TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2014 - Assignment 3
Due date: Monday 13 October at 12.00.

1. Consider again the Duckmaze system from Assignment 1 and 2.

   (a) Use the transformed system from Assignment 2 (Exercise 1b) and the Lyapunov function candidate
   $$V = \frac{1}{2}\left(\tilde{x}_1^2 + m\tilde{x}_2^2\right)$$
   to derive a controller (find $\tilde{u}$) such that
   $$\dot{V} = -(d + k_2)\tilde{x}_2^2$$
   where k_2 is the controller gain.
   (Hint: The resulting closed-loop system should be linear)

   (b) Is the closed-loop system locally/globally asymptotically/exponentially stable at the origin? Investigate all four possibilities and motivate your answers.

   (c) What happens to the system dynamics as k_2 increases? Explain this physically.

   (d) By using the controller in part (a), is it possible to place the poles of the system arbitrarily?

2. For a real matrix Λ we denote $\Lambda \geq 0$ when we mean that the matrix Λ is positive semidefinite and $\Lambda \leq 0$ when it is negative semidefinite. For a real symmetric positive definite matrix P we denote $\lambda_{\min}$ and $\lambda_{\max}$ as its smallest and largest eigenvalue, respectively. Show that the following inequalities
   $$\lambda_{\min} I \leq P \leq \lambda_{\max} I$$
   hold for
   $$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$
   Furthermore show that
   $$\lambda_{\min} \|x\|_2^2 \leq x^T P x \leq \lambda_{\max} \|x\|_2^2$$
   for all x.

3. Exercise 4.9 in Khalil

4. Exercise 4.15 in Khalil. (Hint: $\frac{d}{dz}\int_0^z \psi(u)\, du = \psi(z)$)

5. Consider
   $$\begin{aligned} \dot{x}_1 &= -(x_1 + 2x_2)(x_1 + 2) \\ \dot{x}_2 &= -8x_2(2 + 2x_1 + x_2) \end{aligned}$$

   (a) Using the indirect method (Theorem 4.7 of Khalil), show that the origin is asymptotically stable.

(b) Using the direct method (Theorem 4.1 of Khalil), show that the origin is asymptotically stable.
(Hint: use $\mathcal{D} = \{x \in R^2 | x_1 + 2x_2 + 1 \geq 0 \text{ and } 2x_1 + x_2 + 1 \geq 0\}$ and the Lyapunov function candidate $V = x_1^2 + x_2^2$)

(c) Let $\Omega_c \triangleq \{x \in R^2 | V(x) \leq c\}$. Draw $\mathcal{D}$, $\Omega_{\frac{1}{9}}$ and $\Omega_{6.25}$ together on the plane (you may use `pplane` and select 'Plot level curves' from the 'Solutions' menu). Explain why the trajectory converges to the origin when $x(0) = \left(0, \frac{1}{3}\right)$? Explain also why the trajectory does not converge to the origin when $x(0) = \left(-\frac{4}{3}, 2\right)$ eventhough $x(0)$ belongs to D.

6. Exercise 4.35 in Khalil.

7. Suppose that for each initial condition $x(0)$ the solution of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq \beta(\|x(0)\|, t)$$

for $t \geq 0$ where β is of class $\mathcal{KL}$.
Show that the origin of the system is globally asymptotically stable, i.e.

(a) Show stability for $x = 0$ using the definition of stability and the definition of class-$\mathcal{KL}$ functions.

(b) Show that every trajectory of the system converges to the origin.

8. Consider the system

$$\begin{aligned}\dot{x}_1 &= -\phi(t)x_1 + a\phi(t)x_2 \\ \dot{x}_2 &= b\phi(t)x_1 - ab\phi(t)x_2 - c\psi(t)x_2^3\end{aligned}$$

where a, b and c are positive constants and $\phi(t)$ and $\psi(t)$ are nonnegative, continuous, bounded functions that satisfy

$$\phi(t) \geq \phi_0 > 0, \quad \psi(t) \geq \psi_0 > 0, \quad \forall t \geq 0$$

Show that the origin is globally uniformly asymptotically stable.
(Hint: $V = 0.5\left(bx_1^2 + ax_2^2\right)$)

9. Exercise 4.38 in Khalil. (Hint: use the completion of squares)

10. Exercise 4.45 in Khalil. (Hint: use $V = 0.5\left(x_1^2 + x_2^2\right)$)

11. Consider the Pendulum system with friction as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2\end{aligned}$$

where the friction component is expressed by $(k/m)x_2$. Use the general Lyapunov function

$$V(x) = \frac{1}{2}x^T P x + \frac{g}{l}(1 - \cos x_1)$$

where

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \qquad P = P^T > 0$$

to examine the stability characteristic of the system. In particular, prove that the origin is locally asymptotically stable by an appropriate selection of matrix P?