

TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2014 - Assignment 2
Due date: Monday 22 September at 12.00.

1. Consider again the Duckmaze system from Assignment 1 (Exercise 4) given by

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = -\frac{f_3}{m}x_1^3 - \frac{f_1}{m}x_1 - \frac{d}{m}x_2 - g + \frac{u}{m} \tag{2}$$

(Note: It is possible to do this exercise even if you did not do Assignment 1)

- (a) Is $(0, 0)$ an equilibrium point for $u = 0$?

Since it is desirable to control the position to a desired position x_{1d} , the equilibrium point of (1)–(2) should be placed at $(x_{1d}, 0)$. What conditions must be satisfied for x_1^* and x_2^* and a constant input $u = u_0$ for $(x_{1d}, 0)$ to be an equilibrium point? Find u_0 .

- (b) Split the states and input into stationary values (x_0, u_0) and difference terms (\tilde{x}, \tilde{u}) . You will have $x = x_0 + \tilde{x}$ and $u = u_0 + \tilde{u}$.

Derive the equations for \tilde{x} with \tilde{u} as an input. What is the equilibrium point for $\tilde{u} = 0$?

- (c) Calculate the Jacobian of the system and denote this A . Is A Hurwitz or not? What does this mean related to the stability of the equilibrium point?

2. (a) Consider

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_2^2 \\ \dot{x}_2 &= 2x_1x_2 \end{aligned}$$

Construct the phase portrait of the system. Is the origin stable? Provide your argument with respect to Definition 4.1. on page 112 of Khalil (qualitative argument is enough).

- (b) Use Definition 4.1. (on page 112 of Khalil) to show that the origin of the following system

$$\dot{x} = \alpha x$$

is asymptotically stable for $\alpha < 0$. (Note: in addition to convergence you also have to show quantitatively that for any given ε you could obtain a δ which depends on ε).

3. For the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Comment also on the possibility of a global result. (Hint: see Appendix, at the last page of this assignment)

- (a) The scalar system

$$\dot{x} = -x^3$$

- (b) The system

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1 - x_2^3\end{aligned}$$

- (c) The system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2\end{aligned}$$

(Note: To comment on the possibility of a global result, use simulations.)

- (d) The system

$$\begin{aligned}\dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1)\end{aligned}$$

4. Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1^2 x_2 - 2x_1 x_2 + x_1^2 + 2x_1 \\ \dot{x}_2 &= x_1^3 + 2x_1^2 + x_1^2 x_2 + 2x_1 x_2\end{aligned}$$

Do a change of variables

$$\begin{aligned}z_1 &= x_1 - x_1^* \\ z_2 &= x_2 - x_2^*\end{aligned}$$

where $(x_1^*, x_2^*) = (-1, 1)$ to shift the equilibrium point to the origin.

By using a quadratic Lyapunov function candidate, show that the equilibrium point is asymptotically stable.

Hint I: The resultant system will be of the form $\dot{z} = f(z)(1 - z_1^2)$

Hint II: Remember that close to the origin, the higher order terms will be dominated by lower order terms.

5. Exercise 4.6 in Khalil.

6. Consider again

$$\begin{aligned}\dot{x}_1 &= x_2 + \alpha x_1 (\beta^2 - x_1^2 - x_2^2) \\ \dot{x}_2 &= -x_1 + \alpha x_2 (\beta^2 - x_1^2 - x_2^2)\end{aligned}$$

where $\alpha, \beta > 0$ are constants. Using Chetaev's theorem, show that the origin is unstable!

Hint: $V = 0.5(x_1^2 + x_2^2)$.

7. Consider the system in Figure 1 where the nonlinear function is given by $g(e) = e^3$.

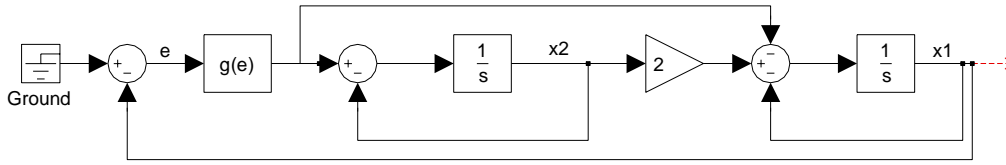


Figure 1: Block diagram of the system

- Find the state space model.
- Show that the origin is asymptotically stable using the Lyapunov function

$$V(x) = x^T P x$$

where

$$P = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

- Sketch an estimate of the region of attraction in the (x_1, x_2) -plane.

8. Consider the system

$$\begin{aligned} \dot{x}_1 &= 4x_1^2 x_2 - f_1(x_1)(x_1^2 + 2x_2^2 - 4) \\ \dot{x}_2 &= -2x_1^3 - f_2(x_2)(x_1^2 + 2x_2^2 - 4) \end{aligned}$$

where the continuous functions f_1 and f_2 have the same sign as their arguments, i.e.

$$\begin{aligned} x_1 f_1(x_1) &> 0 \quad \text{for } x_1 \neq 0 \\ x_2 f_2(x_2) &> 0 \quad \text{for } x_2 \neq 0 \\ f_1(0) &= f_2(0) = 0 \end{aligned}$$

Show that $\{x \in \mathbb{R}^2 | x_1^2 + 2x_2^2 - 4 = 0\}$ and $(x_1, x_2) = (0, 0)$ are invariant sets, and that every trajectory approaches the sets when $t \rightarrow \infty$. Why do you think that the set $\{x \in \mathbb{R}^2 | x_1^2 + 2x_2^2 - 4 = 0\}$ is not a limit cycle?

Hint: Apply Theorem 4.4 on page 128 of Khalil and use $V(x) = (x_1^2 + 2x_2^2 - 4)^2$.

9. Using $V(x) = \alpha x_1^2 + x_2^2$ where $\alpha > 0$ show that the origin of the following system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \alpha x_1 - (x_1 + x_2)^2 x_2 \end{aligned}$$

is globally asymptotically stable!

Appendix

A symmetric matrix $P = P^T$ is positive definite if:

- All eigenvalues of P are greater than zero.

or

- All leading principal minors of P are greater than zero.

Definition: Leading principal minors

Given an $N \times N$ matrix A , a leading principal submatrix of A is a submatrix formed by deleting all but the first n rows and columns. A leading principal minor is the determinant of a leading principal submatrix. Thus, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then the leading principal minors are

$$|a_{11}|, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$