Advanced Algorithms Analysis and Design

By

Nazir Ahmad Zafar

Lecture No 16

Chain Matrix Multiplication Problem using Dynamic Programming

Today Covered

- Chain-Matrix Multiplication
- Problem Analysis
  - Notations
  - Dynamic Algorithm
  - Time Complexity
- Generalization and Applications
- Conclusion

Problem Statement: Chain Matrix Multiplication

Statement: The chain-matrix multiplication problem can be stated as below:

- Given a chain of $[A_1, A_2, \ldots, A_n]$ of n matrices for $i = 1, 2, \ldots, n$, matrix A_i has dimension $p_{i-1} \times p_i$, find the order of multiplication which minimizes the number of scalar multiplications.

Note:

- Order of A_1 is $p_0 \times p_1$,
- Order of A_2 is $p_1 \times p_2$,
- Order of A_3 is $p_2 \times p_3$, etc.
- Order of $A_1 \times A_2 \times A_3$ is $p_0 \times p_3$,
- Order of $A_1 \times A_2 \times \ldots \times A_n$ is $p_0 \times p_n$

Dynamic Programming Solution

Why Dynamic Programming in this problem?

- Problem is of type optimization
- Sub-problems are dependant
- Optimal structure can be characterized and
- Can be defined recursively
- Solution for base cases exits
- Optimal solution can be constructed
- Hence here is dynamic programming

Dynamic Programming Formulation

- Let $A_{i..j} = A_i \cdot A_{i+1} \cdot \cdots A_j$
- Order of $A_i = p_{i-1} \times p_i$, and
- Order of $A_j = p_{j-1} \times p_j$,
- Order of $A_{i..j}$ = rows in A_i x columns in $A_j = p_{i-1} \times p_j$
- At the highest level of parenthesisation,

  $A_{i..j} = A_{i..k} \times A_{k+1..j}$ $\quad\quad\quad\quad$ $i \leq k < j$
- Let m[i, j] = minimum number of multiplications needed to compute $A_{i..j}$, for $1 \leq i \leq j \leq n$
- Objective function = finding minimum number of multiplications needed to compute $A_{1..n}$ i.e. to compute m[1, n]

Mathematical Model

$A_{i..j} = (A_i . A_{i+1} \cdots A_k) . (A_{k+1} . A_{k+2} \cdots A_j) = A_{i..k} \times A_{k+1 \cdot \cdot j}$

$i \leq k < j$

- Order of $A_{i..k} = p_{i-1} \times p_k$, and order of $A_{k+1..j} = p_k \times p_j$,
- m[i, k] = minimum number of multiplications needed to compute $A_{i..k}$
- m[k+1, j] = minimum number of multiplications needed to compute $A_{k+1..j}$
- Mathematical Model

$$m[i, i] = 0$$

$$m[i, j] = \min_{i \leq k < j} (m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j)$$

Example: Dynamic Programming

- Problem: Compute optimal multiplication order for a series of matrices given below

$$\frac{A_1}{10\times100}\cdot\frac{A_2}{100\times5}\cdot\frac{A_3}{5\times50}\cdot\frac{A_4}{50\times20}$$

$P_0 = 10$

$P_1 = 100$

$P_2 = 5$

$P_3 = 50$

$P_4 = 20$

m[1,1]	m[1,2]	m[1,3]	m[1,4]
	m[2,2]	m[2,3]	m[2,4]
		m[3,3]	m[3,4]
			m[4,4]

Main Diagonal

$$m[i,i] = 0, \forall i = 1,\ldots,4$$

$$m[i,j] = \min_{i \le k < j} (m[i,k] + m[k+1,j] + p_{i-1} \cdot p_k \cdot p_j)$$

Main Diagonal

- m[1, 1] = 0
- m[2, 2] = 0
- m[3, 3] = 0
- m[4, 4] = 0

Computing m[1, 2], m[2, 3], m[3, 4]

$$m[i,\ j] = \min_{i \le k < j} (m[i,k] + m[k+1,\ j] + p_{i-1} . p_k . p_j)$$

$$m[1,2] = \min_{1 \le k < 2} (m[1,k] + m[k+1,2] + p_0 . p_k . p_2)$$

$$m[1,2] = \min (m[1,1] + m[2,2] + p_0 . p_1 . p_2)$$

m[1, 2] = 0 + 0 + 10 . 100 . 5

= 5000

s[1, 2] = k = 1

Computing m[2, 3]

$$m[i,\ j] = \min_{i \le k < j} (m[i,k] + m[k+1,\ j] + p_{i-1}.p_k.p_j)$$

$$m[2,3] = \min_{2 \le k < 3} (m[2,k] + m[k+1,3] + p_1.p_k.p_3)$$

$$m[2,3] = \min (m[2,2] + m[3,3] + p_1.p_2.p3)$$

m[2, 3] = 0 + 0 + 100 . 5 . 50

= 25000

s[2, 3] = k = 2

Computing m[3, 4]

$$m[i,\ j] = \min_{i \le k < j} (m[i,k] + m[k+1,\ j] + p_{i-1}.p_k.p_j)$$

$$m[3,4] = \min_{3 \le k < 4} (m[3,k] + m[k+1,4] + p_2.p_k.p_4)$$

$$m[3,4] = \min (m[3,3] + m[4,4] + p_2.p_3.p_4)$$

m[3, 4] = 0 + 0 + 5 . 50 . 20

= 5000

s[3, 4] = k = 3

Computing m[1, 3], m[2, 4]

$$m[i,\ j] = \min_{i \le k < j} (m[i,k] + m[k+1,\ j] + p_{i-1} \cdot p_k \cdot p_j)$$

$$m[1,3] = \min_{1 \le k < 3} (m[1,k] + m[k+1,3] + p_0 \cdot p_k \cdot p_3)$$

$$m[1,3] = \min (m[1,1] + m[2,3] + p_0 \cdot p_1 \cdot p_3,$$

$$m[1,2] + m[3,3] + p_0 \cdot p_2 \cdot p_3))$$

m[1, 3] = min(0+25000+10.100.50, 5000+0+10.5.50)
= min(75000, 2500) = 2500

s[1, 3] = k = 2

Computing m[2, 4]

$$m[i,\ j] = \min_{i \le k < j} (m[i,k] + m[k+1,\ j] + p_{i-1} \cdot p_k \cdot p_j)$$

$$m[2,4] = \min_{2 \le k < 4} (m[2,k] + m[k+1,4] + p_1 \cdot p_k \cdot p_4)$$

$$m[2,4] = \min (m[2,2] + m[3,4] + p_1 \cdot p_2 \cdot p_4,$$

$$m[2,3] + m[4,4] + p_1 \cdot p_3 \cdot p_4))$$

m[2, 4] = min(0+5000+100.5.20, 25000+0+100.50.20)

= min(15000, 35000) = 15000

s[2, 4] = k = 2

Computing m[1, 4]

$$m[i,\ j] = \min_{i \le k < j} (m[i,k] + m[k+1,\ j] + p_{i-1} \cdot p_k \cdot p_j)$$

$$m[1,4] = \min_{1 \le k < 4} (m[1,k] + m[k+1,4] + p_0 \cdot p_k \cdot p_4)$$

$$m[1,4] = \min (m[1,1] + m[2,4] + p_0 \cdot p_1 \cdot p_4,$$

$$m[1,2] + m[3,4] + p_0 \cdot p_2 \cdot p_4, m[1,3] + m[4,4] + p_0 \cdot p_3 \cdot p_4)$$

m[1, 4] = min(0+15000+10.100.20, 5000+5000+
10.5.20, 2500+0+10.50.20)

= min(35000, 11000, 35000) = 11000

s[1, 4] = k = 2

Final Cost Matrix and Its Order of Computation

- Final Cost Matrix

0	5000	2500	11000
	0	25000	15000
		0	5000
			0

- Order of Computation

1	5	8	10
	2	6	9
		3	7
			4

K,s Values Leading Minimum m[i, j]

0	1	2	2
	0	2	2
		0	3
			0

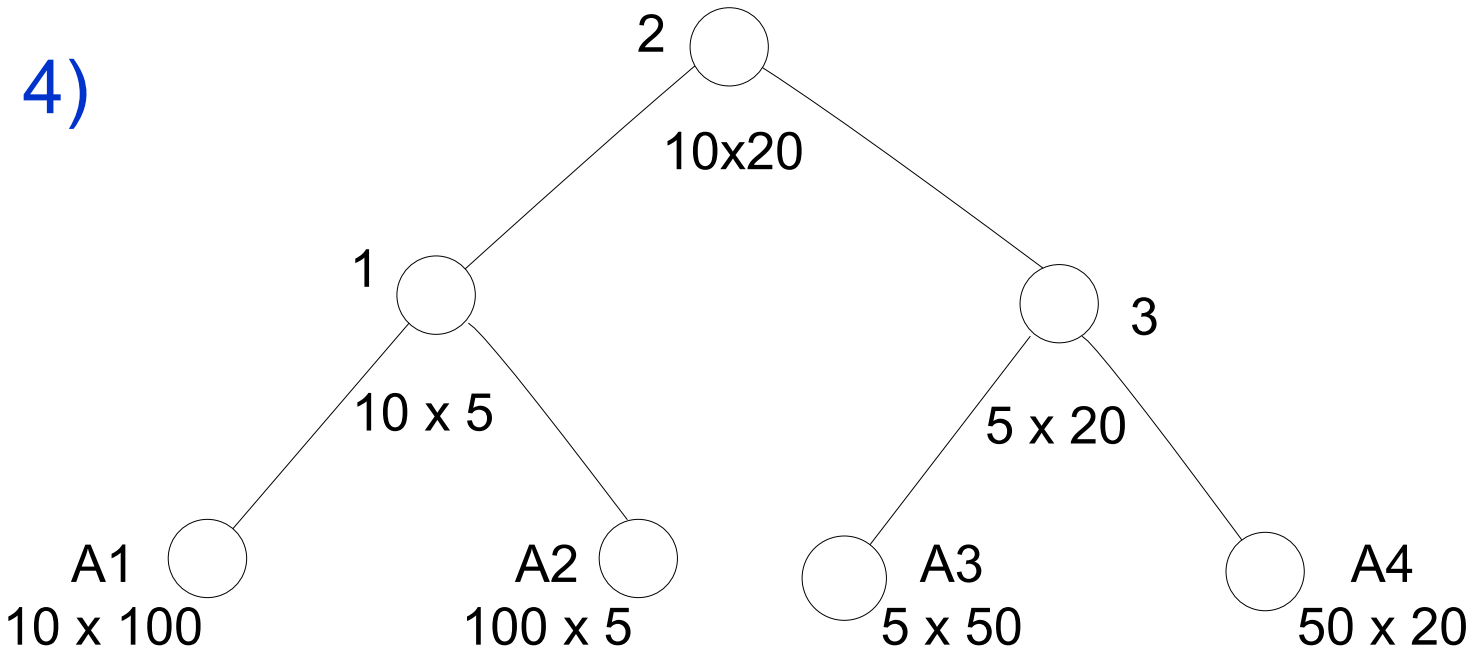
Representing Order using Binary Tree

- The above computation shows that the minimum cost for multiplying those four matrices is 11000.
- The optimal order for multiplication is

$((A_1 . A_2) . (A_3 . A_4))$

For, m(1, 4)

k = 2



Chain-Matrix-Order(p)

```
1.  n ← length[p] – 1
2.  for i ← 1 to n
3.      do m[i, i] ← 0
4.  for l  ← 2 to n,
5.      do for i ← 1 to n-l+1
6.          do j ← i+l-1
7.              m[i, j] ← ∞
8.              for k ← i to j-1
9.                  do q ← m[i, k] + m[k+1, j] + p_{i-1} . p_k . p_j
10.                     if q < m[i, j]
11.                         then m[i, j] = q
12.                             s[i, j] ← k
13. return m and s,                    "l is chain length"
```

m[1,1]	m[1,2]	m[1,3]	m[1,4]
	m[2,2]	m[2,3]	m[2,4]
		m[3,3]	m[3,4]
			m[4,4]

Computational Cost

$$T(n) = n + \sum_{i=1}^{n} \sum_{j=i+1}^{n} (j - i) = \sum_{i=1}^{n} \sum_{k=1}^{n-i} k$$

$$T(n) = n + \sum_{i=1}^{n} \frac{(n-i)(n-i+1)}{2}$$

$$T(n) = n + \frac{1}{2} \sum_{i=1}^{n} (n^2 - 2ni + i^2 + n - i)$$

$$T(n) = n + \frac{1}{2} (\sum_{i=1}^{n} n^2 - \sum_{i=1}^{n} 2ni + \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} n - \sum_{i=1}^{n} i)$$

Computational Cost

$$T(n) = n + \frac{1}{2}(\sum_{i=1}^{n} n^2 - \sum_{i=1}^{n} 2ni + \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} n - \sum_{i=1}^{n} i)$$

$$T(n) = n + \frac{1}{2}(n^2 \sum_{i=1}^{n} 1 - 2n \sum_{i=1}^{n} i + \sum_{i=1}^{n} i^2 + n \sum_{i=1}^{n} 1 - \sum_{i=1}^{n} i)$$

$$T(n) = n + \frac{1}{2}(n^2.n - 2n.\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} + n.n - \frac{n(n+1)}{2})$$

Computational Cost

$$T(n) = n + \frac{1}{2}(n^2.n - 2n.\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6} + n.n - \frac{n(n+1)}{2})$$

$$T(n) = n + \frac{1}{2}(n^3 - n^2(n+1) + \frac{n(n+1)(2n+1)}{6} + n^2 - \frac{n(n+1)}{2})$$

$$T(n) = n + \frac{1}{12}(6n^3 - 6n^3 - 6n^2 + 2n^3 + 3n^2 + n + 6n^2 - 3n^2 - 3n)$$

$$T(n) = \frac{1}{12}(12n + 2n^3 - 2n) = \frac{1}{12}(10n + 2n^3) = \frac{1}{6}(5n + n^3)$$

Cost Comparison Brute Force Dynamic Programming

Dynamic Programming

There are three loop

- The most two loop for i, j, satisfy the condition:
  $1 \leqslant i \leqslant j \leqslant n$
- Cost = ${}^{n}C_2 + n = n(n-1)/2 + n = \Theta(n^2)$
- The third one most inner loop for k satisfies the condition, $i \leqslant k < j$, in worst case, it cost n and
- Hence total cost = $\Theta(n^2 . n) = \Theta(n^3)$

Brute Force Approach

- $P(n) = C(n - 1)\ C(n) \in (4^n/n^{3/2})$

Generalization: Sequence of Objects

- Although this algorithm applies well to the problem of matrix chain multiplication
- Many researchers have noted that it generalizes well to solving a more abstract problem
  - given a linear sequence of objects
  - an associative binary operation on those objects hold
  - the objective to find a way to compute the cost of performing that operation on any two given objects
  - and finally computing the minimum cost for grouping these objects to apply the operation over the entire sequence.
- It is obvious that this problem can be solved using chain matrix multiplication, because there is a one to one correspondence between both problem

Generalization: String Concatenation

- One common special case of chain matrix multiplication problem is string concatenation.
- For example, we are give a list of strings.
  - The cost of concatenating two strings of length m and n is for example O(m + n)
  - Since we need O(m) time to find the end of the first string and O(n) time to copy the second string onto the end of it.
  - Using this cost function, we can write a dynamic programming algorithm to find the fastest way to concatenate a sequence of strings
  - It is possible to concatenate all in time proportional to sum of their lengths, but here we are interested to link this problem with chain matrix multiplication

Generalization: Parallel Processors

- Another generalization is to solve the problem when many parallel processors are available.
- In this case, instead of adding the costs of computing each subsequence, we just take the maximum, because we can do them both simultaneously.
- This can drastically affect both the minimum cost and the final optimal grouping
- But of course more balanced groupings that keep all the processors busy is more favorable solution
- There exists some more sophisticated approaches to solve this problem

Conclusion

- Created some notations to describe mathematical model of the chain matrix multiplication problem
- A recursive model was described
- Based on this model dynamic programming algorithm was designed
- An example was taken for applying model to solve dynamically, constructing optimal solution based on the given information
- Time complexity was computed for the Algorithm
- Applications of chain matrix problem are discussed