Advanced Algorithms Analysis and Design

By

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Lecture No 11

Relations Over Asymptotic Notations

Overview of Previous Lecture

Although Estimation but Useful

- It is not always possible to determine behaviour of an algorithm using Θ-notation.
- For example, given a problem with n inputs, we may have an algorithm to solve it in $a.n^2$ time when n is even and c.n time when n is odd. OR
- We may prove that an algorithm never uses more than $e.n^2$ time and never less than f.n time.
- In either case we can neither claim $\Theta(n)$ nor $\Theta(n^2)$ to be the order of the time usage of the algorithm.
- Big O and Ω notation will allow us to give at least partial information

Reflexive Relation

Definition:

- Let X be a non-empty set and R is a relation over X then R is said to be reflexive if

  $(a, a) \in R, \forall a \in X,$

Example 1:

- Let G be a graph. Let us define a relation R over G as if node x is connected to y then $(x, y) \in G$. Reflexivity is satisfied over G if for every node there is a self loop.

Example 2:

- Let P be a set of all persons, and S be a relation over P such that if $(x, y) \in S$ then x has same birthday as y.
- Of course this relation is reflexive because

  $(x, x) \in S, \quad \forall a \in P,$

Reflexivity Relations over Θ, Ω, O

Example 1

Since, $0 \le f(n) \le cf(n) \quad \forall\ n \ge n_0 = 1, \quad \text{if } c = 1$

Hence $f(n) = O(f(n))$

Example 2

Since, $0 \le cf(n) \le f(n) \quad \forall\ n \ge n_0 = 1, \quad \text{if } c = 1$

Hence $f(n) = \Omega(f(n))$

Example 3

Since, $0 \le c_1 f(n) \le f(n) \le c_2 f(n) \quad \forall\ n \ge n_0 = 1, \text{if } c_1 = c_2 = 1$

Hence $f(n) = \Theta(f(n))$

Note: All the relations, Θ, Ω, *O, are reflexive*

Little o and ω are not Reflexivity Relations

Example

As we can not prove that $f(n) < f(n)$, for any n, and for all $c > 0$

Therefore

1. $f(n) \neq o(f(n))$ and
2. $f(n) \neq \omega(f(n))$

Note :

Hence small o and small omega are not reflexive relations

Symmetry

Definition:

- Let X be a non-empty set and R is a relation over X then R is said to be symmetric if

  $\forall$ a, b $\in$ X, (a, b) $\in$ R $\Rightarrow$ (b, a) $\in$ R

Example 1:

- Let P be a set of persons, and S be a relation over P such that if (x, y) $\in$ S then x has the same sign as y.
- This relation is symmetric because

  (x, y) $\in$ S $\Rightarrow$ (y, x) $\in$ S

Example 2:

- Let P be a set of all persons, and B be a relation over P such that if (x, y) $\in$ B then x is brother of y.
- This relation is not symmetric because

  (Anwer, Sadia) $\in$ B $\Rightarrow$ (Saida, Brother) $\notin$ B

Symmetry over Θ

Property : prove that

$$f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$$

Proof

- Since $f(n) = \Theta(g(n))$ i.e. $f(n) \in \Theta(g(n)) \Rightarrow$
  $\exists$ constants $c_1, c_2 > 0$ and $n_0 \in N$ such that
  $0 \le c_1 g(n) \le f(n) \le c_2 g(n) \quad \forall\, n \ge n_0 \qquad (1)$

$(1) \Rightarrow 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \Rightarrow 0 \le f(n) \le c_2 g(n)$

$\Rightarrow 0 \le (1/c_2) f(n) \le g(n) \qquad (2)$

$(1) \Rightarrow 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \Rightarrow 0 \le c_1 g(n) \le f(n)$

$\Rightarrow 0 \le g(n) \le (1/c_1) f(n) \qquad (3)$

Symmetry over Θ

From (2),(3): $0 \le (1/c_2)f(n) \le g(n) \wedge 0 \le g(n) \le (1/c_1)f(n)$

$\Rightarrow\ 0 \le (1/c_2)f(n) \le g(n) \le (1/c_1)f(n)$

Suppose that $1/c_2 = c_3$, and $1/c_1 = c_4$,

Now the above equation implies that

$$0 \le c_3 f(n) \le g(n) \le c_4 f(n),\ \forall\ n \ge n_0$$

$\Rightarrow g(n) = \Theta(f(n)),\ \forall\ n \ge n_0$

Hence it proves that,

$$f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$$

Exercise:

prove that big O, big omega Ω, little ω, and little o, do not satisfy the symmetry property.

Transitivity

Definition:

- Let X be a non-empty set and R is a relation over X then R is said to be transitive if

  $\forall$ a, b, c $\in$ X, (a, b) $\in$ R $\wedge$ (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R

Example 1:

- Let P be a set of all persons, and B be a relation over P such that if (x, y) $\in$ B then x is brother of y.
- This relation is transitive this is because

  (x, y) $\in$ B $\wedge$ (y, z) $\in$ B $\Rightarrow$ (x, z) $\in$ B

Example 2:

- Let P be a set of all persons, and F be a relation over P such that if (x, y) $\in$ F then x is father of y.
- Of course this relation is not a transitive because if (x, y) $\in$ F $\wedge$ (y, z) $\in$ F $\Rightarrow$ (x, z) $\notin$ F

Transitivity Relation over Θ, Ω, O, o and ω

Prove the following

1. $f(n) = \Theta(g(n))$ & $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$
2. $f(n) = O(g(n))$ & $g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$
3. $f(n) = \Omega(g(n))$ & $g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$
4. $f(n) = o\,(g(n))$ & $g(n) = o\,(h(n)) \Rightarrow f(n) = o\,(h(n))$
5. $f(n) = \omega(g(n))$ & $g(n) = \omega(h(n)) \Rightarrow f(n) = \omega(h(n))$

Note

It is to be noted that all these algorithms complexity measuring notations are in fact relations which satisfy the transitive property.

Transitivity Relation over Θ

Property 1

$$f(n) = \Theta(g(n)) \;\&\; g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$$

Proof

- Since $f(n) = \Theta(g(n))$ i.e. $f(n) \in \Theta(g(n)) \Rightarrow$
  $\exists$ constants $c_1, c_2 > 0$ and $n_{01} \in N$ such that
  $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \forall\, n \geq n_{01} \qquad (1)$

2. Now since $g(n) = \Theta(h(n))$ i.e. $g(n) \in \Theta(h(n)) \Rightarrow$
   $\exists$ constants $c_3, c_4 > 0$ and $n_{02} \in N$ such that
   $0 \leq c_3 h(n) \leq g(n) \leq c_4 h(n) \quad \forall\, n \geq n_{02} \qquad (2)$
3. Now let us suppose that $n_0 = \max(n_{01}, n_{02})$

Transitivity Relation over Θ

4. Now we have to show that $f(n) = \Theta(h(n))$ i.e. we have to prove that

   $\exists$ constants $c_5, c_6 > 0$ and $n_0 \in N$ such that

   $0 \leq c_5 h(n) \leq f(n) \leq c_6 h(n)$?

$(2) \Rightarrow 0 \leq c_3 h(n) \leq g(n) \leq c_4 h(n)$

$\Rightarrow 0 \leq c_3 h(n) \leq g(n)$ (3)

$(1) \Rightarrow 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$

$\Rightarrow 0 \leq c_1 g(n) \leq f(n)$

$\Rightarrow 0 \leq g(n) \leq (1/c_1) f(n)$ (4)

From (3) and (4), $0 \leq c_3 h(n) \leq g(n) \leq (1/c_1) f(n)$

$\Rightarrow 0 \leq c_1 c_3 h(n) \leq f(n)$ (5)

Transitivity Relation over Θ

$(1) \Rightarrow 0 \le c_1 g(n) \le f(n) \le c_2 g(n)$

$\Rightarrow 0 \le f(n) \le c_2 g(n) \Rightarrow 0 \le (1/c_2) f(n) \le g(n) \qquad (6)$

$(2) \Rightarrow 0 \le c_3 h(n) \le g(n) \le c_4 h(n)$

$\Rightarrow 0 \le g(n) \le c_4 h(n) \qquad (7)$

From (6) and (7), $0 \le (1/c_2) f(n) \le g(n) \le (c_4) h(n)$

$\Rightarrow 0 \le (1/c_2) f(n) \le (c_4) h(n)$

$\Rightarrow 0 \le f(n) \le c_2 c_4 h(n) \qquad (8)$

From (5), (8), $0 \le c_1 c_3 h(n) \le f(n) \wedge 0 \le f(n) \le c_2 c_4 h(n)$

$0 \le c_1 c_3 h(n) \le f(n) \le c_2 c_4 h(n)$

$0 \le c_5 h(n) \le f(n) \le c_6 h(n)$

And hence $f(n) = \Theta(h(n)) \qquad \forall\, n \ge n_0$

Transitivity Relation over Big O

Property 2

$$f(n) = O(g(n)) \; \& \; g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$$

Proof

- Since $f(n) = O(g(n))$ i.e. $f(n) \in O(g(n)) \Rightarrow$

  $\exists$ constants $c_1 > 0$ and $n_{01} \in N$ such that

  $0 \leq f(n) \leq c_1 g(n) \quad \forall\, n \geq n_{01} \qquad (1)$

2. Now since $g(n) = O(h(n))$ i.e. $g(n) \in O(h(n)) \Rightarrow$

   $\exists$ constants $c_2 > 0$ and $n_{02} \in N$ such that

   $0 \leq g(n) \leq c_2 h(n) \quad \forall\, n \geq n_{02} \qquad (2)$

3. Now let us suppose that $n_0 = \max(n_{01}, n_{02})$

Transitivity Relation over Big O

Now we have to two equations

$$0 \le f(n) \le c_1 g(n) \qquad \forall\ n \ge n_{01} \qquad (1)$$

$$0 \le g(n) \le c_2 h(n) \qquad \forall\ n \ge n_{02} \qquad (2)$$

$$(2) \Rightarrow 0 \le c_1 g(n) \le c_1 c_2 h(n) \qquad \forall\ n \ge n_{02} \qquad (3)$$

From (1) and (3)

$$0 \le f(n) \le c_1 g(n) \le c_1 c_2 h(n)$$

Now suppose that $c_3 = c_1 c_2$

$$0 \le f(n) \le c_1 c_2 h(n)$$

And hence $f(n) = O(h(n)) \qquad \forall\ n \ge n_0$

Transitivity Relation over Big Ω

Property 3

$$f(n) = \Omega(g(n)) \;\&\; g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$$

Proof

- Since $f(n) = \Omega(g(n)) \Rightarrow$

  $\exists$ constants $c_1 > 0$ and $n_{01} \in N$ such that

  $0 \le c_1 g(n) \le f(n) \quad \forall\, n \ge n_{01} \quad (1)$

2. Now since $g(n) = \Omega(h(n)) \Rightarrow$

   $\exists$ constants $c_2 > 0$ and $n_{02} \in N$ such that

   $0 \le c_2 h(n) \le g(n) \quad \forall\, n \ge n_{02} \quad (2)$

3. Suppose that $n_0 = \max(n_{01}, n_{02})$

Transitivity Relation over Big Ω

4. We have to show that $f(n) = \Omega(h(n))$ i.e. we have to prove that

   $\exists$ constants $c_3 > 0$ and $n_0 \in N$ such that

   $0 \le c_3 h(n) \le f(n) \quad \forall\, n \ge n_0 \quad ?$

$(2) \Rightarrow 0 \le c_2 h(n) \le g(n)$

$(1) \Rightarrow 0 \le c_1 g(n) \le f(n)$

$\Rightarrow 0 \le g(n) \le (1/c_1) f(n) \qquad (3)$

From (2) and (3), $0 \le c_2 h(n) \le g(n) \le (1/c_1) f(n)$

$\Rightarrow 0 \le c_1 c_2 h(n) \le f(n)$ hence $f(n) = \Omega(h(n)), \forall\, n \ge n_0$

Transitivity Relation over *little o*

Property 4

$$f(n) = o(g(n)) \;\&\; g(n) = o(h(n)) \Rightarrow f(n) = o(h(n))$$

Proof

- Since $f(n) = o(g(n))$ i.e. $f(n) \in o(g(n)) \Rightarrow$
  $\exists$ constants $c_1 > 0$ and $n_{01} \in N$ such that
  $0 \leq f(n) < c_1 g(n) \quad \forall\, n \geq n_{01} \qquad (1)$

2. Now since $g(n) = o(h(n))$ i.e. $g(n) \in o(h(n)) \Rightarrow$
   $\exists$ constants $c_2 > 0$ and $n_{02} \in N$ such that
   $0 \leq g(n) < c_2 h(n) \quad \forall\, n \geq n_{02} \qquad (2)$

3. Now let us suppose that $n_0 = \max(n_{01}, n_{02})$

Transitivity Relation over *little o*

Now we have to two equations

$$0 \le f(n) < c_1 g(n) \qquad \forall\, n \ge n_{01} \qquad (1)$$

$$0 \le g(n) < c_2 h(n) \qquad \forall\, n \ge n_{01} \qquad (2)$$

$(2) \Rightarrow 0 \le c_1 g(n) < c_1 c_2 h(n) \qquad \forall\, n \ge n_{02} \qquad (3)$

From (1) and (3)

$$0 \le f(n) \le c_1 g(n) < c_1 c_2 h(n)$$

Now suppose that $c_3 = c_1 c_2$

$$0 \le f(n) < c_1 c_2 h(n)$$

And hence $f(n) = o(h(n)) \qquad \forall\, n \ge n_{01}$

Transitivity Relation over little ω

Property 5

$$f(n) = \omega(g(n)) \;\&\; g(n) = \omega(h(n)) \Rightarrow f(n) = \omega(h(n))$$

Proof

- Since $f(n) = \omega(g(n)) \Rightarrow$

  $\exists$ constants $c_1 > 0$ and $n_{01} \in N$ such that

  $0 \le c_1 g(n) < f(n) \qquad \forall\, n \ge n_{01} \qquad (1)$

2. Now since $g(n) = \omega(h(n)) \Rightarrow$

   $\exists$ constants $c_2 > 0$ and $n_{02} \in N$ such that

   $0 \le c_2 h(n) < g(n) \qquad \forall\, n \ge n_{02} \qquad (2)$

3. Suppose that $n_0 = \max(n_{01}, n_{02})$

Transitivity Relation over little ω

4. We have to show that $f(n) = \omega(h(n))$ i.e. we have to prove that

   $\exists$ constants $c_3 > 0$ and $n_0 \in N$ such that

   $0 \leq c_3 h(n) \leq f(n) \quad \forall\, n \geq n_0 \quad ?$

$(2) \Rightarrow 0 \leq c_2 h(n) < g(n)$

$(1) \Rightarrow 0 \leq c_1 g(n) < f(n)$

$\Rightarrow 0 \leq g(n) < (1/c_1) f(n) \qquad (3)$

From (2) and (3), $0 \leq c_2 h(n) \leq g(n) < (1/c_1) f(n)$

$\Rightarrow 0 \leq c_1 c_2 h(n) < f(n)$ hence $f(n) = \omega(h(n))$, $\forall\, n \geq n_0$

Transpose Symmetry

Property 1

Prove that $f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$

Proof

Since $f(n) = O(g(n)) \Rightarrow$

$\exists$ constants $c > 0$ and $n_0 \in N$ such that

$0 \leq f(n) \leq cg(n) \qquad \forall\ n \geq n_0$

Dividing both side by c

$0 \leq (1/c)f(n) \leq g(n)\ \forall\ n \geq n_0$

Put $1/c = c'$

$0 \leq c'f(n) \leq g(n) \qquad \forall\ n \geq n_0$

Hence, $g(n) = \Omega(f(n))$

Transpose Symmetry

Property 2

Prove that $f(n) = o(g(n)) \Leftrightarrow g(n) = \omega f(n))$

Proof

Since $f(n) = o(g(n)) \Rightarrow$

$\exists$ constants $c > 0$ and $n_0 \in N$ such that

$0 \leq f(n) < cg(n) \qquad \forall\, n \geq n_0$

Dividing both side by c

$0 \leq (1/c)f(n) < g(n) \qquad \forall\, n \geq n_0$

Put $1/c = c'$

$0 \leq c'f(n) < g(n) \qquad \forall\, n \geq n_0$

Hence, $g(n) = \omega(f(n))$

Relation between Θ, Ω , O

Trichotmy property over real numbers

- For any two real numbers *a* and *b*, exactly one of the following must hold: $a < b$, $a = b$, or $a > b$.
- The asymptotic comparision of two functions *f* and *g* and the comparision of two real numbers *a* and *b*.

Trichotmy property over Θ, Ω and *O*

1. $f(n) = O(g(n)) \approx a \le b$
2. $f(n) = \Omega(g(n)) \approx a \ge b$
3. $f(n) = \Theta(g(n)) \approx a = b$
4. $f(n) = o(g(n)) \approx a < b$
5. $f(n) = \omega(g(n)) \approx a > b$

Some Other Standard Notations

Monotonicity

- monotonically increasing if $m \leq n \Rightarrow f(m) \leq f(n)$.
- monotonically decreasing if $m \geq n \Rightarrow f(m) \geq f(n)$.
- strictly increasing if $m < n \Rightarrow f(m) < f(n)$.
- strictly decreasing if $m > n \Rightarrow f(m) > f(n)$.

Polynomials

- Given a positive integer d, a polynomial in n of degree d is a function of the form given below, a_i are coefficient of polynomial.

$$p(n) = \sum_{i=0}^{d} a_i n^i$$

Standard Logarithms Notations

Some Definitions

Exponent

- $x = \log_b a$ is the exponent for $a = b^x$.

Natural log

- $\ln a = \log_e a$

Binary log

- $\lg a = \log_2 a$

Square of log

- $\lg^2 a = (\lg a)^2$

Log of Log

- $\lg \lg a = \lg (\lg a)$

Standard Logarithms Notations

$$a = b^{\log_b a}$$

$$\log_c(ab) = \log_c a + \log_c b$$

$$\log_b a^n = n\log_b a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_b(1/a) = -\log_b a$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_b c} = c^{\log_b a}$$