TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2015 - Solution to Assignment 6

1. (Khalil 13.1) The system is given by

$$
\begin{aligned}
M\ddot{\delta} &= P - D\dot{\delta} - \eta_1 E_q \sin(\delta) \\
\tau \dot{E}_q &= -\eta_2 E_q + \eta_3 \cos(\delta) + E_{FD}
\end{aligned}
$$

which is rewritten in the form $\dot{x} = f(x) + g(x)u$ using

$$
\begin{aligned}
x_1 &= \delta \\
x_2 &= \dot{\delta} \\
x_3 &= E_q \\
u &= E_{FD}
\end{aligned}
$$

This results in the system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \tfrac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \\
\dot{x}_3 &= \tfrac{1}{\tau}\left(-\eta_2 x_3 + \eta_3 \cos(x_1) + u\right)
\end{aligned}
\tag{1}
$$

where it can be seen that

$$
f(x) = \begin{bmatrix} x_2 \\ \tfrac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \\ \tfrac{1}{\tau}\left(-\eta_2 x_3 + \eta_3 \cos(x_1)\right) \end{bmatrix}
$$

$$
g(x) = \begin{bmatrix} 0 \\ 0 \\ \tfrac{1}{\tau} \end{bmatrix}
$$

(a) The output is given by $y = \delta = x_1 = h(x)$. The relative degree is found as

$$
\begin{aligned}
y &= x_1 \\
\dot{y} &= \dot{x}_1 \\
&= x_2 \\
\ddot{y} &= \dot{x}_2 \\
&= \frac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \\
\dddot{y} &= -\frac{D}{M}\dot{x}_2 - \frac{\eta_1}{M}\dot{x}_3 \sin(x_1) - \frac{\eta_1}{M}x_3 \frac{\partial \sin(x_1)}{\partial x_1}\dot{x}_1 \\
&= -\frac{D}{M}\frac{1}{M}\left(P - Dx_2 - \eta_1 x_3 \sin(x_1)\right) \\
&\quad -\frac{\eta_1}{\tau M}\sin(x_1)\left(-\eta_2 x_3 + \eta_3 \cos(x_1) + u\right) \\
&\quad -\frac{\eta_1}{M}x_3 \cos(x_1)x_2
\end{aligned}
$$

And the relative degree of the system is $\rho = 3$.

We have our system on the form

$$\dot{x} = f(x) + g(x)u \tag{2}$$

and would like to transform it to a system on the form

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A_c\xi + B_c\gamma(x)[u - \alpha(x)]$$
$$y = C_c\xi$$

where η is the internal dynamics and ζ the external dynamics. They are both given through the diffeomorphism

$$T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \triangleq \begin{bmatrix} \eta \\ -- \\ \xi \end{bmatrix} \tag{3}$$

where

$$\frac{\partial \phi_i}{\partial x}g(x) = 0, \text{ for } 1 \le i \le n - \rho, \forall \, x \in D_0$$

Since $\rho = n$, one only needs the external part of the system on normal form, ζ. External variables of the normal form is given by evaluating the Lie Derivative of h with respect to f

$$\begin{aligned}
\xi_1 &= h(x) \\
&= x_1 \\
\xi_2 &= L_f h(x) \\
&= x_2 \\
\xi_3 &= L_f^2 h(x) \\
&= \frac{1}{M}(P - Dx_2 - \eta_1 x_3 \sin(x_1))
\end{aligned}$$

The system (1) can therefore be written on normal form as

$$\dot{\xi} = A_c\xi + B_c\gamma(x)[u - \alpha(x)] \tag{4}$$
$$y = C_c\xi \tag{5}$$

where

$$\gamma(x) = L_g L_f^{\rho-1}h(x) \quad \text{and} \quad \alpha(x) = -\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1}h(x)} = -\frac{L_f^\rho h(x)}{\gamma(x)} \tag{6}$$

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This transformation is therefore only valid when $\gamma(x) \neq 0$, which means that

$$
\begin{aligned}
L_g L_f^{\rho-1} h(x) &= L_g L_f^2 h(x) \\
&= -\frac{\eta_1}{\tau M} \sin(x_1) \\
&\neq 0 \ \forall \ x \in D_0
\end{aligned}
$$

where $D_0 = \{x \in R^3 | \sin(x_1) \neq 0\}$. Since the relative degree equals the dimension of the system, we have no internal dynamics and the system is minimum phase.

(b) The output is given by $y = \delta + \zeta \dot{\delta} = x_1 + \zeta x_2 = h(x)$ where $\zeta \neq 0$. The relative degree is obtained from

$$
\begin{aligned}
y &= x_1 + \zeta x_2 \\
\dot{y} &= \dot{x}_1 + \zeta \dot{x}_2 \\
&= x_2 + \zeta \frac{1}{M}(P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\
&= \left(1 - \frac{\zeta D}{M}\right) x_2 - \frac{\zeta \eta_1}{M} x_3 \sin(x_1) + \zeta P \frac{1}{M} \\
\ddot{y} &= \frac{\partial \dot{y}}{\partial x} \dot{x} \\
&= \left[\ -\frac{\zeta \eta_1}{M} x_3 \cos(x_1) \quad \left(1 - \frac{\zeta D}{M}\right) \quad -\frac{\zeta \eta_1}{M} \sin(x_1) \ \right] \dot{x} \\
&= \frac{\zeta \eta_1}{M} x_3 \cos(x_1) x_2 \\
&\quad + \left(1 - \frac{\zeta D}{M}\right) \frac{1}{M}(P - Dx_2 - \eta_1 x_3 \sin(x_1)) \\
&\quad - \frac{\zeta \eta_1}{\tau M} \sin(x_1)(-\eta_2 x_3 + \eta_3 \cos(x_1) + u)
\end{aligned}
$$

And the system thus has relative degree $\rho = 2$.

The region D_0 where the transformation is valid is where $L_g L_f^{\rho-1} h(x) \neq 0$

$$
\begin{aligned}
L_g L_f^{\rho-1} h(x) &= L_g L_f^1 h(x) \\
&= -\frac{\gamma \eta_1}{\tau M} \sin(x_1) \\
&\neq 0 \ \forall \ x \in D_0
\end{aligned}
$$

where $D_0 = \{x \in R^3 | \sin(x_1) \neq 0\}$.

Since $\rho < n$, both internal and external dynamics are needed. The external variables of the normal form is found by evaluating the Lie Derivative of h with

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respect to f

$$\xi_1 = h(x) = x_1 + \zeta x_2$$

$$\xi_2 = L_f h(x) = \frac{\partial h(x)}{\partial x} f(x) = \begin{bmatrix} 1 & \zeta & 0 \end{bmatrix} f(x)$$

$$= x_2 + \frac{\zeta}{M}(P - Dx_2 - \eta_1 x_3 \sin(x_1))$$

The internal dynamics $\eta = \phi(x)$ is chosen to satisfy $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ and the existence of $T^{-1}(x)$ in D_0. It can be verified that $\phi(x) = x_1$ meets these conditions. With $\phi(x) = x_1$ we have that

$$\dot{\eta} = \dot{\phi}(x) = \dot{x}_1 = x_2 = \frac{1}{\zeta}(\xi_1 - \eta) = f_0(\eta, \xi)$$

The system on normal form is thus

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = A_c \xi + B_c \gamma(x)[u - \alpha(x)]$$
$$y = C_c \xi$$

The system is said to be minimum phase if the zero dynamics, $\dot{\eta} = f_0(\eta, 0)$, has an asymptotically stable equilibrium point in the domain of interest. From $\dot{\eta} = f_0(\eta, 0) = -\frac{1}{\gamma}\eta$ it can be recognized that the origin of η is asymptotically stable.

2. (Khalil 13.2) The system is given by

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 - x_3 \\
\dot{x}_2 &= -x_1 x_3 - x_2 + u \\
\dot{x}_3 &= -x_1 + u \\
y &= x_3
\end{aligned}
$$

Rewriting this model on the form $\dot{x} = f(x) + g(x)u$ results in

$$
f(x) = \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 x_3 - x_2 \\ -x_1 \end{bmatrix}
$$

$$
g(x) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
$$

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(a) The relative degree is obtained from

$$
\begin{aligned}
y &= x_3 \\
\dot{y} &= \dot{x}_3 \\
&= -x_1 + u
\end{aligned}
$$

which shows that the system has relative degree 1 in R^3. Hence, the system is input-output linearizable.

(b) The external part of the normal form is given by

$$
\xi_1 = h(x) = x_3
$$

To find the internal dynamics we start by setting up the requirements on $\frac{\partial \phi_i}{\partial x}$

$$
\begin{aligned}
\frac{\partial \phi_1}{\partial x} g(x) &= \left[\begin{array}{ccc} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_1}{\partial x_3} \end{array} \right] g(x) \\
&= \frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_1}{\partial x_3} \\
&= 0 \\
\frac{\partial \phi_2}{\partial x} g(x) &= \left[\begin{array}{ccc} \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_2}{\partial x_3} \end{array} \right] g(x) \\
&= \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_2}{\partial x_3} \\
&= 0
\end{aligned}
$$

By choosing

$$
\begin{aligned}
\phi_1(x) &= x_1 \\
\phi_2(x) &= x_2 - x_3
\end{aligned}
$$

we obtain a global diffeomorphism

$$
\begin{aligned}
T(x) &= \left[\begin{array}{c} x_1 \\ x_2 - x_3 \\ x_3 \end{array} \right] \\
&= \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array} \right] x
\end{aligned}
$$

which is invertable. The system on normal form is

$$
\begin{aligned}
\dot{\eta}_1 &= \dot{x}_1 \\
&= -\eta_1 + \eta_2 \\
\dot{\eta}_2 &= \dot{x}_2 - \dot{x}_3 \\
&= -x_1 x_3 - x_2 + u + x_1 - u \\
&= -\eta_1 \xi_1 - (\eta_2 + x_3) + \eta_1 \\
&= \eta_1 - \eta_2 - \xi_1 - \eta_1 \xi_1 \\
\dot{\xi}_1 &= -\eta_1 + u
\end{aligned}
$$

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Since $L_g L_f^0 h(x) = L_g h(x) = 1$, the transformation is valid in $\mathcal{R}^3$.

(c) To investigate if the system is minimum phase, we analyze the zero dynamics

$$
\dot{\eta} = f_0(\eta, \xi)|_{\xi=0}
$$
$$
= \left[\begin{array}{c} -\eta_1 + \eta_2 \\ \eta_1 - \eta_2 - \xi_1 - \eta_1 \xi_1 \end{array} \right]\Bigg|_{\xi=0}
$$
$$
= \left[\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right] \eta = A\eta
$$

where it can be seen that $\text{eig}(A) = \begin{bmatrix} -2 & 0 \end{bmatrix}^T$. Hence, the origin is not asymptotically stable, and the system is therefore not minimum phase.

3. The system is rewritten as

$$
\dot{x} = f(x) + g(x) u
$$
$$
y = h(x)
$$

where

$$
f(x) = \left[\begin{array}{c} -x_1 \\ x_1 x_2 \\ x_2 \end{array} \right]
$$
$$
g(x) = \left[\begin{array}{c} e^{x_2} \\ 1 \\ 0 \end{array} \right]
$$
$$
h(x) = x_3
$$

(a) The relative degree is found by derivative y with respect to time

$$
y = x_3
$$
$$
\dot{y} = \dot{x}_3 = x_2
$$
$$
\ddot{y} = \dot{x}_2 = x_1 x_2 + u
$$

where it can be seen that the system has a relative degree $\rho = 2$ in $x \in R^2$. The relative degree holds as long as $L_g L_f^{\rho-1} h(x) \neq 0$.

$$
L_g L_f^{\rho-1} h(x) = L_g L_f h(x) = 1 \neq 0 \ \forall \ x \in \mathcal{R}^2
$$

The relative degree thus holds over the entire $\mathcal{R}^3$ space.

(b) The system has a well defined relative degree ρ in the entire $\mathcal{R}^3$, and is therefore input-output linearizable in $\mathcal{R}^3$.

(c) The variables for the external dynamics are found according to

$$\xi_1 = h(x) = x_3$$

$$\xi_2 = L_f h(x) = \frac{\partial h(x)}{\partial x} f = x_2$$

The coordinates for the internal dynamics is chosen such that $T(x)$ is diffeomorphism on $\mathcal{R}^3$ and $\frac{\partial \phi(x)}{\partial x} g(x) = 0$ on $\mathcal{R}^3$, where $\left[\eta, \xi^T\right]^T = [\phi(x), \psi(x)] = T(x)$. In addition to this we require $\phi(0) = 0$ in order to have the origin as equilibrium. We start by calculating

$$\frac{\partial \phi(x)}{\partial x} g(x) = \begin{bmatrix} \frac{\partial \phi(x)}{\partial x_1} & \frac{\partial \phi(x)}{\partial x_2} & \frac{\partial \phi(x)}{\partial x_3} \end{bmatrix} \begin{bmatrix} e^{x_2} \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{\partial \phi(x)}{\partial x_1} e^{x_2} + \frac{\partial \phi(x)}{\partial x_2}$$

$$= 0$$

and based on these calculations we try

$$\frac{\partial \phi(x)}{\partial x_1} = 1$$

$$\frac{\partial \phi(x)}{\partial x_2} = -e^{x_2}$$

which implies that

$$\phi(x) = x_1 - e^{x_2} + c$$

where c is some constant. This constant is chosen to satisfy our requirement $\phi(0) = 0$

$$\phi(0) = -e^0 + c$$

$$= -1 + c$$

$$\Rightarrow c = 1$$

Our resulting coordinate transformation is now given by

$$\begin{bmatrix} \eta \\ \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} x_1 - e^{x_2} + 1 \\ x_3 \\ x_2 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \eta + e^{\xi_2} - 1 \\ \xi_2 \\ \xi_1 \end{bmatrix}$$

Consequently the inverse transformation exists. It follows that $T(x)$ and $T^{-1}(x)$ are continuously differentiable. Hence, $T(x)$ is a diffeomorphism on $\mathcal{R}^3$ and $T(0) = T^{-1}(0) = 0$.

(d) The system may be rewritten as

$$
\begin{aligned}
\dot{\eta} &= \dot{x}_1 - \frac{\partial e^{x_2}}{\partial x_2}\dot{x}_2 \\
&= -x_1 + e^{x_2}u - e^{x_2}\left(x_1 x_2 + u\right) \\
&= -x_1 - e^{x_2}x_1 x_2 \\
&= -\left(\eta + e^{x_2} - 1\right) - e^{x_2}\left(\eta + e^{x_2} - 1\right)x_2 \\
&= \left(1 - \eta - e^{\xi_2}\right) + \left(1 - \eta - e^{\xi_2}\right)e^{\xi_2}\xi_2 \\
&= \left(1 - \eta - e^{\xi_2}\right)\left(1 + e^{\xi_2}\xi_2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\xi} &= A_c\xi + B_c\gamma\left(x\right)\left(u - \alpha\left(x\right)\right) \\
y &= C_c\xi
\end{aligned}
$$

where

$$
\begin{aligned}
A_c &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
B_c &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
C_c &= \begin{bmatrix} 1 & 0 \end{bmatrix} \\
\gamma\left(x\right) &= L_g L_f h\left(x\right) \\
&= 1 \\
\alpha\left(x\right) &= -\frac{L_f^2 h\left(x\right)}{L_g L_f h\left(x\right)} \\
&= -\frac{x_1 x_2}{1} \\
&= -x_1 x_2
\end{aligned}
$$

(e) The zero dynamics is given by

$$
\begin{aligned}
\dot{\eta} &= f_0\left(\eta, \xi\right)|_{\xi=0} \\
&= \left(1 - \eta - e^{\xi_2}\right)\left(1 + e^{\xi_2}\xi_2\right)\big|_{\xi=0} \\
&= \left(1 - \eta - 1\right)\left(1 + 0\right) \\
&= -\eta
\end{aligned}
$$

which has a globally asymptotically stable equilibrium at the origin.

(f) The external dynamics are given by

$$
\dot{\xi} = A_c\xi + B_c\gamma\left(x\right)\left(u - \alpha\left(x\right)\right)
$$

By choosing
$$u = \gamma^{-1}(x) v + \alpha(x)$$
the external dynamics are given by
$$\dot{\xi} = A_c \xi + B_c v$$
Since the system is controllable, $rank\left([B,\ AB]\right) = 2$, it can be stabilized (asymptotically stable) by a control input $v = -K\xi$ where K is chosen such that $(A_c - B_c K)$ is Hurwitz. u is now given by
$$u = -\gamma^{-1}(x) K\xi + \alpha(x)$$
Since $\dot{\eta} = f_0\left(\eta, \xi\right)|_{\xi=0}$ is asymptotically stable, the origin of the entire system is asymptotically stable.

(g) Let
$$\mathcal{R} = \begin{bmatrix} r \\ \dot{r} \end{bmatrix}$$
$$e = \xi - \mathcal{R}$$

Then we can calculate
$$
\begin{aligned}
\dot{e} &= \dot{\xi} - \dot{\mathcal{R}} \\
&= (A_c \xi + B_c v) - (A_c \mathcal{R} + B_c \ddot{r}) \\
&= A_c(\xi - \mathcal{R}) + B_c(v - \ddot{r}) \\
&= A_c e + B_c(v - \ddot{r}) \\
&= A_c e + B_c\left(\gamma(x)\left[u - \alpha(x)\right] - \ddot{r}\right)
\end{aligned}
$$

where the simplified structure $\dot{\mathcal{R}} = A_c \mathcal{R} + B_c \ddot{r}$ is found using the known values of A_c and B_c (see part d).
We can choose the state feedback control
$$u = \gamma^{-1}(x)(v + \ddot{r}) + \alpha(x)$$

The resulting system is
$$
\begin{aligned}
\dot{\eta} &= f_0\left(\eta, e + \mathcal{R}\right) \\
\dot{e} &= A_c e + B_c v
\end{aligned}
$$

and since (A_c, B_c) is controllable, the loop is closed with $v = -Ke$ where K is chosen such that $(A_c - B_c K)$ is Hurwitz. This makes the external dynamics for e exponentially stable.
Since $\dot{\eta} = f_0\left(\eta, \xi\right)|_{\xi=0}$ is asymptotically stable, the origin of the overall closed-loop system is such that for sufficiently small initial conditions $e(0), \eta(0)$ and for $\mathcal{R}(t)$ with sufficiently small $\sup_{t\geq 0}\|\mathcal{R}(t)\|$, all solutions $(\eta(t), e(t))$ of the closed-loop system are bounded and $e(t) \to 0$ as $t \to \infty$.

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4. (Khalil 13.25) The system is given by

$$\dot{x}_1 = x_2 + 2x_1^2$$
$$\dot{x}_2 = x_3 + u$$
$$\dot{x}_3 = x_1 - x_3$$
$$y = x_1$$

Since

$$y = x_1 \Rightarrow \dot{y} = x_2 + 2x_1^2 \Rightarrow \ddot{y} = x_3 + u + 2x_1(x_2 + x_1^2)$$

Therefore, the system has a relative degree 2 in R^3. Let us check the minimum-phase property.

$$y(t) \equiv 0 \Rightarrow \dot{x}_3 = -x_3.$$

Hence, the system is minimum.-phase. Let $e = y - r$.

$$\ddot{e} = \ddot{y} - \ddot{r} = x_3 + u + 2x_1(x_2 + x_1^2) - \ddot{r}$$

Choosing

$$u = -x_3 - 2x_1(x_2 + x_1^2) + \ddot{r} - k_1 e - k_2 \dot{e}$$
$$= -x_3 - 2x_1(x_2 + x_1^2) + \ddot{r} - k_1(y - r) - k_2(x_2 + 2x_1^2 - \dot{r}),$$

where k_1 and k_2 are positive constants. the tracking error e satisfies the equation $\ddot{e} + k_2 \dot{e} + k_1 e = 0$, which shows that $e(t) \to 0$ as $t \to \infty$.

5. (Khalil 14.31) The system is given by

$$\dot{x}_1 = x_2 + a + \left(x_1 - a^{1/3}\right)^3$$
$$\dot{x}_2 = x_1 + u$$

where the first system equation has the virtual input x_2. Choose

$$x_2 = -a - x_1 - \left(x_1 - a^{1/3}\right)^3$$

such that $\dot{x}_1 = -x_1$. Then the Lyapunov function candidate $V_1 = \frac{1}{2}x_1^2$ (which is positive definite and radially unbounded) will have $\dot{V}_1 = x_1\dot{x}_1 = -x_1^2$ which is negative definite.
Augment the virtual input with z, such that

$$x_2 = -a - x_1 - \left(x_1 - a^{1/3}\right)^3 + z$$

then

$$z = x_1 + x_2 + a + \left(x_1 - a^{1/3}\right)^3$$

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and

$$\dot{z} = \dot{x}_1 + \dot{x}_2 + 3\left(x_1 - a^{1/3}\right)^2 \dot{x}_1$$
$$= \left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) + (x_1 + u) + 3\left(x_1 - a^{1/3}\right)^2\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)$$
$$= x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)$$

Calculate

$$\dot{V}_1 = x_1\dot{x}_1 = x_1\left(-x_1 + z\right) = -x_1^2 + x_1 z$$

We may choose a Lyapunov function candidate for the overall system as $V_c = V_1 + \frac{1}{2}z^2$, then

$$\dot{V}_c = \dot{V}_1 + z\dot{z}$$
$$= -x_1^2 + x_1 z + z\left\{x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)\right\}$$
$$= -x_1^2 + z\underbrace{\left\{2x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)\right\}}_{\text{choose}\quad -z}$$
$$= -x_1^2 - z^2$$

where we have chosen

$$-z = 2x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)$$

i.e.

$$u = -z - 2x_1 - \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)$$
$$= -\overbrace{\left(x_1 + x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)}^{z} - 2x_1 - \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)$$
$$= -3x_1 - \left(2 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)$$

such that $\dot{V}_c$ is negative definite. We already know that V_c is positive definite and radially unbounded. Hence, the overall system is globally asymptotically stable (GAS).

Alternative solution:
The system is in the form of (14.53)-(14.54) in Khalil with

$$\begin{aligned} f &= a + \left(x_1 - a^{1/3}\right)^3 \\ g &= 1 \\ f_a &= x_1 \\ g_a &= 1 \end{aligned}$$

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Take

$$\phi\left(x_1\right) = -a - \left(x_1 - a^{1/3}\right)^3 - x_1$$
$$V = \frac{1}{2}x_1^2$$

and use (14.56) in Khalil.

6. Consider $\dot{x}_1 = x_1 x_2 + x_1^2$ with x_2 as a virtual input. Choose a Lyapunov function candidate $V_1(x) = \frac{1}{2}x_1^2$ and calculate

$$\dot{V}_1 = x_1\dot{x}_1 = x_1\left(x_1 x_2 + x_1^2\right)$$

We can enforce $\dot{V}_1 = -x_1^4$ which is negative definite, by choosing the input $x_2 = -x_1 - x_1^2$ (actually, any choice $x_2 = -x_1^{2k} - x_1^2$, $k = 1, 2, 3, \ldots$ will be possible, to get a negative definite $\dot{V}_1$, but for simplification we choose k=1).
Augment the input with z, such that we have $x_2 = -x_1 - x_1^2 + z$ (i.e. $z = x_2 + x_1 + x_1^2$), then

$$\dot{V}_1 = x_1\left(x_1 x_2 + x_1^2\right) = x_1\left(x_1\left(-x_1 - x_1^2 + z\right) + x_1^2\right) = -x_1^4 + x_1^2 z$$

Now choose a Lyapunov function candidate for the complete system $V_c = V_1 + \frac{1}{2}z^2$, which is positive definite and radially unbounded. Then

$$\dot{V}_c = \dot{V}_1 + z\dot{z}$$
$$= -x_1^4 + x_1 z + z\left(u + \left(2x_1 + 1\right)\left(x_1 x_2 + x_1^2\right)\right)$$
$$= -x_1^4 + z\underbrace{\left(x_1 + u + \left(2x_1 + 1\right)\left(x_1 x_2 + x_1^2\right)\right)}_{\text{choose} \quad -z}$$

We can enforce $\dot{V}_c = -x_1^2 - z^2$ (then $\dot{V}_c$ is negative definite), by choosing

$$-z = x_1 + u + \left(2x_1 + 1\right)\left(x_1 x_2 + x_1^2\right)$$
$$u = -x_1 - \left(2x_1 + 1\right)\left(x_1 x_2 + x_1^2\right) - z$$

By inserting z, we get the expression for the stabilizing input

$$u = -x_1 - \left(2x_1 + 1\right)\left(x_1 x_2 + x_1^2\right) - x_2 + x_1 + x_1^2$$
$$= -\left(2x_1 + 1\right)\left(x_1 x_2 + x_1^2\right) - x_2 + x_1^2$$

Since $V_c(x_1, z)$ is continuously differentiable and positive definite, and $\dot{V}_c(x_1, z)$ is negative definite, u asymptotically stabilizes x_1 and z at the origin. Since $z = 0 \rightarrow x_2 = -x_1 - x_1^2$ and $x_1 = 0 \rightarrow -x_1 - x_1^2 = 0$, this means that also x_2 is asymptotically stabilized at the origin. In addition, since $V_c(x_1, z)$ is radially unbounded and there are no singularities in u, the equilibrium point $x = (0, 0)$ is globally asymptotically stable.