TTK4150 Nonlinear Control Systems
Department of Engineering Cybernetics
Norwegian University of Science and Technology
Fall 2014 - Assignment 5
Due date: Monday 10 November at 12.00.

1. Consider a PID controller

$$h(s) = K_p \beta \frac{(1 + T_i s)(1 + T_d s)}{(1 + \beta T_i s)(1 + \alpha T_d s)} \tag{1}$$

as a system with $K_p = 1$, $T_d = 1$, $T_i = 2$, $\beta = 1.5$ and $\alpha = 0.5$.

(a) Show that

$$|h(j\omega)| \leq \frac{K_p \beta}{\alpha} \ \forall \omega \tag{2}$$

(b) Show that

$$\mathrm{Re}\left[h(j\omega)\right] \geq K_p \ \forall \omega \tag{3}$$

For the rest of the exercise, assume that the conditions 2–3 hold for all cases where $K_p > 0$, $0 \leq T_d < T_i$, $1 \leq \beta < \infty$ and $0 < \alpha \leq 1$.

(c) Show that the system is passive (Hint: See Appendix A).

(d) Show that the system is input strictly passive (Hint: See Appendix A).

(e) Show that the system is output strictly passive (Hint: See Appendix A).

(f) Show that the system is zero-state observable.

2. Exercise 6.11 in Khalil (Hint: Use $V(\omega) = \frac{1}{2} J_1 \omega_1^2 + \frac{1}{2} J_2 \omega_2^2 + \frac{1}{2} J_3 \omega_3^2$).

3. Exercise 6.14 in Khalil (Hint: Use $V_1(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right)$ and $V_2(x_3) = \int_0^{x_3} h_2(z)\, dz$).

4. Exercise 6.15 in Khalil (Hint: Use $V_1(x_1, x_2) = \frac{1}{4} x_1^4 + \frac{1}{2} x_2^2$ and $V_2(x_3) = \frac{1}{4} x_3^4$).

5. Exercise 14.43 in Khalil (Hint: Theorem 14.4 in Khalil).

6. In each of the following cases verify the describing function (See Appendix B).
Hint:

$$\int_0^{\pi/2} \sin^u du = \begin{cases} \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} \cdot \frac{\pi}{2} & \text{if } n \geq 2 \text{ and is an even integer} \\ \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{3 \cdot 5 \cdot 7 \cdots n} & \text{if } n \geq 3 \text{ and is an odd integer} \end{cases} \tag{4}$$

(a) Let

$$\psi(y) = y^5$$

then

$$\Psi(a) = \frac{5a^4}{8}$$

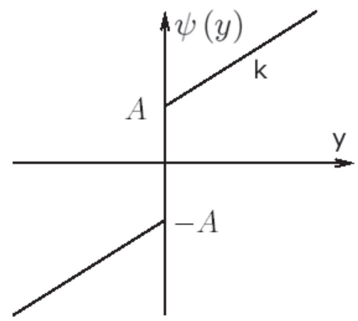

Figure 1: Nonlinearity

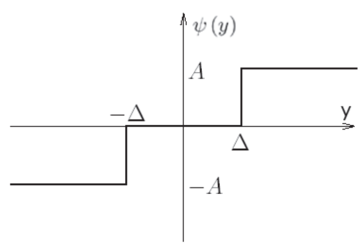

Figure 2: Nonlinearity

(b) Let

$$\psi(y) = y^3 |y|$$

then

$$\Psi(a) = \frac{32a^3}{15\pi}$$

(c) Let the nonlinearity be given by Figure 1, then

$$\Psi(a) = k + \frac{4A}{a\pi}$$

(d) Let the nonlinearity be given by Figure 2 then

$$\Psi(a) = \begin{cases} 0 & \text{when } a \le \Delta \\ \frac{4A}{a\pi}\sqrt{1 - \left(\frac{\Delta}{a}\right)^2} & \text{when } a > \Delta \end{cases}$$

7. Consider the system in Figure 3 where

$$\begin{aligned} h(s) &= \frac{1-s}{s(s+1)} \\ \psi(z) &= z^5 \end{aligned}$$

(a) Justify the use of the describing function method on this system.

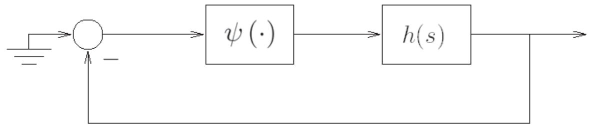

Figure 3: Closed loop system

(b) Use analytic methods to investigate possible periodic solutions. If such a solution exists, estimate its frequency and amplitude.

(c) Use graphical methods to investigate possible periodic solutions. If such a solution exists, estimate its frequency and amplitude.

Appendix A: Passivity

A linear system given by the scalar transfer function $h(s)$ such that all poles p_i satisfy $\mathrm{Re}\left[p_i\right] \leq 0$ is

- passive if $\mathrm{Re}\left[h(j\omega)\right] \geq 0 \quad \forall\omega$ such that $j\omega$ is not a pole.
- input strictly passive if $\mathrm{Re}\left[h(j\omega)\right] \geq \delta > 0 \quad \forall\omega$ such that $j\omega$ is not a pole.
- output strictly passive if $\mathrm{Re}\left[h(j\omega)\right] \geq \epsilon \left|h(j\omega)\right|^2 > 0 \quad \forall\omega$ such that $j\omega$ is not a pole, and for some positive ϵ.

Appendix B: Describing functions

The function $\Psi\left(a\right)$ is called *the describing function* of the nonlinearity ψ, and it is used to find an approximation of this nonlinearity.
Let $\Psi\left(a,\omega\right)$ be the describing function of the nonlinearity $\psi\left(\cdot\right)$. Further, let

$$\begin{aligned} z\left(t\right) &= \psi\left(y\left(t\right)\right) \\ y\left(t\right) &= a\sin\left(\theta\right) \\ \theta &= \omega t \end{aligned}$$

then an approximation of $z\left(t\right)$ is given by

$$z\left(t\right) \approx z_0 + z_1 \sin\left(\theta + \varphi\right)$$

where only the first order Fourier coefficients have been used in the approximation. The various parameters are given by

- $z_0 = \frac{1}{2\pi}\int_0^{2\pi} \psi\left(a\sin\left(\theta\right)\right) d\theta$
- $z_1 = \sqrt{z_{1s}^2 + z_{1c}^2}$
  - $z_{1s} = \frac{1}{\pi}\int_0^{2\pi} \psi\left(a\sin\left(\theta\right)\right)\sin\left(\theta\right) d\theta$
  - $z_{1c} = \frac{1}{\pi}\int_0^{2\pi} \psi\left(a\sin\left(\theta\right)\right)\cos\left(\theta\right) d\theta$
- $\varphi = \arctan\left(\frac{z_{1c}}{z_{1s}}\right)$

In the case of odd, time-invariant, memoryless nonlinearities $\psi\left(\cdot\right)$, the describing function is given as

$$\Psi\left(a\right) = \frac{2}{\pi a}\int_0^{\pi} \psi\left(a\sin\left(\theta\right)\right)\sin\left(\theta\right) d\theta \tag{5}$$