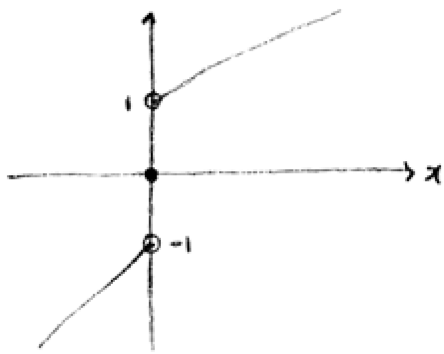
3.1

(2) $f(x) = x + \mathrm{sgn}(x)$



(a) not continuously differentiable at $x=0$

(b) $\|f(x) - f(y)\| = \|x + \mathrm{sgn}(x) - y - \mathrm{sgn}(y)\|$

$$\leqq \|x-y\| + \|\mathrm{sgn}(x) - \mathrm{sgn}(y)\|$$

$$= \|x-y\| + 2$$

$$\leqq L\|x-y\|$$

we can choose L such that $(L-1) \geqq \dfrac{2}{\|x-y\|}$

$\therefore$ Locally Lipschitz on $\mathbb{R}^-$ ($\mathbb{R}^+$)

(c) No

(d) We can't choose constant L

which is independent of x and y.

$\therefore$ No.

3.1

(3) $f(x) = \sin(x)\,\mathrm{sgn}(x)$

(a) $\dfrac{\partial f(x)}{\partial x} = \begin{cases} \cos x & \text{if x} \geq 0 \\ -\cos x & \text{if x} < 0 \end{cases}$

$\therefore$ not continuously differentiable at $x = 0$

(b) $\|f(x) - f(y)\| = \|\sin(x)\,\mathrm{sgn}(x) - \sin(y)\,\mathrm{sgn}(y)\|$

$$= \big\| |\sin(x)| - |\sin(y)| \big\| \qquad \text{(locally)}$$

$$\leq \sigma \|x - y\|$$

$\therefore$ locally Lipschitz

(c) $f(x)$ is continuous

(d) $\dfrac{\partial f(x)}{\partial x}$ is globally uniformly bounded

$\therefore$ globally Lipschitz

(4) $f(x) = -x + a\sin(x)$

$f'(x) = -1 + a\cos(x) \quad \Rightarrow \quad$ continuous & bounded

(a) Yes

(b) Yes

(c) Yes

(d) Yes

3.1

(5) $f(x) = -x + 2|x|$

(a) $$\frac{\partial f(x)}{\partial x} = \begin{cases} 1 & \text{if } x \geq 0 \\ -3 & \text{if } x < 0 \end{cases}$$

$\therefore$ not continuously differentiable at $x = 0$

(b) $\| f(x) - f(y) \| \leq 3\|x - y\|$

$\therefore$ locally Lipschitz

(c) f(x) is continuous

(d) globally Lipschitz

3.6

(a) The solution $x(t)$ is given by

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

$$\|x(t)\| \le \|x_0\| + \int_{t_0}^{t} \|f(s, x(s))\| ds$$

$$\le \|x_0\| + \int_{t_0}^{t} (k_1 + k_2 \|x\|) ds$$

From the Gronwall-Bellman inequality,

$$\|x(t)\| \le \|x_0\| + k_1(t - t_0)$$

$$+ \int_{t_0}^{t} k_2 (\|x_0\| + k_1(s - t_0)) \exp[k_2(t - s)] ds$$

Integrating the righthand side by parts,

$$\|x(t)\| \le \|x_0\| \exp[k_2(t - t_0)] + \frac{k_1}{k_2} \{\exp[k_2(t - t_0)] - 1\}$$

(b) No

3.13
$$\begin{cases} \dot{x}_1 = \tan^{-1}(ax_1) - x_1 x_2 \\ \dot{x}_2 = bx_1^2 - cx_2 \end{cases} \qquad (a_0 = 1,\ b_0 = 0,\ c_0 = 1)$$

- $$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{a}{1+a^2x_1^2} - x_2 & -x_1 \\ 2bx_1 & -c \end{bmatrix} \qquad \frac{\partial f}{\partial \lambda} = \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} & \frac{\partial f}{\partial c} \end{bmatrix} = \begin{bmatrix} \frac{x_1}{1+a^2x_1^2} & 0 & 0 \\ 0 & x_1^2 & -x_2 \end{bmatrix}$$

- $$\left.\frac{\partial f}{\partial x}\right|_{nominal} = \begin{bmatrix} \frac{1}{1+x_1^2} - x_2 & -x_1 \\ 0 & -1 \end{bmatrix} \qquad \left.\frac{\partial f}{\partial \lambda}\right|_{nominal} = \begin{bmatrix} \frac{x_1}{1+x_1^2} & 0 & 0 \\ 0 & x_1^2 & -x_2 \end{bmatrix}$$

- Let $$S = \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} & \frac{\partial x_1}{\partial c} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} & \frac{\partial x_2}{\partial c} \end{bmatrix}\right|_{nominal}$$

$$\dot{S} = \begin{bmatrix} \frac{1}{1+x_1^2} - x_2 & -x_1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix} + \begin{bmatrix} \frac{x_1}{1+x_1^2} & 0 & 0 \\ 0 & x_1^2 & -x_2 \end{bmatrix}, \quad S(t_0) = 0$$

$\therefore$

$$\dot{x}_1 = \tan^{-1}(x_1) - x_1 x_2 \qquad x_1(0) = x_{10}$$

$$\dot{x}_2 = -x_2 \qquad x_2(0) = x_{20}$$

$$\dot{x}_3 = \left(\frac{1}{1+x_1^2} - x_2\right) x_3 - x_1 x_4 + \frac{x_1}{1+x_1^2} \qquad x_3(0) = 0$$

$$\dot{x}_4 = -x_4 \qquad x_4(0) = 0$$

$$\dot{x}_5 = \left(\frac{1}{1+x_1^2} - x_2\right) x_5 - x_1 x_6 \qquad x_5(0) = 0$$

$$\dot{x}_6 = -x_6 + x_1^2 \qquad x_6(0) = 0$$

$$\dot{x}_7 = \left(\frac{1}{1+x_1^2} - x_2\right) x_7 - x_1 x_8 \qquad x_7(0) = 0$$

$$\dot{x}_8 = -x_8 - x_2 \qquad x_8(0) = 0$$

3.18

$$y(t) \le k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)} [k_2 y(\tau) + k_3] d\tau$$

let $z(t) = y(t) e^{\alpha(t-t_0)}$

then

$$z(t) \le k_1 + e^{\alpha(t-t_0)} \int_{t_0}^{t} e^{-\alpha(t-\tau)} \left[k_2 e^{-\alpha(\tau-t_0)} z(\tau) + k_3\right] d\tau$$

$$= \underbrace{k_1 + \frac{k_3}{\alpha}\left(e^{-\alpha(t_0-t)} - 1\right)}_{\lambda(t)} + \int_{t_0}^{t} \underset{\substack{\downarrow \\ \mu(\tau)}}{k_2} z(\tau) d\tau$$

By Gronwall–Bellman inequality,

$$z(t) \le k_1 + \frac{k_3}{\alpha}\left(e^{-\alpha(t_0-t)} - 1\right) + \int_{t_0}^{t} \left[k_1 + \frac{k_3}{\alpha}\left(e^{-\alpha(t_0-\tau)} - 1\right)\right] k_2 \, e^{\int_\tau^t k_2 ds} d\tau$$

$$\vdots$$

$$= k_1 + \frac{k_3}{\alpha}\left(e^{-\alpha(t_0-t)} - 1\right) - \left(k_1 - \frac{k_3}{\alpha}\right)\left(1 - e^{k_2(t-t_0)}\right)$$

$$+ \frac{1}{\alpha - k_2} \frac{k_2 k_3}{\alpha}\left(e^{-\alpha(t_0-t)} - e^{k_2(t-t_0)}\right) \quad (\because \alpha > k_2)$$

In terms of $y(t)$,

$$y(t) \le k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{1}{\alpha-k_2}\left(1 - e^{-(\alpha-k_2)(t-t_0)}\right)$$

3.24

(a) $$V(t,x) = \int_0^1 \frac{\partial V}{\partial x}(t,\sigma x)\,d\sigma\, x$$

$$\le \int_0^1 \left\| \frac{\partial V}{\partial x}(t,\sigma x) \right\| \|x\|\, d\sigma$$

$$\le \int_0^1 c_4 \sigma\, d\sigma\, \|x\|^2 = \frac{1}{2} c_4 \|x\|^2$$

(b) $$c_1\|x\|^2 \le V(t,x) \le \frac{1}{2} c_4 \|x\|^2$$

$$\therefore\ c_1 \le \frac{1}{2} c_4 \rightarrow 2c_1 \le c_4$$

(c) From Lemma 3.1, we obtain

$$\left\| \frac{1}{2\sqrt{V(t,x)}} \frac{\partial V}{\partial x}(t,x) \right\| \le \frac{c_4}{2\sqrt{c_1}}$$

3.27

The following equations are satisfied

$$\dot{x}_1(t) = f(t, x_1) + g(t, x_1)$$

$$\dot{x}_2(t) = f(t, x_2) + g(t, x_2)$$

where $\|g(t, x_1)\| \le \mu_1$, $\|g(t, x_2)\| \le \mu_2$.

Also, the solution $x_1(t)$ and $x_2(t)$ are given by

$$x_1(t) = x_1(a) + \int_a^t f(s, x_1(s))ds + \int_a^t g(s, x_1(s))ds$$

$$x_2(t) = x_2(a) + \int_a^t f(s, x_2(s))ds + \int_a^t g(s, x_2(s))ds$$

$$\therefore \|x_1(t) - x_2(t)\| \le \|x_1(a) - x_2(a)\| + \int_a^t \|f(s, x_1(s)) - f(s, x_2(s))\|ds$$

$$+ \int_a^t \|g(s, x_1(s))\|ds + \int_a^t \|g(s, x_2(s))\|ds$$

$$\le \gamma + (\mu_1 + \mu_2)(t - a) + \int_a^t L\|x_1(s) - x_2(s)\|ds$$

From the Gronwall - Bellman inequality,

$$\|x_1(t) - x_2(t)\| \le \gamma \exp[L(t - a)] + \frac{(\mu_1 + \mu_2)}{L}\{\exp[L(t - a)] - 1\}$$