Basic MHD Equations

Hao Ran

Starting from: April 2024

Contents

This is the lecture notes for Magnetohydrodymacis (MHD) equations, including the physical discussions and equations derivations.

1 Equations and Derivations

1.1 Electromagnetic Equations

Maxwell's Equations

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{1}$$
$$\nabla \cdot \boldsymbol{B} = 0 \tag{2}$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{3}$$
$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{4}$$

where $\boldsymbol{E}$ is the *electric field*, $\boldsymbol{B}$ is the *magnetic field*, $\boldsymbol{j}$ is the *current density*, ρ is the *charge density*, μ is the *magnetic permeability*, ϵ_0 is the *permittivity of free space*, and c is the *speed of light* in vacuum ($\sim 2.998 \times 10^8$ m/s). The four Maxwell equations represent different physical phenomena, the first equation suggests that either currents or time-varying electric fields can generate magnetic fields, the second equation indicates that there are no magnetic monopoles and a magnetic flux tube has a constant strength along its length, and the third and fourth equations imply that either time-varying magnetic fields or electric charges can give rise to electric fields.

Here we also introduce the concept of *Debye Length*, which is a measure of a charge carrier's net electrostatic effect in a solution and how far its electrostatic effect perssists. The potential function for a single electron is

$$\Phi(r) = \frac{e}{4\pi\epsilon_0 r}.$$

Under the influence of a positive charge q, the charge distribution satisfies Boltzmann's distribution:

$$n_e = n_{e0} exp(-\frac{e\Phi(r)}{k_B T_e}) \approx n_{e0}(1 + \frac{e\Phi(r)}{k_B T_e})$$

Assume $n_e = n_{e0} + \delta n_e$, we have:

$$\delta n_e = n_{e0}\frac{e\Phi(r)}{k_B T_e}.$$

Possion's equation claims:

$$\nabla^2 \Phi(r) = -\frac{\rho}{\epsilon_0} = \frac{e^2 n_{e0}}{k_B T_e \epsilon_0}\Phi(r).$$

Combining the boundary conditions $\Phi(r \to \inf) = 0$ and $\Phi(r \to 0) = \frac{q}{4\pi\epsilon_0 r}$, we have the Debye potential function Φ:

$$\Phi(r) = \frac{q}{4\pi\epsilon_0 r} exp(-\frac{r}{\lambda_D}), \tag{5}$$

where

$$\lambda_D = \sqrt{\frac{k_B T_e \epsilon_0}{e^2 n_{e0}}} \tag{6}$$

is the *Debye Length*.

Generalized Ohm's Law

Ohm's Law asserts that the current density is proportional to the total electric field (in a frame of reference moving with the frame), it can be written as:

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{7}$$

where σ is the electrical conductivity.

However, a generalization of Ohm's Law may be more appropriate in some regions of the space. We start from equations of motion for electrons and ions (assumed to be protons):

$$\frac{\partial}{\partial t}(n_e m_e \boldsymbol{u_e}) + \nabla \cdot (n_e m_e \boldsymbol{u_e}\boldsymbol{u_e}) = -\nabla \cdot \boldsymbol{P_e} - e n_e(\boldsymbol{E} + \boldsymbol{u_e} \times \boldsymbol{B}) + \boldsymbol{R_e},$$

$$\frac{\partial}{\partial t}(n_i m_i \boldsymbol{u_i}) + \nabla \cdot (n_i m_i \boldsymbol{u_i}\boldsymbol{u_i}) = -\nabla \cdot \boldsymbol{P_i} + e n_i(\boldsymbol{E} + \boldsymbol{u_i} \times \boldsymbol{B}) + \boldsymbol{R_i}.$$

Multiply the ion equation by m_e/m_i, then substract the electron equation from the ion equation, we have:

$$m_e \frac{\partial}{\partial t}(n_i \boldsymbol{u_i} - n_e \boldsymbol{u_e}) + m_e \nabla \cdot (n_i \boldsymbol{u_i}\boldsymbol{u_i} - n_e \boldsymbol{u_e}\boldsymbol{u_e}) =$$
$$\nabla \cdot (\boldsymbol{P_e} - \frac{m_e}{m_i}\boldsymbol{P_i}) + e[(n_e + \frac{m_e}{m_i}n_i)\boldsymbol{E} + (n_e \boldsymbol{u_e} + \frac{m_e}{m_i}n_i \boldsymbol{u_i}) \times \boldsymbol{B}] - (\boldsymbol{R_e} - \frac{m_e}{m_i}\boldsymbol{R_i}),$$

where $\boldsymbol{R_i} = -\boldsymbol{R_e}$. Note that in MHD regime we have $n_e \approx n_i \approx n$, and we neglect all the $O(m_e/m_i)$ terms, we have:

$$\frac{m_e}{e}\frac{\partial \boldsymbol{j}}{\partial t} + m_e \nabla \cdot [n(\boldsymbol{u_i}\boldsymbol{u_i} - \boldsymbol{u_e}\boldsymbol{u_e})] = \nabla \cdot \boldsymbol{P_e} + ne(\boldsymbol{E} + \boldsymbol{u_e} \times \boldsymbol{B}) - \boldsymbol{R_e}.$$

Replace u_e with $u_e = u_i - \frac{\boldsymbol{j}}{ne} \approx \boldsymbol{u} - \frac{\boldsymbol{j}}{ne}$, where u is the plasma velocity, we have:

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \frac{m_e}{ne^2}\frac{\partial \boldsymbol{j}}{\partial t} - \frac{1}{ne}\nabla \cdot \boldsymbol{P_e} + \frac{\boldsymbol{j} \times \boldsymbol{B}}{ne} + \frac{1}{ne}\boldsymbol{R_e} + \frac{m_e}{ne^2}\nabla \cdot (\boldsymbol{j}\boldsymbol{u} + \boldsymbol{u}\boldsymbol{j} - \frac{\boldsymbol{j}\boldsymbol{j}}{ne}).$$

R_e represents the rate of change of the electron momentum due to the collision with ion, which can be written as:

$$\boldsymbol{R_e} = n m_e \mu_c(\boldsymbol{u_i} - \boldsymbol{u_e}) = \frac{\mu_c m_e}{e}\boldsymbol{j},$$

where μ_c is the collision frequency. By defining the resistivity $\eta = \frac{\mu_c m_e}{ne^2}$, we can write the collision term as

$$\frac{\boldsymbol{R_e}}{ne} = \eta \boldsymbol{j}.$$

Finally, we can write the *generalized Ohm's Law* as:

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j} + \frac{1}{ne}\boldsymbol{j} \times \boldsymbol{B} - \frac{1}{ne}\nabla \cdot \boldsymbol{P_e} + \frac{m_e}{ne^2}[\frac{\partial \boldsymbol{j}}{\partial t} + \nabla \cdot (\boldsymbol{u}\boldsymbol{j} + \boldsymbol{j}\boldsymbol{u} - \frac{\boldsymbol{j}\boldsymbol{j}}{ne})]. \tag{8}$$

Induction Equation

Combining equations (1), (3), and (7), we can easily have:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (-\boldsymbol{v} \times \boldsymbol{B} + \boldsymbol{j}/\sigma) = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) - \nabla \times (\eta \nabla \times \boldsymbol{B}),$$

where $\eta = 1/(\mu\sigma)$ is the *magnetic diffusivity*. Identify that $\nabla \times (\nabla \times \boldsymbol{B}) = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}$, we find the *induction equation*:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}. \tag{9}$$

1.2 Plasma Equations

Mass Continuity

The *equation of mass conservation* usually comes in two forms (equivalent to each other):

$$\frac{d\rho}{dt} + \rho\nabla\cdot\boldsymbol{v} = 0\ (a) \qquad or \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0\ (b), \tag{10}$$

where ρ is the *mass density* and $\boldsymbol{v}$ is the *velocity* of the fluid.

Equation of Motion

Under the condition of electrical neutrality, the *equation of motion* can be written as:

$$\rho\frac{d\boldsymbol{v}}{dt} = -\nabla p + \boldsymbol{j}\times\boldsymbol{B} + \boldsymbol{F}, \tag{11}$$

where p is the *pressure*, $\boldsymbol{j}\times\boldsymbol{B}$ is the *Lorentz force*, and $\boldsymbol{F}$ represents the *external force*. Usually, the external force consists of two parts: the *gravity* ($\boldsymbol{F_g}$) and *viscosity* ($\boldsymbol{F_v}$). The gravity force is:

$$F_g = -\rho g(r)\hat{r},$$

where $g(r) = M(r)G/r^2$ is the *local gravitational acceleration*. The viscous force is

$$F_v = \rho\nu[\nabla^2 + \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{v})],$$

where ν is the *kinematic viscosity*. We can simplify the force to $\boldsymbol{F}_v = \rho\nu\nabla^2\boldsymbol{v}$ when the flow is incompressible.

Perfect Gas Law

The gas pressure is determined by an equation of state, which is taken for simplicity as the *perfect gas law*:

$$p = \frac{\tilde{R}}{\tilde{\mu}}\rho T = \frac{k_B}{m}\rho T = nk_B T, \tag{12}$$

where $\tilde{R}$ is the *specific gas constant*, $\tilde{\mu}$ is the *mean molecular weight*, and T is the *temperature*, n is the *total number* of particles per volume.

1.3 Energy Equations

Different Forms of the Heat Equation

The *heat equation* is the last fundamental equation, which can be written as:

$$\rho T\frac{ds}{dt} = -\mathcal{L}, \tag{13}$$

where $\mathcal{L}$ is the *energy loss function*, representing the net effect of all the sinks and sources of energy in the system. s is the *entropy per unit mass*. The equation states that the rate of increase of heat for a unit volume as it moves in space is due to the net effect of the energy sinks and sources.