Univ Gustave Eiffel - Cosys / Grettia

Reinforcement Learning and Optimal Control - Master 2 SIA

Dynamic Programming

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Dynamic programming

- Solves optimal control problems :
  - calculates optimal policies,
  - given a perfect model of the environment (such as a MDP).
- Drawbacks of DP w.r.t. RL :
  - Assumption of a perfect model.
  - Great computational expenses (computation time & space memory).
- RL can be seen as approximation of Dynamic programming.
- Environment for DP : finite MDP :
  - finite sets $\mathcal{S}, \mathcal{A}, \mathcal{R}$ for states, actions and rewards.
  - the dynamics are given by $p(s', r \mid s, a)$.
- Main idea : Use the value function :

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right], \text{ for all } s \in \mathcal{S},$$

or the action-value function :

$$q_\pi(s, a) \doteq \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s, A_t = a\right].$$

Dynamic programming

- Bellman equation (state value function) :

$$
\begin{aligned}
v_*(s) &= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \,|\, s, a) \Big[r + \gamma v_*(s') \Big],
\end{aligned}
$$

- Bellman equation (state-action value function) :

$$
\begin{aligned}
q_*(s, a) &= \mathbb{E}\Big[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') \;\Big|\; S_t = s, A_t = a \Big] \\
&= \sum_{s',r} p(s', r \,|\, s, a) \Big[r + \gamma \max_{a'} q_*(s', a') \Big],
\end{aligned}
$$

Dynamic programming

Four algorithms

1. Policy evaluation (Prediction)
2. Policy improvement
3. Policy iteration & GPI
4. Value iteration

1 - Policy Evaluation (Prediction)

- Recall the Bellman equation :

$$\begin{aligned} v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \mid S_t = s] \\ &= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} \mid S_t = s] \\ &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\ &= \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big], \end{aligned}$$

- Existence and uniqueness of v_π :
  - Either $\gamma < 1$,
  - Or termination is guaranteed from all states under the policy v_π.
- If the environment is known (the dynamics), the Bellman equation is a linear system of $|\mathcal{S}|$ equations and $|\mathcal{S}|$ variables $v_\pi(s), s \in \mathcal{S}$.

1 - Policy Evaluation (Prediction)

- Iterative policy evaluation :

$$\begin{aligned} v_{k+1}(s) &\doteq \mathbb{E}_\pi[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s] \\ &= \sum_a \pi(a|s) \sum_{s',r} p(s', r \mid s, a)\Big[r + \gamma v_k(s')\Big], \end{aligned}$$

Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$ arbitrarily, for $s \in \mathcal{S}$, and $V(terminal)$ to 0

Loop:
    $\Delta \leftarrow 0$
    Loop for each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

2 - Policy Improvement

- For a given π, suppose we have v_π
- For a given s, what if we select a in s, and thereafter following policy π ?
- The value of this way of behaving is to compute

$$\begin{aligned} q_\pi(s,a) &\doteq \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]. \end{aligned}$$

  and compare it to $v_\pi(s)$.
- If $q_\pi(s,a) \geq v_\pi(s)$, then, it will better to select a every time s is encountered.
- Then, the new policy is better than π.

2 - Policy Improvement

- Let π and π' a pair of deterministic policies, and $s \in \mathcal{S}$, such that :

$$q_\pi(s, \pi'(s)) \geq v_\pi(s).$$

- Then π' must be as good as, or better than π :

$$v_{\pi'}(s) \geq v_\pi(s), \forall s \in \mathcal{S}.$$

- If $q_\pi(s, \pi'(s)) > v_\pi(s)$, then $v_{\pi'}(s) > v_\pi(s), \forall s \in \mathcal{S}$.
- Therefore π' improves π.

2 - Policy improvement

- we have :

$$q_\pi(s, \pi'(s)) \geq v_\pi(s) \Rightarrow v_{\pi'}(s) \geq v_\pi(s), \forall s \in \mathcal{S}.$$

- Proof :

$$\begin{aligned}
v_\pi(s) &\leq q_\pi(s, \pi'(s)) \\
&= \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = \pi'(s)] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\
&\leq \mathbb{E}_{\pi'}[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma \mathbb{E}[R_{t+2} + \gamma v_\pi(S_{t+2}) | S_{t+1}, A_{t+1} = \pi'(S_{t+1})] \mid S_t = s] \\
&= \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 v_\pi(S_{t+2}) \mid S_t = s\big] \\
&\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 v_\pi(S_{t+3}) \mid S_t = s\big] \\
&\vdots \\
&\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \mid S_t = s\big] \\
&= v_{\pi'}(s).
\end{aligned}$$

2 - Policy improvement

- Let us now consider changes at all states, i.e. selecting at each state the action that appears best according to $q_\pi(s, a)$.
- That is to select the new greedy policy π' satisfying :

$$\begin{aligned} \pi'(s) &\doteq \arg\max_a q_\pi(s,a) \\ &= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \arg\max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_\pi(s') \Big], \end{aligned}$$

- Then π' is as good as, or better than π.
- Suppose non that π' is as good as, but not better than π, i.e. $v_{\pi'} = v_\pi$.
- we then have :

$$\begin{aligned} v_{\pi'}(s) &= \max_a \mathbb{E}[R_{t+1} + \gamma v_{\pi'}(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_{\pi'}(s') \Big]. \end{aligned}$$

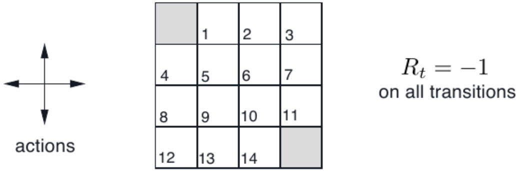
which is the Bellman euation. Therefore π and π' are optimal.

2 - Policy improvement

- Applying the policy improvement to a given policy will converge to the optimal policy.
- We have also convergence with stochastic policies.
- If there are several actions at which the maximum is achieved, then in the stochastic case we need not select a single action from among them.
- Instead, each maximizing action can be given a portion of the probability of being selected in the new greedy policy.
- Any apportioning scheme is allowed as long as all submaximal actions are given zero probability.

2 - Policy improvement

Example 4.1 Consider the 4×4 gridworld shown below.



The nonterminal states are $\mathcal{S} = \{1, 2, \ldots, 14\}$. There are four actions possible in each state, $\mathcal{A} = \{\texttt{up}, \texttt{down}, \texttt{right}, \texttt{left}\}$, which deterministically cause the corresponding state transitions, except that actions that would take the agent off the grid in fact leave the state unchanged. Thus, for instance, $p(6, -1|5, \texttt{right}) = 1$, $p(7, -1|7, \texttt{right}) = 1$, and $p(10, r|5, \texttt{right}) = 0$ for all $r \in \mathcal{R}$. This is an undiscounted, episodic task. The reward is -1 on all transitions until the terminal state is reached. The terminal state is shaded in the figure (although it is shown in two places, it is formally one state). The expected reward function is thus $r(s, a, s') = -1$ for all states s, s' and actions a. Suppose the agent follows the equiprobable random policy (all actions equally likely). The left side of Figure 4.1 shows the sequence of value functions $\{v_k\}$ computed by iterative policy evaluation. The final estimate is in fact v_π, which in this case gives for each state the negation of the expected number of steps from that state until termination. ■

2 - Policy improvement

v_k for the random policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

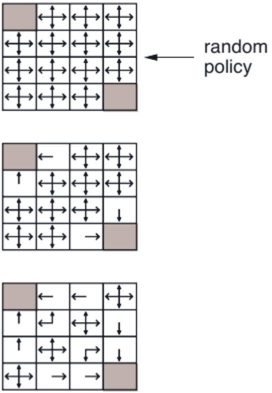
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

greedy policy w.r.t. v_k



Iterative policy evaluation - Corresponding greedy policies

2 - Policy improvement

$k = 3$

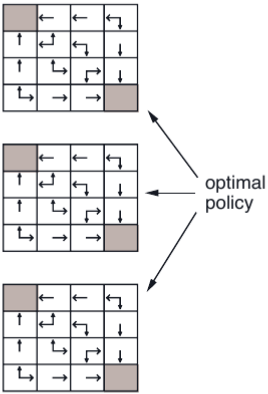
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



Iterative policy evaluation - Corresponding greedy policies

3 - Policy iteration

Once a policy, π, has been improved using v_π to yield a better policy, π', we can then compute $v_{\pi'}$ and improve it again to yield an even better π''. We can thus obtain a sequence of monotonically improving policies and value functions:

$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*,$$

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$; $V(terminal) \doteq 0$

2. Policy Evaluation
   Loop:
   - $\Delta \leftarrow 0$
   - Loop for each $s \in \mathcal{S}$:
     - $v \leftarrow V(s)$
     - $V(s) \leftarrow \sum_{s',r} p(s',r|s,\pi(s))\big[r + \gamma V(s')\big]$
     - $\Delta \leftarrow \max(\Delta, |v - V(s)|)$

   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   - *old-action* $\leftarrow \pi(s)$
   - $\pi(s) \leftarrow \operatorname{argmax}_a \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
   - If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*

   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

4 - Value iteration

- One drawback to policy iteration is that each of its iterations involves policy evaluation.
- If policy evaluation is done iteratively, then convergence exactly to v_π occurs only in the limit.
- The policy evaluation step of policy iteration can be truncated in several ways without losing the convergence guarantees of policy iteration.
- One important special case is when policy evaluation is stopped after just one sweep (one update of each state).
- In this case we call the algorithm : *value iteration*.

$$\begin{aligned} v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_k(s') \Big], \end{aligned}$$

- For arbitrary v_0, the sequence $\{v_k\}$ can be shown to converge to v_* under the same conditions that guarantee the existence of v_*.

4 - Value iteration

$$\begin{aligned} v_{k+1}(s) &\doteq \max_a \mathbb{E}[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s', r \mid s, a) \Big[r + \gamma v_k(s') \Big], \end{aligned}$$

- The value iteration is obtained simply by turning the Bellman optimality equation into an update rule (fixed ponit).
- The value iteration update is identical to the policy evaluation update, except that it requires the maximum to be taken over all actions.
- Like policy evaluation, value iteration formally requires an infinite number of iterations to converge exactly to v_*.
- In practice, we stop once the value function changes by only a small amount in a sweep.

4 - Value iteration

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

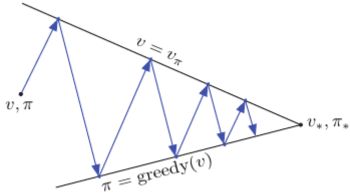
Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $\quad v \leftarrow V(s)$
| $\quad V(s) \leftarrow \max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
| $\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \arg\max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$

Asynchronous Dynamic Programming

- A major drawback to the DP methods is that they involve operations over the entire state set of the MDP.
- If the state set is very large, then even a single sweep can be prohibitively expensive.
- In asynchronous DP algorithms, the values of some states may be updated several times before the values of others are updated once.
- To converge correctly, however, an asynchronous algorithm must continue to update the values of all the states : it can ?t ignore any state after some point in the computation.
- Asynchronous DP algorithms allow great flexibility in selecting states to update.
- In the discounted case ($\gamma < 1$), the sequence could even be random, convergence is guaranteed.

Generalized Policy Iteration (GPI)



- We use the term generalized policy iteration (GPI) to refer to the general idea of letting policy-evaluation and policy-improvement processes interact, independent of the granularity and other details of the two processes.
- Almost all reinforcement learning methods are well described as GPI.
- The value function stabilizes only when it is consistent with the current policy.
- The policy stabilizes only when it is greedy with respect to the current value function.
- If both the value function and the policy stabilze, then the Bellman optimality equation holds.
- Thus the policy and value function are optimal.

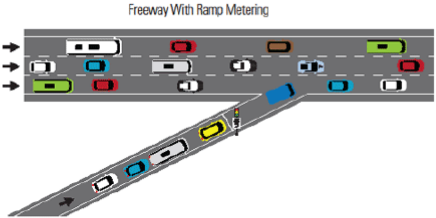
Efficiency of Dynamic Programming

- In practice, DP methods can be used with today's computers to solve MDPs with millions of states.
- Both policy iteration and value iteration are widely used, and it is not clear which, if either, is better in general.
- In practice, these methods usually converge much faster than their theoretical worst-case run times, particularly if they are started with good initial value functions or policies.
- On problems with large state spaces, asynchronous DP methods are often preferred.
- Asynchronous methods and other variations of GPI can be applied in such cases and may find good or optimal policies much faster than synchronous methods can.

The Project

Project 2023-2024 - Reinforcement learning for ramp metring on highways.

The objective of this project is to apply algorithms of Q-learning and Deep Q-learning to learn by numerical simulation the ramp metering control on a highway. We consider a stretch of highway with a given number of lanes (which is a parameter here), with an entering ramp controlled with a traffic light. We use the traffic simulator SUMO (Simulation of Urban Mobility) to simulate the car-following and lane change of all cars. The Q-learning algorithm should control the traffic light at the ramp, in a way that it optimizes the traffic, both on the highway stretch and on the ramp.



https://ops.fhwa.dot.gov

The Project

Possibility of usign the work of Romain Ducrocq :
- Projet 1: Framework DQN: https://github.com/romainducrocq/frameworQ/
- Projet 2: DQN for Intelligent Traffic Signal Control with Partial Detection: https://github.com/romainducrocq/DQN-ITSCwPD/
- **Article sur Arxiv: https://arxiv.org/abs/2109.14337**

State variables :
The state variable representation should reflect the state of traffic on both the highway, and the entering ramp.

Action variables :
The action variable can be for example the proportion of the green light in predefined cycle of the traffic light (cyclic control).

Reward :
The reward modeling should reflect the optimization of traffic on both the highway and the ramp.

Scenarios
Vary the values of the following parameters, in order to cover a maximum number of scenarios

- Length of the highway stretch
- Speed limits on the highway and on the ramp.
- Initial car-density on the highway stretch and on the ramp.
- The car-flow (veh./h.) on the highway, and the inflow from the input ramp.
- Number of lanes
- etc.

Thank you !