TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2017 - Assignment 3

Due date: Thursday 5. October at 11.00.

1. Consider again the system from Assignment 1, Exercise 5 and Assignment 2, Exercise 1.

   (a) Use the transformed system from Assignment 2 (Exercise 1b) given by;

$$\dot{\tilde{x}}_1 = \tilde{x}_2 \tag{1}$$

$$\dot{\tilde{x}}_2 = -\frac{f_3}{m}\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - \frac{f_1}{m}\tilde{x}_1 - \frac{d}{m}\tilde{x}_2 + \frac{\tilde{u}}{m} \tag{2}$$

   and the Lyapunov function candidate

$$V = \frac{1}{2}\left(\tilde{x}_1^2 + m\tilde{x}_2^2\right)$$

   to derive a controller (find $\tilde{u}$) such that

$$\dot{V} = -(d + k_2)\tilde{x}_2^2$$

   where k_2 is the controller gain.
   (Hint: The resulting closed-loop system should be linear)

   (b) Is the closed-loop system locally/globally asymptotically/exponentially stable at the origin? Find the strongest achievable stability result and motivate your answers.

   (c) What happens to the system dynamics as k_2 increases? Explain this physically.

   (d) By using the controller in part (a), is it possible to place the poles of the system arbitrarily?

2. For a real symmetric matrix Λ we denote $\Lambda \geq 0$ when we mean that the matrix Λ is positive semidefinite and $\Lambda \leq 0$ when it is negative semidefinite. For a real symmetric positive definite matrix P we denote $\lambda_{\min}$ and $\lambda_{\max}$ as its smallest and largest eigenvalue, respectively. Show that the following inequalities

$$\lambda_{\min} I \leq P \leq \lambda_{\max} I$$

   hold for

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

   Furthermore show that

$$\lambda_{\min} \|x\|_2^2 \leq x^T P x \leq \lambda_{\max} \|x\|_2^2$$

   for all x.

   Hint I: To show that $\lambda_{\min} x^T x \leq x^T P x$, we need to show that $x^T (P - \lambda_{\min} I) x \geq 0$, i.e. that the matrix $P - \lambda_{\min} I$ is positive semi-definite.
   Hint II: Take one side at the time, but for both proofs at once.

3. Exercise 4.9 in Khalil

4. Exercise 4.15 in Khalil. (Hint: $\frac{d}{dz}\int_0^z \psi(u)\, du = \psi(z)$)

5. Consider

$$\begin{aligned}\dot{x}_1 &= -(x_1+2x_2)(x_1+2)\\ \dot{x}_2 &= -8x_2(2+2x_1+x_2)\end{aligned}$$

(a) Using the indirect method (Theorem 4.7 of Khalil), show that the origin is asymptotically stable.

(b) Using the direct method (Theorem 4.1 of Khalil), show that the origin is asymptotically stable.
(Hint: use $\mathcal{D} = \{x \in R^2 | x_1 + 2x_2 + 1 \geq 0 \text{ and } 2x_1 + x_2 + 1 \geq 0\}$ and the Lyapunov function candidate $V = x_1^2 + x_2^2$)

(c) Let $\Omega_c \triangleq \{x \in R^2 | V(x) \leq c\}$. Draw $\mathcal{D}$, $\Omega_{\frac{1}{9}}$ and $\Omega_{6.25}$ together on the plane (you may use `pplane` and select 'Plot level curves' from the 'Solutions' menu). Explain why the trajectory converges to the origin when $x(0) = \left(0, \frac{1}{3}\right)$? Explain also why the trajectory does not converge to the origin when $x(0) = \left(-\frac{4}{3}, 2\right)$ eventhough $x(0)$ belongs to D.

6. Exercise 4.35 in Khalil.

7. Suppose that for each initial condition $x(0)$ the solution of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq \beta(\|x(0)\|, t)$$

for $t \geq 0$ where β is of class $\mathcal{KL}$.
Show that the origin of the system is globally asymptotically stable, i.e.

(a) Show stability for $x = 0$ using the definition of stability and the definition of class-$\mathcal{KL}$ functions.

(b) Show that every trajectory of the system converges to the origin.

8. Consider the system

$$\begin{aligned}\dot{x}_1 &= -\phi(t)x_1 + a\phi(t)x_2\\ \dot{x}_2 &= b\phi(t)x_1 - ab\phi(t)x_2 - c\psi(t)x_2^3\end{aligned}$$

where a, b and c are positive constants and $\phi(t)$ and $\psi(t)$ are nonnegative, continuous, bounded functions that satisfy

$$\phi(t) \geq \phi_0 > 0, \quad \psi(t) \geq \psi_0 > 0, \quad \forall t \geq 0$$

Show that the origin is globally uniformly asymptotically stable.
(Hint: $V = 0.5(bx_1^2 + ax_2^2)$)

9. Exercise 4.38 in Khalil. (Hint: use the completion of squares)

10. Exercise 4.45 in Khalil. (Hint: use $V = 0.5(x_1^2 + x_2^2)$)

11. Exercise 4.10 in Khalil.

12. Let

$$\begin{aligned} V_1(x_1,x_2,t) &= x_1^2 + \left(1+e^t\right)x_2^2 \\ V_2(x_1,x_2,t) &= \frac{x_1^2+x_2^2}{1+t} \\ V_3(x_1,x_2,t) &= \left(1+\cos^4 t\right)\left(x_1^2+x_2^2\right) \end{aligned}$$

For each of the functions $V_i(x_1,x_2,t)\,,\ i\in\{1,2,3\}$ investigate the properties of positive definite and decrescent.

13. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - c(t)\,x_2 \end{aligned}$$

where the function $c(t)$ is continuous differentiable and satisfies

$$k_1 \le c(t) \le k_2 \text{ and } |\dot{c}(t)| \le k_3 \ \forall t \ge 0$$

and k_i are constants and $k_1 > 0$. Use the Lyapunov function candidate

$$V(x) = \frac{1}{2}\left(x_1^2+x_2^2\right)$$

to show that the origin is uniformly stable and that $x_2 \to 0$ as $t \to \infty$.