EE681 - Nonlinear control

Stability analysis and non linear control of prey and predator system

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Contents

Abstract

This report presents a comparison of a selected nonlinear control technique with the commonly used PD. Models for prey and predator are investigated to check whether stability is achievable. This is gonna be investigated with tools like phase portrait.

1 Introduction

Biological pest control have always been a interesting subject everywhere in the world. This is mostly due to the strong tradition to upkeep a certain animal out of necessity for either food or resources that can later be refined into something useful. In Norway, the agriculture have always been important. Back in the days, when Norway was one of the poorest countries, sheep's and cows were raised for food and clothing. In recent year, there have been a shift towards seafood like salmon. The biggest problem with having such a agriculture are potential predators that feed on the live stocks. For cows and sheep's there are wolves. This report explores the use of nonlinear controller on a prey and predator model which is based on the famous Lotka Volterra equations.

For this project, the stability prospect of the model is gonna be investigated. This will be done with the use of equilibrium point analysis and phase portraits. This is going to be combined with both nonlinear and linear controllers to see whether it is possible to obtain stability where one want it. All of the numbers used in this report is fictional and simplified for the sake of making analysis and implementation.

2 Assumptions

Due to the complexity of Kolmogorov with mutualism, competition and disease, the Lotka Volterra equations solely rely on food. As one can see in equation 1, a big assumption is that the prey always finds food, and if left to be, will always grow with time. This is a valid assumption due to the fact that in this case, most of the preys are household animals which means that they will always have ample with food. Something else which are also assumed is that the predator have limitless appetite, and that the growth of the populations are solely dependent on the size of the two populations.

The biggest constraint on the system is the assumption that neither the input, nor the states can be negative since there are no physical meaning to negative population or negative removal of a population. Large spikes of input are okay, since the removal of large amount of a population would always be possible. Another downfall with the model is the assumption of no perturbation which are common in biological systems in the form of diseases, natural disasters and seasonal effects such as weather and food supply. One can see later that the system is very sensitive to perturbation with the use of phase portrait on the Lotka Volterra model.

3 Mathematical Modeling

Lotka-Volterra

Lotka Volterra equations are usually referred to as the prey and predator equation, which are used to describe the dynamics of biological systems in which two species interact. The original equation can be seen in equation 1 where x_1 is the number of prey, x_2 is the number of predator and $\alpha, \beta, \delta, \gamma$ are positive constants describing the dynamics of interaction between the prey and predator. In this project, $\alpha, \beta, \delta, \gamma$ are set to 1 for simplicity sake.

$$\dot{x_1} = \alpha x_1 - \beta x_1 x_2 \tag{1a}$$
$$\dot{x_2} = \delta x_1 x_2 - \gamma x_2 \tag{1b}$$
$$\tag{1c}$$

The Lotka Volterra equation can basically be looked as conservation of population where the prey have ample food. This means that when there are no predators, the prey population will rise exponentially. The same could be said for the predator where the population size decreases exponentially if left with no food.

Final prey and predator model

In this report, the growth of the predator is the same as the decrease in the prey population since β, δ is set to 1. Normally, this would not be the case, where the prey population parameter would be less than the predator one since in most biological systems the prey population increase faster than the predator population.

The final Lotka Volterra equation can be seen in equation 2. Here a input have been added, in order to cull the predator population if needed.

$$\dot{x_1} = x_1 - x_1 x_2 \tag{2a}$$
$$\dot{x_2} = x_1 x_2 - x_2 - u \tag{2b}$$
$$\tag{2c}$$

4 Analysis

The system is in this case a SISO system, consisting of one input and one output which can be described by:

$$\dot{\mathbf{x}} = f(x) + g(x)u \tag{3a}$$

$$\mathbf{y} = h(x) \tag{3b}$$

where x $\in$R^2 is the state vector, u and y are the control and output respectively. The state vectors can be found in equation 4, while the output and control was derived and shown in equation 6. It is natural in this case to let y $= x_1$ due to it being a household animal and therefore observable.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \alpha x_1 - \beta x_1 x_2 \\ \delta x_1 x_2 - \gamma x_2 \end{bmatrix} \tag{5}$$

$$g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \tag{6}$$

To check if local or global stability is possible for the derived model, one have to determine all equilibria and analyse local and global stability property for all x_{eq}. These can be found by setting $\dot{\boldsymbol{x}} = 0$ and $\boldsymbol{u} = 0$ which gives us:

$$\dot{x} = 0 = \begin{bmatrix} \alpha x_1 - \beta x_1 x_2 \\ \delta x_1 x_2 - \gamma x_2 \end{bmatrix} \tag{7}$$

which gives us the following equation:

$$\alpha x_1 = \beta x_1 x_2 \tag{8a}$$

$$\gamma x_2 = \delta x_1 x_2 \tag{8b}$$

Derived from equation 7 and 8 is first of all the trivial solution for the system at the origin which makes sense. If there are no animals, then the population will never grow unless one supply the population in some kind of unnatural way. The second equilibria can be found at equation 9.

$$(\hat{x_1}, \hat{x_2}) = (0, 0) \tag{9a}$$

$$(\hat{x_1}, \hat{x_2}) = (\frac{\alpha}{\beta}, \frac{\gamma}{\delta}) \tag{9b}$$

The second equilibria is dependent on $\alpha, \beta, \delta, \gamma$. In this case these are all 1, which gives us equilibria at $(\hat{x_1}, \hat{x_2})$ = (1,1). All of the equilibrium's are isolated. By linearization and checking the eigenvalues, one can determine the nature of the equilibrium. Qualitative analysis is done by assuming that the trajectories of the nonlinear system in a small neighbourhood of an equilibrium to be quite close to the linearized trajectory. The jacobian matrix of the system is given by:

$$J(x_1, x_2) = \begin{bmatrix} \alpha - \beta x_2 & -\beta x_1 \\ \delta x_2 & \delta x_1 - \gamma \end{bmatrix} \tag{10}$$

Case 1 ; $(\hat{x_1}, \hat{x_2})$ = (0,0)

Evaluating the jacobian matrix at $(\hat{x_1}, \hat{x_2}) = (0{,}0)$ with the given parameters results into:

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Eigenvalues : \pm 1 \tag{11}$$

$\lambda = \pm 1$ indicates that $(\hat{x_1}, \hat{x_2}) = (0{,}0)$ is a saddle point. Since this is a purely real value, this implies that the equilibrium point is hyperbolic.

Case 2 ; $(\hat{x_1}, \hat{x_2})$ = (1,1)

Evaluating the jacobian matrix at $(\hat{x_1}, \hat{x_2}) = (1{,}1)$ with the given parameters results into:

$$J(1,1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Eigenvalues : \pm i \tag{12}$$

$\lambda = \pm i$ indicates that $(\hat{x_1}, \hat{x_2}) = (1, 1)$ is a center. This implies that the solutions will circle around (1,1) as center with periodic oscillations.

Phase Portrait

To check the equilibria at x_{eq} more accurately a phase portrait of the uncontrolled system is shown in figure 14 with a plot of the trajectory in figure 6. As one can see from figure 6, the trajectory oscillates, where the prey is slightly phase shifted towards the left. This makes sense since whenever there is a huge prey population, the predator will just increase. The oscillations also makes very much sense, since when the predator population gets too large, the prey population will get lowered to the point where the predator population also decreases. This in turns gives the prey population time to recuperate, and then it begins from the start again.

For the phase portrait with no control, one can see that the vector field is heavily pointing toward x_2 when the prey population is large. This makes sense from the discussion before. The magnitude of the vector field seems to get lowered, the less the prey population gets. Some trajectories have been drawn onto the phase portrait. One can identify the equilibra at (1,1) and see that all of the trajectories are oscillating around it. This also shows one of the weaknesses with the Lotka Volterra model, which is that the system is not structurally stable. Small perturbation can destroy these exclusive periodic and bounded solution and bump it into another trajectory. The analysis was only done in the first quadrant due to the constraint of the states both being positive.

5 Simulation

For this assignment, Matlab R2018b and Simulink was used for simulating and plotting the system for the different controllers. In order to calculate and plot the trajectories for phase portraits, ode45 was used as the solver.

6 Linear Control - PD

For reference, a simple PD was tuned and used to gather raw data. The reference for the amount of prey or household animals which are desired are provided by a signal builder in simulink which forces x_{ref} to go from 5 to 7 after 7.5 time steps. In this case, the system is a SISO. The simulink model can be seen in figure 1. The implemented PD controller can be seen in figure 4 with $K_p = 5$ and $K_d = 7$.

The generated trajectory has been plotted and can be seen in figure 7. By properly tuning the controller, the prey population is shown to stabilize first at around 4.4, before bumping up to 6. The predator trajectory smoothly goes from 4 before it stabilize at 1. This happens until 7.5 where it has to bump a bit downwards to give the prey population enough drive for it to achieve its desired steady state. As one can see in figure 7, the prey solution never have steady state at the desired value. This may be due to nonlinearities in the system.

One can fix this problem by increasing K_p to 50. The result of such an increase can be seen in figure 8 where the desired prey reference are achieved perfectly. This will not be realistic, due to the fact that the predator population become negative during the transition at timestep 7.5 where x_{ref} goes from 5 to 7. One can also look at the the input plot in figure 12 that the input goes negative which also breaks the constraint. The input spikes at the transition to over 100, which is non feasible in most cases whereas in the case with lower K_p, the input plot in figure 11 never goes under zero and spikes at a reasonable point.

Further analysis can be done by looking at the updated phase portrait at figure 16 where every trajectory which starts at $x, y > 0$ ends up violating the constraint by ending up in the fourth quadrant where $x_2 < 0$ before converging to x_{eq}. Figure 15 shows the trajectories and vector field for $K_p = 5$. By looking at the figure, one can see that it has reasonable trajectories, and that it only goes into the fourth quadrant when the initial values are too low. The only feasible solution which fullfills the desired steady state is therefor $K_p = 5$. Nonetheless which K_p is used, the vector field have the same qualitative behaviour by pointing downwards before converging into the equilibrium point. A interesting note is that the arrows point upwards if your in the fourth quadrant, which is in our case not so important.

7 Nonlinear Control - Feedback Linearization

Feedback Linearization considers a class of nonlinear systems of the form found in equation 3 and pose the question of whether there exist a state feedback control in the form of the one in equation 13 and a change of variables z = T(x) that transform the nonlinear system into an equivalent linear system. This renders a linear input–output map between the new input and the output. An important assumptions is that the transformation T, must be diffeomorphism with T(0) = 0 for the new system in its "normal form" to be equivalent to the old system. This makes it possible to prove that by choosing the input in the form shown in equation 13 one can achieve stability. Reason behind this transformation from x-coordinates to z-coordinates is that the theory and methodologies which is usually used in linear systems, can be applied once a nonlinear system has been exact linearized.

$$\mathbf{u} = \alpha(x) + \beta(x)v \tag{13}$$

$$\dot{\mathbf{y}} = \dot{x_1} = x_1 - x_1 x_2 \tag{14}$$

$$\ddot{\mathbf{y}} = x_1 - x_1 x_2 + x_1 {x_2}^2 - {x_1}^2 x_2 + x_1 u \tag{15}$$

In order to find the relative degree of the system, the output was differentiated until the input appeared explicitly. The smallest integer for which the input appears is referred to as the relative degree. The result of the derivations can be seen in equation 14 In this case, the relative degree is 2, and the system is therefor feedback linearizable. The relative degree of a system can be seen on as the difference between the degrees of the denominator and numerator polynomials of the transfer function, H(s). Since ρ = n = 2, which implies that the system has no zero dynamics, and by default is said to be a minimum phase. This can also be confirmed by letting y = x_1 = u = 0 which gives us:

$$\dot{\mathbf{x_1}} = 0 \tag{16}$$
$$\dot{\mathbf{x_2}} = -x_2 \tag{17}$$
$$\tag{18}$$

To check if the system is fully linearizable one can check if G has full rank. G can be seen in equation 21, where one can see that the determinant of G is decided by whether x_2 is zero or not, which in this case is not a problem due to the constraint. This implies that $det(G) \neq 0$ and have therefor full rank and is fully linearizable.

$$ad_f g = \begin{bmatrix} -\frac{\partial f}{\partial x} g \end{bmatrix} = \begin{bmatrix} 1 - x_2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - x_2 \\ x_2 \end{bmatrix} \tag{19}$$

$$G = \begin{bmatrix} g & -\frac{\partial f}{\partial x} g \end{bmatrix} = \begin{bmatrix} 1 & 1 - x_2 \\ 0 & x_2 \end{bmatrix}, \quad det(G) = x_2 \tag{20}$$

The diffeormorphism transformation of the system can be done by assigning h(x) = y = x_1 and letting L_fh(x) = $\dot{y} = x_1 - x_1 x_2$. This results into the following transformation matrix:

$$T = \begin{bmatrix} h(x) \\ L_f h(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 - x_1 x_2 \end{bmatrix} \tag{21}$$

The transformation from matrix 21 gives us the following transformed system:

$$\dot{\mathbf{z_1}} = z_2 \tag{22}$$
$$\dot{\mathbf{z_2}} = z_2 - z_2\left(\frac{z_1 - z_2}{z_1}\right) + ({z_1}^2 + z_1 z_2 + z_1 + z_2 + z_1 u) = g(z) + z_1 u \tag{23}$$

One can develop a state feedback control by utilising feedback linearization. Since feedback linearization have already been proved, the controller can be derived by looking at equation 22 and let u = $\frac{-g(z) + v}{z_1}$ which gives:

$$\dot{\mathbf{z_1}} = z_2 \tag{24}$$
$$\dot{\mathbf{z_2}} = v \tag{25}$$

where v = -Kz where K can be constructed so that matrix A in equation 26 is hurwitz and poles in good placements. The poles can be chosen, and the resulting poles will be decided by equation 27. v is in this case a one-dimensional transformed input created by feedback linearization.

$$A_c = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \Rightarrow det(\lambda I - A_c) = \begin{bmatrix} \lambda & -1 \\ k_1 & \lambda + k_2 \end{bmatrix} \tag{26}$$

$$\lambda_{1,2} = \frac{-k_2 \pm \sqrt{{k_2}^2 - 4k_1}}{2} \tag{27}$$

In this case, $k_1, k_2 = (4{,}18)$ which gives poles at $\lambda_{1,2} = \pm 9 + \sqrt{77}$. By defining the error as $x_1 - x_{ref}$, generates the simulink model seen in figure 2 with the controller in figure 5. The structure of the controller is relative complex compared to PD which is shown in figure 4. The generated 2-D line plot of the state trajectories can be seen in figure 9. Compared to the PD plot seen in figure 7, the feedback linearized one is much smoother, quickly converging to 5 before converging to 7 at the transition time. The decrease in the predator solution at the transition time is minimal and very smooth, compared to the PD one.

The resulting input plot can be seen in figure 13 which is fairly good. The peak is somewhere over 5, and is a realistic peak, and much lower than the ones the PD controller generated. It oscillates a bit, but manages to drive the prey trajectory to the desired steady state without much problem.

The dynamic of the system can be further analyzed with the use of phase portraits. The phase portrait of the system with feedback linearization can be seen in 17. This one have a much sharper trajectory down to 1 before slowly converging into a straight line. Compared to the PD phase portrait which is shown in figure 15, the feedback linearized one is much more lenient with the initial conditions and it is just when the predator population starts around 1.8 that the trajectory goes into the fourth quadrant and therefore violates the constraints.

8 Discussion

For comparison, the combined plot of trajectories generated both by PD and FL is plotted on figure 10 where the dotted lines are the solutions generated with the use of feedback linearization and the normal lines are solutions generated with a regular PD controller. The biggest difference is that the PD controlled trajectory never converges to desired steady state. One could always increase K_p, but as discussed earlier is not possible because of the positiveness of states constraint. As shown earlier, by increasing the proportional parameter for the PD, the whole system gets less lenient with different initial values, and those could also lead the trajectory into the fourth quadrant which is inconvenient.

The predator trajectories are quite similar. The only difference is during the settling time, and during the transition. At settling time, one can see that the trajectory produced with feedback linearization converges to steady state much faster and cleaner than the one produced by the PD. The one produced by feedback linearization also have a much smoother transition during timestep $= 7.5$ where it just slightly goes down before slowly converging back to desired steady state. For the PD generated predator trajectory, the decrease in predator is quite large and very abrupt. This leads to a not so smooth transition compared to the one generated by feedback linearization.

This problem can also be seen for the prey trajectories, where the PD one are very abrupt and not as clean as the one produced by feedback linearization. Something which could explain why the PD controller performs worse than FL is the fact that the system is well behaved and very simple in structure. It could had been worse if the order of states had been larger, or if the states had been x^n with n larger than 2.

There is also the question if feedback linearization is better than PD in the sense that x_2 was used in the generation of u_{FL}, and the feedback would therefor be ideal and exact, and would therefore always be better than a PD controller. An argument against that is that one could always construct and observer to estimate the missing state with the use of for example a high gain observer.

Something else that is also very common in agriculture is the direct control of the prey or in this case household population by introducing a input in $\dot{x_1}$ at equation 1. This could had lead to better control of the system, and ultimately led to a smoother trajectory.

9 Conclusion

The main concern of this assignment was to investigate stability for Lotka Volterra equations which has isolated periodic solutions and implement a nonlinear controller in order to achieve a desirable steady state. The chosen nonlinear method was feedback linearization which consist of exact linearization and state feedback control. This was compared with a standard PD regulator for comparison. The result was that the PD controller was lacking and not too robust to changes in initial values. This was investigated and proven with the use of phase portrait. The initial value problem was solved by the introduction of feedback linearization which gave a smooth and satisfactory trajectory. Before using feedback linearization one had to prove that the system could be fully linearizable by checking whether G had full rank or not. Checking if feedback linearization was possible, was checked by looking at the relative degree of the system. The zero dynamic of the system was also investigated to ensure that the controller would drive the solution into the desired steady state. In the end, a normal PID controller could probably be used to control a nonlinear system, but it would always be subpar with a nonlinear based controller which also takes into account the dynamics not around the equilibrium points.

A MATLAB code

A.1 Simulink

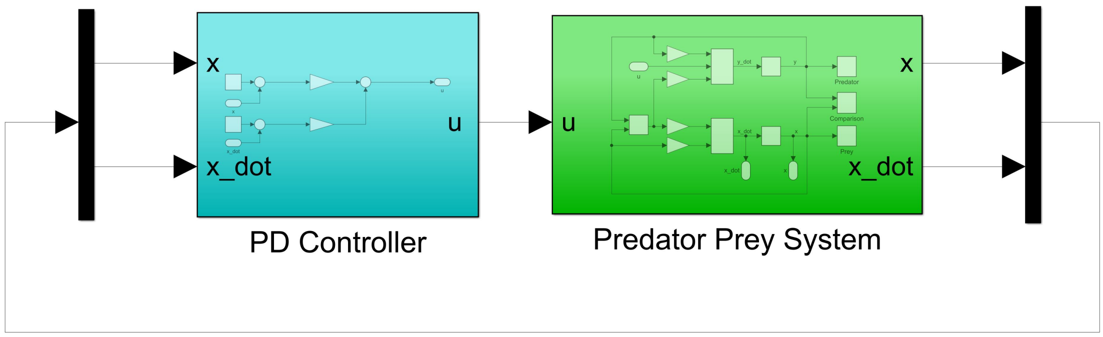

Figure 1: Simulink model with PD control

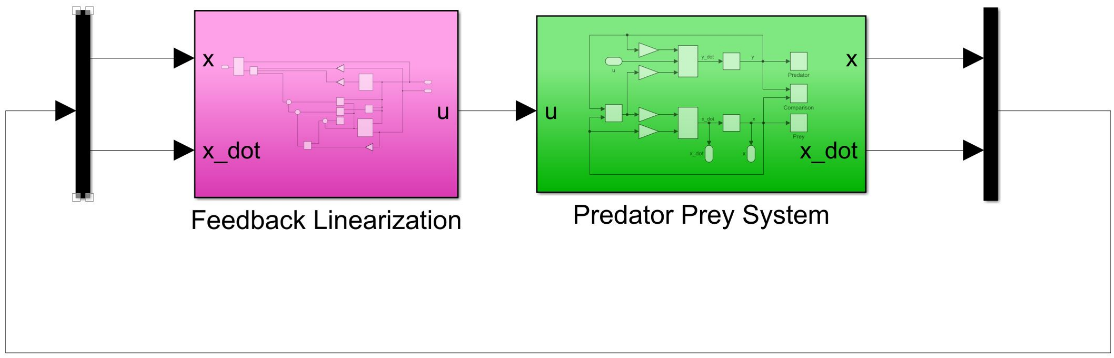

Figure 2: Simulink model with feedback linearization control

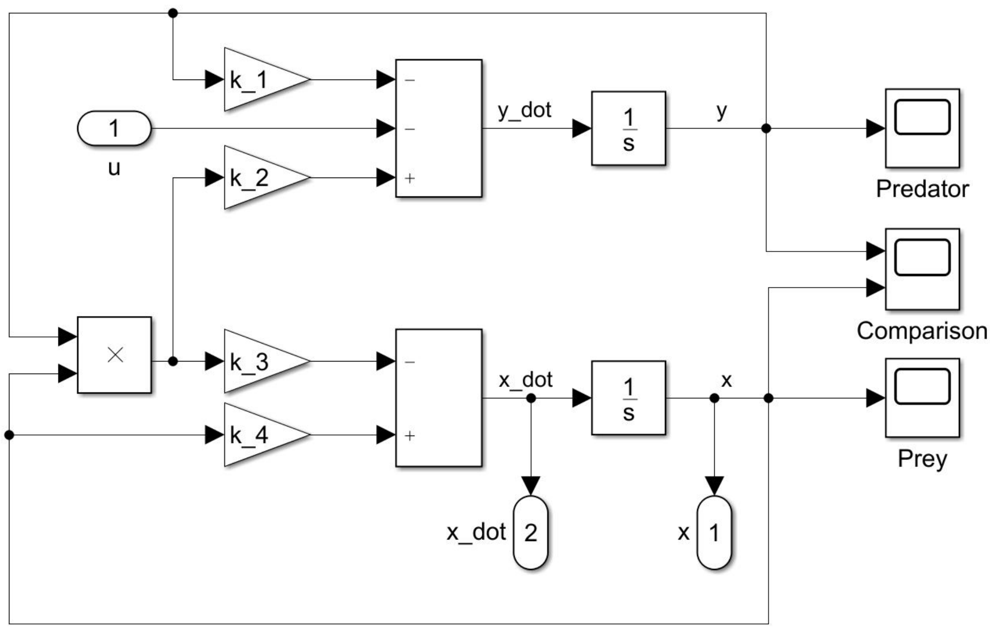

Figure 3: Simulink model of Lotka Volterra

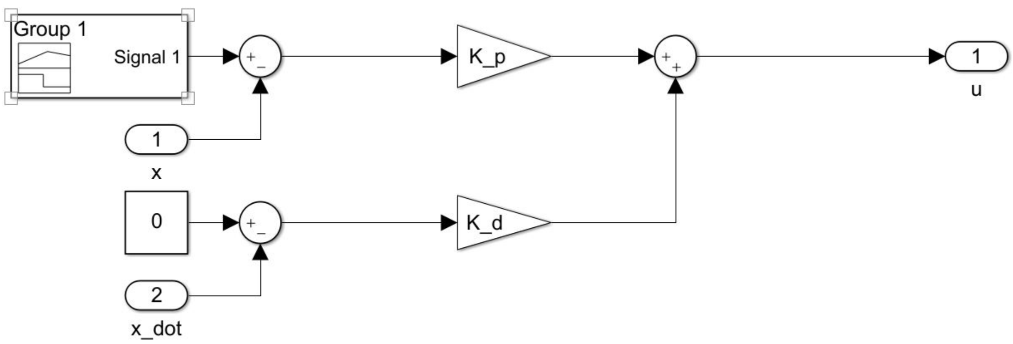

Figure 4: Simulink model of a PD controller

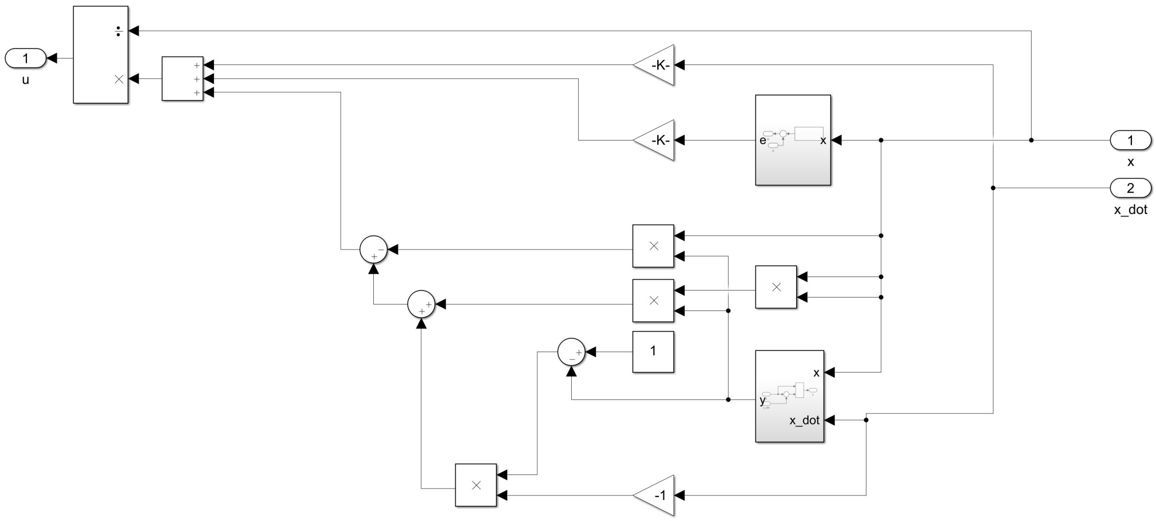

Figure 5: Simulink model of a feedback linearization based controller

B Plots

B.1 Trajectories

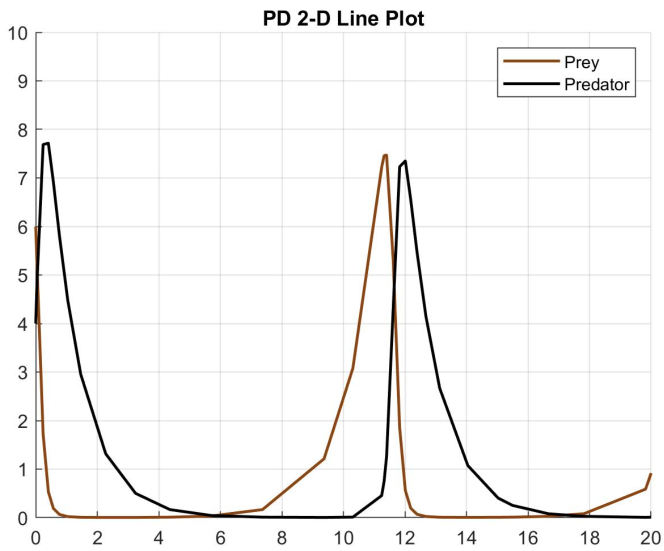

Figure 6: Plot of trajectories with u = 0

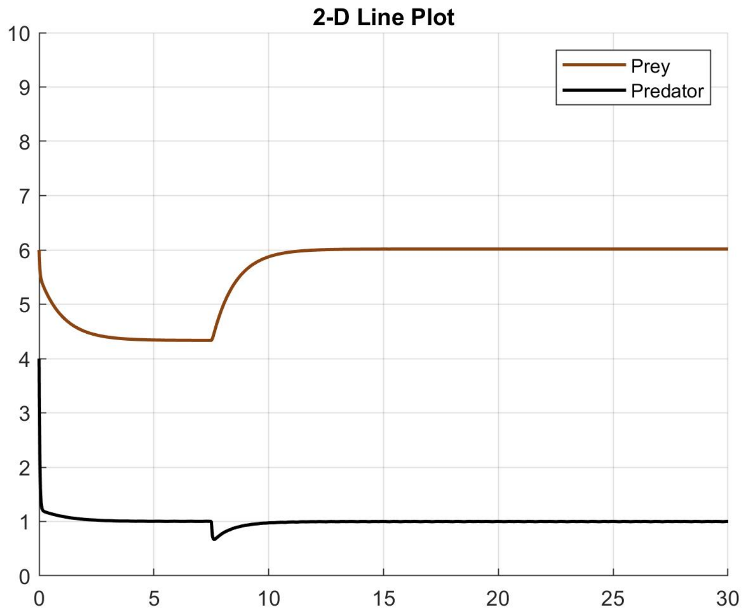

Figure 7: Plot of trajectories generated by PD

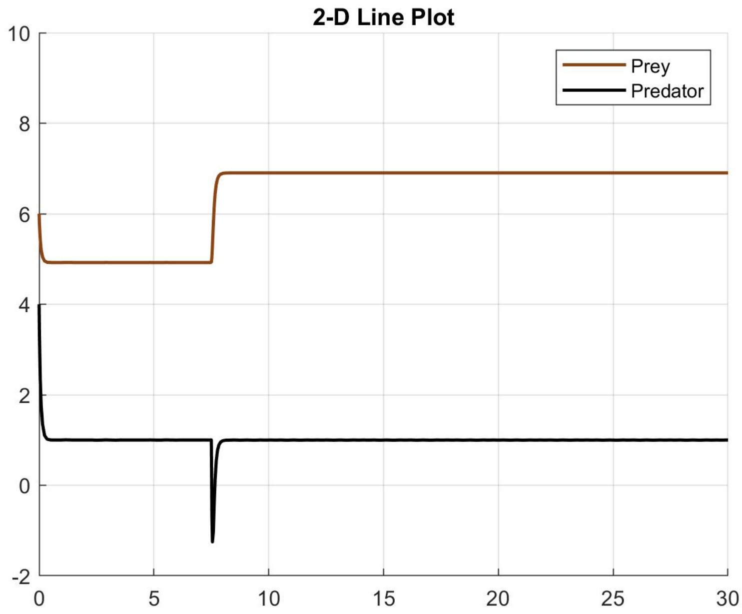

Figure 8: Plot of trajectories generated by PD with $K_p = 50$

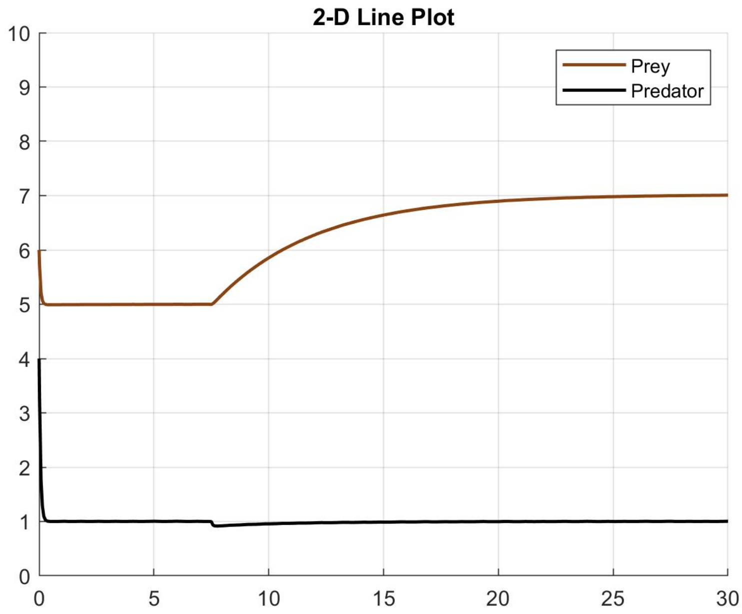

Figure 9: Plot of trajectories generated by FL

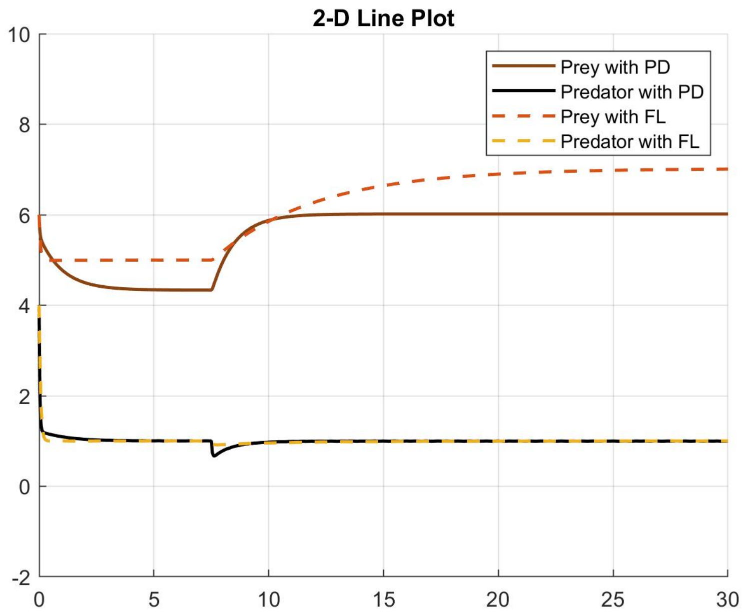

Figure 10: Plot of trajectories generated both by PD and FL

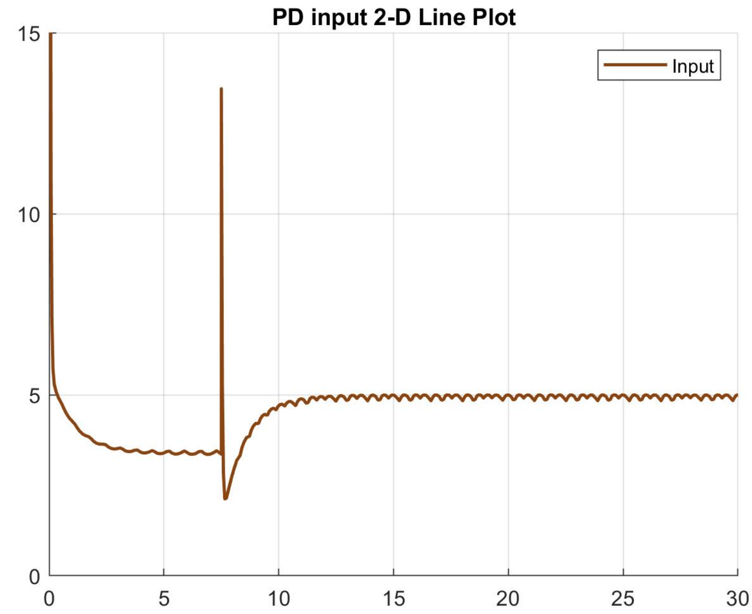

Figure 11: Input plot with PD control

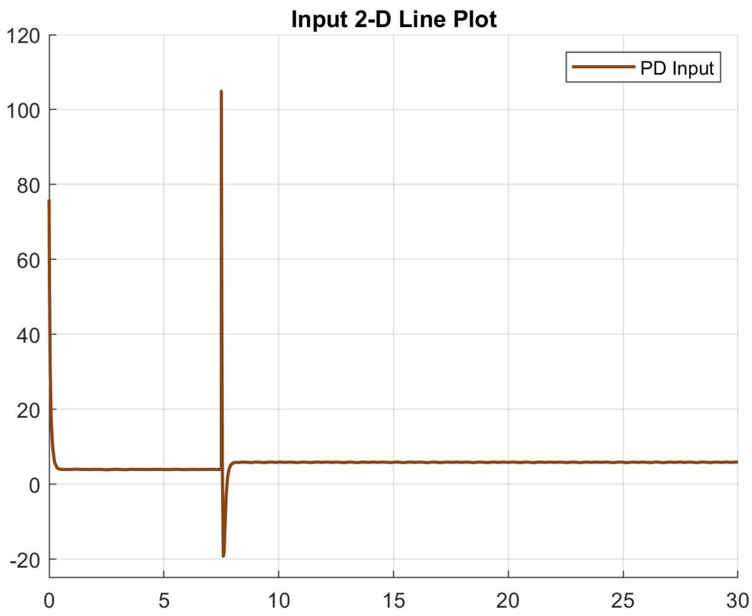

Figure 12: Input plot with PD control with $K_p = 50$

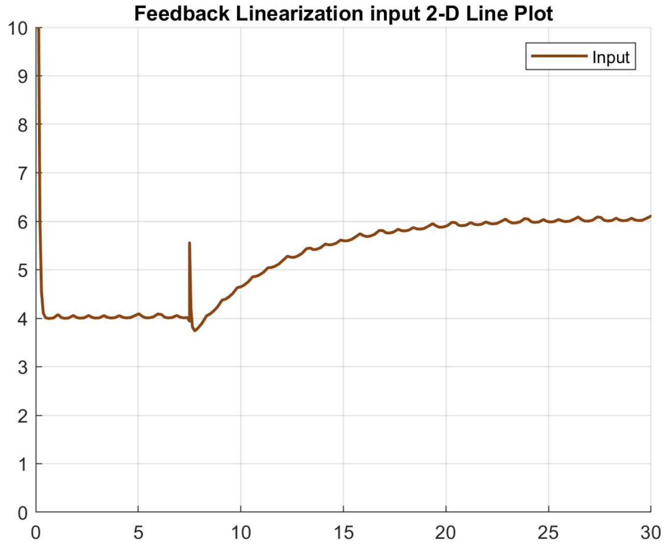

Figure 13: Input plot with feedback linearization based control

B.2 Phase Portraits

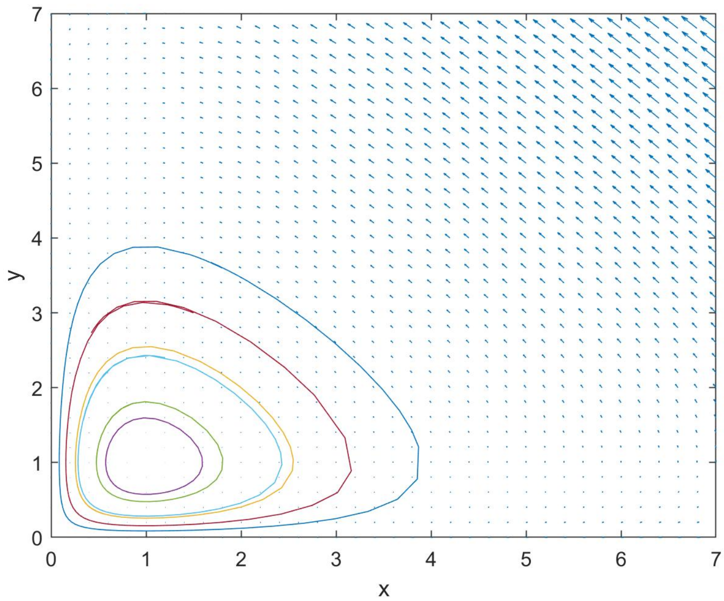

Figure 14: Phase Portrait of system with u = 0

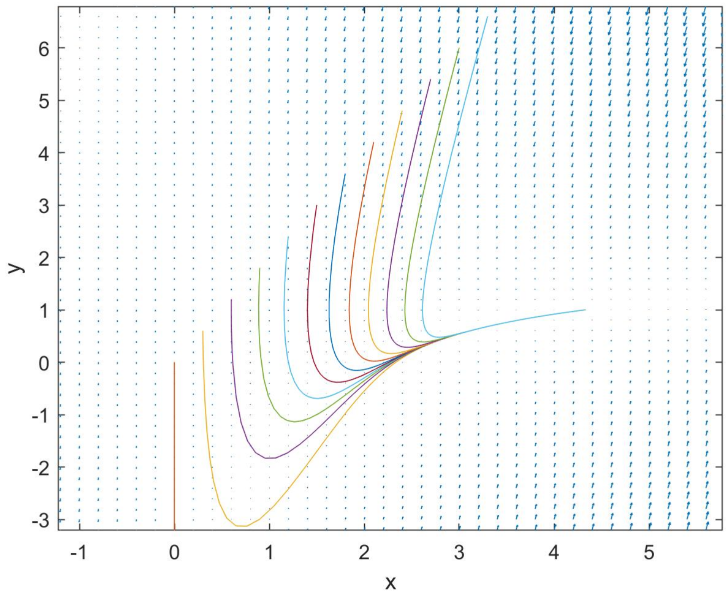

Figure 15: Phase Portrait of system with PD

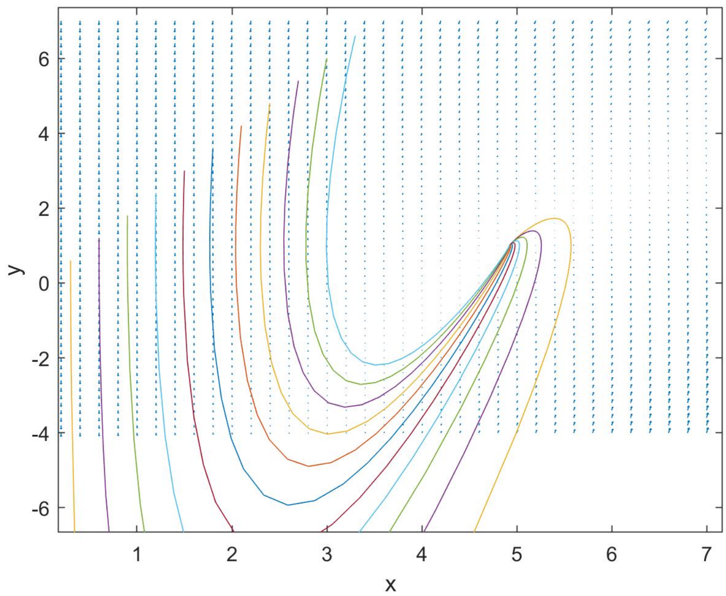

Figure 16: Phase Portrait of system with PD with $K_p = 50$

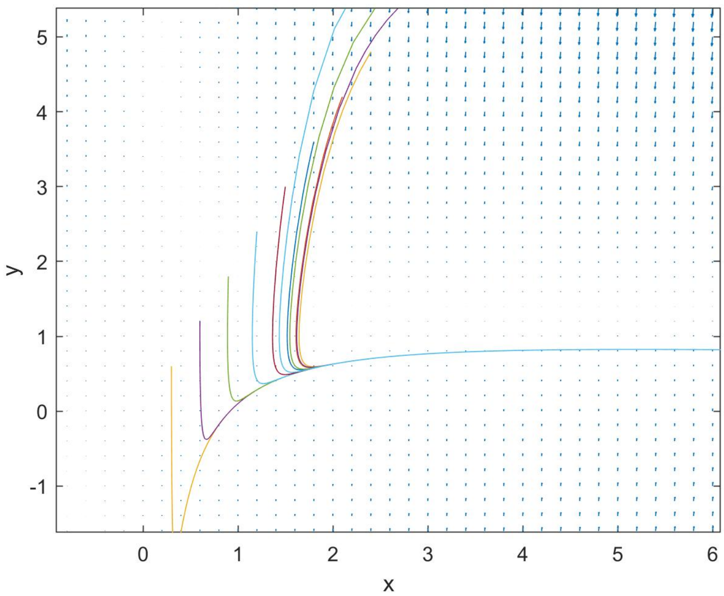

Figure 17: Phase Portrait of system with feedback linearization