TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2014 - Solution to Assignment 1

1. (a)

$$\begin{aligned}
\|x_1\|_1 &= |-1| + |7| = 8 \\
\|x_1\|_2 &= \sqrt{(-1)^2 + 7^2} = \sqrt{50} \\
\|x_1\|_\infty &= \max(|-1|, |7|) = 7 \\
\|x_2\|_1 &= |1| + |-9| + |3| = 13 \\
\|x_2\|_2 &= \sqrt{1^2 + (-9)^2 + 3^2} = \sqrt{91} \\
\|x_2\|_\infty &= \max(|1|, |-9|, |3|) = 9
\end{aligned}$$

(b) $\|\cdot\|_2 > \|\cdot\|_1$ is not possible for any 2D vector. See Fig. 1 - the $\|\cdot\|_2$-norm represents the hypotenuse while the $\|\cdot\|_1$-norm represents the two other sides and will always be longer (Additional note: The $\|\cdot\|_\infty$-norm represents the longest of the two sides that form the $\|\cdot\|_1$-norm).

For any 2D vector of the forms $\begin{bmatrix} a \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ b \end{bmatrix}$, we have $\|\cdot\|_1 = \|\cdot\|_2 = \|\cdot\|_\infty$

See Fig. 1 - when one of the sides is 0, the other side (the $\|\cdot\|_1$-norm) will be equal to the hypotenuse (the $\|\cdot\|_2$-norm).

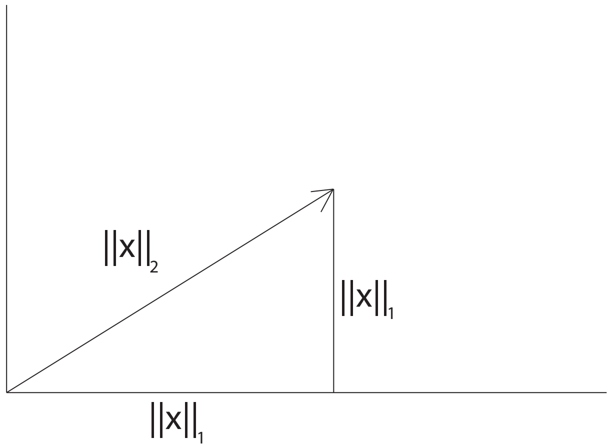

Figure 1: Illustration of 2-norm vs 1-norm.

2. (a)

$$\begin{aligned}
\|A\|_1 &= \max_{1\le j\le n}\sum_{i=1}^{m}|a_{ij}| = \max(3,4,3) = 4\\
\|A\|_\infty &= \max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}| = \max(3,4,3) = 4\\
A^TA &= \begin{bmatrix} 5 & -4 & 1\\ -4 & 6 & -4\\ 1 & -4 & 5\end{bmatrix}\\
\left|\lambda I - A^TA\right| &= \lambda^3 - 16\lambda^2 + 52\lambda - 16 = (\lambda-4)(\lambda^2-12\lambda+4) = 0\\
&\Leftrightarrow \lambda_1 = 4, \lambda_{2,3} = 6\pm 4\sqrt{2}\\
\|A\|_2 &= \left[\lambda_{\max}\left(A^TA\right)\right]^{\frac{1}{2}} = \sqrt{6+4\sqrt{2}} \approx 3.41
\end{aligned}$$

(b)

$$\begin{aligned}
\|A\|_2 &= \sqrt{6+4\sqrt{2}} \quad \|x\|_2 = \sqrt{91} \quad Ax = \begin{bmatrix} 7\\ -14\\ 3\end{bmatrix} \Leftrightarrow \|Ax\|_2 = \sqrt{254}\\
\|Ax\|_2 &\approx 15.94 \quad \le \quad \|A\|_2\,\|x\|_2 \approx 32.57 \quad \blacksquare
\end{aligned}$$

3. For all $x = [x_1, x_2]^T \in \mathbb{R}^2$ and $y = [y_1, y_2]^T \in \mathbb{R}^2$ we have

$$\begin{gathered}
|\langle x, y\rangle| \le \|x\|_2\,\|y\|_2\\
\Leftrightarrow |x_1y_1 + x_2y_2|^2 \le \left(x_1^2+x_2^2\right)\left(y_1^2+y_2^2\right)\\
\Leftrightarrow x_1^2y_1^2 + x_2^2y_2^2 + 2x_1x_2y_1y_2 \le x_1^2y_1^2 + x_2^2y_1^2 + x_1^2y_2^2 + x_2^2y_2^2\\
\Leftrightarrow 0 \le x_2^2y_1^2 + x_1^2y_2^2 - 2x_1x_2y_1y_2\\
\Leftrightarrow 0 \le (x_2y_1 - x_1y_2)^2
\end{gathered}$$

4. (a) To find the linearized model, the Jacobian of the system is calculated at the point $(x_1^*, x_2^*) = (0, 0)$.

$$J = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1\\ -\frac{f_1}{m} & -\frac{d}{m}\end{bmatrix} \tag{1}$$

This gives the linear system

$$\begin{bmatrix} \Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & 1\\ -\frac{f_1}{m} & -\frac{d}{m}\end{bmatrix}\begin{bmatrix} \Delta x_1\\ \Delta x_2\end{bmatrix} + \begin{bmatrix} 0\\ \frac{1}{m}\end{bmatrix}\Delta u \tag{2}$$

where

$$\Delta x_1 = x_1 - x_1^* = x_1 \tag{3}$$

$$\Delta x_2 = x_2 - x_2^* = x_2 \tag{4}$$

$$\Delta u = u - u^* \tag{5}$$

and u^* is the input force in the linearization point. In the following text the delta symbols will be omitted for convenience, i.e. the linear model will be written as

$$\begin{bmatrix} \dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & 1\\ -\frac{f_1}{m} & -\frac{d}{m}\end{bmatrix}\begin{bmatrix} x_1\\ x_2\end{bmatrix} + \begin{bmatrix} 0\\ \frac{1}{m}\end{bmatrix}u \tag{6}$$

(b) (6) can be written as

$$\dot{x} = Ax + bu \tag{7}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{f_1}{m} & -\frac{d}{m} \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \qquad x^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \tag{8}$$

The controllability matrix is given by

$$C = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & -\frac{d}{m^2} \end{bmatrix} \tag{9}$$

Since $rank(C) = 2$ the system is controllable. This means that the poles of the system can be placed arbitrarily.

(c) The model is augmented with an additional state x_3 to include integral action. This gives

$$\dot{x}_1 = x_2 \tag{10}$$

$$\dot{x}_2 = -\frac{f_1}{m}x_1 - \frac{d}{m}x_2 + \frac{u}{m} \tag{11}$$

$$\dot{x}_3 = x_1 - x_{1d} \tag{12}$$

where x_{1d} is the desired position. Having integral action is important to achieve a perfect static performance when modeling errors and constant disturbances are present.

(d) A linear model from the previous question is written in matrix form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{f_1}{m} & -\frac{d}{m} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} x_{1d} \tag{13}$$

Define

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{f_1}{m} & -\frac{d}{m} & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} -1.0 \\ -2.0 \\ -3.0 \end{bmatrix} \tag{14}$$

where P is a vector containing the poles of the system. By using Matlab's `place` command the gains can be calculated.

$$K = \texttt{place}(A, B, P) \tag{15}$$

returns the controller gains

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} 10.0 & 5.0 & 6.0 \end{bmatrix} \tag{16}$$

(e) The solution of the system equations is given by

$$x_1(t) = (1 - e^{-3t} + 3e^{-2t} - 3e^{-t})x_{1d} \tag{17}$$

Then

$$x_1(t) - x_{1d} = (-e^{-3t} + 3e^{-2t} - 3e^{-t})x_{1d} \tag{18}$$

Further,

$$
\begin{aligned}
|x_1(t) - x_{1d}| &= \left|(-e^{-3t} + 3e^{-2t} - 3e^{-t})x_{1d}\right| && (19)\\
&\leq \left(\left|-e^{-3t}\right| + \left|3e^{-2t}\right| + \left|-3e^{-t}\right|\right)|x_{1d}| && (20)\\
&\leq \left(e^{-t} + 3e^{-t} + 3e^{-t}\right)|x_{1d}| && (21)\\
&\leq (1+3+3)|x_{1d}|\, e^{-t} && (22)\\
&\leq 7\,|x_{1d}|\, e^{-t} && (23)
\end{aligned}
$$

where the fact that

$$|a+b| \leq |a| + |b| \tag{24}$$

and

$$e^{-\alpha t} \leq e^{-t} \quad \forall \quad \alpha \geq 1 \tag{25}$$

are applied. This means that

$$k = 7\,|x_{1d}| \text{ and } \lambda = 1 \tag{26}$$

5. (a) If $f(x)$ is globally Lipschitz, the following inequality should be true for any x and y:

$$\frac{|f(x) - f(y)|}{|x-y|} \leq L$$

If we set $y = 0$ and let $x \to y$, i.e. $x \to 0$, we will see that

$$\lim_{y=0, x\to 0} \frac{|f(x) - f(y)|}{|x-y|} = f'(0) = \infty \nleq L$$

The conclusion is that $f(x)$ is not globally Lipschitz.

(b) $f(x)$ is locally Lipschitz for the area $\mathcal{D} = \{\mathbb{R}|x \neq 0\}$. See Lemma 3.2 in Khalil.

6. (a) The pendulum equation with friction and constant input torque (Section 1.2.1) is given by

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 + \frac{1}{ml^2}T \end{bmatrix} \tag{27}$$

The partial derivative of $f(x)$ with respect to x is found as

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & -\frac{k}{m} \end{bmatrix} \tag{28}$$

From (27) and (28) it can be seen that $f(x)$ and $\frac{\partial f(x)}{\partial x}$ are continuous in x on $\mathbb{R}^2$. Using Lemma 3.1 or Lemma 3.2 it can be concluded that f is locally Lipschitz in x the whole state space. Further it can be seen that $\frac{\partial f(x)}{\partial x}$ is bounded on $\mathbb{R}^2$ ($\cos(x_1) \leq 1$ $\forall$ x_1), by which Lemma 3.3 concludes that f is globally Lipschitz.
By Theorem 3.2 in Khalil, we then know that there is global existence and uniqueness of solutions.

(b) The mass-spring equation with linear spring, linear viscous damping, Coulomb friction and zero external force (Section 1.2.3) is given by

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{1}{m}\eta(x_1, x_2) \end{bmatrix}$$

where $\eta(x_1, x_2)$ is discontinuous at $x_2 = 0$. This discontinuity implies that f is not locally Lipschitz at $x_2 = 0$ (any discontinuous function is not locally Lipschitz at the point of discontinuity). Thus f is also not globally Lipschitz, and due to Theorem 3.1 and 3.2 in Khalil we can not conclude about local or global existence and uniqueness of solutions.

(c) The Van der Pol oscillator (Example 2.6) is given by

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 - \varepsilon\left(1 - x_1^2\right) x_2 \end{bmatrix} \tag{29}$$

The partial derivative of $f(x)$ with respect to x is found as

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & -\varepsilon\left(1 - x_1^2\right) \end{bmatrix} \tag{30}$$

From (29) and (30) it can be seen that $f(x)$ and $\frac{\partial f(x)}{\partial x}$ are continuous in x on $\mathbb{R}^2$. Using Lemma 3.1 or Lemma 3.2 it can be concluded that f is locally Lipschitz in x on the whole state space. Further it can be seen that $\frac{\partial f(x)}{\partial x}$ is not globally bounded. It follows from Lemma 3.3 that f is not globally Lipschitz.
By Theorem 3.1 we conclude that there is local existence and uniqueness of solutions.

7. (Khalil 1.4)

Choose the state variables to be $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ and we get

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -M(x_1)^{-1}(C(x_1, x_2) + D)x_2 - M(x_1)^{-1}g(x_1) + M(x_1)^{-1}u \end{cases}$$

Set $f(x_1, x_2) = -M(x_1)^{-1}(C(x_1, x_2) + D)x_2 - M(x_1)^{-1}g(x_1)$ and $P(x_1) = M(x_1)^{-1}$ for simplification, then the state equations will be

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) + P(x_1)u \end{cases}$$

8. By transformation

$$\begin{aligned} r &= \pm\sqrt{x_1^2 + x_2^2} \\ \theta &= \tan^{-1}\frac{x_2}{x_1} \end{aligned}$$

we have

$$\begin{aligned} \dot{r} &= \pm\frac{1}{2}\frac{2x_1\dot{x}_1 + 2x_2\dot{x}_2}{\sqrt{x_1^2 + x_2^2}} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r} \\ &= \frac{x_1\left[x_2 + \alpha x_1\left(\beta^2 - x_1^2 - x_2^2\right)\right] + x_2\left[-x_1 + \alpha x_2\left(\beta^2 - x_1^2 - x_2^2\right)\right]}{r} \\ &= \frac{\alpha\left(x_1^2 + x_2^2\right)\left(\beta^2 - x_1^2 - x_2^2\right)}{r} = \frac{\alpha r^2\left(\beta^2 - r^2\right)}{r} \\ &= \alpha r\left(\beta^2 - r^2\right) \end{aligned}$$

$$
\begin{aligned}
\dot{\theta} &= \frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\left[\frac{\dot{x}_2 x_1 - x_2\dot{x}_1}{x_1^2}\right] = \frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\left[\frac{\dot{x}_2 x_1 - x_2\dot{x}_1}{x_1^2}\right] \\
&= \frac{x_1^2}{x_1^2+x_2^2}\left[\frac{\left[-x_1+\alpha x_2\left(\beta^2-x_1^2-x_2^2\right)\right]x_1 - x_2\left[x_2+\alpha x_1\left(\beta^2-x_1^2-x_2^2\right)\right]}{x_1^2}\right] \\
&= \frac{-x_1^2-x_2^2}{x_1^2+x_2^2} = -1
\end{aligned}
$$

Thus the transformed system is

$$
\begin{cases} \dot{r} = \alpha r\left(\beta^2 - r^2\right) \\ \dot{\theta} = -1 \end{cases}
$$

(Extra note:) By setting $r = \beta$ we have $\dot{r} = 0$ which implies that whenever $r(t_1) = \beta$ it follows that $r(t) = \beta$ for all $t > t_1$. Thus $x_1^2(t) + x_2^2(t) = \beta^2$ is a periodic orbit which only depends on β and is independent of α.

The new system equations can also be found using the expression given in the assignment:

$$
\begin{aligned}
z &= \begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} \sqrt{x_1^2+x_2^2} \\ \tan^{-1}(x_2/x_1) \end{bmatrix} = \psi(x) \\
x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} = \psi^{-1}(z)
\end{aligned}
$$

The partial derivative of $\psi(x)$ is given as

$$
\frac{\partial\psi}{\partial x}(x) = \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{-x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{bmatrix}
$$

which leads to

$$
\frac{\partial\psi}{\partial x}\left(\psi^{-1}(z)\right) = \begin{bmatrix} \frac{r\cos(\theta)}{\sqrt{r^2\cos^2(\theta)+r^2\sin^2(\theta)}} & \frac{r\sin(\theta)}{\sqrt{r^2\cos^2(\theta)+r^2\sin^2(\theta)}} \\ \frac{-r\sin(\theta)}{r^2\cos^2(\theta)+r^2\sin^2(\theta)} & \frac{r\cos(\theta)}{r^2\cos^2(\theta)+r^2\sin^2(\theta)} \end{bmatrix} = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}
$$

and

$$
\begin{aligned}
\dot{z} &= \frac{\partial\psi}{\partial x}\left(\psi^{-1}(z)\right) f\left(\psi^{-1}(z)\right) \\
&= \frac{1}{r}\begin{bmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} r\sin(\theta) + \alpha r\cos(\theta)\left(\beta^2-r^2\right) \\ -r\cos(\theta) + \alpha r\sin(\theta)\left(\beta^2-r^2\right) \end{bmatrix} = \begin{bmatrix} \alpha r\left(\beta^2-r^2\right) \\ -1 \end{bmatrix}
\end{aligned}
$$

9. (a) $\dot{x} = \alpha x \;\rightarrow\; x(t) = C_1 e^{\alpha t}$, where C_1 is a constant. The rabbit population will increase exponentially.

   (b) $\dot{y} = -\gamma y \;\rightarrow\; y(t) = -C_2 e^{\gamma t}$, where C_2 is a constant. The fox population will decrease exponentially.

   (c) Set

$$
\begin{aligned}
x(\alpha - \beta y) &= 0 \\
-y(\gamma - \delta x) &= 0
\end{aligned}
$$

This system has two isolated equilibriums: at $(0,0)$ (which means an extinction of both species) and at $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$. The last solution represents a fixed point at which both populations sustain their current, non-zero numbers, and, in the simplified model, do so indefinitely. The levels of population at which this equilibrium is achieved depend on the chosen values of the parameters.
We may calculate the Jacobian:

$$\begin{aligned} A &= \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix} \end{aligned}$$

In the equilibrium points:

$$\begin{aligned} A\big|_{(x,y)=(0,0)} &= \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix} \\ \left|\lambda I - A\big|_{(x,y)=(0,0)}\right| &= (\lambda - \alpha)(\lambda + \gamma) = 0 \\ &\Rightarrow \quad \lambda_1 = \alpha, \lambda_2 = -\gamma \\ A\big|_{(x,y)=(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})} &= \begin{bmatrix} \alpha - \beta\frac{\alpha}{\beta} & -\beta\frac{\gamma}{\delta} \\ \delta\frac{\alpha}{\beta} & \delta\frac{\gamma}{\delta} - \gamma \end{bmatrix} = \begin{bmatrix} 0 & -\beta\frac{\gamma}{\delta} \\ \delta\frac{\alpha}{\beta} & 0 \end{bmatrix} \\ \left|\lambda I - A\big|_{(x,y)=(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})}\right| &= \lambda^2 + \delta\frac{\alpha}{\beta}\beta\frac{\gamma}{\delta} = \lambda^2 + \gamma\alpha = 0 \\ &\Rightarrow \quad \lambda_{1,2} = \pm i\sqrt{\gamma\alpha} \end{aligned}$$

From the eigenvalues, we see that $(0,0)$ is a saddle point and $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$ is a center point.

(d) A linear system can not have several isolated equilibrium points, because for linear systems of the form $\dot{x} = Ax + b$ there are only the following options:

1. if $|A| \neq 0$, then there is one unique isolated equilibrium point, which is the solution of the equation $Ax + b = 0$
2. if $|A| = 0$, then the equation $Ax+b = 0$ may have either no solutions (in this case there are no equilibrium points), or it may have a whole subspace of solutions (i.e. the equilibrium points are not isolated).

(e) See the phase portrait in Figure 2. A center point is lying in $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}) \approx (76.2, 12.6)$, and a saddle point is lying in the origin, both as expected.

(f) No, it is not possible to cross the lines $x = 0$ and $y = 0$, because they are invariant sets. If the states end up in these sets they will stay there forever, as we discussed in exercise (a)–(b).

(g) Bendixson's criterion states that the system has no periodic orbits lying entirely in a simply connected region $\mathcal{D}$ if $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$ is not identically zero and does not change sign in $\mathcal{D}$. It follows that such sets are given by

$$\begin{aligned} \mathcal{D}_1 &= \left\{\mathbb{R}^2 | \alpha - \beta y - (\gamma - \delta x) > 0\right\} \\ \mathcal{D}_2 &= \left\{\mathbb{R}^2 | \alpha - \beta y - (\gamma - \delta x) < 0\right\} \end{aligned}$$

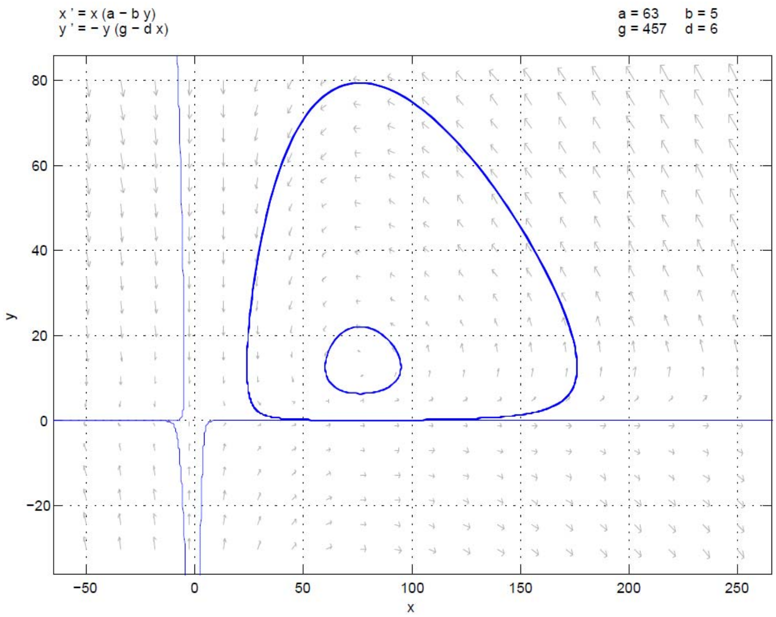

Figure 2: Phase portrait.

10. (a)

$$\begin{aligned} f_1(x_1, x_2) &= ax_1 - x_1 x_2 \\ f_2(x_1, x_2) &= bx_1^2 - cx_2 \\ A &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} a - x_2 & -x_1 \\ 2bx_1 & -c \end{bmatrix} \end{aligned}$$

(b)

$$\begin{aligned} ax_1 - x_1 x_2 &= 0 \\ bx_1^2 - cx_2 &= 0 \end{aligned}$$

The first equation gives equilibrium points for $x_1 = 0$ and $x_2 = a$. Insertion of $x_1 = 0$ into the second equation gives the point $(0, 0)$ and insertion of $x_2 = a$ gives $x_1 = \pm\sqrt{\frac{ac}{b}}$. In total we have three equilibrium points $(0, 0)$, $(\sqrt{\frac{ac}{b}}, a)$ and $(-\sqrt{\frac{ac}{b}}, a)$. To find the type of point, we calculate the Jacobian in each of these points.

$$\begin{aligned} A|_{x=(0,0)} &= \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \\ A|_{x=(\pm\sqrt{\frac{ac}{b}}, a)} &= \begin{bmatrix} 0 & -\sqrt{\frac{ac}{b}} \\ 2\sqrt{abc} & -c \end{bmatrix} \end{aligned}$$

For the point $(0, 0)$ the eigenvalues are $\lambda_1 = a > 0$ and $\lambda_2 = -c < 0$ thus it is a saddle point.
For the points $(\pm\sqrt{\frac{ac}{b}}, a)$ the eigenvalues are

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 8ac}}{2}$$

thus $(\pm\sqrt{\frac{ac}{b}}, a)$ will be stable nodes if $8a < c$ and stable foci if $8a > c$.

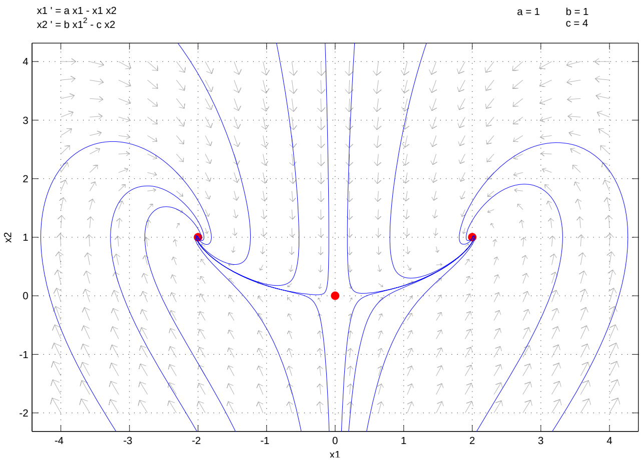

Figure 3: Phase portrait for $a = 1, \quad b = 1, \quad c = 4$.

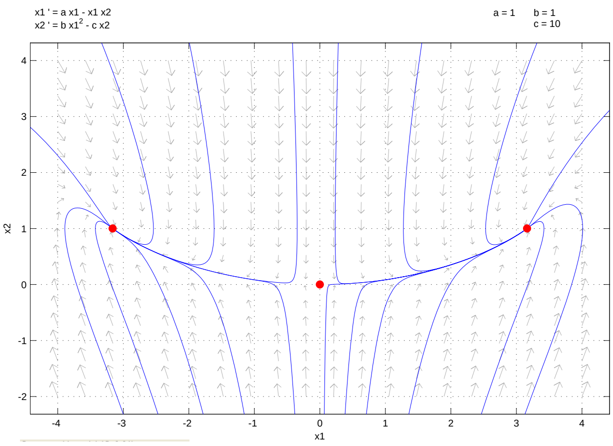

Figure 4: Phase portrait for $a = 1, \quad b = 1, \quad c = 10$.

(c) See phase portaits in Figures 3–4. Wee see that $(0, 0)$ is a saddle point (as expected) and that $(\pm\sqrt{\frac{ac}{b}}, a)$ are stable nodes or foci depending on the values a and c.