TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2017 - Solution to Assignment 6

1. **Design a Model Reference Adaptive Controller (MRAC) for SISO Linear Systems**
The roll dynamics of an aircraft, represented by the scalar first-order ODE:

$$\dot{x} = a_p x + b_p u \tag{1}$$

with the following notations:

x	Aircraft roll rate in stability axes	[radians/s]
u	Total differential aileron-spoiler deflection	[radians]
a_p	Roll damping derivative of the plant	
b_p	Dimensional rolling moment derivative of the plant	, $b_p > 0$

(a) We want to find u on the form:

$$u = a_x x + a_r r$$

such that the aircraft is forced to roll like the reference model, given by the dynamics:

$$\dot{x}_m = a_m x_m + b_m r, \tag{2}$$

when $a_p = -0.8$, $b_p = 1.6$, $a_m = -2$ and $b_m = 2$.
To achieve this, we want x to track $x_m(t)$. We define the tracking error $e = x - x_m(t)$. The tracking error dynamics are then:

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{x}_m \\ &= a_p x + b_p u - (a_m x_m + b_m r) \end{aligned}$$

In order to achieve the tracking error dynamics

$$\dot{e} = a_m e$$

we choose

$$u = \frac{a_m - a_p}{b_p} x + \frac{b_m}{b_p} r$$

which is the answer we were looking for, with the ideal gains $a_x^* = \frac{a_m - a_p}{b_p}$ and $a_r^* = \frac{b_m}{b_p}$. Filling in the numerical values, this gives:

$$\begin{aligned} u &= \frac{-2 - (-0.8)}{1.6} x + \frac{2}{1.6} r \\ &= -\frac{3}{4} x + \frac{5}{4} r \end{aligned}$$

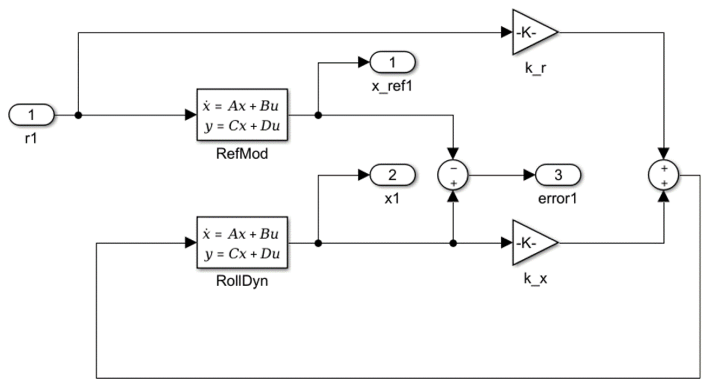


Figure 1: Block diagram of the closed-loop roll dynamics with fixed-gain model reference controller.

(b) Initial values are chosen to be $x(0) = 0$ and $x_m(0) = 0$ for both the reference system and for the plant. The system is modelled as shown in Figure 1.

With $r = 4$, the tracking performance is shown in Figure 2a. With $r = \sin t$, the tracking performance is shown in Figure 2b. The reference model tracking is perfect with both of the reference signals after a stable transient. Had the initial value been different in the reference model from the plant, then the plant roll angle would have needed some time to converge to the value of the reference roll angle. Note that the model reference does not track the commanded roll rate perfectly, as we expected since the reference model is a first order dynamic system with corresponding amplitude and phase shift, as we know from linear frequency response analysis.

(c) Now, the only knowledge about a_p and b_p is that $b_p > 0$. Using an adaptive controller in the form:

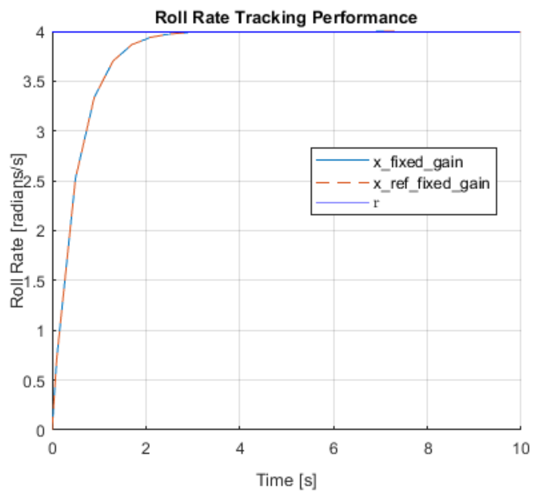
$$u = \hat{a}_x x + \hat{a}_r r, \tag{3}$$

we can insert this controller into (1) to achieve the closed-loop system:

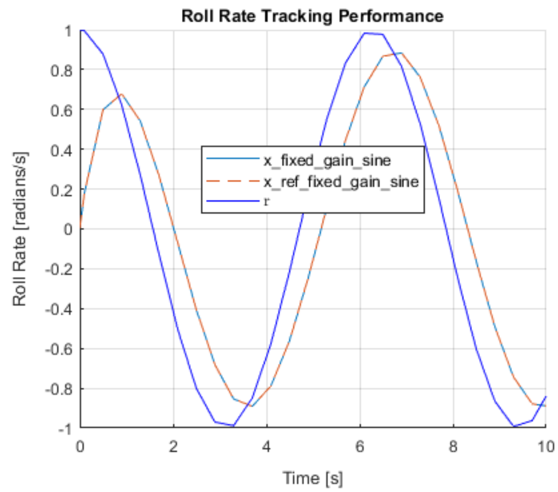
$$\dot{x} = (a_p + \hat{a}_x b_p)\, x + \hat{a}_r b_p r \tag{4}$$

The tracking error is defined as $e = x - x_m$. Our error variables are :

$$\begin{aligned} e &= x - x_m \\ \tilde{a}_x &= \hat{a}_x - \frac{a_m - a_p}{b_p} \\ \tilde{a}_r &= \hat{a}_r - \frac{b_m}{b_p} \end{aligned}$$



(a) Tracking performance using fixed-gain model reference controller.



(b) Tracking performance using fixed-gain model reference controller with error feedback.

Figure 2: Simulations showing tracking error using fixed-gain model reference controller with and without error feedback.

The tracking error dynamics are then found by differentiation:

$$\begin{aligned}
\dot{e} &= \dot{x} - \dot{x}_m \\
&= \left(a_p + \hat{a}_x b_p\right) x + \hat{a}_r b_p r - \left(a_m x_m + b_m r\right) \\
&= a_m x - a_m x + a_m\left(-x_m\right) + \left(a_p + \hat{a}_x b_p\right) x + \left(b_p \hat{a}_r - b_m\right) r \\
&= a_m\left(x - x_m\right) + \left(a_p - a_m + \hat{a}_x b_p\right) x + \left(b_p \hat{a}_r - b_m\right) r \\
&= a_m e + \left(-b_p \frac{a_m - a_p}{b_p} + \hat{a}_x b_p\right) x + \left(b_p \hat{a}_r - b_p \frac{b_m}{b_p}\right) r \\
&= a_m e + b_p\left(\tilde{a}_r r + \tilde{a}_x x\right),
\end{aligned}$$

and knowing that

$$a_x^* = \frac{a_m - a_p}{b_p} \tag{5}$$

$$a_r^* = \frac{b_m}{b_p}, \tag{6}$$

are constant, the parameter error dynamics are

$$\begin{aligned}
\dot{\tilde{a}}_x &= \dot{\hat{a}}_x \\
\dot{\tilde{a}}_r &= \dot{\hat{a}}_r
\end{aligned}$$

and will be determined by the choice of adaptation law in the next step.

(d) We want to show that the tracking error e and the parameter errors $\tilde{a}_x$ and $\tilde{a}_r$ tend to zero. We are provided with a Lyapunov function candidate,

$$V\left(e, \tilde{a}_r, \tilde{a}_x\right) = \frac{e^2}{2} + \frac{|b_p|}{2\gamma_x}\tilde{a}_x^2 + \frac{|b_p|}{2\gamma_r}\tilde{a}_r^2$$

with constant and positive scalar weights γ_x and γ_r. Since this is function is C^1 and positive definite, we need to find a negative definite (or negitive semidefinite) $\dot{V}$. The time derivative of V is:

$$\dot{V}\left(e, \tilde{a}_r, \tilde{a}_x\right) = e\dot{e} + \frac{|b_p|}{\gamma_x}\tilde{a}_x\dot{\tilde{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r\dot{\tilde{a}}_r$$

Substituting the expressions for the error dynamics $\dot{e}$, $\dot{\tilde{a}}_x$ and $\dot{\tilde{a}}_r$ into the previous equation, we get:

$$\begin{aligned}
\dot{V}\left(e, \tilde{a}_r, \tilde{a}_x\right) &= e\left(a_m e + b_p\left(\tilde{a}_r r + \tilde{a}_x x\right)\right) + \frac{|b_p|}{\gamma_x}\tilde{a}_x\dot{\tilde{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r\dot{\tilde{a}}_r \\
&= a_m e^2 + e b_p\left(\tilde{a}_r r + \tilde{a}_x x\right) + \frac{|b_p|}{\gamma_x}\tilde{a}_x\dot{\tilde{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r\dot{\tilde{a}}_r \\
&= a_m e^2 + \tilde{a}_x|b_p|\left(\operatorname{sgn}\left(b_p\right) x e + \frac{\dot{\tilde{a}}_x}{\gamma_x}\right) + \tilde{a}_r|b_p|\left(\operatorname{sgn}\left(b_p\right) r e + \frac{\dot{\tilde{a}}_r}{\gamma_r}\right)
\end{aligned}$$

By choosing the adaptation laws:

$$\begin{aligned}
\dot{\tilde{a}}_x &= -\gamma_x x e \operatorname{sgn}\left(b_p\right) \\
\dot{\tilde{a}}_r &= -\gamma_r r e \operatorname{sgn}\left(b_p\right)
\end{aligned}$$

we obtain

$$\dot{V}\left(e, \tilde{a}_x, \tilde{a}_r\right) = a_m e^2$$

which is negative semidefinite. Since we have a time varying system, we cannot use LaSalle and are left with Barbalat's Lemma: If V is lower bounded, $\dot{V} \leq 0$ and $\ddot{V}$ is uniformly bounded, this ensures that $\dot{V} \to 0$ as $t \to \infty$.

By definition, $V \geq 0$. We can also easily find

$$\begin{aligned}
\ddot{V} &= 2a_m e\dot{e} \\
&= 2a_m e\left(a_m e + b_p\tilde{a}_x x + b_p\tilde{a}_r r\right)
\end{aligned}$$

but we need to find out whether it is uniformly bounded.

$$|\ddot{V}| \leq |2a_m^2 e^2| + |e a_m e b_p \tilde{a}_x x| + |2a_m e b_p \tilde{a}_r r|$$

As a_m and b_p are constants, we need to check e, $\tilde{a}_x$, $\tilde{a}_r$, r and x to see if they are also bounded, to find out if $\ddot{V}$ is bounded. Since V is a nonincreasing function of time,

$$\begin{aligned} e &\quad \text{is bounded} \\ \tilde{a}_x &\quad \text{is bounded} \\ \tilde{a}_r &\quad \text{is bounded} \end{aligned}$$

r is a bounded input by assumption, and as the reference model is a linear globally exponentially stable system, it is input-state-stable (ISS). This means that bounded r produce bounded x_m. As $x = e + x_m$ and both e and x_m are bounded, x is also bounded. The conclusion is that $\ddot{V}$ is bounded, and

$$\begin{aligned} \dot{V} &\rightarrow 0 \\ e &\rightarrow 0 \\ x &\Rightarrow x_m \end{aligned}$$

Using Barbalat's lemma, we have shown that the $\dot{V}$ asymptotically tends to zero, which means that the error dynamics tend to zero and the adaptive controller forces x to track the reference signal x_m asymptotically and globally. This in addition to boundedness prove closed-loop stability.

(e) The system is modelled as shown in Figure 3. We can show the simulations

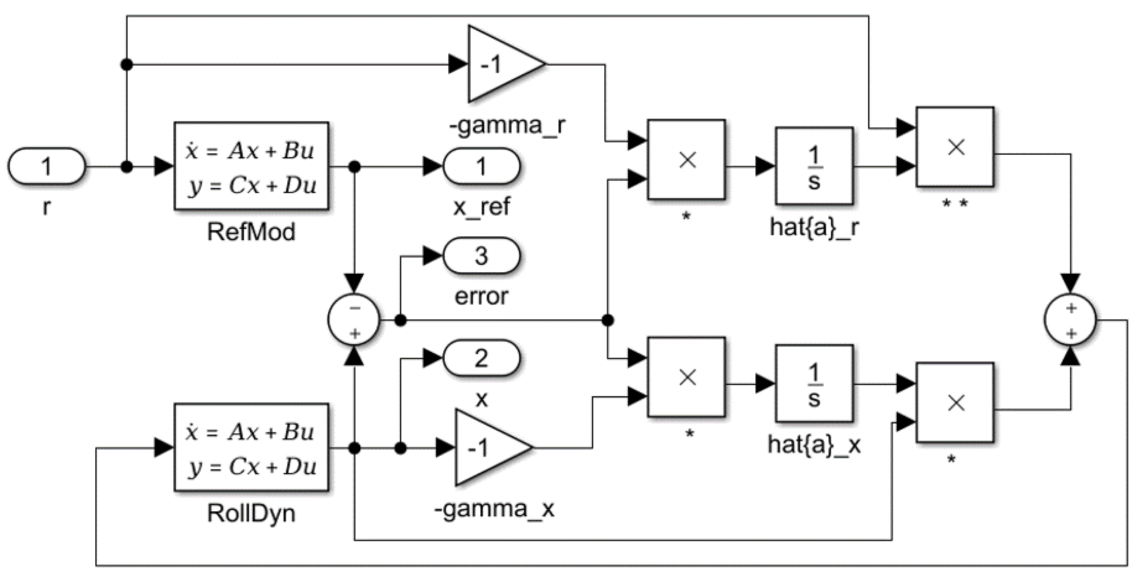


Figure 3: Model reference adaptice controller.

for the tracking performance and parameter estimations in Figure 4 for commanded roll rate $r = 4$, and in Figure 5 for commanded roll rate $r = \sin t$.

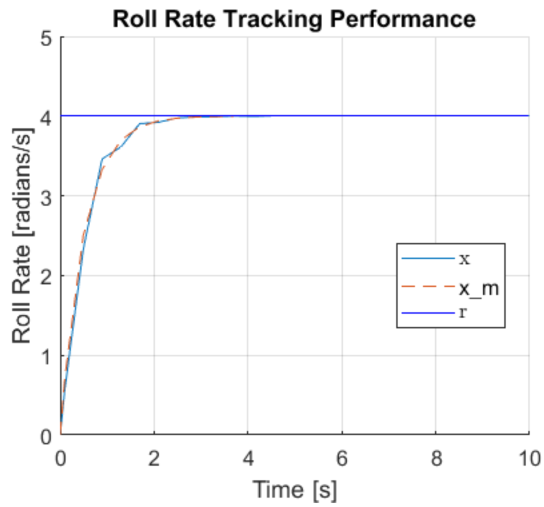


Figure 4: Tracking performance with $r = 4$.

Figure 4 shows that the model reference controller is able to track the commanded roll rate after a stable transient, and that the tracking error for the adaptive controller converges to zero. In Figure 5, we see that the tracking error converges to zero. Note that the model reference does not track the commanded roll rate perfectly, as we expected since the reference model is a first order dynamic system with corresponding amplitude and phase shift, as we know from linear frequency response analysis.
The initial value in the estimation laws of $\hat{a}_x$ and $\hat{a}_r$ are -1 and 1 respectively. The values for γ_x and γ_r were 1.

2. The roll dynamics are now represented by the nonlinear equation:

$$\dot{x} = a_p x + c_p x^3 + b_p u \tag{7}$$

where:

x	Aircraft roll rate in stability axes	[radians/s]
u	Total differential aileron-spoiler deflection	[radians]
a_p	Roll damping derivative	
b_p	Dimensional rolling moment derivative	$b_p > 0$
c_p	Damping constant	

(a) A linear reference model for the plant could be:

$$\dot{x}_m = a_m x + b_m r \tag{8}$$

With perfect tracking, the tracking error dynamics are then:

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{x}_m = 0 \\ &= a_p x + c_p x^3 + b_p u - (a_m x + b_m r) \end{aligned}$$

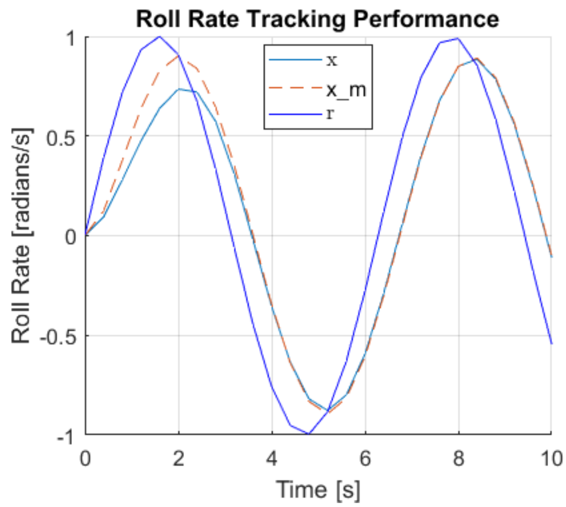


Figure 5: Tracking performance with $r = \sin t$.

In order to achieve the tracking error dynamics

$$\dot{e} = a_m e,$$

we choose

$$u = \frac{a_m - a_p}{b_p} x - \frac{c_p}{b_p} x^3 + \frac{b_m}{b_p}$$

We find that

$$a_x^* = \frac{a_m - a_p}{b_p}, \quad a_f^* = -\frac{c_p}{b_p}, \quad a_r^* = \frac{b_m}{b_p}$$

(b) Since the parameters are unknown, we use estimates for the parameters in the control law found in the previous exercise:

$$u = \hat{a}_x x + \hat{a}_f x^3 + \hat{a}_r r$$

and the closed-loop system becomes

$$\dot{x} = (a_p + \hat{a}_x b_p)\, x + (c_p + \hat{a}_f b_p)\, x^3 + \hat{a}_r b_p r \tag{9}$$

The parameter errors are then:

$$\begin{aligned}
\tilde{a}_x &= \hat{a}_x - a_x^* = \hat{a}_x - \frac{a_m - a_p}{b_p} \\
\tilde{a}_f &= \hat{a}_f - a_f^* = \hat{a}_f + \frac{c_p}{b_p} \\
\tilde{a}_r &= \hat{a}_r - a_r^* = \hat{a}_r - \frac{b_m}{b_p}
\end{aligned}$$

The tracking error dynamics are the found by inserting the controller in the plant dynamics and subtracting the reference model:

$$\begin{aligned}
\dot{e} &= \dot{x} - \dot{x}_m \\
&= \left(a_p + \hat{a}_x b_p\right) x + \left(c_p + \hat{a}_f b_p\right) x^3 + \hat{a}_r b_p r - \left(a_m x_m + b_m r\right) \\
&= a_m x - a_m x + a_m\left(-x_m\right) + \left(a_p + \hat{a}_x b_p\right) x + \left(c_p + \hat{a}_f b_p\right) x^3 + \left(\hat{a}_r b_p - b_m\right) r \\
&= a_m\left(x - x_m\right) + \left(-\frac{a_m - a_p}{b_p} b_p + \hat{a}_x b_p\right) x + \left(\frac{c_p}{b_p} b_p + \hat{a}_f b_p\right) x^3 + \left(\hat{a}_r b_p - \frac{b_m}{b_p} b_p\right) r \\
&= a_m e + b_p \tilde{a}_x x + b_p \tilde{a}_f x^3 + b_p \tilde{a}_r r \\
&= a_m e + b_p\left(\tilde{a}_x x + \tilde{a}_f x^3 + \tilde{a}_r r\right)
\end{aligned}$$

The parameter error dynamics are:

$$\begin{aligned}
\dot{\tilde{a}}_x &= \dot{\hat{a}}_x \\
\dot{\tilde{a}}_f &= \dot{\hat{a}}_f \\
\dot{\tilde{a}}_r &= \dot{\hat{a}}_r
\end{aligned}$$

and will be determined by the choice of adaptation law in the next step.

(c) We now want to show that the tracking error goes to zero globally and asymptotically using the Lyapunov function candidate

$$V\left(e, \tilde{a}_x, \tilde{a}_r, \tilde{a}_f\right) = \frac{e^2}{2} + \frac{|b_p|}{2\gamma_x}\tilde{a}_x^2 + \frac{|b_p|}{2\gamma_r}\tilde{a}_r^2 + \frac{|b_p|}{2\gamma_f}\tilde{a}_f^2$$

and Barbalat's lemma. Again, we need to show that V is lower bounded, that $\dot{V}$ is a nonincreasing function of time and that $\ddot{V}$ is uniformly bounded. As in Exercise 1, $V \geq 0$. Then we derive $\dot{V}$:

$$\dot{V}\left(e, \tilde{a}_r, \tilde{a}_x, \tilde{a}_f\right) = e\dot{e} + \frac{|b_p|}{\gamma_x}\tilde{a}_x\dot{\hat{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r\dot{\hat{a}}_r + \frac{|b_p|}{\gamma_f}\tilde{a}_f\dot{\hat{a}}_f$$

As the ideal parameters are constant, the dynamics for the parameter errors are still expressed by:

$$\begin{aligned}
\dot{\tilde{a}}_x &= \dot{\hat{a}}_x \\
\dot{\tilde{a}}_f &= \dot{\hat{a}}_f \\
\dot{\tilde{a}}_r &= \dot{\hat{a}}_r
\end{aligned}$$

Substituting the expression for $\dot{e}$ into the previous equation, we get:

$$\begin{aligned}
\dot{V}\left(e, \tilde{a}_r, \tilde{a}_x\right) &= e\left(a_m e + b_p\left(\tilde{a}_x x + \tilde{a}_f x^3 + \tilde{a}_r r\right)\right) + \frac{|b_p|}{\gamma_x}\tilde{a}_x\dot{\hat{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r\dot{\hat{a}}_r + \frac{|b_p|}{\gamma_f}\tilde{a}_f\dot{\hat{a}}_f \\
&= a_m e^2 + e b_p\left(\tilde{a}_x x + \tilde{a}_f x^3 + \tilde{a}_r r\right) + \frac{|b_p|}{\gamma_x}\tilde{a}_x\dot{\hat{a}}_x + \frac{|b_p|}{\gamma_r}\tilde{a}_r\dot{\hat{a}}_r + \frac{|b_p|}{\gamma_f}\tilde{a}_f\dot{\hat{a}}_f \\
&= a_m e^2 + \tilde{a}_x|b_p|\left(\operatorname{sgn}\left(b_p\right) x e + \frac{\dot{\hat{a}}_x}{\gamma_x}\right) + \tilde{a}_r|b_p|\left(\operatorname{sgn}\left(b_p\right) r e + \frac{\dot{\hat{a}}_r}{\gamma_r}\right) \\
&\quad + \tilde{a}_f|b_p|\left(\operatorname{sgn}\left(b_p\right) x^3 e + \frac{\dot{\hat{a}}_f}{\gamma_f}\right)
\end{aligned}$$

Selecting the following adaptive laws:

$$\begin{aligned}
\dot{\hat{a}}_x &= -\gamma_x x e \operatorname{sgn}\left(b_p\right) \\
\dot{\hat{a}}_r &= -\gamma_r r e \operatorname{sgn}\left(b_p\right) \\
\dot{\hat{a}}_f &= -\gamma_f f\left(x\right) e \operatorname{sgn}\left(b_p\right),
\end{aligned}$$

we are able to show that $\dot{V} = a_m e^2$, which is negative semidefinite for $a_m < 0$. Using the same arguments as in Exercise 1, we show that

$$\ddot{V} = 2 a_m e \dot{e}$$

is uniformly bounded. This gives us the result that $\dot{V}$ tends to zero asymptotically, which again means that the error dynamics tend to zero and that the adaptive controller forces x to track the reference signal x_m asymptotically and globally.

(d) The system is modelled as shown in Figure 6. The simulations for commanded roll rate signals $r = 4$ and $r = \sin t$ are shown in Figures 7 and 8.

Again, all γ-values are set to 1. The initial value parameter estimates are: $\hat{a}_x(0) = -1$, $\hat{a}_r(0) = 1$ and $\hat{a}_f(0) = -1$.

3. We are given the nonlinear MIMO system

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{10}$$

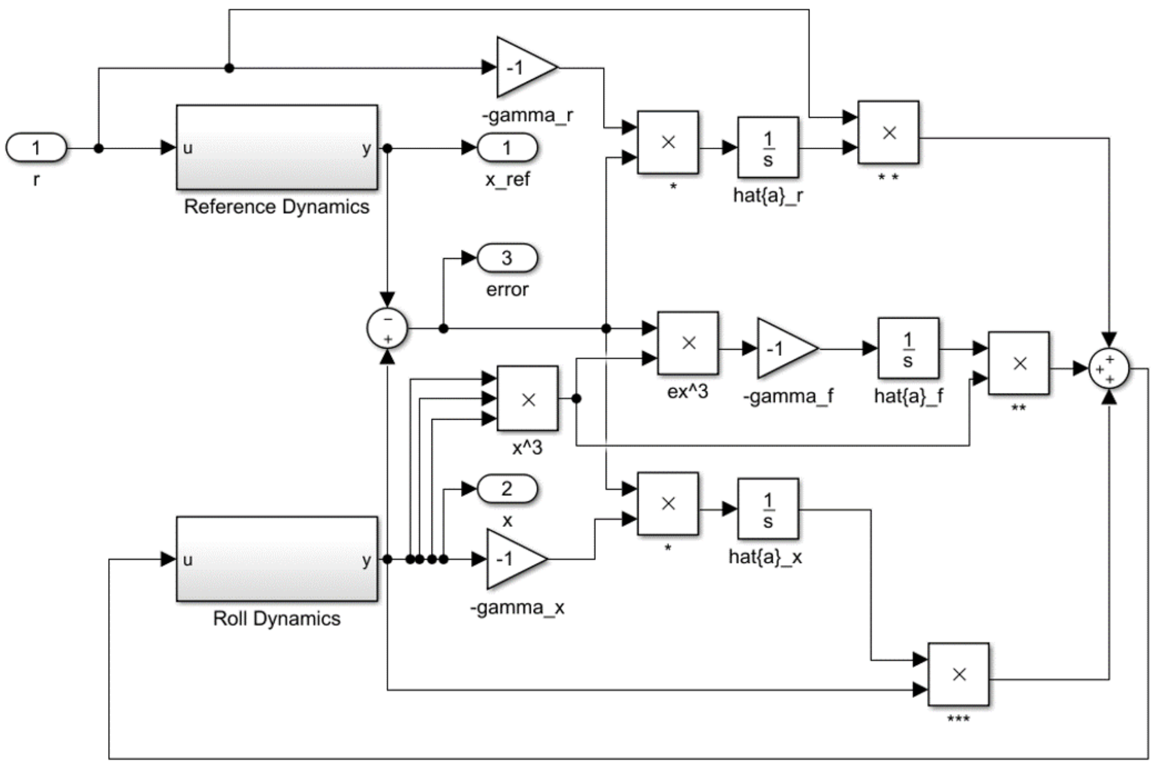


Figure 6: Model reference adaptice controller.

where

$$
\begin{aligned}
H &= \text{Inertia matrix, uniformly positive definite} \\
\theta &= \text{Vector of joint angles} \\
u &= \text{Vector of torques applied at the manipulator joints} \\
C &= \text{Torques} \\
C_{11} &= -h\dot{\theta}_2 \\
C_{12} &= -h\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\
C_{21} &= h\dot{\theta}_1 \\
C_{22} &= 0 \\
H_{11} &= a_1 + 2a_3\cos\theta_2 + 2a_4\sin\theta_2 \\
H_{12} &= H_{21} = a_2 + a_3\cos\theta_2 + a_4\sin\theta_2 \\
H_{22} &= a_2 \\
h &= a_3\sin\theta_2 - a_4\cos\theta_2
\end{aligned}
$$

(a) First, we want to find a linear parametrization to be able to express the system

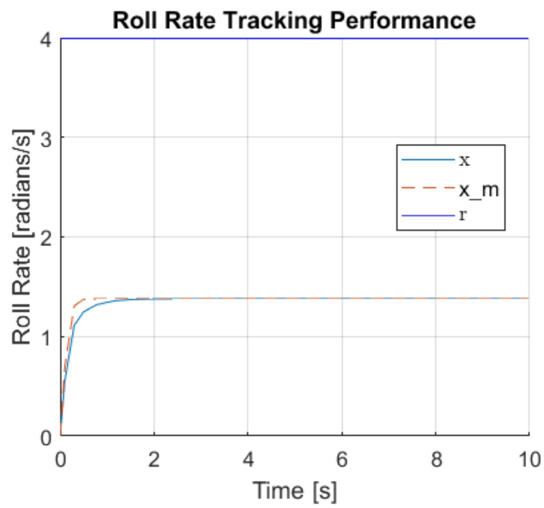


Figure 7: Tracking performance with $r = 4$.

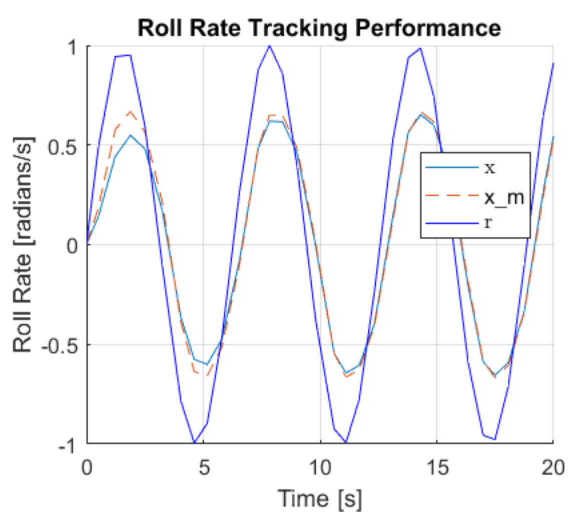


Figure 8: Tracking performance with $r = \sin t$.

as

$$Ya = u$$

We start by writing out the system equation, filling in for matrix C:

$$\begin{aligned}
\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{\theta}_2 & -h\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ h\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\begin{bmatrix} H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 \\ H_{21}\ddot{\theta}_1 + H_{22}\ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{\theta}_1\dot{\theta}_2 - h\dot{\theta}_2\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ h\dot{\theta}_1^2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\begin{bmatrix} H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 - h\dot{\theta}_1\dot{\theta}_2 - h\dot{\theta}_2\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ H_{21}\ddot{\theta}_1 + H_{22}\ddot{\theta}_2 + h\dot{\theta}_1^2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\begin{bmatrix} H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 - 2h\dot{\theta}_1\dot{\theta}_2 - h\dot{\theta}_2^2 \\ H_{21}\ddot{\theta}_1 + H_{22}\ddot{\theta}_2 + h\dot{\theta}_1^2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\end{aligned}$$

Now, filling in for the matrix H, we get:

$$\begin{aligned}
\begin{bmatrix} \left(a_1 + 2a_3\cos\theta_2 + 2a_4\sin\theta_2\right)\ddot{\theta}_1 + \left(a_2 + a_3\cos\theta_2 + a_4\sin\theta_2\right)\ddot{\theta}_2 - 2h\dot{\theta}_1\dot{\theta}_2 - h\dot{\theta}_2^2 \\ \left(a_2 + a_3\cos\theta_2 + a_4\sin\theta_2\right)\ddot{\theta}_1 + a_2\ddot{\theta}_2 + h\dot{\theta}_1^2 \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\begin{bmatrix} \left(a_1 + 2a_3\cos\theta_2 + 2a_4\sin\theta_2\right)\ddot{\theta}_1 + \left(a_2 + a_3\cos\theta_2 + a_4\sin\theta_2\right)\ddot{\theta}_2\,[...] \\ -2\left(a_3\sin\theta_2 - a_4\cos\theta_2\right)\dot{\theta}_1\dot{\theta}_2 - \left(a_3\sin\theta_2 - a_4\cos\theta_2\right)\dot{\theta}_2^2 \\ \\ \left(a_2 + a_3\cos\theta_2 + a_4\sin\theta_2\right)\ddot{\theta}_1 + a_2\ddot{\theta}_2 + \left(a_3\sin\theta_2 - a_4\cos\theta_2\right)\dot{\theta}_1^2 \end{bmatrix} &= \begin{bmatrix} \\ u_1 \\ \\ u_2 \end{bmatrix} \\
\begin{bmatrix} a_1\ddot{\theta}_1 + a_2\ddot{\theta}_2 + a_3\left(\left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)\cos\theta_2 - \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\sin\theta_2\right)[...] \\ +a_4\left(\left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)\sin\theta_2 + \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\cos\theta_2\right) \\ a_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + a_3\left(\ddot{\theta}_1\cos\theta_2 + \dot{\theta}_1^2\sin\theta_2\right) + a_4\left(\ddot{\theta}_1\sin\theta_2 - \dot{\theta}_1^2\cos\theta_2\right) \end{bmatrix} &= \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix}
\end{aligned}$$

Now, choosing the vector of possibly unknown parameters to be

$$a = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}^T$$

and the input vector to be

$$u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$$

we get the regression matrix

$$\begin{aligned}
Y &= \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \end{bmatrix} \\
y_{11} &= \ddot{\theta}_1 \\
y_{12} &= \ddot{\theta}_2 \\
y_{13} &= \left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)\cos\theta_2 - \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\sin\theta_2 \\
y_{14} &= \left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)\sin\theta_2 + \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)\cos\theta_2 \\
y_{21} &= 0 \\
y_{22} &= \ddot{\theta}_1 + \ddot{\theta}_2 \\
y_{23} &= \ddot{\theta}_1\cos\theta_2 + \dot{\theta}_1^2\sin\theta_2 \\
y_{24} &= \ddot{\theta}_1\sin\theta_2 - \dot{\theta}_1^2\cos\theta_2
\end{aligned}$$

Using this notation, we get the model expressed with linear parametrization.

(b) We want to show global uniform asymptotic stability of the plant using the control law:

$$u = H\ddot{\theta}_r + C\dot{\theta}_r - K_p(\theta - \theta_d) - K_d\left(\dot{\theta} - \dot{\theta}_d\right)$$

where

$$\dot{\theta}_r = \dot{\theta}_d - \Lambda(\theta - \theta_d)$$

using

$$\begin{aligned}
V(s) &= \frac{1}{2}\left[s^T H s\right], \\
s &= \dot{\theta} - \dot{\theta}_r
\end{aligned}$$

The error variables are now s and e.
We first write the equations for the error. The tracking error dynamics are found by:

$$\dot{e} = -\Lambda e + s \tag{11}$$

and the dynamics for s are found when we write the closed-loop system when substituting for the control law into the original system:

$$\begin{aligned}
H\dot{s} &= H\ddot{\theta} - H\ddot{\theta}_r \\
&= -C\dot{\theta} + u - H\ddot{\theta}_r \\
&= -C\dot{\theta} + H\ddot{\theta}_r + C\dot{\theta}_r - K_p e - K_d\dot{e} - H\ddot{\theta}_r
\end{aligned}$$

Now, substituting $\dot{\theta} = s + \dot{\theta}_r$ and $K_p = K_d\Lambda$, we get:

$$
\begin{aligned}
H\dot{s} &= -C\left(s + \dot{\theta}_r\right) + H\ddot{\theta}_r + C\dot{\theta}_r - K_p e - K_d\dot{e} - H\ddot{\theta}_r \\
&= -Cs - K_d\lambda e - K_d\dot{e} \\
&= -Cs - K_d\left(\Lambda e + \dot{e}\right)
\end{aligned}
$$

$$
H\dot{s} = -Cs - K_d s \tag{12}
$$

The closed-loop error dynamics 11 and 12 can be recognized as a cascaded system, see Figure 9, where Σ_2 is given by:

$$
H\dot{s} = -Cs - K_d s \tag{13}
$$

with s as an output, and Σ_1 is given by

$$
\dot{e} = -\Lambda e + s \tag{14}
$$

with s as an input, and e as an output.

Lemma 4.7 in Khalil states that if Σ_2 is GUAS and Σ_1 is ISS, then the origin

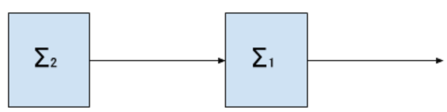


Figure 9: Cascaded system

of the total cascaded system is GUAS. We start by investigating whether Σ_1 is ISS:
Alternative 1:

$$
\dot{e} = -\Lambda e + s = f\left(e, s\right)
$$

$f\left(e, s\right)$ is C^1 and globally Lipschitz in (e, s):

$$
\begin{aligned}
& |f\left(e_2, s\right) - f\left(e_1, s\right)| \\
=\ & |-\Lambda e_2 + s + \Lambda e_1 - s| \\
=\ & |-\Lambda\left(e_2 - e_1\right)| \\
=\ & |\Lambda||e_2 - e_1|, L = |\Lambda|
\end{aligned}
$$

The unforced system

$$
\dot{e} = -\Lambda e
$$

has a GES equilibrium point at $e = 0$. By Lemma 4.6 in Khalil, the system $\dot{e} = \Lambda e + s$ is ISS.

Alternative 2:
Using the Lyapunov function candidate

$$V = \frac{1}{2}e^2,$$

which is C^1 and satisfies

$$\alpha_1 (||x||) \leq V(t, x) \leq \alpha_2 (||x||)$$

with $\alpha_1 (||x||) = \alpha_2 (||x||) = \frac{1}{2}||e||^2$.

$$\begin{aligned} \dot{V} &= e\dot{e} = -\Lambda e^2 + es \\ &= -(\Lambda - \theta) e^2 - \theta e^2 + es, \quad 0 < \theta < \Lambda \end{aligned}$$

When $|es| \leq \theta e^2$, then $-\theta e^2 + es \leq 0$ and thus

$$\dot{V} = -(\Lambda - \theta) e^2 < 0 \quad \forall |e| \geq \frac{|s|}{\theta} > 0$$

By Theorem 4.19 in Khalil the system is ISS with $\gamma(|s|) = \frac{|s|}{\theta}$. Next, we investigate whether Σ_2 is GUAS. To this end, we use the Lyapunov function candidate

$$V(s, \tilde{a}) = \frac{1}{2}\left[s^T H s\right]$$

This is continuous and positive definite, and we need to find out whether its derivative is negative definite.

$$\begin{aligned} \dot{V} &= s^T H \dot{s} + \frac{1}{2} s^T \dot{H} s \\ &= s^T (-Cs - K_d s) + \frac{1}{2} s^T \dot{H} s \\ &= -s^T K_d s - s^T C s + \frac{1}{2} s^T \dot{H} s \end{aligned}$$

Now we use the fact that because of skew-symmetry, $z^T \left(\dot{H} - 2C\right) z = 0, \quad \forall z \in R^2$.

$$\dot{V} = -s^T K_d s < 0 \text{ in } s$$

Theorem 4.9 in Khalil thus gives that Σ_2 is GUAS, which together with Σ_1 being ISS means that the cascaded system is GUAS, and thus $(e, \dot{e}) \to 0$ asymptotically.

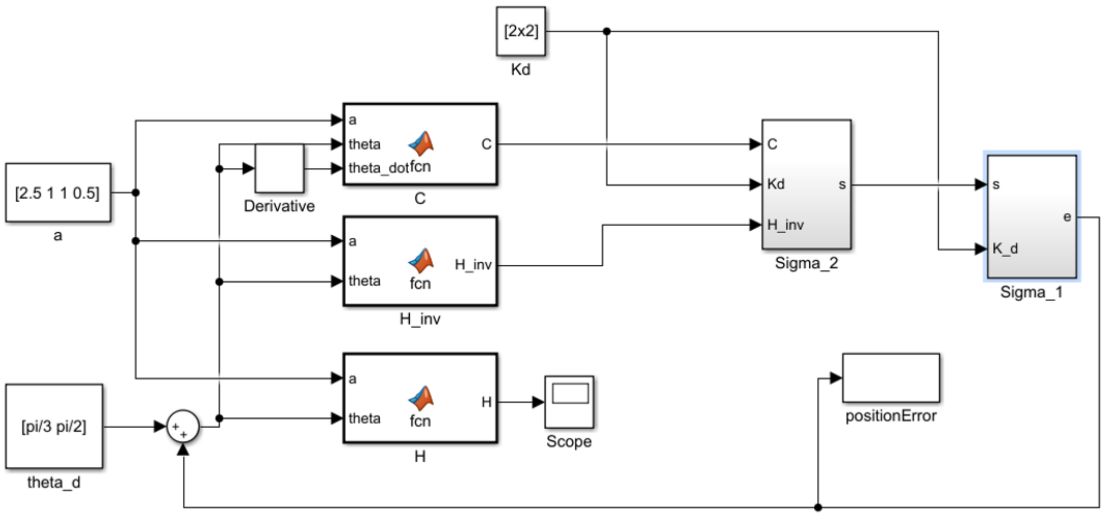


Figure 10: The MIMO system

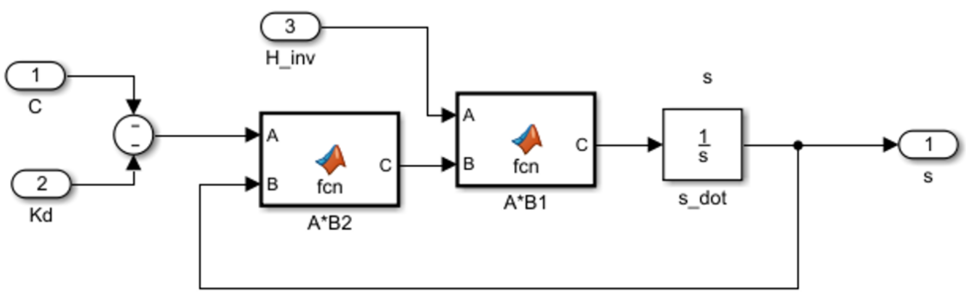


Figure 11: Subsystem 2

(c) The system is modelled as shown in Figures 10-12. As initial value for the error vector (indicating the initial position of the manipulator), $\left(-\frac{\pi}{3} \quad -\frac{\pi}{2}\right)$ is chosen. The initial value for the s vector was chosen to zero. The tracking error is shown in Figure 13. The tracking error shows asymptotic convergence. The values were set to $K_d = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$ and $K_p = \begin{bmatrix} 2000 & 0 \\ 0 & 2000 \end{bmatrix}$.

(d) Now, the plant parameters a_i are unknown, but assumed to be constant. We want to adjust the control law from the previous subtask, and replace the parameters with their estimates:

$$u = \hat{H}(\theta)\,\ddot{\theta}_r + \hat{C}\left(\theta, \dot{\theta}\right)\dot{\theta}_r - K_p\left(\theta - \theta_d\right) - K_d\left(\dot{\theta} - \dot{\theta}_d\right)$$

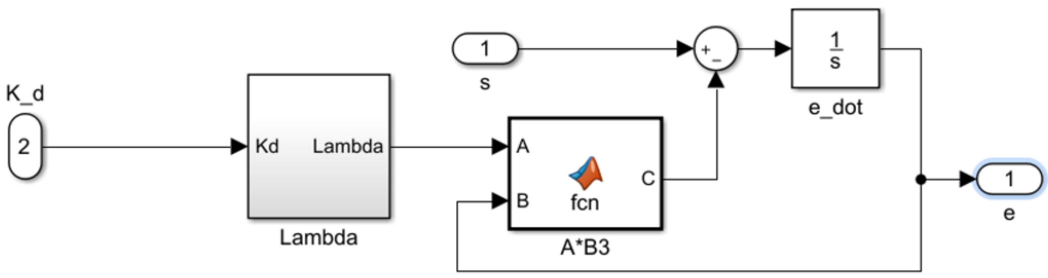


Figure 12: Subsystem 1

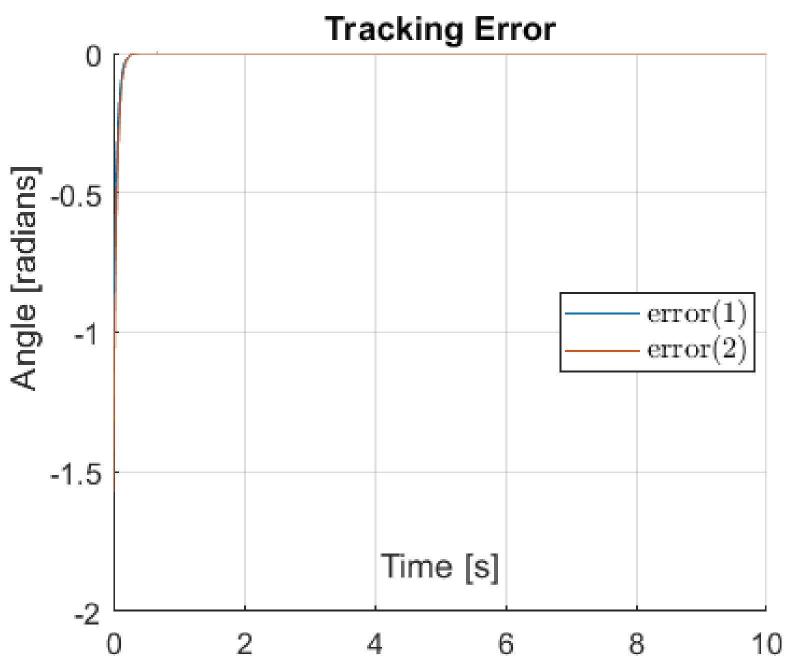


Figure 13: The tracking error

Then we define

$$\hat{H}(\theta)\ddot{\theta}_r + \hat{C}\left(\theta,\dot{\theta}\right)\dot{\theta}_r = Y\left(\theta,\dot{\theta},\theta_r,\dot{\theta}_r,\ddot{\theta}\right)\hat{a}$$

and get

$$u = Y\hat{a} - K_p e - K_d\dot{e}$$

where the error variables are

$$\begin{aligned} e &= \theta - \theta_d \\ s &= \dot{\theta} - \dot{\theta}_r \\ \tilde{a} &= \hat{a} - a \end{aligned}$$

Now, we want to find the adaptation law and prove that asymptotic trajectory tracking error is achieved globally for the closed-loop system, and that the estimation error is bounded. We first write the closed-loop error dynamics, using the error variables $(e, s, \tilde{a})$

$$\begin{aligned} H\dot{s} &= H\ddot{\theta} - H\ddot{\theta}_r \\ &= -C\dot{\theta} + u - H\ddot{\theta}_r \\ &= -C\dot{\theta} + Y\hat{a} - K_p e - K_d\dot{e} - H\ddot{\theta}_r \end{aligned}$$

Now, substituting $\dot{\theta} = s + \dot{\theta}_r$ and $K_p = K_d\Lambda$, we get:

$$\begin{aligned} H\dot{s} &= -C\left(s + \dot{\theta}_r\right) + Y\hat{a} - K_d(\Lambda e + \dot{e}) - H\ddot{\theta}_r \\ &= -Cs - \left(H\ddot{\theta}_r + C\dot{\theta}_r\right) + Y\hat{a} - K_d s \\ &= -Cs - K_d s + Y(\hat{a} - a) \\ &= -(C + K_d)s + Y\tilde{a} \end{aligned}$$

The closed-loop error dynamics, we have the error dynamics are thus:

$$\begin{aligned} \dot{e} &= -\Lambda e + s && (15) \\ H\dot{s} &= -Cs - K_d s + Y\tilde{a} && (16) \\ \dot{\tilde{a}} &= \dot{\hat{a}} && (17) \end{aligned}$$

The closed-loop error dynamics 15, 16 and 17 can be recognized as a cascaded system, where Σ_2 is given by:

$$\begin{aligned} H\dot{s} &= -(C + K_d)s + Y\tilde{a} \\ \dot{\tilde{a}} &= \dot{\hat{a}} \end{aligned}$$

and Σ_1 is given by:

$$\dot{e} = -\Lambda e + s$$

which we already know is ISS. Using Lemma 4.7 in Khalil, we need to find out whether Σ_2 is GUAS to find out if the whole cascaded system is GUAS. We use the Lyapunov function candidate

$$V(s, \tilde{a}) = \frac{1}{2}\left[s^T H s + \tilde{a}^T \Gamma^{-1} \tilde{a}\right]$$

which we know is continuous and positive definite, where $\Gamma = \Gamma^T > 0$ is constant. We want to see if $\dot{V} < 0$.

$$\begin{aligned}
\dot{V} &= s^T\left(-(C + K_d)s + Y\tilde{a}\right) + \frac{1}{2}s^T \dot{H} s + \tilde{a}^T \Gamma^{-1} \dot{\tilde{a}} \\
&= s^T\left(\frac{1}{2}\dot{H} - C\right)s - s^T K_d s + \tilde{a}^T\left(Y^T s + \Gamma^{-1}\dot{\hat{a}}\right)
\end{aligned}$$

where $s^T\left(\frac{1}{2}\dot{H} - C\right)s = 0$ because of skew symmetry. Remember that the transpose of a scalar is the scalar, such that $s^T Y \tilde{a} = \tilde{a}^T Y^T s$. By choosing:

$$\dot{\hat{a}} = -\Gamma Y^T s$$

we obtain

$$\dot{V} = -s^T K_d s \leq 0$$

Now, as this is only negative semidefinite, we utilize Barbalat's Lemma to further investigate the stability. Note that Σ_2 is a time-varying system since $Y\left(\theta_r(t), \dot{\theta}_r(t), \dot{\theta}(t), \theta(t)\right)$ contains time-varying functions (which are not the system states). Barbalat's lemma says that if V is lower bounded, $\dot{V} \leq 0$ and $\ddot{V}$ is uniformly bounded, then $\dot{V} \to 0$ as $t \to \infty$.
By definition, $V \geq 0$. We can also easily find

$$\begin{aligned}
\ddot{V} &= -2s^T K_d \dot{s} \\
&= -2s^T K_d\left(-H^{-1}(C + K_d)s + H^{-1}Y\tilde{a}\right) \\
&= 2s^T K_d H^{-1}(C + K_d)s - 2s^T K_d H^{-1} Y\tilde{a}
\end{aligned}$$

By the triangle inequality,

$$|\ddot{V}| \leq |2s^T K_d H^{-1}(C + K_d)s| + |2s^T K_d H^{-1} Y\tilde{a}|$$

showing that $\ddot{V}$ is bounded if s, $\tilde{a}$, H and Y are bounded. Furthermore, Y is bounded if H, C, $\ddot{\theta}_r$ and $\dot{\theta}_r$ are bounded. H is bounded if $\cos\theta_2$ and $\sin\theta_2$ are bounded, which they are by definition. C is bounded if $\dot{\theta}$, $\cos\theta$ and $\sin\theta$ are bounded. This leaves us with finding out whether s, $\tilde{a}$, $\ddot{\theta}_r$, $\dot{\theta}_r$ and $\dot{\theta}$ are bounded.

θ_d, $\dot{\theta}_d$ and $\ddot{\theta}_d$ are bounded by assumption. As $\dot{V} \leq 0$, V is a nonincreasing

function of time, and thus s and $\tilde{a}$ are bounded by $V(0)$. As Σ_1 is ISS and s is bounded, e is also bounded.
As e and θ_d are bounded, θ is also bounded:

$$\begin{aligned} e &= \theta - \theta_d \\ \rightarrow \theta &= e + \theta_d \end{aligned}$$

As both e and s are bounded, also $\dot{e}$ is bounded:

$$\begin{aligned} s &= \dot{e} + \Lambda e \\ \rightarrow \dot{e} &= s - \Lambda e \end{aligned}$$

As $\dot{e}$ and $\dot{\theta}_d$ are bounded, also $\dot{\theta}$ is bounded:

$$\begin{aligned} \dot{e} &= \dot{\theta} - \dot{\theta}_d \\ \rightarrow \dot{\theta} &= \dot{e} + \dot{\theta}_d \end{aligned}$$

As s and $\dot{\theta}$ are bounded, then also $\dot{\theta}_r$ is bounded:

$$\begin{aligned} s &= \dot{\theta} - \dot{\theta}_r \\ \rightarrow \dot{\theta}_r &= \dot{\theta} - s \end{aligned}$$

As $\ddot{\theta}_d$ is bounded, $\dot{\theta}$ is bounded and $\dot{\theta}_d$ is bounded, then also $\ddot{\theta}_r$ is bounded:

$$\ddot{\theta}_r = \ddot{\theta}_d - \Lambda \dot{e}$$

The conclusion is that $\ddot{V}$ is bounded, and thus Barbalat's lemma gives that $\dot{V} \rightarrow 0$, which implies that $s \rightarrow 0$. Since Σ_1 is ISS, this implies that $\theta \rightarrow \theta_d$. This shows that we have asymptotic tracking for the closed-loop system Also, the estimation error $\tilde{a}$ is bounded.

(e) To simulate the adaptive control system, we need to calculate the Y-matrix that is given by the equation:

$$Y\hat{a} = \hat{H}\ddot{\theta}_r + \hat{C}\dot{\theta}_r$$

We start by writing out the system equation, filling in for matrix C:

$$\begin{aligned}
\begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_{r1} \\ \ddot{\theta}_{r2} \end{bmatrix} + \begin{bmatrix} -\hat{h}\dot{\theta}_2 & -\hat{h}\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ \hat{h}\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_{r1} \\ \dot{\theta}_{r2} \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\begin{bmatrix} \hat{H}_{11}\ddot{\theta}_{r1} + \hat{H}_{12}\ddot{\theta}_{r2} \\ \hat{H}_{21}\ddot{\theta}_{r1} + \hat{H}_{22}\ddot{\theta}_{r2} \end{bmatrix} + \begin{bmatrix} -\hat{h}\dot{\theta}_2\dot{\theta}_{r1} - \hat{h}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_{r2} \\ \hat{h}\dot{\theta}_1\dot{\theta}_{r1} \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\begin{bmatrix} \hat{H}_{11}\ddot{\theta}_{r1} + \hat{H}_{12}\ddot{\theta}_{r2} - \hat{h}\dot{\theta}_2\dot{\theta}_{r1} - \hat{h}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_{r2} \\ \hat{H}_{21}\ddot{\theta}_{r1} + \hat{H}_{22}\ddot{\theta}_{r2} + \hat{h}\dot{\theta}_1\dot{\theta}_{r1} \end{bmatrix} &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\begin{bmatrix} \hat{H}_{11}\ddot{\theta}_{r1} + \hat{H}_{12}\ddot{\theta}_{r2} - (\hat{a}_3 \sin\theta_2 - \hat{a}_4 \cos\theta_2)\dot{\theta}_2\dot{\theta}_{r1} [...] \\ - (\hat{a}_3 \sin\theta_2 - \hat{a}_4 \cos\theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_{r2} \\ \hat{H}_{21}\ddot{\theta}_{r1} + \hat{H}_{22}\ddot{\theta}_{r2} + (\hat{a}_3 \sin\theta_2 - \hat{a}_4 \cos\theta_2)\dot{\theta}_1\dot{\theta}_{r1} \end{bmatrix} &= \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix}
\end{aligned}$$

Now, filling in for the matrix H, we get:

$$\begin{bmatrix} (\hat{a}_1 + 2\hat{a}_3\cos\theta_2 + 2\hat{a}_4\sin\theta_2)\,\ddot{\theta}_{r1} + (\hat{a}_2 + \hat{a}_3\cos\theta_2 + \hat{a}_4\sin\theta_2)\,\ddot{\theta}_{r2}\,[...] \\ -(\hat{a}_3\sin\theta_2 - \hat{a}_4\cos\theta_2)\,\dot{\theta}_2\dot{\theta}_{r1} - (\hat{a}_3\sin\theta_2 - \hat{a}_4\cos\theta_2)\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_{r2} \\ (\hat{a}_2 + \hat{a}_3\cos\theta_2 + \hat{a}_4\sin\theta_2)\,\ddot{\theta}_{r1} + \hat{a}_2\ddot{\theta}_{r2} + (\hat{a}_3\sin\theta_2 - \hat{a}_4\cos\theta_2)\,\dot{\theta}_1\dot{\theta}_{r1} \end{bmatrix} = \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{a}_1\ddot{\theta}_{r1} + \hat{a}_2\ddot{\theta}_{r2} + \hat{a}_3\left(2\cos\theta_2\ddot{\theta}_{r1} + \cos\theta_2\ddot{\theta}_{r2} - \sin\theta_2\dot{\theta}_2\dot{\theta}_{r1} - \sin\theta_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_{r2}\right)[...] \\ +\hat{a}_4\left(2\sin\theta_2\ddot{\theta}_{r1} + \sin\theta_2\ddot{\theta}_{r2} - \cos\theta_2\dot{\theta}_2\dot{\theta}_{r1} - \cos\theta_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_{r2}\right) \\ \hat{a}_2\left(\ddot{\theta}_{r1} + \ddot{\theta}_{r2}\right) + \hat{a}_3\left(\cos\theta_2\ddot{\theta}_{r1} + \sin\theta_2\dot{\theta}_1\dot{\theta}_{r1}\right) + \hat{a}_4\left(\sin\theta_2\ddot{\theta}_{r1} - \cos\theta_2\dot{\theta}_1\dot{\theta}_{r1}\right) \end{bmatrix} = \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{a}_1\ddot{\theta}_{r1} + \hat{a}_2\ddot{\theta}_{r2} + \hat{a}_3\left(\cos\theta_2\left(2\ddot{\theta}_{r1} + \ddot{\theta}_{r2}\right) - \sin\theta_2\left(\dot{\theta}_2\dot{\theta}_{r1} + \left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_{r2}\right)\right)[...] \\ +\hat{a}_4\left(\sin\theta_2\left(2\ddot{\theta}_{r1} + \ddot{\theta}_{r2}\right) - \cos\theta_2\left(\dot{\theta}_2\dot{\theta}_{r1} + \left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_{r2}\right)\right) \\ \hat{a}_2\left(\ddot{\theta}_{r1} + \ddot{\theta}_{r2}\right) + \hat{a}_3\left(\cos\theta_2\ddot{\theta}_{r1} + \sin\theta_2\dot{\theta}_1\dot{\theta}_{r1}\right) + \hat{a}_4\left(\sin\theta_2\ddot{\theta}_{r1} - \cos\theta_2\dot{\theta}_1\dot{\theta}_{r1}\right) \end{bmatrix} = \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix}$$

And finally, with $\hat{a} = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \hat{a}_4 \end{bmatrix}$, we find that the regression matrix

$$\begin{aligned}
Y &= \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} \\ y_{21} & y_{22} & y_{23} & y_{24} \end{bmatrix} \\
y_{11} &= \ddot{\theta}_{r1} \\
y_{12} &= \ddot{\theta}_{r2} \\
y_{13} &= \cos\theta_2\left(2\ddot{\theta}_{r1} + \ddot{\theta}_{r2}\right) - \sin\theta_2\left(\dot{\theta}_2\dot{\theta}_{r1} + \left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_{r2}\right) \\
y_{14} &= \sin\theta_2\left(2\ddot{\theta}_{r1} + \ddot{\theta}_{r2}\right) - \cos\theta_2\left(\dot{\theta}_2\dot{\theta}_{r1} + \left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_{r2}\right) \\
y_{21} &= 0 \\
y_{22} &= \ddot{\theta}_{r1} + \ddot{\theta}_{r2} \\
y_{23} &= \cos\theta_2\ddot{\theta}_{r1} + \sin\theta_2\dot{\theta}_1\dot{\theta}_{r1} \\
y_{24} &= \sin\theta_2\ddot{\theta}_{r1} - \cos\theta_2\dot{\theta}_1\dot{\theta}_{r1}
\end{aligned}$$

The system is modelled as shown in Figures 14-16. As initial value for the error vector, $\left(-\frac{\pi}{3}, -\frac{\pi}{2}\right)$ is chosen. The initial value for the s vector was chosen to zero, and the initial value of $\hat{a}$ was chosen to zero. The tracking error is shown in Figure 17, and the angles and their reference values are shown in Figure 18. The simulations confirm asymptotic convergence of the tracking error.

4. (Khalil 14.31) The system is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 + a + \left(x_1 - a^{1/3}\right)^3 \\
\dot{x}_2 &= x_1 + u
\end{aligned}$$

where the first system equation has the virtual input x_2. Choose

$$x_2 = -a - x_1 - \left(x_1 - a^{1/3}\right)^3$$

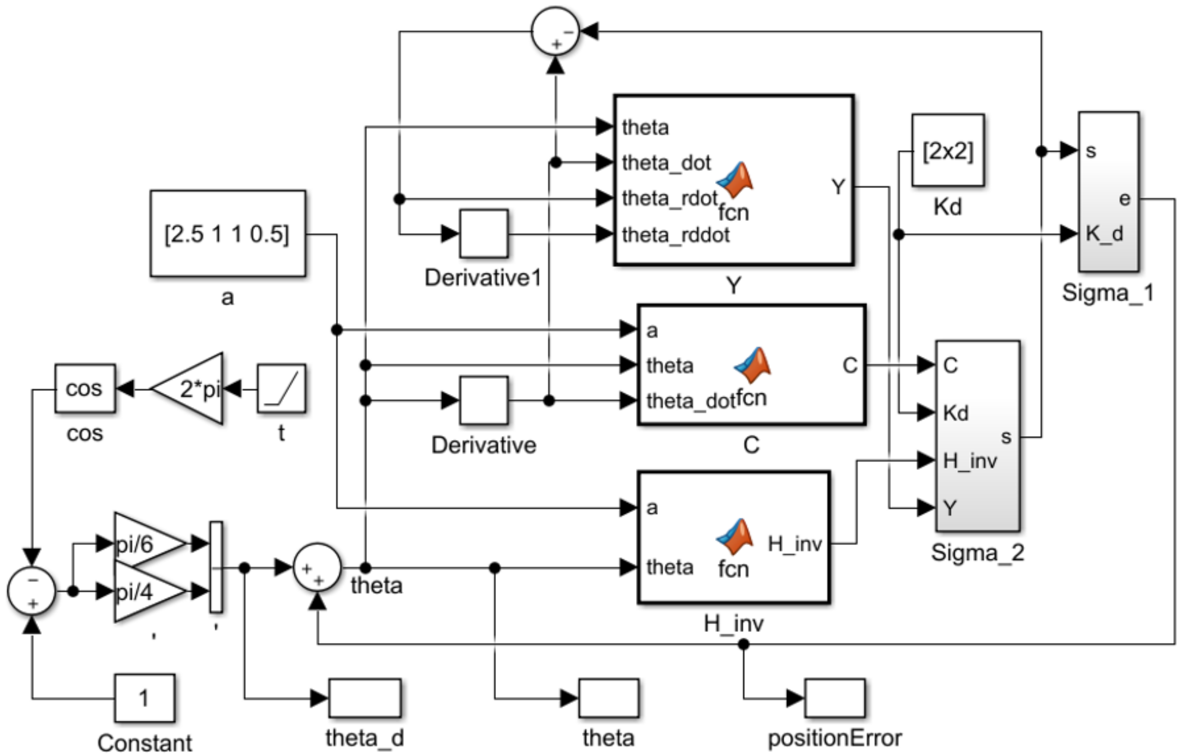


Figure 14: The MIMO system

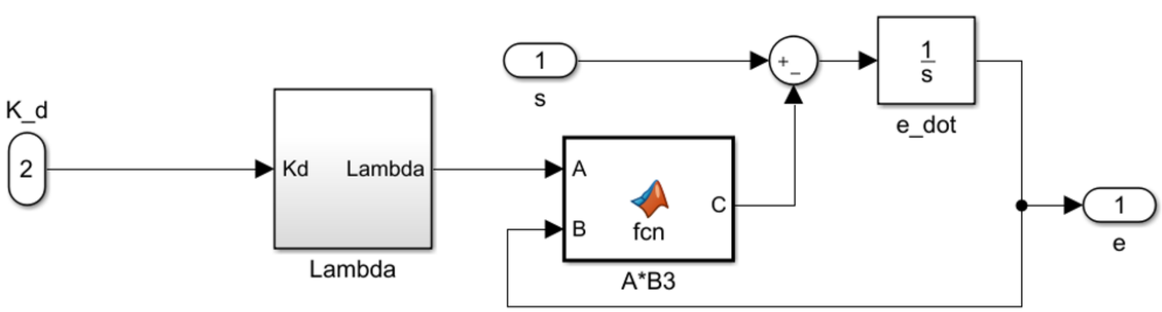


Figure 15: Subsystem 1

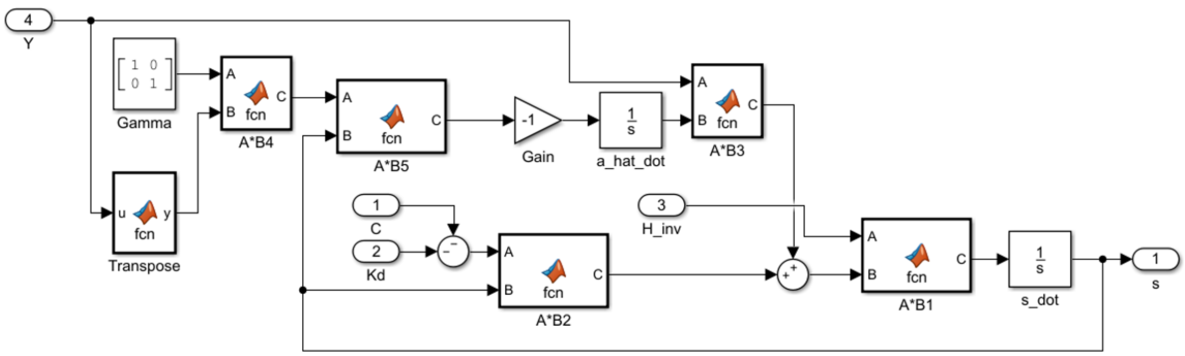


Figure 16: Subsystem 2

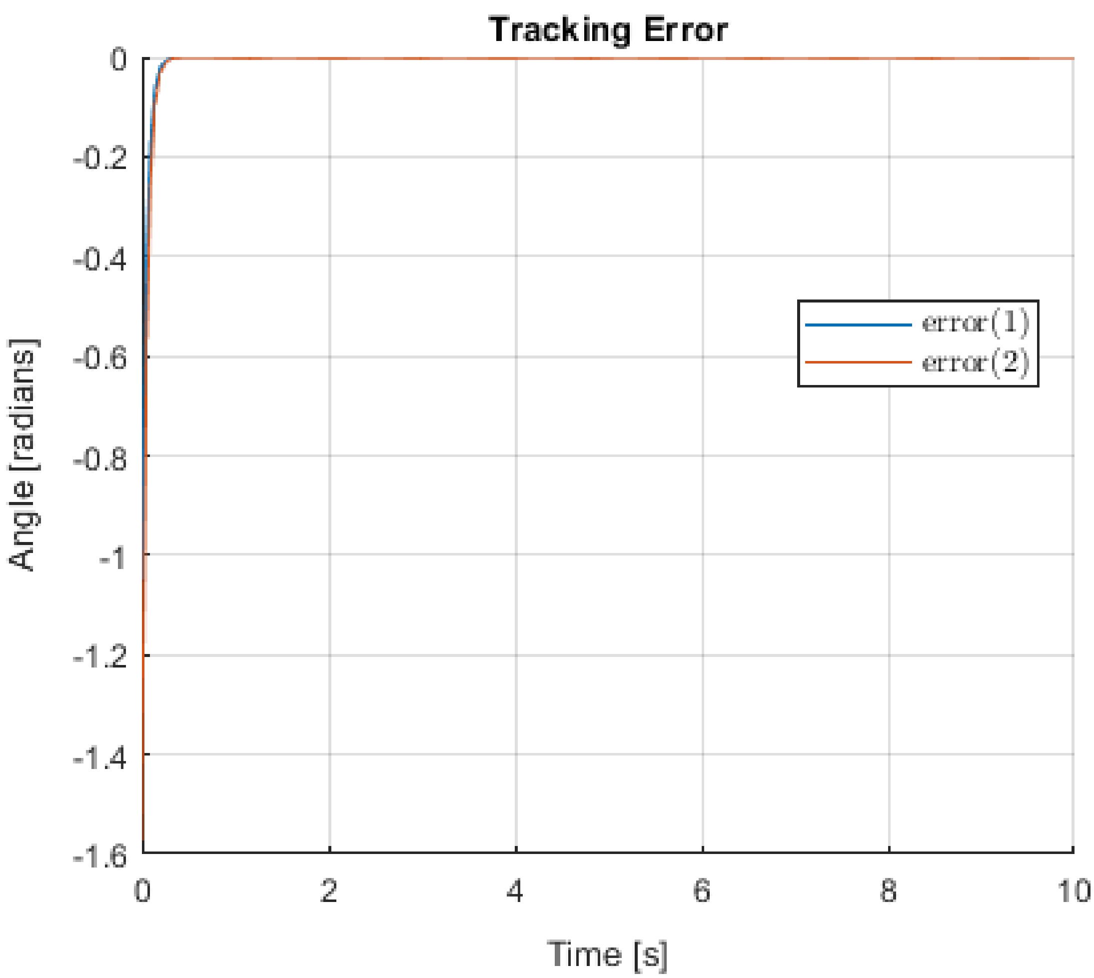


Figure 17: The tracking error

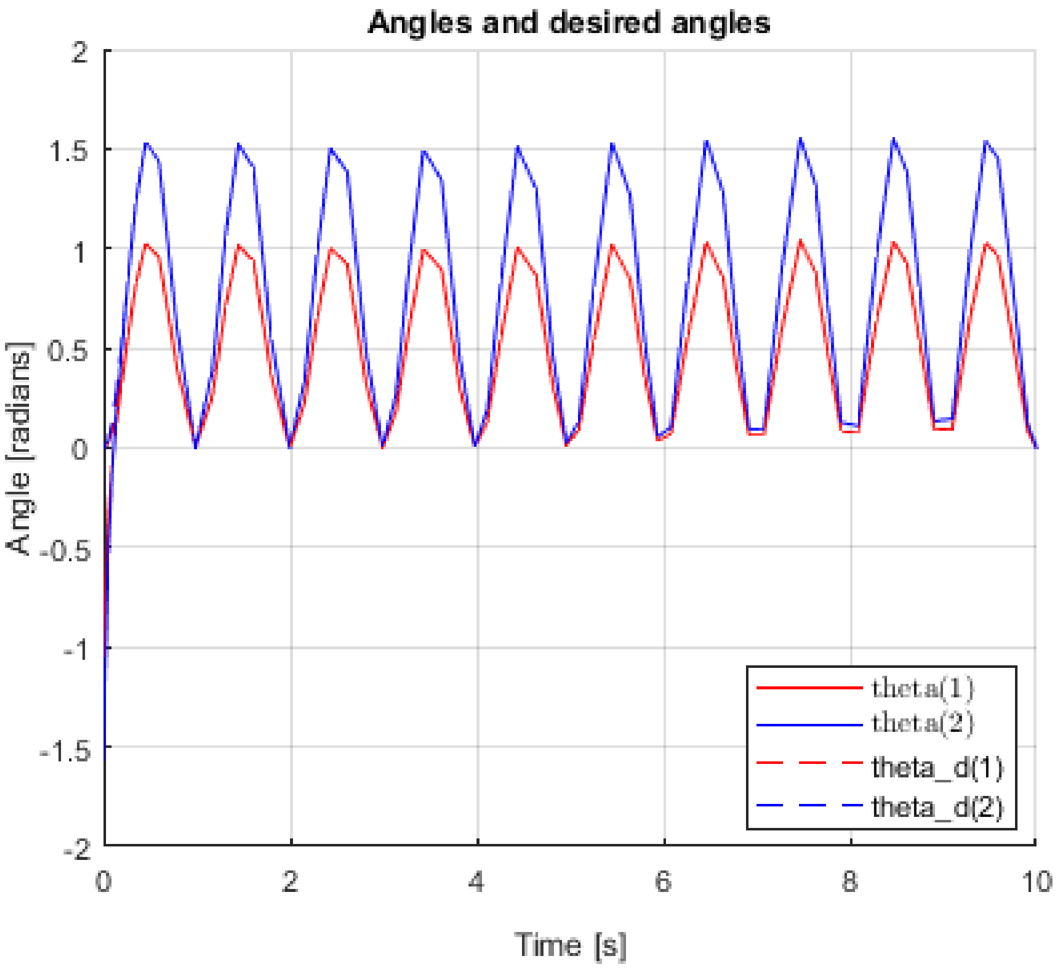


Figure 18: The desired and measured angles

such that $\dot{x}_1 = -x_1$. Then the Lyapunov function candidate $V_1 = \frac{1}{2}x_1^2$ (which is positive definite and radially unbounded) will have $\dot{V}_1 = x_1\dot{x}_1 = -x_1^2$ which is negative definite.
Augment the virtual input with z, such that

$$x_2 = -a - x_1 - \left(x_1 - a^{1/3}\right)^3 + z$$

then

$$z = x_1 + x_2 + a + \left(x_1 - a^{1/3}\right)^3$$

and

$$\begin{aligned}
\dot{z} &= \dot{x}_1 + \dot{x}_2 + 3\left(x_1 - a^{1/3}\right)^2 \dot{x}_1 \\
&= \left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) + (x_1 + u) + 3\left(x_1 - a^{1/3}\right)^2 \left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) \\
&= x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)
\end{aligned}$$

Calculate

$$\dot{V}_1 = x_1\dot{x}_1 = x_1(-x_1 + z) = -x_1^2 + x_1 z$$

We may choose a Lyapunov function candidate for the overall system as $V_c = V_1 + \frac{1}{2}z^2$, then

$$\begin{aligned}
\dot{V}_c &= \dot{V}_1 + z\dot{z} \\
&= -x_1^2 + x_1 z + z\left\{x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)\right\} \\
&= -x_1^2 + z\underbrace{\left\{2x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)\right\}}_{\text{choose} \quad -z} \\
&= -x_1^2 - z^2
\end{aligned}$$

where we have chosen

$$-z = 2x_1 + u + \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)$$

i.e.

$$\begin{aligned}
u &= -z - 2x_1 - \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) \\
&= -\overbrace{\left(x_1 + x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)}^{z} - 2x_1 - \left(1 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right) \\
&= -3x_1 - \left(2 + 3\left(x_1 - a^{1/3}\right)^2\right)\left(x_2 + a + \left(x_1 - a^{1/3}\right)^3\right)
\end{aligned}$$

such that $\dot{V}_c$ is negative definite. We already know that V_c is positive definite and radially unbounded. Hence, the overall system is globally asymptotically stable (GAS).

Alternative solution:
The system is in the form of (14.53)-(14.54) in Khalil with

$$\begin{aligned} f &= a + \left(x_1 - a^{1/3}\right)^3 \\ g &= 1 \\ f_a &= x_1 \\ g_a &= 1 \end{aligned}$$

Take

$$\begin{aligned} \phi(x_1) &= -a - \left(x_1 - a^{1/3}\right)^3 - x_1 \\ V &= \frac{1}{2}x_1^2 \end{aligned}$$

and use (14.56) in Khalil.

5. Consider $\dot{x}_1 = x_1x_2 + x_1^2$ with x_2 as a virtual input. Choose a Lyapunov function candidate $V_1(x) = \frac{1}{2}x_1^2$ and calculate

$$\dot{V}_1 = x_1\dot{x}_1 = x_1\left(x_1x_2 + x_1^2\right)$$

We can enforce $\dot{V}_1 = -x_1^4$ which is negative definite, by choosing the input $x_2 = -x_1 - x_1^2$ (actually, any choice $x_2 = -x_1^{2k} - x_1^2$, $k = 1, 2, 3, \ldots$ will be possible, to get a negative definite $\dot{V}_1$, but for simplification we choose k=1).
Augment the input with z, such that we have $x_2 = -x_1 - x_1^2 + z$ (i.e. $z = x_2 + x_1 + x_1^2$), then

$$\dot{V}_1 = x_1\left(x_1x_2 + x_1^2\right) = x_1\left(x_1\left(-x_1 - x_1^2 + z\right) + x_1^2\right) = -x_1^4 + x_1^2 z$$

Now choose a Lyapunov function candidate for the complete system $V_c = V_1 + \frac{1}{2}z^2$, which is positive definite and radially unbounded. Then

$$\begin{aligned} \dot{V}_c &= \dot{V}_1 + z\dot{z} \\ &= -x_1^4 + x_1 z + z\left(u + (2x_1 + 1)\left(x_1x_2 + x_1^2\right)\right) \\ &= -x_1^4 + z\underbrace{\left(x_1 + u + (2x_1 + 1)\left(x_1x_2 + x_1^2\right)\right)}_{\text{choose} \quad -z} \end{aligned}$$

We can enforce $\dot{V}_c = -x_1^2 - z^2$ (then $\dot{V}_c$ is negative definite), by choosing

$$\begin{aligned} -z &= x_1 + u + (2x_1 + 1)\left(x_1x_2 + x_1^2\right) \\ u &= -x_1 - (2x_1 + 1)\left(x_1x_2 + x_1^2\right) - z \end{aligned}$$

By inserting z, we get the expression for the stabilizing input

$$\begin{aligned} u &= -x_1 - (2x_1 + 1)\left(x_1x_2 + x_1^2\right) - x_2 - x_1 - x_1^2 \\ &= -(2x_1 + 1)\left(x_1x_2 + x_1^2\right) - x_2 - 2x_1 - x_1^2 \end{aligned}$$

Since $V_c(x_1, z)$ is continuously differentiable and positive definite, and $\dot{V}_c(x_1, z)$ is negative definite, u asymptotically stabilizes x_1 and z at the origin. Since $z = 0 \rightarrow x_2 = -x_1 - x_1^2$ and $x_1 = 0 \rightarrow -x_1 - x_1^2 = 0$, this means that also x_2 is asymptotically stabilized at the origin. In addition, since $V_c(x_1, z)$ is radially unbounded and there are no singularities in u, the equilibrium point $x = (0, 0)$ is globally asymptotically stable.