Chapter 2

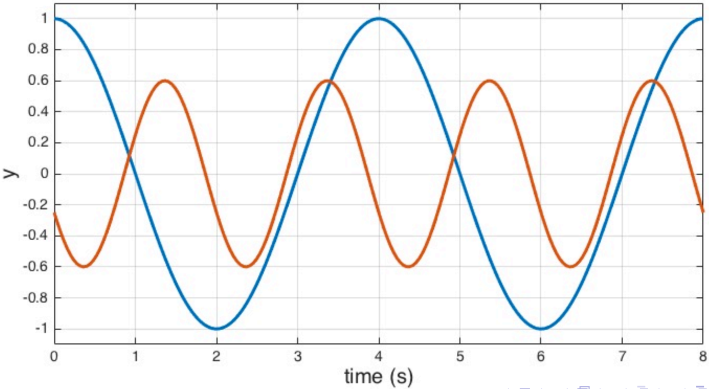
Fourier transform

Periodic functions

Let's consider a function (or as we shall call it a signal)

$$s(t) = A\cos(\omega t + \phi)$$

blue : $A = 1,\ \omega = 2\pi\frac{1}{4},\ \phi = 0$ red : $A = 0.6,\ \omega = 2\pi\frac{1}{2},\ \phi = 2$



The time delay τ can be calculated by:

$$s_1(t) = s_2(t-\tau) = \cos(\omega t) = \cos(\omega(t-\tau)+\phi) \Rightarrow \omega\tau = \phi \Rightarrow \tau = \frac{\phi}{\omega}$$

Def.: Given a signal $s(t) = A\cos(\omega t + \phi)$ the magnitude A is called the amplitude of the signal.

Orthogonal functions

Def.: Two functions $f(x)$ and $g(x)$ are called orthogonal if their scalar product is equal to zero:

$$\langle f, g \rangle = \int_a^b f^*(x) g(x)\, dx = 0$$

Problem 2.2a Show that $\{\cos nx, \sin nx\}$ $(n \geq 0)$ are orthogonal in $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} \cos nx \sin nx\, dx = -\frac{1}{n} \int_{-\pi}^{\pi} \cos nx\, d \cos nx = -\frac{\cos^2 nx}{2n} \Big|_{-\pi}^{\pi} = 0$$

Problem 2.5a Find Fourier series for $s(t) = |t|$ on $[-\pi, \pi]$.

We note that this is an even function. Thus, $b_n = 0$. Let's find a_n:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |t| \, dt = \frac{1}{\pi} \int_0^{\pi} t \, dt = \frac{1}{\pi} \frac{\pi^2}{2} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |t| \cos(nt) \, dt = \frac{2}{\pi} \int_0^{\pi} t \cos(nt) \, dt = [\text{by parts}] =$$

$$= \frac{2}{\pi n} \left[t \sin(nt) \Big|_0^{\pi} - \int_0^{\pi} \sin(nt) \, dt \right] = \frac{2}{\pi n^2} \cos(nt) \Big|_0^{\pi} = \frac{2[(-1)^n - 1]}{\pi n^2} =$$

$$= -\frac{4}{\pi n^2} \begin{cases} 0, & \text{if } \; n = 2k \\ 1, & \text{if } \; n = 2k+1 \end{cases}$$

Matlab script for Problem 2.5a

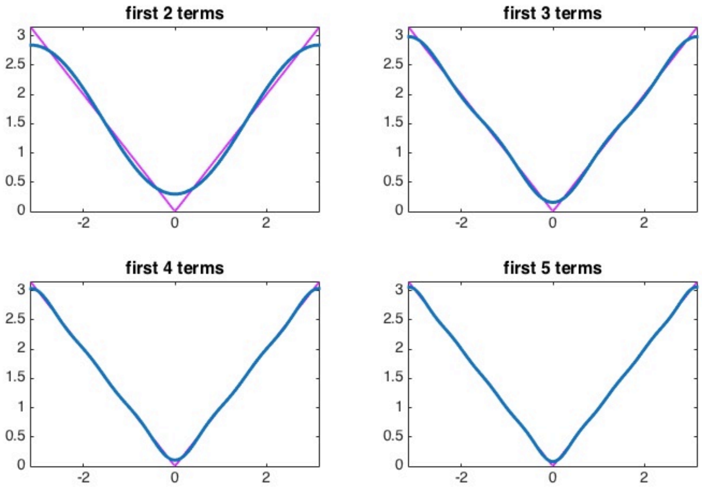
```
%% Fourier Series for s(t) = |t| on [-pi,pi]
t = linspace(-pi,pi,200); % time
s = abs(t);               % signal

a0 = pi/2; f0 = ones(1,200); % constant term

k = (1:5)';
a = -4./(pi*(2*k-1).^2);  % values for a_n
e = cos((2*k-1)*t);       % cos(nt) functions

% Make drawing
figure('color','w','position',[100 100 800 500])
sp = a0*f0;
for k = 1:4
    subplot(2,2,k)
    plot(t,s,'m','LineWidth',2)
    hold on
    sp = sp + a(k)*f(k,:);
```

Graphic representation for Problem 2.5a



$s(t)$ is defined on the interval $[0, T]$. The Fourier series:

$$s(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi n}{T} t\right) + b_n \sin\left(\frac{2\pi n}{T} t\right)\right)$$

$$a_n = \frac{2}{T} \int_0^T s(t) \cos\left(\frac{2\pi n}{T} t\right)\, dt$$

$$b_n = \frac{2}{T} \int_0^T s(t) \sin\left(\frac{2\pi n}{T} t\right)\, dt$$

$$a_0 = \frac{1}{T} \int_0^T s(t)\, dt$$

Theorem (*Convergence*). If a Fourier series of a function $s(t)$ with period T converges uniformly, then the series converges to $s(t)$.

Convergence of Fourier series

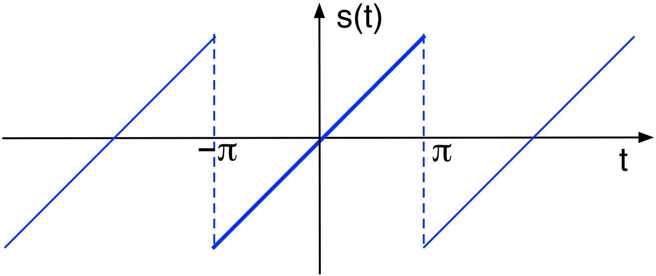
Theorem: Let $s : \mathbb{R} \to \mathbb{R}$, $f(t+T) = f(t)$ be a piecewise continuous (finite number of step-type discontinuities). If for $t \in [0, T]$ there exist lateral derivatives $s'(t^-)$ and $s'(t^+)$, then its Fourier series converges pointwise to

$$\frac{s(t^+) + s(t^-)}{2}$$

Note: If $s(t)$ is derivable, then its Fourier series converges pointwise to it. Indeed, $(s(t^+) + s(t^-))/2 = s(t)$.

Problem 2.7.4 $s(t) = t,\ t \in (-\pi, \pi]$

1. Draw the function



2. The existence of a Fourier series: It is derivable in $(-\pi, \pi)$, thus its Fourier series converges on this interval pointwise to $s(t)$. At $t = \pi$ the series will converge to $(\pi - \pi)/2 = 0$.

3. Fourier series: It's an odd function, thus $a_n = 0$.

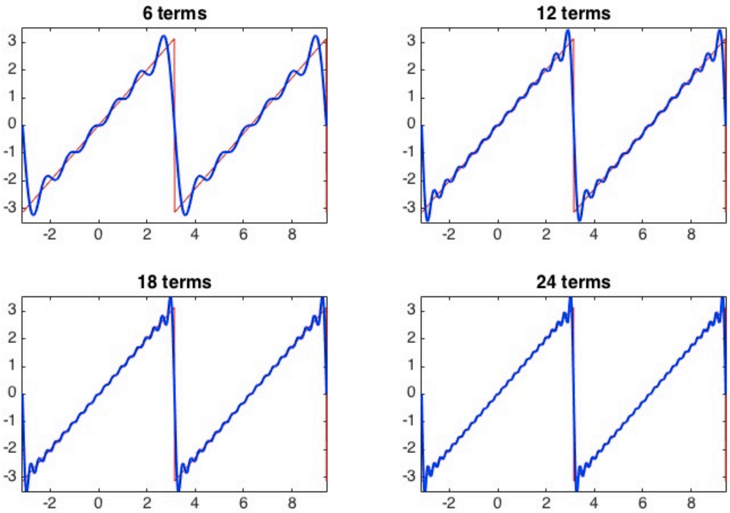
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)\,dt = [u = t,\ \sin(nt)dt = dv, v = -\cos(nt)/n] =$$

$$= -\frac{t\cos(nt)}{\pi n}\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos(nt)\,dt = -\frac{2(-1)^n}{n}$$

Thus on $t \in (-\pi, \pi)$

$$s(t) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(nt)}{n}$$

For $t = \pi$ we have $\sin(n\pi) = 0$, therefore, the series is equal to 0. This coincides with the Theorem (see item 2).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12 terms
18 terms
24 terms
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