Systems of Equations

- So far we have just looked at a single equation. What if we had a <u>system</u> of equations?
- Suppose we have the set:

$$f_1(x_1, \ldots , x_n) = 0$$

$$\ldots \qquad \text{or} \qquad \mathbf{f}(\mathbf{x}) = \mathbf{0}$$

$$f_n(x_1, \ldots , x_n) = 0$$

Systems of Equations

- We wish to determine $\mathbf{x} = (x_1, \dots, x_n)$ such that $\mathbf{f}(\mathbf{x}) = (f_1, f_2, \dots, f_n) = \mathbf{0}$
- We shall use Newton's method: expand $\mathbf{f}(\mathbf{x})$ in a Taylor series about $\mathbf{x}^{(k)}$.
- Note: $x^{(k)}$ is the k^{th} guess for the root.
- So… (written notes)

So: $f(\underset{\sim}{x}) = \underset{\sim}{f}(\underset{\sim}{x}^{(k)}) + \nabla \underset{\sim}{f}\Big|_{\underset{\sim}{x}^{(k)}} \cdot (\underset{\sim}{x} - \underset{\sim}{x}^{(k)}) + \cdots$

The matrix $\underset{\sim}{\nabla} \underset{\sim}{f}$ is known as the <u>Jacobian</u> of $\underset{\sim}{f}$:

$$\underset{\approx}{J} \equiv \underset{\sim}{\nabla} \underset{\sim}{f}$$

or,

$$\underset{\approx}{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

We truncate the Taylor series after the linear term & set it equal to zero to get the next guess!

$$\underset{\sim}{0} = \underset{\sim}{f}^k + \underset{\approx}{J}^k (\underset{\sim}{x}^{k+1} - \underset{\sim}{x}^k)$$

Thus:

$$\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^k - [\underset{\approx}{J}^k]^{-1} \underset{\sim}{f}^k$$

Actually, this should be solved using Gaussian elimination.

Systems of Nonlinear Equations

- If the Jacobian is singular, Newton's method will fail to converge.
- This is equivalent to the first deriv. $f'(x_k) = 0$ for a one-dimensional Newton's method problem.
- Let's do a simple example:

$$f_1(x_1,x_2) = x_1 x_2 - x_2^3 - 1 = 0$$

$$f_2(x_1,x_2) = x_1^2 x_2 + x_2 - 5 = 0$$

- What is the Jacobian? (handwritten notes…)

Systems of Nonlinear Equations

- The solution will converge to (2,1) after about 8 iterations.
- Note that it doesn't go straight to the solution!
- We <u>started</u> at $x_1 = 2$, which was the <u>solution</u>, but we moved <u>away</u> from it at first!
- Why? → because locally the tangent may not point in the correct direction.

Systems of Nonlinear Equations

- Okay, how do we improve on Newton's method?
- We wish to guard against failure to converge.
- We shall require that:
  1. The algorithm makes progress to the solution at every step.
  2. The steps are never too large.

  Thus we require that:

$$||\mathbf{f}(\mathbf{x}^{(k+1)})||_2 < ||\mathbf{f}(\mathbf{x}^{(k)})||_2$$

  and that:

$$||\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}||_2 < \delta$$

  where δ is picked by the algorithm.

Systems of Nonlinear Equations

- If the truncated Taylor series (including the linear term) is a <u>good</u> approximation to the function, we make δ large, if not we reduce δ.
- Suppose we have:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{p}$$

Correction at step k

- From Newton's method we expect:

$$\mathbf{p} = -(\mathbf{J}^{(k)})^{-1} \mathbf{f}^{(k)}$$

but this may not satisfy $||\mathbf{p}||_2 < \delta$

Systems of Nonlinear Equations

- In addition, if $\mathbf{J}^{(k)}$ is singular, $\mathbf{p}$ may not exist!
- Instead, let's solve the constrained optimization problem:

$$\text{find } \mathbf{p} \text{ that minimizes } ||\mathbf{J}^{(k)}\,\mathbf{p} + \mathbf{f}^{(k)}||_2$$

$$\text{subject to } ||\mathbf{p}||_2 < \delta$$

- This works even for singular $\mathbf{J}^{(k)}$!
- This is a bit different from the linear regression problem. Linear regression is an unconstrained optimization problem, what we have is a constrained problem.
- We'll look into solving this in a later lecture.

Systems of Nonlinear Equations

- If we have such a solver, we would

1. If $\mathbf{f}(\mathbf{x}^{(k)}) = \mathbf{0}$ then stop
2. Calculate $\mathbf{p}$ via constrained optimization solution.
3. If $||\mathbf{f}(\mathbf{x}^{(k)} + \mathbf{p})||_2 < ||\mathbf{f}(\mathbf{x}^{(k)})||_2$ then accept step and go to (1).

   You may also want to increase δ.
4. If not, then reduce δ and go back to step (2).

- This sort of algorithm will reliably find roots, and will be trapped by local minima.

- Key: start closer to the answer!

Optimization (Ch. 8)

- In optimization we are trying to minimize or maximize an objective function, usually subject to a set of constraints.
- This is known as a constrained optimization problem.
- If there are no constraints, then we have an unconstrained optimization problem.
- Example: suppose we are designing soup cans. We want to minimize the metal usage for volume enclosed, e.g., minimize surface/volume ratio.
- Obviously a sphere would do this, but it's difficult to make and open.

Optimization (Ch. 8)

- Instead, we choose a cylindrical shape:

  $$\text{Metal Area} = \underbrace{2\pi r^2}_{\text{2 ends}} + \underbrace{2\pi r h}_{\text{sides}} = A$$

  $$\text{Volume} = \pi r^2 h = V$$

- We wish to minimize A subject to the that V = 16 oz (for instance).
- This is a 2-D nonlinear constrained optimization problem.

Optimization (Ch. 8)

- We can convert this to a 1-D unconstrained problem by recognizing:

  $$h = V/\pi r^2$$

- Therefore, $A = 2\pi r^2 + 2V/r$
- Which has the solution $r = 4.22$, $h = 8.45$ cm.
- Thus, $A = 336$ cm^2.
- Actually, cans don’t fit this ideal, as there are additional constraints such as labeling and packing considerations.

Optimization (Ch. 8)

- We have the general problem:
  minimize $F(\mathbf{x}) = F(x_1, x_2, \ldots, x_n)$
  over some domain S in n-dimensional space.
- For the can problem,
  $$F(x_1,x_2) = 2\pi x_1^2 + 2\pi x_1 x_2$$
  subject to $x \varepsilon S = \{(x_1,x_2) | 2\pi x_1^2 x_2 = V\}$
- If $S = R_N$ (all space) then problem is <u>unconstrained</u>.
- $\mathbf{x} \varepsilon S$ are the <u>feasible</u> solutions.

Optimization (Ch. 8)

- Least squares data fitting is an <u>unconstrained</u> optimization problem.
- If we have some point **x** such that **grad**(F) = **0** then **x** is a <u>critical point</u>.
- It may be a minimum, maximum, or a saddle point!
- We define a point **x*** to be a <u>local minimum</u> if:

$$\mathbf{x}^* \in S \text{ such that } F(\mathbf{x}^*) < F(\mathbf{x}^* + \delta)$$

Optimization (Ch. 8)

- **x*** is a global minimum if

  $F(\mathbf{x}^*) < F(\mathbf{x}), \ \mathbf{x} \in S$ (for all $\mathbf{x}^*$)

  You can't guarantee to find the global minimum!

One-Dimensional Optimization

- There are three basic approaches analogous to the three root finding techniques. They are:
  1. Newton’s Method
  2. Successive Parabolic Interpolation
  3. Golden Search (Fibonacci Search)

Q1: Which of the following is NOT needed to begin using the bisection method?

A. A continuous function f(x)
B. Endpoints of the interval: [a,b]
C. $f(a)*f(b) < 0$
D. Initial slope of f(x)

Q2: If $f(a)f(b) < 0$, that indicates the following:

A. There is one root in the interval.
B. There are no roots in the interval.
C. There is at least one root in the interval.
D. There is a local minimum in the interval.

Q2: If f(a)f(b) < 0, that indicates the following:

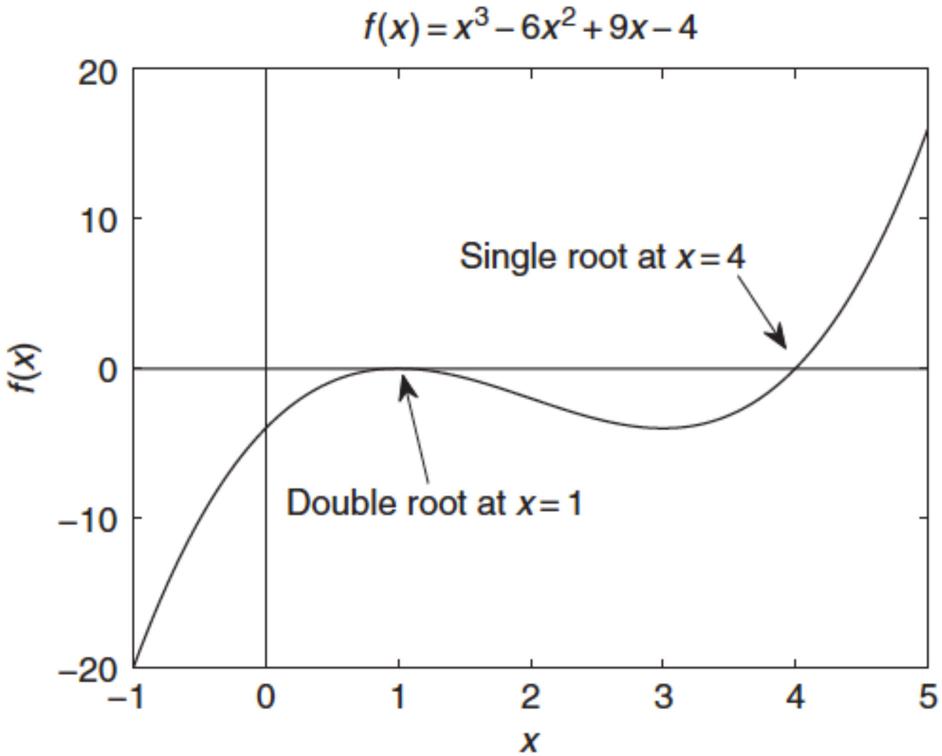
A. There is one root in the interval.

B. There are no roots in the interval.

C. There is at least one root in the interval.

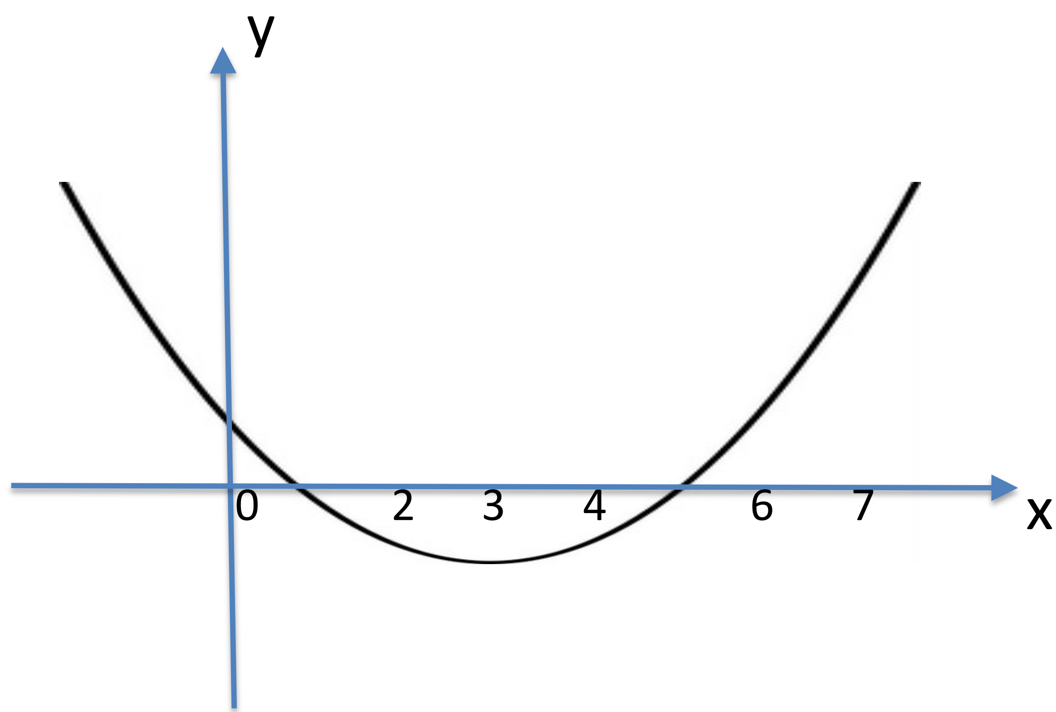
D. There is a local minimum in the interval.

[a=0, b=5]



Q3: At which initial value of x will Newton's method fail to find the root?

A. 0
B. 2
C. 3
D. 6
E. 7



Q4: How would you use the built-in Matlab function fzero to solve for the root of $f(x) = x^4 + e^x - 4$ that is closest to 2.5?

A. fzero(x^4 +exp(x)-4,2.5)
B. fzero(‘x^4+e^x-4’,2.5)
C. fzero(‘x^4+exp(x)-4’,2.5)
D. fzero(“x^4+exp(x)-4”,2.5)
E. fzero(‘x^4+e(x)-4’,2.5)

Poiseuille flow

- **Steady flow in a rigid cylindrical tube**
  - **Pressure gradient** $F_{\mathrm{p}} = 2\pi r(p_1 - p_2)\delta r$
  - **Viscous force** $F_{\mathrm{v}} = -\frac{\partial}{\partial r}(2\pi r L \mu \frac{\partial v}{\partial r})\delta r$

The forces are equal and opposite:

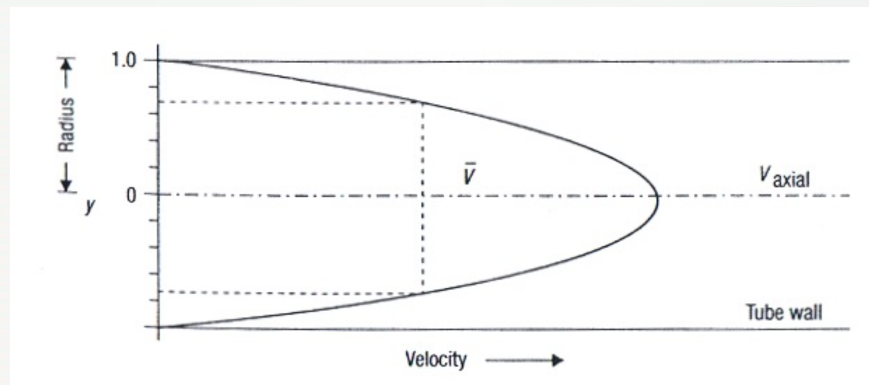
$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{p_1 - p_2}{\mu L} = 0$$

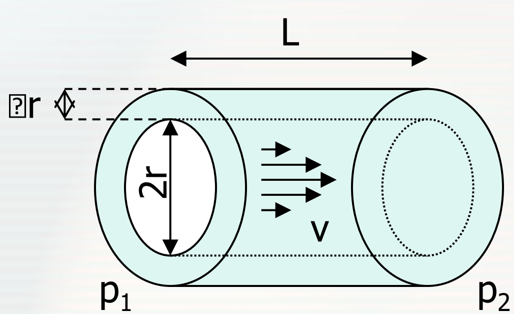
$$v(r) = -r^2 \frac{p_1 - p_2}{4\mu L} + A \ln r + B$$

v(r=R)=0

v(r=0)≠∞

$$\boxed{v(r) = \frac{p_1 - p_2}{4\mu L}(R^2 - r^2)}$$





$$Q = \int_0^R 2\pi v(r) r dr = \pi R^4 \frac{p_1 - p_2}{8\mu L}$$

volume flow

$$\bar{v} = \frac{Q}{\pi R^2} = R^2 \frac{p_1 - p_2}{8\mu L} = \frac{1}{2} v(r=0) = \frac{v_{\max}}{2}$$

average velocity

Physical properties of blood

BLOOD =

plasma + blood cells

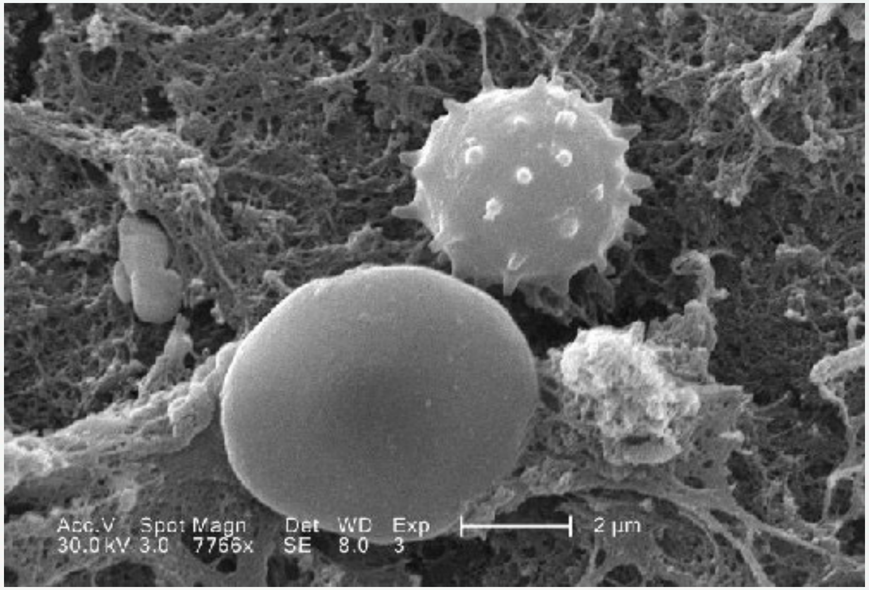
(55%) (45%)

⇩ electrolyte solution containing 8% of proteins

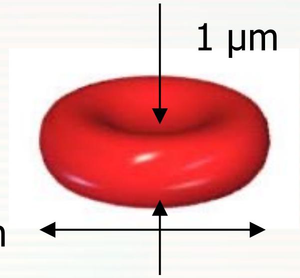
⇩ Red blood cells (95%)

White blood cells (0.13%)

Platelets (4.9%)



RBC:

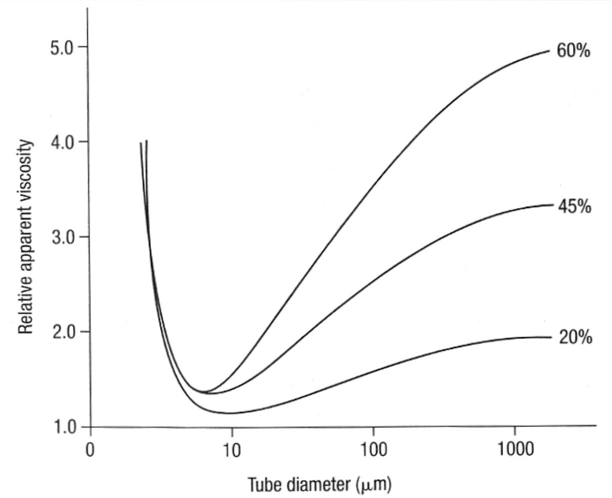


Reference values

	PLASMA	WHOLE BLOOD
density	**1035 kg/m^3**	**1056 kg/m^3**
viscosity	**1.3×10^{-3} Pa s**	**3.5×10^{-3} Pa s**

Viscosity

- **Viscosity varies with samples**
  - **variations in species**
  - **variations in proteins and RBC**
- **Temperature dependent**
  - **decrease with increasing T**



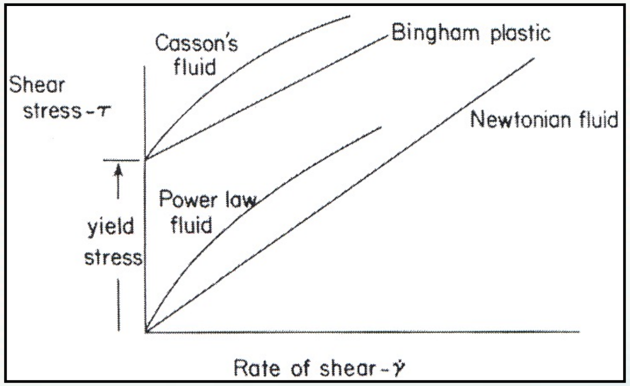
In small tubes the blood viscosity has a very low value because of a cell-free zone near the wall.

⇕

Fahraeus-Lindqvist effect

- **Blood**
  - **a non-Newtonian fluid at low shear rates (the agreggates of RBC)**
  - **a Newtonian fluid above shear rates of 50 s^{-1}**
  - **Casson's equation**

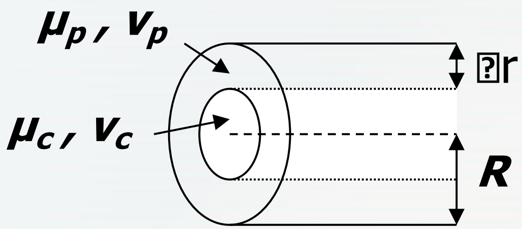
$$\sqrt{\tau} = \sqrt{\tau_0} + K_c \sqrt{dv/dr}$$



Fahraeus-Lindqvist Effect

Cell-free marginal layer model

- Core region μ_c, v_c, 0?r?R-?
- Cell-free plasma region near the wall μ_p, v_p, R-??r? R



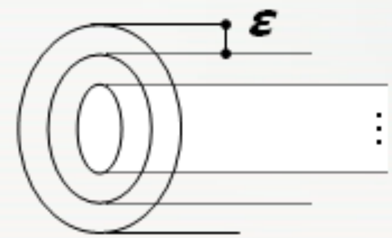
$$-\frac{\Delta p}{L} = \frac{1}{r}\frac{d}{dr}\left(\mu r \frac{dv}{dr}\right)$$

- the volume flow

$$Q = \frac{\pi R^4 \Delta p}{8L} \underbrace{\frac{1}{\mu_p}\left(1-(1-\delta/R)^4(1-\mu_p/\mu_c)\right)}_{1/\mu}$$

The Sigma effect theory

- velocity profile is not continuous
- small tubes (N red blood cells move abreast)



- the volume flow is rewritten

$$Q = \frac{\pi \Delta p}{2\mu L}\int_0^R r^3 dr$$

- N concentric laminae, each of thickness ε

$$Q = \frac{\pi \Delta p}{2\mu L}\sum_{n=1}^{N}(n\varepsilon^3)\varepsilon = \frac{\pi \Delta p R^4}{8L}\underbrace{\frac{1}{\mu}\left(1+\frac{\varepsilon}{R}\right)}_{1/\mu}$$