

**TTK4150 Nonlinear Control Systems**  
**Department of Engineering Cybernetics**  
**Norwegian University of Science and Technology**  
**Fall 2017 - Solution to Assignment 2**

1. (a) By using the fact that the derivatives are zero in an equilibrium point, the following equations must be true

$$\begin{aligned} 0 &= \dot{x}_2 \\ 0 &= -\frac{f_3}{m}x_1^{*3} - \frac{f_1}{m}x_1^* - \frac{d}{m}x_2^* - g \\ &= -\frac{f_3}{m}x_1^{*3} - \frac{f_1}{m}x_1^* - g \end{aligned}$$

Inserting  $(0,0)$  into the above equations leads to an illegal expression since  $g$  is not zero but  $9.81m/s^2$ . Therefore  $(0,0)$  is not an equilibrium point.

With  $u = u_0$  we have the equations

$$0 = \dot{x}_2 \tag{1}$$

$$0 = -\frac{f_3}{m}x_1^{*3} - \frac{f_1}{m}x_1^* - g + \frac{u_0}{m} \tag{2}$$

Inserting  $(x_1^*, x_2^*) = (x_{1d}, 0)$  into 2 gives

$$0 = -f_3x_{1d}^3 - f_1x_{1d} - mg + u_0$$

thus

$$u_0 = f_3x_{1d}^3 + f_1x_{1d} + mg \tag{3}$$

- (b) In the original system equations we insert  $x_1 = \tilde{x}_1 + x_{1d}$ ,  $\tilde{x}_2 = x_2$  and  $u = u_0 + \tilde{u}$ . We have

$$\begin{aligned} \dot{\tilde{x}}_1 &= \dot{x}_2 = \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= -\frac{f_3}{m}x_1^3 - \frac{f_1}{m}x_1 - \frac{d}{m}x_2 - g + \frac{u}{m} \\ &= -\frac{f_3}{m}(\tilde{x}_1 + x_{1d})^3 - \frac{f_1}{m}(\tilde{x}_1 + x_{1d}) - \frac{d}{m}\tilde{x}_2 - g + \frac{(f_3x_{1d}^3 + f_1x_{1d} + mg + \tilde{u})}{m} \end{aligned}$$

The resulting system equations are

$$\dot{\tilde{x}}_1 = \tilde{x}_2 \tag{4}$$

$$m\dot{\tilde{x}}_2 = -f_3[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3] - f_1\tilde{x}_1 - d\tilde{x}_2 + \tilde{u} \tag{5}$$

In the equilibrium point for  $\tilde{u} = 0$  we have

$$\begin{aligned} 0 &= \tilde{x}_2^* \\ 0 &= -f_3[(\tilde{x}_1^* + x_{1d})^3 - x_{1d}^3] - f_1\tilde{x}_1^* - d\tilde{x}_2^* \\ &= -f_3[(\tilde{x}_1^* + x_{1d})^3 - x_{1d}^3] - f_1\tilde{x}_1^* \end{aligned}$$

The equilibrium point is now in the origin.

(c) The Jacobian is calculated for (4)–(5).

$$A = \left[ \begin{array}{cc} \frac{\partial f_1}{\partial \tilde{x}_1} & \frac{\partial f_1}{\partial \tilde{x}_2} \\ \frac{\partial f_2}{\partial \tilde{x}_1} & \frac{\partial f_2}{\partial \tilde{x}_2} \end{array} \right] \bigg|_{x=(0,0)} = \left[ \begin{array}{cc} 0 & 1 \\ -\frac{3f_3x_{1d}^2+f_1}{m} & -\frac{d}{m} \end{array} \right] \quad (6)$$

(Note that you may instead calculate the Jacobian for the original system, as long as you use the correct equilibrium point for this system.)

To find out whether  $A$  is Hurwitz or not, the eigenvalues of the matrix must be calculated

$$\lambda I - A = \left[ \begin{array}{cc} \lambda & 0 \\ 0 & \lambda \end{array} \right] - \left[ \begin{array}{cc} 0 & 1 \\ -\frac{3f_3x_{1d}^2+f_1}{m} & -\frac{d}{m} \end{array} \right] = \left[ \begin{array}{cc} \lambda & -1 \\ \frac{3f_3x_{1d}^2+f_1}{m} & \lambda + \frac{d}{m} \end{array} \right] \quad (7)$$

$$|\lambda I - A| = \lambda\left(\lambda + \frac{d}{m}\right) + \frac{3f_3x_{1d}^2+f_1}{m} \quad (8)$$

$$= \lambda^2 + \frac{d}{m}\lambda + \frac{3f_3x_{1d}^2+f_1}{m} \quad (9)$$

The eigenvalues of  $A$  are thus given as

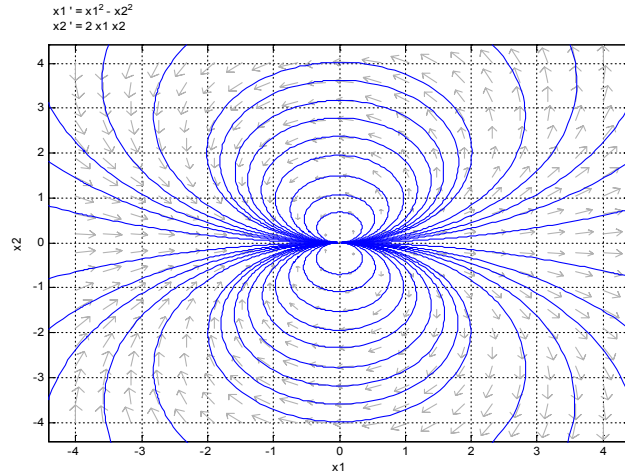
$$\lambda = \frac{1}{2} \left( -\frac{d}{m} \pm \sqrt{\left(\frac{d}{m}\right)^2 - \frac{4(3f_3x_{1d}^2+f_1)}{m}} \right) \quad (10)$$

Since  $f_1, f_3, x_{1d}^2, m > 0$

$$\frac{d}{m} > \sqrt{\left(\frac{d}{m}\right)^2 - \frac{4(3f_3x_{1d}^2+f_1)}{m}} \quad (11)$$

and the eigenvalues will always lie in the left half plane which means that  $A$  is Hurwitz. This means that  $(0, 0)$  of (4)–(5) is locally asymptotically stable.

2. (a) The phase portrait can be seen in Figure 1.



**Figure 1:** Phase portrait

The origin is not stable in the sense of Lyapunov. Given any  $\varepsilon > 0$ , no matter how small a  $\delta$  we choose for the region of initial condition there always some initial conditions close to the  $x_1$ -axis which will exit the  $\varepsilon$ -region before converging to the origin.

- (b) We need to show two conditions for asymptotic stability. First we need to show that for any given  $\varepsilon > 0$  we could always find a  $\delta > 0$  such that  $\|x(0)\| < \delta \implies \|x(t)\| < \varepsilon, \forall t \geq 0$ . Furthermore we need to show that when the initial condition is on some domain every trajectory converges to the origin.

The solution is given by  $x(t) = e^{\alpha t}x(0)$ . We then have  $|x(t)| \leq |x(0)|$  for all  $t \geq 0$  since  $\alpha < 0$ . Given any  $\varepsilon > 0$ , choose  $\delta = \varepsilon$  to show that for all  $|x(0)| < \delta = \varepsilon$  it follows that  $|x(t)| < \varepsilon, \forall t \geq 0$ . Thus the origin is stable. From the solution it is easy to see that for any  $\delta$  we have

$$|x(0)| < \delta \implies \lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} e^{\alpha t}x(0) = 0.$$

as  $\alpha < 0$ .

3. Remark: the answer to this exercise may vary a lot depending on the chosen parameters of Lyapunov function and the domain  $D$ . One possible solution set is presented below.

- (a) The scalar system is given by  $\dot{x} = -x^3$  with equilibrium point at the origin. Suppose  $V(x) = px^2$  where  $p > 0$ . Then  $V$  is positive definite. Taking the derivative along the trajectory we have  $\dot{V} = 2px\dot{x} = 2px(-x^3) = -2px^4$  which is negative definite. Hence the origin is asymptotically stable. Further  $V$  is radially unbounded which implies that the origin is globally asymptotically stable.
- (b) The system is given by

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1 - x_2^3\end{aligned}$$

where it can be seen that the equilibrium point is given by

$$(x_1^*, x_2^*) = (0, 0)$$

A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^T Px, \quad P = P^T$$

which is positive definite if and only if all the leading principal minors of  $P$  are positive, that is

$$\begin{aligned}p_{11} &> 0 \\ p_{11}p_{22} - p_{12}^2 &> 0\end{aligned}$$

(and it follows that  $p_{22} > 0$ ). The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T Px \\ &= \begin{bmatrix} -x_1 - x_2 \\ x_1 - x_2^3 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= -p_{11}x_1x_2 - p_{12}x_1x_2 + p_{22}x_1x_2 - p_{11}x_1^2 + p_{12}x_1^2 - p_{12}x_2^2 - p_{22}x_2^4 - p_{12}x_1x_2^3 \\ &= -(p_{11} - p_{12})x_1^2 - (p_{11} + p_{12} - p_{22})x_1x_2 - p_{12}x_2^2 - p_{22}x_2^4 - p_{12}x_1x_2^3\end{aligned}$$

In order to eliminate the undesirable terms,  $p_i$  is chosen according to

$$\begin{aligned} p_{11} + p_{12} - p_{22} &= 0 \\ p_{12} &= 0 \\ \Rightarrow p_{11} &= p_{22} \end{aligned}$$

which fulfill the requirements imposed in order to guarantee  $V(x)$  to be positive definite. The derivative of  $V(x)$  with respect to time is

$$\begin{aligned} \dot{V}(x) &= -p_{11}x_1^2 - p_{11}x_2^4 \\ &< 0 \quad \forall x \in R^2 - \{0\} \end{aligned}$$

Since  $V(x)$  is radially unbounded it follows that the origin is globally asymptotically stable.

(c) The system is given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

where it can be seen that the equilibrium point is  $(x_1^*, x_2^*) = (0, 0)$ . A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^T P x, \quad P = P^T$$

which is positive definite if and only if all the leading principal minors of  $P$  are positive

$$\begin{aligned} p_{11} &> 0 \\ p_{11}p_{22} - p_{12}^2 &> 0 \end{aligned}$$

(and it follows that  $p_{22} > 0$ ). The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x \\ &= \begin{bmatrix} -x_1 + x_2^2 \\ -x_2 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} -x_1 + x_2^2 \\ -x_2 \end{bmatrix}^T \begin{bmatrix} p_{11}x_1 + p_{12}x_2 \\ p_{12}x_1 + p_{22}x_2 \end{bmatrix} \\ &= (-x_1 + x_2^2)(p_{11}x_1 + p_{12}x_2) - x_2(p_{12}x_1 + p_{22}x_2) \\ &= p_{12}x_2^3 - p_{11}x_1^2 - p_{22}x_2^2 - 2p_{12}x_1x_2 + p_{11}x_1x_2^2 \end{aligned}$$

By choosing  $p_{12} = 0$ , the term  $x_2^3$  and  $x_1x_2$  vanishes and the derivative is rewritten as

$$\begin{aligned} \dot{V}(x) &= -p_{11}x_1^2 - p_{22}x_2^2 + p_{11}x_1x_2^2 \\ &= -p_{11}x_1^2 - (p_{22} - p_{11}x_1)x_2^2 \\ &= -p_{11}x_1^2 - p_{11}\left(\frac{p_{22}}{p_{11}} - x_1\right)x_2^2 \\ &< 0, \quad \forall \frac{p_{22}}{p_{11}} > x_1 \end{aligned}$$

By taking  $D = \left\{x \in R^n | x_1 < \frac{p_{22}}{p_{11}}\right\}$ , where  $\frac{p_{22}}{p_{11}}$  may be chosen arbitrary large, it follows that the equilibrium point is asymptotically stable.

(d) The system is given by

$$\begin{aligned}\dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1)\end{aligned}$$

where it can be seen that the equilibrium points are given by

$$(x_1^*, x_2^*) = (0, 0)$$

and the set

$$x_1^{*2} + x_2^{*2} = 1$$

This implies that the origin can not be globally asymptotically stable, since any initial condition in  $\{x \in R^2 | x_1^2 + x_2^2 = 1\}$  implies that the solution stays in that set. A general quadratic Lyapunov function candidate is given by

$$V(x) = \frac{1}{2}x^T Px, \quad P = P^T$$

which is positive definite if and only if all leading principal minors of  $P$  are positive, that is

$$\begin{aligned}p_{11} &> 0 \\ p_{11}p_{22} - p_{12}^2 &> 0\end{aligned}$$

(and it follows that  $p_{22} > 0$ ). The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T Px \\ &= \begin{bmatrix} (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= (2x_1x_2p_{12} - x_1x_2p_{11} + x_1x_2p_{22} + x_1^2p_{11} + x_1^2p_{12} - x_2^2p_{12} + x_2^2p_{22})(x_1^2 + x_2^2 - 1) \\ &= x^T \begin{bmatrix} p_{11} + p_{12} & p_{12} - \frac{1}{2}p_{11} + \frac{1}{2}p_{22} \\ p_{12} - \frac{1}{2}p_{11} + \frac{1}{2}p_{22} & p_{22} - p_{12} \end{bmatrix} x (x_1^2 + x_2^2 - 1) \\ &= x^T Qx (x_1^2 + x_2^2 - 1)\end{aligned}$$

By choosing  $Q$  such that  $x^T Qx > 0 \quad \forall x \neq 0$  and taking  $D = \{x \in R^2 | x_1^2 + x_2^2 < 1\}$ , it can be seen that

$$\dot{V}(x) < 0 \quad \forall x \in D$$

Choosing  $p_{12} = 0$ , the matrix  $P$  is positive definite if and only if

$$\begin{aligned}p_{11} &> 0 \\ p_{22} &> 0\end{aligned}$$

and the matrix  $Q$  is positive definite if and only if

$$\begin{aligned}p_{11} &> 0 \\ p_{22} &> 0\end{aligned}$$

Thus by choosing  $p_{11}, p_{22} > 0$  and  $p_{12} = 0$  we have shown that the origin of the system is asymptotically stable.

4. We first introduce a change of variables

$$\begin{aligned} z_1 &= x_1 - x_1^* \\ z_2 &= x_2 - x_2^* \end{aligned}$$

where  $(x_1^*, x_2^*) = (-1, 1)$ . We could write our differential equations as

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 = -x_1^2 x_2 - 2x_1 x_2 + x_1^2 + 2x_1 \\ &= -(z_1 - 1)^2 (z_2 + 1) - 2(z_1 - 1)(z_2 + 1) + (z_1 - 1)^2 + 2(z_1 - 1) \\ &= z_2 (1 - z_1^2) \end{aligned}$$

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 = x_1^3 + 2x_1^2 + x_1^2 x_2 + 2x_1 x_2 \\ &= (z_1 - 1)^3 + 2(z_1 - 1)^2 + (z_1 - 1)^2 (z_2 + 1) + 2(z_1 - 1)(z_2 + 1) \\ &= -(z_1 + z_2 - z_1^3 - z_1^2 z_2) \\ &= -(z_1 + z_2) (1 - z_1^2) \end{aligned}$$

and the new system becomes

$$\dot{z} = \begin{bmatrix} z_2 \\ -(z_1 + z_2) \end{bmatrix} (1 - z_1^2) \quad (12)$$

Let our Lyapunov function candidate be in the form  $V(z) = z^T P z$  where  $z^T = [z_1 \ z_2]$  and

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

is a positive definite symmetric matrix. Each component on the matrix  $P$  will be assigned later to a specific value such that our function candidate  $V$  becomes a Lyapunov function (in this case  $P$  does NOT have to be UNIQUE). Expanding the Lyapunov function candidate we have  $V(z) = p_{11} z_1^2 + 2p_{12} z_1 z_2 + p_{22} z_2^2$ . Taking the derivative of the function along the trajectory we have

$$\begin{aligned} \dot{V} &= 2p_{11} z_1 \dot{z}_1 + 2p_{12} \dot{z}_1 z_2 + 2p_{12} z_1 \dot{z}_2 + 2p_{22} z_2 \dot{z}_2 \\ &= 2p_{11} z_1 z_2 (1 - z_1^2) + 2p_{12} z_2^2 (1 - z_1^2) - 2p_{12} (z_1^2 + z_1 z_2) (1 - z_1^2) - 2p_{22} (z_1 z_2 + z_2^2) (1 - z_1^2) \\ &= 2p_{11} z_1 z_2 - 2p_{11} z_1^3 z_2 + 2p_{12} z_2^2 - 2p_{12} z_2^2 z_1^2 - 2p_{12} z_1^2 + 2p_{12} z_1^4 - 2p_{12} z_1 z_2 \\ &\quad + 2p_{12} z_1^3 z_2 - 2p_{22} z_1 z_2 + 2p_{22} z_1^3 z_2 - 2p_{22} z_2^2 + 2p_{22} z_2^2 z_1^2 \\ &= -2p_{12} z_1^2 + 2(p_{11} - p_{12} - p_{22}) z_1 z_2 - 2(p_{22} - p_{12}) z_2^2 + H.O.T. \end{aligned}$$

where the higher order term is given by

$$H.O.T. = -2p_{11} z_1^3 z_2 - 2p_{12} z_2^2 z_1^2 + 2p_{12} z_1^4 + 2p_{12} z_1^3 z_2 + 2p_{22} z_1^3 z_2 + 2p_{22} z_2^2 z_1^2$$

We could rewrite the derivative as

$$\dot{V} = -z^T Q z + H.O.T.$$

where

$$Q = \begin{bmatrix} 2p_{12} & -p_{11} + p_{12} + p_{22} \\ -p_{11} + p_{12} + p_{22} & 2(p_{22} - p_{12}) \end{bmatrix}$$

To prove asymptotic stability it is enough to require our function candidate  $V$  to be a Lyapunov function on a neighborhood of the origin (does not have to be in the global

region since we are not talking about global asymptotic stability). This means that it is enough to show  $\dot{V}$  to be negative definite on a neighborhood of the origin. From this point of view we could see that near the origin, the quadratic term  $z^T Q z$  dominates the higher order term  $H.O.T.$  Hence,  $\dot{V}$  will be negative definite in the neighborhood of the origin if the term  $-z^T Q z$  is negative definite. In this case it is enough to show that  $Q$  is a positive definite matrix. Thus a sufficient condition for asymptotic stability is to have  $P$  and  $Q$  to be positive definite. By checking all the leading principle minors of  $P$  and  $Q$  to be positive we might end up having nonunique solution of  $P$  and  $Q$ . One choice is by setting

$$p_{11} = 3, \quad p_{12} = 1, \quad p_{22} = 2$$

where we have

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Since  $P$  and  $Q$  are positive definite matrices then  $(z_1, z_2) = (0, 0)$  is asymptotically stable, or equivalently  $(x_1, x_2) = (-1, 1)$  is asymptotically stable.

Another, and perhaps more intuitive approach, would be to utilize that  $\dot{V}$  can be written:

$$\begin{aligned} \dot{V} &= (1 - z_1^2) \begin{bmatrix} z_2 \\ -(z_1 + z_2) \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= (1 - z_1^2) (z_1 z_2 p_{11} - z_1 (z_1 + z_2) p_{12} + z_2^2 p_{12} - z_2 (z_1 + z_2) p_{22}) \\ &= - (1 - z_1^2) (z_1^2 (p_{12}) + z_1 z_2 (-p_{11} + p_{12} + p_{22}) + z_2^2 (p_{12} + p_{22})) \\ &= - (1 - z_1^2) z^T Q z \end{aligned}$$

$$Q = \begin{bmatrix} p_{12} & \frac{1}{2}(-p_{11} + p_{12} + p_{22}) \\ \frac{1}{2}(-p_{11} + p_{12} + p_{22}) & p_{12} + p_{22} \end{bmatrix}$$

We can thus see that  $\dot{V}$  is negative definite on a domain  $D = \{z \in R^2 | 1 - z_1^2 > 0\}$  under the condition that  $z^T Q z > 0 \forall z \neq 0$ , i.e. that  $Q$  is positive definite. Furthermore,  $P$  still needs to be positive definite for  $V$  to be a Lyapunov function candidate.

$$\begin{aligned} P > 0 : \quad & p_{11} > 0 \quad p_{11} p_{22} > p_{12}^2 \Rightarrow p_{22} > 0 \\ Q > 0 : \quad & p_{12} > 0 \quad p_{12} (p_{12} + p_{22}) - \frac{1}{4} (-p_{11} + p_{12} + p_{22})^2 > 0 \\ & p_{12} (p_{12} + p_{22}) > \frac{1}{4} (p_{12} + p_{22} - p_{11})^2 \end{aligned}$$

Choosing  $p_{12} = 1 > 0 \Rightarrow p_{11} p_{22} > 1$  and  $1 (1 + p_{22}) > \frac{1}{4} (1 + p_{22} - p_{11})^2$ .

Choosing  $p_{22} = 2 > 0 \Rightarrow p_{11} > \frac{1}{2}$  and  $1 (1 + 2) > \frac{1}{4} (1 + 2 - p_{11})^2 \Rightarrow 12 > (3 - p_{11}^2)$ .

Choosing  $p_{11} = 3 > 0 \Rightarrow 3 > \frac{1}{2}$  and  $12 > 0$ .

To conclude: By using the Lyapunov function  $V(z) = z^T P z$ ,  $P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$  we have shown that the equilibrium is asymptotically stable.

5. We have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(x_1 + x_2) - h(x_1 + x_2) \end{aligned}$$

We want to determine the gradient,  $g(x)$ , of the Lyapunov function so that

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} \quad (13)$$

and

$$\dot{V}(x) = g^T(x)f(x) < 0 \quad \forall \quad x \neq 0 \quad (14)$$

$$V(x) = \int_0^x g^T(y)dy > 0 \quad \forall \quad x \neq 0 \quad (15)$$

Let

$$g(x) = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

where  $\alpha, \beta, \gamma, \delta > 0$ . The symmetry requirement (13) gives

$$\beta = \gamma$$

The derivative of  $V$  along the trajectories of the system is now given by

$$\begin{aligned} \dot{V}(x) &= g(x)^T f(x) \\ &= \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \beta x_1 + \delta x_2 \end{bmatrix}^T \begin{bmatrix} x_2 \\ -(x_1 + x_2) - h(x_1 + x_2) \end{bmatrix} \\ &= (\alpha x_1 + \beta x_2)x_2 + (\beta x_1 + \delta x_2)(-(x_1 + x_2) - h(x_1 + x_2)) \end{aligned}$$

Taking  $\beta = \delta$

$$\begin{aligned} \dot{V}(x) &= (\alpha x_1 + \beta x_2)x_2 + \beta(x_1 + x_2)(-(x_1 + x_2) - h(x_1 + x_2)) \\ &= (\alpha x_1 + \beta x_2)x_2 - \beta(x_1 + x_2)^2 - \beta(x_1 + x_2)h(x_1 + x_2) \\ &= \alpha x_1 x_2 + \beta x_2^2 - \beta(x_1^2 + 2x_1 x_2 + x_2^2) - \beta(x_1 + x_2)h(x_1 + x_2) \\ &= \alpha x_1 x_2 + \beta x_2^2 - \beta x_1^2 - \beta 2x_1 x_2 - \beta x_2^2 - \beta(x_1 + x_2)h(x_1 + x_2) \\ &= -\beta x_1^2 - (2\beta - \alpha)x_1 x_2 - \beta(x_1 + x_2)h(x_1 + x_2) \end{aligned}$$

Taking  $\beta = \frac{1}{2}\alpha$  in order to get rid of the  $x_1 x_2$ -term

$$\begin{aligned} \dot{V}(x) &= -\beta x_1^2 - \beta(x_1 + x_2)h(x_1 + x_2) \\ &< 0 \quad \forall x \in R^2 \end{aligned}$$

since  $zh(z) > 0 \quad \forall \quad z \neq 0$  and  $\beta > 0$ . The function  $V$  is constructed by

$$\begin{aligned} V(x) &= \int_0^{x_1} g_1(y_1, 0) dy_1 + \int_0^{x_2} g_2(x_1, y_2) dy_2 \\ V(x) &= \int_0^{x_1} \alpha y_1 dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2) dy_2 \\ &= \alpha \left[ \frac{1}{2} y_1^2 \right]_0^{x_1} + \gamma x_1 [y_2]_0^{x_2} + \delta \left[ \frac{1}{2} y_2^2 \right]_0^{x_2} \\ &= \frac{1}{2} \alpha x_1^2 + \gamma x_1 x_2 + \frac{1}{2} \delta x_2^2 \\ &= \beta x_1^2 + \beta x_1 x_2 + \frac{\beta}{2} x_2^2 \\ &= x^T P x \end{aligned}$$



where

$$P = \begin{bmatrix} \beta & \frac{\beta}{2} \\ \frac{\beta}{2} & \frac{\beta}{2} \end{bmatrix}$$

and

$$\begin{aligned} \beta &> 0 \\ \frac{\beta^2}{2} - \frac{\beta^2}{4} &= \frac{\beta^2}{4} > 0 \end{aligned}$$

which implies that  $P > 0$  (and  $V(x)$  is positive definite on  $R^2$  and radially unbounded). By Theorem 4.2 it is concluded that the origin is globally asymptotically stable.

6. The pendulum system is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \end{aligned}$$

A general quadratic Lyapunov function candidate is given by

$$\begin{aligned} V(x) &= \frac{1}{2} x^T P x + \frac{g}{l} (1 - \cos x_1) \\ &= \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{g}{l} (1 - \cos x_1) \end{aligned}$$

The quadratic form  $\frac{1}{2} x^T P x$  is positive definite if and only if all the leading principal minors of  $P$  are positive

$$\begin{aligned} p_{11} &> 0 \\ p_{11} p_{22} - p_{12}^2 &> 0 \end{aligned} \tag{16}$$

The derivative of the Lyapunov function candidate along the trajectories of the system is given by

$$\begin{aligned} \dot{V}(x) &= x^T P \dot{x} + \frac{g}{l} \dot{x}_1 \sin x_1 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \frac{g}{l} \dot{x}_1 \sin x_1 \\ &= (x_1 p_{11} + x_2 p_{12}) \dot{x}_1 + (x_1 p_{12} + x_2 p_{22}) \dot{x}_2 + \frac{g}{l} \sin x_1 \cdot \dot{x}_1 \\ &= \left[ p_{11} x_1 + p_{12} x_2 + \frac{g}{l} \sin x_1 \right] x_2 + (p_{12} x_1 + p_{22} x_2) \left[ -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 \right] \\ &= \frac{g}{l} (1 - p_{22}) x_2 \sin x_1 - \frac{g}{l} p_{12} x_1 \sin x_1 + \left[ p_{11} - p_{12} \frac{k}{m} \right] x_1 x_2 + \left[ p_{12} - p_{22} \frac{k}{m} \right] x_2^2 \end{aligned}$$

The elements  $p_{11}$ ,  $p_{12}$ , and  $p_{22}$  should be selected such that  $\dot{V}(x)$  becomes negative definite. The signs of  $x_2 \sin x_1$  and  $x_1 x_2$  change based on the quadrant of  $x_1$  and  $x_2$  and therefore, these two elements should be eliminated. This happens by choosing  $p_{22} = 1$  and  $p_{11} = (k/m)p_{12}$ . Then, from (16),  $p_{12}$  should satisfy  $0 < p_{12} < (k/m)$  to have a positive definite  $V$ . By choosing  $p_{12} = \frac{1}{2}(k/m)$

$$\dot{V}(x) = -\frac{1}{2} \frac{g}{l} \frac{k}{m} x_1 \sin x_1 - \frac{1}{2} \frac{k}{m} x_2^2$$

Obviously,  $x_1 \sin x_1 > 0$  for all  $-\pi < x_1 < \pi$ . Therefore, by selecting

$$B_r = \{x \in R^2 \mid |x_1| < \pi\},$$

it concludes that  $V(x)$  is positive definite and  $\dot{V}(x)$  is negative definite in  $B_r$ . Thus, we can conclude that the origin is locally asymptotically stable.

7. The function  $V(x) = 0.5(x_1^2 + x_2^2)$  is positive definite at all points which are not in the origin. Then

$$\begin{aligned}\dot{V} &= x_1 \dot{x}_1 + x_2 \dot{x}_2 \\ &= x_1 (x_2 + \alpha x_1 (\beta^2 - x_1^2 - x_2^2)) + x_2 (-x_1 + \alpha x_2 (\beta^2 - x_1^2 - x_2^2)) \\ &= \alpha (x_1^2 + x_2^2) (\beta^2 - x_1^2 - x_2^2)\end{aligned}$$

Defining

$$U \triangleq \{x \in R^2 \mid \|x\|_2 \leq r, 0 < r < \beta\}$$

which is nonempty, it follows that  $V$  and  $\dot{V}$  are positive definite in  $U$ . By the Chetaev's theorem the origin is unstable.

8. (a) From the figure it can be seen that

$$\begin{aligned}\dot{x}_1 &= -g(e) + 2x_2 - x_1 \\ \dot{x}_2 &= g(e) - x_2 \\ e &= -x_1\end{aligned}$$

and the system is given by

$$\begin{aligned}\dot{x}_1 &= x_1^3 + 2x_2 - x_1 \\ \dot{x}_2 &= -x_1^3 - x_2\end{aligned}$$

- (b) Clearly the function  $V(x)$  is positive definite and radially unbounded. The derivative of  $V(x)$  along the trajectories of the system is given by

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} = 2x^T P \dot{x} \\ &= -x_1^2 - x_2^2 - 2x_1^3 x_2 \\ &= -x_1^2 - x_2^2 - 2x^T \begin{bmatrix} 0 & \frac{1}{2}x_1^2 \\ \frac{1}{2}x_1^2 & 0 \end{bmatrix} x \\ &= -x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x - x^T \begin{bmatrix} 0 & x_1^2 \\ x_1^2 & 0 \end{bmatrix} x \\ &= -x^T \begin{bmatrix} 1 & x_1^2 \\ x_1^2 & 1 \end{bmatrix} x \\ &= -x^T Q(x) x\end{aligned}$$

where positive definiteness of  $Q(x)$  implies that the origin is asymptotically stable. In order for  $Q(x)$  to be positive definite, it is required that all its leading principal minors are positive. This imposes the requirements

$$\begin{aligned}1 &> 0 \\ 1 - x_1^4 &> 0\end{aligned}$$

Taking  $D = \{x \in R^2 \mid |x_1| < 1\}$  and applying Theorem 4.1, the origin is asymptotically stable.

- (c) Since  $V(x)$  is radially unbounded it is known that the set  $\Omega_c = \{x \in R^2 | V(x) \leq c\}$ , where  $c$  is chosen such that  $|x_1| < 1 \quad \forall x \in \Omega_c$ , is positively invariant. The constant  $c$  is obtained by

$$\begin{aligned} c &= \min_{|x_1|=1} V(x) = \min_{|x_1|=1} x^T P x \\ &= \min_{|x_1|=1} \left( \frac{1}{2} x_1^2 + x_1 x_2 + \frac{3}{2} x_2^2 \right) \\ &= \min \begin{cases} \frac{1}{2} + x_2 + \frac{3}{2} x_2^2, & \text{if } x_1 = 1 \\ \frac{1}{2} - x_2 + \frac{3}{2} x_2^2, & \text{if } x_1 = -1 \end{cases} \end{aligned}$$

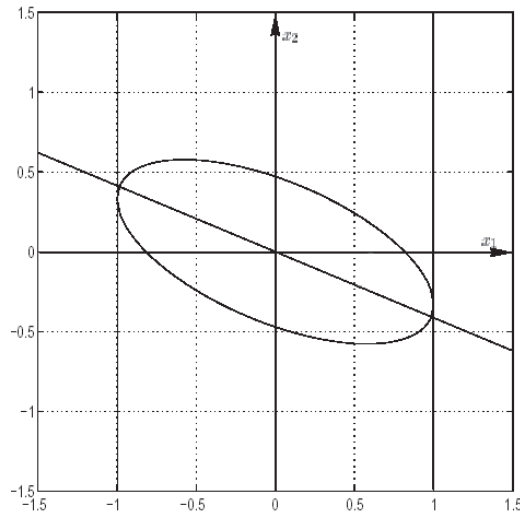
The minimum of  $V(x)$  along  $x_1 = 1$  and  $x_1 = -1$  is found through

$$\begin{aligned} \frac{\partial}{\partial x_2} \left( \frac{1}{2} + x_2 + \frac{3}{2} x_2^2 \right) &= 1 + 3x_2 = 0 \\ \frac{\partial}{\partial x_2} \left( \frac{1}{2} - x_2 + \frac{3}{2} x_2^2 \right) &= -1 + 3x_2 = 0 \end{aligned}$$

which implies that  $(-1, \frac{1}{3})$  and  $(1, -\frac{1}{3})$  are candidates for minimum:

$$\begin{aligned} c &= \min_{|x_1|=1} V(x) \\ &= \min_{x \in \{(-1, \frac{1}{3}), (1, -\frac{1}{3})\}} V(x) \\ &= \min \left\{ V\left(-1, \frac{1}{3}\right), V\left(1, -\frac{1}{3}\right) \right\} \\ &= \frac{1}{3} \end{aligned}$$

Take  $\Omega = \Omega_{\frac{1}{3}}$ ,  $E = \{x \in \Omega | \dot{V}(x) = 0\} = (0, 0) = M$ . This means that  $(0, 0)$  is the largest invariant set in  $E$ , and with respect to Theorem 4.4  $\Omega$  may be taken as an estimate of the region of attraction. Figure 2 shows a plot of the region of attraction.



**Figure 2:** Region of attraction

9. The system is given by

$$\begin{aligned}\dot{x}_1 &= 4x_1^2x_2 - f_1(x_1)(x_1^2 + 2x_2^2 - 4) \\ \dot{x}_2 &= -2x_1^3 - f_2(x_2)(x_1^2 + 2x_2^2 - 4)\end{aligned}$$

As the functions  $f_1$  and  $f_2$  have the same sign as their arguments it follows that  $x_1f_1(x_1) \geq 0$ ,  $x_2f_2(x_2) \geq 0$ ,  $f_1(0) = f_2(0) = 0$ . Then it is easy to see that there are three equilibrium points, i.e.  $(0, 0)$ ,  $(0, \pm\sqrt{2})$ . Next we want to show that  $\{x \in R^2 | x_1^2 + 2x_2^2 - 4 = 0\}$  is an invariant set. We define a new variable  $z \triangleq x_1^2 + 2x_2^2 - 4$ . The derivative of  $z$  with respect to time is

$$\begin{aligned}\dot{z} &= 2x_1\dot{x}_1 + 4x_2\dot{x}_2 \\ &= 2x_1[4x_1^2x_2 - f_1(x_1)(x_1^2 + 2x_2^2 - 4)] \\ &\quad + 4x_2[-2x_1^3 - f_2(x_2)(x_1^2 + 2x_2^2 - 4)] \\ &= -2x_1f_1(x_1)(x_1^2 + 2x_2^2 - 4) - 4x_2f_2(x_2)(x_1^2 + 2x_2^2 - 4) \\ &= -(2x_1f_1(x_1) + 4x_2f_2(x_2))(x_1^2 + 2x_2^2 - 4) \\ &= -2(x_1f_1(x_1) + 2x_2f_2(x_2))z\end{aligned}$$

where it can be seen that  $z = 0$  is an equilibrium point for this differential equation. This means that the relationship between  $x_1$  and  $x_2$  as given by  $z$  stays constant if  $z = 0$  initially, and  $\{x \in R^2 | x_1^2 + 2x_2^2 - 4 = 0\}$  is an invariant set for the system. Consider the function

$$V(x) = (x_1^2 + 2x_2^2 - 4)^2$$

which is radially unbounded. The derivative of  $V$  with respect to time is

$$\begin{aligned}\dot{V} &= 2(x_1^2 + 2x_2^2 - 4)(2x_1\dot{x}_1 + 4x_2\dot{x}_2) \\ &= -4(x_1f_1(x_1) + 2x_2f_2(x_2))(x_1^2 + 2x_2^2 - 4)^2\end{aligned}$$

which is negative semidefinite. Let  $D = R^2$ . The set  $\Omega_c = \{x \in R^2 | V(x) \leq c, \dot{V}(x) \leq 0\} = \{x \in R^2 | V(x) \leq c\}$  is a compact positively invariant set for any finite  $c > 0$  due to the radially unboundedness of  $V(x)$ . Let  $\Omega = \Omega_c$ , the set  $E$  is given by

$$\begin{aligned}E &= \{x \in \Omega | \dot{V}(x) = 0\} \\ &= \{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\} \cup \{x \in \Omega | x_1f_1(x_1) + 2x_2f_2(x_2) = 0\} \\ &= \{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\} \cup (0, 0)\end{aligned}$$

Since both  $\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\}$  and  $(0, 0)$  are invariant sets (recall that the origin is an equilibrium point), the largest invariant set in  $E$  is given by  $M = E$ , and by Theorem 4.4 it can be concluded that every solution starting in  $\Omega$  approaches  $\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\}$  or the origin as  $t \rightarrow \infty$ . However, the set  $\{x \in \Omega | x_1^2 + 2x_2^2 - 4 = 0\}$  is not a limit cycle since it contains equilibrium points  $(0, \pm\sqrt{2})$ .

10. From  $V(x) = \alpha x_1^2 + x_2^2$  and

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2\end{aligned}$$

we have

$$\begin{aligned}
\dot{V} &= 2\alpha x_1 \dot{x}_1 + 2x_2 \dot{x}_2 \\
&= 2\alpha x_1 x_2 + 2x_2 [-x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2] \\
&= -2x_2^2 - 2x_2^2 x_1^2 - 4x_1 x_2^3 - 2x_2^4 \\
&= -2x_2^2 [1 + x_1^2 + 2x_1 x_2 + x_2^2] \\
&= -2x_2^2 [1 + (x_1 + x_2)^2]
\end{aligned}$$

Thus  $\dot{V}$  is negative semidefinite since  $\dot{V}(x) = 0$  for  $x = (x_1, 0)$  where  $x_1 \in R$ . When  $\dot{V} = 0$  we should have  $x_2 = 0$  and also  $\dot{x}_2 = 0$ . From  $\dot{x}_2 = 0$  we should have  $-x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2 = 0$  or equivalently  $\alpha x_1 = 0$  since  $x_2 = 0$ . It follows that  $\dot{V}(x) = 0$  identically on  $x = (0, 0)$ . Hence, by Corollary 4.1, the origin is asymptotically stable. Furthermore since  $V$  is radially unbounded, by Corollary 4.2 we conclude that the origin is globally asymptotically stable.