Quantitative and Functional Imaging
BME 4420/7450

<u>Image Noise</u>

Suppose you acquire an image and find the image intensity at a given point to be I. This value likely represents a combination of the true (noise-free) value, I_0, and random noise. We can model this with the relation

$$I = I_0 + \varepsilon \tag{1.1}$$

where ε is a random variable with mean value μ and variance σ^2.

The *expectation value* of a random quantity is the mean over an infinite number of samples. It is usually denoted with angle brackets, $\langle \ldots \rangle$, so the expectation value of x is written $\langle x \rangle$. We have

$$\begin{aligned} \mu &= \langle \varepsilon \rangle \\ \sigma^2 &= \left\langle (\varepsilon - \mu)^2 \right\rangle \end{aligned} \tag{1.2}$$

If ε_x and ε_y are two random quantities, their *covariance* is defined as

$$\sigma_{xy}^2 = \left\langle (\varepsilon_x - \mu_x) \cdot (\varepsilon_y - \mu_y) \right\rangle \tag{1.3}$$

where

$$\begin{aligned} \mu_x &= \langle \varepsilon_x \rangle \\ \mu_y &= \langle \varepsilon_y \rangle \end{aligned} \tag{1.4}$$

Random noise introduces errors in signal measurements. These affect quantities calculated from image data. So the question arises, how can we estimate the errors in calculated quantities? It turns out that all we need are 1) knowledge of how the quantities are calculated from images and 2) an estimate of the noise variance.

<u>Propagation of Errors</u>

Consider a function, $f(x, y)$, of two variables that can include some level of random noise (*x* and *y* can represent position or any other measurable quantities). The value of the function at $(x + \varepsilon_x, y + \varepsilon_y)$ near (x, y) is

$$f(x + \varepsilon_x, y + \varepsilon_y) \simeq f(x, y) + \varepsilon_x \cdot \frac{\partial f}{\partial x} + \varepsilon_y \cdot \frac{\partial f}{\partial y} \tag{1.5}$$

using a Taylor Series expansion to first order. If we select random points in a small region around (x, y), the mean of the function will be

$$\left\langle f\left(x+\varepsilon_x, y+\varepsilon_y\right)\right\rangle = f\left(x,y\right)+\left\langle \varepsilon_x\right\rangle \cdot \frac{\partial f}{\partial x}+\left\langle \varepsilon_y\right\rangle \cdot \frac{\partial f}{\partial y} \tag{1.6}$$

If the random variables ε_x and ε_y have zero mean,

$$\left\langle \varepsilon_x\right\rangle = \left\langle \varepsilon_y\right\rangle = 0\,, \tag{1.7}$$

then

$$\left\langle f\left(x+\varepsilon_x, y+\varepsilon_y\right)\right\rangle = f\left(x,y\right) \tag{1.8}$$

In words, the expectation value of the function sampled at the noisy points equals the noise-free value.

The difference in the function's value from the mean is

$$f\left(x+\varepsilon_x, y+\varepsilon_y\right) - f\left(x,y\right) = \varepsilon_x \cdot \frac{\partial f}{\partial x}+\varepsilon_y \cdot \frac{\partial f}{\partial y} \tag{1.9}$$

Squaring both sides of this equation and taking the expectation values gives

$$\left\langle \left[f\left(x+\varepsilon_x, y+\varepsilon_y\right) - f\left(x,y\right)\right]^2\right\rangle = \left\langle \left(\varepsilon_x \cdot \frac{\partial f}{\partial x}+\varepsilon_y \cdot \frac{\partial f}{\partial y}\right)^2\right\rangle \tag{1.10}$$

The quantity on the left-hand side of this equation is the variance of the function values, σ_f^2. Expanding the right-hand side yields

$$\sigma_f^2 = \left\langle \varepsilon_x^2 \cdot \left(\frac{\partial f}{\partial x}\right)^2 + 2\varepsilon_x\varepsilon_y \cdot \left(\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right) + \varepsilon_y^2 \cdot \left(\frac{\partial f}{\partial y}\right)^2 \right\rangle \tag{1.11}$$

If ε_x and ε_y have zero mean, then

$$\sigma_x^2 = \left\langle \varepsilon_x^2\right\rangle, \quad \sigma_y^2 = \left\langle \varepsilon_y^2\right\rangle, \quad \sigma_{xy}^2 = \left\langle \varepsilon_x\varepsilon_y\right\rangle \tag{1.12}$$

In this case,

$$\sigma_f^2 = \sigma_x^2 \cdot \left(\frac{\partial f}{\partial x}\right)^2 + 2\sigma_{xy}^2 \cdot \left(\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\right) + \sigma_y^2 \cdot \left(\frac{\partial f}{\partial y}\right)^2 \tag{1.13}$$

This relation shows how variance in the x and y variables propagates to variance in the function f.

If two random variables are independent, then their covariance is zero. If ε_x and ε_y are independent, then we have

$$\sigma_f^2 = \sigma_x^2 \cdot \left(\frac{\partial f}{\partial x} \right)^2 + \sigma_y^2 \cdot \left(\frac{\partial f}{\partial y} \right)^2 \tag{1.14}$$

Hence, independent sources of noise add “in quadrature” (i.e., as the sum of the squares). This equation shows that the noise in a function *f* comes from the noise in each of its arguments (*x* and *y* in this case), weighted by the change in *f* produced by a small change in the argument (i.e., the partial derivative).