TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
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Fall 2015 - Solution to Assignment 4

1. (Khalil 4.54)

(1) The system is not input-to-state stable (ISS) since with $u(t) \equiv c < -1$ and $x(0) > 1$ we have $x(t) \to \infty$ for $t \to \infty$.

(2) Let $V(x) = \frac{1}{2}x^2$ which is positive definite and decrescent. Then

$$\begin{aligned}\dot{V} &= -x^4 - ux^4 - x^6 \leq -x^6 + |u|\, x^4 \\ \dot{V} &\leq -(1-\theta)\, x^6 - \theta x^6 + |u|\, x^4\end{aligned}$$

where $0 < \theta < 1$, and

$$\dot{V} \leq -(1-\theta)\, x^6 \quad \forall \quad \|x\| \geq \rho(|u|) > 0$$

where

$$\rho(|u|) = \sqrt{\frac{|u|}{\theta}}$$

By Theorem 4.19 in Khalil, the system is ISS.

(3) The system is not ISS since with $u(t) \equiv 1$ and $x(0) > 0$ we have $x(t) \to \infty$ for $t \to \infty$.

(4) With $u(t) \equiv 0$, the origin of $\dot{x} = x - x^3$ is unstable. Hence, the system is not ISS.

2. (Khalil 4.55)

(1) The system is given by

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -x_1^3 - x_2 + u\end{aligned}$$

Let $V(x)$ be given by

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

which is a $\mathcal{K}_\infty$ function. The time derivative along the trajectories of the system is

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1\left(-x_1 + x_1^2 x_2\right) + x_2\left(-x_1^3 - x_2 + u\right) \\ &= -x_1^2 + x_1^3 x_2 - x_1^3 x_2 - x_2^2 + ux_2 \\ &= -x_1^2 - x_2^2 + ux_2 \\ &= -\|x\|_2^2 + ux_2\end{aligned}$$

and upper bounded as

$$\begin{aligned}
\dot{V}(x) &\leq -\|x\|_2^2 + |ux_2| \\
&= -\|x\|_2^2 + |u|\,|x_2| \\
&\leq -\|x\|_2^2 + |u|\,\|x\|_2 \\
&= -\|x\|_2^2 + |u|\,\|x\|_2 + \theta\|x\|_2^2 - \theta\|x\|_2^2 \\
&= -(1-\theta)\|x\|_2^2 + |u|\,\|x\|_2 - \theta\|x\|_2^2 \\
&= -(1-\theta)\|x\|_2^2 - (\theta\|x\|_2 - |u|)\|x\|_2 \\
&\leq -(1-\theta)\|x\|_2^2 \quad \forall\ \theta\|x\|_2 - |u| \geq 0 \\
&= -(1-\theta)\|x\|_2^2 \quad \forall\ \ \|x\|_2 \geq \frac{|u|}{\theta}
\end{aligned}$$

where $\theta \in (0,1)$. Hence, by Theorem 4.19, the system is input-to-state stable (ISS) with $\rho(|u|) = \frac{|u|}{\theta}$.

(2) The system is given by

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= -x_1^3 - x_2 + u
\end{aligned}$$

Let $V(x)$ be given by

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$$

The time derivative along the trajectories of the system is calculated as

$$\begin{aligned}
\dot{V}(x) &= x_1^3\dot{x}_1 + x_2\dot{x}_2 \\
&= x_1^3(-x_1 + x_2) + x_2\left(-x_1^3 - x_2 + u\right) \\
&= -x_1^4 + x_1^3x_2 - x_1^3x_2 - x_2^2 + ux_2 \\
&= -x_1^4 - x_2^2 + ux_2
\end{aligned}$$

and upper bounded as

$$\begin{aligned}
\dot{V}(x) &= -x_1^4 - (1-\theta)x_2^2 + ux_2 - \theta x_2^2 \\
&\leq -x_1^4 - (1-\theta)x_2^2 \ \forall\,|x_2| \geq \frac{|u|}{\theta}
\end{aligned} \tag{1}$$

where $\theta \in (0,1)$. When $|x_2| \leq \frac{|u|}{\theta}$ have that

$$\begin{aligned}
\dot{V}(x) &= -x_1^4 - x_2^2 + ux_2 \\
&\leq -x_1^4 - x_2^2 + |x_2|\,|u| \\
&\leq -x_1^4 - x_2^2 + \frac{|u|^2}{\theta} \\
&= -(1-\theta)x_1^4 - x_2^2 - \left(\theta x_1^4 - \frac{|u|^2}{\theta}\right) \\
&\leq -(1-\theta)x_1^4 - x_2^2 \ \forall\,|x_1| \geq \sqrt{\frac{|u|}{\theta}}
\end{aligned} \tag{2}$$

By using (1) and (2) it follows that

$$\dot{V}(x) \leq -(1-\theta)\left(x_1^4 + x_2^2\right) \ \forall \|x\|_\infty \geq \rho(|u|)$$

where

$$\rho(r) = \max\left(\frac{r}{\theta}, \sqrt{\frac{r}{\theta}}\right)$$

Hence, the system is ISS.

(4) With $u = 0$ the system is given by

$$\begin{aligned}\dot{x}_1 &= (x_1 - x_2)\left(x_1^2 - 1\right)\\ \dot{x}_2 &= (x_1 + x_2)\left(x_1^2 - 1\right)\end{aligned}$$

and it can be seen that it has an equilibrium set $\{x_1^2 = 1\}$. Hence, the origin is not globally asymptotically stable. It follows that the system is not ISS.

(5) The unforced system ($u = 0$) has equilibrium points $(-1, -1)$, $(0, 0)$ and $(1, 1)$. Hence, the origin is not globally asymptotically stable. Consequently, the system is not ISS.

3. (Khalil 4.56) The system is given by

$$\dot{x}_1 = -x_1^3 + x_2 \tag{3}$$
$$\dot{x}_2 = -x_2^3 \tag{4}$$

We first show that the system $\dot{x}_1 = -x_1^3 + u$ is ISS using Theorem 4.19 with $V = \frac{1}{2}x_1^2$.

$$\begin{aligned}\dot{V} &= x_1\left(-x_1^3 + u\right) = -x_1^4 + x_1 u\\ &= -(1-\theta)x_1^4 - \theta x_1^4 + x_1 u\\ &\leq -(1-\theta)x_1^4 - \theta x_1^4 + |x|\,|u|\\ &\leq -(1-\theta)x_1^4 \ \ \forall \ \ |x_1| \geq \left(\frac{|u|}{\theta}\right)^{1/3}\end{aligned}$$

where $0 < \theta < 1$. The system $\dot{x}_1 = -x_1^3 + u$ is thus ISS.

Next we show that the system $\dot{x}_2 = -x_2^3$ is GAS at the origin using Theorem 4.2 with $V = \frac{1}{2}x_2^2$.

$$\dot{V} = -x_2^4 < 0 \ \ \forall \ \ x \neq 0$$

Since V also is radially unbounded in x_2, the system $\dot{x}_2 = -x_2^3$ is GAS at the origin. Hence, by Lemma 4.7 the cascade system (3) - (4) is GAS at the origin.

4. (Khalil 5.3)

(a) Let $\alpha(r) = r^{1/3}$; α is a class $\mathcal{K}_\infty$ function. We have

$$|y| \leq |u|^{1/3} \Longrightarrow \|y_\tau\|_{\mathcal{L}_\infty} \leq \left(\|u_\tau\|_{\mathcal{L}_\infty}\right)^{1/3} \Longrightarrow \|y_\tau\|_{\mathcal{L}_\infty} \leq \alpha\left(\|u_\tau\|_{\mathcal{L}_\infty}\right).$$

Hence the system is $\mathcal{L}_\infty$ stable with zero bias.

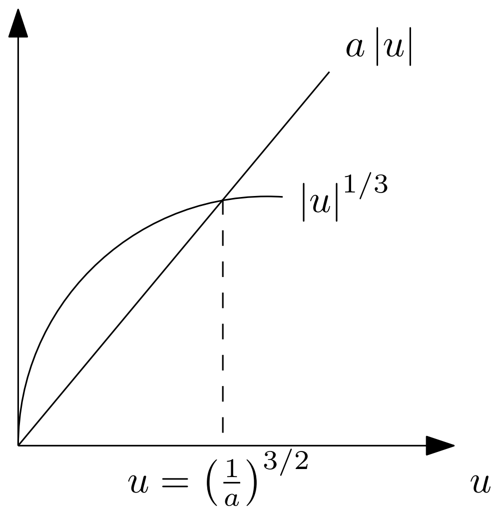

Figure 1: $y = a|u|$ and $y = |u|^{1/3}$.

(b) The two curves $|y| = |u|^{1/3}$ and $|y| = a|u|$ intersect at the point $|u| = (1/a)^{3/2}$. See Figure 1. Therefore, for $|u| \le (1/a)^{3/2}$ we have

$$|y| \le |u|^{1/3} \le (1/a)^{3/2 \cdot 1/3} = (1/a)^{1/2}$$

while for $|u| > (1/a)^{3/2}$ we have

$$|y| \le a|u|$$

Thus

$$|y| \le a|u| + (1/a)^{1/2}, \quad \forall |u| \ge 0.$$

Setting $\gamma = a$ and $\beta = (1/a)^{1/2}$ we obtain

$$\|y_\tau\|_{\mathcal{L}_\infty} \le \gamma \|u_\tau\|_{\mathcal{L}_\infty} + \beta.$$

(c) To show finite-gain stability we must use nonzero bias. This example shows that a nonzero bias term may be used to achieve finite-gain stability in situations where it is not possible to have finite-gain stability with zero bias.

5. (Khalil 5.4)

(1) Since the system is globally Lipschitz we can say that $h(0) = 0 \Longrightarrow |h(u)| \le L|u|, \forall u$. For $p = \infty$ we have

$$\sup_{t \ge 0} |y(t)| \le L \sup_{t \ge 0} |u(t)|$$

which shows that the system is finite-gain $\mathcal{L}_\infty$ stable with zero bias. For $p \in [1, \infty)$ we have

$$\int_0^\tau |y(t)|^p \, dt \le L^p \int_0^\tau |u(t)|^p \, dt \Longrightarrow \|y_\tau\|_{\mathcal{L}_p} \le L \|u_\tau\|_{\mathcal{L}_p}.$$

Hence for each $p \in [1, \infty)$ the system is finite gain $\mathcal{L}_p$ stable with zero bias.

(2) Let $|h(0)| = k > 0$. Then $|h(u)| \le L|u| + k$. For $p = \infty$ we have

$$\sup_{t\ge 0} |y(t)| \le L \sup_{t\ge 0} |u(t)| + k$$

which shows that the system is finite gain $\mathcal{L}_\infty$ stable. For $p \in [1,\infty)$ the integral $\int_0^\tau (L|u(t)| + k)^p \, dt$ diverges as $\tau \to \infty$. The system is not $\mathcal{L}_p$ stable for $p \in [1,\infty)$ as it can be seen by taking $u(t) \equiv 0$.

6. (Khalil 5.20) The closed-loop transfer functions are given by

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s+2} & \frac{-1}{s+2} \\ \frac{1}{s+2} & \frac{s+1}{(s-1)(s+2)} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} , \quad \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s+2} & \frac{-(s+1)}{(s-1)(s+2)} \\ \frac{s-1}{s+2} & \frac{s+1}{s+2} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

The closed-loop transfer function from (u_1, u_2) to (y_1, y_2) (or (e_1, e_2)) has four components. Due to pole-zero cancellation of the unstable pole $s = 1$, three of these components do not contain the unstable pole; thus, each component by itself is input-output stable. If we restrict our attention to any one of these components, we miss the unstable hidden mode. By studying all four components we will be sure that unstable hidden modes must appear in at least one component.

7. (Khalil 9.12) The system is given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + (x_1 + a)x_2 \\ \dot{x}_2 &= -x_1(x_1 + a) + bx_2, \quad a \ne 0 \end{aligned}$$

(a) Let $b = 0$. Let $V(x)$ be given by

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

which is a $\mathcal{K}_\infty$ function. The time derivative along the trajectories of the system is

$$\begin{aligned} \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1(-x_1 + (x_1 + a)x_2) + x_2(-x_1(x_1 + a)) \\ &= -x_1^2 + x_1x_2(x_1 + a) - x_1x_2(x_1 + a) \\ &= -x_1^2 \end{aligned}$$

If $\dot{V}(x) = 0 \Rightarrow x_1(t) \equiv 0 \Rightarrow ax_2(t) \equiv 0 \Rightarrow x_2(t) = 0$. Thus, the origin is globally asymptotically stable. To investigate exponential stability, the system is linearised at $x = 0$.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\begin{bmatrix} -1 + x_2 & x_1 + a \\ -2x_1 - a & 0 \end{bmatrix}\right|_{x=0} = \begin{bmatrix} -1 & a \\ -a & 0 \end{bmatrix}$$

The characteristic equation of A is $\lambda^2 + \lambda + a^2 = 0$. Hence, A is Hurwitz and the origin is exponential stable.

(b) Let $b > 0$. $A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} -1 & a \\ -a & b \end{bmatrix}$. The characteristic equation of A is $\lambda^2 + (1+b)\lambda + a^2 - b = 0$. Hence, A is Hurwitz if $b < \min\{1, a^2\}$.

(c) For $b > 0$. To find the equilibrium points set

$$\begin{aligned} 0 &= -x_1 + (x_1 + a)x_2 \\ 0 &= -x_1(x_1 + a) + bx_2, \end{aligned}$$

then by solving two equation with two unknowns the equilibrium points are found to be

$$(0,0), \quad \left(-a + \sqrt{b}, \frac{-a + \sqrt{b}}{\sqrt{b}}\right), \quad \left(-a - \sqrt{b}, \frac{a + \sqrt{b}}{\sqrt{b}}\right).$$

Since the system has multiple equilibria, the origin is not global asymptotically stable.

(d) While we cannot apply Lemma 9.1 because the Jacobian matrix is not global bounded, the lemma hints that the origin of the nominal system is not global exponential stable because if it was so we would have expected the origin of the perturbed system to be global exponential stable as well.

8. (Khalil 9.13) The scalar system

$$\dot{x} = \frac{-x}{1 + x^2}$$

and $V(x) = x^4$ has been given.

(a) $V(x) = x^4$ satisfies (9.11) with $\alpha_1(r) = \alpha_2(r) = r^4$. Next, we check if $\alpha_3(r) = \frac{4r^4}{1+r^2}$ satisfies (9.12).

$$\dot{V}(x) = 4x^3\dot{x} = -\frac{4x^4}{1 + x^2} \Rightarrow \dot{V}(x) \le -\alpha_3(r), \text{ with } \alpha_3(r) = \frac{4r^4}{1 + r^2}$$

Finally, the condition of (9.13) is checked.

$$\left|\frac{dV}{dx}\right| = 4|x|^3 \le \alpha_4(|x|), \text{ with } \alpha_4(r) = 4r^3.$$

(b) $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_4(\cdot)$ are clearly class $\mathcal{K}_\infty$ functions. $\alpha_3(r)$ is monotonically increasing and $\alpha_3(r) \to \infty$ as $r \to \infty$. Hence, $\alpha_3(\cdot)$ belongs to class $\mathcal{K}_\infty$.

(c) By applying the right-hand side of (9.14) it can be seen that

$$\frac{\alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{\alpha_4(r)} = \frac{\alpha_3(r)}{\alpha_4(r)} = \frac{4r^4}{4r^3(1 + r^2)} = \frac{r}{1 + r^2} \to 0 \text{ as } r \to \infty$$

(d) The perturbed system $\dot{x} = \frac{-x}{1+x^2} + \delta$, where $\delta > 0$ is given. Since $\frac{-x}{1+x^2}$ is a bounded function, we need to show what maximum value, the function can take, is and if $\delta > \max\left(\frac{-x}{1+x^2}\right)$ then the solution $x(t)$ escapes to ∞.

$$\left|\frac{x}{1+x^2}\right| = \frac{|x|}{1+x^2} \leq \frac{1}{2}, \quad \forall |x|$$
$$\delta > \frac{1}{2} \Rightarrow \dot{x} = \frac{-x}{1+x^2} + \delta > 0.$$

Hence, the solution $x(t)$ escapes to ∞ for any initial state $x(0)$.

9. (Khalil 10.9)

(1) The system is given as

$$\dot{x} = \varepsilon(x - x^2)\sin^2 t$$

Applying (10.25)

$$f_{av}(x) = \frac{1}{\pi}\int_0^\pi (x - x^2)\sin^2 t\, dt = \frac{1}{\pi}\int_0^\pi \sin^2 t\, dt(x - x^2) = \frac{1}{2}(x - x^2)$$

The average system $\dot{x} = \varepsilon(x - x^2)$ has equilibrium points at $x = 0$ and $x = 1$. The Jacobian functions at these points are $\varepsilon/2$ and $-\varepsilon/2$. Thus the equilibrium point $x = 1$ is exponential stable. By Theorem 10.4, we can conclude that, for sufficient small ε, the system the an exponentially stable periodic solution of period π in an $O(\varepsilon)$ neighbourhood of $x = 1$. However, $x = 1$ is an equilibrium point of the original system. Thus, the periodic solution is the trivial solution $x(t) \equiv 1$, and the equilibrium point $x = 1$ is exponentially stable for sufficiently small ε. Moreover, for initial states sufficiently near $x = 1$, $x(t, \varepsilon) = x_{av}(t, \varepsilon) + O(\varepsilon)$ for all $t \geq 0$.

(2) The system is given as

$$\dot{x} = \varepsilon\left(x\cos^2 t - \frac{1}{2}x^2\right)$$

Applying (10.25)

$$f_{av}(x) = \frac{1}{\pi}\int_0^\pi \left(x\cos^2 t - \frac{1}{2}x^2\right) dt = \frac{1}{2}(x - x^2)$$

This is the same average function as part (1). The rest of solution is similar to part (1), except that in the current case the periodic solution is nontrivial.

10. (Khalil 10.10)

(1) The system is given as

$$\dot{x}_1 = \varepsilon x_2$$
$$\dot{x}_2 = -\varepsilon(1 + 2\sin t)x_2 - \varepsilon(1 + \cos t)\sin x_1$$

$$f_{av}(x) = \frac{1}{2\pi}\int_0^{2\pi}\begin{bmatrix} x_2 \\ -(1 + 2\sin t)x_2 - (1 + \cos t)\sin x_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2 - \sin x_1 \end{bmatrix}$$

The average system has an equilibrium point at the origin. Linearisation at the origin yields $\frac{\partial f_{av}}{\partial x}\Big|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, which is Hurwitz. Thus, the origin of the average system is exponentially stable. It follows that, for sufficiently small ε, the original system has a unique exponentially stable periodic solution in the neighbourhood of the origin. But the origin is an equilibrium point of the original system. Hence, the periodic solution is the trivial solution, which shows that the origin is exponentially stable.

(2) The system is given as

$$\dot{x}_1 = \varepsilon[(-1 + 1.5\cos^2 t)x_1 + (1 - 1.5\sin t\cos t)x_2]$$
$$\dot{x}_2 = \varepsilon[(-1 - 1.5\sin t\cos t)x_1 + (-1 + 1.5\sin^2 t)x_2]$$

$$f_{av}(x) = \frac{1}{2\pi}\int_0^{2\pi}\begin{bmatrix} (-1 + 1.5\cos^2 t)x_1 + (1 - 1.5\sin t\cos t)x_2 \\ (-1 - 1.5\sin t\cos t)x_1 + (-1 + 1.5\sin^2 t)x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 1 \\ -1 & -\frac{1}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The matrix on the right-hand side is Hurwitz. Hence, the origin of the average system is exponentially stable. Noting that the origin is also an equilibrium point for the original system, we conclude, as in part (1), that the origin is exponentially stable for sufficiently small ε.