TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2015 - Solution to Assignment 5

1. (Khalil 14.43) Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + v \\ y &= x_2\end{aligned}$$

With $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$

$$\dot{V} = x_1^3 x_2 - x_1^3 x_2 + x_2 v = yv$$

Hence the system is passive. With $v = 0$

$$y(t) \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

Thus the system is zero-state observable. Therefore, we can globally stabilize the system by $v = -\phi(y)$ for any $\phi(y)$ such that

$$\begin{gathered}\phi(y) \text{ is locally Lipschitz} \\ \phi(0) = 0 \\ y\phi(y) > 0 \text{ for all } y \neq 0\end{gathered}$$

Pick $\phi(y) = -\psi(-y)$, then

$$\begin{gathered}\phi(y) \text{ is locally Lipschitz} \\ \phi(0) = -\psi(-0) = 0 \\ y\phi(y) = -y\psi(-y) > 0 \text{ for all } -y \neq 0\end{gathered}$$

Thus $v = \psi(-y)$ is the stabilizing feedback which means $u = -y$.

2. (Khalil 13.1) The system is given by

$$\begin{aligned}M\ddot{\delta} &= P - D\dot{\delta} - \eta_1 E_q \sin(\delta) \\ \tau\dot{E}_q &= -\eta_2 E_q + \eta_3 \cos(\delta) + E_{FD}\end{aligned}$$

which is rewritten in the form $\dot{x} = f(x) + g(x)u$ using

$$\begin{aligned}x_1 &= \delta \\ x_2 &= \dot{\delta} \\ x_3 &= E_q \\ u &= E_{FD}\end{aligned}$$

This results in the system

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \frac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin\left(x_1\right)\right) \\ \dot{x}_3 & = & \frac{1}{\tau}\left(-\eta_2 x_3 + \eta_3 \cos\left(x_1\right) + u\right) \end{array} \tag{1}$$

where it can be seen that

$$\begin{array}{rcl} f\left(x\right) & = & \begin{bmatrix} x_2 \\ \frac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin\left(x_1\right)\right) \\ \frac{1}{\tau}\left(-\eta_2 x_3 + \eta_3 \cos\left(x_1\right)\right) \end{bmatrix} \\ g\left(x\right) & = & \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix} \end{array}$$

(a) The output is given by $y = \delta = x_1 = h\left(x\right)$. The relative degree is found as

$$\begin{array}{rcl} y & = & x_1 \\ \dot{y} & = & \dot{x}_1 \\ & = & x_2 \\ \ddot{y} & = & \dot{x}_2 \\ & = & \dfrac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin\left(x_1\right)\right) \\ \dddot{y} & = & -\dfrac{D}{M}\dot{x}_2 - \dfrac{\eta_1}{M}\dot{x}_3 \sin\left(x_1\right) - \dfrac{\eta_1}{M}x_3 \dfrac{\partial \sin\left(x_1\right)}{\partial x_1}\dot{x}_1 \\ & = & -\dfrac{D}{M}\dfrac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin\left(x_1\right)\right) \\ & & -\dfrac{\eta_1}{\tau M}\sin\left(x_1\right)\left(-\eta_2 x_3 + \eta_3 \cos\left(x_1\right) + u\right) \\ & & -\dfrac{\eta_1}{M}x_3 \cos\left(x_1\right) x_2 \end{array}$$

And the relative degree of the system is $\rho = 3$.
We have our system on the form

$$\dot{x} = f\left(x\right) + g\left(x\right) u \tag{2}$$

and would like to transform it to a system on the form

$$\begin{aligned} &\dot{\eta} = f_0\left(\eta, \xi\right) \\ &\dot{\xi} = A_c \xi + B_c \gamma\left(x\right)\left[u - \alpha\left(x\right)\right] \\ &y = C_c \xi \end{aligned}$$

where η is the internal dynamics and ζ the external dynamics. They are both given through the diffeomorphism

$$T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \triangleq \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} \tag{3}$$

where

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \text{ for } 1 \leq i \leq n-\rho, \; \forall \; x \in D_0$$

Since $\rho = n$, one only needs the external part of the system on normal form, ζ. External variables of the normal form is given by evaluating the Lie Derivative of h with respect to f

$$\begin{aligned} \xi_1 &= h(x) \\ &= x_1 \\ \xi_2 &= L_f h(x) \\ &= x_2 \\ \xi_3 &= L_f^2 h(x) \\ &= \frac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \end{aligned}$$

The system (1) can therefore be written on normal form as

$$\dot{\xi} = A_c \xi + B_c \gamma(x)\left[u - \alpha(x)\right] \tag{4}$$

$$y = C_c \xi \tag{5}$$

where

$$\gamma(x) = L_g L_f^{\rho-1} h(x) \quad \text{and} \quad \alpha(x) = -\frac{L_f^{\rho}h(x)}{L_g L_f^{\rho-1}h(x)} = -\frac{L_f^{\rho}h(x)}{\gamma(x)} \tag{6}$$

This transformation is therefore only valid when $\gamma(x) \neq 0$, which means that

$$\begin{aligned} L_g L_f^{\rho-1} h(x) &= L_g L_f^2 h(x) \\ &= -\frac{\eta_1}{\tau M} \sin(x_1) \\ &\neq 0 \; \forall \; x \in D_0 \end{aligned}$$

where $D_0 = \{x \in R^3 | \sin(x_1) \neq 0\}$. Since the relative degree equals the dimension of the system, we have no internal dynamics and the system is minimum phase.

(b) The output is given by $y = \delta + \zeta\dot{\delta} = x_1 + \zeta x_2 = h(x)$ where $\zeta \neq 0$. The relative degree is obtained from

$$
\begin{aligned}
y &= x_1 + \zeta x_2 \\
\dot{y} &= \dot{x}_1 + \zeta \dot{x}_2 \\
&= x_2 + \zeta \frac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \\
&= \left(1 - \frac{\zeta D}{M}\right) x_2 - \frac{\zeta \eta_1}{M} x_3 \sin(x_1) + \zeta P \frac{1}{M} \\
\ddot{y} &= \frac{\partial \dot{y}}{\partial x}\dot{x} \\
&= \begin{bmatrix} -\frac{\zeta\eta_1}{M} x_3 \cos(x_1) & \left(1 - \frac{\zeta D}{M}\right) & -\frac{\zeta\eta_1}{M}\sin(x_1) \end{bmatrix} \dot{x} \\
&= \frac{\zeta\eta_1}{M} x_3 \cos(x_1)\, x_2 \\
&+ \left(1 - \frac{\zeta D}{M}\right) \frac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \\
&- \frac{\zeta\eta_1}{\tau M}\sin(x_1)\left(-\eta_2 x_3 + \eta_3 \cos(x_1) + u\right)
\end{aligned}
$$

And the system thus has relative degree $\rho = 2$.
The region D_0 where the transformation is valid is where $L_g L_f^{\rho-1} h(x) \neq 0$

$$
\begin{aligned}
L_g L_f^{\rho-1} h(x) &= L_g L_f^1 h(x) \\
&= -\frac{\gamma\eta_1}{\tau M}\sin(x_1) \\
&\neq 0 \ \forall \ x \in D_0
\end{aligned}
$$

where $D_0 = \{x \in R^3 | \sin(x_1) \neq 0\}$.
Since $\rho < n$, both internal and external dynamics are needed. The external variables of the normal form is found by evaluating the Lie Derivative of h with respect to f

$$
\begin{aligned}
\xi_1 &= h(x) = x_1 + \zeta x_2 \\
\xi_2 &= L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = \begin{bmatrix} 1 & \zeta & 0 \end{bmatrix} f(x) \\
&= x_2 + \frac{\zeta}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right)
\end{aligned}
$$

The internal dynamics $\eta = \phi(x)$ is chosen to satisfy $\frac{\partial\phi(x)}{\partial x} g(x) = 0$ and the existence of $T^{-1}(x)$ in D_0. It can be verified that $\phi(x) = x_1$ meets these conditions. With $\phi(x) = x_1$ we have that

$$
\dot{\eta} = \dot{\phi}(x) = \dot{x}_1 = x_2 = \frac{1}{\zeta}(\xi_1 - \eta) = f_0(\eta, \xi)
$$

The system on normal form is thus

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A_c\xi + B_c\gamma(x)[u - \alpha(x)]$$
$$y = C_c\xi$$

The system is said to be minimum phase if the zero dynamics, $\dot{\eta} = f_0(\eta, 0)$, has an asymptotically stable equilibrium point in the domain of interest. From $\dot{\eta} = f_0(\eta, 0) = -\frac{1}{\gamma}\eta$ it can be recognized that the origin of η is asymptotically stable.

3. (Khalil 13.2) The system is given by

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 - x_3 \\
\dot{x}_2 &= -x_1 x_3 - x_2 + u \\
\dot{x}_3 &= -x_1 + u \\
y &= x_3
\end{aligned}$$

Rewriting this model on the form $\dot{x} = f(x) + g(x)u$ results in

$$\begin{aligned}
f(x) &= \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 x_3 - x_2 \\ -x_1 \end{bmatrix} \\
g(x) &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$

(a) The relative degree is obtained from

$$\begin{aligned}
y &= x_3 \\
\dot{y} &= \dot{x}_3 \\
&= -x_1 + u
\end{aligned}$$

which shows that the system has relative degree 1 in R^3. Hence, the system is input-output linearizable.

(b) The external part of the normal form is given by

$$\xi_1 = h(x) = x_3$$

To find the internal dynamics we start by setting up the requirements on $\frac{\partial \phi_i}{\partial x}$

$$\begin{aligned}
\frac{\partial \phi_1}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_1}{\partial x_3} \end{bmatrix} g(x) \\
&= \frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_1}{\partial x_3} \\
&= 0 \\
\frac{\partial \phi_2}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_2}{\partial x_3} \end{bmatrix} g(x) \\
&= \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_2}{\partial x_3} \\
&= 0
\end{aligned}$$

By choosing

$$\begin{aligned}
\phi_1(x) &= x_1 \\
\phi_2(x) &= x_2 - x_3
\end{aligned}$$

we obtain a global diffeomorphism

$$\begin{aligned}
T(x) &= \begin{bmatrix} x_1 \\ x_2 - x_3 \\ x_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} x
\end{aligned}$$

which is invertable. The system on normal form is

$$\begin{aligned}
\dot{\eta}_1 &= \dot{x}_1 \\
&= -\eta_1 + \eta_2 \\
\dot{\eta}_2 &= \dot{x}_2 - \dot{x}_3 \\
&= -x_1 x_3 - x_2 + u + x_1 - u \\
&= -\eta_1 \xi_1 - (\eta_2 + x_3) + \eta_1 \\
&= \eta_1 - \eta_2 - \xi_1 - \eta_1 \xi_1 \\
\dot{\xi}_1 &= -\eta_1 + u
\end{aligned}$$

Since $L_g L_f^0 h(x) = L_g h(x) = 1$, the transformation is valid in $\mathcal{R}^3$.

(c) To investigate if the system is minimum phase, we analyze the zero dynamics

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi)\big|_{\xi=0} \\
&= \left.\begin{bmatrix} -\eta_1 + \eta_2 \\ \eta_1 - \eta_2 - \xi_1 - \eta_1 \xi_1 \end{bmatrix}\right|_{\xi=0} \\
&= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \eta = A\eta
\end{aligned}$$

where it can be seen that $\text{eig}(A) = \begin{bmatrix} -2 & 0 \end{bmatrix}^T$. Hence, the origin is not asymptotically stable, and the system is therefore not minimum phase.

4. The system is rewritten as

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{aligned}
$$

where

$$
\begin{aligned}
f(x) &= \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} \\
g(x) &= \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} \\
h(x) &= x_3
\end{aligned}
$$

(a) The relative degree is found by derivative y with respect to time

$$
\begin{aligned}
y &= x_3 \\
\dot{y} &= \dot{x}_3 = x_2 \\
\ddot{y} &= \dot{x}_2 = x_1 x_2 + u
\end{aligned}
$$

where it can be seen that the system has a relative degree $\rho = 2$ in $x \in R^2$. The relative degree holds as long as $L_g L_f^{\rho-1} h(x) \neq 0$.

$$
L_g L_f^{\rho-1} h(x) = L_g L_f h(x) = 1 \neq 0 \ \forall \ x \in \mathcal{R}^2
$$

The relative degree thus holds over the entire $\mathcal{R}^3$ space.

(b) The system has a well defined relative degree ρ in the entire $\mathcal{R}^3$, and is therefore input-output linearizable in $\mathcal{R}^3$.

(c) The variables for the external dynamics are found according to

$$
\begin{aligned}
\xi_1 &= h(x) = x_3 \\
\xi_2 &= L_f h(x) = \frac{\partial h(x)}{\partial x} f = x_2
\end{aligned}
$$

The coordinates for the internal dynamics is chosen such that $T(x)$ is diffeomorphism on $\mathcal{R}^3$ and $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ on $\mathcal{R}^3$, where $\left[\eta, \xi^T\right]^T = [\phi(x), \psi(x)] = T(x)$.

In addition to this we require $\phi(0) = 0$ in order to have the origin as equilibrium. We start by calculating

$$\begin{aligned}\frac{\partial \phi(x)}{\partial x} g(x) &= \begin{bmatrix} \frac{\partial \phi(x)}{\partial x_1} & \frac{\partial \phi(x)}{\partial x_2} & \frac{\partial \phi(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix} \\ &= \frac{\partial \phi(x)}{\partial x_1} e^{x_2} + \frac{\partial \phi(x)}{\partial x_2} \\ &= 0\end{aligned}$$

and based on these calculations we try

$$\begin{aligned}\frac{\partial \phi(x)}{\partial x_1} &= 1 \\ \frac{\partial \phi(x)}{\partial x_2} &= -e^{x_2}\end{aligned}$$

which implies that

$$\phi(x) = x_1 - e^{x_2} + c$$

where c is some constant. This constant is chosen to satisfy our requirement $\phi(0) = 0$

$$\begin{aligned}\phi(0) &= -e^0 + c \\ &= -1 + c \\ &\Rightarrow c = 1\end{aligned}$$

Our resulting coordinate transformation is now given by

$$\begin{bmatrix} \eta \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} x_1 - e^{x_2} + 1 \\ x_3 \\ x_2 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \eta + e^{\xi_2} - 1 \\ \xi_2 \\ \xi_1 \end{bmatrix}$$

Consequently the inverse transformation exists. It follows that $T(x)$ and $T^{-1}(x)$ are continuously differentiable. Hence, $T(x)$ is a diffeomorphism on $\mathcal{R}^3$ and $T(0) = T^{-1}(0) = 0$.

(d) The system may be rewritten as

$$\begin{aligned}
\dot{\eta} &= \dot{x}_1 - \frac{\partial e^{x_2}}{\partial x_2}\dot{x}_2 \\
&= -x_1 + e^{x_2}u - e^{x_2}\left(x_1x_2 + u\right) \\
&= -x_1 - e^{x_2}x_1x_2 \\
&= -\left(\eta + e^{x_2} - 1\right) - e^{x_2}\left(\eta + e^{x_2} - 1\right)x_2 \\
&= \left(1 - \eta - e^{\xi_2}\right) + \left(1 - \eta - e^{\xi_2}\right)e^{\xi_2}\xi_2 \\
&= \left(1 - \eta - e^{\xi_2}\right)\left(1 + e^{\xi_2}\xi_2\right)
\end{aligned}$$

and

$$\begin{aligned}
\dot{\xi} &= A_c\xi + B_c\gamma(x)\left(u - \alpha(x)\right) \\
y &= C_c\xi
\end{aligned}$$

where

$$\begin{aligned}
A_c &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
B_c &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
C_c &= \begin{bmatrix} 1 & 0 \end{bmatrix} \\
\gamma(x) &= L_gL_fh(x) \\
&= 1 \\
\alpha(x) &= -\frac{L_f^2h(x)}{L_gL_fh(x)} \\
&= -\frac{x_1x_2}{1} \\
&= -x_1x_2
\end{aligned}$$

(e) The zero dynamics is given by

$$\begin{aligned}
\dot{\eta} &= \left. f_0(\eta, \xi)\right|_{\xi=0} \\
&= \left.\left(1 - \eta - e^{\xi_2}\right)\left(1 + e^{\xi_2}\xi_2\right)\right|_{\xi=0} \\
&= (1 - \eta - 1)(1 + 0) \\
&= -\eta
\end{aligned}$$

which has a globally asymptotically stable equilibrium at the origin.

(f) The external dynamics are given by

$$\dot{\xi} = A_c\xi + B_c\gamma(x)\left(u - \alpha(x)\right)$$

By choosing

$$u = \gamma^{-1}(x)\, v + \alpha(x)$$

the external dynamics are given by

$$\dot{\xi} = A_c \xi + B_c v$$

Since the system is controllable, $rank\left([B,\ AB]\right) = 2$, it can be stabilized (asymptotically stable) by a control input $v = -K\xi$ where K is chosen such that $(A_c - B_c K)$ is Hurwitz. u is now given by

$$u = -\gamma^{-1}(x)\, K\xi + \alpha(x)$$

Since $\dot{\eta} = f_0(\eta, \xi)|_{\xi=0}$ is asymptotically stable, the origin of the entire system is asymptotically stable.

(g) Let

$$\begin{aligned} \mathcal{R} &= \begin{bmatrix} r \\ \dot{r} \end{bmatrix} \\ e &= \xi - \mathcal{R} \end{aligned}$$

Then we can calculate

$$\begin{aligned} \dot{e} &= \dot{\xi} - \dot{\mathcal{R}} \\ &= (A_c\xi + B_c v) - (A_c\mathcal{R} + B_c\ddot{r}) \\ &= A_c(\xi - \mathcal{R}) + B_c(v - \ddot{r}) \\ &= A_c e + B_c(v - \ddot{r}) \\ &= A_c e + B_c(\gamma(x)\,[u - \alpha(x)] - \ddot{r}) \end{aligned}$$

where the simplified structure $\dot{\mathcal{R}} = A_c\mathcal{R} + B_c\ddot{r}$ is found using the known values of A_c and B_c (see part d).
We can choose the state feedback control

$$u = \gamma^{-1}(x)\,(v + \ddot{r}) + \alpha(x)$$

The resulting system is

$$\begin{aligned} \dot{\eta} &= f_0(\eta, e + \mathcal{R}) \\ \dot{e} &= A_c e + B_c v \end{aligned}$$

and since (A_c, B_c) is controllable, the loop is closed with $v = -Ke$ where K is chosen such that $(A_c - B_c K)$ is Hurwitz. This makes the external dynamics for e exponentially stable.
Since $\dot{\eta} = f_0(\eta, \xi)|_{\xi=0}$ is asymptotically stable, the origin of the overall closed-loop system is such that for sufficiently small initial conditions $e(0), \eta(0)$ and for $\mathcal{R}(t)$ with sufficiently small $\sup_{t\geq 0}\|\mathcal{R}(t)\|$, all solutions $(\eta(t), e(t))$ of the closed-loop system are bounded and $e(t) \to 0$ as $t \to \infty$.

5. (Khalil 13.25) The system is given by

$$\begin{aligned}\dot{x}_1 &= x_2 + 2x_1^2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_3 \\ y &= x_1\end{aligned}$$

Since

$$y = x_1 \Rightarrow \dot{y} = x_2 + 2x_1^2 \Rightarrow \ddot{y} = x_3 + u + 2x_1(x_2 + x_1^2)$$

Therefore, the system has a relative degree 2 in R^3. Let us check the minimum-phase property.

$$y(t) \equiv 0 \Rightarrow \dot{x}_3 = -x_3.$$

Hence, the system is minimum.-phase. Let $e = y - r$.

$$\ddot{e} = \ddot{y} - \ddot{r} = x_3 + u + 2x_1(x_2 + x_1^2) - \ddot{r}$$

Choosing

$$\begin{aligned}u &= -x_3 - 2x_1(x_2 + x_1^2) + \ddot{r} - k_1 e - k_2 \dot{e} \\ &= -x_3 - 2x_1(x_2 + x_1^2) + \ddot{r} - k_1(y - r) - k_2(x_2 + 2x_1^2 - \dot{r}),\end{aligned}$$

where k_1 and k_2 are positive constants. the tracking error $e$satisfies the equation $\ddot{e} + k_2\dot{e} + k_1 e = 0$, which shows that $e(t) \to 0$ as $t \to \infty$.