Guideline for ALPS

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Contents

1 Introduction

The Arbitrary Linear Plasma Solver (ALPS; *Verscharen et al. 2018*[1]) is a software package for the calculation of the dispersion relation in a hot (even relativistic) plasma. It allows arbitrary number of particle species with arbitrary gyro-tropic equilibrium distribution functions for any direction of wave propagation with respect to the background field.

2 Mathematical Basis Derivation

We start from the mathematical basis for the calculation of the hot-plasma dispersion relation. The kinetic wave dispersion relation in a hot plasma is based on the linearized set of Maxwell's equations and the linearized Vlasov equation. The expression of a wave or instability associated with a first-order perturbation δf_j in the distribution function of species j about a prescribed time-averaged background distribution function f_{j0} is:

$$f_j(\boldsymbol{r},\boldsymbol{p},t) = f_{0j}(\boldsymbol{p}) + \delta f_j(\boldsymbol{r},\boldsymbol{p},t), \tag{1}$$

where $\boldsymbol{r}$ is the spatial coordinate and $\boldsymbol{p}$ is the momentum. Other magnetic properties can be written in a similar way:

$$\begin{aligned}
\boldsymbol{E} &= \delta\boldsymbol{E}(\boldsymbol{r},t) \\
\boldsymbol{B} &= B_0\boldsymbol{e_z} + \delta\boldsymbol{B}(\boldsymbol{x},t) \\
\rho &= \delta\rho(\boldsymbol{x},t) \\
\boldsymbol{J} &= \delta\boldsymbol{J}(\boldsymbol{x},t).
\end{aligned} \tag{2}$$

Substitute the above expressions into the Vlasov's set of equations, we have:

$$\begin{aligned}
&\frac{\partial(\delta f_j)}{\partial t} + \boldsymbol{v}\cdot\frac{\partial(\delta f_j)}{\partial r} + \frac{q_j}{c}(\boldsymbol{v}\times B_0\boldsymbol{e_z})\frac{\partial(\delta f_j)}{\partial \boldsymbol{p}} = -q_j(\delta\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\delta\boldsymbol{B})\frac{\partial f_{j0}}{\partial \boldsymbol{p}}, \\
&\nabla\cdot\delta\boldsymbol{E} = 4\pi\delta\rho, \qquad \nabla\cdot(\delta\boldsymbol{B}) = 0, \\
&\nabla\times(\delta\boldsymbol{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\delta B), \qquad \nabla\times(\delta\boldsymbol{B}) = \frac{1}{c}\frac{\partial}{\partial t}(\delta\boldsymbol{E}) + \frac{4\pi}{c}(\delta\boldsymbol{J}).
\end{aligned} \tag{3}$$

We have the normal mode:

$$\begin{aligned}
\delta f_j(\boldsymbol{r},\boldsymbol{p},t) &= \delta f_j(\boldsymbol{p})\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t), \\
\delta\boldsymbol{E}(\boldsymbol{r},t) &= \delta\boldsymbol{E}\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t), \\
\delta\boldsymbol{B}(\boldsymbol{r},t) &= \delta\boldsymbol{B}\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t), \\
\delta\boldsymbol{J}(\boldsymbol{r},t) &= \delta\boldsymbol{J}\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t) = \sum_j \frac{q_j}{m_j}\int \delta f_j(\boldsymbol{p})\boldsymbol{p}d^3v,
\end{aligned} \tag{4}$$

Note that we did not include density ρ in this set of equations. This is because we could replace ρ by $\boldsymbol{J}$ in the linearized equations $\nabla\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0 \rightarrow i\boldsymbol{k}\cdot(\delta\boldsymbol{J}) - i\omega(\delta\rho) = 0 \rightarrow \delta\rho = \frac{1}{\omega}\boldsymbol{k}\cdot(\delta\boldsymbol{J})$.

Insert eqs. (4) into eqs. (3), we can write the distribution perturbation as (Note that the LHS of the 1st equation of eqs. (3) is $\frac{d(\delta f_j)}{dt}$):

$$\begin{aligned}
&\delta f_j(\boldsymbol{r},\boldsymbol{p},t) - \delta f_j[\boldsymbol{r}'(t'),\boldsymbol{v}'(t'),t']_{t'=-\infty} \\
&= -q_j\int_{-\infty}^{t} dt'\{\delta\boldsymbol{E}_1[\boldsymbol{r}'(t'),t'] + \frac{1}{c}\boldsymbol{v}'(t')\times\delta\boldsymbol{B}[\boldsymbol{r}'(t')]\}\cdot\frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'}.
\end{aligned} \tag{5}$$

We obtain:

$$
\begin{aligned}
&\delta f_{jk}(\boldsymbol{p})e^{-1\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \delta f_{jk}[\boldsymbol{p}'(t')]e^{-i\omega t'+i\boldsymbol{k}\cdot\boldsymbol{r}'}|_{t'=-\infty} \\
&= -q_j \int_{-\infty}^{t} dt'[\delta\boldsymbol{E}_k + \frac{1}{c}\boldsymbol{v}'(t') \times (\delta\boldsymbol{B}_k)]e^{-i\omega t'+i\boldsymbol{k}\cdot\boldsymbol{r}'} \cdot \frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'} \\
&= -q_j e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} \int_{-\infty}^{t} dt'[\delta\boldsymbol{E}_k + \frac{1}{c}\boldsymbol{v}'(t') \times (\delta\boldsymbol{B}_k)] \cdot \frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'} \cdot e^{-i\omega(t'-t)+i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{r})}.
\end{aligned} \tag{6}
$$

We assume $\omega_i(= Im(\omega)) > 0$, then $\delta f_{jk}[\boldsymbol{v}'(t')]e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}|_{t'=-\infty} = 0$ and eq 6 becomes:

$$
\begin{aligned}
&\delta f_{jk}(\boldsymbol{p})e^{i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} = \\
&q_j e^{-i\omega t+I\boldsymbol{k}\cdot\boldsymbol{r}} \int_{-\infty}^{t} dt'[\delta\boldsymbol{E}_k + \frac{1}{c}\boldsymbol{v}'(t') \times (\delta\boldsymbol{B}_k)] \cdot \frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'} e^{-i\omega(t'-t)+i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{r})}.
\end{aligned} \tag{7}
$$

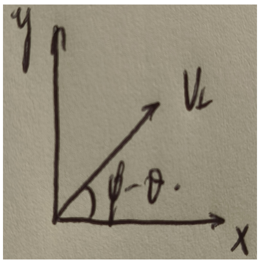
Factoring out $e^{i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}$, we have:

$$
\delta f_{jk}(\boldsymbol{p}) = q_j \int_{-\infty}^{t} dt'[\delta\boldsymbol{E}_k + \frac{1}{c}\boldsymbol{v}'(t') \times (\delta\boldsymbol{B}_k)] \cdot \frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'} e^{-i\omega(t'-t)+i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{r})}. \tag{8}
$$

Define $\tau = t - t'$, we may rewrite eq. 8 in terms of τ:

$$
\delta f_{jk}(\boldsymbol{p}) = -q_j \int_{-\infty}^{0} d\tau[\delta\boldsymbol{E}_k + \frac{1}{c}\boldsymbol{v}'(\tau) \times (\delta\boldsymbol{B}_k)] \cdot \frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'} \cdot e^{-i\omega\tau+I\boldsymbol{k}[\boldsymbol{r}'(\tau)-\boldsymbol{r}]} \tag{9}
$$

We also have the orbital equations:



$$
\begin{aligned}
&v_x'(\tau) = v_\perp cos(\phi - \theta + \Omega_j\tau) \\
&v_y'(\tau) = v_\parallel sin(\phi - \theta - \Omega_j\tau) \\
&v_z'(\tau) = v_z \\
&x'(\tau) = x - \frac{v_\perp}{\Omega_j} sin(\phi - \theta - \Omega_j\tau) + \frac{v_\perp}{\Omega_j} sin(\phi - \theta) \\
&y'(\tau) = y + \frac{v_\perp}{\Omega_j} cos(\phi - \theta + \Omega_j\tau) - \frac{v_\perp}{\Omega_j} cos(\phi - \theta) \\
&z'(\tau) = z + v_z\tau
\end{aligned} \tag{10}
$$

Without losing generality, we can write $\boldsymbol{k} = k_\perp\boldsymbol{e}_x + k_z\boldsymbol{e}_z$. Using eqs. 10, we have: $\boldsymbol{k}\cdot[\boldsymbol{r}'(\tau) - \boldsymbol{r}] = k_\perp[x'(\tau) - x] + k_z[z'(\tau) - z] = \frac{k_\perp v_\perp}{\Omega_j}[sin(\phi - \theta - \Omega_j\tau) - sin(\phi)] + k_z v_z\tau$. From this, we obtain:

$$
e^{i\omega\tau+i\boldsymbol{k}\cdot[\boldsymbol{x}'(\tau)-\boldsymbol{x}]} = e^{-i(\omega-k_z v_z)\tau+i\frac{k_\perp v_\perp}{\Omega_j}[sin(\phi-\theta\Omega_j\tau)-sin(\phi-\theta)]} = e^{-\alpha}. \tag{11}
$$

Since $f_{j0}(\boldsymbol{v}') = f_{j0}(v'_\perp, v'_z)$ and $v'_\perp(= v_\perp)$, $v'_z(= v_z)$ are constants of motion, we have:

$$\frac{\partial f_{j0}(\boldsymbol{p}')}{\partial \boldsymbol{p}'} = \frac{\partial f_{j0}(\boldsymbol{p})}{\partial \boldsymbol{p}} = \frac{\partial f_{j0}}{\partial p_\perp}\frac{\boldsymbol{p}_\perp}{p_\perp} + \frac{\partial f_{0j}}{\partial p_z}\frac{\boldsymbol{p}_z}{p_z} = 2\frac{\partial f_{j0}}{\partial p_\perp^2}\boldsymbol{p}_\perp + 2\frac{\partial f_{j0}}{\partial p_\parallel^2}\boldsymbol{p}_\parallel \tag{12}$$

Therefore, we have the following:

$$\begin{aligned} \delta \boldsymbol{E}_k \cdot \frac{\partial f_{j0}(\boldsymbol{p}')}{\partial \boldsymbol{p}'} &= 2(\delta E_{kx}\boldsymbol{e}_x + \delta E_{ky}\boldsymbol{e}_y + \delta E_{kz}\boldsymbol{e}_z) \cdot (\frac{\partial f_{j0}}{\partial p_\perp^2}\boldsymbol{v}_\perp + \frac{\partial f_{j0}}{\partial v_z^2}\boldsymbol{v}_z) \\ &= 2\frac{\partial f_{j0}}{\partial p_\perp^2}(\delta E_{kx}v_x + \delta E_{ky}v_y) + 2\delta E_{kz}v_z\frac{\partial f_{j0}}{\partial p_z^2} \end{aligned} \tag{13}$$

From Faraday's Law, we have:

$$\nabla \times (\delta \boldsymbol{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\delta B) \Rightarrow i\boldsymbol{k} \times (\delta \boldsymbol{E}) = \frac{-i}{c}\omega(\delta \boldsymbol{B}_k) \Rightarrow \delta \boldsymbol{B}_k = \frac{c}{\omega}\boldsymbol{k} \times (\delta \boldsymbol{E}_k), \tag{14}$$

thus we may write:

$$\boldsymbol{v} \times (\delta \boldsymbol{B}_k) = \frac{c}{\omega}\boldsymbol{v} \times (\boldsymbol{k} \times (\delta \boldsymbol{E}_k)) = \frac{c}{\omega}[(\boldsymbol{v} \cdot (\delta \boldsymbol{E}_k))\boldsymbol{k} - (\boldsymbol{v} \cdot \boldsymbol{k})(\delta \boldsymbol{E}_k)]. \tag{15}$$

Therefore, we have:

$$\begin{aligned} &\frac{1}{c}(\boldsymbol{v} \times (\delta \boldsymbol{B}_k)) \cdot \frac{\partial f_{j0}}{\partial \boldsymbol{p}'} = \frac{2}{\omega}[(\boldsymbol{v} \cdot (\delta \boldsymbol{E}_k))\boldsymbol{k} - (\boldsymbol{k} \cdot \boldsymbol{v})(\delta \boldsymbol{E}_k)] \cdot (\frac{\partial f_{j0}}{\partial v_\perp^2}\boldsymbol{v}_\perp + v_z\frac{\partial f_{j0}}{\partial v_z^2}\boldsymbol{e}_z) \\ &= \frac{2}{\omega}\{\frac{\partial f_{j0}}{\partial v_\perp^2}[(v_x\delta E_{kx}) + v_y\delta E_{ky}) + v_z\delta E_{kz})k_\perp v_x - (k_\perp v_x + k_z v_z) \cdot (\delta E_{kx}v_x + \delta E_{ky}v_y)] \\ &+ \frac{\partial f_{j0}}{\partial v_z^2}[(v_x\delta E_{kx}) + v_y\delta E_{ky}) + v_z\delta E_{kz})k_z v_z - (k_\perp v_x + k_z v_z) \cdot v_z(\delta E_{kz})]\} \\ &= \frac{2}{\omega}\{\frac{\partial f_{j0}}{\partial v_\perp^2}(-k_z\delta E_{kx}v_x - k_z\delta E_{ky}v_y + k_\perp\delta E_{kz}v_x)v_z + \frac{\partial f_{j0}}{\partial v_z^2}(v_x\delta E_{kx}k_z + v_y\delta E_{ky}k_z - k_\perp v_x\delta E_{kz})v_z\} \end{aligned} \tag{16}$$

Then we can write:

$$\begin{aligned} &[\delta \boldsymbol{E}_k + \frac{1}{c}(\boldsymbol{v} \times \delta B_k)]\frac{\partial f_{j0}}{\partial \boldsymbol{p}'} = 2v'_x X + 2v'_y Y + 2v'_z Z \\ &= 2v_\perp cos(\phi - \theta + \Omega_j)X + 2v_\perp sin(\phi - \theta + \Omega_j)Y + 2v_z Z. \end{aligned} \tag{17}$$

where:

$$\begin{aligned} X &= \delta Ekx\frac{\partial f_{j0}}{\partial v_\perp^2} + \frac{v_z}{\omega}(k_z\delta E_{kx} - k_\perp\delta E_{kz})(\frac{\partial f_{j0}}{\partial v_z^2} - \frac{\partial f_{j0}}{\partial v_\perp^2}) \\ Y &= \delta E_{ky}\frac{\partial f_{j0}}{\partial v_\perp^2} + \frac{v_z}{\omega}k_z\delta E_{ky}(\frac{\partial f_{j0}}{\partial v_z^2} - \frac{\partial f_{j0}}{\partial v_\perp^2}) \\ Z &= \delta E_{kz}\frac{\partial f_{j0}}{\partial v_z^2} \end{aligned} \tag{18}$$

Reorganize the sequence, we can write:

$$\delta f_j = -q_j e^{i\boldsymbol{k}\cdot\boldsymbol{r}-I\omega t}\int_0^{\infty} d\tau e^{i\alpha}\{\delta E_{kx}Ucos(\phi+\Omega_j\tau)+\delta E_{ky}Usin(\phi+\Omega_j\tau)$$
$$+\delta E_{kz}[\frac{\partial f_{0j}}{\partial p_{\parallel}} - Vcos(\phi-\theta+\Omega_j\tau)]\}, \tag{19}$$

where:

$$\alpha = -\frac{k_{\perp}v_{\perp}}{\Omega_j}[sin(\phi-\theta+\Omega_j\tau)-sin(\phi-\theta)]+(\omega-k_{\parallel v_{\parallel}})\tau$$
$$U = \frac{\partial f_{0j}}{\partial p_{\perp}} + \frac{k_{\parallel}}{\omega}(v_{\perp}\frac{\partial f_{0j}}{\partial p_{\parallel}} - v_{\parallel}\frac{\partial f_{0j}}{\partial p_{\perp}}) \tag{20}$$
$$V = \frac{k_{\perp}}{\omega}(v_{\perp}\frac{\partial f_{0j}}{\partial p_{\parallel} - v_{\parallel}\frac{\partial f_{0j}}{\partial p_{\perp}}}).$$

Here,

$$\Omega_j = (q_j B_0)/(m_j c\sqrt{1+(p_{\perp}^2+p_{\parallel}^2)/m_j^2c^2} \tag{21}$$

is the relativistic gyrofrequency.

Note the Bessel Function:

$$e^{\pm iasin(x)} = \sum_{n=-\infty}^{+\infty} J_n(a)e^{\pm inx}. \tag{22}$$

We may write:

$$e^{-\frac{k_{\perp}v_{\perp}}{\Omega}sin(\Omega\tau+\phi)} = \sum_{n=-\infty}^{\infty} J_n(\frac{k_{\perp}v_{\perp}}{\Omega})e^{-in(\phi+\Omega\tau)}$$
$$e^{-\frac{k_{\perp}v_{\perp}}{\Omega}sin(\Omega\tau+\phi)} = \sum_{m=-\infty}^{\infty} J_m(\frac{k_{\perp}v_{\perp}}{\Omega})e^{in\phi} \tag{23}$$

Therefore we have:

$$e^{i\alpha} = e^{i\{-\frac{k_{\perp}v_{\perp}}{\Omega_j}[sin(\phi-\theta+\Omega_j\tau)-sin(\phi-\theta)]+(\omega-k_{\parallel v_{\parallel}})\tau\}}$$
$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} J_n(\frac{k_{\perp}v_{\perp}}{\Omega})J_m(\frac{k_{\perp}v_{\perp}}{\Omega})e^{-(\omega-n\Omega-k_{\parallel}v_{\parallel})\tau}e^{i(m-n)\phi} \tag{24}$$

On the other hand, we have:

$$\boldsymbol{J} = \sum_j \boldsymbol{J}_j = \sum_j q_j\int d^3\boldsymbol{v}\boldsymbol{v}\delta f_j = \bar{\bar{\sigma}}\cdot\delta\boldsymbol{E}, \tag{25}$$

where $\bar{\bar{\sigma}}$ is the dielectric tensor that can be written as:

$$\bar{\bar{\sigma}} = \begin{pmatrix} \sigma_{x1} & \sigma_{x2} & \sigma_{x3} \\ \sigma_{y1} & \sigma_{y2} & \sigma_{y3} \\ \sigma_{z1} & \sigma_{z2} & \sigma_{z3}. \end{pmatrix} \tag{26}$$

Combined with Eqs. 10, we can expand Eq. 25 in the following format:

$$
\begin{aligned}
\sum_{i=1}^{3}\sigma_{xi}\delta E_i &= \sum_j q_j \int d^2\boldsymbol{v} v_\perp \int_0^{2\pi} d\phi cos(\Omega\tau+\phi)\delta f \\
\sum_{i=1}^{3}\sigma_{yi}\delta E_i &= \sum_j q_j \int d^2\boldsymbol{v} v_\perp \int_0^{2\pi} d\phi sin(\Omega\tau+\phi)\delta f \\
\sum_{i=1}^{3}\sigma_{zi}\delta E_i &= \sum_j q_j \int d^2\boldsymbol{v} v_\parallel \int_0^{2\pi} d\phi \delta f,
\end{aligned} \tag{27}
$$

where $\int d^2\boldsymbol{v} = \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty v_\perp dv_\perp$.

As an example, we show the calculation of σ_{x1} here, the rest of the components can be derived in a similar way. By comparing the coefficient of E_x, we have

$$
\begin{aligned}
\sigma_{x1} = &-\sum_j \frac{q_j^2}{m_j}\int d^2\boldsymbol{v} v_\perp \int_0^\infty d\tau \sum_m\sum_n e^{i(\omega-n\Omega-k_\parallel v_\parallel)\tau} J_m(\xi_j)J_n(\xi_j) \\
&\cdot U\int_0^{2\pi} d\phi cos(\phi+\Omega\tau)cos(\phi)e^{i(m-n)\phi},
\end{aligned} \tag{28}
$$

where the argument ξ_j equals to $\frac{k_\perp v_\perp}{\Omega}$.

Look at the integration on ϕ:

$$
\begin{aligned}
&\int_0^{2\pi} d\phi cos(\Omega\tau+\phi)cos\phi e^{i(m-n)\phi} = \int_0^{2\pi} d\phi[cos(\Omega\tau)cos^2\phi - sin(\Omega-\tau)cos\phi sin\phi]e^{i(m-n)\phi} \\
&= \frac{cos(\Omega\tau)}{2}\int_0^{2\pi} d\phi(1+cos(2\phi))\phi e^{i(m-n)\phi} - \frac{sin(\Omega\phi)}{2}\int_0^{2\pi} d\phi sin(2\phi)e^{i(m-n)\phi}
\end{aligned}
$$

The first item goes;

$$
\begin{aligned}
&\int_0^{2\pi} d\phi(1+cos(2\phi))e^{i(m-n)\phi} = 2\pi\delta_{m,n} + \int_0^{2\pi} d\phi\frac{e^{i2\phi}+e^{-i2\phi}}{2}e^{i(m-n)\phi} \\
&= 2\pi\delta_{m,n} + \pi(\delta_{m,n-2}+\delta_{m,n+2}).
\end{aligned}
$$

The second item goes:

$$
\int_0^{2\pi} d\phi sin(2\phi)e^{i(m-n)\phi} = \int_0^{2\pi} d\phi\frac{e^{-2i\phi}-e^{2i\phi}}{2i}e^{i(m-n)\phi} = -i\pi(\delta_{m,n+2}-\delta_{m,n-2}).
$$

Here, $\delta_{m,n}$ is the Kronecker delta, which is 1 if $m=n$ and 0 otherwise.

With the above results combined, we have:

$$
\begin{aligned}
&\int_0^{2\pi} d\phi cos(\Omega\tau+\phi)cos\phi e^{i(m-n)\phi} \\
&= \frac{\pi}{2}\{2cos(\Omega\tau)\delta_{m,n} + cos(\Omega\tau)[\delta_{m,n-2}+\delta_{m,n+2}] + isin(\Omega\tau)[\delta_{m,n+2}-\delta_{m,n-2}]\} \\
&= \frac{\pi}{2}[e^{i\Omega\tau}(\delta_{m,n-2}+\delta_{m,n+2}) + e^{-i\Omega\tau}(\delta_{m,n}+\delta_{m,n+2})].
\end{aligned} \tag{29}
$$

Now we can write:

$$\sigma_{x1} = -\frac{1}{4}\sum_j \frac{q_j^2}{m_j}\int d^3\boldsymbol{v}v_\perp \int_0^\infty d\tau U \sum_{m,n} J_m(\xi_j)J_n(\xi_j)[e^{i(\omega-(n-1)\delta\Omega-k_\parallel v_\parallel)\tau}(\delta_{m,n-2}+\delta_{m,n}) \\ + e^{(\omega-(n+1)\Omega-k_\parallel v_\parallel)\tau}(\delta_{m,n}+\delta_{m,n+2})] \tag{30}$$

Given the fact that n and m range from $-\infty$ to ∞, we can safely replace n with n+1 and n with n-1 in the first and second items in the above equation, therefore we have:

$$\begin{aligned} &\sum_{m,n} J_m(\xi_j)J_n(\xi_j)[e^{i(\omega-(n-1)\delta\Omega-k_\parallel v_\parallel)\tau}(\delta_{m,n-2}+\delta_{m,n}) + e^{(\omega-(n+1)\Omega-k_\parallel v_\parallel)\tau}(\delta_{m,n}+\delta_{m,n+2})] \\ &= \sum_{m,n} e^{i(\omega-n\Omega-k_\parallel v_\parallel)\tau}(\delta_{m,n-1}+\delta_{m,n+1})[J_mJ_{n+1}+J_mJ_{n-1}] \\ &= \sum_{m,n} e^{i(m-n\Omega-k_\parallel v_\parallel)\tau}\frac{2n}{\xi_j}(\delta_{,m,n-1}+\delta_{m,n+1})J_mJ_n \\ &= \sum_{m,n} e^{i(m-n\Omega-k_\parallel v_\parallel)\tau}\frac{2n}{\xi_j}(J_mJ_{n-1}+J_mJ_{n+1}) \\ &= \sum_{n} e^{i(\omega-n\Omega-k_\parallel v_\parallel)\tau}(\frac{2n}{\xi_j})^2J_n^2. \end{aligned} \tag{31}$$

Note that the relation $J_{l+1}(z)+J_{l-1}(z) = \frac{2l}{z}J_l^2(z)$ is used here. Another important relation will also be used in the following content: $J_{l+1}(z)-J_{l-1}(z) = -2lJ_l^2(z)$.

The τ integration goes:

$$\int_0^\infty d\tau e^{i(\omega-n\Omega-k_\parallel v_\parallel)\tau} = -\frac{1}{i(\omega-n\Omega-k_\parallel v_\parallel)}. \tag{32}$$

Here, please note that ω has a small imaginary part, which makes the integrated 0 when $\tau\to\infty$.

In this way, we can write:

$$\sigma_{x1} = -i\sum_j \frac{q_j^2}{m_j^2}\sum_{n=-\infty}^{\infty}\int d^3\boldsymbol{v}\frac{Up_\perp}{\omega-n\Omega-k_\parallel v_\parallel}\sum_n(\frac{nJ_n(\xi_j)}{\xi_j})^2 \tag{33}$$

Note that the dielectric tensor is $\bar{\bar{\epsilon}} = \bar{\bar{I}} + \frac{\bar{\bar{\sigma}}}{i\omega}$, we have:

$$K_{x1} = \frac{\epsilon_{x1}}{\epsilon_0} = 1 + \sum_j \frac{q_j^2}{\omega^2\epsilon_0 m_j^2}\sum_{n=-\infty}^{\infty}\int d^3\boldsymbol{v}\frac{p_\perp U}{\omega-n\Omega-k_\parallel v_\parallel}(\frac{nJ_n(\xi_j)}{\xi_j})^2. \tag{34}$$

By repeating the same process, we have the full dielectric tensor as:

$$K_{ij} = \delta_{ij} + \sum_s \frac{q_j^2}{\omega^2\epsilon_0 m_j^2}\sum_{n=-\infty}^{+\infty}\int d^3\boldsymbol{v}\frac{\bar{\bar{S}}_{n,ij}}{\omega-n\Omega-k_\parallel v_\parallel}. \tag{35}$$

$\bar{\bar{S}}_n =$ is the susceptibility tensor, it can be expressed as:

$$\bar{\bar{S}}_n = \begin{pmatrix} p_\perp(\frac{nJ_n}{\xi_j})^2M & ip_\perp(\frac{n}{\xi_j})J_nJ_n'M & p_\perp(\frac{n}{\xi_j})J_n^2W \\ -ip_\perp(\frac{n}{\xi})J_nJ_n'M & p_\perp(J_n')^2M & -ip_\perp J_nJ_n'W \\ p_\parallel(\frac{n}{\xi_j})J_n^2M & ip_\perp J_nJ_n'M & p_\parallel J_n^2W \end{pmatrix}, \tag{36}$$

where:

$$
\begin{aligned}
M &= \omega U = (\omega - k_\parallel v_\parallel)\frac{\partial f_{0j}}{\partial p_\perp} + v_\perp k_\parallel \frac{\partial f_{0j}}{\partial p_\parallel} \\
W &= n\Omega \frac{v_\parallel}{v_\perp}\frac{\partial f_{0j}}{\partial p_\perp} + (\omega - n\Omega)\frac{\partial f_{0j}}{\partial v_\parallel}.
\end{aligned}
\tag{37}
$$

This expression is equivalent to the expression in *Verscharen et al. (2018)* [1].

3 Code Example

3.1 Generate Different f-0 Tables with ALPS

3.2 Generate f-0 Table from Observations

3.3 Important Tips

References

[1] Daniel Verscharen, Kristopher G Klein, Benjamin DG Chandran, Michael L Stevens, Chadi S Salem, and Stuart D Bale. Alps: the arbitrary linear plasma solver. *Journal of Plasma Physics*, 84(4):905840403, 2018.