4.3

(2)

$$\bullet V(x) = \frac{1}{2}x_1^{\,2} + \frac{1}{2}x_2^{\,2}$$

$$\rightarrow \dot{V}(x) = -(x_1^{\,2} + x_2^{\,2})(1 - x_1^{\,2} - x_2^{\,2})$$

$$D = \{x \mid x_1^{\,2} + x_2^{\,2} < 1\}, \quad \dot{V}(x) < 0, x \in D$$

(4)

$$\bullet V(x) = x_1^{\,2} + \frac{1}{2}x_2^{\,2} \rightarrow \dot{V}(x) = -2x_1^{\,2} - x_2^{\,4}$$

4.13

(1)

$$\dot{x}_1 = x_1^3 + x_1^2 x_2$$

$$\dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

$$\text{set } V(x) = \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2$$

(i) V(x) is continuously differentiable

(ii) $V(0) = 0$

(iii) $V(x_0) > 0$ (i.e $x_0 = (1,0)$)

Let $B_r = \{x \in R^2 \,|\, \|x\| < r\}$, $U = \{x \in B_r \,|\, V(x) > 0\}$

$$\dot{V}(x) = x_1^4 + 2x_2 x_1^3 + x_2^2(1 - x_1 - x_2)$$

For small $r > 0$, $\dot{V}(x) > 0$ in U

$\therefore$ unstable

4.13

(2)
$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = x_1^6 - x_2^3$$
$$set\ V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$
(i) V(x) is continuously differentiable

(ii) V(0) = 0

(iii) $\mathrm{V}(\mathrm{x}_0) > 0$ (i.e $\mathrm{x}_0 = (1,0)$)

$$Let\ \mathrm{B}_{\mathrm{r}} = \{x \in R^2 \big| \|x\| < 1\},$$
$$\Gamma = \{0 \le x_1 \le 1\} \cap \{x_2 \ge x_1^3\} \cap \{x_2 \le x_1^2\}$$
$$\dot{V}(x) = x_1(x_2 - x_1^3) + x_2(x_1^2 - x_2)(x_1^4 + x_1^2 x_2 + x_2^2)$$
(a) Γ is nonempty

(b) $\dot{V}(x) > 0$ in set Γ except at (0,0), (1,1)

(c) $V(x) > 0$ in set Γ except at (0,0)

$$Define\ \mathrm{U} = \{\mathrm{x} \in \mathrm{B}_{\mathrm{r}} \big| V(x) > 0\} \subset \Gamma$$
$\therefore$ unstable

4.29

(a) $\text{eq. pts} \rightarrow (0,0), (2,6), (-2,-6)$

(b) $(0,0) \rightarrow \text{unstable}$

$(2,6) \rightarrow \text{asym. stable}$

$(-2,-6) \rightarrow \text{asym. stable}$

(c)

(i) Around eq. pt (2,6)

We use the direct Lyapunov method.

For simplicity, let $z_1 = x_1 - 2$ and $z_2 = x_2 - 6$

then system becomes

$$\dot{z}_1 = -11z_1 - 6z_1{}^2 - z_1{}^3 + z_2$$

$$\dot{z}_2 = 3z_1 - z_2$$

Let $V(z) = \frac{1}{2}z_1{}^2 + \frac{1}{2}z_2{}^2 \to p.d$ on R^2

$$\dot{V}(z) = -11z_1{}^2 - 6z_1{}^3 - z_1{}^4 + 4z_1 z_2 - z_2{}^2$$

$$= -11z_1{}^2 - 6z_1{}^3 - z_1{}^4 + \frac{4}{\varepsilon^2}z_1{}^2 - (\frac{2}{\varepsilon}z_1 - \varepsilon z_2)^2 - (1-\varepsilon^2)z_2{}^2$$

where $0 < \varepsilon^2 < 1$

$\dot{V}(z) \approx -7z_1{}^2 - 6z_1{}^3 - z_1{}^4 - (1-\varepsilon^2)z_2{}^2$ (for $\varepsilon^2 \approx 1$)

$\dot{V}(z)$ is n.d. on $z_1 > -3+\sqrt{2}$ and any z_2

Let $\Omega_c = \{z \mid V(z) \le c\}$

To make Ω_c be inside the set $D = \{z_1 > -3+\sqrt{2},\ \forall z_2\}$

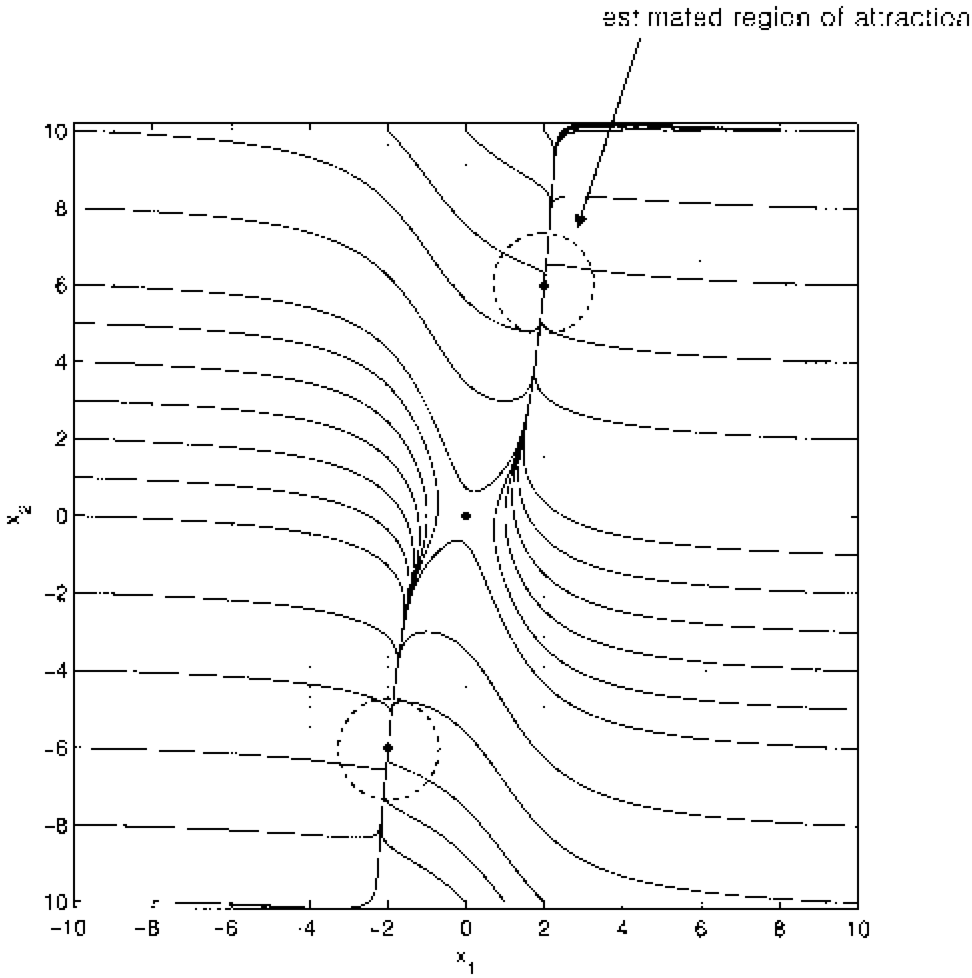
Let $c = \min_{\partial D} V(z) = \frac{1}{2}(z_1{}^2 + z_2{}^2)\Big|_{z_1=-3+\sqrt{2},\, z_2=0}$

$$= \frac{1}{2}(-3+\sqrt{2})^2 \approx 1.2574$$

(ii) Around eq. pt (-2,-6), we get the similar result. [$c \approx 1.2574$]

4.29

(d)



est mated region of attraction
10
8
6
4
2
0
-2
-4
-6
-8
10
−10
−8
-6
-4
-2
0
2
4
6
8
10
x_2
x_1

4.32 We do the easiest approach:

i)Linearization: $A=\left.\frac{\partial \mathrm{f}}{\partial x}\right|_{x=0}$

ii)Check the eigenvalues of A

(2)*asym.stable*

(4)*asym.stable*

4.37

(2) $V(x) = x_1^2 + x_2^2$

$$\dot{V}(x) = 2x_1\{-x_1 + \alpha(t)x_2\} + 2x_2\{-\alpha(t)x_1 - 2x_2\}$$

$$= -2x_1^2 - 4x_2^2 \le -2x_1^2 - 2x_2^2 = -2\|x\|_2^2 \quad \therefore \text{ exp. stable}$$

(4)

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}ax_2^2$$

Let $c_1 = \min\{1, a\}, c_2 = \max\{1, a\}$

$$c_1\|x\|_2^2 \le V(x) \le c_2\|x\|_2^2$$

$$\dot{V}(x) = -x_1^2 - 2ax_2^2 + a\alpha(t)x_1x_2$$

$$= -1/4x_1^2 + a\alpha(t)x_1x_2 - (a\alpha(t)x_2)^2 + (a\alpha(t)x_2)^2$$

$$-3/4x_1^2 - 2ax_2^2 \qquad -(1/2x_1^2 - a\alpha(t)x_2)^2$$

$$= -\cancel{(1/2x_1^2 - (a\alpha(t)x_2)^2)} - 3/4x_1^2 - (2a - (a\alpha(t))^2)x_2^2$$

$$\le -3/4x_1^2 - a(2 - a\alpha(t)^2)x_2^2$$

Since $\alpha(t)$ is bounded, $\alpha(t)^2 \le k$.

If we choose a between $0<a<2/k$, then $2 - a\alpha(t)^2 > 0$.

$$\therefore \dot{V}(x) \le -\min\{3/4, a(2 - a\alpha(t)^2)\}\|x\|_2^2$$

$\therefore$ exp. stable

4.45

(1)

$$V(x) = \frac{1}{2}x_1^{\ 2} + \frac{1}{2}x_2^{\ 2} \to \frac{1}{4}(x_1^{\ 2} + x_2^{\ 2}) \le V(x) \le (x_1^{\ 2} + x_2^{\ 2})$$

$$\dot{V}(x) = -g(t)(x_1^{\ 4} + x_2^{\ 4}) \le -k(x_1^{\ 4} + x_2^{\ 4})$$

$\therefore$ uniformly asymptotically stable

(2)

Let's find

$$\frac{\partial f}{\partial x}(t,0) = \begin{bmatrix} 0 & h(t) \\ -h(t) & 0 \end{bmatrix}$$

Choose $V = \frac{1}{2}x_1^{\ 2} + \frac{1}{2}x_2^{\ 2}$.

It means that the solutions stating on $V(x) = C$ remain on $V(x) = C, \forall t$. Thus the origin is not exponentially stable. Using Th.4.15, we conclude that the origin of the nonlinear system is not exponentially stable.

(3) Yes

(4) No