14.1

$$\dot{x}_1 = x_2 + \sin x_1$$

$$\dot{x}_2 = \theta_1 x_1^2 + (1+\theta_2)u, \quad |\theta_1| \le 1, \quad |\theta_2| \le \frac{1}{2}$$

(a)

$$x_2 = -kx_1$$

$$x_1\dot{x}_1 = x_1(-kx_1 + \sin x_1) \le -(k-1)x_1^2 \quad (k > 1)$$

$$s = kx_1 + x_2$$

$$\dot{s} = k(x_2 + \sin x_1) + (\theta_1 x_1^2 + (1+\theta_2)u)$$

$$u = -k(x_2 + \mathrm{sinx}_1) + v$$

$$\dot{s} = \underbrace{\theta_1 x_1^2 - k\theta_2(x_2 + \sin x_1)}_{\Delta} + (1+\theta_2)v$$

$$|\Delta| \le \underbrace{|\theta_1|\, x_1^2 + |k|\,|\theta_2|\,(|x_2| + |\sin x_1|)}_{\beta(\mathbf{x}) = 2x_1^2 + |k|\frac{1}{2}(|x_2| + |\sin x_1|) + \beta_0 \quad (\beta_0 > 0)}$$

$$v = -a\beta(x)\,\mathrm{sgn}(s) \quad (a = 2)$$

14.1

(b)

$$
\begin{aligned}
&e_1 = x_1 - r \\
&\dot{e}_1 = \dot{x}_1 - \dot{r} = x_2 + \sin x_1 - \dot{r} = e_2 \\
&\dot{e}_2 = \theta_1 {x_1}^2 + (1+\theta_2)u - \ddot{r} + (x_2 + \sin x_1)\cos x_1 \\
&\quad\;\; = \theta_1 (e_1 + r)^2 (e_2 + \dot{r})\cos(e_1 + r) + (1+\theta_2)u - \ddot{r}
\end{aligned}
$$

$$
\begin{aligned}
&s = ke_1 + e_2 \\
&\dot{s} = ke_1 + \theta_1 (e_1 + r)^2 + (e_2 + \dot{r})\cos(e_1 + r) + (1+\theta_2)u - \ddot{r} \\
&u = -ke_1 - (e_2 + \dot{r})\cos(e_1 + r) + \ddot{r}
\end{aligned}
$$

$$
\therefore \dot{s} = \underbrace{\theta_1 (e_1 + r)^2 - \theta_2 (e_2 + \dot{r})\cos(e_1 + r) - k\theta_2 e_2}_{\Delta} + \theta_2 \ddot{r} + (1+\theta_2)v
$$

$$
|\Delta| \le 2(e_1 + r)^2 + \frac{1}{2}|e_2 + \dot{r}||\cos(e_1 + r)| + k\frac{1}{2}|\mathrm{e}_2| + \frac{1}{2}|\ddot{r}| = \beta_1(x)
$$

Note that $r, \dot{r}, \ddot{r}$ *are bounded, Thus,*

$$
\begin{aligned}
&\beta(x) = \beta_1(x) + \beta_0 \quad (\beta_0 > 0) \\
&v = -a\beta(x)\,\mathrm{sgn}(s) \quad (a = 2)
\end{aligned}
$$

14.5

$$\theta = x_1$$
$$\dot{\theta} = x_2$$

$$\dot{x}_1 = x_2$$
$$\begin{aligned}\dot{x}_2 &= -\frac{k}{m}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{h(t)}{l}\cos\theta + \frac{T}{ml^2} \\ &= -\frac{k}{m}x_2 - \frac{g}{l}\sin x_1 + \frac{h(t)}{l}\cos x_1 + \frac{1}{ml^2}u\end{aligned}$$

$$\downarrow$$

$$\begin{cases}\dot{x}_1 = x_2 \\ \dot{x}_2 = h(x) + g(x)u\end{cases} \quad where \quad \begin{cases} h(x) = -\frac{k}{m}x_2 - \frac{g}{l}\sin x_1 + \frac{h(t)}{l}\cos x_1 \\ g(x) = \frac{1}{ml^2}\end{cases}$$

· *Sliding manifold*

$$s = a_1 x_1 + x_2 \quad with \quad a_1 = 1$$

$$\dot{s} = a_1 x_2 + h(x) + g(x)u$$

$$u = -\frac{a_1 x_1 + \hat{h}(x)}{\hat{g}(x)} + v \quad where \quad \begin{cases}\hat{h}(x) = -\frac{\hat{k}}{\hat{m}}x_2 - \frac{g}{\hat{l}}\sin x_1 \\ \hat{g}(x) = \frac{1}{\hat{m}\hat{l}^2}\end{cases}$$

$$\therefore \dot{s} = a_1\left(1 - \frac{g(x)}{\hat{g}(x)}\right)x_2 + h(x) - \frac{g(x)}{\hat{g}(x)}\hat{h}(x) + g(x)v \triangleq \delta(x) + g(x)v$$

$$\left|\frac{\delta(x)}{g(x)}\right| \le \rho(x)$$

$$v = -\beta(x)\,\mathrm{sgn}(s) \quad where \quad \beta(x) \ge \rho(x) + \beta.$$

In the above switching controller v, to use continuous sliding mode controller, we change the signum function to the saturation function as follows:

$$v = -\beta(x) sat\left(\frac{s}{\epsilon}\right)$$

when we set $\epsilon = 0.01$, *we achieve the ultimate boundedness with* $|\theta| \le 0.01$ *and* $|\dot{\theta}| \le 0.01$.

$$\therefore u = -\frac{a_1 x_1 + \hat{h}(x)}{\hat{g}(x)} + v$$

$$\text{where} \begin{cases} a_1 = 1 \\ \hat{h}(x) = -\dfrac{\hat{k}}{\hat{m}} x_2 - \dfrac{g}{\hat{l}} \sin x_1 \\ \hat{g}(x) = \dfrac{1}{\hat{m}\hat{l}^2} \\ v = -\beta(x) sat\left(\dfrac{s}{\epsilon}\right) \quad with \quad \epsilon = 0.01 \end{cases}$$

14.8

$$\dot{y} = \frac{1}{A(y)}(u - c\sqrt{y}), \quad \dot{\sigma} = y - r$$

$$\rho_1 > 0, \rho_2 > 0, \rho_3 \geq 0, \text{ and } 0 \leq \rho_4 < 1$$

$$\rightarrow \rho_1 \leq A(y) \leq \rho_2, \quad |\hat{c} - c| \leq \rho_3, \quad \text{and} \quad \left| \frac{A(y) - \hat{A}(\mathrm{y})}{A(y)} \right| \leq \rho_4$$

Let $x = y - r$

$$\dot{\sigma} = x$$

then $\dot{x} = \dfrac{1}{A(x+r)}(u - c\sqrt{x+r})$

Sliding surface $\rightarrow x + \sigma$

then $\dot{s} = \dot{x} + \dot{\sigma} = \dfrac{1}{A(x+r)}(u - c\sqrt{x+r}) + x = \dfrac{1}{A(x+r)}u - \dfrac{c\sqrt{x+r}}{A(x+r)} + x$

Here, $\rho_1 \leq A(x+r) \leq \rho_2$ *means* $A(x+r) > 0$

thus, $V(\mathrm{s}) = \dfrac{1}{2}s^2$

$$\dot{V}(s) = s\dot{s} = s\frac{1}{A(x+r)}u + s\left\{-\frac{c\sqrt{x+r}}{A(x+r)} + x\right\}$$

$u = -\beta(\mathrm{x})\mathrm{sgn}(\mathrm{s})$ *where* $\beta(x)$ *is large and has same sign as* $A(x+r)$ $(\beta(x) > 0)$

then $\dot{V}(s) < 0$ *on* $\mathrm{s} = 0, \quad \dot{\sigma} = -\sigma \quad \Rightarrow \quad \sigma \rightarrow 0$ *as* $t \rightarrow \infty$

14. 14

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_2 - \frac{g}{l}\sin x_1 + \frac{h(t)}{l}\cos x_1 + \frac{1}{ml^2}u$$

nominal stabilizing feedback controller.

$$\Psi(x) = -\hat{m}\hat{l}^2\left(-\frac{\hat{k}}{\hat{m}}x_2 - \frac{g}{\hat{l}}\sin x_1\right) - \hat{m}\hat{l}^2(k_1x_1 + k_2x_2)$$

where k_1 *and* k_2 *are chosen such that*

$$A_c = A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \text{ is Hurwitz}$$

$$|\delta| \le \rho_1 \|\mathrm{x}\|_2 + k_0|v| + H\cdot\hat{m}\hat{l}^2 \quad \text{where} \quad \left|\frac{h(t)}{l}\right| \le H$$

we have $\rho(x) = \rho_1\|\mathrm{x}\|_2 + H\cdot\hat{m}\hat{l}^2$

$$\Rightarrow u = \Psi(x) - \eta(x)\,\mathrm{sgn}(w)$$

$$\text{where} \begin{cases} w = 2x^TPB, \quad A_c^TP + PA_c = -I, \quad B = \begin{bmatrix} 0 \\ \dfrac{1}{ml^2} \end{bmatrix} \\ \eta(x) \ge \dfrac{\rho_1\|x\|_2 + H\cdot\hat{m}\hat{l}^2}{1-k_0} \end{cases}$$

14. 15

Similarly, we obtain

$$\dot{\sigma} = x$$

$$\dot{x} = \frac{1}{A(x+r)}(u - c\sqrt{x+r})$$

then

$$\dot{x} = \frac{1}{A(x+r)}u - \frac{c\sqrt{x+r}}{A(x+r)}$$

$$= \frac{u}{\hat{A}(x+r)} + \underbrace{\left(\frac{1}{A(x+r)} - \frac{1}{\hat{A}(x+r)}\right)u - \frac{c\sqrt{x+r}}{A(x+r)}}_{\le \rho(\mathrm{x}) + \mathrm{k}_0 \|u\|, \quad 0 \le k_0 < 1}$$

$$\Rightarrow u = k_1\sigma + k_2x_2 - \eta(x)\,\mathrm{sgn}(w)$$

where

$$(i)\ A_c = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix} \text{ is Hurwitz}$$

$$(ii)\ w = 2x^T PB, \text{ where } x = [\sigma\ x],\ A_c^T P + PA_c = -I, \quad B = \begin{bmatrix} 0 \\ \dfrac{c\sqrt{x+r}}{A(x+r)} \end{bmatrix}$$

$$(iii)\ \eta(x) \ge \frac{\rho}{1-k_0}$$

14.29

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_2 - \frac{g}{l}\sin x_1 + \frac{h(t)}{l}\cos x_1 + \frac{1}{ml^2}u$$

Backstepping

1) Virtual control

$$x_2 = \phi(x_1) = -x_1$$

2) Let $z = x_2 + x_1$

$$\dot{z} = \dot{x}_1 + \dot{x}_2$$
$$= -x_1 + z - \frac{k}{m}x_2 - \frac{g}{l}\sin x_1 + \frac{h(t)}{l}\cos x_1 + \frac{1}{ml^2}u$$
$$= -x_1 + z - \frac{k}{m}z + \frac{k}{m}x_1 - \frac{g}{l}\sin x_1 + \frac{h(t)}{l}\cos x_1 + \frac{1}{ml^2}u$$
$$= -\left(1-\frac{k}{m}\right)x_1 - \frac{g}{l}\sin x_1 + \frac{h(t)}{l}\cos x_1 + \left(1-\frac{k}{m}\right)z + \frac{1}{ml^2}u$$

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}z^2$$
$$\dot{V} = x_1(-x_1+z) + z\left\{\underbrace{-\left(1-\frac{k}{m}\right)x_1 - \frac{g}{l}\sin x_1 + \frac{h(t)}{l}\cos x_1 + \left(1-\frac{k}{m}\right)z}_{\triangleq\, \alpha(x_1,z)} + \frac{1}{ml^2}u\right\}$$

$$= -x_1^2 + x_1 z + z\left\{\alpha(x_1,z) + \frac{1}{ml^2}u\right\}$$

$$\Rightarrow u = ml^2\{-\alpha(x_1,z) - x_1 - z\}$$