TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2017 - Assignment 6

Due date: Monday 27 November at 11.00.

1. **Design a Model Reference Adaptive Controller (MRAC) for SISO Linear Systems**

   Consider the roll dynamics of a conventional aircraft using differential motion of ailerons and spoilers, approximated by a scalar first-order ordinary differential equation in the form:

$$\dot{x} = a_p x + b_p u \tag{1}$$

   with the following notations:

x	Aircraft roll rate in stability axes	[radians/s]
u	Total differential aileron-spoiler deflection	[radians]
a_p	Roll damping derivative	
b_p	Dimensional rolling moment derivative	, $b_p > 0$

   (a) Given the roll damping $a_p = -0.8\,(s^{-1})$ and the aileron effectiveness $b_p = 1.6\,(s^{-1})$, develop a fixed-gain model reference controller in the form of

$$u = a_x x + a_r r \tag{2}$$

   to recover the reference model dynamics

$$\dot{x}_m = a_m x_m + b_m r, \tag{3}$$

   with $a_m = -2$, $b_m = 2$.

   (b) For each of the bounded roll rate commands

   1. r = 4
   2. r = sin(t)

   simulate the closed-loop system response. Add plots of the tracking performances and comment on the results.

   (c) Assume that the constant roll dynamics parameters (a,b) are unknown and that only the sign of b is known to be positive. Using the same reference model parameters and an adaptive controller on the form of $u = \hat{a}_x x + \hat{a}_r r$, derive the error dynamics in the form:

$$\dot{e} = a_m e + b_p\left(\tilde{a}_x x + \tilde{a}_r r\right) \tag{4}$$

   where the parameter errors:

$$\begin{aligned}\tilde{a}_x &= \hat{a}_x - a_x^* \\ \tilde{a}_r &= \hat{a}_r - a_r^*\end{aligned}$$

are defined by the difference between the estimated gain and the ideal parameters.

(d) Using the Lyapunov function candidate

$$V\left(e, \tilde{a}_x, \tilde{a}_r\right) = \frac{e^2}{2} + \frac{|b_p|}{2\gamma_x}\tilde{a}_x^2 + \frac{|b_p|}{2\gamma_r}\tilde{a}_r^2$$

and Barbalat's lemma to show that the tracking error e tends to zero globally and asymptotically. Select the following adaptation laws:

$$\begin{aligned} \dot{\hat{a}}_x &= -\gamma_x x e \ \mathrm{sgn}\left(b_p\right) \\ \dot{\hat{a}}_r &= -\gamma_r r e \ \mathrm{sgn}\left(b_p\right) \end{aligned}$$

(e) For each of the bounded roll rate commands

1. r = 4
2. r = sin(t)

simulate the closed-loop system response with the MRAC. Experiment and find rates of adaptation that gives the wanted performance. Compare the fixed-gain versus MRAC controller tracking performances and comment on your results.

2. **Design an MRAC for SISO Nonlinear Systems**
Now, let us include nonlinear damping in the roll dynamics:

$$\dot{x} = a_p x + c_p x^3 + b_p u \tag{5}$$

where:

x	Aircraft roll rate in stability axes	[radians/s]
u	Total differential aileron-spoiler deflection	[radians]
a_p	Roll damping derivative	
b_p	Dimensional rolling moment derivative	$b_p > 0$
c_p	Damping constant	

(a) Assume that the constant roll dynamics data (a_p,b_p,c_p) are **known** and that only the sign of b_p is known to be positive, as in Exercise 1(d). Specify a desired closed-loop behaviour by a linear reference model and derive the control law that leads to perfect tracking of the reference model and find the expressions for a_x, a_r and a_f.

(b) Assume that the parameter gains are **unknown**. Using the control law derived in the previous exercise (2(a)), replace the parameters with their estimates, and derive the tracking error dynamics:

$$\dot{e} = a_m e + b_p\left(\tilde{a}_x x + \tilde{a}_f x^3 + \tilde{a}_r r\right)$$

and write down the expressions for $\dot{\tilde{a}}_f$, $\dot{\tilde{a}}_r$ and $\dot{\tilde{a}}_x$.

(c) Using the Lyapunov function candidate

$$V\left(e,\tilde{a}_x,\tilde{a}_r,\tilde{a}_f\right)=\frac{e^2}{2}+\frac{|b_p|}{2\gamma_x}\tilde{a}_x^2+\frac{|b_p|}{2\gamma_r}\tilde{a}_r^2+\frac{|b_p|}{2\gamma_f}\tilde{a}_f^2$$

and Barbalat's lemma to show that the tracking error e tends to zero asymptotically and globally. Select the following adaptive laws:

$$\begin{aligned}\dot{\hat{a}}_x &= -\gamma_x x e \text{ sgn}\left(b_p\right)\\ \dot{\hat{a}}_r &= -\gamma_r r e \text{ sgn}\left(b_p\right)\\ \dot{\hat{a}}_f &= -\gamma_f f\left(x\right) e \text{ sgn}\left(b_p\right),\end{aligned}$$

(d) For each of the bounded roll rate commands

1. r = 4
2. r = sin(t)

simulate the closed-loop system response and comment on the results. As in Exercise 1, $a_p=-0.8\,(s^{-1})$,$b_p=1.6\,(s^{-1})$ and $a_m=-2\,(s^{-1})$,$b_m=2\,(s^{-1})$. Let $c_p=-1.2\,(s^{-1})$ and $c_m=-2\,(s^{-1})$. Add plots of the tracking performances.

3. **Design of Adaptive Tracking Controller for a Class of MIMO Nonlinear Systems**

Consider the two-link manipulator illustrated in Figure 1, whose dynamics can be written explicitly as

$$\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}+\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}=\begin{bmatrix}u_1\\ u_2\end{bmatrix} \tag{6}$$

where

$$\begin{aligned}H &= \text{Inertia matrix, uniformly positive definite}\\ \theta &= \text{Vector of joint angles}\\ u &= \text{Vector of torques applied at the manipulator joints}\\ C &= \text{Torques}\\ C_{11} &= -h\dot{\theta}_2\\ C_{12} &= -h\left(\dot{\theta}_1+\dot{\theta}_2\right)\\ C_{21} &= h\dot{\theta}_1\\ C_{22} &= 0\\ H_{11} &= a_1+2a_3\cos\theta_2+2a_4\sin\theta_2\\ H_{12} &= H_{21}=a_2+a_3\cos\theta_2+a_4\sin\theta_2\\ H_{22} &= a_2\\ h &= a_3\sin\theta_2-a_4\cos\theta_2\end{aligned}$$

where a_i are functions of length, mass and inertia of the two links.

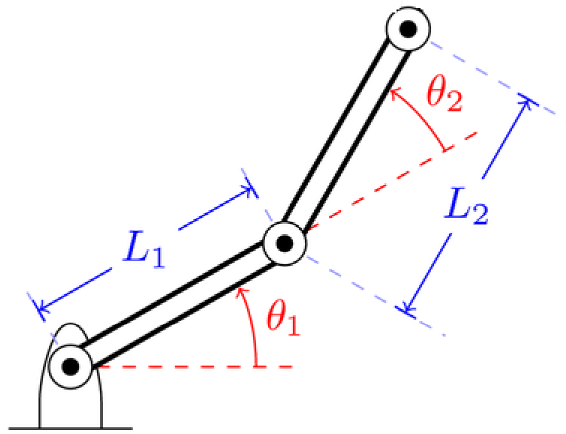

Figure 1: Two-link robotic manipulator.

(a) Find a regression matrix for the model with an unknown parameter vector such that the dynamics can be expressed as

$$Ya = u$$

(b) Given a bounded trajectory $\theta_d(t) = \begin{bmatrix} \theta_{1d} \\ \theta_{2d} \end{bmatrix}$ of desired joint angles where $\theta_d\,(t)$, $\dot{\theta}_d\,(t)$ and $\ddot{\theta}_d\,(t)$ are bounded. Define $e = \theta - \theta_d$ and $\dot{e} = \dot{\theta} - \dot{\theta}_d$. Show that the tracking control law

$$u \;=\; H\ddot{\theta}_r + C\dot{\theta}_r - K_p\,(\theta - \theta_d) - K_d\left(\dot{\theta} - \dot{\theta}_d\right)$$

where

$$\dot{\theta}_r \;=\; \dot{\theta}_d - \Lambda\,(\theta - \theta_d)\,,$$

where $\Lambda = K_d^{-1}K_p$, K_p and K_d are symmetric positive definite matrix, makes $(e, \dot{e}) = (0, 0)$ globally uniformly asymptotically stable if all the plant parameters were known, using the Lyapunov function candidate

$$\begin{aligned} V\,(s) &= \frac{1}{2}\left[s^T H s\right] \\ s &= \dot{\theta} - \dot{\theta}_r \end{aligned}$$

A useful property of the system is that $z^T \left(\dot{H} - 2C\right) z = 0, \quad \forall z \in R^2$.

(c) With initial values $x_1(0) = 0$ and $x_2(0) = 0$, and the parameter values $a = \begin{bmatrix} 2.5 & 1 & 1 & 0.5 \end{bmatrix}^T$, simulate the position errors and control torques under PD control with $x_{d1} = \frac{\pi}{3}$ and $x_{d2} = \frac{\pi}{2}$. Include the position error plots and comment on the stability.

(d) The plant parameters a_i found in 3.(a) are all uncertain, and can be assumed constant. Make the control law from the previous subtask adaptive by replacing the parameter vector a with an estimate $\hat{a}$. Find an adaptation law and prove that asymptotic trajectory tracking is achieved globally for the resulting closed-loop system, and also that the estimation error is bounded, using the Lyapunov function candidate

$$
\begin{aligned}
V(s, \tilde{a}) &= \frac{1}{2}\left[s^T H s + \tilde{a}^T \Gamma^{-1} \tilde{a}\right], \\
s &= \dot{\theta} - \dot{\theta}_r, \\
\tilde{a} &= \hat{a} - a
\end{aligned}
$$

where Γ is a symmetric positive definite matrix, and Barbalat's lemma.

(e) With the desired trajectory

$$
\begin{aligned}
x_{1d} &= \frac{\pi}{6}\left(1 - \cos(2\pi t)\right) \\
x_{2d} &= \frac{\pi}{4}\left(1 - \cos(2\pi t)\right)
\end{aligned}
$$

simulate the tracking errors and control torques.

4. Exercise 14.31 in Khalil.

5. Use the backstepping method to stabilize

$$
\begin{aligned}
\dot{x}_1 &= x_1 x_2 + x_1^2 \\
\dot{x}_2 &= u
\end{aligned}
$$