TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2014 - Assignment 1

Due date: Monday 1 September at 12.00.

1. (a) Calculate the vector norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ for the following vectors:

$$x_1 = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ -9 \\ 3 \end{bmatrix}$$

(Hint: See Appendix A in Khalil)

   (b) For a 2D vector, is $\|\cdot\|_2 > \|\cdot\|_1$ possible?
   Characterize all 2D vectors for which the $\|\cdot\|_2$ norm is equal to the $\|\cdot\|_1$ norm. Explain graphically.

2. Let

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}$$

   (a) Calculate the matrix norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ for matrix A.

   (b) The following statement is true for any norm:

$$\|Ax\| \leq \|A\| \, \|x\|$$

   Show that it is true for the given matrix A and vector x, using the $\|\cdot\|_2$ norm.

3. For any $x, y \in \mathbb{R}^2$ define an inner product

$$\langle x, y \rangle \triangleq x^T y.$$

Show that

$$|\langle x, y \rangle| \leq \|x\|_2 \, \|y\|_2 \, .$$

4. Consider the following nonlinear mass-damper-spring system (see Figure 1) with the following notations.

x_1	Position of the box. This is zero when the spring is in its natural shape.
x_2	Velocity of the box.
x_3	$\int(x_1 - x_{1d})dt$
x_{1d}	The desired position.
F_s	Spring force ($F_s = f_3 x_1^3 + f_1 x_1$).
F_d	Damping force ($F_d = d\dot{x}_1$).
f_1	Spring constant (linear effect).
f_3	Spring constant (nonlinear effect).
d	Damping coefficient.
m	Mass of the box.
g	Gravity constant.
w	A disturbing force.
u	Input force to the system.

and w is assumed to be unknown and will not be considered in the development of the system equations. By using Newton's second law of motion the following mathematical model is obtained

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = -\frac{f_3}{m}x_1^3 - \frac{f_1}{m}x_1 - \frac{d}{m}x_2 - g + \frac{u}{m} \tag{2}$$

This model is nonlinear. In this course you will learn how to analyze such nonlinear systems and design controllers. However, with your current knowledge of linear control systems you are already able to design a simple controller by linearizing the system. That is the main purpose of this exercise. It will also allow you to recall important material from the courses TTK4105 Control Engineering and TTK4115 Linear System Theory.

(a) Linearize the system (1)-(2) about the point $(x_1^*, x_2^*) = (0, 0)$. If you do not remember how to linearize, you should recover this from the book of the courses TTK4105 Control Engineering and TTK4115 Linear System Theory.

(b) Show that the system is controllable. What does this mean in connection with pole placement?

(c) Augment the linear model to include integral action. Call the new state x_3. Explain what is achieved by doing this.

(d) Design a linear state feedback controller $u = -k_1x_1 - k_2x_2 - k_3x_3$ that places the poles of the system in

$$p_1 = -1,\ p_2 = -2,\ p_3 = -3$$

(Hint: You may use the Matlab command `place`)
k_1, k_2, and k_3 are the controller gains. Use the following numerical values: $f_1 = 1$, $f_3 = 1$, $d = 1$, $g = 9.81$, $m = 1$.

(e) The solution of the augmented model with the gains calculated in the previous question is (the initial conditions of the states have been set to zero)

$$x_1(t) = (1 - e^{-3t} + 3e^{-2t} - 3e^{-t})x_{1d} \tag{3}$$

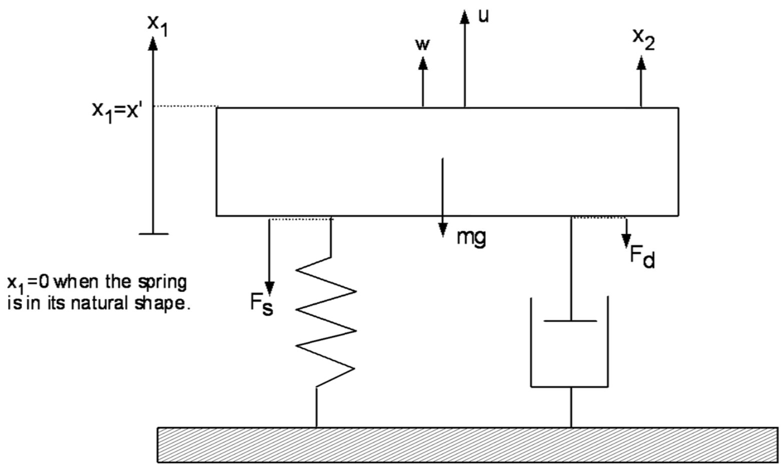

Figure 1: Sketch of the mass-damper-spring system (as indicated by x' the system position is measured at the top of the mass).

Use this equation to show that

$$|x_1(t) - x_{1d}| \leq ke^{-\lambda t} \tag{4}$$

where k and λ are constants greater than zero. Find k and λ.

5. Let

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \sqrt{x} & \text{else} \end{cases}$$

   (a) Is $f(x)$ globally Lipschitz? Explain.

   (b) Find the area for which $f(x)$ is locally Lipschitz.

6. For the following systems, find whether
   (1) f is locally Lipschitz in x on the whole state space
   (2) f is globally Lipschitz
   Based on this information, say for which systems we can conclude about local or global existence and uniqueness of solutions.
   The systems are:

   (a) The pendulum equation with friction and constant input torque (see Section 1.2.1 in Khalil)

   (b) The mass-spring equation with linear spring, linear viscous damping, Coulomb friction and no external force (see Section 1.2.3 in Khalil)

   (c) The Van der Pol oscillator (Example 2.6 in Khalil)

7. Exercise 1.4 in Khalil

8. In some cases, nonlinear systems $\dot{x} = f(x)$ with a complicated $f(x)$ can be written in a simpler form by changing the coordinate vector x into a new coordinate vector $z = \psi(x)$, where ψ is a continuously differentiable function (such that $\psi^{-1}(z)$ exists and is also continuously differentiable).
   With the new coordinate vector the corresponding system equations can be obtained by

$$\dot{z} = \frac{\partial \psi}{\partial x}(x)\dot{x} = \frac{\partial \psi}{\partial x}(x) f(x)\bigg|_{x=\psi^{-1}(z)} = \frac{\partial \psi}{\partial x}(\psi^{-1}(z)) f(\psi^{-1}(z))$$

   Although this expression may look more complicated, in some cases the resulting system can be simpler and more suitable for analysis than the original system, as illustrated in the following exercise.

   Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + \alpha x_1 \left(\beta^2 - x_1^2 - x_2^2\right) \\ \dot{x}_2 &= -x_1 + \alpha x_2 \left(\beta^2 - x_1^2 - x_2^2\right) \end{aligned}$$

   where $\alpha, \beta \geq 0$ are constants.
   Show that using polar coordinates ($x_1 = r\cos\theta, x_2 = r\sin\theta$) we can rewrite the original system in terms of the new coordinates, to get a simpler equation set:

$$\begin{aligned} \dot{r} &= \alpha r \left(\beta^2 - r^2\right) \\ \dot{\theta} &= -1 \end{aligned}$$

9. Using (nonlinear) differential equations we can not only describe and analyze technical systems, but also biological, economical and other systems.
A simplified representation of populations of foxes and rabbits is given by the following equation set:

$$\begin{aligned}\dot{x} &= x(\alpha - \beta y)\\ \dot{y} &= -y(\gamma - \delta x)\end{aligned}$$

where

$$\begin{aligned}x &: \text{ Number of rabbits}\\ \alpha x &: \text{ Natural reproduction of rabbits}\\ \beta xy &: \text{ Rate of predation}\\ y &: \text{ Number of foxes}\\ \gamma y &: \text{ Natural death of foxes}\\ \delta xy &: \text{ Growth rate for foxes}\\ \alpha, \beta, \gamma, \delta &: \text{ Parameters representing the interaction of the two species}\end{aligned}$$

We know that $x \geq 0, y \geq 0$, since the number of animals can not be negative.
Note that the rate of predation is quite similar to the growth rate for foxes, but they contain different parameters (as the fox population growth is not *necessarily* equal to the rate at which it consumes the rabbits).

(a) If initially $x > 0, y = 0$, what would happen to the rabbit population?

(b) If initially $x = 0, y > 0$, what would happen to the fox population?

(c) Find the equilibrium points of the system, and determine the type of each equilibrium point when all parameters are positive.

(d) Is the existence of several isolated equilibrium points possible for linear systems? Why/why not?

(e) Construct the phase portait for $x \geq 0, y \geq 0$ and discuss the qualitative behavior of the system. Choose the values $\alpha = 63, \beta = 5, \gamma = 457, \delta = 6$. Use the Matlab phase portrait program `pplane` from `http://math.rice.edu/~dfield/`.

(f) If initially $x > 0$ and $y > 0$, is it possible to arrive at $x < 0$ or $y < 0$? Explain your answer.

(g) Using Bendixson's criterion (Lemma 2.2), specify a *simply connected* region in which *no* periodic orbits are lying entirely. You may describe the region using inequalities. (See definiton in Khalil at p. 67.)

10. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= ax_1 - x_1x_2 \\ \dot{x}_2 &= bx_1^2 - cx_2\end{aligned}$$

where $a > 0$, $b > 0$, $c > 0$.

(a) Find the Jacobian of the right-hand side of the system.

(b) Find all equilibrium points of the system. Determine the type of each isolated equilibrium point for all values of $a > 0$, $b > 0$, $c > 0$.

(c) For each of the following cases, construct the phase portait and discuss the qualitative behavior of the system.

1. $a = 1, \quad b = 1, \quad c = 4$
2. $a = 1, \quad b = 1, \quad c = 10$

Do the phase portraits confirm the placement and type of equilibrium points which you found in (b)?