Math 237: Discrete Math

Ramsfield

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1 5.3 Weak Induction

Two harder types of problems that can be solved using weak induction:

1.1 Divisibility Problems

EX: Prove that for all integers $n \geq 0$, $4|(5^n - 1)$
Proof By Induction: For the base case, we have $5^n - 1 = 5^0 - 1 = 1 - 1 = 0$ and $4|0$ $(0 = 4 * k\ k \in Z)$
now assume $n \geq 0$ is arbitrary and $4|(5^n - 1)5$
[NTS: $4|5^{n+1} - 1$] Then by definition there is an integer k such that $5^n - 1 = 4k$
[NTS: $5^{n+1} - 1 = 4m$ for some $m \in Z$]
$5^{n+1} - 1 = 5^1 * 5^n - 1 = (4+1)5^n - 1 = 4 * 5^n + 5^n - 1$ Substitute 4k for $5^n - 1$
$4 * 5^n + 4k = 4(5^n + k)$ and $(5^n + k) \in Z$ because $n, k \in Z$
$\therefore 4|(5^n - 1)$ for all $n \geq 0$

1.2 Proving Inequalities

Facts about inequalities:
$a \leq a$ for any $a \in R$
if $a \leq b$ and $c \in R$, then $a + c \leq b + c$
To prove that $2 < 10$, it suffices to prove that $2 < 3 < 7 < 9 < 10$. Note that the fact that $2 < 11$ is true but useless.
If $a \leq b$ and $c \geq 0$ then $ac \leq bc$

EX: Prove that for all integers $n \geq 0$, $1 + 3n \leq 4^n$ **Proof By Induction:** For the base case, $n = 0$, then $1 + 3n = 1 + 3(0) \leq 4^0 = 1$ or $1 \leq 1$ which is true.
For the inductive hypothesis $n \geq 0$ is arbitrary, and assume $1 + 3n \leq 4^n$ [NTS: $1 + 3(n+1) \leq 4^{n+1}$
Now $1 + 3(n+1) = 1 + 3n + 3$ which is $\leq 4^n + 3$ by the inductive hypothesis.
$4^{n+1} = 4 * 4^n = (3+1)4^n = 3 * 4^n + 4^n$
$\leq 4^n = 3 * 4^n$
$= 4^n(1+3)$
$= 4 * 4^n$

$= 4^{n+1}$
$\therefore 1 + 3(n+1) \leq 4^{n+1}$ when $1 + 3n \leq 4^n$

2 5.4 Strong Induction

Sometimes we need `Strong Induction` to prove a statement $\forall n \in S[P(n)]$, $S = n_0, n_0+1, ... \in Z$
This is also a 3 step processes:
1. Prove the `base case` by proving that $P(n_0)$ is true.
2. Assume the inductive hypothesis, which involves taking an arbitrary $n \in S$ and assuming $P(k)$ is true for all $n_0 \leq k \leq n$
3. Now prove the inductive step, which in this case says $P(n_0) \wedge P(n_0+1) \wedge P(n_0+2) \wedge \cdots \wedge P(n) \rightarrow P(n+1)$

EX: Prove that for every integer $n \geq 2$, n can be written as a product of a finite number of primes.
Proof By Induction For the base case, when $n = 2$ we have that n is a "product" of a single prime 2, so the claim holds.
Now assume $n \geq 2$ is arbitrary and the claim holds for any integer $2 \leq k \leq n$
[NTS: The claim holds for n+1]
If $n+1$ is prime, then we're done!
If $n+1$ is not prime (composite), it is composite, so there is some integer m that divides $n+1$ such that $2 \leq m \leq n$. Furthermore, $\frac{n+1}{m} \in Z$ and $2 \leq \frac{n+1}{m} \leq n$ Therefore both m and $\frac{n+1}{m}$ are a product of a finite number of primes. This means $n+1 = \frac{n+1}{m} * m$ is also.

EX: For the recursive sequence, $b_1 = 4$, $b_2 = 12$, $b_k = b_{k-1} + b_{k-2}$, $k \geq 3$
Prove that $4|b_n \forall n \geq 1$.
Proof By Induction: For the base case, note that $b_1 = 4 = 4 * 1$ and $b_2 = 12 = 4 * 3$ so $4|b_1$ and $4|b_2$
For the inductive hypothesis, assume $n \geq 3$ and that $4|b_k$ for all $1 \leq k \leq n-1$
[NTS: $4|b_n$]
Then by definition $b_n = b_{n-1} + b_{n-2}$ and since $n \geq 3$, we know that $1 \leq n-2 \leq n-1$ and $1 \leq n-1 \leq n-1$ Therefore, by the inductive hypothesis we have $4|b_{n-2}$ and $4|b_{n-1}$. So $b_{n-2} = 4q$ for some $q \in Z$ and $b_{n-1} = 4s$ for some $s \in Z$.
Meaning $b_n = 4q + 4s$, $b_n = 4(q+s)$ and $(q+s) \in Z$ because $q, s \in Z$
$\therefore \mathbf{4|b_n}$ **while** $\boldsymbol{n \geq 3}$

EX: Prove that $\forall n \geq 1$, C_n is even where $C_1 = 4$, $C_2 = 10$ and $C_n = C_{n-1} + 2C_{n-2}$ $\forall n \geq 3$
Proof by Strong Induction: Base case: Check everything up to the base case. For $n = 1$ $C_1 = 4$ and 4 is even because $4 = 2(2)$. For $n = 2$ $C_2 = 10$ and $10 = 2(5)$. For $n = 3$ $C_3 = 10 + 2(4) = 2(5+4)$ which is even.
Inductive Hypothesis: Let $n \geq 3$ be arbitrary and assume C_k is even whenever

$1 \le k \le n$
Inductive Step: [NTS: C_{n+1} is even] Then $C_{n+1} = C_n + 2C_{n-1}$ By the inductive hypothesis, C_n and C_{n-1} are even. $\therefore C_n = 2s$ and $C_{n-1} = 2t$ for some $s, t \in \mathbb{Z}$
$\therefore C_{n+1} = 2(s + 2t)$ which is even

3 6.1 Set Theory

In Set Theory, we always have a **universal set** that we consider to be the set containing all sets under consideration. We will denote our universal set by X (the book uses U). Given sets A and B inside X, the basic notion is that of **set containment**. We say A is a **subset** of B and write $A \subseteq B$ if the following holds:

$$\forall x \in X[x \in A \Rightarrow x \in B]$$

What does it mean if A is not a subset of B? In this case we write $A \nsubseteq B$ and this is true if the negation of the previous statement holds.
Make sure to remember $\sim (\forall x[P(x) \Rightarrow Q(x)]) \equiv \exists x[P(x) \wedge (\ Q(x))]$
For set theory we can write:

$$\exists x \in X[x \in A \wedge x \notin B]$$

We say sets A and B are **equal** and write $A = B$ if $A \subseteq B$ and $B \subseteq A$.
Now we want to find the analogues of $\vee$, $\wedge$, $\sim$ in set theory.
The analogue of $\wedge$ is **intersection**, and for sets A and B the intersection of A and B is the set

$$A \cap B = x \in X | x \in A \wedge x \in B$$

The analogue for $\vee$ is **union**. The union of A and B is the set

$$A \cup B = x \in X | x \in A \vee x \in B$$

The analogue of $\sim$ is the **complement**. The complement of A (in X) is the set

$$A^c = x \in X | x \notin A$$

There is also refinement of this called the **relative complement** or **set difference**. Given sets A and B, the complement of A in B is the set

$$B - A = x \in B | x \notin A$$

Now that we have our "dictionary" between logic and set theory, we see that e.g. every logical equivalence in Theorem 2.1.1 on page 35 is a statement about set equality:
EX: $\sim (\sim P) \equiv P$ (Double Negation) looks like $(A^{c^c}) = A$ This is the **Double Complement Law**

EX: De Morgan's Law says $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$ so in set theory it looks like

$$(A \cup B)^C = A^c \cap B^c$$

This is one of **De Morgan's Laws** for sets
Read page 342 to review **interval notation** $((a, b), [a, b], etc)$
The **empty set** $\emptyset$ is the unique subset of X that contains no elements.
Facts:
$X^c = \emptyset$
$\emptyset^c = X$
$\emptyset \subseteq A$ for any $A \subseteq X$
$A \subseteq A$ for any $A \subseteq X$

Two sets A and B are **disjoint** if $A \cap B = \emptyset$
The **power set** of a set A is the set $P(A) = B|B \subseteq A$
EX: $P(\emptyset) = \{\emptyset\}$
EX: $P(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Prove: if A has n elements, $P(a)$ has 2^n elements
Proof By Induction: Base case, $P(n = 0)$ or set is $P(\emptyset)$
$P(\emptyset) = \{\emptyset\} = 1 = 2^n = 2^0 = 1$
`Inductive Hypothesis`: Assume $P(k) = 2^k$ for all $0 \leq k \leq n - 1$ [NTS: $P(k + 1) = 2^{k+1}$]
$P(k + 1) = P(k) + P(1) = 2^k * 2^1$ For the left side
$2^{k+1} = 2^k * 2^1$
$\therefore P(n) = 2^n$

Quiz on Thurs: 4.4, 4.6, 5.1

4 Set Notation, cont.

6.2 Set Proofs

EX: Prove that if A,B,C are sets and $A \subseteq B$, then $A \cap C \subseteq B \cap C$
Direct Proof: Assume A, B, C are sets and $A \subseteq B$ [NTS: $A \cap C \subseteq B \cap C$]
Assume $x \in A \cap C$ [NTS: $x \in B \cap C$]
Then $x \in A$ and $x \in C$. [NTS: $x \in B$ and $x \in C$] but since, if $x \in C$, then $x \in C$ we only need to show that if $x \in A$, $x \in B$
Since $x \in A$ and $A \subseteq B$, $x \in B$
and since $x \in B$ and $x \in C$, $x \in B \cap C$

Claim: Prove that for any sets A, B, if $A \subseteq B$, then $B^c \subseteq A^c$
Direct Proof by Contradiction: Assume A, B are sets and $A \subseteq B$.[NTS: $B^c \subseteq A^c$]
Assume $x \in B^c$

By definition of complement, if $x \in B^c, x \notin B$
Suppose for the sake of contradiction $x \in A$. Then $x \in B$ since $A \subseteq B$. This is a contradiction since we know $x \notin B \therefore x \notin A$ By definition if $x \notin A, x \in A^c$
$\therefore$ if $x \in B^c$ then, $x \in A^c$ which is the definition of a subset.
$\therefore B^c \subseteq A^c$

5 More Set Proofs

Claim: Prove that for any sets A, B, if $A \subseteq B$, then $B^c \subseteq A^c$
Direct Proof by Contrapositive: Assume A, B are sets and $A \subseteq B$.[NTS: $B^c \subseteq A^c$]
Then for every $x \in X$, if $x \in A$ then $x \in B$
Taking the contrapositive, we see that for all $x \in X$, if $x \notin B$ then $x \notin A$
By definition of complement, this says for any $x \in X$, if $x \in B^c$ then $x \in A^c$
$\therefore B^c \subseteq A^c$

Stronger Theorem:
For all sets A, B, $A \subseteq B \iff B^c \subseteq A^c$
Proof: $A \subseteq B \iff \forall x \in X[x \in A \to x \in B]$
$\iff \forall x \in X[x \notin B \to x \notin A]$
$\iff \forall x \in X[x \in B^c \to x \in A^c]$
$\iff B^c \subseteq A^c$

EX: Prove that for any sets A, B, C, $A \times (B \cup C) = (A \times B) \cup (A \times C)$
Proof: Let A, B, C be arbitrary sets [NTS: $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$]
$\subseteq$ Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. If $y \in B$ then $(x, y) \in A \times B$
Which means that $(x, y) \in (A \times B) \cup (A \times C)$
Similarly, if $y \in C$, then $(x, y) \in A \times C$, so $(x, y) \in (A \times B) \cup (A \times C)$
$\therefore A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
$\supseteq$: Assume $(x, y) \in (A \times B) \cup (A \times C)$
Then $(x, y) \in A \times B$ or $(x, y) \in A \times C$
If $(x, y) \in A \times B$ then $x \in A$ and $y \in B$
Therefore $y \in B \cup C$ so $(x, y) \in A \times (B \cup C)$
Similarly, if $(x, y) \in A \times C$ then $x \in A$ and $y \in C$ so $y \in B \cup C$ and $(x, y) \in A \times (B \cup C)$
$\therefore (A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$
$\therefore A \times (B \cup C) = (A \times B) \cup (A \times C)$

Same theory, using if and only if statements
$(x, y) \in A \times (B \cup C) \iff (x \in A) \wedge (y \in B \cup C)$ (The definition of cartesian product)
$\iff (x \in A) \wedge [(y \in B) \vee (y \in C)]$(Definition of Union)
$\iff ((x \in A) \wedge (x \in B) \vee ((x \in A) \wedge (x \in C))$(By the distributive law)

$\iff [(x,y) \in A \times B] \vee [(x,y) \in A \times C]$(Definition of Cartesian product)
$\iff (x,y) \in (A \times B) \cup (A \times C)$(By definition of union)

Theorem 6.2.1
For all sets A, B
1. $A \cup B \subseteq A$ and $A \cup B \subseteq B$
2. $A \subseteq A \cup B$ and $B \subseteq A \cup B$
3. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

6.3 Disproof and Algebraic Proofs

EX: Fine a counterexample to disprove this claim
For all sets A, B, C:

$$(A \cap B) \cup C = A \cap (B \cup C)$$

Counterexample: $A = \{1,2,3\}$ $B = \{4\}$ $C = \{4,5,6\}$
$(A \cap B) \cup C = \emptyset \cup C = C$
$A \cap (B \cup C) = A \cap C = \emptyset$ But $C \neq \emptyset$
So, $(A \cap B) \cup C \neq A \cap (B \cup C)$

6 6.3 Continued

The symmetric difference of A and B is the set $A \Delta B = (A - B) \cup (B - A)$, or everything that's in A and not in B, and everything that's in B and not in A.
$A \Delta B \equiv (A \cup B) - (A \cap B)$
EX: Prove that for any set A, $A \Delta A = \emptyset$
Proof: By definition, $A \Delta A = (A - A) \cup (A - A)$
This reduces to $A - A$ since for any set X, $X \cup X = X$
And $x \in (A - A) \iff x \in A \wedge x \notin A$ but this is a contradiction! $\therefore$ there is no such x, i.e. $A - A = \emptyset$
$\therefore A \Delta A = \emptyset$

6.1 Chapter 7

6.2 7.1 Definitions of functions

Recall that a relation $R \subseteq A \times B$ is a function from A to B if
1) $\forall a \in A \exists b \in B[(a,b) \in R]$
2) $\forall a \in A[(a,b_1) \in R \wedge (a,b_2) \in R \rightarrow b_1 = b_2]$
In this case, we normally write T as $F : A \rightarrow B$ and write $b = f(a)$ when $(a,b) \in R$
A is the domain of F.
B is the co-domain of F.
The range of F is the set $F(A) = \{F(a) \in B | a \in A\} \subseteq B$
Given any $b \in B$ the Inverse Image or preimage of b under F is the set $F^{-1}(b) =$

$\{a \in A | f(a) \in B\}$
Note that if $f(A) \neq B$ then for any $b \in B - F(a)$, $f^{-1}(b) = \emptyset$
EX: let $D = \{finite\ subsets\ of\ \mathbb{N}\}$ and define $T : \mathbb{N} \to D$, $T(n) = \{positive\ divisors\ of\ n\}$
For instance, $T(5) = \{1, 5\}$
$T(24) = \{1, 2, 3, 4, 6, 8, 12, 24\}$
$T^{-1}(1, 5) = \{5\}$
$T^{-1}(1, 2, 3) = \{\emptyset\}$
EX: Let $A = \{1, 2, 3, 4, 5\}$ and define

$$F : \wp(A) \to \mathbb{Z}$$

$F(x) = \{0\ if\ x\ contains\ even\ elements\ 1\ if\ x\ contains\ odd\ elements\}$
$F(A) = 1$ because A contains 5 elements and 5 is odd
$F(\emptyset) = 0$ because $\emptyset$ contains 0 elements and 0 is even
The range F is $F(\wp(A)) = \{0, 1\}$
$F^{-1}(2) = \emptyset$

7 7.2 1-1/Onto functions

A function $f : A \to B$ is 1-1 (one to one) if the following holds:
If whenever $f(x) = f(y)$ in B, $x = y$ in A
f is onto B if for any $b \in B$ there exists an $a \in A$ such that $b = f(a)$ You can think of One to One as the opposite of the Existance axiom for functions (There is only one y for every x)
You can think of onto as the Uniqueness axiom opposite (There is a y for every x)

EX: Prove that the function $F : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 2$ is 1-1 and onto $\mathbb{R}$
A test to help figure this out: Graph the function. Use a horizontal line to make sure it only intersects at one point
Proof: One to One: Suppose $x, y \in \mathbb{R}$ such that $f(x) = f(y)$ [NTS: x=y]
Then by definition $3x + 2 = 3y + 2 == 3x = 3y == x = y$
$\therefore f$ is one to one.
Onto: let $y \in \mathbb{R}$[NTS $\exists x \in \mathbb{R}$ such that $f(x) = y$]
`Scratch Work` Need $x \in \mathbb{R}$ such that $y = f(x) = 3x + 2$ Solve for x. $y = 3x + 2 => \frac{y-2}{3} = x$
let $x = \frac{y-2}{3}$. Then $f(x) = f(\frac{y-2}{3}) = 3(\frac{y-2}{3}) + 2 = y - 2 + 2 = y$
$\therefore f$ is onto.

Exercise: Let $f : \mathbb{Z} \to \mathbb{Z}$ and $f(x) = 3x + 2$
Prove or disprove this is 1-1 and onto
1-1: Suppose $x, y \in \mathbb{Z}$ and we need to show by definition that $f(x) = f(y)$ implies $x = y$
$3x + 2 = 3y + 2 => 3x = 3y => x = y$
$\therefore f$ is one to one

Onto: By definition of onto, $\forall y \in \mathbb{Z} \exists x \in \mathbb{Z}$ such that $f(x) = y$. So to disprove, we need to show that $\exists y \in \mathbb{Z} \forall x \in \mathbb{Z}$ such that $f(x) \neq y$. Let $y = 1$ this means $1 = 3x + 2$ which means $3x = -1$ and $x = \frac{-1}{3}$ but $\frac{-1}{3} \notin \mathbb{Z}$ and $1 \in \mathbb{Z}$
$\therefore f$ is not onto
The range is actually equal to: $\{3k + 2 | k \in \mathbb{Z}\} \notin \mathbb{Z}$

8 7.3 Composition and inverses

Quiz topics: 5.2, 5.3, 5.4 **INDUCTION** 2 proof problems. 1 Weak induction 1 strong induction

Let $F : X \to Y$, $g : Y' \to Z$ be functions such that $F(X) \subseteq Y' \subseteq Y$
The composition of g with F is the function $goF : X \to Z$, $(goF)(x) = g(F(x))$
EX: $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = 2x+4$ and $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = x^2 = 1$
Then $goF : \mathbb{R} \to \mathbb{R}$, $(goF)(x) = g(F(x))$, $g(2x + 4) = (2x + 4)^2 - 1 = 4x^2 + 16x + 15$ In this case, you can actually compose the other way:
$Fog : \mathbb{R} \to \mathbb{R}$ such that $(fog)(x) = f(g(x)) = f(x^2-1) = 2(x^2-1)+4 = 2x^2+2$
Note that foG is not equal to goF, in other words, composition is not commutative. And usually is not even defined both ways

What function properties are preserved under composition?
EX: if $F : X \to Y$ and $g : Y \to Z$ are 1-1 functions, is $gof : X \to Z$ also 1-1?
What would a proof look like?
Assume $x, y \in X$ such that $gof(x) = gof(y)$ [NTS: $x = y$]
By definition, $g(f(x)) = g(f(y))$. Since g is 1-1, this means $f(x) = f(y)$ and since f is 1-1, $x = y$
$\therefore gof$ is 1-1

This means that 1-1 is conserved in composition
What about onto?

EX: If $F : X \to Y$, $g : Y \to Z$ and both are onto, is $goF : X \to Z$ onto?
Recall that $F : X \to Y$ is onto Y if $\forall y \in Y \exists x \in X[y = F(x)]$
if goF is onto, then we need to show that for any $z \in Z$ there exists an $x \in X$ such that $z = goF(x)$
Since g is onto, $z = g(y)$ for some $y \in Y$. but F is also onto, so $y = f(x)$ for some $x \in X$.
$\therefore z = g(y) = g(f(x))$
$\therefore goF$ is onto Z
This means that onto is conserved in composition

Given a set X that is nonempty, the Identity function on X is $I_x : X \to X$ such that $I_x(x) = x \; \forall x \in X$
Note that if $f : X \to Y$ then $foI_x = f$ and $I_y of = f$

If $f : X \to Y$ is both 1-1 and onto, then f is called a bijection or is said to be bijective
In this case, there is a unique function $F^{-1} : Y \to X$ called the inverse of f such that $F^{-1}(y) = x \iff F(x) = y$
I.E. $FoF^{-1} = I_y$ and $F^{-1}oF = I_x$

EX: $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x - 1$. This is a bijection. Find the inverse function: F^{-1}
Set $y = 2x - 1$ and then solve for x. So $x = \frac{y+1}{2}$
Reverse the roles of x and y to get $F^{-1}(x) = \frac{x+1}{2}$
Check: $fof^{-1}(x) = f(f^{-1}(x))$
and the opposite

9 7.4: Cardinality

Cardinal Numbers $= 0, 1, 2, 3, ..., n, ..., \aleph_0, \aleph$
The cardinal numbers are an ordered set that keep track of the possible sizes of sets. We will write $|A|$ for the Cardinality of the set A. $|\emptyset| = 0$ by definition and for each $n \in \mathbb{N}$, $|\{1, 2, 3, ..., n\}| = n$ by definition. So a set A is finite if it is in bijection with the set $\{1, 2, 3, ..., n\}$ for some $n \in \mathbb{N}$ or $A = \emptyset$.
If A is not finite, then A is Infinite.
EX: $\mathbb{N}$ is an infinite set by the Archimedian principle which says there is no largest natural number. $\therefore \nexists$ bijection $F : \mathbb{N} \to \{1, 2, 3, ..., n\}$ for any $n \in \mathbb{N}$.
We set $|\mathbb{N}| = \aleph_0 =$ "Aleph nought"

In general, we say two non-empty sets have the same cardinality if $\exists$ bijection $F : A \to B$
EX: $|\mathbb{Z}| = |\mathbb{N}|$
We need a bijection from $\mathbb{Z}$ to $\mathbb{N}$. Say $F : \mathbb{Z} \to \mathbb{N}$
$\mathbb{Z} = ..., -3, -2, -1, 0, 1, 2, 3, ...$
$\mathbb{N} = 1, 2, 3, 4, 5, 6, ...$
$f(0) = 1, f(1) = 2, f(-1) = 3, f(2) = 4, f(-2) = 5, f(3) = 6, ...$

For the Real Numbers, there is infinitely many numbers between each integer. But we still do not achieve a larger infinite.
EX: $|\mathbb{Q}| = |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$
Idea: To see that $\mathbb{N}$ and $\mathbb{Q}^+$= positive rational are in bijections, you can make a one to one and onto function.
"Punchline"
EX: $|\mathbb{R}| \neq |\mathbb{N}|$ We can show that $|(0, 1)| \neq |\mathbb{N}$ using Cantor's Diagmelization method
Suppose $|(0, 1)| = |\mathbb{N}$. then $\exists$ a bijection $f : \mathbb{N} \to (0, 1)$ so we can list all $r \in (0, 1)$ as a sequence. $a_1 = f(1)$, $a_2 = f(2)$...
Using decimal notation, we have the following. $a_1 = 0.d_{n1}d_{n2}....d_{nn}$
Let $c = 0.c_1c_2c_3c_4$

10 Ch 8: Equivalence Relations

8.1 Intro

Again, recall that a relation from a set A to a set B is any subset $R \subseteq A \times B$. Now given such an R, the inverse relation for R is $R^{-1} = \{(b,a) \in B \times A | (a,b) \in R\} \subseteq B \times A$

EX: $A = \{1,2,3\} B = \{a,b,c\}$
$R = \{(2,c),(1,b),(3,b)\} \subseteq A \times B$
this means that $R^{-1} = \{(c,2),(b,1),(b,3)\} \subseteq B \times A$
This R^{-1} is not a function because it violates the uniqueness axiom, but it is still a relation from $B \times A$.

If $A = B$, then a relation $R \subseteq A \times A$. This is called a relation on A

8.2 Equivalence relations

Given a relationship $R \subseteq A \times A$ on a set A, there are 3 properties to consider:
`1. Reflexivity` R is reflexive if $(a,a) \in R$ for every $a \in A$
R is reflexice is the diagonal of A, $\{(a,a) \in A \times A | a \in A\}$ is contained in R
`2. Symmetric` R is symmetric if for any $(a,b) \in A$, $(a,b) \in R \iff (b,a) \in R$
`3. Transitive` R is transitive if for any $a,b,c \in A$ if $(a,b) \in R \wedge (b,c)]inR \rightarrow (a,c) \in R$

A relation $R \subseteq A \times A$ on a set A satisfying all three properties is called an equivalence relation on A.

EX: $A = \{1,2,3\} R = \{(1,2),(2,1),(1,1)\} \subseteq A \times A$
1. Is R reflexive?
No, $(1,1) \in R$ but $(2,2),(3,3) \notin R$
2. Is R symmetric?
Yes, $(1,2)$ is symmetric to $(2,1)$ and $(1,1)$ is symmetric with itself
3. Is R transitive?
No $(2,1),(1,2) \in R$ but $(2,2) \notin R$

EX: $A = \mathbb{R}, R = \{(a,b) \in \mathbb{R} \times \mathbb{R} | a = b\}$
Then $R = \{(a,a) | a \in \mathbb{R}\}$ So it's just the diagonal of $\mathbb{R}$
1. Is R reflexive?
Yes! Basically, by definition.
2. is R symmetric?
Yes!
3. is R transitive?
Yes! Because for any $(a,b) \in R$ then $a = b$, and for any $(b,c) \in R$ then $b = c$ $\therefore a = c$ and $(a,a) \in R$
The diagonal will always give you an equivalence relation.

EX: Let A be a set and define a relation $R \subseteq \wp(A) \times \wp(A)$ by
$(X, Y) \in R \iff X \subseteq Y$
`1.` Is R reflexive?
[NTS: $\forall X \subseteq A[(X, X) \in R]$]
let $X \in \wp(A)$, so that $X \subseteq A$. Then $(X, X) \in R \iff X \subseteq X$
since every set is a subset of itself, this is true for all $X \in \wp(A) \therefore R$ is reflexive.
`2.` is R symmetric?
[NTS: for any $X, Y \in \wp(A), (X, Y) \in R \iff (Y, X) \in R$]
Suppose $X, Y \in \wp(A)$ such that $(X, Y) \in R$. By definition, $X \subseteq Y \subseteq A$. Is $(Y, X) \in R$? This is true only if $Y \subseteq X$. This is not true in general. In fact, $X \subseteq Y \wedge Y \subseteq X \iff X = Y$. So, unless $A = \emptyset$, R is not symmetric. `3.` Is R transitive?
If $(X, Y) \in R$ and $(Y, Z) \in R$ then $X \subseteq Y$ and $Y \subseteq Z$. This means that $X \subseteq Z$. $\therefore (X, Z) \in R$

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