2.1

(1)

- $\dfrac{\partial f}{\partial x} = \begin{bmatrix} -1 + 6x_1^2 & 1 \\ -1 & -1 \end{bmatrix}$

- equilibrium points :

  $x_s^1 = (0, 0) \rightarrow$ stable focus

  $x_s^2 = (1, -1) \rightarrow$ saddle

  $x_s^3 = (-1, 1) \rightarrow$ saddle

(3)

- $\dfrac{\partial f}{\partial x} = \begin{bmatrix} 1 - x_1 - \dfrac{2x_2}{1+x_1} + \left(-1 + \dfrac{2x_2}{(1+x_1)^2}\right)x_1 & -\dfrac{2x_1}{1+x_1} \\ \dfrac{x_2^2}{(1+x_1)^2} & 2 - \dfrac{x_2}{1+x_1} - \dfrac{x_2}{1+x_1} \end{bmatrix}$

  $x_s^1 = (0, 0) \rightarrow$ unstable node

  $x_s^2 = (0, 2) \rightarrow$ stable node

  $x_s^3 = (1, 0) \rightarrow$ saddle

  $x_s^4 = (-3, -4) \rightarrow$ saddle

(5) ① Isolated equilibrium point

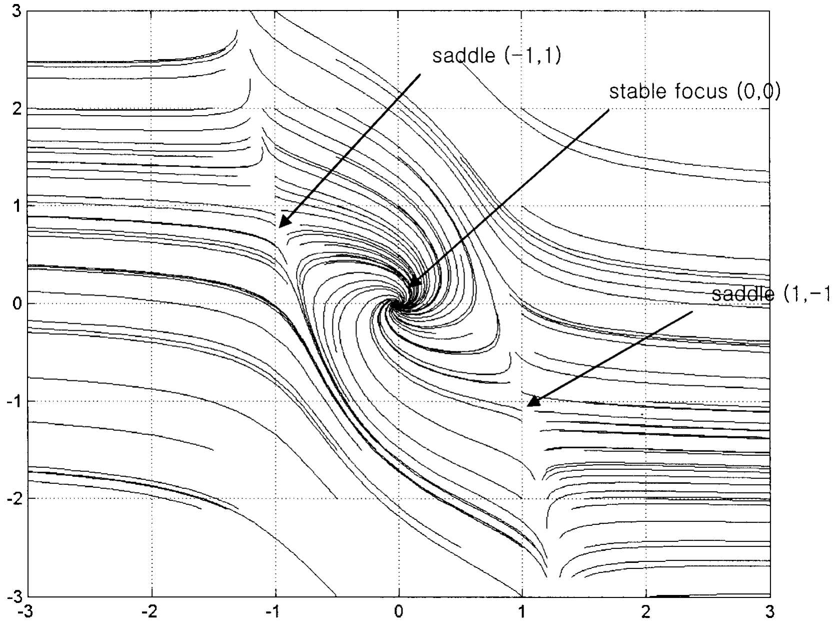
$x_s = (0,0) \rightarrow$ unstable focus

② Nonisolated equilibrium points

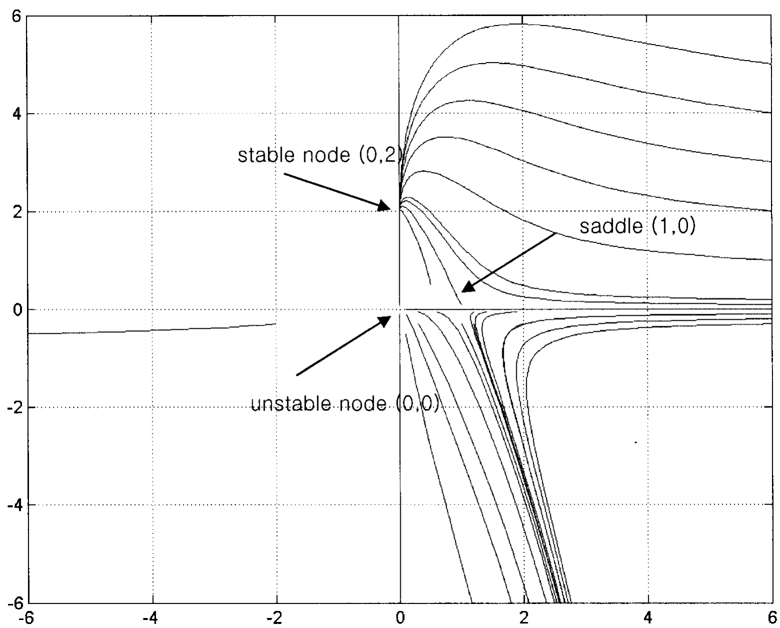
There are many nonisolated equilibrium points represented by $S = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$.

2.3

(1)

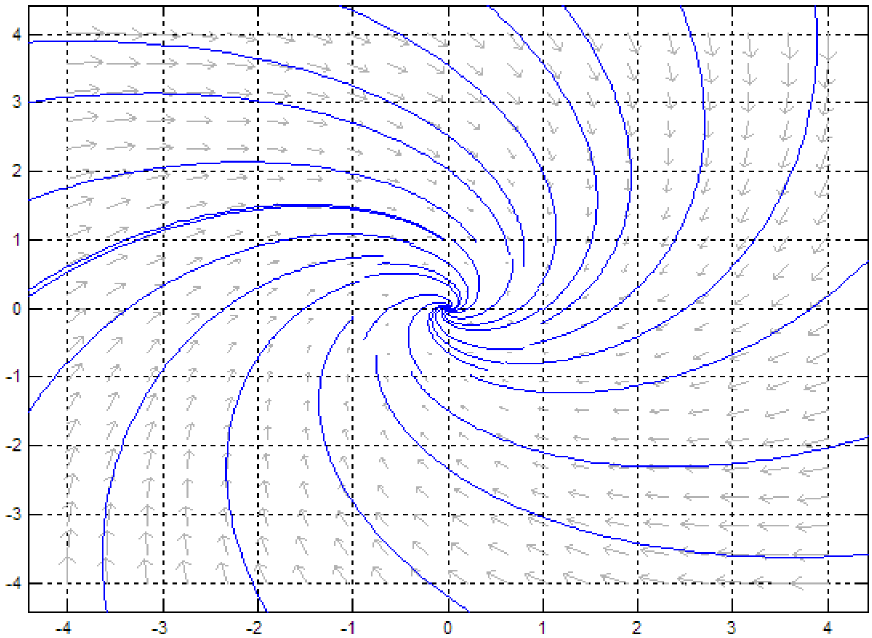

saddle (−1,1)
stable focus (0,0)
saddle (1,−1)

(3)

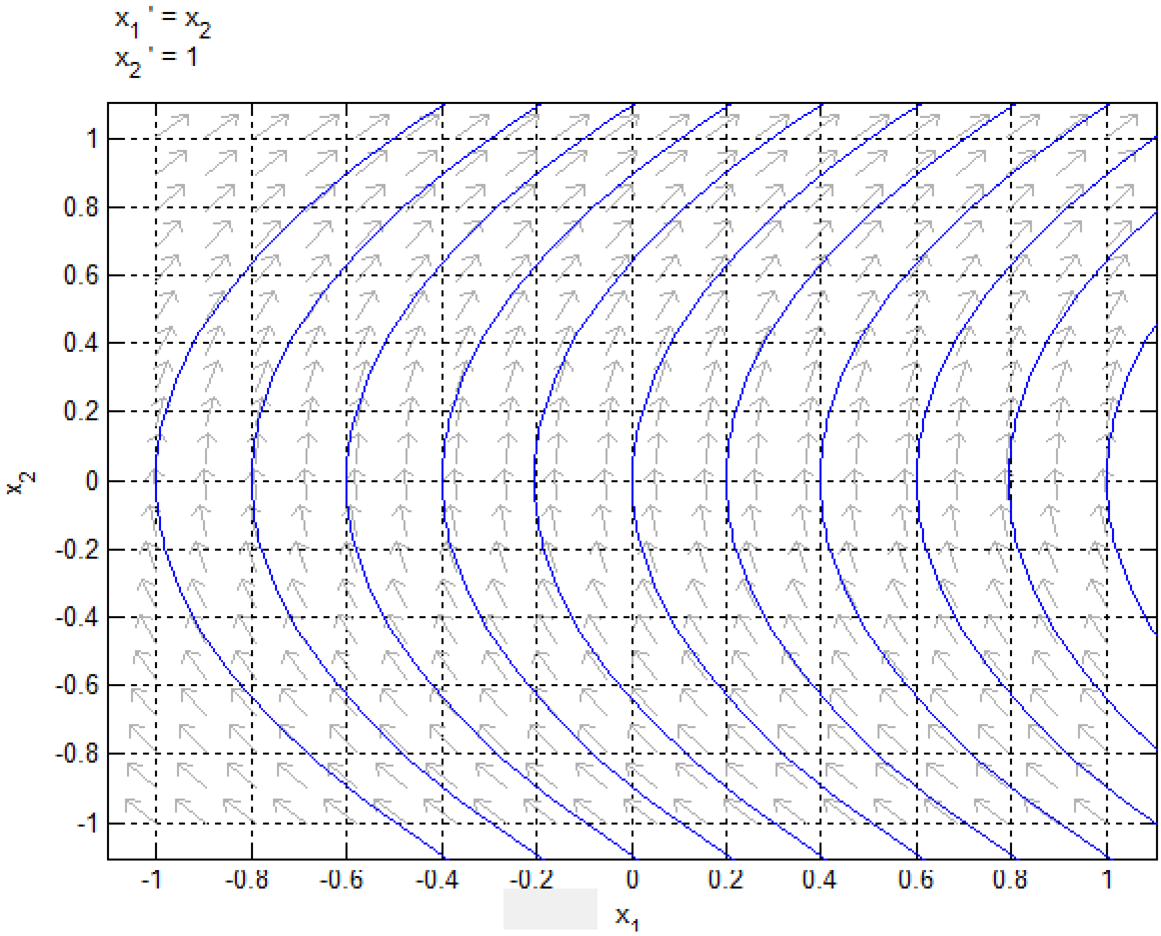

stable node (0,2)
saddle (1,0)
unstable node (0,0)

2.3

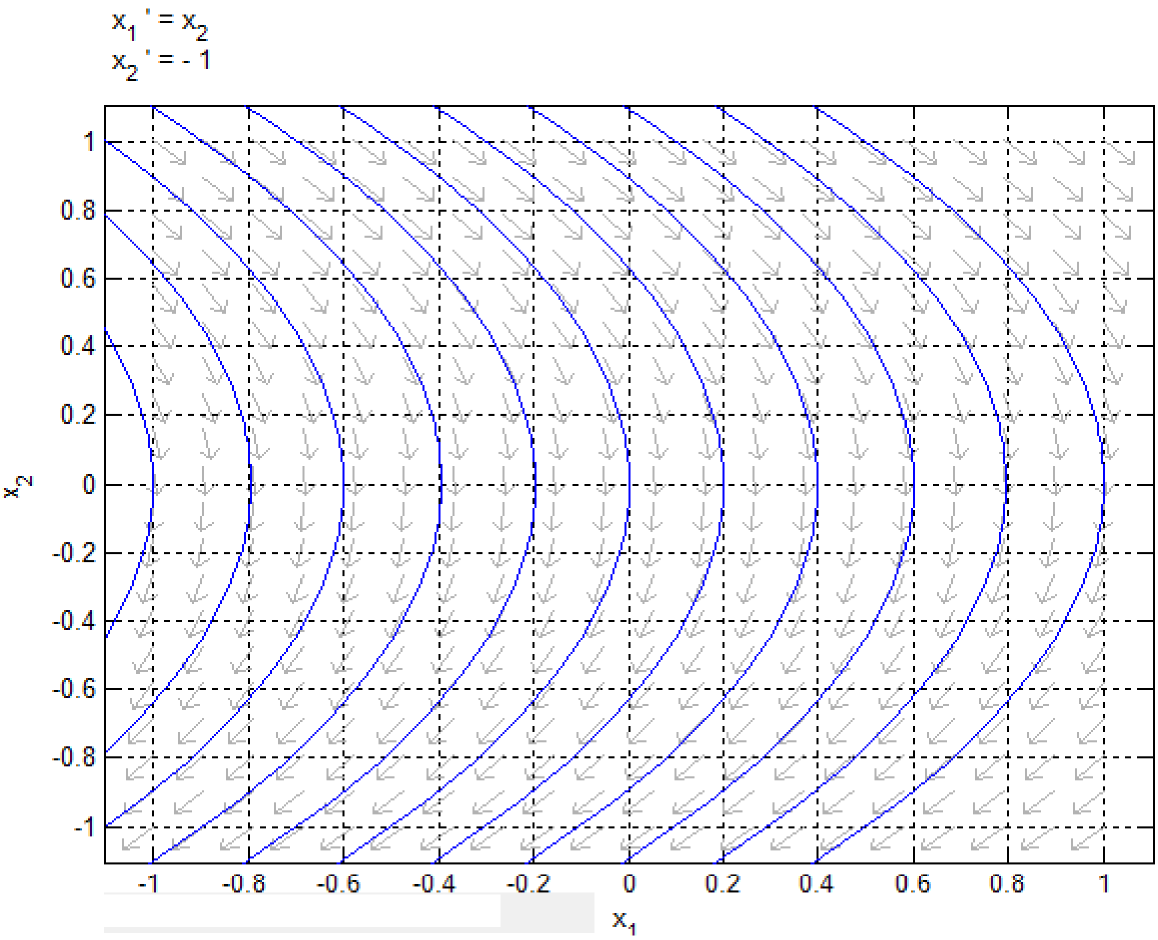
(5)



2.15 (a)

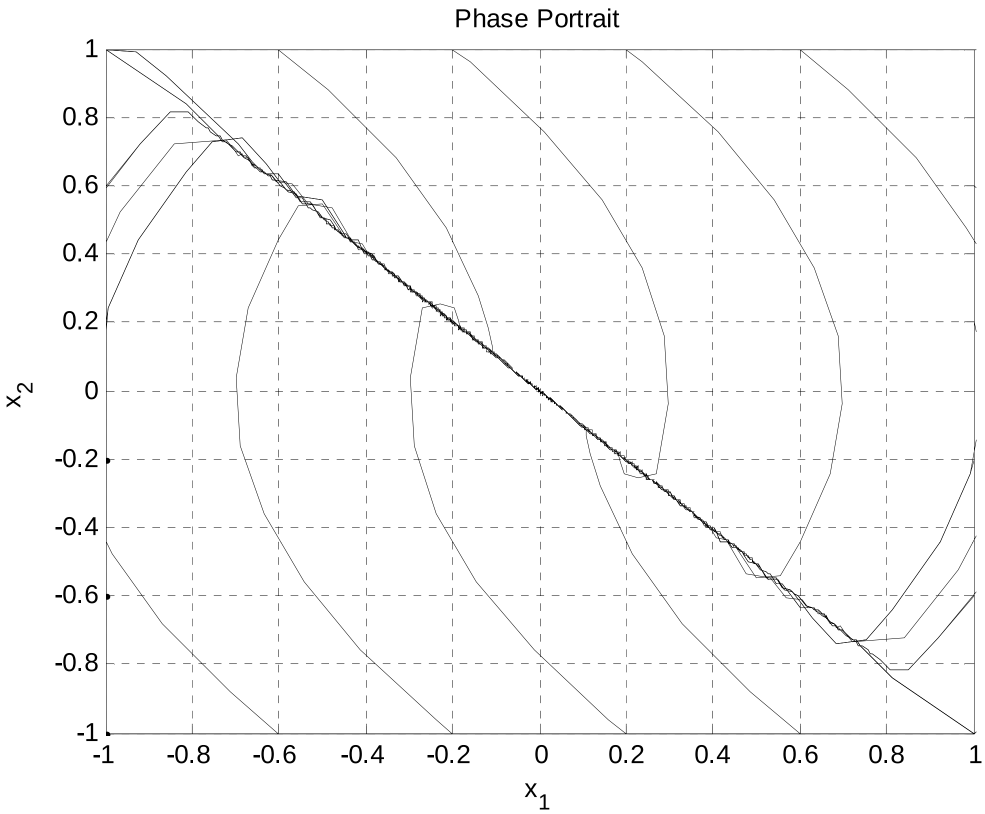


2.15 (b)



2.15 (c)

$$u = \begin{cases} -1 & \text{if} \quad x_1 + x_2 \geq 0 \\ 1 & \text{if} \quad x_1 + x_2 < 0 \end{cases}$$



2.17

(1)

Let $x_1 = y,\ x_2 = \dot{x}_1,\ \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \varepsilon x_2 (1 - x_1{}^2 - x_2{}^2) \end{cases}$

Let $M = \{x \mid c_1 \le V(x) \le c_2\}$ where $V(x) = x_1{}^2 + x_2{}^2$ and $c_2 > c_1 > 0$

$f(x)\nabla V(x) = 2\varepsilon x_2{}^2 (1 - x_1{}^2 - x_2{}^2)$

By setting $c_1 = \dfrac{1}{2},\ c_2 = \dfrac{3}{2}$

$$\begin{cases} f(x)\nabla V(x) > 0 \quad \text{on } V(x) = c_1 \\ f(x)\nabla V(x) < 0 \quad \text{on } V(x) = c_2 \end{cases}$$

Moreover, M does not contain any equlibrium point.

$\therefore$ M contains a periodic orbit

2.17

(3)

We need to find a closed boundary set M so that M contains no equilibrium point and positively invariant.

Consider $V = 3x_1^2 + 2x_1x_2 + 2x_2^2$.

$$\dot{V} = 6x_1x_2 + 2x_2^2 + 2(x_1 + 2x_2)(-x_1 + x_2) - 4(x_1 + 2x_2)^2 x_2^2$$
$$= -2(x_1^2 + x_2^2) + 1 - (1 - 2x_2(x_1 + 2x_2))^2$$
$$\leq -2(x_1^2 + x_2^2) + 1 \leq 0$$

if $x_1^2 + x_2^2 \geq \frac{1}{2}$.

Thus construct the surface $V(x) = c$ so that this surface contains the circle $\{x_1^2 + x_2^2 = \frac{1}{2}\}$ in it.

Then all trajectories stating in $M = \{V(x) \leq c\}$ stay in M.

Moreover, linearization at 0 shows that the origin is unstable.

Therefore by P - B Theorem, there is a periodic solution.

2.20

(1) Using Bendixson criterion

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -1 + a \neq 0$$

=> no change of sign $(\because a \neq 1$ and $a:$ constant)

$\therefore$ no limit cycle

(3) Using Bendixson criterion

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -{x_2}^2 \neq 0$$

i) $D_1 = \{(x_1, x_2) | x_2 < 0, (x_1, x_2) \in R^2\}$

ii) $D_2 = \{(x_1, x_2) | x_2 > 0, (x_1, x_2) \in R^2\}$

=> in D_1 and D_2, no change of sign

$$(\because \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \neq 0)$$

$\therefore$ The system has no periodic orbits lying entierly in D_1 or D_2.

iii) across D_1 and D_2

$$x_1 = 0 \rightarrow \begin{cases} \dot{x}_1 = 1 \\ \dot{x}_2 = 0 \end{cases}, \quad x_2 = 0 \rightarrow \begin{cases} \dot{x}_1 = 1 \\ \dot{x}_2 = x_1 \end{cases}$$

no closed curve $\rightarrow$ no limit cycle

2.20

(5) We use Index theorem.

$$\text{eq. pts} \rightarrow \begin{bmatrix} x_1 = n\pi, (n = 0, \pm 1, \pm 2 \cdots) \\ x_2 = 0 \end{bmatrix}$$

$$n = 2k (k = 0,1,2 \cdots) \rightarrow \begin{bmatrix} 2kn \\ 0 \end{bmatrix} \rightarrow \lambda = \pm 1 : \text{saddle}$$

$$n = 2k + 1 (k = 0,1,2 \cdots) \rightarrow \begin{bmatrix} 2(k+1)n \\ 0 \end{bmatrix} \rightarrow \lambda = \pm 1 : \text{saddle}$$

$\therefore$ no limit cycle

2.22

(a) Let $V(x) = x_2$

Then $V(x) > 0$ in $\forall x \in D$ and $V(x) = 0$ on ∂D

$$\dot{V}(x)\Big|_{V(x)=0} = bx_1^{\,2} \geq 0$$

Hence the trajectories on the boundary D

must move into D

$\therefore$ D : positively invariant set

(b)

$\nabla f(x) = a - c - x_2 < 0, \;\; \forall x \in D$

(no change of sign)

From (a), D is shown to be a positively

invariant set.

$\therefore$ No periodic orbit through any point $x \in D$.