately produces $\{p_1, p_2\}$, the closest pair in S_1, and $\{q_1, q_2\}$, the closest pair in S_2.
- Let δ be the smallest distance found so far:

  $\delta = \min(|p_2 - p_1|, |q_2 - q_1|)$
- The closest pair in S is either $\{p_1, p_2\}$ or $\{q_1, q_2\}$ or some $\{p_3, q_3\}$ with $p_3 \in S_1$ and $q_3 \in S_2$.



1-D Closest Pair – Divide & Conquer

- To check for such a point $\{p_3, q_3\}$, is it necessary to test every possible pair of points in S_1 and S_2?
- Note that if $\{p_3, q_3\}$ is to be closer than δ (i.e., $|q_3 - p_3| < \delta$), then both p_3 and q_3 must be within δ of m.
- Because δ is the distance between the closest pair in either S_1 or S_2, a semi-closed interval of length δ can contain at most 1 point.
- For the same reason, there can be at most 1 point of S_2 within δ of m
- So, the number of distance computations needed to check for a closest pair $\{p_3, q_3\}$ with $p_3 \in S_1$ and $q_3 \in S_2$ is 1, not $O(N^2)$.
- Thus a divide-and-conquer algorithm can solve 1-dimensional CLOSEST-PAIR in $O(N \log N)$ time.

1-D Closest Pair – Divide & Conquer

Divide-and-conquer for *d* = 1

procedure CPAIR1(S)

Input: $X[1{:}N]$, N points of S in one dimension.

Output: δ, the distance between the two closest points.

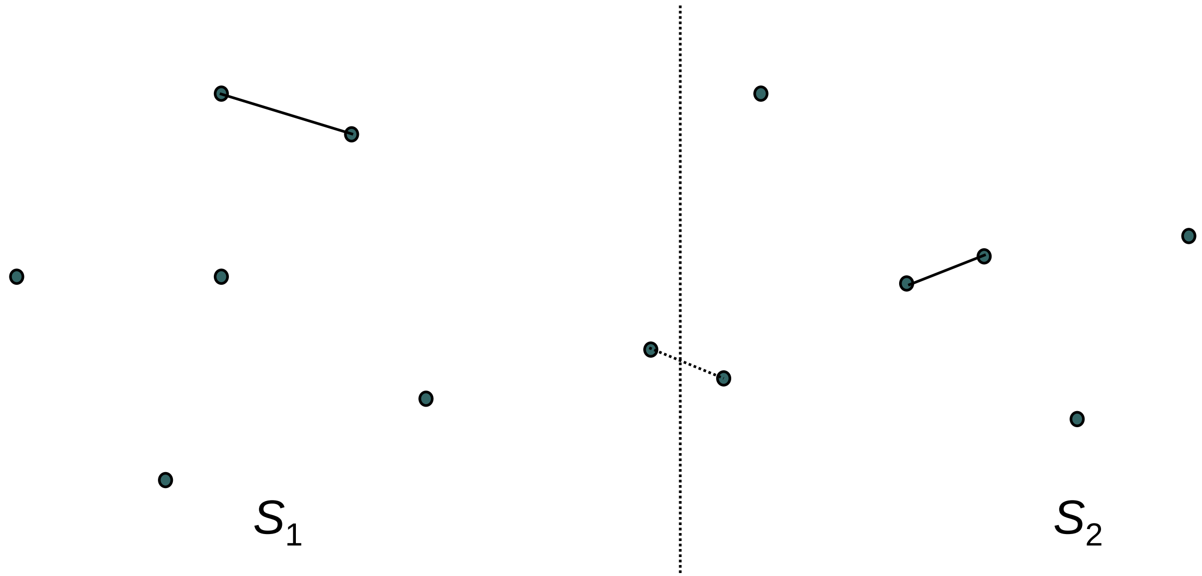
begin
    if ($|S| = 2$) then
        $\delta = |X[2] - X[1]|$
    else if ($|S| = 1$) then
        $\delta = \infty$
    else
        begin
            Construct(S_1, S_2) /* $S_1 = \{p: p \le m\}$, $S_2 = \{p: p > m\}$ */
            δ_1 = CPAIR1(S_1)
            δ_2 = CPAIR1(S_2)
            $p = \max(S_1)$
            $q = \min(S_2)$
            $\delta = \min(\delta_1, \delta_2, q - p)$
        end
    endif
    return δ
end

Closest Pair – Divide & Conquer

- Divide the problem into two equal-sized sub problems
- Solve those sub problems recursively
- Merge the sub problem solutions into an overall solution

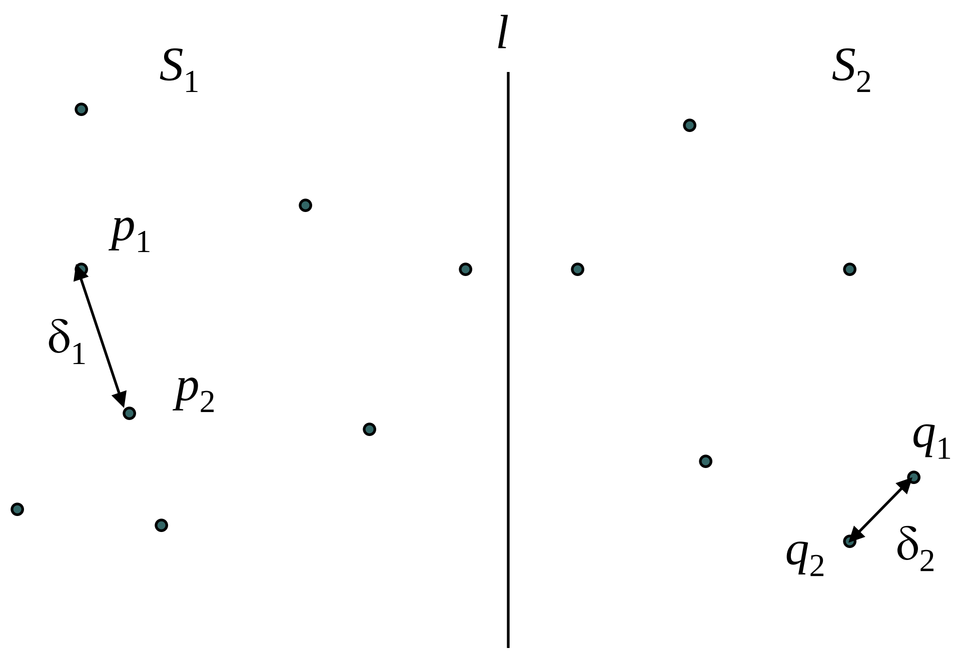
Closest Pair – Divide & Conquer

- Assume that we have solutions for sub problems S1, S2.
- How can we merge in a time-efficient way?
  - The closest pair can consist of one point from S_1 and another from S_2
  - Testing all possibilities requires: $O(n/2) \cdot O(n/2) \in O(n^2)$
  - Not good enough



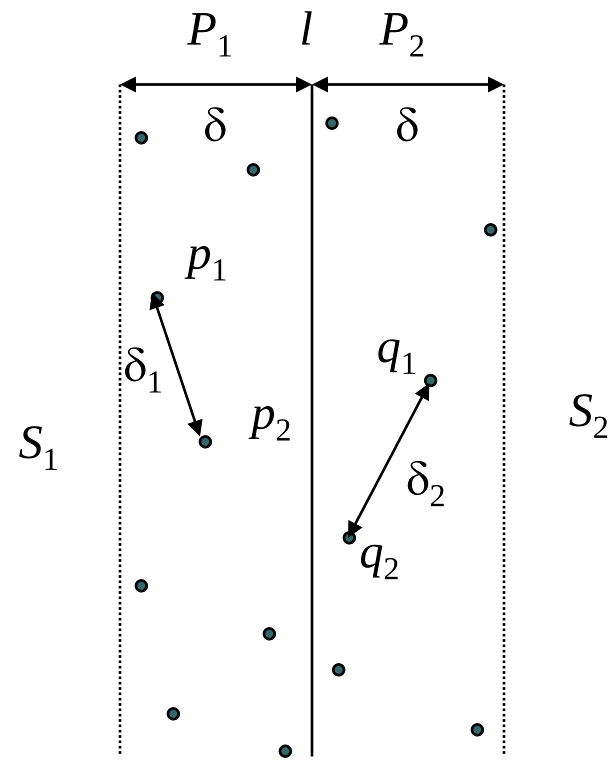
Closest Pair – Divide & Conquer

- Partition two dimensional set S into subsets S_1 and S_2 by a vertical line l at the median x coordinate of S.
- Solve the problem recursively on S_1 and S_2.
- Let $\{p1, p2\}$ be the closest pair in S_1 and $\{q1, q2\}$ in S_2.
- Let δ_1 = distance($p1$,$p2$) and δ_2 = distance($q1$,$q2$)
- Let $\delta = \min(\delta_1, \delta_2)$



Closest Pair – Divide & Conquer

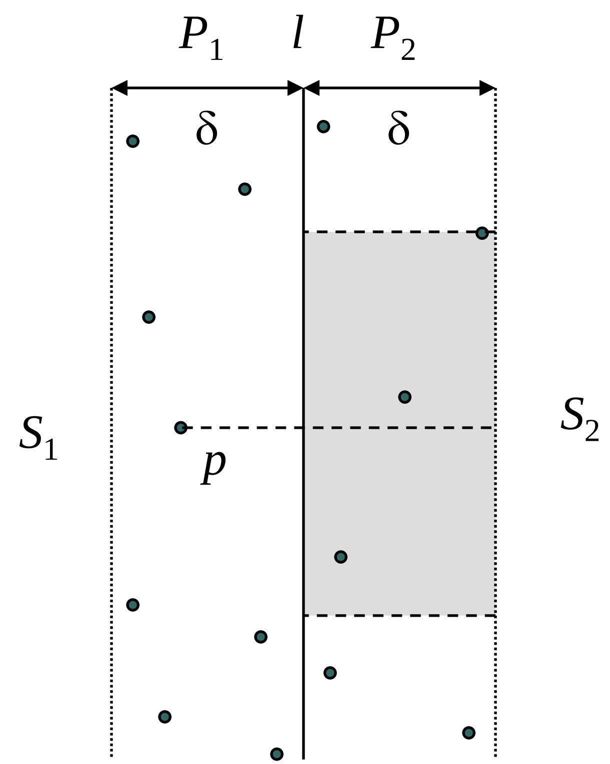
- In order to merge we have to determine if exists a pair of points {*p*, *q*} where $p \in S_1$, $q \in S_2$ and distance(*p*, *q*) < δ.
- If so, *p* and *q* must both be within δ of *l*.
- Let P_1 and P_2 be vertical regions of the plane of width δ on either side of *l*.
- If {*p*, *q*} exists, *p* must be within P_1 and *q* within P_2.
- However, every point in S_1 and S_2 may be a candidate, as long as each is within δ of *l*, which implies: $O(n/2) \cdot O(n/2) = O(n^2)$



<u>Can we do better ?</u>

Closest Pair – Divide & Conquer

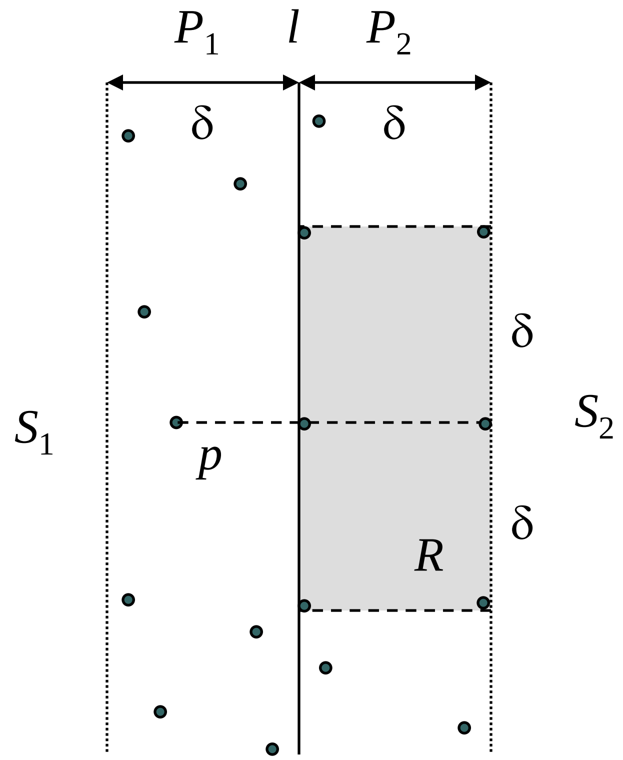
- For a point *p in* P_1, which portion of P_2 should be checked?
- We only need to check the points that are within δ of p.
- Thus we can limit the portion of P2.
- The points to consider for a point p must lie within $\delta \times 2\delta$ rectangle R.
- At most, how many points are there in rectangle R?



Closest Pair – Divide & Conquer

How many points are there in rectangle *R*?

- Since no two points can be closer than δ, there can only be at most 6 points
- Therefore, $6 \cdot O(n/2) \in O(n)$
- Thus, the time complexity is
  - $O(n \log n)$



How do we know which 6 points to check?

Closest Pair – Divide & Conquer

How do we know which 6 points to check?

- Project p and all the points of S_2 within P_2 onto l.
- Only the points within δ of p in the y projection need to be considered (max of 6 points).
- After sorting the points on y coordinate we can find the points by scanning the sorted lists. Points are sorted by y coordinates.
- To prevent resorting in O(n log n) in each merge, two previously sorted lists are merged in O(n).

Time Complexity: O(n log n)

Thank you!