TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2017 - Assignment 4
Due date: Monday 30 October at 16.00.

1. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 + x_2|\cos x_2|\end{aligned}$$

Use the Lyapunov function candidate $V(x) = x^T P x, P = P^T$, where P is positive definite, and Young's inequality to prove global asymptotic stability of the origin.

2. Consider the system

$$\begin{aligned}\dot{x}_1 &= -2x_2 \\ \dot{x}_2 &= 2x_1 + \left(x_1^2 - 4\right) x_2\end{aligned}$$

Use the Lyapunov function candidate $V(x) = x^T P x$ where

$$P = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

and the Cauchy-Schwarz inequality to show that the origin is asymptotically stable.

3. Exercise 4.54 in Khalil.
Hint: If a system is ISS, then:

   (a) for $u(t) \equiv 0$ the origin is globally asymptotically stable.

   (b) for a bounded input $u(t)$, every solution $x(t)$ is bounded.

   If one of these is not satisfied, the system can **not** be ISS.

4. Exercise 4.55 no. (1), (2), (4) and (5) in Khalil.
Hint for part (2): Read example 4.27 before doing this exercise.
Hint for part (4): For $u(t) \equiv 0$ an ISS system needs to have a globally asymptotically stable origin. This requires the absence of other equilibria.

5. Exercise 4.56 in Khalil.

6. Exercise 5.3 in Khalil.

7. Exercise 5.4 in Khalil.

8. Exercise 5.20 in Khalil.

9. Exercise 6.2 in Khalil.
10. Exercise 6.4 in Khalil.
11. Consider again the system from the previous assignments.
    (a) Consider the transformed system from Assignment 2 (Exercise 1b):

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2 & (1) \\ m\dot{\tilde{x}}_2 &= -f_3\left[(\tilde{x}_1 + x_{1d})^3 - x_{1d}^3\right] - f_1\tilde{x}_1 - d\tilde{x}_2 + \tilde{u} & (2) \end{aligned}$$

    Define the output

$$y = \tilde{x}_2 \tag{3}$$

    As in Assignment 3 (Exercise 1a), use $V = \frac{1}{2}\left(\tilde{x}_1^2 + m\tilde{x}_2^2\right)$ as Lyapunov function candidate.
    Outline a control law that makes the system passive from the new control input v to the output y (in Khalil this technique is described as feedback passivation).
    Note: This topic (Chapter 14.4: Passivity-based Control) will be covered in the lectures later on - this problem is however an easy introduction which you will be able to solve.
    (b) Is the system zero state observable?
    (c) Explain why the origin can be globally stabilized. Derive a controller that globally stabilizes the origin.
    (d) An unknown constant disturbance w is acting on the system, i.e. the system equations are changed to

$$\begin{aligned} \dot{x}_1 &= x_2 & (4) \\ \dot{x}_2 &= -\frac{f_3}{m}x_1^3 - \frac{f_1}{m}x_1 - \frac{d}{m}x_2 - g + \frac{u}{m} + \frac{w}{m} & (5) \end{aligned}$$

    Do we still have $x_1^* = \lim_{t\to\infty} x_1 = x_{1d}$?

    When investigating passivity for interconnected systems, the first step is often to try a storage function as a sum of the storage functions for the interconnected systems.
12. Exercise 6.6 in Khalil.
    (Note: In this problem, for output strictly passivity you can assume that $y_i^T\rho_i(y_i) \geq \delta_i y_i^T y_i$ for some positive δ_i)
13. Exercise 6.1 in Khalil.
14. Consider a PID controller

$$h(s) = K_p\beta\frac{(1+T_is)(1+T_ds)}{(1+\beta T_is)(1+\alpha T_ds)} \tag{6}$$

    as a system with $K_p = 1$, $T_d = 1$, $T_i = 2$, $\beta = 1.5$ and $\alpha = 0.5$.

(a) Show that
$$|h(j\omega)| \leq \frac{K_p\beta}{\alpha} \ \forall\omega \tag{7}$$

(b) Show that
$$\mathrm{Re}\left[h(j\omega)\right] \geq K_p \ \forall\omega \tag{8}$$

For the rest of the exercise, assume that the conditions 7–8 hold for all cases where $K_p > 0$, $0 \leq T_d < T_i$, $1 \leq \beta < \infty$ and $0 < \alpha \leq 1$.

(c) Show that the system is passive (Hint: See Appendix A).

(d) Show that the system is input strictly passive (Hint: See Appendix A).

(e) Show that the system is output strictly passive (Hint: See Appendix A).

(f) Show that the system is zero-state observable.

15. Exercise 6.11 in Khalil (Hint: Use $V(\omega) = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2$).
16. Exercise 6.14 in Khalil (Hint: Use $V_1(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ and $V_2(x_3) = \int_0^{x_3} h_2(z)\,dz$).
17. Exercise 6.15 in Khalil (Hint: Use $V_1(x_1, x_2) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ and $V_2(x_3) = \frac{1}{4}x_3^4$).

Appendix A: Passivity

A linear system given by the scalar transfer function $h(s)$ such that all poles p_i satisfy $\mathrm{Re}\left[p_i\right] \leq 0$ is

- passive if $\mathrm{Re}\left[h(j\omega)\right] \geq 0 \quad \forall\omega$ such that $j\omega$ is not a pole.
- input strictly passive if $\mathrm{Re}\left[h(j\omega)\right] \geq \delta > 0 \quad \forall\omega$ such that $j\omega$ is not a pole.
- output strictly passive if $\mathrm{Re}\left[h(j\omega)\right] \geq \epsilon\left|h(j\omega)\right|^2 > 0 \quad \forall\omega$ such that $j\omega$ is not a pole, and for some positive ϵ.