1

Given the relationship between the Christoffel symbol $\Gamma^{\sigma}_{\mu\nu}$ and the metric tensor $g_{\mu\nu}$ as:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

and the metric tensor given by the relationship:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^{2\alpha(t,r)} dt^2 + e^{2\beta(t,r)} dr^2 + r^2(d\theta^2 + Sin^2\theta\, d\phi^2)$$

Calculate the following Christoffel symbols:
[We use labels: $(0,1,2,3) \leftrightarrow (t, r, \theta, \phi)$]

(i) Γ^{0}_{00}

(ii) Γ^{0}_{01}

(iii) Γ^{0}_{11}

(iv) Γ^{2}_{12}

(v) Γ^{3}_{23}

(3 points for each part.)

2

Given the relationship between the Riemann Tensor $R^{\rho}_{\sigma\mu\nu}$ and the Christoffel symbols as:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

and the following metric relationship:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = a^2(d\theta^2 + Sin^2\theta\, d\phi^2)$$

where a is a constant.

and the following non-vanishing Christoffel symbols:

$$\Gamma^{\theta}_{\phi\phi} = -Sin\theta\, Cos\theta$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = Cot\theta$$

[All other Christoffel symbols are zero.]

Determine the following (given the above information):

(i) $R^{\theta}{}_{\phi\theta\phi}$

(ii) $R_{\theta\phi\theta\phi}$

(iii) $R_{\phi\phi}$

(iv) $R_{\theta\theta}$

(v) $R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi}$

(3 points for each part.)

3

.. Consider the following expression for the metric $g_{\mu\nu}$ which describes the geometry outside a Black Hole of mass M : **[15 points]**

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 Sin^2\theta\, d\phi^2$$

Note: M is the mass of the Black Hole and is a constant. (t, r, θ, ϕ) are the space-time coordinates.

For this metric, determine the following connection coefficient $\Gamma^{\gamma}_{\beta\mu}$.

[You may use the expression: $\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\mu} + \frac{\partial g_{\alpha\mu}}{\partial x^\beta} - \frac{\partial g_{\beta\mu}}{\partial x^\alpha}\right)$]

(i) Γ^{t}_{rt} (ii) Γ^{r}_{rr} (iii) Γ^{r}_{tt} (iv) $\Gamma^{r}_{\theta\theta}$

(v) $\Gamma^{r}_{\phi\phi}$ [3 points for each of the 5 parts in this question 1]

4

During the history of the Universe, just after the Quark Hadron transition, the Pions$(\pi^{\pm}, \pi^0)$ and Muons$(\mu^{\pm})$ annihilate. [17 points]

(a) Calculate the degrees of freedom g_* corresponding to $\pi^+,\ \pi^- and \pi^0$that disappear as a result of this. [3 points]

(b) Calculate the degrees of freedom g_*that correspond to $\mu^+ and \mu^-$ that disappear as a result of this. [3 points]

(c) If the number of effective degrees of freedom g_*after the Quark Hadron transition was g_*= 17.25, calculate the number of effective degrees of freedom g_* remaining after the Pions and Muons annihilate. [3 points]

5

Use the Friedmann Equation and the relationship between the energy density ρ and temperature T to express the relationship between H (Hubble Parameter) and T (Temperature) as: **[8 points]**

$$H = AG^n g_*^b T^c$$

(i) What is the value of A (simplify to a single number to 3 significant figures)? [2 points]

(ii) What is the value of b? [3 points]

(iii) What is the value of c? [2 points]

(iv) What is the value of n? [1 point]

You can use the fact that we live in a flat Universe (i.e. k = 0 in the Friedmann Equation).

Consider fermions which have a chemical potential μ, temperature T and mass m such that $\mu \gg T$ and $\mu \gg m$. [18 points]

In this case evaluate:

(i) The energy density ρ and express it in the form:

$$\rho = gK_1\mu^{a_1}$$

(a) What is the value of a_1? [3 points]

(b) What is the value of K_1? [3 points]

(ii) The number density n and express it in the form:

$$n = gK_2\mu^{a_2}$$

(a) What is the value of a_2? [3 points]

(b) What is the value of K_2? [3 points]

(iii) The pressure P and express it in the form:

$$P = gK_3\mu^{a_3}$$

(a) What is the value of a_3? [3 points]

(b) What is the value of K_3? [3 points]

You may use the relationships:

$$\rho = \frac{g}{(2\pi)^3}\int f(\vec{p})\,E(\vec{p})d^3p$$

$$n = \frac{g}{(2\pi)^3}\int f(\vec{p})\,d^3p$$

$$P = \frac{g}{(2\pi)^3}\int \frac{|\vec{p}|^2}{3E} f(\vec{p})\,d^3p$$

and $f(\vec{p}) = \frac{1}{exp\left[\frac{(E-\mu)}{T}\right]+1}$ for fermions with $E^2 = |\vec{p}|^2 + m^2$

(i)

Use the First Law of Thermodynamics:

$$\frac{d}{dt}(\rho R^3) = -p\frac{d}{dt}(R^3)$$

and the equation of state :

$$p = \omega\rho$$

to arrive at a relationship between ρ and R of the form:

$$ln\ \rho = ln\ K + E\ ln\ R$$

where $K, E\ and\ \omega$ are constants.

What is the value of E expressed in terms of ω ?

(ii)

For radiation, $\omega = \frac{1}{3}$

What is the value of A in the expression for $\rho = C R^{-A}$? (where C is Constant)

(iii)

For matter, $\omega = 0$

What is the value of B in the expression $\rho = C R^{-B}$? (where C is Constant)

(iv)

For vacuum, $\omega = -1$

What is the value of D in the expression $\rho = C R^{-D}$? (where C is Constant)

(v)

For a Closed Universe, what is the condition for $\Omega = \frac{\rho}{\rho_C}$, where ρ_c is the critical density?

Q.1 The Christoffel symbol $\Gamma^{\alpha}_{\mu\nu}$ can be expressed in terms of the metric tensor by the relationship: [15 points]

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right)$$

Here α, β, μ, ν range over values 0, 1, 2, 3 for the space time coordinates. Use this relationship and the metric tensor $(g_{\mu\nu})$ given by the relationship:

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = -(1+2\phi)dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)$$

to determine the following Christoffel Symbols to first order in ϕ (you can neglect product terms in ϕ of the form ϕ^p with $p \geq 2$) : [Note here that the coordinates are labelled as $(X^0, X^1, X^2, X^3) \equiv (t, x, y, z)$ and c = 1] [Recall that to linear order in : $(1+\phi)^{-n} = 1 - n\phi + O(\phi^2) = 1 - n\phi$, to linear order in ϕ.]

(i) Γ^{0}_{00}

(ii) Γ^{0}_{01}

(iii) Γ^{1}_{00}

(iv) Γ^{2}_{00}

(v) Γ^{3}_{00}

Q.2 Consider the 2-D Surface of a Sphere embedded in 3 Space Dimensions.
In this case the differential element of length is given by:

$dl^2 = dx^2 + dy^2 + dz^2$

And the surface of the sphere is defined by

$x^2 + y^2 + z^2 = R^2$

Now introduce the polar coordinates through the relationships:

$x = r'Cos\theta \,;\, y = r'Sin\theta$.

Express the differential length element as:

$dl^2 = F(R, r')dr'^2 + G(R, r')d\theta^2$

(i) Evaluate and simplify $F(R, r')$ as much as possible. [8 points]

(ii) Evaluate and simplify $G(R, r')$ as much as possible. [7 points]