5.11

(1)

Let $V = \frac{1}{2}(x_1{}^2 + x_2{}^2)$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$

$$= x_1(-x_1 + x_1{}^2 x_2) + x_2(-x_1{}^3 - x_2 + u)$$

$$\le -\|x\|_2{}^2 + \|x\|_2 |u|$$

$$= -(1-\theta)\|x\|_2{}^2 - \theta\|x\|_2{}^2 + \|x\|_2 |u|, \quad 0 < \theta < 1$$

$$\le -(1-\theta)\|x\|_2{}^2, \ \forall \|x\|_2 \ge \frac{|u|}{\theta}$$

Thus, by Theorem 4.19, the system is input to state stable.

$\|h(t,x,u)\| \le \alpha_1(\|x\|)$ is satisfied where α_1 is class K.

$\therefore$ by Theorem 5.3, L_∞ is stable

5.11

(3)

Let $V = \frac{1}{2}(x_1^2 + x_2^2)$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$

$$= -x_1^2 - x_2^2 + (x_1^2 + x_2^2)\|x\|_2^2 - x_1 u(1 - \|x\|_2^2)$$

$$\leq -\|x\|_2^2 + \|x\|_2^4 + \|x\|_2 |u|(1 - \|x\|_2^2), \qquad \forall \|x\|_2 < 1$$

$$\leq -(1-\theta)\|x\|_2^2(1 - \|x\|_2^2) - \theta\|x\|_2^2(1 - \|x\|_2^2) + \|x\|_2 |u|(1 - \|x\|_2^2), \quad 0 < \theta < 1$$

$$\leq -(1-\theta)\|x\|_2^2, \ \forall 1 > \|x\|_2 \geq \frac{|u|}{\theta}$$

Thus, by Theorem 4.19, the system is locally input to state stable.

$\|h(t,x,u)\| \leq \alpha_1(\|x\|)$ is satisfied where α_1 is class K.

$\therefore$ by Theorem 5.2, small- signal L_∞ stable.

5.11

(5)

Let $V = \frac{1}{2}(x_1^{\,2} + x_2^{\,2})$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$
$$= -x_1^{\,2} + x_1^{\,3}x_2 + x_1x_2 - x_2^{\,2} + x_2u$$
$$\le -\|x\|_2^{\,2} + x_1x_2(x_1^{\,2}+1) + \|x\|_2\,|u|$$
$$\le -\|x\|_2^{\,2} + \frac{1}{2}\|x\|_2^{\,2}(\|x\|_2^{\,2}+1) + \|x\|_2\,|u|$$
$$\le -\frac{1}{2}(1-\theta)\|x\|_2^{\,2}(1-\|x\|_2^{\,2}) - \frac{1}{2}\theta\|x\|_2^{\,2}(1-\|x\|_2^{\,2}) + \|x\|_2\,|u|, \quad 0<\theta<1$$
$$\le -(1-\theta)\|x\|_2^{\,2}, \ \forall 1 > \|x\|_2 \ge \frac{2|u|}{\theta}$$

Thus, by Theorem 4.19, the system is locally input to state stable.

$\|h(t,x,u)\| \le \alpha_1(\|x\|) + \alpha_2(\|u\|)$ is satisfied where α_1 and α_2 are class K.

$\therefore$ by Theorem 5.2, small- signal L_∞ stable.

5.16

(a) Let $V(x) = \frac{1}{2}(x_1^2 + x_2^2) + \int_0^{x_1} \sigma(s)\,ds$

- Using $\sigma(s) \le L|s|$, we have

$$\frac{1}{2}\|x\|_2^2 \le V(x) \le \frac{1+L}{2}\|x\|_2^2$$

- $$\begin{aligned}\frac{\partial V}{\partial x} f(x) &\le -x_1^2 - x_2^2 + |x_2||u| \\ &\le -\|x\|_2^2 + \|x\|_2|u| \\ &= -(1-\theta)\|x\|_2^2 - \|x\|_2(\theta\|x\|_2 - |u|) \qquad 0<\theta<1 \\ &\le -(1-\theta)\|x\|_2^2 \qquad \forall\ \|x\|_2 \ge \frac{|u|}{\theta}\end{aligned}$$

Thus, by theorem 4.19, the system is input-to-state stable

$\|h(t,x,u)\| \le \alpha_1(\|x\|)$ is satisfied where α_1 is class K.

∴ by theorem 5.3, $\mathcal{L}_\infty$ stable

(b) Use theorem 5.5

$$H(V,f,g,h,\gamma) \overset{\text{def}}{=} \frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x} g(x) g^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h^T(x) h(x) \le 0$$

$$\left\{\begin{array}{ll} \frac{\partial V}{\partial x} f(x) = -x_1^2 - x_2^2 - x_1\sigma(x_1) & g(x)g^T(x) = \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix} \\ \frac{\partial V}{\partial x} = [x_1 + \sigma(x_1) \quad x_2] & h^T(x)h(x) = x_2^2 \end{array}\right\}$$

By substitution, we have

$$-x_1^2 - x_2^2 - x_1\sigma(x_1) + \frac{1}{2\gamma^2}x_2^2 + \frac{1}{2}x_2^2 \le 0$$

$$-x_1^2 - x_1\sigma(x_1) - x_2^2\left(1 - \frac{1}{2\gamma^2} - \frac{1}{2}\right) \le 0$$

$$1 - \frac{1}{2\gamma^2} - \frac{1}{2} \ge 0 \implies \gamma \ge 1$$

∴ system is finite-gain $\mathcal{L}_2$ stable, $\mathcal{L}_2$ gain ≤ 1

8.6

(2)

$$\text{eq.pt. : } \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow A = \left.\frac{\partial f}{\partial x}\right|_{0,0} = \left.\begin{bmatrix} 2ax_1 & -2x_2 \\ 2x_1 + x_2 & -1 + x_1 \end{bmatrix}\right|_{0,0} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow T = T^{-1} = I_{2\times 2}$$

$$\begin{bmatrix} y \\ z \end{bmatrix} = Tx = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{y} = ay^2 - z^2 \Rightarrow A_1 = 0,\ g_1(y,z) = ay^2 - z^2$$

$$\dot{z} = -z + y^2 + yz \Rightarrow A_2 = -1,\ g_2(y,z) = y^2 + yz$$

- $z = h(y),\ h(0) = 0,\ \dfrac{\partial h}{\partial y}(0) = 0$
- center manifold equation

$$0 = -h(y) + g_2(y,z) - \frac{\partial h}{\partial y}(y)[g_1(y,z)]$$

$$= -h(y) + (y^2 + yh(y)) - \frac{\partial h}{\partial y}(y)[ay^2 - h(y)^2]$$

$$\therefore\ h(y) = y^2 + O(|y|^3)$$

$$\underline{\dot{y} = ay^2 + O(|y|^3)}$$

$$\cancel{\dot{y} = -(y^2 + O(|y|^3))^2 = -y^4 + O(|y|^5)}$$

$\therefore$ unstable.

(6),(8) : the same method (2) (ref, lecture note 6-1~6-8)

8.6 (8)

$$eq.pt.: \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow A = \frac{\partial f}{\partial x}\bigg|_{0,0} = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since we know the eigenvalues of A have imaginary parts

(0 and -0.5 $\pm j\dfrac{\sqrt{3}}{2}$), we don't need to have the A as Jordan

form. Alternatively, we divide A matrix to two sub-parts: stable part and uncertain part. Thus, we take T as follows:

$$T = T^{-1} = I_{3\times 3}$$

$$\begin{bmatrix} y \\ z_1 \\ z_2 \end{bmatrix} = Tx = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Hence, we show the following system.

$$\begin{bmatrix} \dot{y} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} (z_1 - y)^2 \\ (z_1 - y)^2 \\ 0 \end{bmatrix}$$

$$\bullet z = h(y),\ \mathrm{h}(0) = 0,\ \frac{\partial h}{\partial \mathrm{y}}(0) = 0$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} h_1(y) \\ h_2(y) \end{bmatrix} = \begin{bmatrix} h_{13}y^2 + O(|y|^3) \\ h_{23}y^2 + O(|y|^3) \end{bmatrix}$$

•center manifold equation

$$0=\begin{bmatrix}-2 & -3\\ 1 & 1\end{bmatrix}\begin{bmatrix}h_1(y)\\ h_2(y)\end{bmatrix}+\begin{bmatrix}(h_1(y)-y)^2\\ 0\end{bmatrix}-\begin{bmatrix}\dfrac{\partial h_1}{\partial \mathrm{y}}(y)\\ \dfrac{\partial h_2}{\partial \mathrm{y}}(y)\end{bmatrix}\begin{bmatrix}(h_1(y)-y)^2\\ 0\end{bmatrix}$$

From the above equation, we show following two equtions:

$i)\ (-2h_{13}-3h_{23})y^2+y^2+O(|y|^3)=0$

$ii)\ (h_{13}+h_{23})y^2+O(|y|^3)=0$

since $(h_1(y)-y)^2=y^2+O(|y|^3),\ \dfrac{\partial h_1}{\partial \mathrm{y}}(y)(h_1(y)-y)^2=O(|y|^3).$

Hence, since $h_{13}=-1$ and $h_{23}=1,$ we have

$$\begin{bmatrix}z_1\\ z_2\end{bmatrix}=\begin{bmatrix}h_1(y)\\ h_2(y)\end{bmatrix}=\begin{bmatrix}-y^2+O(|y|^3)\\ y^2+O(|y|^3)\end{bmatrix}$$

Finally,

$\dot{y}=y^2+O(|y|^3)$

$\Rightarrow$ we conclude that this system is unstable

8.15

See Example 8-10.

(a) $\dot{V}(x) = -(2x_2 + x_1)^2 + 2x_2{}^2(2x_2 + x_1)^2 - 3x_1{}^2 - 6x_2{}^2$

$$\le -(2x_2 + x_1)^2(1 - 2x_2{}^2)$$

$$1 - 2x_2{}^2 > 0 \rightarrow -\frac{1}{\sqrt{2}} < x_2 < \frac{1}{\sqrt{2}}$$

$\therefore$ asymptotically stable

(b) i) $x_2 = 1 \rightarrow \dot{V}(x) = -2(x_1 - 1)^2$

ii) $x_2 = -1 \rightarrow \dot{V}(x) = -2(x_1 + 1)^2$

Let $\sigma = x_2$

i) $\sigma = 1 \rightarrow x_1 \ge -1$

ii) $\sigma = -1 \rightarrow x_1 \le 1$

$\therefore$ (-1,1), (1,-1)

$V(\text{-}1,1) = 5,\ V(1,\text{-}1) = 5$