Guideline for ALPS

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Contents

1 Introduction

The Arbitrary Linear Plasma Solver (ALPS; *Verscharen et al. 2018*[5]) is a software package for the calculation of the dispersion relation in a hot (even relativistic) plasma. It allows an arbitrary number of particle species with arbitrary gyro-tropic equilibrium distribution functions for any direction of wave propagation with respect to the background field.

In this file, we will present

2 Mathematical Basis Derivation

A mathematical derivation is presented in *Stix 1992* [4], which is already quite clear and comprehensive. Here, we show another detailed approach based on the author's understanding. We start from the mathematical basis for the calculation of the hot-plasma dispersion relation. The kinetic wave dispersion relation in a hot plasma is based on the linearized set of Maxwell's equations and the linearized Vlasov equation. The expression of a wave or instability associated with a first-order perturbation δf_j in the distribution function of species j about a prescribed time-averaged background distribution function f_{j0} is:

$$f_j(\boldsymbol{r},\boldsymbol{p},t) = f_{0j}(\boldsymbol{p}) + \delta f_j(\boldsymbol{r},\boldsymbol{p},t), \tag{2.1}$$

where $\boldsymbol{r}$ is the spatial coordinate and $\boldsymbol{p}$ is the momentum. Other magnetic properties can be written in a similar way:

$$\begin{aligned}
\boldsymbol{E} &= \delta\boldsymbol{E}(\boldsymbol{r},t)\\
\boldsymbol{B} &= B_0\boldsymbol{e_z} + \delta\boldsymbol{B}(\boldsymbol{x},t)\\
\rho &= \delta\rho(\boldsymbol{x},t)\\
\boldsymbol{J} &= \delta\boldsymbol{J}(\boldsymbol{x},t).
\end{aligned} \tag{2.2}$$

Substitute the above expressions into the Vlasov's set of equations, we have:

$$\begin{aligned}
&\frac{\partial(\delta f_j)}{\partial t} + \boldsymbol{v}\cdot\frac{\partial(\delta f_j)}{\partial r} + \frac{q_j}{c}(\boldsymbol{v}\times B_0\boldsymbol{e_z})\frac{\partial(\delta f_j)}{\partial \boldsymbol{p}} = -q_j(\delta\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\delta\boldsymbol{B})\frac{\partial f_{j0}}{\partial\boldsymbol{p}},\\
&\nabla\cdot\delta\boldsymbol{E} = 4\pi\delta\rho, \qquad \nabla\cdot(\delta\boldsymbol{B}) = 0,\\
&\nabla\times(\delta\boldsymbol{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\delta B), \qquad \nabla\times(\delta\boldsymbol{B}) = \frac{1}{c}\frac{\partial}{\partial t}(\delta\boldsymbol{E}) + \frac{4\pi}{c}(\delta\boldsymbol{J}).
\end{aligned} \tag{2.3}$$

We have the normal mode:

$$\begin{aligned}
\delta f_j(\boldsymbol{r},\boldsymbol{p},t) &= \delta f_j(\boldsymbol{p})\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t),\\
\delta\boldsymbol{E}(\boldsymbol{r},t) &= \delta\boldsymbol{E}\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t),\\
\delta\boldsymbol{B}(\boldsymbol{r},t) &= \delta\boldsymbol{B}\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t),\\
\delta\boldsymbol{J}(\boldsymbol{r},t) &= \delta\boldsymbol{J}\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t) = \sum_j \frac{q_j}{m_j}\int \delta f_j(\boldsymbol{p})\boldsymbol{p}d^3v,
\end{aligned} \tag{2.4}$$

Note that we did not include density ρ in this set of equations. This is because we could replace ρ by $\boldsymbol{J}$ in the linearized equations $\nabla\cdot\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0 \rightarrow i\boldsymbol{k}\cdot(\delta\boldsymbol{J}) - i\omega(\delta\rho) = 0 \rightarrow \delta\rho = \frac{1}{\omega}\boldsymbol{k}\cdot(\delta\boldsymbol{J})$.

Insert eqs. (2.4) into eqs. (2.3), we can write the distribution perturbation as (Note that the LHS of the 1st equation of eqs. (2.3) is $\frac{d(\delta f_j)}{dt}$):

$$\begin{aligned}
&\delta f_j(\boldsymbol{r},\boldsymbol{p},t) - \delta f_j[\boldsymbol{r}'(t'),\boldsymbol{v}'(t'),t']_{t'=-\infty} \\
&= -q_j\int_{-\infty}^{t} dt'\{\delta\boldsymbol{E}_1[\boldsymbol{r}'(t'),t'] + \frac{1}{c}\boldsymbol{v}'(t')\times\delta\boldsymbol{B}[\boldsymbol{r}'(t')]\}\cdot\frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'}.
\end{aligned} \tag{2.5}$$

We obtain:

$$\begin{aligned}
&\delta f_{jk}(\boldsymbol{p})e^{-1\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} - \delta f_{jk}[\boldsymbol{p}'(t')]e^{-i\omega t'+i\boldsymbol{k}\cdot\boldsymbol{r}'}|_{t'=-\infty} \\
&= -q_j\int_{-\infty}^{t} dt'[\delta\boldsymbol{E}_k + \frac{1}{c}\boldsymbol{v}'(t')\times(\delta\boldsymbol{B}_k)]e^{-i\omega t'+i\boldsymbol{k}\cdot\boldsymbol{r}'}\cdot\frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'} \\
&= -q_j e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}\int_{-\infty}^{t} dt'[\delta\boldsymbol{E}_k + \frac{1}{c}\boldsymbol{v}'(t')\times(\delta\boldsymbol{B}_k)]\cdot\frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'}\cdot e^{-i\omega(t'-t)+i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{r})}.
\end{aligned} \tag{2.6}$$

We assume $\omega_i(=Im(\omega))>0$, then $\delta f_{jk}[\boldsymbol{v}'(t')]e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}|_{t'=-\infty}=0$ and eq 2.6 becomes:

$$\begin{aligned}
&\delta f_{jk}(\boldsymbol{p})e^{i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}} = \\
&q_j e^{-i\omega t+I\boldsymbol{k}\cdot\boldsymbol{r}}\int_{-\infty}^{t} dt'[\delta\boldsymbol{E}_k + \frac{1}{c}\boldsymbol{v}'(t')\times(\delta\boldsymbol{B}_k)]\cdot\frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'}e^{-i\omega(t'-t)+i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{r})}.
\end{aligned} \tag{2.7}$$

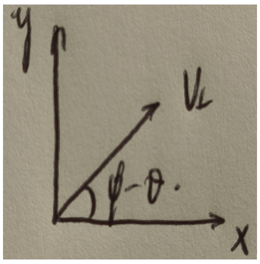
Factoring out $e^{i\omega t+i\boldsymbol{k}\cdot\boldsymbol{r}}$, we have:

$$\delta f_{jk}(\boldsymbol{p}) = q_j\int_{-\infty}^{t} dt'[\delta\boldsymbol{E}_k + \frac{1}{c}\boldsymbol{v}'(t')\times(\delta\boldsymbol{B}_k)]\cdot\frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'}e^{-i\omega(t'-t)+i\boldsymbol{k}\cdot(\boldsymbol{r}'-\boldsymbol{r})}. \tag{2.8}$$

Define $\tau = t-t'$, we may rewrite eq. 2.8 in terms of τ:

$$\delta f_{jk}(\boldsymbol{p}) = -q_j\int_{-\infty}^{0} d\tau[\delta\boldsymbol{E}_k + \frac{1}{c}\boldsymbol{v}'(\tau)\times(\delta\boldsymbol{B}_k)]\cdot\frac{\partial f_{j0}(\boldsymbol{p}')}{\partial\boldsymbol{p}'}\cdot e^{-i\omega\tau+I\boldsymbol{k}[\boldsymbol{r}'(\tau)-\boldsymbol{r}]} \tag{2.9}$$

We also have the orbital equations:



$$\begin{aligned}
&v_x'(\tau) = v_\perp cos(\phi-\theta+\Omega_j\tau) \\
&v_y'(\tau) = v_\parallel sin(\phi-\theta-\Omega_j\tau) \\
&v_z'(\tau) = v_z \\
&x'(\tau) = x - \frac{v_\perp}{\Omega_j}sin(\phi-\theta-\Omega_j\tau) + \frac{v_\perp}{\Omega_j}sin(\phi-\theta) \\
&y'(\tau) = y + \frac{v_\perp}{\Omega_j}cos(\phi-\theta+\Omega_j\tau) - \frac{v_\perp}{\Omega_j}cos(\phi-\theta) \\
&z'(\tau) = z + v_z\tau
\end{aligned} \tag{2.10}$$

Without losing generality, we can write $\boldsymbol{k} = k_\perp \boldsymbol{e}_x + k_z \boldsymbol{e}_z$. Using eqs. 2.10, we have: $\boldsymbol{k} \cdot [\boldsymbol{r}'(\tau) - \boldsymbol{r}] = k_\perp[x'(\tau) - x] + k_z[z'(\tau) - z] = \frac{k_\perp v_\perp}{\Omega_j}[sin(\phi - \theta - \Omega_j \tau) - sin(\phi)] + k_z v_z \tau$. From this, we obtain:

$$e^{i\omega\tau + i\boldsymbol{k}\cdot[\boldsymbol{x}'(\tau) - \boldsymbol{x}]} = e^{-i(\omega - k_z v_z)\tau + i\frac{k_\perp v_\perp}{\Omega_j}[sin(\phi - \theta\Omega_j \tau) - sin(\phi - \theta)]} = e^{-\alpha}. \tag{2.11}$$

Since $f_{j0}(\boldsymbol{v}') = f_{j0}(v'_\perp, v'_z)$ and $v'_\perp (= v_\perp)$, $v'_z (= v_z)$ are constants of motion, we have:

$$\frac{\partial f_{j0}(\boldsymbol{p}')}{\partial \boldsymbol{p}'} = \frac{\partial f_{j0}(\boldsymbol{p})}{\partial \boldsymbol{p}} = \frac{\partial f_{j0}}{\partial p_\perp}\frac{\boldsymbol{p}_\perp}{p_\perp} + \frac{\partial f_{0j}}{\partial p_z}\frac{\boldsymbol{p}_z}{p_z} = 2\frac{\partial f_{j0}}{\partial p_\perp^2}\boldsymbol{p}_\perp + 2\frac{\partial f_{j0}}{\partial p_\parallel^2}\boldsymbol{p}_\parallel \tag{2.12}$$

Therefore, we have the following:

$$\begin{aligned} \delta \boldsymbol{E}_k \cdot \frac{\partial f_{j0}(\boldsymbol{p}')}{\partial \boldsymbol{p}'} &= 2(\delta E_{kx}\boldsymbol{e}_x + \delta E_{ky}\boldsymbol{e}_y + \delta E_{kz}\boldsymbol{e}_z) \cdot (\frac{\partial f_{j0}}{\partial p_\perp^2}\boldsymbol{v}_\perp + \frac{\partial f_{j0}}{\partial v_z^2}\boldsymbol{v}_z) \\ &= 2\frac{\partial f_{j0}}{\partial p_\perp^2}(\delta E_{kx} v_x + \delta E_{ky} v_y) + 2\delta E_{kz} v_z \frac{\partial f_{j0}}{\partial p_z^2} \end{aligned} \tag{2.13}$$

From Faraday's Law, we have:

$$\nabla \times (\delta \boldsymbol{E}) = -\frac{1}{c}\frac{\partial}{\partial t}(\delta B) \Rightarrow i\boldsymbol{k} \times (\delta \boldsymbol{E}) = \frac{-i}{c}\omega(\delta \boldsymbol{B}_k) \Rightarrow \delta \boldsymbol{B}_k = \frac{c}{\omega}\boldsymbol{k} \times (\delta \boldsymbol{E}_k), \tag{2.14}$$

thus we may write:

$$\boldsymbol{v} \times (\delta \boldsymbol{B}_k) = \frac{c}{\omega}\boldsymbol{v} \times (\boldsymbol{k} \times (\delta \boldsymbol{E}_k)) = \frac{c}{\omega}[(\boldsymbol{v} \cdot (\delta \boldsymbol{E}_k))\boldsymbol{k} - (\boldsymbol{v} \cdot \boldsymbol{k})(\delta \boldsymbol{E}_k)]. \tag{2.15}$$

Therefore, we have:

$$\begin{aligned} &\frac{1}{c}(\boldsymbol{v} \times (\delta \boldsymbol{B}_k)) \cdot \frac{\partial f_{j0}}{\partial \boldsymbol{p}'} = \frac{2}{\omega}[(\boldsymbol{v} \cdot (\delta \boldsymbol{E}_k))\boldsymbol{k} - (\boldsymbol{k} \cdot \boldsymbol{v})(\delta \boldsymbol{E}_k)] \cdot (\frac{\partial f_{j0}}{\partial v_\perp^2}\boldsymbol{v}_\perp + v_z \frac{\partial f_{j0}}{\partial v_z^2}\boldsymbol{e}_z) \\ &= \frac{2}{\omega}\{\frac{\partial f_{j0}}{\partial v_\perp^2}[(v_x \delta E_{kx}) + v_y \delta E_{ky}) + v_z \delta E_{kz})k_\perp v_x - (k_\perp v_x + k_z v_z) \cdot (\delta E_{kx} v_x + \delta E_{ky} v_y)] \\ &+ \frac{\partial f_{j0}}{\partial v_z^2}[(v_x \delta E_{kx}) + v_y \delta E_{ky}) + v_z \delta E_{kz})k_z v_z - (k_\perp v_x + k_z v_z) \cdot v_z(\delta E_{kz})]\} \\ &= \frac{2}{\omega}\{\frac{\partial f_{j0}}{\partial v_\perp^2}(-k_z \delta E_{kx} v_x - k_z \delta E_{ky} v_y + k_\perp \delta E_{kz} v_x)v_z + \frac{\partial f_{j0}}{\partial v_z^2}(v_x \delta E_{kx} k_z + v_y \delta E_{ky} k_z - k_\perp v_x \delta E_{kz})v_z\} \end{aligned} \tag{2.16}$$

Then we can write:

$$\begin{aligned} &[\delta \boldsymbol{E}_k + \frac{1}{c}(\boldsymbol{v} \times \delta B_k)]\frac{\partial f_{j0}}{\partial \boldsymbol{p}'} = 2v'_x X + 2v'_y Y + 2v'_z Z \\ &= 2v_\perp cos(\phi - \theta + \Omega_j)X + 2v_\perp sin(\phi - \theta + \Omega_j)Y + 2v_z Z. \end{aligned} \tag{2.17}$$

where:

$$\begin{aligned} X &= \delta Ekx \frac{\partial f_{j0}}{\partial v_\perp^2} + \frac{v_z}{\omega}(k_z \delta E_{kx} - k_\perp \delta E_{kz})(\frac{\partial f_{j0}}{\partial v_z^2} - \frac{\partial f_{j0}}{\partial v_\perp^2}) \\ Y &= \delta E_{ky}\frac{\partial f_{j0}}{\partial v_\perp^2} + \frac{v_z}{\omega}k_z \delta E_{ky}(\frac{\partial f_{j0}}{\partial v_z^2} - \frac{\partial f_{j0}}{\partial v_\perp^2}) \\ Z &= \delta E_{kz}\frac{\partial f_{j0}}{\partial v_z^2} \end{aligned} \tag{2.18}$$

Reorganize the sequence, we can write:

$$\begin{aligned}\delta f_j = -q_j e^{i\boldsymbol{k}\cdot\boldsymbol{r}-I\omega t}\int_0^\infty d\tau e^{i\alpha}\{\delta E_{kx}Ucos(\phi+\Omega_j\tau)+\delta E_{ky}Usin(\phi+\Omega_j\tau)\\ +\delta E_{kz}[\frac{\partial f_{0j}}{\partial p_\parallel}-Vcos(\phi-\theta+\Omega_j\tau)]\},\end{aligned}\tag{2.19}$$

where:

$$\begin{aligned}\alpha &= -\frac{k_\perp v_\perp}{\Omega_j}[sin(\phi-\theta+\Omega_j\tau)-sin(\phi-\theta)]+(\omega-k_{\parallel v_\parallel})\tau\\ U &= \frac{\partial f_{0j}}{\partial p_\perp}+\frac{k_\parallel}{\omega}(v_\perp\frac{\partial f_{0j}}{\partial p_\parallel}-v_\parallel\frac{\partial f_{0j}}{\partial p_\perp})\\ V &= \frac{k_\perp}{\omega}(v_\perp\frac{\partial f_{0j}}{\partial p_\parallel - v_\parallel\frac{\partial f_{0j}}{\partial p_\perp}}).\end{aligned}\tag{2.20}$$

Here,

$$\Omega_j = (q_j B_0)/(m_j c\sqrt{1+(p_\perp^2+p_\parallel^2)/m_j^2c^2}\tag{2.21}$$

is the relativistic gyrofrequency.

Note the Bessel Function:

$$e^{\pm ia sin(x)} = \sum_{n=-\infty}^{+\infty} J_n(a)e^{\pm inx}.\tag{2.22}$$

We may write:

$$\begin{aligned}e^{-\frac{k_\perp v_\perp}{\Omega}sin(\Omega\tau+\phi)} &= \sum_{n=-\infty}^{\infty} J_n(\frac{k_\perp v_\perp}{\Omega})e^{-in(\phi+\Omega\tau)}\\ e^{-\frac{k_\perp v_\perp}{\Omega}sin(\Omega\tau+\phi)} &= \sum_{m=-\infty}^{\infty} J_m(\frac{k_\perp v_\perp}{\Omega})e^{in\phi}\end{aligned}\tag{2.23}$$

Therefore we have:

$$\begin{aligned}e^{i\alpha} &= e^{i\{-\frac{k_\perp v_\perp}{\Omega_j}[sin(\phi-\theta+\Omega_j\tau)-sin(\phi-\theta)]+(\omega-k_{\parallel v_\parallel})\tau\}}\\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} J_n(\frac{k_\perp v_\perp}{\Omega})J_m(\frac{k_\perp v_\perp}{\Omega})e^{-(\omega-n\Omega-k_\parallel v_\parallel)\tau}e^{i(m-n)\phi}\end{aligned}\tag{2.24}$$

On the other hand, we have:

$$\boldsymbol{J} = \sum_j \boldsymbol{J}_j = \sum_j q_j\int d^3\boldsymbol{v}\boldsymbol{v}\delta f_j = \bar{\bar{\sigma}}\cdot\delta\boldsymbol{E},\tag{2.25}$$

where $\bar{\bar{\sigma}}$ is the dielectric tensor that can be written as:

$$\bar{\bar{\sigma}} = \begin{pmatrix}\sigma_{x1} & \sigma_{x2} & \sigma_{x3}\\ \sigma_{y1} & \sigma_{y2} & \sigma_{y3}\\ \sigma_{z1} & \sigma_{z2} & \sigma_{z3}.\end{pmatrix}\tag{2.26}$$

Combined with Eqs. 2.10, we can expand Eq. 2.25 in the following format:

$$\begin{aligned}
\sum_{i=1}^{3}\sigma_{xi}\delta E_i &= \sum_j q_j \int d^2\boldsymbol{v}v_\perp \int_0^{2\pi} d\phi cos(\Omega\tau+\phi)\delta f \\
\sum_{i=1}^{3}\sigma_{yi}\delta E_i &= \sum_j q_j \int d^2\boldsymbol{v}v_\perp \int_0^{2\pi} d\phi sin(\Omega\tau+\phi)\delta f \\
\sum_{i=1}^{3}\sigma_{zi}\delta E_i &= \sum_j q_j \int d^2\boldsymbol{v}v_\parallel \int_0^{2\pi} d\phi\delta f,
\end{aligned} \tag{2.27}$$

where $\int d^2\boldsymbol{v} = \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty v_\perp dv_\perp$.

As an example, we show the calculation of σ_{x1} here, the rest of the components can be derived in a similar way. By comparing the coefficient of E_x, we have

$$\begin{aligned}
\sigma_{x1} = &-\sum_j \frac{q_j^2}{m_j}\int d^2\boldsymbol{v}v_\perp \int_0^\infty d\tau \sum_m\sum_n e^{i(\omega-n\Omega-k_\parallel v_\parallel)\tau}J_m(\xi_j)J_n(\xi_j) \\
&\cdot U\int_0^{2\pi} d\phi cos(\phi+\Omega\tau)cos(\phi)e^{i(m-n)\phi},
\end{aligned} \tag{2.28}$$

where the argument ξ_j equals to $\frac{k_\perp v_\perp}{\Omega}$.

Look at the integration on ϕ:

$$\begin{aligned}
&\int_0^{2\pi} d\phi cos(\Omega\tau+\phi)cos\phi e^{i(m-n)\phi} = \int_0^{2\pi} d\phi[cos(\Omega\tau)cos^2\phi - sin(\Omega-\tau)cos\phi sin\phi]e^{i(m-n)\phi} \\
&= \frac{cos(\Omega\tau)}{2}\int_0^{2\pi} d\phi(1+cos(2\phi))\phi e^{i(m-n)\phi} - \frac{sin(\Omega\phi)}{2}\int_0^{2\pi} d\phi sin(2\phi)e^{i(m-n)\phi}
\end{aligned}$$

The first item goes;

$$\begin{aligned}
&\int_0^{2\pi} d\phi(1+cos(2\phi))e^{i(m-n)\phi} = 2\pi\delta_{m,n} + \int_0^{2\pi} d\phi\frac{e^{i2\phi}+e^{-i2\phi}}{2}e^{i(m-n)\phi} \\
&= 2\pi\delta_{m,n} + \pi(\delta_{m,n-2}+\delta_{m,n+2}).
\end{aligned}$$

The second item goes:

$$\int_0^{2\pi} d\phi sin(2\phi)e^{i(m-n)\phi} = \int_0^{2\pi} d\phi\frac{e^{-2i\phi}-e^{2i\phi}}{2i}e^{i(m-n)\phi} = -i\pi(\delta_{m,n+2}-\delta_{m,n-2}).$$

Here, $\delta_{m,n}$ is the Kronecker delta, which is 1 if $m=n$ and 0 otherwise.

With the above results combined, we have:

$$\begin{aligned}
&\int_0^{2\pi} d\phi cos(\Omega\tau+\phi)cos\phi e^{i(m-n)\phi} \\
&= \frac{\pi}{2}\{2cos(\Omega\tau)\delta_{m,n} + cos(\Omega\tau)[\delta_{m,n-2}+\delta_{m,n+2}] + isin(\Omega\tau)[\delta_{m,n+2}-\delta_{m,n-2}]\} \\
&= \frac{\pi}{2}[e^{i\Omega\tau}(\delta_{m,n-2}+\delta_{m,n+2}) + e^{-i\Omega\tau}(\delta_{m,n}+\delta_{m,n+2})].
\end{aligned} \tag{2.29}$$

Now we can write:

$$\sigma_{x1} = -\frac{1}{4}\sum_j \frac{q_j^2}{m_j}\int d^3\boldsymbol{v} v_\perp \int_0^\infty d\tau U \sum_{m,n} J_m(\xi_j)J_n(\xi_j)[e^{i(\omega-(n-1)\delta\Omega-k_\parallel v_\parallel)\tau}(\delta_{m,n-2}+\delta_{m,n}) \\ + e^{(\omega-(n+1)\Omega-k_\parallel v_\parallel)\tau}(\delta_{m,n}+\delta_{m,n+2})] \tag{2.30}$$

Given the fact that n and m range from $-\infty$ to ∞, we can safely replace n with n+1 and n with n-1 in the first and second items in the above equation, therefore we have:

$$\begin{aligned}
&\sum_{m,n} J_m(\xi_j)J_n(\xi_j)[e^{i(\omega-(n-1)\delta\Omega-k_\parallel v_\parallel)\tau}(\delta_{m,n-2}+\delta_{m,n}) + e^{(\omega-(n+1)\Omega-k_\parallel v_\parallel)\tau}(\delta_{m,n}+\delta_{m,n+2})] \\
&= \sum_{m,n} e^{i(\omega-n\Omega-k_\parallel v_\parallel)\tau}(\delta_{m,n-1}+\delta_{m,n+1})[J_mJ_{n+1}+J_mJ_{n-1}] \\
&= \sum_{m,n} e^{i(m-n\Omega-k_\parallel v_\parallel)\tau}\frac{2n}{\xi_j}(\delta_{,m,n-1}+\delta_{m,n+1})J_mJ_n \\
&= \sum_{m,n} e^{i(m-n\Omega-k_\parallel v_\parallel)\tau}\frac{2n}{\xi_j}(J_mJ_{n-1}+J_mJ_{n+1}) \\
&= \sum_{n} e^{i(\omega-n\Omega-k_\parallel v_\parallel)\tau}(\frac{2n}{\xi_j})^2 J_n^2.
\end{aligned} \tag{2.31}$$

Note that the relation $J_{l+1}(z)+J_{l-1}(z) = \frac{2l}{z}J_l^2(z)$ is used here. Another important relation will also be used in the following content: $J_{l+1}(z)-J_{l-1}(z) = -2lJ_l^2(z)$.

The τ integration goes:

$$\int_0^\infty d\tau e^{i(\omega-n\Omega-k_\parallel v_\parallel)\tau} = -\frac{1}{i(\omega-n\Omega-k_\parallel v_\parallel)}. \tag{2.32}$$

Here, please note that ω has a small imaginary part, which makes the integrated 0 when $\tau\to\infty$.

In this way, we can write:

$$\sigma_{x1} = -i\sum_j \frac{q_j^2}{m_j^2}\sum_{n=-\infty}^{\infty}\int d^3\boldsymbol{v}\frac{Up_\perp}{\omega-n\Omega-k_\parallel v_\parallel}\sum_n(\frac{nJ_n(\xi_j)}{\xi_j})^2 \tag{2.33}$$

Note that the dielectric tensor is $\bar{\bar{\epsilon}} = \bar{\bar{I}} + \frac{\bar{\bar{\sigma}}}{i\omega}$, we have:

$$K_{x1} = \frac{\epsilon_{x1}}{\epsilon_0} = 1 + \sum_j \frac{q_j^2}{\omega^2\epsilon_0 m_j^2}\sum_{n=-\infty}^{\infty}\int d^3\boldsymbol{v}\frac{p_\perp U}{\omega-n\Omega-k_\parallel v_\parallel}(\frac{nJ_n(\xi_j)}{\xi_j})^2. \tag{2.34}$$

By repeating the same process, we have the full dielectric tensor as:

$$K_{ij} = \delta_{ij} + \sum_s \frac{q_j^2}{\omega^2\epsilon_0 m_j^2}\sum_{n=-\infty}^{+\infty}\int d^3\boldsymbol{v}\frac{\bar{\bar{S}}_{n,ij}}{\omega-n\Omega-k_\parallel v_\parallel}. \tag{2.35}$$

$\bar{\bar{S}}_n =$ is the susceptibility tensor, it can be expressed as:

$$\bar{\bar{S}}_n = \begin{pmatrix} p_\perp(\frac{nJ_n}{\xi_j})^2M & ip_\perp(\frac{n}{\xi_j})J_nJ_n'M & p_\perp(\frac{n}{\xi_j})J_n^2W \\ -ip_\perp(\frac{n}{\xi})J_nJ_n'M & p_\perp(J_n')^2M & -ip_\perp J_nJ_n'W \\ p_\parallel(\frac{n}{\xi_j})J_n^2M & ip_\perp J_nJ_n'M & p_\parallel J_n^2W \end{pmatrix}, \tag{2.36}$$

where:

$$
\begin{aligned}
M &= \omega U = (\omega - k_{\parallel} v_{\parallel}) \frac{\partial f_{0j}}{\partial p_{\perp}} + v_{\perp} k_{\parallel} \frac{\partial f_{0j}}{\partial p_{\parallel}} \\
W &= n\Omega \frac{v_{\parallel}}{v_{\perp}} \frac{\partial f_{0j}}{\partial p_{\perp}} + (\omega - n\Omega) \frac{\partial f_{0j}}{\partial v_{\parallel}}.
\end{aligned}
\tag{2.37}
$$

This expression is equivalent to the expression in *Verscharen et al. (2018)* [5].

3 Apply ALPS to Solar Orbiter Observations

The Solar Orbiter mission aims to study the Sun and its heliosphere at unprecedented close distances and angles (*Muller et al. (2020)* [2]).

3.1 PAS Data Processing

The SWA Proton-Alpha System (SWA-PAS) is designed to measure the 3D velocity distribution functions (VDFs) of solar wind protons and alpha particles with both high time resolution and high energy resolution (*Owen et al. 2020* [3]). However, PAS can only measure the particle flux as a function of the energy per charge ratio of the particles. This raises ambiguity when different ion species are involved, as in the case of solar wind, where protons constitute the majority of the population ($\sim 95\%$), alpha particles are a minor component ($\sim 4\%$), and heavier ions are present in trace amounts ($\sim 1\%$). Alpha particles travel at a similar speed to protons, but their mass-to-charge ratio is twice that of protons. Consequently, we can observe a two-peak shape in the energy spectrum of the solar wind, with the first peak corresponding to protons and the second peak corresponding to alpha particles ($\sqrt{2}\mathrm{V}_{\alpha}$). Figure 1 shows an example.

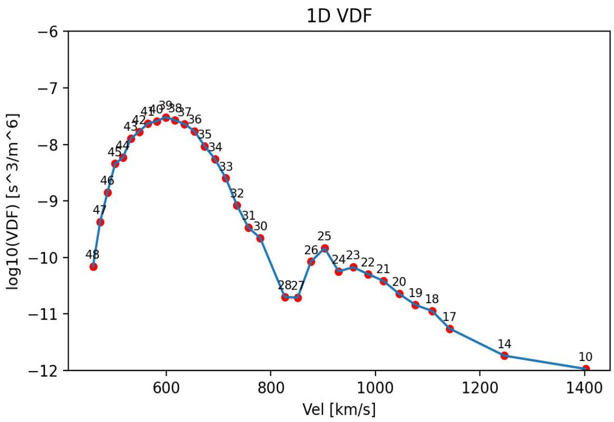


Figure 1: An example of a typical 1D VDF measured by PAS (2022-02-26T17:10:01). Numbers indicate the energy bin number.

3.1.1 Separation of Protons and Alpha Particles

ALPS requires the VDFs of different ion species as input. Therefore, it is necessary to separate the proton and alpha particle populations measured by PAS. *De Marco et al. (2023)* [1] proposed an innovative method to distinguish the ion populations in the solar wind using a Gaussian mixture model (GMM). Here we start with the principle of GMM, and then we will introduce the application of GMM to the separation of protons and alpha particles in PAS data with examples.

3.1.2 Gaussian Mixture Model

A Gaussian Mixture Model (GMM) is a probabilistic model that represents a dataset as a mixture of multiple Gaussian distributions (identified by $k \in \{1, ..., k\}$), enabling the identification of underlying clusters or components within the data. Each Gaussian k consists of the following parameters:

- A mean μ that defines its center.
- A covariance matrix Σ that defines its width.
- A mixing coefficient π that defines the weight of the Gaussian in the mixture (how big or small).

In general, we can use the above parameters to define the Gaussian density function:

$$\mathcal{N}(\boldsymbol{x}|\mu, \Sigma) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} exp(-\frac{1}{2}(\boldsymbol{x} - \mu)^T \Sigma^{-1}(\boldsymbol{x} - \mu)), \tag{3.1}$$

where $\boldsymbol{x}$ represents the data points, D is the number of dimensions. Apply a log to the above equation, we have:

$$ln\mathcal{N}(\boldsymbol{x}|\mu, \Sigma) = -\frac{D}{2}ln(2\pi) - \frac{1}{2}ln|\Sigma| - \frac{1}{2}(\boldsymbol{x} - \mu)^T \Sigma^{-1}(\boldsymbol{x} - \mu). \tag{3.2}$$

Let us move on to some initial derivations. Suppose that we want to know what is the probability that a data point $\boldsymbol{x}$ comes from Gaussian k, we can express it as:

$$p(z_{nk} = 1|\boldsymbol{x}_n), \tag{3.3}$$

which reads "*given a data point $\boldsymbol{x}$, what is the probability it came from Gaussian k ?*" Here, z is a latent variable that indicates the Gaussian from which the data point $\boldsymbol{x}$ comes. It is one when $\boldsymbol{x}$ came from Gaussian k and zero otherwise. Likewise, we can state:

$$\pi_k = p(z_k = 1), \tag{3.4}$$

which means that the overall probability of observing a point that comes from Gaussian k is actually equivalent to the mixing coefficient for that Gaussian (the bigger the Gaussian is, the more possible). Therefore, we have:

$$p(\boldsymbol{z}) = p(z_1 = 1)^{z1} p(z_2 = 1)^{z2} ... p(z_k = 1)^{zk} = \prod_{k=1}^{K} \pi_k^{z_k}, \tag{3.5}$$

where $\boldsymbol{z} = \{z_1, ..., z_k\}$ contains all possible latent variables. This turns out to be actually the Gaussian function itself:

$$p(\boldsymbol{x}_n|\boldsymbol{z}) = \prod_{k=1}^{K} \mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k)^{z_k}. \tag{3.6}$$

Now, we can determine our initial aim: what is the possibility of z given our observation $\boldsymbol{x}$? From the product rule of possibilities, we know that:

$$p(\boldsymbol{x}_n, \boldsymbol{z}) = p(\boldsymbol{x}_n|\boldsymbol{z})p(\boldsymbol{z}). \tag{3.7}$$

Marginalizing over $\boldsymbol{z}$, we have:

$$p(\boldsymbol{x}_n) = \sum_{k=1}^{K} p(\boldsymbol{x}_n|\boldsymbol{z})p(\boldsymbol{z}) = \sum_{k=1}^{K} \pi_k \mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k). \tag{3.8}$$

This is the equation that defines a Gaussian Mixture, and it already depends on the parameters that we have mentioned earlier. The maximum likelihood of the model can be expressed as the joint probability of all observations $\boldsymbol{x}_n$:

$$p(\boldsymbol{X}) = \prod_{n=1}^{N} p(\boldsymbol{x}_n) = \prod_{n=1}^{N} \sum_{k=1}^{K} \pi_k \mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k). \tag{3.9}$$

Similarly, we can apply a log to the above equation:

$$lnp(\boldsymbol{X}) = \sum_{n=1}^{N} ln \sum_{k=1}^{K} \pi_k \mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k). \tag{3.10}$$

Before we differentiate the equation to find the optimal parameters, we need to remove the logarithm. From Bayes rule, we know that:

$$p(z_k = 1|\boldsymbol{x}_n) = \frac{p(\boldsymbol{x}_n|z_k = 1)}{\sum_{j=1}^{K} p(\boldsymbol{x}_n|z_j = 1)p(z_j = 1)}. \tag{3.11}$$

From our earlier derivations, we know that:

$$p(z_k = 1) = \pi_k, \qquad p(\boldsymbol{x}_n|z_k = 1) = \mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k). \tag{3.12}$$

So, we can write:

$$p(z_k = 1|\boldsymbol{x}_n) = \frac{\pi_k \mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k)}{\sum_{j=1}^{K} \pi_j \mathcal{N}(\boldsymbol{x}_n|\mu_j, \Sigma_j)} = \gamma(z_{nk}). \tag{3.13}$$

This is exactly what we are looking for, it represents the possibility of z_k given the observation $\boldsymbol{x}_n$.

Expectation-Maximization Algorithm Let the parameters of our model be $\theta = \{\pi_k, \mu_k, \Sigma_k\}$. Let us first initialize the parameters accordingly. Then, the second step is to evaluate the expectation of the log-likelihood function. We evaluate:

$$Q(\theta^*, \theta) = \mathbb{E}[lnp(\boldsymbol{X}, \boldsymbol{Z}|\theta^*)] = \sum_{\boldsymbol{Z}} p(\boldsymbol{Z}|\boldsymbol{X}, \theta) lnp(\boldsymbol{X}, \boldsymbol{Z}|\theta^*) = \sum_{\boldsymbol{Z}} \gamma(z_{nk}) lnp(\boldsymbol{X}, \boldsymbol{Z}|\theta^*). \tag{3.14}$$

On the other hand, we have:

$$P(\boldsymbol{X}, \boldsymbol{Z}|\theta^*) = \prod_{n=1}^{N} \prod_{k=1}^{K} \pi^{z_{nk}} \mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k)^{z_{nk}}. \tag{3.15}$$

Here, the left side represents the probability of obtaining $\boldsymbol{x}$ and $\boldsymbol{z}$ with a given θ^*, and the right side ensures that only the component k corresponding to $z_{nk} = 1$ contributes to the probability. Therefore, we have:

$$lnp(\boldsymbol{X}, \boldsymbol{Z}|\theta^*) = \sum_{n=1}^{N}\sum_{k=1}^{K} z_{nk}[ln\pi_k + ln\mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k)] \tag{3.16}$$

Finally, we insert eq. 3.16 into eq. 3.14, we have:

$$Q(\theta^*, \theta) = \sum_{n=1}^{N}\sum_{k=1}^{K} \gamma(z_{nk})[ln\pi_k + ln\mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k)]. \tag{3.17}$$

Here, Q is the function that we want to maximize, representing a log-likelihood.

Now we are at our final step, the maximization of the function. Find the revised parameters θ^* using:

$$\theta^* = \arg\max_{\theta} Q(\theta^*, \theta), \tag{3.18}$$

where Q is given by eq. 3.17. Given that $\sum_{k=1}^{K} = 1$, we can rewrite eq. 3.17 as:

$$Q(\theta^*, \theta) = \sum_{n=1}^{N}\sum_{k=1}^{K} \gamma(z_{nk})[ln\pi_k + ln\mathcal{N}(\boldsymbol{x}_n|\mu_k, \Sigma_k)] - \lambda(\sum_{k=1}^{K}\pi_k - 1). \tag{3.19}$$

Differentiate the maximum likelihood function with respect to π_k, we have:

$$\frac{\partial Q(\theta, \theta^*)}{\partial \pi_k} = \sum_{n=1}^{N}\frac{\gamma(z_{nk})}{\pi_k} - \lambda = 0 \Rightarrow \sum_{k=1}^{K}\gamma(z_{nk}) = \pi_k\lambda \Rightarrow \sum_{k=1}^{K}\sum_{n=1}^{N}\gamma(z_{nk}) = \sum_{k=1}^{K}\pi_k\lambda. \tag{3.20}$$

Note that: sum π over all k is 1, and sum γ over all k is also 1. Therefore, we have $\lambda = N$. Similarly, differentiate Q with respect to μ and Σ, we get:

$$\mu_k^* = \frac{\sum_{n=1}^{N}\gamma(z_{nk})\boldsymbol{x}_n}{\sum_{n=1}^{N}\gamma(z_{nk})}, \qquad \Sigma_k^* = \frac{\sum_{n=1}^{N}\gamma(z_{nk})(\boldsymbol{x}_n - \mu_k^*)(\boldsymbol{x}_n - \mu_k^*)^T}{\sum_{n=1}^{N}\gamma(z_{nk})}. \tag{3.21}$$

That's it! We can now use the revised value as a new θ, and apply them to the next iteration to obtain the result!

3.1.3 Implementation of this Method in Python

3.2 EAS Data Processing

References

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