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1. Hoare Triples

Task 1.1

Let S = while $i < n$ do $x := x * i$; $i := i+1$ od

a. $\{i=1, n=6\} \models [i<n]\ S\ [i=n]$

when we apply this value to S,

$\Sigma_0 = \{i=1, n=6\}$

$\Sigma_1 = \{i=2, n=6\}$

$\Sigma_2 = \{i=3, n=12\}$ and goes on

→ Here the pre-condition satisfied since $i<n$

→ The loop(s) will goes on without terminating

→ The post condition will not be satisfied which is $i=x$.

Hence the above total Correctness triple is unsatisfied.

b. $\{i=-1, n=5\} \models \{i<n\}\ S\ \{i \geq 0 \wedge n \leq 0\}$

when we apply this to S,

$\Sigma_0 = \{i=-1, x=5\}$

$\Sigma_1 = \{i=0, n=-5\}$

→ The ~~post con~~ pre-condition is satisfied since $i<n$

→ Statement terminates with $\{i=0, n=-5\}$

→ And the post condition $\{i \geq 0 \wedge n \leq 0\}$ is satisfied

Hence the above partial correctness triple is Satisfied

c. $\{i=1, x=0\} \models \{i<n\}\ S\ \{i=n\}$

when we apply this value to S,

$\Sigma_0 = \{i=1, n=0\}$

→ Here the pre-condition is failed

→ Program is terminated since it did not enter the loop

→ And the post condition is also failed, since $i \neq n$

Hence the partial correctness triple is satisfied ~~state~~ in the given state since pre-condition is failed and it doesn't matter if the program is executed.

d. $\{i=1, n=2\} \models \{x=k\}\ S\ \{n=k!\}$

when we apply this value to S,

$\Sigma_0 = \{i=1, n=2, k=2\}$

$\Sigma_\varphi = \{i=2, n=2, k=2\}$

→ Here the precondition is satisfied.

→ Program is terminated since i is not less than x at $\{i=2, n=2\}$

→ And the post-condition is also satisfied since $\{n=k!\}$

Hence the partial correctness triple is satisfied

~~Task 1.2.~~

~~S = while $i<n$ do $x := x*i$; $i := i+1$ od~~

~~a) $\{T\}\ S\ \{n>0\}$~~

e. $\{i=1, n=6\} \models \{T\}\ S\ \{\exists k. x = k!\}$

when we apply this value to s,

$\Sigma_0 = \{i=1, x=6\}$

$\Sigma_1 = \{i=1, x=6\}$

$\Sigma_2 = \{i=3, n=6\}$ and goes on.

→ Here the pre-condition is satisfied

→ The program doesn't terminate. So post condition won't be checked.

→ Hence the Partial Correctness triple is satisfied

Task 1.2

$S =$ while $i<n$ do $x := x+i$; $i := i+1$ od

a) $\{T\}\ S\ \{n>0\}$

→ Not Valid

Counter example:

lets take $\{x=2, i=0\}$

$\Sigma_0 = \{i=0, x=2\}$, $\Sigma_1 = \{x=0, i=1\}$

→ Here precondition is satisfied, program gets terminated but the post condition fails since $n=0$.

Fix

To make this Valid lets fix the post condition

Post condition $\Rightarrow \{i > x\}$

so the triple will be $\{T\}\ S\ \{i > n\}$

b) $\{x = k\}\ S\ \{x = k\}$

$\rightarrow$ Not Valid

~~Co~~unter example:

lets take $\{x = 2,\ i = -1\}$ and after loop $x := -2$ and

Post condition fails.

Fix $\rightarrow$ fixing the program

S = while $i < n$ do $x := x$; $i = i + 1$ od

example

$\{i = 1, x = 2, k = 2\}$ after loop $\{i = 2, x = 2, k = 2\} \ldots$

$\rightarrow$ Pre-condition satisfied, program ends and post condition satisfied so, $\{x = k\}$ while $i < n$ do $x := x$; $i = i+1$ od $\{x = k\}$

c) $[i = 1 \wedge n = k \wedge n > 0]\ S\ [i = k \wedge x = k\ !]$

$\rightarrow$ Not Valid

Counter example

$\{i = 1, k = 5, n = 5\}$ after loop $\{i = 2, k = 5, x = 5\}$

$\rightarrow$ Precondition satisfied, program executes ~~but~~ but post condition failed.

Fix

Fix the post condition. to $[x = k]$. and triple will be $[i = 1 \wedge x = k \wedge n > 0]\ S\ [x = k]$

example

$\{i = 1, k = 5, n = 5\}$ after loop $\{i = 2, k = 5, x = 5\}$

Task 1.3

Precondition $\Rightarrow$ $m = 2k \wedge \exists_k \in \mathbb{Z}. k \leq 0 \wedge r = 1$

So triple would be,

$[m = 2k \wedge \exists_k \in \mathbb{Z}. k \leq 0 \wedge r = 1]$ $n := -m$; while $n \neq 0$ do

$r := r * -3$; $n := n - 1$ od $[r = 3^{-m}]$

example:

lets take $\{m = -2, n = 2, r = 1\}$

$\Sigma_0 = \{m = -2, n = 2, r = 1\}$

$\Sigma_1 = \{m = -2, n = 1, r = -3\}$

$\Sigma_2 = \{m = -2, n = 0, r = 9\}$

→ Here the precondition is satified with given value.

→ Program terminates with $\{m = -2, n = 0, r = 9\}$

→ Post condition is satisfied since $3^{-m} = r \left(3^{-(-2)} = 9\right)$

Task 1.4

$[P_1]\ x := sqrt(x)/y\ [T]$

$P_1 = [x \geq 0 \wedge y \neq 0]$

→ with this pre-condition the statement always executes without error and x will be updated.

→ Also the post condition is always true.

Hence $[x \geq 0 \wedge y \neq 0]\ x := sqrt(x)/y\ [T]$

Task 1.5

a) $\nexists [P_2]\ S_2\ [T]$

Program:

$[x > 0]$ while $x > 0$ do $x := x + 1$ od $[T]$

→ Here the pre condition is satisfied

→ ~~Since~~ the statement will never terminate since x will be always greater than 0

→ Post condition is also always true.

→ Hence the total Correctness triple is invalid.

b.

Since I have used while in previous ans, I am writing different S,

$[x > 0 \wedge y = 0] \ x := x/y \ [T]$

→ Here the precondition is $x > 0$ and $y = 0$

→ The statement will be executed and end in error.

→ Hence the total Correctness triple is invalid.

Task 1.6

a.

S = while $x > 0$ do if even(x) then $x := 5x+1$ else $x := x/2$ fi od.

triple valid ⇒ $[x > 3] \ S \ [x \le 1]$

<u>example</u>.

lets take $x = 4$

$\cancel{\Sigma_0 = \{4\}, \ \Sigma_1 = \{21\}, \ \Sigma_2 =}$

$\Sigma_0 = \{x = 4\}$, $\Sigma_1 = \{x = 21\}$, $\Sigma_2 = \{x = 10.5\}$

$\Sigma_3 = \{x = 5.25\}$, $\Sigma_4 = \{x = 2.625\}$, $\Sigma_5 = \{x = 1.3125\}$

$\Sigma_6 = \{x = 0.65\}$

→ Here the pre-condition is satisfied.

→ Program terminates when $x = 0.065$

→ And the post condition is satisfied.

→ Hence the triple Valid for total Correctness triple.

b.

$S =$ while $x > 1$ do if even(x) then $x := 5x + 1$ else

$x := x/2$ fi od

Valid triple $\{x \leq 3\}$ S $\{F\}$

~~Here the~~

~~example~~

~~let take~~

→ Here the pre-condition will Satisfy

→ Statement will execute but never terminate

→ But the post-condition is always false.

→ Hence the Partial Correctness triple is Valid when $x \leq 3$ since the Statement will never execute

2. Substitution

Task 2.1

a) $[y+2/y]\ \exists_z . \forall_x . (x + y \geq z + y)$

$= \exists_z . [y+2/y] . \forall_x\ (x + y \geq z + y)$

$= \exists_z . \forall_x . [y+2/y]\ (x + y \geq z + y)$

$= \exists_z . \forall_x . ((x + (y+2)) \geq (z + (y+2)))$

$= \exists_z . \forall_x . (x + y + 2 \geq z + y + 2)$

b) $[y+2/x]\ \exists_z . \forall_x . (x + y \geq z + y)$

$= \exists_z . [y+2/x] . \forall_x \ (x + y \geq z + y)$

$= \exists_z . \forall_x \ (x + y \geq z + y)$ (Since no free variable x, there is not Substitution)

c) $[x+2/y]\ \exists_z . \forall_x \ (x + y \geq z + y)$

$= \exists_z . [x+2/y] . \forall_x \ (x + y \geq z + y)$

In this to avoid the variable capture, we need to do α conversion on bound variable x, and renaming it to w.

$= \exists_z . [x+2/y]\ \forall_w \ (w + y \geq z + y)$

$= \exists_z . \forall_w \ [x+2/y] (w + y \geq z + y)$

$= \exists_z . \forall_w \ (w + (x+2) \geq z + (x+2))$

$= \exists_z . \forall_w \ (w + x + 2 \geq z + x + 2)$

d. $[z/x]\ (x \geq z \rightarrow (\exists_z . \forall_x . x + y \geq z + y) \wedge y > z)$

after applying α conversion on bound variables x, z with w, v respectively.

$= [z/x]\ (x \geq z \rightarrow (\exists_v . \forall_w . w + y \geq v + y) \wedge y > z)$

$= (z \geq z \rightarrow (\exists_v . \forall_w . w + y \geq v + y) \wedge y > z)$

$= (z \geq z \rightarrow (\exists_v . \forall_w . w \geq v) \wedge y > z)$

$= (z \geq z \rightarrow (\exists_v . \forall_w . w \geq v) \wedge y > z)$

e. $[z/x]\ (x \geq z \rightarrow (\exists_x . x + y \geq z + y) \wedge y > z)$

after applying α conversion on bound variable x with w.

$= [z/x]\ (x \geq z \rightarrow (\exists_w . w + y \geq z + y) \wedge y > z)$

$= (z \geq z \rightarrow (\exists_w . w \geq z) \wedge y > z)$

3. proofs and proof outlines.

Task 3.1

a. given $size(a) = 2$

S = if $a[0] > a[1]$ then $m := a[0]$; else $m := a[1]$ fi

b. $\{T\}\ S\ \{m = max(a[0], a[1])\}$

to expand,

$\{T\}$ if $a[0] > a[1]$ then $m := a[0]$; else $m := a[1]$ fi
$\{(m = a[0] \land a[0] \geq a[1]) \lor (m = a[1] \land a[1] > a[0])\}$

c. Hilbert-Style ~~proof outline~~ for the above S = if $a[0] > a[1]$ then $m := a[0]$ else $m := a[1]$ fi

1. $\{a[0] \geq a[1]\}\ m := a[0]\ \{m = a[0] \land a[0] \geq a[1]\}$ Assign
2. $\{T \land a[0] \geq a[1]\}\ m := a[0]\ \{m = a[0] \land a[0] > a[1]\}$ Conseq(1)
3. $\{a[0] < a[1]\}\ m := a[1]\ \{m = a[1] \land a[1] > a[0]\}$ Assign
4. $\{T \land a[0] < a[1]\}\ m := a[1]\ \{m = a[1] \land a[1] > a[0]\}$ Conseq(3)
5. $\{T\}$ if $a[0] > a[1]$ then $m := a[0]$ else $m := a[1]$ fi
$\{(m = a[0] \land a[0] > a[1]) \lor (m = a[1] \land a[1] > a[0])\}$ if (1, 2)

d.

$\{T\}$

if $a[0] > a[1]$ then	$\{T \wedge a[0] > a[1]\}$ (or) $\{a[0] = a[0] \wedge a[0] > a[1]\}$
$m := a[0]$	$\{m = a[0] \wedge m > a[1]\}$
else	$\{T \wedge (a[1] > a[0])\}$ or $\{a[1] = a[1] \wedge a[1] > a[0]\}$
$m := a[1]$	$\{m = a[1] \wedge a[0] < a[1]\}$
fi	$\{(m = a[0] \wedge a[0] > a[1]) \vee (m = a[1] \wedge a[0] < a[1])\}$

Task 3.2

1. $\{x \geq 0 \wedge x \% 3 = 0\}\ s := x\ \{s \geq 0 \wedge s \% 3 = 0\}$ Assign.

2. $\{x \geq 0 \wedge x \% 3 = 0\}\ s := x\ \{s \% 3 = 0\}$ Conseq (1)

3. $\{x - 1 \geq 0 \wedge x - 1 \% 3 = 0\}\ s := x - 1\ \{s \geq 0 \wedge s \% 3 = 0\}$ Assign.

4. $\{x \geq 0 \wedge x \% 3 \neq 0 \wedge x \% 3 = 1\}\ s := x - 1\ \{s \% 3 = 0\}$ Conseq (3)

5. $\{x - 2 \geq 0 \wedge x - 2 \% 3 = 0\}\ s := x - 2\ \{s \geq 0 \wedge s \% 3 = 0\}$ Assign.

6. $\{x \geq 0 \wedge x \% 3 \neq 0 \wedge x \% 3 \neq 1\}\ s := x - 2\ \{s \% 3 = 0\}$ Conseq (5)

7. $\{x \geq 0 \wedge x \% 3 \neq 0\}$ if $x \% 3 = 1$ then $s := x - 1$ else $s := x - 2$ fi $\{s \% 3 = 0\}$ if (4, 6)

8. $\{x \geq 0\}$ if $x \% 3 = 0$ then $s := x$ else if $x \% 3 = 1$ then $s := x - 1$ else $s := x - 2$ fi fi $\{s \% 3 = 0\}$ if (2, 8)

4. It took me approximately 14 hours to complete the assignment.